mathematics

IMPACT FACTOR 2.3

CITESCORE 4.0

Article

Solutions of the Mathieu–Hill Equation for a Trapped-Ion Harmonic Oscillator—A Qualitative Discussion

Bogdan M. Mihalcea

MDPI

https://doi.org/10.3390/math12192963

mathematics

MDPI

Article

Solutions of the Mathieu–Hill Equation for a Trapped-Ion Harmonic Oscillator—A Qualitative Discussion

Bogdan M. Mihalcea

National Institute for Laser, Plasma and Radiation Physics (INFLPR), Atomiştilor Str. Nr. 409, 077125 Măgurele, Romania; bogdan.mihalcea@inflpr.ro

Abstract: We investigate solutions of the classical Mathieu–Hill (MH) equation that characterizes the dynamics of trapped ions. The analytical model we introduce demonstrates the equations of motion are equivalent to those of a harmonic oscillator (HO). Two independent approaches are used, based on two classes of complex solutions of the MH equation. This paper addresses both a damped HO and parametric oscillator (PO) for an ion confined in an electrodynamic (Paul) trap, along with stability and instability regions for the associated periodic orbits.

Keywords: Mathieu–Hill equation; Floquet theory; Sturm–Liouville theorem; electrodynamic trap; stability diagram

MSC: 37M10

check for updates

Citation: Mihalcea, B.M. Solutions of the Mathieu–Hill Equation for a Trapped-Ion Harmonic Oscillator—A Qualitative Discussion. *Mathematics* **2024**, *12*, 2963. https://doi.org/10.3390/math12192963

Academic Editor: Jonathan Blackledge

Received: 26 August 2024
Revised: 20 September 2024
Accepted: 21 September 2024
Published: 24 September 2024

1. Introduction

Parametric excitation of second-order nonlinear systems [1,2] has been the subject of intense investigations [3,4], amongst which we mention the method of normal forms. Ref. [5] uses such a method to investigate the dynamical stability of a nonlinear system characterized by a nonlinear Mathieu–Duffing equation of motion [1,6–11] in case of parametric excitation [12]. It was demonstrated that the intrinsic nonlinear nature of the system induces a subharmonic region that was not previously reported for systems characterized by linear MH equations [8,13]. Normal modes analysis is also used in [14] to describe dynamical stability for trapped ion systems.

Nanoelectromechanical systems (NEMS) enable performing tests on the fundaments of quantum mechanics by studying the transition from classical to quantum behaviour of a driven nonlinear Duffing resonator [15]. The numerical solutions of the equations of motion associated to the resonator demonstrate that the quantum Wigner function slowly deviates from the corresponding classical phase-space probability density. Thus, nonlinearity is the cause of such differences which provide experimental evidence that NEMS resonators are excellent systems for quantum simulation and investigations of nonlinearity. On the other hand, ion traps are versatile tools that enable interdisciplinary investigations on nonlinearity [16–19].

Ref. [20] introduces a method to calculate the wave functions and energies of ground state and low-lying excited states of quantum multibody systems by employing the deep neural network and the unsupervised machine learning (ML) technique. A simple method of symmetrization for bosonic systems and antisymmetrization for fermionic systems is also proposed, in order to perform calculations in case of many-particle systems consisting of identical particles.

Investigations of the stability of physical systems that undergo periodic parametric driving [21], such as ions confined by oscillating electric fields (Paul traps) [22], is a subject of vivid scientific interest. The behaviour of these systems can be better explained based on an approach that employs the pseudopotential approximation, which explains well trapping of charged particles within a quadratic potential and resonances that arise out of parametric excitation [14]. In the pseudopotential approximation (for large values of the radiofrequency—RF), both in the classical and quantum cases, the time periodic Hamilton function that characterizes an ion confined within an electrodynamic trap is replaced by an autonomous (time-independent) Hamilton function. Furthermore, linear ion traps (LIT) that operate at two RF values represent a versatile tool to simultaneously confine two ion species of interest [23]. Parametric excitation is also employed to carry out mass-selective removal of ions out of an electrodynamic trap [24–26].

It is well established that a single trapped ion can be employed as a paradigm to test a choice of essential physical models realized as time-dependent (quantum) harmonic oscillators (QHO) [21,27–42]. A numerical modelling for a segmented Paul trap that exhibits four blade electrodes that generate the trapping field and a pair of two biasing rods, where the latter are employed to compensate micromotion [43], is performed in ref. [44]. Such an approach delivers enhanced optical access for both fluorescence spectroscopy and individual ion addressing, a mandatory requirement to achieve ion crystals or perform quantum engineering and manipulation [45] of trapped ion quantum states [46,47]. Applications span atomic clocks [48–50], which are excellent tools to search for physics Beyond the Standard Model (BSM) [51,52] or to test novel quantum technologies (QT) based on ultracold trapped ions [53,54]. Multipole linear Paul traps (LPTs) are also versatile tools to investigate cold ion-atom collisions and thus provide new experimental evidence of the phenomena involved [55–58].

A matter of utmost importance is the issue of quantum-enhanced measurements that exhibit potential to enhance high-precision sensing. Applications span areas such as ultraprecise optical clocks [50,59,60], realization of high-fidelity quantum logic gates [61], many-body quantum enhanced sensors [62,63] or measurements on the time-variations of the fundamental constants of nature at the cosmological scale [64]. Quantum sensors suffer from sensitivity loss in the presence of quantum noise [65]. To mitigate such an effect, different quantum error-correcting codes have been tested and implemented. Ultracold trapped ions demonstrate to be a promising platform for both quantum sensing and quantum error correction experiments [39,53,66].

Ref. [2] investigates Hamiltonian dynamics [67] of a single QHO [21,68,69] in the presence of dissipation and parametric driving, while it establishes that a time-dependent parametric frequency incessantly drives the system out of its dynamical equilibrium state. In addition, fine-tuning of the system parameters enables one to control the contention between dissipation and parametric driving in a similar way to which Ref. [16] investigates the competition between multipole anharmonicities of the trap electric potential and a periodic kicking term (laser field), driving the system from a regular motion regime towards chaotic dynamics [17]. To achieve control of a QHO, a key issue lies in characterizing and suppressing the inherent noise [70,71]. The intrinsic noise spectrum of a trapped ion can be characterized and experimental demonstrations of QHO control are recently reported in literature [39].

The paper is intended as a follow-up of the review paper recently published in Photonics [72]. Section 2 starts from the analytical model introduced in [73]. The paper investigates the issue of the solutions of the MH equation that explains ion dynamics within an electrodynamic ion trap (EIT). We show such an equation can be expressed as a HO equation and suggest two classes of complex solutions. It is well established that the MH equation solution can be expressed as a Hill series [74–77] and we analyse such cases. The first method we use yields a system of linear equations that enables one to determine the constants in the Fourier series solution. In case of the second method suggested, we derive a recursive relationship between these coefficients for a particular solution of the MH equation. Both approaches are original. Section 3 applies the results previously derived in Section 2 and supplies an analytical model to characterize ion dynamics as bound or unbound. By considering the residual interaction we supply the solutions of the MH equation. Finally, we demonstrate the MH equation can be regarded as a HO equation.

In Appendix A.1 we show a trapped ion can be assimilated with a kicked-damped HO model and discuss solutions depending on the discriminant of the equation. In Appendix B we analyse solutions of the MH equation for an ion confined within an electrodynamic (or RF) trap, assimilated with a parametric oscillator (PO).

The results in the paper apply to electrodynamic traps employed in ultra-high-resolution spectroscopy, mass spectrometry (MS) and in the domain of quantum technologies (QT) based on ultracold trapped ions [78–81], with an emphasis on quantum sensors and quantum metrology.

2. Solutions of the Mathieu–Hill Equation

An iconic paper that reports on an elaborate analysis and discussion of the transition of a few ion crystals into ion clouds is ref. [73]. The paper demonstrates how the crystalline phase prevails until the Mathieu stability limit is attained, emphasizing that the corresponding transition cannot be regarded as an order $\rightarrow$ chaos transition which occurs when a control parameter reaches a critical value. In addition, in the vicinity of this limit the system exhibits sensitivity to perturbations, which enables the experimenter to investigate crystal melting substantially forward the instability limit. The influence of the ion micromotion on ion crystal stability is also discussed in [73], where numerical modelling is used to characterize dynamical stability and discriminate between four possible regimes. Representations of large Coulomb crystals composed of trapped ions are reported in [55], where Molecular Dynamics (MD) numerical modelling is employed to account for cold ion–atom elastic collisions that occur between Coulomb crystals and very light virtual atoms.

A review of strongly coupled Coulomb systems consisting of ultracold trapped ions is performed in [82], whereas 2D crystals in zig-zag configurations and structural transitions are discussed in [83–85]. Recent approaches report direct observation of micromotion for a many-body Coulomb crystal consisting of ultracold trapped ions, while novel techniques to measure the micromotion amplitude are implemented [86]. A numerical modelling is employed to determine phase transitions in small-ion Coulomb crystals confined in an electrodynamic ion trap (EIT) [87], where the phase transition points are introduced as extremes of the interpolated functions employed. The novel conceptualization of fractal quasi-Coulomb crystals is introduced in [88], for the case of surface electrodynamic traps (SET) characterized by a Cantor Dust electrode configuration. New experimental evidence of orientational melting that arises in a 2D crystal consisting of up to 15 ions is reported in [89].

An analytical and numerical modelling focused on dynamical stability in the case of trapped ion systems is performed in [90] by extending the model introduced in [73,91]. Numerical modelling illustrates that the associated dynamics is either quasiperiodic or periodic, depending on the initial conditions. The evolution in time for a system of two coupled oscillators levitated in an RF trap is described using an analytical model that depends on the chosen control parameters, while ion dynamics is shown to be integrable only for discrete values of the ratio between the axial and the radial frequencies of the ion secular motion. A qualitative discussion of the system dynamical stability is then performed by employing the Morse theory. The results are further applied to many body strongly coupled Coulomb systems (trapped ions), locally investigated in the vicinity of equilibrium configurations that designate ordered structures. These equilibrium configurations exhibit a large interest for Coulomb ion crystals (atomic clocks) or to achieve quantum information processing (QIP).

For example, the Kibble–Zurek mechanism (KZM) generally describes nonequilibrium dynamics along with topological defect occurrence for physical systems that undergo second-order phase transitions. A generalized KZM that explains defect emergence in trapped ion systems is discussed in [92,93], where the analytical models proposed enable investigation of KZ physics in the case of inhomogeneous systems.

