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# Pseudo-Riemann's quartics in Finsler's geometry—two-dimensional case

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**Abstract.** Finsler's geometry usually describes an extension of Riemmann's geometry into a direction-dependent geometric structure. Historically, the well-known Riemann's quartic length element example served as the inspiration for this construction. Surprisingly, the covariant Fresnel equation—a fundamental dispersion relation in solid-state electrodynamics—emerges as the exact same quartic expression. As a result, Riemann's quartic length expression can be regarded of as a mathematical representation of a well-known physical phenomenon. In this study, we offer numerous Riemann's quartic examples that show Finsler's geometry, even in the situation of a positive definite Euclidean signature space, is too restrictive for many applications. The strong axioms of Finsler's geometry are violated in a substantially greater number of distinctive subsets for the spaces having an indefinite (Minkowski) signature. We suggest a weaker characterization of Finsler's structure based on explicitly calculated two-dimensional examples. In tangential vector space, this concept permits singular subsets. Only open subsets of a manifold's tangent bundle are required to satisfy the strong axioms of Finsler's geometry. We demonstrate the distinctive unique subsets of the Riemann's quartic in two dimensions and briefly discuss their possible physical origins.

## 1. Introduction

In 1854, Riemann developed his theory of higher dimension geometry and delivered in 1854 his famous Habilitation lecture at Göttingen entitled *"Über die Hypothesen welche der Geometrie zu Grunde liegen"* (On the hypotheses which underlie geometry). His advisor and examiner, Gauss, was greatly impressed by the ideas presented in this work. The paper was only published by Dedekind in 1868, two years after Riemann's death.

Riemann wrote: *"ds is the square root of an always positive integral homogeneous function of the second order of the quantities dx."* In other words, Riemann's construction is started with the quadratic length element

$$ds=\sqrt{g_{ij}(x)dx^idx^j}\,, \tag{1}$$

where $g_{ij}(x)$ is a symmetric positive definite tensor. Nowadays this length element serves as a well-known basis of Riemannian geometry. Its indefinite version with the tensor $g_{ij}(x)$ of the Lorentz signature provides a mathematical background of General Relativity.

In his Habilitation lecture, Riemann continued: *"…The next case in simplicity includes those manifolds in which the line-element may be expressed as the fourth root of a quartic differential expression. The investigation of this more general kind would require no really different principles, but would take considerable time and throw little new light on the theory of* *space, especially as the results cannot be geometrically expressed, I restrict myself, therefore, to those manifolds in which the line-element is expressed as the square root of a quadratic differential expression..."* We denote the quartic Riemann's expression as

$$ds = \sqrt[4]{M_{ijkl}(x)dx^i dx^j dx^k dx^l}\,. \tag{2}$$

The well-known extension of the quartic Riemann's line element is Finler's geometry that deals with a general expression for the line elements

$$ds = F(x, dx). \tag{3}$$

Here $F(x, dx)$ is an arbitrary smooth positive function that is first-order homogeneous in its second argument. After intensive study of Finsler, Cartan, Chern, and many others, this subject turned into a solid mathematical theory, see [1], [2] and the references given therein. The classical Finsler's geometry is restricted to the positive defined case, i.e., to the Euclidean signature.

Various physics applications motivate study of Finsler geometry in the *pseudo-Euclidean* domain. The well-known example from classical physics is the Randers metric that can be viewed as an attempt of geometrization of the Lagrangian of a test electric charge. Possible pseudo-Finsler modifications of gravity were discussed in [6]. Also in quantum field theories, Finsler's geometry is a rather popular construction. In particular, in [8], [9] pseudo-Finsler geometry is applied as a Lorentz symmetry violating model. In [10] Finsler-type modification of the Coulomb law and corresponding additional splitting of hydrogen energy levels was calculated. This result provides a way for an experimental verification of Finsler's structure. Another example is the covariant premetric dispersion relation in electromagnetism [5] that is presented as a general fourth order polynomial equation of the form

$$G^{ijkl} q_i q_j q_k q_l = 0, \tag{4}$$

where $q_i$ is the wave covector, while the tensor $G^{ijkl}$ is constructed from the characteristic parameters of the media such as *electric permittivity* and *magnetic impermeability.* Due to the well-observed birefringence effect in anisotropic optics, the covariant premetric electromagnetism must be considered as a viable theory. Remarkably, the covariant premetric dispersion relation has exactly the same form as the one exemplified by Riemann in Eq.(2).

This paper is aimed to discuss how Riemann's quadric

$$Q(x, v) = M_{ijkl}(x) v^i v^j v^k v^l \tag{5}$$

can be squeezed into the basic definitions of Finsler's metric. Our conclusion is that the for physics applications Finsler's geometry must be extended to include various singular subsets.

The paper is organized as follows: In Sect. 2, we provide the classical definition of Finsler's geometry and discuss the reasoning for its strong restriction. In Sect. 3, we apply Finsler geometry in the Euclidean domain to Riemann's quadric. We observe that even in this classical subject, the Finsler metric is degenerated on various conics of tangential vector space. So the basic definition must be modified. In Sect. 4, we discuss the extension of Finsler geometry into the non-Euclidean domain. In contrast to the familiar pure Lorentzian case, this construction is well-defined and differentiable only in a certain *conical subset of the tangent space.* Sect. 6 is devoted to a novel weakened definition of Finsler structure. We require this structure to be defined on an open subspace of the tangent bundle. We explicitly identify the singular subsets where some of the restrictions of the basic definition are broken down. In our opinion, these subsets provide important information about the physics of the corresponding processes. In the concluding section, we discuss the possible extensions and applications of the weekend Finsler's structure.

## 2. Finsler's construction

A generalization of Riemann's quadratic (1) and quartic (3) linear elements to a general linear element was evaluated by Finsler (1918). In this section, we briefly discuss the basic definitions of Finsler's geometry. For the details, see, e.g., [1], [2] and the references given therein.

### 2.1. Basic definitions

A *Finsler manifold* is a differentiable manifold $M$ endowed with a *Finsler function* $F(x,v)$ of two different sets of variables: a point $x \in M$ and a vector $v \in T_xM$. Finsler's function is assumed to satisfy the following conditions:

(F1) *Continuity:* $F(x,v)$ is defined and continuous on the entire tangent bundle $TM$, i.e., for all $x$ and $v$.

(F2) *Positive definiteness:* For all $x$ and all $v$, Finsler's function is non-negative, $F(x,v) \geq 0$. Equality holds only at the origin $v = 0$.

(F3) *Positive homogeneity:* For all $\lambda > 0$ and all $v \neq 0$, the first order homogeneity equation holds,

$$F(x, \lambda v) = \lambda F(x,v). \tag{6}$$

(F4) *Smoothness:* For all points $x$ and for all *nonzero* vectors $v$, the function $F(x,v)$ is smooth in its second argument (at least to the class $C^4$).

(F5) *Finsler's metric:* Finsler's metric tensor is defined as the Hessian of the Finsler function square $F(x,v)^2$

$$f_{ij}(x,v) = \frac{1}{2}\,\frac{\partial^2 F(x,v)^2}{\partial v^i \partial v^j}\,. \tag{7}$$

The metric is required to be non-degenerate for all points $x$ and for all vectors $v \neq 0$.

