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Extrinsic curvature in geometrothermodynamics

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ABSTRACT

Based on the first law of black hole thermodynamics, we yield the metric of geometrothermodynamics. Choosing a hypersurface of constant extensive variable, the extrinsic curvature scalar is calculated and general correspondence of singularities between the extrinsic curvature scalar and the specific heat is discovered. This correspondence is further shown to exist in the constant intensive variable ensemble. We also show that extrinsic curvature scalar can reflect thermodynamic stability information, in spite of considering fluctuations of different thermodynamic variables or choosing different hypersurfaces.

1. Introduction

It has been proved that black holes possess temperature and entropy [1]. Investigations on black hole thermodynamics surged thereafter. Laws of black hole mechanics was set up [2] and then the followings are thermodynamics [3,4]. Further studies exhibit that there are substantial phase structures and critical phenomena for black holes [5,6]. Traditionally, we obtain information of phase transitions for black hole through analyzing its temperature, free energy and specific heat [7,8].

Black holes, originating from collapsed objects, are special spacetime regions with strong gravity [9]. Our consensus nowaday is "Gravitational Interaction=Spacetime Curvature". Can we ask "Thermodynamic Interaction=Thermodynamic Curvature"? Geometrization of black hole thermodynamics seems intriguing and enlightening.

Gibbs [10], Carathéodory [11], Fisher [12] and Rao[13] are forerunners on planting geometry perceptions into thermodynamics. Thanks to Weinhold [14] and Ruppeiner [15,16], Riemannian metric structure is introduced to the study of black hole phase transitions. Afterwards, a Riemannian structure obeying Legendre invariance was introduced in equilibrium space by Quevedo [17]. It is proved that universal correspondence exists between intrinsic Ricci scalar and phase transition points [18,19]. Recently, thermodynamic extrinsic curvature of a certain hypersurface is raised and calculated for (phantom) Reissner–Nordström–(A)dS black hole using Ruppeiner metric [20], it is found that the extrinsic curvature shares the same divergent points and signs with specific heat on a constant electricity Q hypersurface.

Within this paper, Quevedo thermodynamic geometry which is Legendre transformation invariant is briefly introduced in Sec. 2. In Sec. 3, we explore a formulation for the thermodynamic extrinsic curvature of black holes, and generally prove that there exists correspondence between thermodynamic extrinsic curvature and phase transition points for black holes, based on the first law of thermodynamics. In Sec. 4, we will present our thought on relation between extrinsic curvature and thermodynamic stability. Sec. 5 is devoted to our conclusion.

2. Quevedo thermodynamic geometry

One needs a thermodynamic potential Ξ, a set of n extensive variables E^a $(a = 1, 2, \cdots, n)$, and their dual intensive variables I^a to describe a thermodynamic system which holds n degrees of freedom in the equilibrium thermodynamics. The thermodynamic phase space $\mathcal{T}$ then has coordinates $Z^A = (\Xi, E^a, I^a)$ where $a = 1, 2, \cdots, n$. The Legendre transformations defined in $\mathcal{T}$ is given by

$$(\Xi, E^a, I^a) \to (\tilde{\Xi}, \tilde{E}^a, \tilde{I}^a), \tag{1}$$

$$\Xi = \tilde{\Xi} - \delta_{kl}\tilde{E}^a\tilde{I}^a,\ E^i = -\tilde{I}^i,\ E^j = \tilde{E}^j,\ I^i = \tilde{E}^i,\ I^j = \tilde{I}^j, \tag{2}$$

where $i \cup j$ is any disjoint decomposition of the set of indices $\{1, 2, \cdots, n\}$ and $k, l = 1, 2, \cdots, i$. Particularly, we can obtain the total Legendre transformation for $i = \{1, 2, \cdots, n\}$. In the thermodynamic phase space $\mathcal{T}$, a canonical contact structure determined by the fundamental Gibbs 1-form $\Theta = d\Xi - \delta_{ab}I^b dE^a$ exists. A equi-

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librium space $\mathcal{E}$ defined by the embedding map $\varphi : \mathcal{E} \to \mathcal{T}$ with constraints

$$\varphi^*(\Theta) = 0, \text{ i.e. } d\Xi = I_a dE^a \text{ and } I_a = \frac{\partial \Xi}{\partial E^a} \tag{3}$$

can be defined. Here φ^* is the pullback of φ. At the same time, these two expressions can be viewed as the first law of thermodynamics and the equilibrium conditions, respectively.