We consider the MH equation for an ion confined in an EIT [72,73]

$$\ddot{x} + f(\tau)x = 0\,,\quad f(\tau) = a + 2q\cos(2\tau)\,. \tag{1}$$

Equation (1) is a linear differential equation with a periodic coefficient, and it is assumed that f is a periodic function $f(\tau + T_0) = f(\tau)$ with the particular case $T_0 = \pi$. We introduce

$$\begin{cases} a_z = -2a_x = -2a_y = -\frac{8eU_0}{m\Omega^2\left(r_0^2+2z_0^2\right)} \\ q_z = -2q_x = -2q_y = \frac{4eV_0}{m\Omega^2\left(r_0^2+2z_0^2\right)} \end{cases}, \tag{2}$$

where e is the elementary electric charge, m denotes the ion mass, $r^2 = r_0^2 + 2z_0^2$ with r_0 and z_0 the trap semiaxes, while U_0 stands for the d.c. trapping voltage applied to the trap endcap electrodes. In addition, $a_x = -a_y$ and $q_x = -q_y$. Because we discuss a Paul (RF) trap $\tau = \Omega t/2$ is the dimensionless time [73,78], whilst Ω is the RF of the oscillating trapping voltage (denoted as V_0) supplied to the cylindrical electrodes in case of a 2D linear Paul trap, or to the ring electrode for a classical hyperbolic 3D geometry. Moreover, the function $f : \mathbb{R} \to \mathbb{R}$ is real and continuous for period $T_0 > 0$. We also analyse cases in which the function f is a Fourier series [72,74,94,95].

$$f(\tau) = A_0 + \sum_{k>0} A_k \cos k\tau + \sum_{k>0} B_k \sin k\tau\,, \tag{3}$$

where A_0 is a known constant, whilst the A_k and B_k coefficients are considered as real

$$\sum_{k>0}\left(A_k^2 + B_k^2\right) < \infty\,,\quad A_k, B_k \in \mathbb{R}\,. \tag{4}$$

The Floquet theory [77] explains the general solution (which is a complex function) can be cast as [73]

$$x_1 = Q(\tau)\Phi(\tau)\,, \tag{5}$$

where the function

$$Q(\tau) = e^{i\mu\tau}\,,\quad \mu \in \mathbb{C}\,, \tag{6}$$

is associated with the slow (or secular) ion motion, while $\Phi(\tau) = \Phi(\tau + T_0)$ represents a complex, periodical and twice-differentiable function that characterizes the micromotion [72,96]. In the case of a MH equation ion dynamics is stable as long as the Floquet exponent is a complex number $\mu \in \mathbb{C}$ [97,98]. Hence, the solution is written as

$$x = e^{i\mu\tau}\Phi(\tau) + e^{-i\mu\tau}\Phi^*(\tau) \in \mathbb{R}\,, \tag{7}$$

which depends on the coefficients A_k and B_k, with $(k > 0)$ and A_0. $\Phi^*(\tau)$ denotes the complex conjugate of the previously introduced $\Phi(\tau)$ function. One can also expand $\Phi(\tau)$ as a Fourier series

$$\Phi(\tau) = C_0 + \sum_{k>0} C_k \cos k\tau + \sum_{k>0} D_k \sin k\tau\,. \tag{8}$$

If

$$\sum_{k>0}\left(A_k^2 + B_k^2\right) \ll A_0\,,$$

then Equation (1) can be expressed as the equation of a harmonic oscillator (HO) [40–42,99,100]

$$\ddot{x} + A_0 x = 0\,, \tag{9}$$

Further on, we illustrate two different methods to solve the MH equation:

1. One chooses a solution expressed as $x = x_2 + x_2^*$, where x_2^* is the complex conjugate of x_2 so that

$$x_2 = e^{i\mu\tau}(C_0 + C_1\cos\tau + D_1\sin\tau)\,. \tag{10}$$

A double differentiation with respect to τ yields

$$\dot{x}_2 = e^{i\mu\tau}[i\mu C_0 + \cos\tau(D_1 + i\mu C_1) + \sin\tau(i\mu D_1 - C_1)] \,, \tag{11}$$

$$\ddot{x}_2 \;=\; e^{i\mu\tau}\left\{-\mu^2 C_0 + \cos\tau\left[2i\mu D_1 - C_1\left(1+\mu^2\right)\right] - \sin\tau\left[D_1\left(1+\mu^2\right) + 2i\mu C_1\right]\right\} \;. \tag{12}$$

We now revert to Equation (1) and introduce the expression $f(\tau)$ supplied by Equation (3)

$$\ddot{x}_2 + (A_0 + A_1\cos\tau + B_1\sin\tau)x_2 = 0 \,. \tag{13}$$

Equation (13) is also verified for the complex conjugate x_2^*. Then, one uses Equations (11) and (12) along with Equation (13) to further derive

$$\begin{aligned} -\mu^2 C_0 + \cos\tau\left[2i\mu D_1 - C_1\left(1+\mu^2\right)\right] - \sin\tau\left[D_1\left(1+\mu^2\right) + 2i\mu C_1\right] \\ + (A_0 + A_1\cos\tau + B_1\sin\tau)(C_0 + C_1\cos\tau + D_1\sin\tau) = 0 \,. \end{aligned} \tag{14}$$

Equation (14) can also be cast as

$$\begin{aligned} -\mu^2 C_0 + A_0 C_0 + A_1 C_1\cos^2\tau + B_1 D_1\sin^2\tau + (A_1 D_1 + B_1 C_1)\sin\tau\cos\tau \\ + \cos\tau\left[A_0 C_1 + A_1 C_0 - C_1\left(1+\mu^2\right) + 2i\mu D_1\right] \\ + \sin\tau\left[A_0 D_1 + B_1 C_0 - 2i\mu C_1 - D_1\left(1+\mu^2\right)\right] \,. \end{aligned} \tag{15}$$

One uses [101–104]

$$\sin^2\tau = \frac{1-\cos 2\tau}{2} \,, \quad \cos^2\tau = \frac{1+\cos 2\tau}{2} \,, \quad \sin 2\tau = 2\sin\tau\cos\tau \,. \tag{16}$$

Finally, by performing the calculus in Equation (14) with $k > 1$, one infers the following system of equations:

$$\begin{cases} -\mu^2 C_0 + A_0 C_0 + \frac{1}{2}A_1 C_1 + \frac{1}{2}B_1 D_1 = 0 \\ B_1 C_0 + A_0 D_1 - D_1\left(1+\mu^2\right) - 2i\mu C_1 = 0 \\ A_1 C_0 + A_0 C_1 + 2i\mu D_1 - C_1\left(1+\mu^2\right) = 0 \end{cases} \tag{17}$$

Using this system of equations one can determine the coefficients C_0, C_1 and D_1, respectively. Hence, one determines the solution x_2 from Equation (10) and implicitly the solution x in Equation (9). Hence, an ion confined within an electrodynamic trap can be treated as a HO and the method introduced above allows one to derive the MH equation that characterizes the associated dynamics.

2. As demonstrated in [77] one chooses a solution of the form $x_2 = e^{i\mu\tau}\Phi$. Therefore,

$$\dot{x}_2 = e^{i\mu\tau}\left(i\mu\Phi + \dot{\Phi}\right) \tag{18}$$

$$\ddot{x}_2 = e^{i\mu\tau}\left(\ddot{\Phi} + 2i\mu\dot{\Phi} - \mu^2\Phi\right) \tag{19}$$

We introduce the solution $x = x_2 + x_2^*$ in the MH Equation (1) and derive

$$\begin{aligned} e^{i\mu\tau}\left(\ddot{\Phi} + 2i\mu\dot{\Phi} - \mu^2\Phi\right) + e^{-i\mu\tau}\left(\ddot{\Phi} - 2i\mu\dot{\Phi} - \mu^2\Phi\right) \\ + f(\tau)\left(e^{i\mu\tau} + e^{-i\mu\tau}\right)\Phi = 0 \,. \end{aligned} \tag{20}$$

By performing a Fourier series expansion [105] one can write

$$\Phi = \sum_{k\in\mathbb{Z}} c_k e^{ik\tau} \ , \ f = \sum_{k\in\mathbb{Z}} a_k e^{ik\tau} \ , \ a_k^* = a_{-k} \ , \tag{21}$$

where a_k are arbitrary constants and $e^{i\mu\tau} = \cos\mu\tau + i\sin\mu\tau$. Consequently, Equation (20) becomes

$$\sum_{k\in\mathbb{Z}} c_k \left[-k^2 - 2\mu k - \mu^2\right] e^{i(\mu+k)\tau} + \sum_{k\in\mathbb{Z}} \left(-k^2 + 2\mu k - \mu^2\right) e^{i(k-\mu)\tau} + \sum_{p\in\mathbb{Z}} a_p \sum_{k\in\mathbb{Z}} c_k e^{i(k+\mu+p)} + \sum_{p\in\mathbb{Z}} a_p \sum_{k\in\mathbb{Z}} c_k e^{i(k+p-\mu)\tau} = 0 \, . \tag{22}$$

It can be noticed that the equation above contains terms such as $e^{i\mu\tau}\left(\sum\limits_{k+p=n} a_p c_k\right) e^{in\tau}$. One denotes $b_n = \sum\limits_{k+p=n} a_p c_k = \sum\limits_{p\in\mathbb{Z}} a_p c_{n-p}$. As a result, Equation (22) can be recast as

$$e^{i\mu\tau} E + e^{-i\mu\tau} E^* = 0 \, , \tag{23}$$

with

$$\begin{cases} E = \sum_{k\in\mathbb{Z}} \left[-c_k (k+\mu)^2 + b_k\right] e^{ik\tau} \, , \\ E^* = \sum_{k\in\mathbb{Z}} \left[-c_k (k+\mu)^2 + b_k\right] e^{-ik\tau} . \end{cases} \tag{24}$$

By using Equations (23) and (24) one obtains

$$c_k (k+\mu)^2 = b_k = \sum_{p\in\mathbb{Z}} a_{k-p} c_p \ , \ \ k \in \mathbb{Z} \, , \tag{25}$$

which is amenable to

$$\sum_{p\in\mathbb{Z}} \left[a_{k-p} - (k+\mu)^2 \delta_{kp}\right] c_p = 0 \, , \tag{26}$$

where δ_{kp} stands for the Kronecker delta function. One further denotes

$$A = \left[a_{k-p} - (k+\mu)^2 \delta_{kp}\right] . \tag{27}$$

Further on we introduce

$$C = \begin{pmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix} . \tag{28}$$