(F6) *Euclidean signature:* The signature of the metric $f_{ij}(x,v)$ is assumed to be Euclidean. It means that at every point $x \in M$ there is a basis where the metric is equal to the unit matrix $f_{ij}(x,v) = \mathrm{diag}(1,1,1,1)$.

### 2.2. Motivations and comments

Some motivations for the conditions presented in the definition of the Finsler structure can be given as follows. The first group of conditions describes the norm (length) of an arbitrary vector $v \in T_xM$ at an arbitrary point $x \in M$.

(F1) Continuity condition guarantees that a line element (length of a vector) $F(x,v)$ is defined (and final) for every vector $v$ attached at an arbitrary point $x$.

(F2) Positiveness of $F(x;v)$ is connected to the Euclidean signature of the space. It guarantees a positive length for an arbitrary non-zero vector $v$. This condition must be modified in the Minkowskian signature space, see Sect. 6. In particular, for a smooth curve $x(s) \in M$ with a tangent vector $v = \dot{x}(s)$, the Finsler function $F(x,v)$ assigns a positive length

$$\ell = \int_{s_1}^{s_2} F\big(x(s), \dot{x}(s)\big)\, ds\,. \tag{8}$$

(F3) The positive homogeneity of the degree one provides proper physical dimension of the length function. Moreover, it guarantees that the length of an arbitrary curve is invariant under positive oriented reparametrization of the curve.

The second group of conditions provides an additional structure on the tangential vector space $T_xM$ (and on the total manifold $M$). The condition (F5) introduces a scalar product of two vectors $u$ and $w$ that depends on a chosen vector $v$. The corresponding metric tensor (even being dependent on $v$) opens a room for applying the standard tools of differential geometry—connections, curvature tensors, and so on.

(F4) The smoothness condition permits the derivatives of the function $F(x, v)$ to be singular only for $v = 0$. This restriction is necessary because already in the simplest case (1), the derivative of the square root expression is not defined for $v = 0$. One formulates the condition as differentiability of $F(x, v)$ on a subset of a bundle $TM\backslash\{0\}$, where $\{0\}$ is a null section. In fact, only the derivatives of the Lagrangian $L(x, v) = F(x, v)^2$ are in use in Eq.(15). Thus it is possible to require a bit weaker condition—smoothness of the square $F(x, v)^2$ instead of the function $F(x, v)$ itself. Such modification is useful for Riemann's quadratic expression. We will see that this condition is related to the Euclidean (elliptic) structure on the manifold. In the pseudo-Euclidean case, it must be modified to permit additional singular sets.

(F5) The Euclidean metric construction can be written in a coordinate-free form:
For each nonzero $v \in T_pM$, the Hessian of $F^2(x, v)$ is applied at two independent vectors $u$ and $w$,

$$\mathbf{f}_v(u, w) := \frac{1}{2}\,\frac{\partial^2}{\partial s \partial t} F(x, v + su + tw)^2\bigg|_{s=t=0} \tag{9}$$

As a result, the tensor $f_{ij}$ is symmetric. Moreover, with respect to the direction variable the metric function turns out to be a homogeneous function of the zero order.

$$f_{ij}(x, \lambda v) = f_{ij}(x, v). \tag{10}$$

Consequently, the corresponding norm $\sqrt{f_{ij} w^i w^j}$ of an arbitrary vector $w^i$ has a proper physical dimension of length. In the simplest case of Finsler's function that is quadratic in the direction variable, Eq. (15) yields the ordinary Riemann metric. These properties (together with positive definiteness) justify the choice of the Hessian as a metric tensor.

(F6) The Euclidean signature condition require the matrix $f_{ij}$ to be positive definite for an arbitrary vector $u$

$$f_{ij}(x, v) u^i u^j \geq 0. \tag{11}$$

Under a general linear transformation of the basis, the components of the matrix $f_{ij}$ are transformed by the standard tensor rule. Thus $f_{ij}$ is referred to as *Finsler's metric.* This metric-type tensor depends not only on the point $x \in M$ but also on the directions given by the tangent vector $v \in T_xM$. Recall that $f_{ij}(x, v)$ is not necessarily defined for $v = 0$. Using Finsler's metric, the scalar product of two arbitrary vectors $u$ and $w$ can be defined in the standard way

$$(u, w)|_{f(v)} = f_{ij}(x, v) u^i w^j\,. \tag{12}$$

For a given Finsler's function, this expression can be treated as a *scalar product of vectrors $u, w$ with respect to the chosen vector $v$.* Correspondingly, a *norm of a vector $u$ with respect to the chosen vector $v$* is defined as

$$|u|^2_{f(v)} = (u, u)|_{f(v)} = f_{ij}(x, v) u^i u^j\,. \tag{13}$$

Due to the homogeneity condition, the norm of the vector $v$ calculated with respect to the vector $v$ itself is equal to the square of the Finsler function

$$|v|^2_{f(v)} = (v, v)|_{f(v)} = f_{ij}(x, v) v^i v^j = F(x, v)^2\,. \tag{14}$$

Deviation of the Finsler structure from the Riemann one is characterized by the *Cartan tensor*

$$C_{ijk} = \frac{1}{2}\frac{\partial}{\partial v^k} f_{ij}(x, v) = \frac{1}{4}\frac{\partial^3 F(x, v)^2}{\partial v^i \partial v^j \partial v^k}\,. \tag{15}$$

When $C_{ijk} = 0$ for all vectors $v$ and for all points $x$, the metric $f_{ij}$ is independent of the direction, hence the space is pure Riemannian.

Note that most of Finsler's functions, in particular those that are derived from pseudo-Riemann quartics, provide metrics that are degenerate on some subset of the vectors $v$. In mathematical texts, the metric condition is extended to a weaker subadditivity inequality condition. It is a generalization of the standard triangle inequality of Euclidean geometry.

(F5`–F6`) *Subadditivity:*

$$F(x, v + w) \leq F(x, v) + F(x, w) \quad \text{for all} \quad x \in M \quad \text{and all} \quad v, w \in T_xM. \tag{16}$$

In this case, one does not need the metric construction at all. This extension, however, does not conform with the standard construction of Riemann's geometry (metric, connection, curvature, etc.). Moreover, it cannot be extended to the pseudo-Riemannian case.

## 3. Two-dimensional positive definite quartics

In order to clarify the conditions presented in the definition of Finsler's structure, we consider some explicit 2-dimensional examples. We use Beem's construction [Beem(1970)] with the second-order homogeneous Lagrangian $L(x, v)$ instead of the first-order homogeneous Finsler's function $F(x, v)$ itself. The dependence of a point $x$ in the Lagrangian $L(x, v)$ is suppressed while the direction variable is presented in the form $v = (v^1, v^2) = (\alpha, \beta)$. Note that the 2-dimensional case has some special similarity to the 4-dimensional one. In both cases, a metrics $f_{ij}$ of the Euclidean signature have to satisfy the condition

$$\det f_{ij} > 0. \tag{17}$$

In the 2-dimensional case, this condition is necessary and sufficient since it extracts the metrics of the signatures $(+1, +1)$ and $(-1, -1)$, so that both can be considered as Euclidean. In the 4-dimensional case, (17) is only a necessary condition since it is satisfied by the metrics of the signature $(+1, +1, +1, +1)$ and $(-1, -1, +1, +1)$. In this case, we need additional algebraic inequalities in order to extract the Euclidean signature. For the Lorentzian signature, the condition

$$\det f_{ij} < 0 \tag{18}$$

is necessary and sufficient in two-dimensional and four-dimensional cases (but not in the three-dimensional case).

