Negative Even Grade mKdV Hierarchy and its Soliton Solutions

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Abstract. In this paper we discuss the algebraic construction of the mKdV hierarchy in terms of an affine Lie algebra $\hat{sl}(2)$. An interesting novelty araises from the negative even grade sector of the affine algebra leading to nonlinear integro-differential equations admiting non-trivial vacuum configuration. These solitons solutions are constructed systematically from generalization of the dressing method based on non zero vacua. The sub-hierarchies admiting such class of solutions are classified.

1. Introduction

A subclass of non linear integrable models underlined by an affine $\hat{sl}(2)$ Lie algebra is well known to be connected to the mKdV equation. These are in fact higher flows obtained from the zero curvature representation [1] and by general algebraic arguments these flows are restricted to be related to certain positive odd grade generators. Another subclass of nonlinear integrable models containing for instance, the sinh-Gordon model may be formulated, also within the zero curvature representation, but now associated to negative odd generators. The relation between the mKdV and sine-Gordon models was observed some time ago [2], [3] in terms of conservation laws and by more algebraic arguments in [4]. In this paper we review an universal algebraic approach for constructing these models an show that there is a third consistent subclass of models associated to *negative even* grade generators. These later are shown to admit non trivial vacuum configuration.

In this notes we start by reviewing, in section 2, the algebraic construction of the mkdV hierarchy and show how the *positive and negative odd* sub-hierarchis naturally arises from the zero curvature representation. We then discuss the compatibility of the *negative even* sub-hierarchy and develop explicitly the first nontrivial equation. This is an integro-differential nonlinear equation proposed in [5] using recursion operator techniques and later developed further within the algebraic approach in [6]. Contrary to the known cases of positive and negative odd sub-hierarchies, it is observed that a constant (non-zero) configuration is a solution of such equation (non-trivial vacuum solution). This fact introduces extra terms within the zero curvature representation of the vacuum which in turn, induces deformation in the dressing method and in the vertex operators for constructing soliton solutions. In section 3 we discuss in detail the generalization required within the dressing formalism to incorporate this class of solutions and construct explicitly the one solitons solution. Finally in section 4 we discuss and classify the sub-hierarchies admiting non-trivial vacuum solutions.

2. The mKdV Hierarchy

In this section we review the basic ingredients for constructing the mkdV hierarchy. Let us start with an affine algebra $\mathcal{G} = \hat{sl}(2)$ with generatos $\{h^{(m)} = \lambda^m h, E_{\pm\alpha}^{(m)} = \lambda^m E_{\pm\alpha}\}$ satisfying

$$[h^{(m)}, E_{\pm\alpha}^{(n)}] = \pm 2E_{\pm\alpha}^{(n)}, \qquad [E_{\alpha}^{(m)} E_{-\alpha}^{(n)}] = h^{(m+n)} \tag{1}$$

A second important ingredient is a grading operator $Q = 2\lambda\frac{d}{d\lambda} + \frac{1}{2}h$ which decomposes the affine algebra $\mathcal{G}$ into graded subspaces $\mathcal{G} = \oplus\mathcal{G}_a$ such that $[Q, \mathcal{G}_a] = a\mathcal{G}_a$, where,

$$\begin{aligned} \mathcal{G}_{2m} &= \{h^{(m)} = \lambda^m h\}, \\ \mathcal{G}_{2m+1} &= \{\lambda^m (E_\alpha + \lambda E_{-\alpha}),\ \lambda^m (E_\alpha - \lambda E_{-\alpha})\} \end{aligned} \tag{2}$$

for $m = 0, \pm 1, \pm 2, \cdots$ and $[\mathcal{G}_a, \mathcal{G}_b] \subset \mathcal{G}_{a+b}$. The integrable hierarchy is then specified by a choice of a semi-simple element $E = E^{(1)}$, where

$$E^{(2n+1)} = \lambda^n (E_\alpha + \lambda E_{-\alpha}) \tag{3}$$

which decomposes $\mathcal{G} = \mathcal{K} + \mathcal{M}$ where $\mathcal{K} = \{x, /[x, E] = 0\}$ is the Kernel of E and $\mathcal{M}$ is its complement. From (2) it follows that

$$\mathcal{K} = \mathcal{K}_{2n+1} = \{\lambda^n (E_\alpha + \lambda E_{-\alpha})\} \tag{4}$$

has grade $2n + 1$. We assume that E is *semi-simple* in the sense that this second decomposition is such that