Then, $A \cdot C = 0$ where C is the column matrix introduced above. One further infers

$$\det A = 0 \, , \tag{29}$$

which is exactly the MH equation that determines the Floquet exponent μ as a function of the a_k coefficients (particularly as a function of the eigenvalue a and q parameter). It is assumed that $a_k = 0$ for $|k| > 2$ (Mathieu case) and $a_1 = a_{-1} = 0$. In such a case

$$f(\tau) = a_0 + a_2 e^{2i\tau} + a_{-2} e^{-2i\tau} \, . \tag{30}$$

As explained above, one also supposes that $a_2 = a_{-2} = q$ and $a_0 = a$. Then

$$f(\tau) = a + 2q\cos 2\tau\ , \tag{31}$$

which is exactly the classical MH equation. Based on Equation (25) one derives

$$c_k(k+\mu)^2 = a_0c_k + c_{k-2}a_2 + a_{-2}c_{k+2} = ac_k + q(c_{k+2} + c_{k-2})\ , \tag{32}$$

with $q \neq 0$ and

$$\begin{cases} c_{k+2} = \alpha c_k + \beta c_{k-2}\ ,\ \text{if } k > 0\ , \\ c_{k-2} = \gamma c_k + \zeta c_{k-2}\ ,\ \text{if } k < 0\ . \end{cases} \tag{33}$$

where α, β, γ, and ζ are known coefficients. When $|c_k| \sim 0$ and $|k| > 2$, the column matrix C [described by Equation (28)] can be recast as

$$C = \begin{pmatrix} \vdots \\ c_{-2} \\ 0 \\ c_0 \\ 0 \\ c_2 \\ \vdots \end{pmatrix}, \tag{34}$$

where it is assumed that $c_1 = c_{-1} = 0$ is an initial condition (hypothesis). In such case the function Φ [in Equation (21)] can be expressed as

$$\Phi = c_0 - 2c_2\cos 2\tau\ , \tag{35}$$

with $c_2 = c_{-2}$. Then, for $k = 0$ one finds

$$c_0\mu^2 = ac_0 + 2c_2q \longmapsto \mu^2 = a + 2\frac{c_2}{c_0}q\ . \tag{36}$$

We consider the initial conditions

$$\begin{cases} c_0 = x_0\ , \\ 2\frac{c_2}{c_0} = \frac{q}{2}\ . \end{cases} \tag{37}$$

Under these circumstances Equation (36) is recast as

$$\mu^2 = a + \frac{q^2}{2}\ . \tag{38}$$

We return to Equation (1) and express f as

$$f = \mu^2 + f_{res}\ . \tag{39}$$

Then, one uses Equation (38) and the residual interaction function f_{res} is

$$f_{res} = f - \mu^2\ ,\ \ f_{res} = 2q\cos 2\tau - \frac{q^2}{2}\ , \tag{40}$$

which we discuss below.

Discussion

When $|k| > 1$, then $a_k = 0$. In case of the HO [6,8,41,72] $a_0 > 0$, and from Equation (32) one infers

$$c_k(k+\mu)^2 = a_0c_k\ , \tag{41}$$

and we distinguish two distinct cases

(i) $k = 0 \Rightarrow a_0 = \mu^2$ with $c_0 \neq 0$.

(ii) $c_k \neq 0 \Rightarrow (k+\mu)^2 = \mu^2 = a_0 \Rightarrow k+\mu = \pm\mu$ and the two solutions of the equations are

$$\begin{cases} k_1 = 0\,, \\ k_2 = -2\mu \notin \mathbb{Z}\,, \end{cases} \tag{42}$$

where the latter case is discarded. Then, $\Phi = c_0$ and

$$x = e^{i\mu\tau} c_0 + e^{-i\mu\tau} c_0^*\,, \tag{43}$$

which stands for the equation of the HO for $\mu \in \mathbb{R}$, while c_0^* stands for the complex conjugate of c_0. Hence, the MH equation turns into the equation of a classical HO

$$\ddot{x} + \mu^2 x = 0\,. \tag{44}$$

It is demonstrated that one can use the time-dependent variational principle (TDVP) [106–108] and then express the Hamilton equations of motion for the unidimensional case in order to derive the quantum equation of motion in the Husimi (Q) representation, for a boson enclosed in a nonlinear ion trap [37,109]. This equation of motion is fully consistent with the one that portrays a perturbed classical oscillator. Hence, a phase-space formulation of quantum mechanics, e.g., the Husimi or Wigner representation, discloses the composition of the corresponding phase-space [36] and establishes a correlation between classical and quantum dynamics for this class of mesoscopic systems [110,111].

3. Stability Diagram of the MH Equation

The standard (canonical) form of the Mathieu equation for an ion confined within an electrodynamic trap, characterized by the parameters a and q, is [9,72,74,76,95,112,113]

$$\frac{d^2 w}{d\tau^2} + [a - 2q\cos 2\tau] w = 0\,. \tag{45}$$

We demonstrated in [72] that this is a Sturm–Liouville system [114] characterized by periodic boundary conditions, while the system exhibits solutions exclusively for particular values of the eigenvalue a, which depend upon q. It is also established that other solutions exist for every pair (a, q), but we consider only periodic solutions [105].

In order to characterize ion dynamics, one of the possible approaches is to separate the ion micromotion from the secular motion [73]. The MH equation can be recast as (see Equation (1))

$$\ddot{x} + f x = 0\,, \text{ where one introduces } f = \mu^2 + f_{res}\,, \tag{46}$$

where f_{res} describes the residual interaction that accounts for the ion micromotion. In addition,

$$\frac{1}{T_0}\int_0^{T_0} f_{res} d\tau = 0\,, \quad T_0 = \pi\,, \tag{47}$$

so the micromotion term averages out during a period of the RF drive which leads us to the case of an autonomous Hamilton function for the trapped ion. By separating the micromotion Equation (46) changes accordingly

$$\ddot{x} + \mu^2 x + f_{res} x = 0\,, \tag{48}$$

which stands for the equation of the HO [8,21,36,37,100,115] of period $2\pi/\sqrt{|\mu|}$, with $\mu \in \mathbb{R}$, $2\mu \notin \mathbb{Z}$. It can be observed that ion dynamics is decomposed into a slowly oscillating part (the secular motion) at frequency μ (see Equation (38)) and a fast parametric drive (the micromotion) f_{res} at frequency Ω. The secular ion motion (or the macromotion) is characterized by the equation

$$\ddot{x} + \mu^2 x = 0\,, \tag{49}$$

where μ is the Floquet exponent [74,77]. We now revert to Equation (25) and choose $k = 0$. Consequently,

$$\mu^2 = a_0 + \frac{c_{-2}}{c_0} a_2 + \frac{c_2}{c_0} a_{-2} \,. \tag{50}$$

From Equation (46) one derives $f_{res} = f - \mu^2$, which we multiply by x. We consider the solution $x = x_2 + x_2^*$ and use Equation (21). Hence,

$$f_{res} x = \left(\sum_k a_k e^{ik\tau} - \mu^2 \right) \left(e^{i\mu\tau} \sum_p c_p e^{ip\tau} + e^{-i\mu\tau} \sum_p c_p e^{-ip\tau} \right) . \tag{51}$$

One denotes

$$\phi(\tau) = \sum_p c_p e^{ip\tau} \,, \;\; \phi^*(\tau) = \sum_p c_p e^{-ip\tau} \,, \tag{52}$$

where $\phi^*(\tau)$ stands for the complex conjugate of $\phi(\tau)$. Because

$$\int_0^{2\pi} e^{ik\tau} d\tau = \frac{1}{ik} e^{ik\tau} |_0^{2\pi} = 0 \,, \;\; \text{when } k \neq 0 \,, \tag{53}$$

$$\int_0^{\pi} e^{ik\tau} d\tau = \frac{1}{ik} \left(e^{ik\pi} - 1 \right) = \begin{cases} 0, & k \text{ even} \\ 2i/k, & k \text{ odd} \end{cases} \tag{54}$$

We revert to Equation (47) and denote

$$\begin{aligned} I = \int_0^{\pi} f_{res} x \, d\tau &= \sum_{p,k} a_k c_p \left[\frac{e^{i(k+p+\mu)\pi} - 1}{i(\mu + k + p)} - \frac{e^{-i(k+p+\mu)\pi} - 1}{i(\mu + k + p)} \right] \\ &\quad - \mu^2 \sum_p c_p \frac{1}{i(\mu + p)} \left[e^{i(\mu+p)\pi} - e^{-i(\mu+p)\pi} \right] \\ &= \sum_{p,k} a_k c_p \frac{2 \sin(\mu + k + p)\pi}{\mu + k + p} - \mu^2 \sum_p \frac{2 \sin(\mu + p)\pi}{\mu + p} \,. \end{aligned} \tag{55}$$

As demonstrated in [72], by choosing $a_0 = a$, $a_2 = a_{-2} = 2q$, $|a_k| = 0$ for $|k| > 2$, one derives [73]

$$\begin{aligned} I = &\left\{ \left(\sum_p c_p \frac{2 \sin(\mu + p)\pi}{\mu + p} \right) \left(a_0 - \mu^2 \right) + 2q \sum_p c_p \left[\frac{2 \sin(\mu + p)\pi}{\mu + 2 + p} + \frac{2 \sin(\mu + p)\pi}{\mu - 2 + p} \right] \right\} \\ &= 2 \sum_p c_p \sin(\mu + p)\pi \left[\frac{a_0}{\mu + p} + \frac{2q}{\mu + 2 + p} + \frac{2q}{\mu - 2 + p} \right] = 0 \,. \end{aligned} \tag{56}$$

By assuming $c_1 = c_{-1} = 0$; $|c_k| = 0$ for $|k| > 2$ and $c_2 = c_{-2}$ for values $p = -2, 0, 2$, and the micromotion term averages out (see [73])

$$\begin{aligned} &c_0 \left[\frac{a_0 - \mu^2}{\mu} + q \left(\frac{1}{\mu + 2} + \frac{1}{\mu - 2} \right) \right] \\ &\quad + c_2 \left[\frac{a_0 - \mu^2}{\mu} + q \left(\frac{1}{\mu + 4} + \frac{1}{\mu} \right) \right] \\ &\quad + c_{-2} \left[\frac{a_0 - \mu^2}{\mu} + q \left(\frac{1}{\mu - 4} + \frac{1}{\mu} \right) \right] = 0 \,, \end{aligned} \tag{57}$$

which is further amenable to

$$\frac{c_2}{c_0} = - \frac{2(a_0^2 - \mu) + 2q \left(2 + \frac{\mu}{\mu+2} + \frac{\mu}{\mu-2} \right)}{a_0 - \mu^2 + 2q \left(\frac{\mu}{\mu+2} + \frac{\mu}{\mu-2} \right)} \,. \tag{58}$$

In cases when q is very small [72,73,95]

$$a_0 = \mu^2 + \frac{q^2}{2(\mu^2-1)} + \frac{(5\mu^2+7)q^4}{32(\mu^2-1)^3(\mu^2-4)} + \ldots \,, \quad \mu \neq 1,2,3\,, \tag{59}$$

and the even solution can be expressed as

$$\frac{c_2}{c_0} = \frac{q}{4(\mu+1)} + \frac{(\mu^2+4\mu+7)q^3}{128(\mu+1)^3(\mu+2)(\mu-1)} + \ldots \,, \quad \mu \neq 1,2\,, \tag{60}$$

which completely determines the MH equation for an ion confined in a Paul trap, while it stands as the main result of the paper. In ref. [73] the values used are

$$a = \mu^2 - \frac{1}{2}q^2\,, \quad \frac{c_2}{c_0} = \frac{q}{4}\,, \tag{61}$$

as the value of the Floquet exponent μ was chosen to be small, and implicitly the characteristic value (eigenvalue) a and parameter q in the MH equation as well. Both of them are real-valued, as previously emphasized in Section 1. Nevertheless, this hypothesis does not coincide with the values used in [73].