### 3.1. *Square of Euclidean quadric*

The simplest positive definite quartic expression is presented as the square of the Euclidean quadratic

$$Q(v) = (\alpha^2 + \beta^2)^2 \,. \tag{19}$$

The unique possible Lagrangian for this expression is given by

$$L(v) = \alpha^2 + \beta^2 \,. \tag{20}$$

This expression is positive for all non-zero vectors $v$ and vanishes only at the origin. Moreover, it is continuous and smooth for all $v \in V$. The corresponding Finsler metric is Riemannian

$$f_{ij} = \text{diag}(1, 1) \tag{21}$$

and well-defined for all vectors in the plan. Finsler's function

$$F(v) = \sqrt{\alpha^2 + \beta^2} \tag{22}$$

is well-defined for all $v$ and smooth everywhere except the origin $v = (\alpha, \beta) = 0$.

### *3.2. One more positive defined quartic*

As an example of a non-trivial Riemann's quartic of the Euclidean type, we consider the expression of the form

$$Q(v) = \alpha^4 + \beta^4. \tag{23}$$

The unique possible Lagrangian for this quartic is given by

$$L(v) = L(\alpha, \beta) = \sqrt{\alpha^4 + \beta^4}\,. \tag{24}$$

This expression is well-defined and continuous for all vectors $v$ and has the required second order of homogeneity. Note that the expression (24) can be regarded as a special case of a quartic decomposable into a product of two positive definite quadratic factors

$$L(v) = L(\alpha, \beta) = \sqrt{(\alpha^2 - \sqrt{2}\alpha\beta + \beta^2)(\alpha^2 + \sqrt{2}\alpha\beta + \beta^2)}\,. \tag{25}$$

The vector of the partial derivatives of the Lagrangian (24)

$$L_{,i} = \frac{2}{\sqrt{\alpha^4 + \beta^4}} \begin{pmatrix} \alpha^3 \\ \beta^3 \end{pmatrix} \tag{26}$$

is continuous and non-zero for all $v \neq 0$. At the origin $v = (\alpha, \beta) = 0$, the function $L(v)$ is non-differentiable. Calculating the Hessian of $L(v)$, we obtain Finsler's metric in the form

$$f_{ij} = \frac{1}{(\alpha^4 + \beta^4)^{3/2}} \begin{pmatrix} \alpha^2(\alpha^4 + 3\beta^4) & -2\alpha^3\beta^3 \\ -2\alpha^3\beta^3 & (3\alpha^4 + \beta^4)\beta^2 \end{pmatrix}. \tag{27}$$

The determinant of this matrix

$$\det(f_{ij}) = \frac{3\alpha^2\beta^2}{\alpha^4 + \beta^4} \tag{28}$$

is non-negative for all $v$. It is singular only at the origin $v = (\alpha, \beta) = 0$. Moreover it vanishes on two separate axes $\alpha = 0$ and $\beta = 0$, i.e., for the vectors $v = (0, \beta)$ and $v = (\alpha, 0)$, respectively. Beyond these axes, the metric tensor is positive definite. In particular, the norm of an arbitrary nonzero vector $u = (x, y)$ with respect to the metric (27) reads

$$||u||_v^2 = \frac{1}{(\alpha^4 + \beta^4)^{3/2}} \left(\alpha^2(\alpha^4 + 3\beta^4)x^2 - 4\alpha^3\beta^3 xy + (3\alpha^4 + \beta^4)\beta^2 y^2\right), \tag{29}$$

or, equivalently,

$$||u||_v^2 = \frac{1}{(\alpha^4 + \beta^4)^{3/2}} \left((\alpha^3 x + \beta^3 y)^2 + 3\alpha^2\beta^2(\beta x - \alpha y)^2\right). \tag{30}$$

With respect to the new coordinates $(x, y) \to (\xi, \eta)$

$$\xi = \frac{\alpha^3 x + \beta^3 y}{(\alpha^4 + \beta^4)^{3/4}}, \qquad \eta = \frac{\sqrt{3}\alpha\beta(\beta x - \alpha y)}{(\alpha^4 + \beta^4)^{3/4}} \tag{31}$$

the norm expression is given in the standard Euclidean form

$$||u||_v^2 = \xi^2 + \eta^2\,. \tag{32}$$

In order to have in (31) a reversible transformation of coordinates we have to require

$$\frac{1}{(\alpha^4+\beta^4)^{3/2}}\det\begin{pmatrix}\alpha^3 & \beta^3\\ \sqrt{3}\alpha\beta^2 & -\sqrt{3}\alpha^2\beta\end{pmatrix}=-\frac{\sqrt{3}\alpha\beta}{\sqrt{\alpha^4+\beta^4}}\neq 0. \tag{33}$$

This requirement holds only for $\alpha\beta\neq 0$ that is in correspondence with the expression for the determinant (28). The expression (32) is positive beyond the axes. However on $\alpha$-axis, i.e., for $\beta=0$, it takes the value $||u||_v^2=x^2$ that is zero for nonzero vectors of the form $u=(0,y)$. A similar singular behavior emerges also in the direction based on the basis-element vectors $v=(0,\beta)$.

Due to two peculiar directions in which the geometry degenerates, Finsler's metric specified for the Lagrangian (24) cannot be regarded as pure Finslerian.

## 4. Two-dimensional indefinite quartics

In this section, we consider several two-dimensional examples of the indefinite quartic

$$Q(x,v)=M_{ijkl}v^iv^jv^kv^l. \tag{34}$$

In this case, the indices change in the range $i,j,\dots=1,2$. We use the presentation $v^i=(\alpha,\beta)$. The quartic $Q(x,v)$ is indefinite–it can have positive, negative, or zero value for different directions of the vector $v$.

In the 2-dimensional case (as well as the 4-dimensional case) the metrics $f_{ij}$ of Lorentzian signature is distinguished by an unique condition

$$\det f_{ij}<0. \tag{35}$$

In other dimensions, we need more algebraic inequalities. Having in the mind 4-dimensional Lorentz signature of the form $(-1,+1,+1,+1)$ we refer to the vectors of positive/negative Lagrangians (squared Finsler's norms) as *spacelike* and *timelike*, respectively. The *null vectors* are vectors with a zero Lagrangian.

### *4.1. Square of Lorentzian quadric*

We start with a simplest quartic that is presented as a square of an indefinite quadratic

$$Q(v)=(\alpha^2-\beta^2)^2\,. \tag{36}$$

This expression is positive but vanishes for non-zero vectors of the form $v=(\alpha,\pm\beta)$.