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \qquad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M}, \qquad [\mathcal{M}, \mathcal{M}] \subset \mathcal{K}.$$

We now define the Lax operator

$$L = \partial + E + A_0, \qquad A_0 \in \mathcal{M} \in \mathcal{G}_0 \tag{5}$$

where $A_0 = vh^{(0)}$ and $v = v(x, t)$ is the physical field.

2.1. Zero Curvature for the Positive Hierarchy

Let us consider the zero curvature representation for the positive hierarchy

$$[\partial_x + E^{(1)} + A_0, \partial_{t_N} + D^{(N)} + D^{(N-1)} + \cdots + D^{(0)}] = 0, \tag{6}$$

where $D^{(j)} \in \mathcal{G}_j$. Eqn. (6) can be decomposed according to the graded structure as

$$\begin{aligned} [E, D^{(N)}] &= 0 \\ [E, D^{(N-1)}] + \partial_x D^{(N)} &= 0 \\ \vdots \quad &= \quad \vdots \\ [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_N} A_0 &= 0 \rightarrow \text{Eq. of motion} \end{aligned} \tag{7}$$

and the unknown $D^{(j)}$'s can be solved starting from the highest to the lowest grade projections. Notice that, in particular the highest grade equation, namely $[E, D^{(N)}] = 0$ implies $D^{(N)} \in \mathcal{K}$

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and since from (4), $\mathcal{K} \in \mathcal{G}_{2n+1}$, $N = 2n+1$. The component within the $\mathcal{M}$ of the zero grade projection leads to the evolutions equations according to time $t = t_{2n+1}$. Examples are,

$$N = 3 \qquad 4v_{t_3} = v_{3x} - 6v^2 v_x, \qquad mKdV$$

$$N = 5 \qquad 16v_{t_5} = v_{5x} - 10v^2 v_{3x} - 40 v v_x v_{2x} - 10 v_x^3 + 30 v^4 v_x,$$

$$\begin{aligned} N = 7 \qquad 64v_{t_7} &= v_{7x} - 182 v_x v_{2x}^2 - 126 v_x^2 v_{3x} - 140 v v_{2x} v_{3x} \\ &- 84 v v_x v_{4x} - 14 v^2 v_{5x} + 420 v^2 v_{3x} + 560 v^3 v_x v_{2x} \\ &+ 70 v^4 v_{3x} - 140 v^6 v_x \\ \cdots etc & \end{aligned}$$

2.2. Negative mKdV Hierarchy

For the negative mKdV hierarchy let us propose the following form for the zero curvature representation

$$[\partial_x + E^{(1)} + A_0, \partial_{t_{-N}} + D^{(-N)} + D^{(-N+1)} + \cdots + D^{(-1)}] = 0. \tag{8}$$

Differently from the positive hierarchy case, the lowest grade projection now yields,

$$\partial_x D^{(-N)} + [A_0, D^{(-N)}] = 0$$

a nonlocal equation for $D^{(-N)}$. Having solved for $D^{(-N)}$, the second lowest projection of grade $-N+1$, leads to

$$\partial_x D^{(-N+1)} + [A_0, D^{(-N+1)}] + [E^{(1)}, D^{(-N)}] = 0$$

which determines $D^{(-N+1)}$. The proccess follows recursively until we reach the zero grade projection

$$\partial_{t_{-N}} A_0 + [E^{(1)}, D^{(-1)}] = 0 \tag{9}$$

which yields the evolution equation for field A_0 according to time $t = t_{-N}$ Notice that in this case there is no condition upon N.

