23 The finite subgroups of $SU(3)$

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Abstract The finite subgroups of SU(3) are frequently used in particle physics. Though they were classified already at the beginning of the 20th century, there have been many new and interesting developments in the last few years. In this article we will list the finite subgroups of SU(3) and summarize some of their properties.

23.1 Introduction

Particle physics offers a wide range of applications for the theory of finite groups, and in particular the finite subgroups of SU(3) have been intensively studied in the past. The wide range of applications of SU(3)-subgroups covers different fields such as hadron physics and computational tools in lattice QCD. The field of particle physics which has made the most intensive use of the finite subgroups of SU(3) in the recent years is flavour physics, where finite SU(3)-subgroups are frequently used as symmetries in the quark, lepton and scalar sector [1, 2].

The classification of the finite subgroups of SU(3) presented in this article is based on the work of H.F. Blichfeldt as published in the famous book [3]. A short summary of the history of the contributions to the analysis of the finite subgroups of SU(3) (from a physicist's perspective) can be found in the introduction of [4].

There is a lot of literature covering aspects of the finite subgroups of SU(3). Apart from the classic textbook [3] we refer the reader to the review articles [5–7] and references therein.

23.2 The finite subgroups of SU(3)

In 1916 H.F. Blichfeldt classified the finite subgroups of SU(3) into the following five classes [3].

(A) Abelian groups.

(B) Finite subgroups of SU(3) with faithful two-dimensional representations.

(C) The groups $C(n, a, b)$ generated by the matrices

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F(n, a, b) = \mathrm{diag}(\eta^a, \eta^b, \eta^{-a-b}), \tag{23.1}$$

where $\eta = \exp(2\pi i/n)$, $n \in \mathbb{N}\backslash\{0, 1\}$ and $a, b \in \{0, \ldots, n-1\}$.

(D) The groups $D(n, a, b; d, r, s)$ generated by E, $F(n, a, b)$ and

$$\widetilde{G}(d, r, s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix}, \tag{23.2}$$

where $\delta = \exp(2\pi i/d)$, $d \in \mathbb{N}\backslash\{0\}$ and $r, s \in \{0, \ldots, d-1\}$.

(E) Six exceptional finite subgroups of SU(3):

- $\Sigma(60) \cong A_5$, $\Sigma(168) \cong \mathrm{PSL}(2, 7)$,
- $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$, $\Sigma(216 \times 3)$ and $\Sigma(360 \times 3)$,

as well as the direct products $\Sigma(60) \times \mathbb{Z}_3$ and $\Sigma(168) \times \mathbb{Z}_3$.

In the following we will go through these five types of groups and dwell a bit on the structures of their members.

(A) Abelian groups The possible structures of the Abelian finite subgroups of SU(3) are strongly restricted by the following theorem [4].

Theorem 1. Every finite Abelian subgroup $\mathcal{G}$ of SU(3) is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_p$, where

$$m = \max_{a \in \mathcal{G}} \mathrm{ord}(a) \tag{23.3}$$

and p is a divisor of m.

Thus every finite Abelian subgroup of SU(3) is either a cyclic group or a direct product of two cyclic groups. Examples for cyclic subgroups of SU(3) are the three-dimensional rotation groups about one axis. An example for a direct product of two cyclic groups is Klein's four group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(B) Groups with two-dimensional faithful representations Suppose we are given a finite group possessing a two-dimensional faithful representation (i.e. a finite subgroup of U(2)), then via the homomorphism

$$A \mapsto \begin{pmatrix} \det A^* & 0 \\ 0 & A \end{pmatrix} \in \mathrm{SU}(3) \qquad (A \in \mathrm{U}(2)) \tag{23.4}$$

we can construct an isomorphic finite subgroup of SU(3) [6]. In this way every finite subgroup of U(2) can be interpreted as a finite subgroup of SU(3). Examples for SU(3)-subgroups possessing a two-dimensional faithful representation are

- the dihedral groups D_n and
- the double covers $\widetilde{T}$, $\widetilde{O}$, $\widetilde{I}$, $\widetilde{D}_n$ of the finite three-dimensional rotation groups.[1]

[1] The finite three-dimensional rotation groups (SO(3)-subgroups) are [8]: the rotation groups about one axis (cyclic groups), the dihedral groups D_n, the tetrahedral group $T \cong A_4$, the octahedral group $O \cong S_4$ and the icosahedral group $I \cong A_5$.