Define the Lagrangian in the standard (special relativistic) form

$$L(v)=\alpha^2-\beta^2\,. \tag{37}$$

This function is smooth for all vectors $v$. The Finsler metric has the standard Lorentz form

$$f_{ij}(v)=\mathrm{diag}(1,-1)\,. \tag{38}$$

Note that three characteristic sets can be determined

$$\mathcal{A} = \{v\in T_xM\big||\alpha|>|\beta|\}\implies \text{spacelike vectors} \tag{39}$$
$$\mathcal{B} = \{v\in T_xM\big||\alpha|<|\beta|\}\implies \text{timelike vectors} \tag{40}$$
$$\mathcal{C} = \{v\in T_xM\big||\alpha|=|\beta|\}\implies \text{null vectors} \tag{41}$$

Consider different possible definitions of the Finsler function (norm). In fact they are not related to the specific choice of the Lagrangian.

(1) Finsler's function of the form

$$F(v)=\sqrt{\alpha^2-\beta^2} \tag{42}$$

is defined only in the region satisfying $|\alpha|\geq|\beta|$ and differentiable in the open conic $|\alpha|>|\beta|$.

(2) Another possibility is to define the Finsler function in the absolute value form

$$F(v)=\sqrt{|\alpha^2-\beta^2|}\,. \tag{43}$$

Recall that this construction was proposed by Beem, see (84). This function is defined for all vectors $v$ and differentiable besides the set $\mathcal{C}$. It cannot, however, separate the time-like and space-like vectors.

(3) The third definition is based on the proposal in Sect. 5.

$$F(v)=\sqrt{|\alpha^2-\beta^2|}\,\mathrm{sgn}L(v)=\begin{cases} \sqrt{\alpha^2-\beta^2} & \text{for} \quad |\alpha|\geq|\beta|; \\ -\sqrt{\beta^2-\alpha^2} & \text{for} \quad |\alpha|<|\beta|. \end{cases} \tag{44}$$

In this case, the norm of the space-like vectors is positive and of the time-like vectors is negative. The norm of the null-vectors is zero.

### *4.2. Lorentzian non-birefringent space-time*

We consider now a Lorentzian-type quartic of the form

$$Q(v)=\alpha^4-\beta^4\,. \tag{45}$$

This expression is well-defined and smooth for all vectors $v=(\alpha,\beta)$. It can be expressed as a product of two quadrics–one Euclidean and one Lorentzian

$$Q(v)=(\alpha^2+\beta^2)(\alpha^2-\beta^2)\,. \tag{46}$$

Our first task is to construct a proper Lagrangian. Let us consider different options:

(1) Define the Lagrangian of the form

$$L(v)=L(\alpha,\beta)=\sqrt{\alpha^4-\beta^4}=\sqrt{(\alpha^2-\beta^2)(\alpha^2+\beta^2)} \tag{47}$$

This function is definite and continuous only in the region $|\alpha|\geq|\beta|$ and smooth in its interior $|\alpha|>|\beta|$. Then, the characteristic sets are

$$\begin{aligned} \mathcal{A} &= \{v\in T_xM \big| |\alpha|>|\beta|\} \implies \text{spacelike vectors} && (48)\\ \mathcal{B} &= \emptyset \implies \text{timelike vectors} && (49)\\ \mathcal{C} &= \{v\in T_xM \big| |\alpha|=|\beta|\} \implies \text{null vectors} && (50) \end{aligned}$$

The gradient of this function is given by

$$L_{,i}=\frac{2}{\sqrt{\alpha^4-\beta^4}}\begin{pmatrix}\alpha^3\\ -\beta^3\end{pmatrix}. \tag{51}$$

This expression is continuous in the region $\mathcal{A}$. The same is true for all higher derivatives of $L(v)$. Consequently for all positive integers $i$, the sets of differentiability coincide with the set of positive values of $L(v)$, i.e., $\mathcal{D}_i = \mathcal{A}$ . The Hessian yields Finsler's metric of the form

$$f_{ij} = \frac{1}{(\alpha^4-\beta^4)^{3/2}} \begin{pmatrix} \alpha^2(\alpha^4-3\beta^4) & 2\alpha^3\beta^3 \\ 2\alpha^3\beta^3 & \beta^2(\beta^4-3\alpha^4) \end{pmatrix} . \tag{52}$$

The determinant of this metric,

$$f = \frac{-3\alpha^2\beta^2}{\alpha^4-\beta^4} , \tag{53}$$

is non-positive in the set $\mathcal{A}$. It is degenerate on the strength line $\beta = 0$ (the line $\alpha = 0$ is beyond the set $\mathcal{A}$) . It means that the set of degeneration is defined as $\mathcal{E} = \{v \in T_xM | \beta = 0\}$. Besides this singular set, the metric is Lorentzian, i.e., $\mathcal{G} = \{v \in T_xM | \beta \neq 0\}$.

For the norm of an arbitrary vector $u = (x, y)$, we have

$$||u||_v^2 = \frac{1}{(\alpha^4-\beta^4)^{3/2}} \left(\alpha^2(\alpha^4-3\beta^4)x^2 + 4\alpha^3\beta^3 xy + \beta^2(\beta^4-3\alpha^4)y^2\right) , \tag{54}$$

or, equivalently,

$$||u||_v^2 = \frac{1}{(\alpha^4-\beta^4)^{3/2}} \left((\alpha^3 x + \beta^3 y)^2 - 3\alpha^2\beta^2(\beta x - \alpha y)^2\right) , \tag{55}$$

Let us define new coordinates $(x, y) \to (\xi, \eta)$

$$\xi = \frac{\alpha^3 x + \beta^3 y}{(\alpha^4-\beta^4)^{3/4}}, \qquad \eta = \frac{\sqrt{3}\alpha\beta(\beta x - \alpha y)}{(\alpha^4-\beta^4)^{3/4}} . \tag{56}$$

Then the norm of the vector $u$ is given in the standard Lorentzian form

$$||u||_v^2 = \xi^2 - \eta^2 . \tag{57}$$

Note that the transformation of the coordinates is applicable only in the regions where it is revertible. So we have to require

$$\det \begin{pmatrix} \alpha^3 & \beta^3 \\ \sqrt{3}\alpha\beta^2 & -\sqrt{3}\alpha^2\beta \end{pmatrix} = -\sqrt{3}\alpha\beta(\alpha^4+\beta^4) \neq 0 . \tag{58}$$

It means that the transformations are available only for $\alpha\beta \neq 0$.