One can define a Legendre transformations invariant metric G in the space $\mathcal{T}$. There is one typical metric [21,22]

$$G = \Theta^2 + \left(\delta_{ab} E^a I^b\right)\left(\eta_{cd} dE^c dI^d\right), \tag{4}$$
$$\delta_{ab} = \mathrm{diag}(1, 1, \cdots, 1), \eta_{ab} = \mathrm{diag}(-1, 1, \cdots, 1),$$

which can describe black hole systems with second order phase transitions. One can obtain a thermodynamic metric which is invariant under Legendre transformations for the equilibrium space $\mathcal{E}$ as

$$\begin{aligned} g \equiv \varphi^*(G) &= \left(\delta_a^b E^a \frac{\partial \Xi}{\partial E^b}\right)\left(\eta_{cd} dE^c \delta^{df} dE^e \frac{\partial^2 \Xi}{\partial E^e \partial E^f}\right) \\ &= \left(E^c \frac{\partial \Xi}{\partial E^c}\right)\left(\eta_{ab}\delta^{bc} \frac{\partial^2 \Xi}{\partial E^c \partial E^d} dE^a dE^d\right). \end{aligned} \tag{5}$$

By using this metric to describe the equilibrium manifold $\mathcal{E}$, we aim to extract information of the thermodynamic properties of black hole systems.

3. Extrinsic curvature and phase transition

3.1. Extrinsic curvature in constant extensive variable ensemble

Firstly, we want to discuss our problem in constant generalized displacement ensemble, say, constant X ensemble as follows. As is known, once the thermodynamic potential is chosen as the mass, universal first law of thermodynamics for any black hole can be expressed as

$$dM = TdS + YdX + \sum_i y_i dx_i, \tag{6}$$

where X, x_i are generalized displacements, such as electric charge (Q), angular momentum (J), etc., Y, y_j are generalized forces, such as electrical potential (Φ), angular velocity (Ω), etc.

One can then yield the thermodynamic metric [22]

$$\begin{aligned} g_{\mu\nu} &= (SM_S + XM_X + \sum_i x_i M_{x_i}) \\ &\quad \times \left(-M_{SS}dS^2 + M_{XX}dX^2 + \sum_{i,j} M_{x_i x_j} dx_i dx_j\right) \\ &\equiv g_{SS}dS^2 + g_{XX}dX^2 + g_{\xi_\mu \xi_\nu} d\xi^\mu d\xi^\nu, \end{aligned} \tag{7}$$

where $M_S = \left(\frac{\partial M}{\partial S}\right)_{X,x_i}$, $M_X = \left(\frac{\partial M}{\partial X}\right)_{S,x_i}$, $M_{x_i} = \left(\frac{\partial M}{\partial x_i}\right)_{S,X}$, $M_{SS} = \left(\frac{\partial^2 M}{\partial S^2}\right)_{X,x_i}$, $M_{XX} = \left(\frac{\partial^2 M}{\partial X^2}\right)_{S,x_i}$ and $M_{x_i x_j} = \left(\frac{\partial^2 S}{\partial X_i X_j}\right)_{S,X}$.

After choosing an $X = \text{Const.}$ hypersurface, we can define a restricted function Ψ as

$$\Psi = X - \text{Const.} \tag{8}$$

Then the normalized normal covector is

$$n_\alpha = \frac{\Psi_{,\alpha}}{\sqrt{|g^{ab}\Psi_{,a}\Psi_{,b}|}} = \left(0, \frac{1}{\sqrt{|g^{XX}|}}, 0, \cdots, 0\right). \tag{9}$$

Letting $n^\alpha \Psi_{,\alpha} > 0$, so n^α points to the direction of increasing Ψ. The normal vector is

$$\begin{aligned} n^\beta = g^{\alpha\beta} n_\alpha &= \begin{bmatrix} g_{SS} & 0 & \cdots & 0 \\ 0 & g_{XX} & \cdots & g_{Xx_i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g_{x_i X} & \cdots & g_{x_i x_i} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{|g^{XX}|}} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \frac{g^{XX}}{\sqrt{|g^{XX}|}} \\ \frac{g^{x_1 X}}{\sqrt{|g^{XX}|}} \\ \vdots \\ \frac{g^{x_i X}}{\sqrt{|g^{XX}|}} \end{bmatrix} = \begin{bmatrix} g^{SS} & 0 & \cdots & 0 \\ 0 & g^{XX} & \cdots & g^{Xx_i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g^{x_i X} & \cdots & g^{x_i x_i} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{|g^{XX}|}} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \frac{g^{XX}}{\sqrt{|g^{XX}|}} \\ \frac{g^{x_1 X}}{\sqrt{|g^{XX}|}} \\ \vdots \\ \frac{g^{x_i X}}{\sqrt{|g^{XX}|}} \end{bmatrix}. \end{aligned} \tag{10}$$