The simplest example is to take $N = 1$ when the zero curvature decomposes into

$$\begin{aligned} \partial_x D^{(-1)} + [A_0, D^{(-1)}] &= 0, \\ \partial_{t_{-1}} A_0 - [E^{(1)}, D^{(-1)}] &= 0. \end{aligned} \tag{10}$$

In order to solve the first equation, we define the zero grade group element $B = \exp(\mathcal{G}_0)$ and define

$$D^{(-1)} = B^{-1} E^{(-1)} B, \qquad A_0 = B^{-1}\partial_x B, \tag{11}$$

Under such parametrization the second eqn. (11) becomes the well known Leznov-Saveliev equation,

$$\partial_{t_{-1}} \left(B^{-1} \partial_x B \right) = [E^{(1)}, B^{-1} E^{(-1)} B] \tag{12}$$

which for $\hat{sl}(2)$ with principal gradation $Q = 2\lambda\frac{d}{d\lambda} + \frac{1}{2}h$, yields the sinh-Gordon equation (*relativistic*)

$$\partial_{t_{-1}}\partial_x\phi = e^{2\phi} - e^{-2\phi}, \qquad B = e^{\phi h}. \tag{13}$$

where $t_{-1} = z, x = \bar{z}, A_0 = vh = \partial_x\phi h$. An important fact to point out is that $\phi = 0$ corresponds to the vacuun solution of eqn. (13).

The next simple example is for $N = 2$ when the zero curvature decomposes as (see [6])

$$\begin{aligned} \partial_x D^{(-2)} + [A_0, D^{(-2)}] &= 0, \\ \partial_x D^{(-1)} + [A_0, D^{(-1)}] + [E^{(1)}, D^{(-2)}] &= 0, \\ \partial_{t_{-2}} A_0 - [E^{(1)}, D^{(-1)}] &= 0. \end{aligned} \tag{14}$$

Propose solution of the form

$$\begin{aligned} D^{(-2)} &= c_{-2}\lambda^{-1}h, \\ D^{(-1)} &= a_{-1}\left(\lambda^{-1}E_\alpha + E_{-\alpha}\right) + b_{-1}\left(\lambda^{-1}E_\alpha - E_{-\alpha}\right). \end{aligned}$$

we find $c_{-2} = const$ and

$$\begin{aligned} a_{-1} + b_{-1} &= 2c_{-2}\exp(-2d^{-1}v)d^{-1}\left(\exp(2d^{-1}v)\right), \\ a_{-1} - b_{-1} &= -2c_{-2}\exp(2d^{-1}v)d^{-1}\left(\exp(-2d^{-1}v)\right), \end{aligned}$$

Equation of motion is

$$\partial_{t_{-2}}v + 2c_{-2}e^{-2d^{-1}v}d^{-1}\left(e^{2d^{-1}v}\right) + 2c_{-2}e^{2d^{-1}v}d^{-1}\left(e^{-2d^{-1}v}\right) = 0. \tag{15}$$

where $d^{-1}f = \int^x f(x')dx'$. Oposite to the previous example $v = 0$ is not solution of (15), however, it is interesting to notice that $v = v_0 = \text{const.} \neq 0$ is a solution.

3. Dressing Method

In order to employ the dressing method to construct systematically soliton solutions, we need to consider the full Kac-Moody algebra with central terms to ensure the existence of highest weight states, i.e.,

$$\begin{aligned} [h^{(m)}, h^{(n)}] &= 2m\delta_{m+n,0}\hat{c}, \\ [h^{(m)}, E^{(n)}_{\pm\alpha}] &= \pm 2E^{(m+n)}_{\pm\alpha}, \\ [E^{(m)}_{\alpha}, E^{(n)}_{-\alpha}] &= h^{(m+n)} + m\delta_{m+n,0}\hat{c} \end{aligned}$$

together with the derivation operator $\hat{d}$ such that,

$$[\hat{d}, T^{(n)}_a] = nT^{(n)}_a, \qquad T^{(n)}_a = \{h^{(n)},\ E^{(n)}_{\pm\alpha}\}.$$

The grading operator now reads $Q = 2\hat{d} + 1/2h^{(0)}$.