(2) Let us consider a Lagrangian of the absolute value form

$$L(v) = L(\alpha, \beta) = \sqrt{|\alpha^4-\beta^4|} = \sqrt{|\alpha^2-\beta^2|(\alpha^2+\beta^2)} \tag{59}$$

The function is non-negative in the whole plane and strictly positive in open conic regions beside the strength lines $|\alpha| = |\beta|$. Then, the characteristic sets are

$$\mathcal{A} = \{v \in T_xM \big| |\alpha| \neq |\beta|\} \implies \text{spacelike vectors} \tag{60}$$

$$\mathcal{B} = \emptyset \implies \text{timelike vectors} \tag{61}$$

$$\mathcal{C} = \{v \in T_xM \big| |\alpha| = |\beta|\} \implies \text{null vectors.} \tag{62}$$

In the regions $|\alpha| > |\beta|$ and $|\alpha| < |\beta|$, the gradient of this function is given by

$$L_{,i} = \frac{2}{\sqrt{\alpha^4 - \beta^4}} \begin{pmatrix} \alpha^3 \\ -\beta^3 \end{pmatrix}, \qquad \text{and} \qquad L_{,i} = \frac{2}{\sqrt{-\alpha^4 + \beta^4}} \begin{pmatrix} -\alpha^3 \\ \beta^3 \end{pmatrix}, \tag{63}$$

respectively. The Hessian yields Finsler's metric of the form

$$f_{ij} = \frac{1}{|\alpha^4 - \beta^4|^{3/2}} \begin{pmatrix} \alpha^2(\alpha^4 - 3\beta^4) & 2\alpha^3\beta^3 \\ 2\alpha^3\beta^3 & \beta^2(\beta^4 - 3\alpha^4) \end{pmatrix}. \tag{64}$$

The determinant of this metric reads

$$f = \frac{-3\alpha^2\beta^2}{|\alpha^4 - \beta^4|} \tag{65}$$

Consequently for $\alpha, \beta \neq 0$ the determinant is negative. In a 2-dimensional space (and in a 4-dimensional space, as well) this requirement is sufficient to deduce that the metric is Lorentzian. For this Lagrangian, the norm of an arbitrary vector can be also written in a pure Lorentzian form $||u||_v^2 = \xi^2 - \eta^2$. This form is derived by the transformations (56) with an absolute value function added into the denominator.

(3) Let us consider a Lagrangian of the form

$$L(v) = L(\alpha, \beta) = \sqrt{|\alpha^4 - \beta^4|}\,\mathrm{sgn}(Q) = \begin{cases} \sqrt{\alpha^4 - \beta^4} & \text{for} \quad |\alpha| \geq |\beta|; \\ -\sqrt{-\alpha^4 + \beta^4} & \text{for} \quad |\alpha| < |\beta|. \end{cases} \tag{66}$$

The function is definite and continuous in the whole plane. The characteristic sets are

$$\mathcal{A} = \{v \in T_xM \big| |\alpha| > |\beta|\} \implies \text{spacelike vectors} \tag{67}$$
$$\mathcal{B} = \{v \in T_xM \big| |\alpha| < |\beta|\} \implies \text{timelike vectors} \tag{68}$$
$$\mathcal{C} = \{v \in T_xM \big| |\alpha| = |\beta|\} \implies \text{null vectors.} \tag{69}$$

and strictly positive in open conic regions beside the strength lines $|\alpha| = |\beta|$. In the regions $|\alpha| > |\beta|$ and $|\alpha| < |\beta|$, the gradient of this function is given by

$$L_{,i} = \frac{2}{\sqrt{\alpha^4 - \beta^4}} \begin{pmatrix} \alpha^3 \\ -\beta^3 \end{pmatrix}, \qquad \text{and} \qquad L_{,i} = \frac{2}{\sqrt{-\alpha^4 + \beta^4}} \begin{pmatrix} \alpha^3 \\ -\beta^3 \end{pmatrix}, \tag{70}$$

respectively. The Hessian yields Finsler's metric of the form

$$f_{ij} = \frac{1}{|\alpha^4 - \beta^4|^{3/2}} \begin{pmatrix} \alpha^2(\alpha^4 - 3\beta^4) & 2\alpha^3\beta^3 \\ 2\alpha^3\beta^3 & \beta^2(\beta^4 - 3\alpha^4) \end{pmatrix} \mathrm{sgn}(Q). \tag{71}$$

The determinant of this metric

$$f = \frac{-3\alpha^2\beta^2}{|\alpha^4 - \beta^4|} \tag{72}$$

is definite and negative almost everywhere. It vanishes only on the lines $|\alpha| = 0$ and $|\beta| = 0$ and degenerates on the lines $|\alpha| = |\beta|$. In this metric the squared norm of a vector can be also transformed into the standard Lorentzian form $||u||_v^2 = \xi^2 - \eta^2$.

## 5. Quartic as a product of two quadratics

Let us consider an example of a quartic that is expanded into a product of two factors

$$Q(v)=Q(\alpha,\beta)=\left(g_{ij}v^iv^j\right)\left(h_{ij}v^iv^j\right) \tag{73}$$

with two matrices $g_{ij}$ and $h_{ij}$. Since these two matrices can be transformed simultaneously into the diagonal form, we come to expressions of three possible type

### *5.1. Euclid × Euclid*

Let us consider a quartic expression

$$Q(v)=Q(\alpha,\beta)=(\alpha^2+\beta^2)(\alpha^2+k\beta^2)\,, \tag{74}$$

where $k$ is a dimensionless *positive* numerical parameter. Thus we have a positive-definite quartic. For this quartic function, the unique possible Lagrangian is given by

$$L(v)=L(\alpha,\beta)=\sqrt{(\alpha^2+\beta^2)(\alpha^2+k\beta^2)}\,. \tag{75}$$

This function is definite and continuous for all vectors $v=(\alpha,\beta)$. The gradient of $L(v)$ takes the form

$$L_{,i}=\frac{1}{\sqrt{(\alpha^2+\beta^2)(\alpha^2+k\beta^2)}}\begin{pmatrix}2\alpha^3+(k+1)\alpha\beta^2\\(k+1)\alpha^2\beta+2k\beta^3\end{pmatrix}. \tag{76}$$

Thus the first order derivatives (together with the higher order ones) are defined on the whole plane except the origin. The Hessian yields Finsler's metric of the form

$$f_{ij}=\frac{1}{2\sqrt{(\alpha^2+\beta^2)^3(\alpha^2+k\beta^2)^3}}\cdot \begin{pmatrix}2\alpha^6+3(k+1)\alpha^4\beta^2+6k\alpha^2\beta^4+(k^2+k)\beta^6 & (k-1)^2\alpha^3\beta^3\\(k-1)^2\alpha^3\beta^3 & (k+1)\alpha^6+6k\alpha^4\beta^2+3(k^2+k)\alpha^2\beta^4+2k^2\beta^6\end{pmatrix}. \tag{77}$$

We calculate the determinant of this metric by applying the Wolfram computation system. It is given by

$$\det(f_{ij})=\frac{2(1+k)\alpha^4-(k^2-10k+1)\alpha^2\beta^2+2k(k+1)\beta^4}{4(\alpha^2+\beta^2)(\alpha^2+k\beta^2)} \tag{78}$$

Observe that for $k=1$, we have here the standard Euclidean metric $f_{ij}=\mathrm{diag}(1,1)$, that is independent on the direction $v=(\alpha,\beta)$. For $k=-1$, we have the quartic expressions $Q(v)=\alpha^4-\beta^4$ with the corresponding Finsler metric that we consider later.

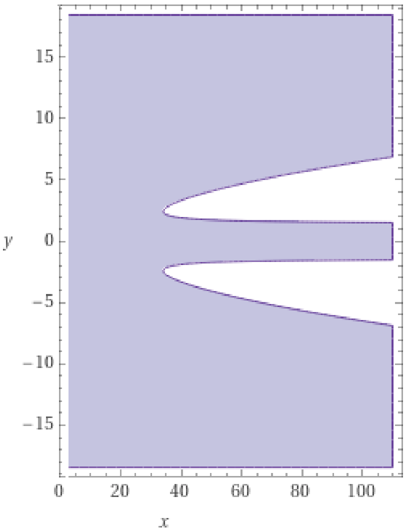


**Figure 1.** In the plan with the axes $x = k$ and $y = \alpha/\beta$, we depict the area (in blue) of positive determinant $\det(f_{ij}) > 0$. In this case, the metric is Euclidean. Starting with the sufficiently big values of the parameter $k \sim 30$ the determinant is negative for small angles of the vector $v$. In this area, the space is Lorentzian.