It was demonstrated in [72] that for ion trap operating points that lie within the first stability region of the MH equation and for small values of the eigenvalue a and q parameter, the coefficients c_k rapidly converge to zero. Under such conditions higher harmonics become insignificant and the fundamental frequency prevails. It was also demonstrated that the solution of the MH equation is the sum between a shift that corresponds to the fundamental frequency and a correction due to the occurrence of higher harmonics associated with ion dynamics. The problem of evaluating the maximum amplitude of stable oscillations under given initial conditions was approached. The maximum is characterized by upper bounds determined by the relative position of the electrodes. For optimum trap operating parameters it is essential to consider all axes for which the equation of motion is of Mathieu type. A nonzero solution can be expressed as the combination of two linear dependent functions, which form a fundamental solution of the MH eq. Ion trajectories are investigated by building a family of ellipses, while the transfer matrix is supplied along with the maximum amplitude of stable oscillations [72].

The frontiers of the stability diagrams are defined by the equations [75,76,95,112]

$$a_0(q) = -\frac{q^2}{2} + \frac{7q^4}{128} - \frac{29q^6}{2304} + \frac{68687q^8}{18874368} + \ldots \,, \tag{62}$$

$$a_1(-q) = b_1(q) = 1 - q - \frac{q^2}{8} + \frac{q^3}{64} - \frac{q^4}{1536} - \frac{11q^5}{36864} + \frac{49q^6}{589824} - \frac{55q^7}{9437184} - \frac{83q^8}{35389440} + \ldots \,, \tag{63}$$

$$a_2(q) = 4 + \frac{5q^2}{12} - \frac{763q^4}{13824} + \frac{1002401q^6}{79626240} - \frac{1669068401q^8}{458647142400} + \ldots \,, \tag{64}$$

$$b_2(q) = 4 - \frac{q^2}{12} + \frac{5q^4}{13824} - \frac{289q^6}{79626240} + \frac{21391q^8}{458647142400} + \ldots \,. \tag{65}$$

When $a, |q| \ll 1$ and $r \geq 7$ (case when a_r is approximately equal to b_r), the characteristic values of the frontiers of the stability region are described by the following power series approximation [8,76]

$$a_r, b_r = r^2 + \frac{q^2}{2(r^2-1)} + \frac{(5r^2+7)q^4}{32(r^2-1)^3(r^2-4)} + \frac{(9r^4+58r^2+29)q^6}{64(r^2-1)^5(r^2-4)(r^2-9)} + \ldots \,. \tag{66}$$

The analytical modelling performed in this paper is valid for all electrodynamic trap designs. It is essential to establish whether ion trajectories are stable or unstable depending on the experimental conditions, as well as whether the boundaries are stable and unstable regions associated with the ion dynamics. It is established that these limits correspond to combinations of cosine and sine elliptic series, as shown in [8,72,74,77,94,95]. Such value solutions of the MH equations are periodic but not bound, while they determine the point at which the ion trajectory becomes unbound.

4. Results

This paper suggests particular solutions of the classical MH equation with complex solutions for ions confined in electrodynamic ion traps (EITs) by employing the Floquet theory [74,77]. We analyse cases when the MH equation for a trapped ion can be cast as a Fourier series whose coefficients are determined by the analytical model introduced, which turns it into the equation of a HO. Thus, a system of linear equations is derived that enables one to determine the constants in the Fourier series solution.

The second method suggested delivers the solution of the MH equation for trapped ion systems. We derive a recursive relation that yields the Floquet coefficient as a function of the a_k coefficients in the series expansions that we use. Ion dynamics within an electrodynamic (Paul) trap are shown to be consistent with the equation of motion for a classical HO. Finally, we apply these methods in Section 3 for an electrodynamic trap and we extend the model in [73]. Then, we derive the expression for a_0 and the coefficients of the even solution as a function of the Floquet coefficient for very low values of the q parameter. Hence, we completely determine the solution of the MH equation and the frontiers of the stability diagram.

The HO with damping is discussed in Appendix A, while the parametric HO is examined in Appendix B. It is demonstrated the solutions are linearly independent, while they also establish a fundamental system. We shortly review the Hill method to determine the Floquet coefficient [72,74,77,116,117] and supply the equation that defines the stability frontiers for the MH equation. A very short comment on EITs that operate under Standard Atmospheric Temperature and Pressure (SATP) conditions ends this paper.

The pseudopotential approximation can be safely used when $\Omega \gg 1$, which means the time scale of the secular motion is considerably larger than the time scale associated with the micromotion. In the case of the pseudopotential approximation for a Paul (RF) trap, the electric potential is described by a polynomial of rank 2. When considering the micromotion, stability regions result that are qualitatively similar to those of the Mathieu equation but with frontiers that are shifted depending on the anharmonicity parameter $\lambda(\beta)$, as shown in [72] and in the literature [118].

Applications of the methods used in this paper span MS, high-resolution spectroscopy, and trapping of ion crystals, where the latter is of extreme importance for optical atomic clocks, ultra-high-resolution spectroscopy, quantum metrology, and quantum information processing (QIP) with ultracold ions [42,46,60,119,120].

Funding: This research was funded by the Ministry of Research, Innovation and Digitalization, under the Romanian National Core Program LAPLAS VII—Contract No. 30N/2023.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Acknowledgments: The author acknowledges helpful discussions and comments on the text from Yair Zarmi, Ben-Gurion Univ. of the Negev. I thank him for his assistance and kindness.

Conflicts of Interest: The author declares no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

2D	Two-Dimensional
3D	Three-Dimensional
BSM	Beyond the Standard Model
DIT	Digital Ion Trap
EIT	Electrodynamic Ion Trap
ESI-MS	Electrospray Mass Spectrometry
HO	Harmonic Oscillator
IT	Ion Trap
KZ	Kibble-Zurek
LIT	Linear Ion Trap
LPT	Linear Paul Trap
MD	Molecular Dynamics
MS	Mass Spectrometry
PO	Parametric Oscillator
QIP	Quantum Information Processing
QT	Quantum Technologies
RF	Radiofrequency
SATP	Standard Atmospheric Temperature and Pressure
TDVP	Time Dependent Variational Principle

Appendix A. Harmonic Oscillator (HO)

Appendix A.1. Harmonic Oscillator with Damping

It is generally known that a trapped ion can be assimilated with a kicked-damped HO model for an ion trap [36,121] or a parametric oscillator [16,69,122].

We consider the equation of motion for a damped HO (unidimensional case) [2,123]

$$m\ddot{x} = -kx - \alpha\dot{x} \,, \tag{A1}$$

where k denotes the elastic constant and $\alpha > 0$ is the damping coefficient. Finally, one derives

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0 \,, \ \text{ with } \frac{k}{m} = \omega_0^2 \,, \ 2\lambda = \frac{\alpha}{m} \,. \tag{A2}$$

A solution to this equation is $x = e^{rt}$, where r is either real $r \in \mathbb{R}$ or complex $r \in \mathbb{C}$. Then, Equation (A2) is cast as

$$r^2 + 2\lambda r + \omega_0^2 = 0 \,. \tag{A3}$$

Further on, we shortly discuss the solutions of Equation (A3).

1. *Case 1—Overdamping*

   We denote

   $$\Delta = 4\lambda^2 - 4\omega_0^2 > 0 \,, \tag{A4}$$

   Then, a solution of Equation (A2) is

   $$x = C_1 e^{r_1 t} + C_2 e^{r_2 t} = C_1 e^{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)t} \,, \tag{A5}$$

   which describes damped oscillations that are nonperiodical, with C_1, C_2 constants of integration.

2. *Case 2—Critical damping*

   $$\Delta = 0 \Rightarrow \lambda = \omega_0 \,. \tag{A6}$$

   In such a case the solution is expressed as

$$r_1 = r_2 = -\lambda \Rightarrow x = (B + Ct)e^{-\lambda t} \, , \tag{A7}$$

with B, C constants.

3. *Case 3—Underdamped oscillations* $\Delta < 0$
The solution is cast as

$$x = C_3 e^{r_1' t} + C_4 e^{r_2' t} = e^{-\lambda t}\left(C_3 e^{i\sqrt{\omega_0^2 - \lambda^2} t} + C_4 e^{-i\sqrt{\omega_0^2 - \lambda^2} t}\right) \, , \tag{A8}$$

with $C_3 = C_4^*$ and C_4^* is the complex conjugate of C_4. Considering that

$$e^{ix} = \cos x + i \sin x \Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2} \, , \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \, , \tag{A9}$$

Equation (A8) can be cast as

$$x = e^{-\lambda t}\left(D_3 \cos \sqrt{\omega_0^2 - x^2 t} + D_4 \sin \sqrt{\omega_0^2 - x^2 t}\right) \, , \tag{A10}$$

with D_3, D_4 constants.

Appendix B. Parametric Harmonic Oscillator—Floquet's Coefficient—Hill's Method

We start from the equation

$$\frac{d^2 x}{dt^2} + \omega^2(t) x = 0 \, , \quad \omega(t + T) = \omega(t) \, , \tag{A11}$$

which is a Mathieu–Hill (MH)-type differential equation with periodic coefficients of period $T = \pi$. We choose

$$\omega^2(t) = a - 2q \cos 2t \, . \tag{A12}$$

The Floquet theory [74,75,77] states that Equation (A11) exhibits a solution of the form [113]

$$x_1 = e^{\mu t} P(t) \, , \tag{A13}$$

where $P(t)$ stands for a periodic function of period π, while μ denotes the Floquet characteristic exponent [74,75,77,95]. It is obvious that

$$x_2 = e^{-\mu t} P(-t) \, , \tag{A14}$$

is also a solution of Equation (A11), as the latter is invariant to the change $t \to -t$. Both functions $P(t)$ and $P(-t)$ are periodic, with period π. Generally, the solutions x_1 (described by Equation (A13) and x_2 (described by Equation (A14)) are linearly independent. They establish a fundamental system of solutions for Equation (A11). There is only one exception, the case of Mathieu periodic functions when $i\mu$ is an integer, which is discussed in [72]. Besides that any solution is expressed as

$$x = c_1 x_1 + c_2 x_2 \, , \quad c_1, c_2 = const \, , \tag{A15}$$

with [72,113]

$$x_1 = \sum_{n=-\infty}^{\infty} c_n e^{(\mu + 2in)t} \, . \tag{A16}$$

There are several methods to determine the Floquet coefficient μ, amongst which we distinguish Hill's method and the Lindemann–Stieltjes method [77,94,95,113,116,117]. Both of them require an elaborate analysis because μ is not a simple function of q. For example, one could assign a real value to the q parameter and consider that a varies from $-\infty$ to ∞. In this case μ switches between real and complex values, while changes occur when the Hill–Mathieu notation given by Equation (A23), namely $\triangle(0) \sin^2\left(\frac{1}{2}\pi\sqrt{a}\right)$, passes through the values of 0 and 1. This is an outcome of the fact that $\triangle(0)$ is a complex expression of a and q [77].