The groups of type (C) The groups $C(n,a,b)$ are generated by the permutation matrix E and a diagonal matrix $F(n,a,b)$–see equation (23.1). The subgroup $N(n,a,b)$ of all diagonal matrices is generated by

$$F(n,a,b) \quad \text{and} \quad EF(n,a,b)E^{-1}, \tag{23.5}$$

from which follows that $N(n,a,b)$ is a normal subgroup of $C(n,a,b)$. Therefore the groups of type (C) have the structure of a semi-direct product

$$C(n,a,b) \cong N(n,a,b) \rtimes \mathbb{Z}_3, \tag{23.6}$$

where the $\mathbb{Z}_3$-subgroup is generated by E. Since $N(n,a,b)$ is an Abelian finite subgroup of SU(3), we can use theorem 1 to arrive at

$$C(n,a,b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3. \tag{23.7}$$

There are two important special cases emerging from (23.7):

- $p = 1 \Rightarrow$ Groups of the type[2] $T_m \cong \mathbb{Z}_m \rtimes \mathbb{Z}_3$.
- $p = m \Rightarrow$ Groups of the type $(\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes \mathbb{Z}_3 \cong \Delta(3m^2)$.

Examples for groups of type (C) are well-known groups such as $A_4 \cong \Delta(12)$, $\Delta(27)$, T_7 and T_{13}. However, there are also groups of type (C) which are neither of the form T_m, nor of the form $\Delta(3m^2)$. The smallest example, which is not a direct product, is the group [4]

$$C(9,1,1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3. \tag{23.8}$$

The groups of type (D) The groups $D(n,a,b;d,r,s)$ are generated by the generators E and $F(n,a,b)$ of (C) and the additional generator $\widetilde{G}(d,r,s)$–see equation (23.2). It was shown in [6] that by means of a unitary transformation one can obtain a different set of generators consisting of three diagonal matrices and the two S_3-generators

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{23.9}$$

Thus, as in the case of (C), the subgroup $N(n,a,b;d,r,s)$ of diagonal matrices is an invariant subgroup, and the structure of the groups of type (D) is found to be

$$D(n,a,b;d,r,s) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes S_3, \tag{23.10}$$

where $N(n,a,b;d,r,s) \cong \mathbb{Z}_m \times \mathbb{Z}_p$ and S_3 is generated by E and B.

For the special case of $p = m$ we obtain the groups $(\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes S_3 \cong \Delta(6m^2)$. Thus the groups of type (D) comprise the well-known groups $S_4 \cong \Delta(24)$, $\Delta(54)$ and $\Delta(96)$. The smallest group of type (D), which is neither a direct product, nor of the form $\Delta(6m^2)$, is [4]

$$D(9,1,1;2,1,1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes S_3. \tag{23.11}$$

[2] m must be a product of powers of primes of the form $6k+1$, $k \in \mathbb{N}$ [5].

(E) The exceptional finite subgroups of SU(3) This set of groups collects all finite subgroups of SU(3) which do not fall into one of the categories (A)–(D). Among these groups are the two simple groups $\Sigma(60) \cong I \cong A_5$ and $\Sigma(168) \cong \mathrm{PSL}(2,7)$. For a detailed treatment of these two groups we refer the reader to [9]. Also the direct products $\Sigma(60) \times \mathbb{Z}_3$ and $\Sigma(168) \times \mathbb{Z}_3$ are finite subgroups of SU(3).

The remaining four exceptional groups are $\Sigma(36\times3)$, $\Sigma(72\times3)$, $\Sigma(216\times3)$ and $\Sigma(360\times3)$. Their generators can be found in [3]. For a detailed study of the first three groups we refer the reader to [10]. The largest exceptional finite SU(3)-subgroup $\Sigma(360\times3)$ possesses only one non-trivial invariant subgroup, namely the center $\{\mathbb{1}, \omega\mathbb{1}, \omega^2\mathbb{1}\} \cong \mathbb{Z}_3$ of SU(3) ($\omega = \exp(2\pi i/3)$). The corresponding factor group $\Sigma(360) \equiv \Sigma(360\times3)/\mathbb{Z}_3$ is isomorphic to the permutation group A_6 [11]. The character table of $\Sigma(360\times3)$ can be found in [7]. $\Sigma(60) \cong I \cong A_5$ is a subgroup of $\Sigma(360\times3)$.

23.3 Representations of the finite subgroups of SU(3)

By definition a finite subgroup of SU(3) possesses at least one three-dimensional representation of determinant one. This representation is not necessarily irreducible, however many SU(3)-subgroups possess three-dimensional irreps.

- The groups of type (A) are Abelian, which implies that all their irreps are one-dimensional.
- Much less is known about the representations of the groups of type (B), which are the finite subgroups of U(2). By definition they possess at least one two-dimensional faithful representation. The dihedral groups D_n and their double covers possess only one- and two-dimensional irreps [12], while the double covers $\widetilde{T}$, $\widetilde{O}$ and $\widetilde{I}$ of the rotation groups T, O and I possess also three- and higher-dimensional irreps [6].
- It was shown in [6] that the groups of type (C) possess only one- and three-dimensional irreps.
- Also the dimensions of the irreps of the groups of type (D) can be determined in general. A group of type (D) can possess one-, two-, three- and six-dimensional irreps [6].
- All of the exceptional finite subgroups (E) of SU(3) possess three-dimensional irreps. Since $\Sigma(60) \cong I \cong A_5$ and $\Sigma(168) \cong \mathrm{PSL}(2,7)$ are simple, all their non-trivial irreps are faithful. Detailed information on the irreps of $\Sigma(36\times3)$, $\Sigma(72\times3)$ and $\Sigma(216\times3)$ can be found in [10]. The largest exceptional SU(3)-subgroup $\Sigma(360\times3)$ possesses irreps of dimensions 1,3,5,6,8,9,10 and 15 [7].