The zero curvature condition implies pure gauge connections,

$$A_x = T^{-1}\partial_x T = E + A_0$$

and

$$A_{t_n} = T^{-1}\partial_{t_n}T = D^{(n)} + \cdots + D^{(0)}$$

or

$$A_{t_{-n}} = T^{-1}\partial_{t_{-n}}T = D^{(-n)} + \cdots + D^{(-1)},$$

according to positive or negative hierarchies respectively.

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3.1. Dressing for zero vacua solution

Suppose there exists a vacuum solution $v = 0$ satisfying

$$A_{x,vac} = E^{(1)} - t_k\delta_{k+1,0}\hat{c}, \qquad A_{t_k,vac} = E^{(k)},$$

The solution for $A_{x,vac} = T_0^{-1}\partial_x T_0$ and $A_{t_k,vac} = T_0^{-1}\partial_{t_k} T_0$ is therefore given by

$$T_0 = \exp(xE^{(1)})\exp(t_k E^{(k)}).$$

The dressing method is based on the assumption of the existence of two gauge transformations, generated by $\Theta_\pm$, mapping the vacuum into non trivial configuration, i.e.

$$\begin{aligned} A_x &= (\Theta_\pm)^{-1}A_{x,vac}\Theta_\pm + (\Theta_\pm)^{-1}\partial_x\Theta_\pm, \\ A_{t_k} &= (\Theta_\pm)^{-1}A_{t_k,vac}\Theta_\pm + (\Theta_\pm)^{-1}\partial_{t_k}\Theta_\pm. \end{aligned}$$

where $\Theta_\pm$ are group elements of the form

$$\Theta_-^{-1} = e^{p(-1)}e^{p(-2)}\dots, \qquad \Theta_+^{-1} = e^{q(0)}e^{q(1)}e^{q(2)}\dots,$$

$p^{(-i)}$ and $q^{(i)}$ are linear combinations of grade $(-i)$ and (i) generators, respectively.

As a consequence we relate

$$\Theta_-\Theta_+^{-1} = T_0^{-1}gT_0$$

where g is an arbitrary constant group element. For instance, from

$$A_x = E^{(1)} + B^{-1}\partial_x B + \partial_x\nu\hat{c} - t_k\delta_{k+1,0}\hat{c} = (\Theta_+)^{-1}(\partial_x + E)\Theta_+,$$

we find,

$$e^{q(0)} = B^{-1}e^{-\nu\hat{c}}$$

and henceforth

$$\dots e^{-p(-2)}e^{-p(-1)}B^{-1}e^{-\nu\hat{c}}e^{q(1)}e^{q(2)}\dots = T_0^{-1}gT_0$$

hence,

$$< \lambda'|B^{-1}|\lambda > e^{-\nu} = < \lambda'|T_0^{-1}gT_0|\lambda >$$

where $|\lambda >$ and $< \lambda'|$ are annihilated by $\mathcal{G}_>$ and $\mathcal{G}_<$, respectively. Explicit space time dependence for the field in $\mathcal{G}_0$, is given by choosing specific matrix elements

$$\begin{aligned} e^{-\nu} &= < \lambda_0|T_0^{-1}gT_0|\lambda_0 > \equiv \tau_0, \\ e^{-\phi-\nu} &= < \lambda_1|T_0^{-1}gT_0|\lambda_1 > \equiv \tau_1. \end{aligned}$$

i.e.,

$$e^{\phi} = ln\,(\tau_0/\tau_1)$$

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Suppose we now write the constant group element g as

$$g = \exp\{F(\gamma)\},$$

where γ is a complex parameter and we choose $F(\gamma)$ to be an eigenstate of $E^{(k)}$, i.e.