In Fig.(5.1), we present numerical presentation of the regions where the determinant $\det(f_{ij})$ considering as a function of two variables $k$ and $\alpha/\beta$ is positive or negative. On the boundary curves, the metric is degenerated. The asymptotic lines $y = \alpha/\beta = \sqrt{2}$ are visual.

For large values of the parameter $k$, the determinant approaches the asymptotic expansion

$$\det(f_{ij}) \sim \left(\frac{2\beta^2 - \alpha^2}{\alpha^2 + \beta^2}\right) \frac{k}{4} \, . \tag{79}$$

It means that for the large values of $k$ and sufficiently small tangent of the vector $v$, such as $\beta/\alpha < 1/\sqrt{2}$, there is a region of the negative determinant, i.e., of the Lorentzian signature. In the exterior open domain, the metric is of the Euclidean signature. These domains are bounded by the strength lines where the metric is degenerated.

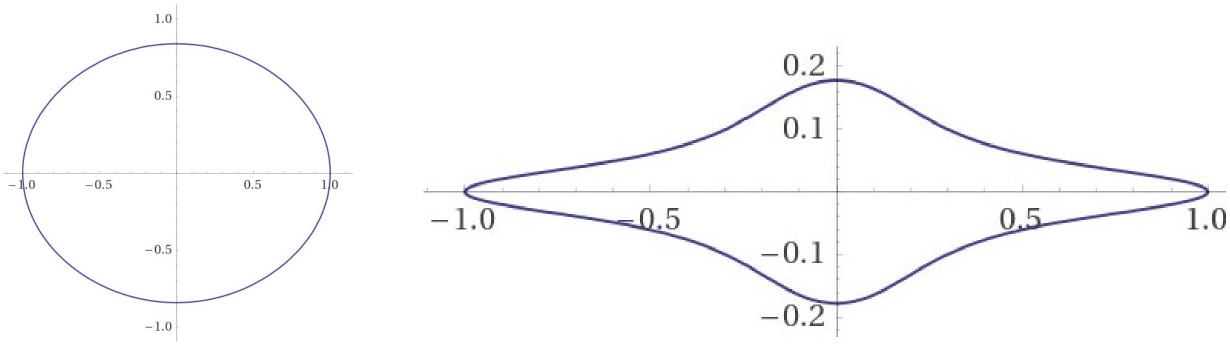


**Figure 2.** The graphs present the solutions of the equation $Q(v) = 1$ for different values of the parameter $k$. For small values of the parameter $k$, (the first graph) the curve is convex. For big values of the parameter $k$, (the second graph) the curve has convex and concave parts.

Note a non-trivial behavior of Finsler's metric in the Lorentzian regime. Since

$$f_{ij} v^i v^j = L(v) > 0, \tag{80}$$

the squared line element is positive for every $v \neq 0$.

### *5.2. Euclid × Lorentz*

Let us consider a quartic expression

$$Q(v) = Q(\alpha, \beta) = (\alpha^2 + \beta^2)(\alpha^2 - k\beta^2)\,, \tag{81}$$

with a dimensionless *positive* numerical parameter $k$. Thus we have a product of a positive-definite quadratic and indefinite one. The Lagrangian corresponding to this quartic

$$L(v) = L(\alpha, \beta) = \sqrt{(\alpha^2 + \beta^2)|\alpha^2 - k\beta^2|}\,, \tag{82}$$

The gradient of $L(v)$ takes the form

$$L_{,i} = \frac{1}{\sqrt{(\alpha^2 + \beta^2)|\alpha^2 - k\beta^2|}} \begin{pmatrix} 2\alpha^3 + (1-k)\alpha\beta^2 \\ (1-k)\alpha^2\beta - 2k\beta^3 \end{pmatrix} \mathrm{sgn}(\alpha^2 - k\beta^2). \tag{83}$$

Thus the first order derivatives (together with the higher order ones) are defined on the whole plane except the pair of strength lines $\alpha^2 = k\beta^2$.

The Hessian yields Finsler's metric of the form

$$f_{ij} = \frac{1}{2\sqrt{(\alpha^2 + \beta^2)|\alpha^2 - k\beta^2||^3}} \cdot \begin{pmatrix} 2\alpha^6 + 3(1-k)\alpha^4\beta^2 - 6k\alpha^2\beta^4 + (k^2 - k)\beta^6 & (k+1)^2\alpha^3\beta^3 \\ (k+1)^2\alpha^3\beta^3 & (1-k)\alpha^6 - 6k\alpha^4\beta^2 + 3(k^2-k)\alpha^2\beta^4 + 2k^2\beta^6 \end{pmatrix}. \tag{84}$$

The determinant of this matrix takes the form

$$\det(f_{ij}) = \frac{2(1-k)\alpha^4 - (k^2 + 10k + 1)\alpha^2\beta^2 - 2k(1-k)\beta^4}{4(\alpha^2 + \beta^2)|\alpha^2 - k\beta^2|} \tag{85}$$

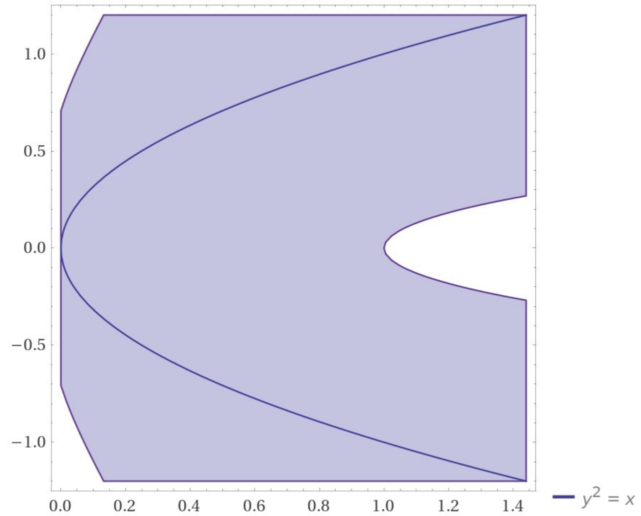


**Figure 3.** The solution of the inequality $\det(f_{ij}) < 0$ is presented in the blue area. The white area corresponds to the positive defined metric. The metric is degenerated at the line $\alpha^2 = k\beta^2$.