Let us finish this section with a theorem, which can be very helpful when one looks for finite groups which possess irreps of a given dimension.

Theorem 2. The dimension of an irrep of a finite group is a divisor of the order of the group.

A collection of helpful theorems including references to their proofs can be found in [6].

23.4 Summary and outlook

In this work we reviewed the structure of the finite subgroups of SU(3). A summary of these results can be found in figure 23.1, which shows the finite subgroups of SU(3) classified into the five types defined in [3]. Among the five types, we studied (A), (C) and (D) in more detail.

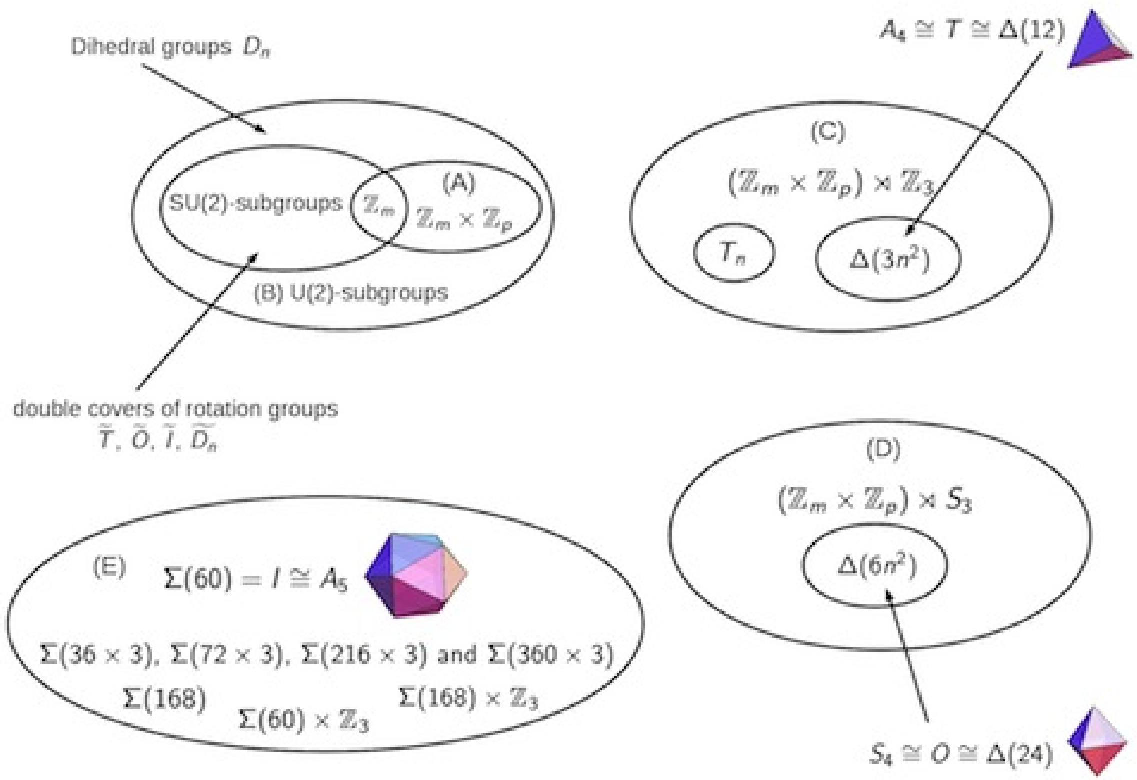

Figure 23.1: The finite subgroups of SU(3) as presented in section 23.2.

The Abelian finite subgroups of SU(3) were found to have the structure of a direct product $\mathbb{Z}_m \times \mathbb{Z}_p$. This insight allowed to determine the general structure of the hitherto not very well-known SU(3)-subgroups of type (C) and (D) which is similar to the one of the well-known series $\Delta(3n^2)$ and $\Delta(6n^2)$, which are subseries of (C) and (D), respectively.

The finite subgroups of SU(3) comprise an interesting field of study, especially with respect to their application as symmetries in particle physics. An interesting question frequently arising in the context of flavour physics is the breaking of a group to one of its subgroups. For works dealing with this question–especially in the context of SU(3)-subgroups–we refer the reader to [13, 14].

Finally we would like to mention two very helpful tools for studying finite groups, namely the computer algebra system GAP [15] and the SmallGroups library [16], a GAP-package which provides valuable information on all finite groups up to order 2000. Two examples for works where these tools have been successfully used are [17, 18].

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