Appendix B.1. Hill's Method

By using Hill's method one derives a homogeneous system of linear equations [72]

$$c_n + \gamma_n(c_{n-1} + c_{n+1}) = 0\,, \quad n = 0, \pm 1, \pm 2, \dots\,. \tag{A17}$$

with

$$\gamma_n = \frac{q}{(2n - i\mu)} - a\,. \tag{A18}$$

The infinite determinant of the system is [72,77,113,117]

$$\triangle(\mu) = \begin{vmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & \gamma_{-2}(\mu) & 0 & 0 & 0 & \dots \\ \dots & \gamma_{-1}(\mu) & 1 & \gamma_{-1}(\mu) & 0 & 0 & \dots \\ \dots & 0 & \gamma_0(\mu) & 1 & \gamma_0(\mu) & 0 & \dots \\ \dots & 0 & 0 & \gamma_1(\mu) & 1 & \gamma_1(\mu) & \dots \\ \dots & 0 & 0 & 0 & \gamma_2(\mu) & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}, \tag{A19}$$

and the Floquet exponent is determined by the equation

$$\triangle(\mu) = 0\,. \tag{A20}$$

The infinite determinant described by Equation (A19) is absolutely convergent and it represents a meromorphic function [124] of the Floquet exponent μ [74,77,94,116], with simple poles for $\mu = \pm i\left(\sqrt{a} + 2s\right),\ ; s = 0, \pm 1, \pm 2, \dots$ [72,77]. Therefore,

$$\triangle(\mu) - \frac{C}{\cosh(\pi\mu) - \cos(\pi\sqrt{a})}\,, \tag{A21}$$

is an even and periodic meromorphic function. If the constant C is determined so that the function described by Equation (A21) has no pole at $\mu = i\sqrt{a}$, hence it will exhibit no other pole which makes it a constant. Because $\triangle(\mu) \to 1$ as $\mu \to \infty$, such a constant is equal to 1. To derive the expression of C one chooses $\mu = 0$ and infers [72,75,113]

$$\begin{aligned} \triangle(\mu) &= 1 - \frac{[1 - \triangle(0)]\left[1 - \cos(\pi\sqrt{a})\right]}{\cosh(\pi\mu) - \cos(\pi\sqrt{a})} \\ &= \frac{\cosh(\pi\mu) - 1 + \triangle(0)\left[1 - \cos(\pi\sqrt{a})\right]}{\cosh(\mu\pi) - \cos(\pi\sqrt{a})}\,. \end{aligned} \tag{A22}$$

Because the Floquet exponent is determined by the Equation (A20), one obtains

$$\cosh(\pi\mu) = 1 + 2\triangle(0)\sin^2\left(\frac{1}{2}\pi\sqrt{a}\right). \tag{A23}$$

Equation (A23) defines the stability frontiers of the MH equation when $\cosh(\pi\mu) = \pm 1$.

A numerical simulation is employed to construct the Mathieu stability plot in [125] for any arbitrary toroidal ion trap mass analyzer. The toroidal multipole coefficients of the traps are evaluated and then used to infer the Mathieu eq. parameters, a and q. These parameters are further used to illustrate the stability plot, predicting the secular frequency of ion motion and evaluating nonlinear resonances. Ref. [72] analyzes the dynamical stability of the MH equation for a trapped ion, when the trap operating points $a, q \in \mathbb{R}$ are located within the first stability region, focused on the associated dynamics is bounded or unbounded.

The issue of stability and instability intervals for the MH equation is very well explained in [75,95]. Further on, we review the analysis performed in [113]. If $a, q \in \mathbb{R}$ are both real, then Equation (A23) shows that $\cosh(\pi\mu) \in \mathbb{R}$. When $-1 < \cosh(\pi\mu) < 1$, then $\mu \in \mathbb{C}$ is a complex number, $i\mu$ is not an integer and any solution of the MH equation is bounded along the real axis x. The domains of stability located within the parameter plane (a, q) are characterized by $-1 < \cosh(\pi\mu) < 1$. When $\cosh(\pi\mu) > 1$ one can consider the Floquet exponent μ as real and nonzero. On the other hand, if $\cosh(\pi\mu) < -1$ one may consider $\mu - i$ as real and nonzero. In both cases, the solution of the MH equation is not bounded along the real x axis. Unstable regions of the MH equations are characterized by $\cosh(\pi\mu) > 1$ or $\cosh(\pi\mu) < -1$. The stable and unstable regions are separated by curves characterized by $\cosh(\pi\mu) = \pm 1$, where one solution is bounded and periodic, while the general solution stays unbounded [113].

One can distinguish among several cases [72,74,94,95]:

- $\mu \in \mathbb{C}$ (pure imaginary) and $i\mu \notin \mathbb{Z}$;
- The frontiers of the stability domains are defined by $i\mu \notin Z$ (not integer);
- The associated dynamics are unstable when $\mu \in \mathbb{R}$ or $\mu - i \in \mathbb{R}$, with $\mu = i\theta$.

The stability of an ion with few crystals is first discussed in [73], based on a well-established analytical model that is frequently used in the literature to describe regular and nonlinear ion dynamics in 3D QIT. The numerical modeling of quantum manifestation of order and chaos for ions confined in a Paul trap is explored in [126], where it is established that quasienergy state statistics serve to discriminate between integrable and nonintegrable quantum dynamics. Double-well dynamics for systems made of two trapped ions (in either a Paul or a Penning trap) are investigated in [91], where the RF trapping voltage effect towards enhancing or altering quantum transport in the chaotic separatrix layer is also examined. The dynamical stability of two-ion crystals in a Paul trap is explored in [127], based on the pseudopotential approximation [79,81,128]. The nonlinear dynamics of ions confined in a RF trap are reported in [129], where a new deterministic melting region is reported, along with crystallization in a secondary Mathieu stability region. The irregular dynamics of single-ion dynamics in an EIT with axial symmetry is approached in [130] by employing both analytical and numerical modeling.

The dynamics of a trapped ion in a mass spectrometer, which undergoes the action of both quadrupole RF and dipolar DC excitation, are investigated in ref. [131] for a classical quadrupole 3D trap that exhibits hyperbolic geometry. It is demonstrated that dipolar excitation qualitatively changes the associated dynamics. The equation of motion is shown to be a classical MH equation that is perturbed with a constant inhomogeneous term, along with a small quadratic nonlinearity. An experimental approach implements motional parametric excitation and reports coherent spin–motion coupling of ions obtained by employing a spin-dependent force [62] for 2D crystals consisting of 100 ^{9}Be ions confined in a Penning trap.

Appendix B.2. Electrodynamic Ion Traps (EIT) Operating under SATP Conditions—Damping Case

We shortly discuss the case of an electrodynamic trap that operates under Standard Atmospheric Temperature and Pressure (SATP) conditions [111,132–135]. The equation of motion is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2(t)x = 0\,, \tag{A24}$$

where the second term characterizes friction (damping) in air, while λ stands for the damping coefficient. A solution to this equation is

$$x = e^{\rho t} f \dot{x} = e^{\rho t}\left(\rho f + \dot{f}\right), \ddot{x} = e^{\rho t}\left(\rho^2 f + 2\rho f + \dot{f} + \ddot{f}\right), \tag{A25}$$

where f is a periodic, time-dependent function. By introducing Equations (A25) in Equation (A24) and choosing $\rho = -\lambda$, one derives

$$\ddot{f} + \left(\omega^2 - \lambda^2\right) = 0\,, \tag{A26}$$

which is exactly an MH-type equation.

One of the first papers that reports the trapping of macroscopic dust particles under SATP conditions and the occurrence of ordered structures is ref. [136], which emphasizes that friction in air results in an efficient *cooling* of the trapped particle. Ordered structures are also reported in [132]. The dynamics of damped single-charged particles levitated in a Paul trap are investigated in [137] based on an analytical model. The modified stability diagrams in the (a, q) parameter space are derived, demonstrating that stable regions are not only enlarged but also shifted [72,118].

A multipole linear RF trap [138] is described in [139], which delivers an effective potential that characterizes three additional stable quasi-equilibrium points. The trap is used to levitate a group of charged silicate microspheres under SATP condition. The set-up exhibits a strong dependence on the RF field modulation and effective potential.