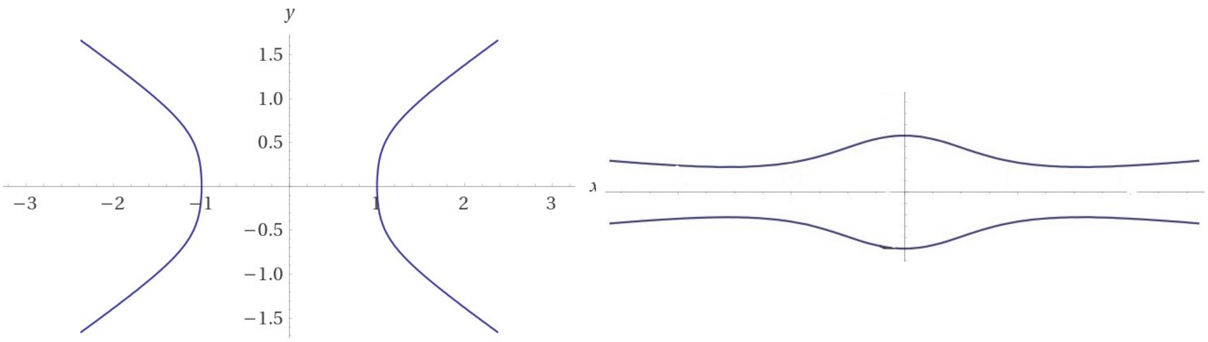


**Figure 4.** Unit spheres in the metric corresponding to the product quartic (81) for $k = 2$ and $k = 0.01$, respectively. The first graph is concave. On the second graph, the change of the convexity is visual.

### *5.3. Lorentz × Lorentz*

Let us consider a quartic expression

$$Q(v) = Q(\alpha,\beta) = (\alpha^2-\beta^2)(\alpha^2-k\beta^2)\,, \tag{86}$$

with a dimensionless *positive* numerical parameter $k$. We have here a product of two indefinite factors. Consider a Lagrangian for this quadric in the form

$$L(v) = L(\alpha,\beta) = \sqrt{|(\alpha^2-\beta^2)(\alpha^2-k\beta^2)|}\,\mathrm{sgn}Q\,. \tag{87}$$

The gradient of the Lagrangian reads

$$L_{,i} = \frac{1}{\sqrt{|(\alpha^2-\beta^2)(\alpha^2-k\beta^2)|}}\begin{pmatrix} 2\alpha^3-(1+k)\alpha\beta^2 \\ -(1+k)\alpha^2\beta+2k\beta^3\end{pmatrix}\mathrm{sgn}\,Q. \tag{88}$$

Finsler's metric is given by

$$f_{ij}=\frac{\mathrm{sgn}\,Q}{2\sqrt{|(\alpha^2-\beta^2)(\alpha^2-k\beta^2)|^3}}\cdot \begin{pmatrix} 2\alpha^6-3(1+k)\alpha^4\beta^2+6k\alpha^2\beta^4-(k^2+k)\beta^6 & (k-1)^2\alpha^3\beta^3 \\ (k-1)^2\alpha^3\beta^3 & -(k+1)\alpha^6+6k\alpha^4\beta^2-3(k^2+k)\alpha^2\beta^4+2k^2\beta^6\end{pmatrix}. \tag{89}$$

The determinant of this metric es expressed as

$$\det(f_{ij}) = -\frac{2(k+1)\alpha^4+(k^2-10k+1)\alpha^2\beta^2+2k(k+1)\beta^4}{4|(\alpha^2-\beta^2)(\alpha^2-k\beta^2)|} \tag{90}$$

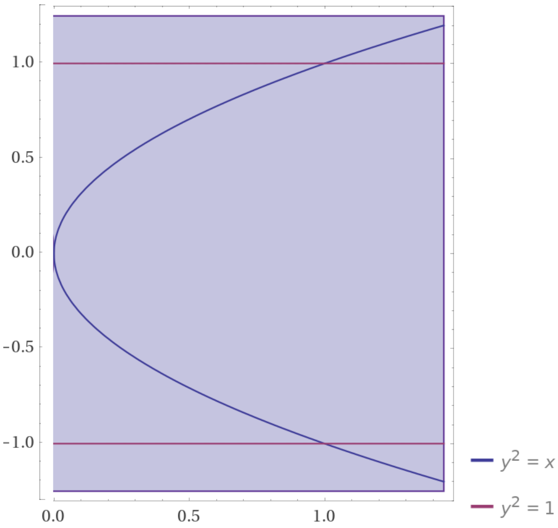


**Figure 5.** In the plan with the axes $x = k$ and $y = \alpha/\beta$, we depict the area (in blue) of the negative determinant $\det(f_{ij}) < 0$. In this case, the metric is Lorentzian. The metric is degenerates on the lines $\alpha^2 = \beta^2$ and $\alpha^2 = k\beta^2$

This determinant expression is strictly negative for all positive values of the parameter $k$ and almost for all vectors $v$. The metric degenerates on the hypersurfaces $\alpha^2 = \beta^2$ and $\alpha^2 = k\beta^2$. It means that the space is Lorentzian beyond the degenerate surfaces.

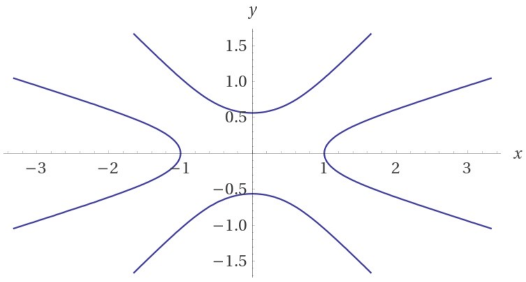


**Figure 6.** For all values of the parameter $k$, the unit spheres are hyperbolas, as in the ordinary Lorentz space.

## 6. Singular conics on two-dimensional space

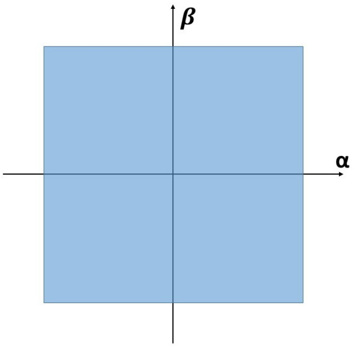
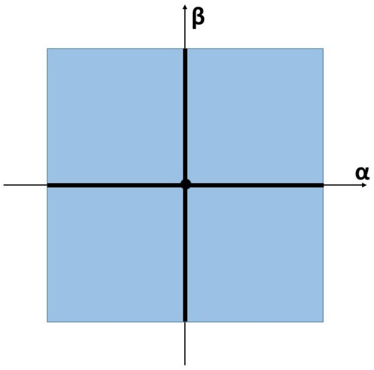
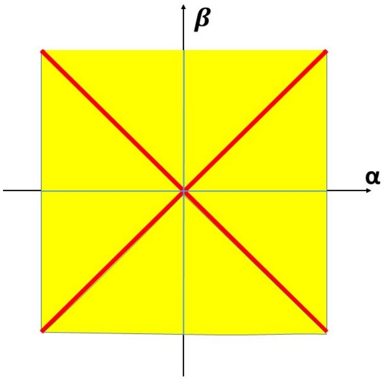
In this section, we present the characteristic sets $\mathcal{A}, \cdots, \mathcal{H}$ for a number of the two-dimensional examples of Riemann quartic that were calculated above.