$$[E^{(k)}, F(\gamma)] = f^{(k)}(\gamma) F(\gamma)$$

where $f^{(k)}$ are specific functions of γ. It therefore follows that

$$T_0^{-1} g T_0 = \exp\{\rho(\gamma) F(\gamma)\}$$

where

$$\rho(\gamma) = \exp\{-t_k f^{(k)}(\gamma) - x f^{(1)}(\gamma)\}.$$

For more general cases in which

$$g = \exp\{F_1(\gamma_1)\} \exp\{F_2(\gamma_2)\} \ldots \exp\{F_N(\gamma_N)\}$$

with

$$[E^{(k)}, F_i(\gamma_i)] = f_i^{(k)}(\gamma_i) F_i(\gamma_i)$$

we find

$$T_0^{-1} g T_0 = \exp\{\rho_1(\gamma_1) F_1(\gamma_1)\} \exp\{\rho_2(\gamma_2) F_2(\gamma_2)\} \ldots \exp\{\rho_N(\gamma_N) F_N(\gamma_N)\}$$

where

$$\rho_i(\gamma_i) = \exp\{-t_k f_i^{(k)}(\gamma_i) - x f_i^{(1)}(\gamma_i)\}.$$

The specific eigenstate, in this case of $\hat{sl}(2)$, is given by

$$F(\gamma) = \sum_{n=-\infty}^{\infty} \left(h^{(n)} - \frac{1}{2}\delta_{n,0}\hat{c}\right)\gamma^{-2n} + \left(E_\alpha^{(n)} - E_{-\alpha}^{(n+1)}\right)\gamma^{-2n-1}$$

whose eigenvalues are obtained from

$$[E^{(k)}, F(\gamma)] = -2\gamma^k F(\gamma).$$

An important property of Vertex operators $F(\gamma)$ is nilpotency, i.e., It can be shown that

$$F(\gamma)^2 = 0$$

which implies truncation

$$\begin{aligned} T_0^{-1} g T_0 &= \exp\{\rho_1 F_1\} \exp\{\rho_2 F_2\} \ldots \exp\{\rho_N F_N\} \\ &= (1 + \rho_1 F_1)(1 + \rho_2 F_2) \cdots (1 + \rho_N F_N) \end{aligned}$$

for $\rho_i \equiv \rho_i(\gamma_i)$, $\quad F_i \equiv F_i(\gamma_i)$.

Examples soliton solution

$$\begin{aligned} \phi_{1-sol} &= ln\left(\frac{1 - a_1\rho_1}{1 + a_1\rho_1}\right) \\ \phi_{2-sol} &= ln\left(\frac{1 - a_1\rho_1 - a_2\rho_2 + a_1a_2\rho_1\rho_2}{1 + a_1\rho_1 + a_2\rho_2 + a_1a_2a_{12}\rho_1\rho_2}\right) \\ \vdots &= \vdots \end{aligned} \tag{16}$$

$a_{12} = \left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)$ and

$$\rho_i(\gamma_i) = \exp\{2t_k\gamma_i^k + 2x\gamma_i\}. \tag{17}$$

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3.2. Dressing Solution for non-trivial vacuun configuration