References

1. Brouwers, J.J.H. Asymptotic solutions for Mathieu instability under random parametric excitation and nonlinear damping. *Physica D* **2011**, *240*, 990–1000. [CrossRef]
2. Chaki, S.; Bhattacherjee, A. Role of dissipation in the stability of a parametrically driven quantum harmonic oscillator. *J. Korean Phy. Soc.* **2021**, *79*, 600–605. [CrossRef]
3. Nayfeh, A.H.; Mook, D.T. *Nonlinear Oscillations*; Wiley Classics Library, Wiley-VCH: Weinheim, Germany, 2008. [CrossRef]
4. Kovacic, I.; Brenner, M.J. (Eds.) *The Duffing Equation: Nonlinear Oscillations and their Behaviour*; Theoretical, Computational, and Statistical Physics; Wiley: Chichester, West Sussex, 2011. [CrossRef]
5. Mond, M.; Cederbaum, G.; Khan, P.B.; Zarmi, Y. Stability Analysis Of The Non-Linear Mathieu Equation. *J. Sound Vib.* **1993**, *167*, 77–89. [CrossRef]
6. Nayfeh, A.H.; Balachandran, B. *Applied Nonlinear Dynamics: Analytical, Computational, and Experimental Methods*; Wiley Series in Nonlinear Science; Wiley-VCH: Weinheim, Germany, 2008. [CrossRef]
7. Landa, H.; Drewsen, M.; Reznik, B.; Retzker, A. Classical and quantum modes of coupled Mathieu equations. *J. Phys. A Math. Theor.* **2012**, *45*, 455305. [CrossRef]
8. Kovacic, I.; Rand, R.; Sah, S.M. Mathieu's Equation and Its Generalizations: Overview of Stability Charts and Their Features. *Appl. Mech. Rev.* **2018**, *70*, 020802. [CrossRef]
9. Daniel, D.J. Exact solutions of Mathieu's equation. *Prog. Theor. Exp. Phys.* **2020**, *2020*, 043A01. [CrossRef]
10. Karličić, D.; Chatterjee, T.; Cajić, M.; Adhikari, S. Parametrically amplified Mathieu-Duffing nonlinear energy harvesters. *J. Sound Vib.* **2020**, *488*, 115677. [CrossRef]
11. Azimi, M. Stability and bifurcation of Mathieu–Duffing equation. *Int. J. Nonlin. Mech.* **2022**, *144*, 104049. [CrossRef]
12. El-Dib, Y.O. An innovative efficient approach to solving damped Mathieu–Duffing equation with the non-perturbative technique. *Commun. Nonlinear Sci. Numer. Simul.* **2024**, *128*, 107590. [CrossRef]
13. Rand, R.H. CISM Course: Time-Periodic Systems. 5–9 September 2016. Available online: http://audiophile.tam.cornell.edu/randpdf/rand_mathieu_CISM.pdf (accessed on 20 September 2024).
14. Trypogeorgos, D.; Foot, C.J. Cotrapping different species in ion traps using multiple radio frequencies. *Phys. Rev. A* **2016**, *94*, 023609. [CrossRef]
15. Katz, I.; Retzker, A.; Straub, R.; Lifshitz, R. Signatures for a Classical to Quantum Transition of a Driven Nonlinear Nanomechanical Resonator. *Phys. Rev. Lett.* **2007**, *99*, 040404. [CrossRef] [PubMed]
16. Mihalcea, B.M.; Vişan, G.G. Nonlinear ion trap stability analysis. *Phys. Scr.* **2010**, *T140*, 014057. [CrossRef]
17. Akerman, N.; Kotler, S.; Glickman, Y.; Dallal, Y.; Keselman, A.; Ozeri, R. Single-ion nonlinear mechanical oscillator. *Phys. Rev. A* **2010**, *82*, 061402(R). [CrossRef]
18. Doroudi, A. Application of a Modified Homotopy Perturbation Method for Calculation of Secular Axial Frequencies in a Nonlinear Ion Trap with Hexapole, Octopole and Decapole Superpositions. *J. Bioanal. Biomed.* **2012**, *4*, 85–91. [CrossRef]
19. Wu, H.Y.; Xie, Y.; Wan, W.; Chen, L.; Zhou, F.; Feng, M. A complicated Duffing oscillator in the surface-electrode ion trap. *Appl. Phys. B* **2014**, *114*, 81–88. [CrossRef]
20. Naito, T.; Naito, H.; Hashimoto, K. Multi-body wave function of ground and low-lying excited states using unornamented deep neural networks. *Phys. Rev. Res.* **2023**, *5*, 033189. [CrossRef]
21. Tibaduiza, D.M.; Pires, L.; Rego, A.L.C.; Szilard, D.; Zarro, C.; Farina, C. Efficient algebraic solution for a time-dependent quantum harmonic oscillator. *Phys. Scr.* **2020**, *95*, 105102. [CrossRef]
22. Xiong, C.; Zhou, X.; Zhang, N.; Zhan, L.; Chen, Y.; Nie, Z. Nonlinear Ion Harmonics in the Paul Trap with Added Octopole Field: Theoretical Characterization and New Insight into Nonlinear Resonance Effect. *J. Am. Soc. Mass Spectrom.* **2016**, *27*, 344–351. [CrossRef]

23. Foot, C.J.; Trypogeorgos, D.; Bentine, E.; Gardner, A.; Keller, M. Two-frequency operation of a Paul trap to optimise confinement of two species of ions. *Int. J. Mass. Spectrom.* **2018**, *430*, 117–125. [CrossRef]
24. Londry, F.A.; Hager, J.W. Mass selective axial ion ejection from a linear quadrupole ion trap. *J. Am. Soc. Mass Spectrom.* **2003**, *14*, 1130–1147. [CrossRef]
25. Reece, M.P.; Huntley, A.P.; Moon, A.M.; Reilly, P.T.A. Digital Mass Analysis in a Linear Ion Trap without Auxiliary Waveforms. *J. Am. Soc. Mass Spectrom.* **2019**, *31*, 103–108. [CrossRef] [PubMed]
26. Schmidt, J.; Hönig, D.; Weckesser, P.; Thielemann, F.; Schaetz, T.; Karpa, L. Mass-selective removal of ions from Paul traps using parametric excitation. *Appl. Phys.B* **2020**, *126*, 176. [CrossRef] [PubMed]
27. Lewis, H.R.; Riesenfeld, W.B. An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field. *J. Math. Phys.* **1969**, *10*, 1458–1473. [CrossRef]
28. Combescure, M. The quantum stability problem for some class of time-dependent hamiltonians. *Ann. Phys.* **1988**, *185*, 86–110. [CrossRef]
29. Combescure, M. Crystallization of trapped ions—A quantum approach. *Ann. Phys.* **1990**, *204*, 113–123. [CrossRef]
30. Gheorghe, V.N.; Vedel, F. Quantum dynamics of trapped ions. *Phys. Rev. A* **1992**, *45*, 4828–4831. [CrossRef]
31. Stenholm, S. Quantum Motion in a Paul Trap. *J. Mod. Opt.* **1992**, *39*, 279–290. [CrossRef]
32. Glauber, R.J. The Quantum Mechanics of Particles in Time-Dependent Quadrupole Fields. In *Recent Developments in Quantum Optics*; Inguva, R., Ed.; Plenum Press: New York, NY, USA, 1993; pp. 1–13.
33. Gardiner, S.A.; Cirac, J.I.; Zoller, P. Quantum Chaos in an Ion Trap: The Delta-Kicked Harmonic Oscillator. *Phys. Rev. Lett.* **1997**, *79*, 4790–4793. [CrossRef]
34. Dodonov, V.V.; Man'ko, V.I.; Rosa, L. Quantum singular oscillator as a model of a two-ion trap: An amplification of transition probabilities due to small-time variations of the binding potential. *Phys. Rev. A* **1998**, *57*, 2851–2858. [CrossRef]
35. Pedrosa, I.A.; Rosas, A.; Guedes, I. Exact quantum motion of a particle trapped by oscillating fields. *J. Phys. A Math. Gen.* **2005**, *38*, 7757–7763. [CrossRef]
36. Menicucci, N.C.; Milburn, G.J. Single trapped ion as a time-dependent harmonic oscillator. *Phys. Rev. A* **2007**, *76*, 052105. [CrossRef]
37. Mihalcea, B.M. Nonlinear harmonic boson oscillator. *Phys. Scr.* **2010**, *T140*, 014056. [CrossRef]
38. Wittemer, M.; Schröder, J.P.; Hakelberg, F.; Kiefer, P.; Fey, C.; Schuetzhold, R.; Warring, U.; Schaetz, T. Trapped-ion toolkit for studies of quantum harmonic oscillators under extreme conditions. *Phil. Trans. R. Soc. A* **2020**, *378*, 20190230. [CrossRef]
39. Keller, J.; Hou, P.Y.; McCormick, K.C.; Cole, D.C.; Erickson, S.D.; Wu, J.J.; Wilson, A.C.; Leibfried, D. Quantum Harmonic Oscillator Spectrum Analyzers. *Phys. Rev. Lett.* **2021**, *126*, 250507. [CrossRef] [PubMed]
40. Mihalcea, B.M. Quasienergy operators and generalized squeezed states for systems of trapped ions. *Ann. Phys.* **2022**, *442*, 169826. [CrossRef]
41. Coelho, S.S.; Queiroz, L.; Alves, D.T. Exact Solution of a Time-Dependent Quantum Harmonic Oscillator with Two Frequency Jumps via the Lewis-Riesenfeld Dynamical Invariant Method. *Entropy* **2022**, *24*, 1851. [CrossRef]
42. García Herrera, E.; Torres-Leal, F.; Rodríguez-Lara, B.M. Continuous-time quantum harmonic oscillator state engineering. *New J. Phys.* **2023**, *25*, 123045. [CrossRef]
43. Kato, A.; Goel, A.; Lee, R.; Ye, Z.; Karki, S.; Liu, J.J.; Nomerotski, A.; Blinov, B.B. Two-tone Doppler cooling of radial two-dimensional crystals in a radio-frequency ion trap. *Phys. Rev. A* **2022**, *105*, 023101. [CrossRef]
44. Hong, J.; Kim, M.; Lee, H.; Lee, M. Numerical investigation of a segmented-blade ion trap with biasing rods. *Appl. Phys. B* **2022**, *129*, 16. [CrossRef]
45. Colombo, S.; Pedrozo-Peñafiel, E.; Adiyatullin, A.F.; Li, Z.; Mendez, E.; Shu, C.; Vuletić, V. Time-reversal-based quantum metrology with many-body entangled states. *Nat. Phys.* **2022**, *18*, 925–930. [CrossRef]
46. Wineland, D.J. Nobel Lecture: Superposition, entanglement, and raising Schrödinger's cat. *Rev. Mod. Phys.* **2013**, *85*, 1103–1114. [CrossRef]
47. Wolf, F.; Shi, C.; Heip, J.C.; Gessner, M.; Pezzè, L.; Smerzi, A.; Schulte, M.; Hammerer, K.; Schmidt, P.O. Motional Fock states for quantum-enhanced amplitude and phase measurements with trapped ions. *Nat. Commun.* **2019**, *10*, 2929. [CrossRef] [PubMed]
48. Lamata, L.; Mezzacapo, A.; Casanova, J.; Solano, E. A high-precision segmented Paul trap with minimized micromotion for an optical multiple-ion clock. *EPJ Quantum Technol.* **2014**, *1*, 9. [CrossRef]
49. Martínez-Lahuerta, V.J.; Eilers, S.; Mehlstäubler, T.E.; Schmidt, P.O.; Hammerer, K. Ab initio quantum theory of mass defect and time dilation in trapped-ion optical clocks. *Phys. Rev. A* **2022**, *106*, 032803. [CrossRef]
50. Leibrandt, D.R.; Porsev, S.G.; Cheung, C.; Safronova, M.S. Prospects of a thousand-ion Sn^{2+} Coulomb-crystal clock with sub-10^{-19} inaccuracy. *Nat. Comm.* **2024**, *15*, 5663. [CrossRef] [PubMed]
51. Schkolnik, V.; Budker, D.; Fartmann, O.; Flambaum, V.; Hollberg, L.; Kalaydzhyan, T.; Kolkowitz, S.; Krutzik, M.; Ludlow, A.; Newbury, N.; et al. Optical atomic clock aboard an Earth-orbiting space station (OACESS): Enhancing searches for physics beyond the standard model in space. *Quantum Sci. Technol.* **2022**, *8*, 014003. [CrossRef]
52. Tsai, Y.D.; Eby, J.; Safronova, M.S. Direct detection of ultralight dark matter bound to the Sun with space quantum sensors. *Nat. Astron.* **2023**, *7*, 113–121. [CrossRef]
53. Reiter, F.; Sørensen, A.S.; Zoller, P.; Muschik, C.A. Dissipative quantum error correction and application to quantum sensing with trapped ions. *Nat. Comm.* **2017**, *8*, 1822. [CrossRef]