The corresponding extrinsic curvature is

$$\begin{aligned} K = h^{ab}K_{ab} = n^\alpha_{;\alpha} &= \frac{1}{\sqrt{|g|}} \partial_k(\sqrt{|g|} n^k) \\ &= \frac{1}{\sqrt{|g|}} \partial_X \left(\sqrt{|g|} \times \frac{g^{XX}}{\sqrt{|g^{XX}|}}\right) \\ &\quad + \frac{1}{\sqrt{|g|}} \sum_i \partial_{x_i} \left(\sqrt{|g|} \times \frac{g^{x_i X}}{\sqrt{|g^{XX}|}}\right) \\ &\backsim \frac{\partial_X \sqrt{|g|}}{\sqrt{|g|}} \times \frac{g^{XX}}{\sqrt{|g^{XX}|}} + \sum_i \frac{\partial_{x_i}\sqrt{|g|}}{\sqrt{|g|}} \times \frac{g^{x_i X}}{\sqrt{|g^{XX}|}} \\ &= \mathrm{Sgn}(g) \cdot \mathrm{Sgn}(g^{XX}) \cdot \frac{\sqrt{|g^{XX}|}\partial_X g}{2|g|} + \mathrm{Sgn}(g) \cdot \frac{\partial_X g}{2|g|} \times \frac{g^{x_i X}}{\sqrt{|g^{XX}|}} \\ &= \mathrm{Sgn}(g^{XX}) \cdot \frac{\sqrt{|g^{XX}|}\partial_X g}{2g} + \frac{\partial_X g}{2g} \times \frac{g^{x_i X}}{\sqrt{|g^{XX}|}}, \end{aligned} \tag{11}$$

where Sgn is the sign function, non-related terms in the fourth line are neglected and

$$g = \mathrm{Det}(g_{\mu\nu}) = g_{SS} \times \mathrm{Det} \begin{bmatrix} g_{XX} & \cdots & g_{Xx_i} \\ \vdots & \ddots & \vdots \\ g_{x_i X} & \cdots & g_{x_i x_i} \end{bmatrix} \propto M_{SS}. \tag{12}$$

The heat capacity diverges at the points where the phase transitions of black holes take place [23]. It is reminiscent of the definition of the specific heat

$$C_{Xx_i} = T\left(\frac{\partial S}{\partial T}\right)_{X,x_i} = \frac{T}{\left(\frac{\partial T}{\partial S}\right)_{X,x_i}} = TM_{SS}^{-1}, \tag{13}$$

which means that the critical point in the phase transition is located at the point where M_{SS} vanishes and the specific heat diverges. Thinking of the extrinsic curvature, we can find that it also diverges at critical points.

3.2. Extrinsic curvature in constant intensive variable ensemble

One common sense is that thermodynamic properties, including the phase structure, depend on the choice of statistical ensemble. A question arises that whether extrinsic curvature is able to reflect the phase transition information correctly while thermodynamic ensemble is transferred. We will study the extrinsic curvature at the constant generalized force ensemble. In this case, we define a new thermodynamic potential $\bar{M}$ using the Legendre transformation

$$\bar{M} = M - XY - \sum_i x_i y_i, \tag{14}$$

then we can obtain

$$d\bar{M} = TdS - XdY - \sum_i x_i dy_i. \tag{15}$$

The thermodynamic metric can be obtained as

$$\begin{aligned} g_{\mu\nu} &= (S\bar{M}_S + Y\bar{M}_Y + \sum_i y_i \bar{M}_{y_i}) \\ &\quad \times \left(-\bar{M}_{SS}dS^2 + \bar{M}_{YY}dY^2 + \sum_{i,j}\bar{M}_{y_iy_j}dy_idy_j\right) \\ &\equiv g_{SS}dS^2 + g_{YY}dY^2 + g_{\xi_\mu\xi_\nu}d\xi^\mu d\xi^\nu, \end{aligned} \tag{16}$$