Propose the simplest non trivial vacuum configuration

$$\begin{aligned} A_{x,vac} &= \left(E_{\alpha}^{(0)} + E_{-\alpha}^{(1)}\right) + v_0 h^{(0)} - \frac{1}{v_0} t_{-2m} \delta_{m-1,0} \hat{c}, \\ A_{t_{-2m},vac} &= \frac{1}{v_0}\left(E_{\alpha}^{(-m)} + E_{-\alpha}^{(1-m)}\right) + h^{(-m)} \end{aligned}$$

with $v_0 = const. \neq 0$. It is straightforward to verify the zero curvature equation

$$[\partial_x + A_{x,vac}, \partial_{t_{-2m}} + A_{t_{-2m},vac}] = 0.$$

This *nontrivial vacuum* leads to the following deformation, i.e., $A_{x,vac} = T_0^{-1}\partial_x T_0$ and $A_{t_k,vac} = T_0^{-1}\partial_{t_k} T_0$,

$$\begin{aligned} T_0 = \exp\left\{x\left(E_{\alpha}^{(0)} + E_{-\alpha}^{(1)} + v_0 h^{(0)}\right)\right\} \cdot \\ \exp\left\{\frac{t_{-2m}}{v_0}\left(E_{\alpha}^{(-m)} + E_{-\alpha}^{(1-m)} + v_0 h^{(-1)}\right)\right\}. \end{aligned}$$

which leads to

$$e^{q(0)} = B^{-1} e^{x v_0 h^{(0)}} e^{-\nu \hat{c}}.$$

The solution is then given by

$$\begin{aligned} e^{-\nu} &= <\lambda_0|T_0^{-1} g T_0|\lambda_0> \equiv \tau_0, \\ e^{-\phi + x v_0 - \nu} &= <\lambda_1|T_0^{-1} g T_0|\lambda_1> \equiv \tau_1 \end{aligned}$$

and hence,

$$v = v_0 - \partial_x \ln\left(\frac{\tau_0}{\tau_1}\right), \qquad v = \partial_x \phi. \tag{18}$$

In order to construct explicit soliton solutions we need a simultaneous eigenstate of

$$\begin{aligned} b_1 &\equiv E_{\alpha}^{(0)} + E_{-\alpha}^{(1)} + v_0 h^{(0)}, \\ b_{-2m} &\equiv \left(E_{\alpha}^{(-m)} + E_{-\alpha}^{(-m+1)} + v_0 h^{(-m)}\right) \end{aligned}$$

Let

$$\begin{aligned} F(\gamma, v_0) = \sum_{n=-\infty}^{\infty} \left(\gamma^2 - v_0^2\right)^{-n} [h^{(n)} &+ \frac{v_0 - \gamma}{2\gamma}\delta_{n,0}\hat{c} + E_{\alpha}^{(n)}(\gamma + v_0)^{-1} \\ &- E_{-\alpha}^{(n+1)}(\gamma - v_0)^{-1}]. \end{aligned}$$

be our deformed vertex operator. A direct calculation shows that

$$\begin{aligned} [b_1, F(\gamma, v_0)] &= -2\gamma F(\gamma, v_0), \\ [b_{-2m}, F(\gamma, v_0)] &= -2\gamma\left(\gamma^2 - v_0^2\right)^{-m} F(\gamma, v_0). \end{aligned}$$

Therefore we find,

$$\rho(\gamma, v_0) = \exp\left\{2\gamma x + \frac{2\gamma t_{-2m}}{v_0\left(\gamma^2 - v_0^2\right)^m}\right\}. \tag{19}$$

if we take,

$$g = \exp\{F(\gamma, v_0)\}$$

We find the soliton solution from (18) with

$$\tau_0 = 1 + C_0 \rho(\gamma, v_0), \qquad \tau_1 = 1 + C_1 \rho(\gamma, v_0)$$

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4. Models with non-trivial vacua

In this section we consider non-trivial vacum configuration, i.e., $v = v_0 \neq= 0$ and discuss which models within the hierarchy satisfy the zero curvature representation. Let us consider the following cases