54. Kaiser, R.; Leduc, M.; Perrin, H. (Eds.) *Ultra-Cold Atoms, Ions, Molecules and Quantum Technologies*; EDP Sciences: Les Ulis, France, 2022. [CrossRef]
55. Okada, K.; Wada, M.; Takayanagi, T.; Ohtani, S.; Schuessler, H.A. Characterization of ion Coulomb crystals in a linear Paul trap. *Phys. Rev. A* **2010**, *81*, 013420. [CrossRef]
56. Bahrami, A.; Müller, M.; Drechsler, M.; Joger, J.; Gerritsma, R.; Schmidt-Kaler, F. Operation of a Microfabricated Planar Ion-Trap for Studies of a Yb^{+}–Rb Hybrid Quantum System. *Phys. Status Solidi B* **2019**, *256*, 1800647. [CrossRef]
57. Perego, E.; Duca, L.; Sias, C. Electro-Optical Ion Trap for Experiments with Atom-Ion Quantum Hybrid Systems. *Appl. Sci.* **2020**, *10*, 2222. [CrossRef]
58. Niranjan, M.; Prakash, A.; Rangwala, S.A. Analysis of Multipolar Linear Paul Traps for Ion–Atom Ultracold Collision Experiments. *Atoms* **2021**, *9*, 38. [CrossRef]
59. Coles, P. Pushing the Limits of Quantum Sensing with Variational Quantum Circuits. *Physics* **2021**, *14*, 172. [CrossRef]
60. Peik, E. Optical Atomic Clocks. In *Photonic Quantum Technologies*; Benyoucef, M., Ed.; Wiley: Hoboken, NJ, USA, 2023; Chapter 14, pp. 333–348. [CrossRef]
61. Mehta, K.K.; Zhang, C.; Malinowski, M.; Nguyen, T.L.; Stadler, M.; Home, J.P. Integrated optical multi-ion quantum logic. *Nature* **2020**, *586*, 533–537. [CrossRef]
62. Affolter, M.; Ge, W.; Bullock, B.; Burd, S.C.; Gilmore, K.A.; Lilieholm, J.F.; Carter, A.L.; Bollinger, J.J. Toward improved quantum simulations and sensing with trapped two-dimensional ion crystals via parametric amplification. *Phys. Rev. A* **2023**, *107*, 032425. [CrossRef]
63. Ye, J.; Zoller, P. Essay: Quantum Sensing with Atomic, Molecular, and Optical Platforms for Fundamental Physics. *Phys. Rev. Lett.* **2024**, *132*, 190001. [CrossRef] [PubMed]
64. Barontini, G.; Boyer, V.; Calmet, X.; Fitch, N.J.; Forgan, E.M.; Godun, R.M.; Goldwin, J.; Guarrera, V.; Hill, I.R.; Jeong, M.; et al. QSNET, a network of clock for measuring the stability of fundamental constants. In *Quantum Technology: Driving Commercialisation of an Enabling Science II*; Padgett, M.J., Bongs, K., Fedrizzi, A., Politi, A., Eds.; International Society for Optics and Photonics, SPIE: Bellingham, WA, USA, 2021; Volume 11881, pp. 63–66. [CrossRef]
65. Wolf, F.; Schmidt, P.O. Quantum sensing of oscillating electric fields with trapped ions. *Meas. Sens.* **2021**, *18*, 100271. [CrossRef]
66. Fountas, P.N.; Poggio, M.; Willitsch, S. Classical and quantum dynamics of a trapped ion coupled to a charged nanowire. *New J. Phys.* **2019**, *21*, 013030. [CrossRef]
67. Vilasi, G. *Hamiltonian Dynamics*; World Scientific: Singapore, 2001. [CrossRef]
68. Mihalcea, B.M. Study of quasiclassical dynamics of trapped ions using the coherent state formalism and associated algebraic groups. *Rom. J. Phys.* **2017**, *62*, 113. Available online: https://www.nipne.ro/rjp/2017_62_5-6/RomJPhys.62.113.pdf (accessed on 20 September 2024).
69. Onah, F.E.; Garcíía Herrera, E.; Ruelas-Galván, J.A.; Juárez Rangel, G.; Real Norzagaray, E.; Rodríguez-Lara, B.M. A quadratic time-dependent quantum harmonic oscillator. *Sci. Rep.* **2023**, *13*, 8312. [CrossRef]
70. Lemmer, A.; Plenio, M.B.; Bermudez, A. Noise studies of driven geometric phase gates with trapped ions. In *Ion Traps for Tomorrow's Applications*; Knoop, M., Marzoli, I., Morigi, G., Eds.; International School of Physics "Enrico Fermi", Institute of Physics, Societa Italiana di Fisica: Bologna, Italy, 2015; Volume 189, pp. 229–244. [CrossRef]
71. Wang, Y.X.; Liu, S.C.; Cheng, L.; Peng, L.Y. Systematic investigations on ion dynamics with noises in Paul trap. *J. Phys. A Math. Theor.* **2023**, *56*, 465302. [CrossRef]
72. Mihalcea, B.M. Mathieu–Hill Equation Stability Analysis for Trapped Ions: Anharmonic Corrections for Nonlinear Electrodynamic Traps. *Photonics* **2024**, *11*, 551. [CrossRef]
73. Blümel, R.; Kappler, C.; Quint, W.; Walther, H. Chaos and order of laser-cooled ions in a Paul trap. *Phys. Rev. A* **1989**, *40*, 808–823. [CrossRef] [PubMed]
74. Brillouin, L. A practical method for solving Hill's equation. *Quart. Appl. Math.* **1948**, *6*, 167–178. [CrossRef]
75. Magnus, W.; Winkler, S. *Hill's Equation*; Interscience Tracts in Pure and Applied Mathematics; Wiley: New York, NY, USA, 1966; Volume 20. [CrossRef]
76. Wolf, G. Mathieu Functions and Hill's Equation. In *NIST Handbook of Mathematical Functions*; Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W., Eds.; NIST & Cambridge University Press: New York, NY, USA, 2010; Chapter 28, pp. 651–681.
77. Whittaker, E.T.; Watson, G.N. *A Course of Modern Analysis*, 5th ed.; Moll, V.H., Ed.; Cambridge University Press: Cambridge, UK, 2021. [CrossRef]
78. Major, F.G.; Gheorghe, V.N.; Werth, G. *Charged Particle Traps: Physics and Techniques of Charged Particle Field Confinement*; Springer Series on Atomic, Optical and Plasma Physics; Springer: Berlin/Heidelberg, Germany, 2005; Volume 37. [CrossRef]
79. Knoop, M.; Madsen, N.; Thompson, R.C. (Eds.) *Physics with Trapped Charged Particles: Lectures from the Les Houches Winter School*; Imperial College Press & World Scientific: London, UK, 2014. [CrossRef]
80. Knoop, M.; Marzoli, I.; Morigi, G. (Eds.) *Ion Traps for Tomorrow Applications*; Proceedings of the International School of Physics Enrico Fermi; Società Italiana di Fisica and IOS Press: Bologna, Italy, 2015; Volume 189.
81. Kajita, M. *Ion Traps*; IOP Publishing: Bristol, UK, 2022. [CrossRef]
82. Drewsen, M. Ion Coulomb crystals and their applications. In *Ion Traps for Tomorrow's Applications*; Knoop, M., Marzoli, I., Morigi, G., Eds.; International School of Physics "Enrico Fermi", Institute of Physics, Societa Italiana di Fisica: Bologna, Italy, 2015; Volume 189, pp. 81–101. [CrossRef]