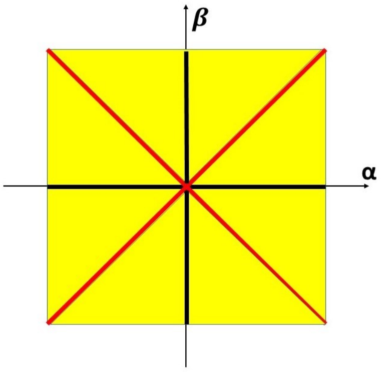
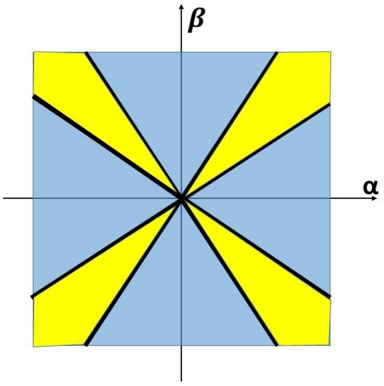
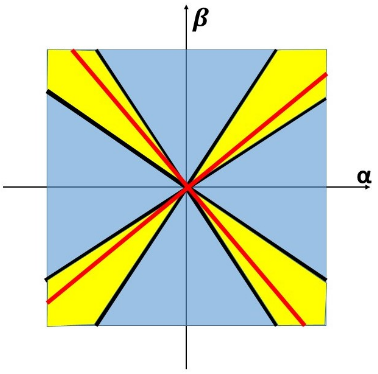
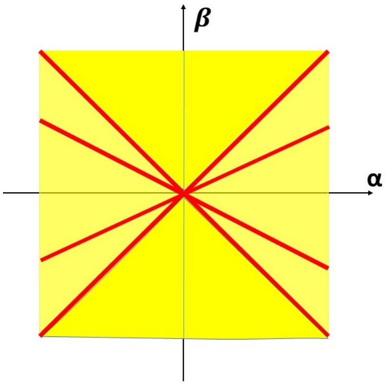
Let us look at a few general characteristics of the Finsler building made of homogeneous quartics. While the characteristic sets $\mathcal{A}, \cdots, \mathcal{H}$ are all generally distinct from one another, some of them have become the same in the case of Riemann's quartic. In particular,

- For indefinite quartics, conic singularities appear from the absolute-value definition of the Lagrangian, i.e. on the subset $Q(x, v) = 0$.
- Since Riemann's quartic is an analytic function in $v$, the non-differential singularities $\mathcal{D}_i$ appear only on the null-vectors conic, $Q(x, v) = 0$, then $\mathcal{C} = \mathcal{D}$. Moreover, these singularities are the same for the all order of derivation, i.e., $\mathcal{D} = \mathcal{D}_i$ for all $i$.

- In two-dimensional case, we cannot distinguish between the sets $\mathcal{G}$ of Lorentzian signature and the sets $\mathcal{H}$ of mixed signature.

In the following table, we denote by $V$ the tangent vector space at a general point without the origin $V = T_xM\backslash\{0\}$. The Lagrangian is defined due to Eq.(66). The parametrization $v = (\alpha, \beta)$ is used. The areas of the positive-definite Finsler's metric are given in blue, while of the indefinite metric in yellow. Black lines describe the point where the metric is degenerated. On the red lines, the metric is not defined.

| **Quartic** | **Characteristic Sets** | **Diagram** |
|---|---|---|
| $(\alpha^2+\beta^2)^2$ | $\mathcal{A} = V$<br>$\mathcal{B} = \varnothing;$<br>$\mathcal{C} = \{0\}$<br>$\mathcal{D} = \varnothing;$<br>$\mathcal{E} = \varnothing;$<br>$\mathcal{F} = V;$<br>$\mathcal{G} = \varnothing$ |  |
| $\alpha^4+\beta^4$ | $\mathcal{A} = V$<br>$\mathcal{B} = \varnothing;$<br>$\mathcal{C} = \{0\}$<br>$\mathcal{D} = V;$<br>$\mathcal{E} = \{v \in V \vert \alpha\beta = 0\};$<br>$\mathcal{F} = V\backslash\mathcal{E};$<br>$\mathcal{G} = \varnothing$ |  |
| $(\alpha^2-\beta^2)^2$ | $\mathcal{A} = V$<br>$\mathcal{B} = \varnothing;$<br>$\mathcal{C} = \{v \in V \vert \alpha = \pm\beta\}$<br>$\mathcal{D} = \mathcal{C};$<br>$\mathcal{E} = \varnothing;$<br>$\mathcal{F} = \varnothing$<br>$\mathcal{G} = V\backslash\mathcal{C}$ |  |

| **Quartic** | **Characteristic Sets** | **Diagram** |
|---|---|---|
| $\alpha^4 - \beta^4$ | $\mathcal{A} = \{v \in V \vert \alpha^2 > \beta^2\};$<br>$\mathcal{B} = \{v \in V \vert \alpha^2 < \beta^2\};$<br>$\mathcal{C} = \{v \in V \vert \alpha^2 = \beta^2\};$<br>$\mathcal{D} = \mathcal{C};$<br>$\mathcal{E} = \{v \in V \vert \alpha\beta = 0\};$<br>$\mathcal{F} = \varnothing$<br>$\mathcal{G} = V \backslash (\mathcal{C} \cup \mathcal{E})$ |  |
| $(\alpha^2 + \beta^2)(\alpha^2 + k\beta^2)$ | $\mathcal{A} = V$<br>$\mathcal{B} = \varnothing;$<br>$\mathcal{C} = \{0\}$<br>$\mathcal{D} = \varnothing;$<br>$\mathcal{E} \neq \varnothing;$<br>$\mathcal{F} \neq \varnothing;$<br>$\mathcal{G} \neq \varnothing$ |  |
| $(\alpha^2 + \beta^2)(\alpha^2 - k\beta^2)$ | $\mathcal{A} = \{v \in V \vert \alpha^2 > k\beta^2\}$<br>$\mathcal{B} = \{v \in V \vert \alpha^2 < k\beta^2\};$<br>$\mathcal{C} = \{v \in V \vert \alpha^2 = k\beta^2\}$<br>$\mathcal{D} = \mathcal{C};$<br>$\mathcal{E} \neq \varnothing;$<br>$\mathcal{F} \neq \varnothing;$<br>$\mathcal{G} \neq \varnothing$ |  |
| $(\alpha^2 - \beta^2)(\alpha^2 - k\beta^2)$ | $\mathcal{A} = V$<br>$\mathcal{B} = \varnothing;$<br>$\mathcal{C} = \{v \in V \vert \alpha = \pm\beta\}$<br>$\mathcal{D} = \mathcal{C};$<br>$\mathcal{E} = \varnothing;$<br>$\mathcal{F} = \varnothing$<br>$\mathcal{G} = V \backslash \mathcal{C}$ |  |

## 7. Conclusion

We discussed how Riemann's quartic and Finsler's geometric construction are related. We focused on Finsler's metric for various positive-definite and indefinite two-dimensional quartics. The dimensional constraint enables explicit calculations. Our results are as follows:

- For positive-definite quartics:
  - metric tensor is definite over the entire space (without the origin);
  - metric degenerates at conic subsets of the tangential space in general;
  - metrics of different signatures can coexist on the same tangential plane.
- For indefinite quartics:
  - Lagrangian is defined on the entire space;
  - there are conic subsets of the tangential space where metric tensor degenerates;
  - on some conic null space where the quartic vanishes, the metric tensor is singular;
  - signature of metric can vary over the tangential space.

The future line of consideration is to study the relations between the singular sets of Finsler's structure and observable phenomena of electromagnetic wave propagation.

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