- **Negative Even Hierarchy**
The zero curvature representation in the vacuum is

$$[\partial_x + E + v_0 h^{(0)}, \partial_{t_{-2m}} + D^{(-2m)}_{vac} + D^{(-2m+1)}_{vac} + \cdots + D^{(-2)}_{vac} + D^{(-1)}_{vac}] = 0$$

with solution given as

$$D^{(-2m)}_{vac} = \lambda^{-m} h^{(0)} \qquad D^{(-2m+1)}_{vac} = \frac{\lambda^{-m} E}{v_0},$$
$$\vdots \qquad \vdots$$
$$D^{(-2)}_{vac} = \lambda^{-1} h^{(0)} \qquad D^{(-1)}_{vac} = \frac{\lambda^{-1} E}{v_0},$$

Since the r.h.s. of zero curvature equation requires $2m$ terms which combine together to form

$$D^{(-2i)}_{vac} + D^{(-2i+1)}_{vac} = \lambda^{-i}(h^{(0)} + \frac{E}{v_0})$$

such that

$$[\partial_x + E + v_0 h^{(0)}, \partial_{t_{-2m}} + \lambda^{-m}(h^{(0)} + \frac{E}{v_0}) + \cdots + \lambda^{-1}(h^{(0)} + \frac{E}{v_0})] = 0$$

all models within the negative even sub-hierarchy admit non-trivial vacuum configuration.

- **Positive Odd Hierarchy**
This case contains the mKdV equation, etc and the zero curvature representation in the vacuum is

$$[\partial_x + E + v_0 h^{(0)}, \partial_{t_{2m+1}} + D^{(2m+1)}_{vac} + D^{(2m)}_{vac} + \cdots + D^{(1)}_{vac} + D^{(0)}_{vac}] = 0$$

with solution

$$D^{(2m+1)}_{vac} = \frac{\lambda^m E}{v_0} \lambda^m h^{(0)} \qquad D^{(2m)}_{vac} = \lambda^m h^{(0)},$$
$$\vdots \qquad \vdots$$
$$D^{(1)}_{vac} = \frac{\lambda E}{v_0} \qquad D^{(0)}_{vac} = h^{(0)},$$

The r.h.s. of zero curvature equation requires $2m$ terms which combine together to form

$$D^{(2i+1)}_{vac} + D^{(2i)}_{vac} = \lambda^i (h^{(0)} + \frac{E}{v_0})$$

such that

$$[\partial_x + E + v_0 h^{(0)}, \partial_{t_{2m+1}} + \lambda^m(h^{(0)} + \frac{E}{v_0}) + \cdots + \lambda^0(h^{(0)} + \frac{E}{v_0})] = 0$$

and therofore all models within the positive odd sub-hierarchy admit solutions with non-trivial vacuum configuration.

- **Negative Odd Hierarchy**
  This is a subcalss of models which contains the sinh-Gordon and the zero curvature representation is

$$[\partial_x + E + v_0 h^{(0)} \quad , \quad \partial_{t_{-2m-1}} + D_{vac}^{(-2m-1)} \\ + \quad D_{vac}^{(-2m)} + D_{vac}^{(-2m+1)} + \cdots + D_{vac}^{(-2)} + D_{vac}^{(-1)}] = 0$$

  Notice the r.h.s. of zero curvature equation has $2m+1$-terms that they cannot combine into terms proportional to $E + v_0 h^{(0)}$ and henceforth the negative odd hierarchy does not allow constant vacuum solutions.

5. Conclusions and Outlook

The general algebraic structure of the mKdV hierarchy has lead to the construction of sub-hierarchies and to the classification of its soliton solutions with trivial and non trivial vacuum. It is interesting to point out that within each sector, the soliton solutions for diferent members (equations) of the hierarchy are all related by keeping the general form (16) or (18), varying only from the time evolution parameter in (17) or (19).

More complicated examples may be constructed following the same line of reasoning by considering higher rank affine Lie algebras and generalized mixed gradations (see for instance [7]). On the other hand, it would be interesting to develop the dressing method to construct periodic solutions for the mkdV equation. These are known to be written in terms of ratios of Jacobi Theta functions and under certain specific limit, yield the soliton solutions constructed in (16).

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