83. Drewsen, M. The linear-zigzag structural transition in cold ion chains. In *Ion Traps for Tomorrow's Applications*; Knoop, M., Marzoli, I., Morigi, G., Eds.; International School of Physics "Enrico Fermi", Institute of Physics, Societa Italiana di Fisica: Bologna, Italy, 2015; Volume 189, pp. 103–114. [CrossRef]
84. Joshi, M.K.; Fabre, A.; Maier, C.; Brydges, T.; Kiesenhofer, D.; Hainzer, H.; Blatt, R.; Roos, C.F. Polarization-gradient cooling of 1D and 2D ion Coulomb crystals. *New J. Phys.* **2020**, *22*, 103013. [CrossRef]
85. Spivey, R.F.; Inlek, I.V.; Jia, Z.; Crain, S.; Sun, K.; Kim, J.; Vrijsen, G.; Fang, C.; Fitzgerald, C.; Kross, S.; et al. High-Stability Cryogenic System for Quantum Computing with Compact Packaged Ion Traps. *IEEE Trans. Quant. Eng.* **2022**, *3*, 1–11. [CrossRef]
86. Zhukas, L.A.; Millican, M.J.; Svihra, P.; Nomerotski, A.; Blinov, B.B. Direct Observation of Ion Micromotion in a Linear Paul Trap. *Phys. Rev. A* **2021**, *103*, 023105. [CrossRef]
87. Rudyi, S.S.; Romanova, A.V.; Rozhdestvensky, Y.V. Numerical analysis of phase transitions in ion Coulomb crystals. *Comput. Part. Mech.* **2023**, *11*, 329–337. [CrossRef]
88. Rudyi, S.; Ivanov, A.; Shcherbinin, D. Fractal Quasi-Coulomb Crystals in Ion Trap with Cantor Dust Electrode Configuration. *Fractal Fract.* **2023**, *7*, 686. [CrossRef]
89. Duca, L.; Mizukami, N.; Perego, E.; Inguscio, M.; Sias, C. Orientational Melting in a Mesoscopic System of Charged Particles. *Phys. Rev. Lett.* **2023**, *131*, 083602. [CrossRef]
90. Mihalcea, B.M.; Lynch, S. Investigations on Dynamical Stability in 3D Quadrupole Ion Traps. *Appl. Sci.* **2021**, *11*, 2938. [CrossRef]
91. Farrelly, D.; Howard, J.E. Double-well dynamics of two ions in the Paul and Penning traps. *Phys. Rev. A* **1994**, *49*, 1494–1497. [CrossRef]
92. Burgermeister, T.; Mehlstäubler, T.E. Creation and dynamics of topological defects in ion Coulomb crystals. In *Ion Traps for Tomorrow's Applications*; Knoop, M., Marzoli, I., Morigi, G., Eds.; International School of Physics "Enrico Fermi", Institute of Physics, Societa Italiana di Fisica: Bologna, Italy, 2015; Volume 189, pp. 115–125. [CrossRef]
93. Wen, W.; Zhu, S.; Xie, Y.; Ou, B.; Wu, W.; Chen, P.; Gong, M.; Guo, G. Generalized Kibble-Zurek mechanism for defects formation in trapped ions. *Sci. China Phys. Mech. Astron.* **2023**, *66*, 280311. [CrossRef]
94. Blanch, G. On the Computation of Mathieu Functions. *J. Math. Phys.* **1946**, *25*, 1–20. [CrossRef]
95. McLachlan, N.W. *Theory and Application of Mathieu Functions*; Dover Publications: New York, NY, USA, 1964; Volume 1233.
96. Pedregosa-Gutierrez, J.; Champenois, C.; Kamsap, M.R.; Hagel, G.; Houssin, M.; Knoop, M. Symmetry breaking in linear multipole traps. *J. Mod. Opt.* **2018**, *65*, 529–537. [CrossRef]
97. Brewer, R.G.; Hoffnagle, J.; DeVoe, R.G.; Reyna, L.; Henshaw, W. Collision-induced two-ion chaos. *Nature* **1990**, *344*, 305–309. [CrossRef]
98. Hoffnagle, J.; Brewer, R.G. On the Frequency-Locked Orbits of Two Particles in a Paul Trap. *Science* **1994**, *265*, 213–215. [CrossRef] [PubMed]
99. Mihalcea, B.M. Squeezed coherent states of motion for ions confined in quadrupole and octupole ion traps. *Ann. Phys.* **2018**, *388*, 100–113. [CrossRef]
100. Coelho, S.S.; Queiroz, L.; Alves, D.T. Squeezing equivalence of quantum harmonic oscillators under different frequency modulations. *Phys. Scr.* **2024**, *99*, 085104. [CrossRef]
101. Hassani, S. *Mathematical Methods: For Students of Physics and Related Fields*, 2nd ed.; Physics and Astronomy; Springer: New York, NY, USA, 2009. [CrossRef]
102. Riley, K.F.; Hobson, M.P. *Essential Mathematical Methods for the Physical Sciences*; Cambridge University Press: Cambridge, UK, 2011.
103. Hassani, S. *Mathematical Physics: A Modern Introduction to Its Foundations*, 2nd ed.; Physics and Astronomy; Springer: Cham, Switzerland, 2013. [CrossRef]
104. Pence, T.J.; Wichman, I.S. *Essential Mathematics for Engineers and Scientists*; Cambridge University Press: Cambridge, UK, 2020. [CrossRef]
105. Richards, D. *Advanced Mathematical Methods with Maple*; Cambridge University Press: Cambridge, UK, 2009.
106. Kramer, P.; Saraceno, M. (Eds.) *Geometry of the Time-Dependent Variational Principle in Quantum Mechanics*; Lecture Notes in Physics; Springer: Berlin/Heidelberg, Germany, 1981; Volume 140. [CrossRef]
107. Ceausescu, V.; Gheorghe, A. Classical limit and quantization of hamiltonian systems. In *Symmetries and Semiclassical Features of Nuclear Dynamics*; Raduta, A.A., Ed.; Lecture Notes in Physics; Springer: Berlin/Heidelberg, Germany, 1987; Volume 279, pp. 69–117. [CrossRef]
108. Mihalcea, B. Coherent States for Trapped Ions: Applications in Quantum Optics and Precision Measurements. In *CPT and Lorentz Symmetry*; World Scientific: Singapore, 2023; pp. 192–195. [CrossRef]
109. Mihalcea, B.M. Semiclassical dynamics for an ion confined within a nonlinear electromagnetic trap. *Phys. Scr.* **2011**, *T143*, 014018. [CrossRef]
110. Schlipf, S. From a Single Ion to a Mesoscopic System: Crystalization of Ions in Paul Traps. In *Coherent and Collective Interactions of Particles and Radiation Beams*; Aspect, A., Barletta, W., Bonifacio, R., Eds.; Proceedings of the International School Physics "Enrico Fermi"; IOS Press: Tepper Drive Clifton, VA, USA, 1996; Volume 131, pp. 61–99. [CrossRef]
111. Mihalcea, B.M.; Filinov, V.S.; Syrovatka, R.A.; Vasilyak, L.M. The physics and applications of strongly coupled Coulomb systems (plasmas) levitated in electrodynamic traps. *Phys. Rep.* **2023**, *1016*, 1–103. [CrossRef]

112. Abramowitz, M.; Stegun, I.A. (Eds.) *Mathieu Functions*; Applied Mathematical Series; US Department of Commerce, National Bureau of Standards: Washington, DC, USA, 1972; Volume 55, Chapter 20, pp. 722–750.
113. Bateman, J. *Higher Transcendental Functions*; Robert E. Krieger Publishing: Malabar, FL, USA, 1981; Volume 3, Chapter 16, pp. 91–166.
114. Moussa, R.A. Generalization of Ince's Equation. *J. Appl. Math. Phys.* **2014**, *2*, 1171–1182. [CrossRef]
115. Tibaduiza, D.M.; Pires, L.; Szilard, D.; Zarro, C.A.D.; Farina, C.; Rego, A.L.C. A Time-Dependent Harmonic Oscillator with Two Frequency Jumps: An Exact Algebraic Solution. *Braz. J. Phys.* **2020**, *50*, 634–646. [CrossRef]
116. Sträng, J.E. On the characteristic exponents of Floquet solutions to the Mathieu equation. *Bull. Acad. R. Belg.* **2005**, *16*, 269–287. [CrossRef]
117. Jones, T. Mathieu's Equations and the ideal RF-Paul Trap. Available online: http://einstein.drexel.edu/~tim/open/mat/mat.pdf (accessed on 20 September 2024).
118. Sevugarajan, S.; Menon, A.G. Transition curves and *iso*-β_u lines in nonlinear Paul traps. *Int. J. Mass Spectrom.* **2002**, *218*, 181–196. [CrossRef]
119. Katz, O.; Cetina, M.; Monroe, C. Programmable *N*-Body Interactions with Trapped Ions. *PRX Quantum* **2023**, *4*, 030311. [CrossRef]
120. Spampinato, A.; Stacey, J.; Mulholland, S.; Robertson, B.I.; Klein, H.A.; Huang, G.; Barwood, G.P.; Gill, P. An ion trap design for a space-deployable strontium-ion optical clock. *Proc. R. Soc. A* **2024**, *480*, 20230593. [CrossRef]
121. van Eijkelenborg, M.A.; Dholakia, K.; Storkey, M.E.M.; Segal, D.M.; Thompson, R.C. A driven, trapped, laser cooled ion cloud: A forced damped oscillator. *Opt. Commun.* **1999**, *159*, 169–176. [CrossRef]
122. Mihalcea, B.M. A quantum parametric oscillator in a radiofrequency trap. *Phys. Scr.* **2009**, *T135*, 014006. [CrossRef]
123. Zhang, L.; Zhang, W. Lie transformation method on quantum state evolution of a general time-dependent driven and damped parametric oscillator. *Ann. Phys.* **2016**, *373*, 424–455. [CrossRef]
124. Weyl, H. *Meromorphic Functions and Analytic Curves*; Annals of Mathematics Studies; Princeton University Press—De Gruyter: Princeton, NJ, USA, 2016; Volume 12. [CrossRef]
125. Kotana, A.N.; Mohanty, A.K. Computation of Mathieu stability plot for an arbitrary toroidal ion trap mass analyser. *Int. J. Mass Spectrom.* **2017**, *414*, 13–22. [CrossRef]
126. Moore, M.; Blümel, R. Quantum manifestations of order and chaos in the Paul trap. *Phys. Rev. A* **1993**, *48*, 3082–3091. [CrossRef]
127. Hoffnagle, J.A.; Brewer, R.G. Stability of two-ion crystals in the Paul trap: A comparison of exact and pseudopotential calculations. *Appl. Phys. B* **1995**, *60*, 113–117. [CrossRef]
128. Paul, W. Electromagnetic traps for charged and neutral particles. *Rev. Mod. Phys.* **1990**, *62*, 531–540. [CrossRef]
129. Blümel, R. Nonlinear dynamics of trapped ions. *Phys. Scr.* **1995**, *T59*, 369–379. [CrossRef]
130. Blümel, R.; Bonneville, E.; Carmichael, A. Chaos and bifurcations in ion traps of cylindrical and spherical design. *Phys. Rev. E* **1998**, *57*, 1511–1518. [CrossRef]
131. Tandel, D.D.; Chatterjee, A.; Mohanty, A.K. Quadrupole ion trap with dipolar DC excitation: Motivation, nonlinear dynamics, and simple formulas. *Nonlinear Dyn.* **2023**, *111*, 15837–15852. [CrossRef]
132. Gheorghe, V.N.; Giurgiu, L.; Stoican, O.; Cacicovschi, D.; Molnar, R.; Mihalcea, B. Ordered Structures in a Variable Length AC Trap. *Acta Phys. Pol. A* **1998**, *93*, 625–629. [CrossRef]
133. Kulkarni, P.; Baron, P.A.; Willeke, K. (Eds.) *Aerosol Measurement: Principles, Techniques and Applications*, 3rd ed.; Wiley: Hoboken, NJ, USA, 2011. [CrossRef]
134. Syrovatka, R.A.; Filinov, V.S.; Vasilyak, L.M.; Pecherkin, V.Y.; Deputatova, L.V.; Vladimirov, V.I.; Popel, O.S.; Tarasenko, A.B. Cleaning dielectric surfaces by the electrical fields of the linear electrodynamic Paul trap. *J. Electrostat.* **2021**, *112*, 103583. [CrossRef]
135. Rybin, V.; Shcherbinin, D.; Semynin, M.; Gavenchuk, A.; Zakharov, V.; Ivanov, A.; Rozhdestvensky, Y.; Rudyi, S. Novel nonlinear damping identification method: Simultaneous size, mass, charge and density measurements of a microparticle in quadrupole trap. *Powder Technol.* **2023**, *427*, 118717. [CrossRef]
136. Winter, H.; Ortjohann, H.W. Simple demonstration of storing macroscopic particles in a "Paul trap". *Am. J. Phys.* **1991**, *59*, 807–813. [CrossRef] [PubMed]
137. Hasegawa, T.; Uehara, K. Dynamics of a single particle in a Paul trap in the presence of the damping force. *Appl. Phys. B* **1995**, *61*, 159–163. [CrossRef]
138. Mihalcea, B.M.; Visan, G.T.; Giurgiu, L.C.; Radan, S. Optimization of ion trap geometries and of the signal to noise ratio for high resolution spectroscopy. *J. Optoelectron. Adv. Mat.* **2008**, *10*, 1994–1998. Available online: https://old.joam.inoe.ro/download.php?idu=1538 (accessed on 20 September 2024).
139. Rudyi, S.S.; Vovk, T.A.; Kosternoi, I.A.; Rybin, V.V.; Rozhdestvensky, Y.V. Single-phase multipole radiofrequency trap. *AIP Adv.* **2020**, *10*, 085016. [CrossRef]