where $\bar{M}_S = \left(\frac{\partial\bar{M}}{\partial S}\right)_{X,x_i}$, $\bar{M}_Y = \left(\frac{\partial\bar{M}}{\partial Y}\right)_{S,x_i}$, $\bar{M}_{x_i} = \left(\frac{\partial\bar{M}}{\partial x_i}\right)_{S,X}$, $\bar{M}_{SS} = \left(\frac{\partial^2\bar{M}}{\partial S^2}\right)_{X,x_i}$, $\bar{M}_{YY} = \left(\frac{\partial^2\bar{M}}{\partial Y^2}\right)_{S,x_i}$ and $\bar{M}_{x_ix_j} = \left(\frac{\partial^2 S}{\partial X_i X_j}\right)_{S,Y}$.

In this thermodynamic space, the normal covector of the constant Y hypersurface is

$$n_\alpha = \frac{\Psi_{,\alpha}}{\sqrt{|g^{ab}\Psi_{,a}\Psi_{,b}|}} = \left(0, \frac{1}{\sqrt{|g^{YY}|}}, 0, \cdots, 0\right), \tag{17}$$

where $\Psi = Y - \text{Const}$. The normal vector is

$$\begin{aligned} n^\beta = g^{\alpha\beta}n_\alpha &= \begin{bmatrix} g^{SS} & 0 & \cdots & 0 \\ 0 & g^{YY} & \cdots & g^{Yy_i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g^{y_iY} & \cdots & g^{y_iy_i} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{|g^{YY}|}} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \frac{g^{YY}}{\sqrt{|g^{YY}|}} \\ \frac{g^{y_1Y}}{\sqrt{|g^{YY}|}} \\ \vdots \\ \frac{g^{y_iY}}{\sqrt{|g^{YY}|}} \end{bmatrix}. \end{aligned} \tag{18}$$

Being similar to the former case, one can obtain the extrinsic curvature

$$\begin{aligned} K &= \frac{1}{\sqrt{|g|}}\partial_Y\left(\sqrt{|g|} \times \frac{g^{YY}}{\sqrt{|g^{YY}|}}\right) \\ &\quad + \frac{1}{\sqrt{|g|}}\sum_i \partial_{y_i}\left(\sqrt{|g|} \times \frac{g^{y_iY}}{\sqrt{|g^{YY}|}}\right) \\ &\backsim \text{Sgn}(g)\cdot\text{Sgn}(g^{YY})\cdot\frac{\sqrt{|g^{YY}|}\partial_Y g}{2|g|} + \text{Sgn}(g)\cdot\frac{\partial_Y g}{2|g|} \times \frac{g^{x_iY}}{\sqrt{|g^{YY}|}} \\ &= \text{Sgn}(g^{YY})\cdot\frac{\sqrt{|g^{YY}|}\partial_Y g}{2g} + \frac{\partial_Y g}{2g} \times \frac{g^{x_iY}}{\sqrt{|g^{YY}|}}, \end{aligned} \tag{19}$$

where

$$g = \text{Det}(g_{\mu\nu}) = g_{SS} \times \text{Det}\begin{bmatrix} g_{YY} & \cdots & g_{Yy_i} \\ \vdots & \ddots & \vdots \\ g_{y_iY} & \cdots & g_{y_iy_i} \end{bmatrix} \propto \bar{M}_{SS}. \tag{20}$$

The specific heat takes the form

$$C_{Y,y_i} = -\frac{1}{T^2 S_{\bar{M}\bar{M}}} = T\bar{M}_{SS}^{-1}. \tag{21}$$

As is shown, the extrinsic curvature K and the specific heat again share the same term $\bar{M}_{SS}$ in the denominator, which verifies that the extrinsic curvature reflects information of phase structure, even though statistical ensemble changes.

3.3. How about choosing a different hypersurface?

We know from Eq. (13) that in order to obtain the specific heat, one must keep X and x_i invariant. We have verified that the extrinsic curvature at const X hypersurface diverges where the specific heat does. Here we want to show that this correspondence also holds when the $\alpha_1 X + \sum_j \alpha_j x_j = \text{Const.}$ hypersurface is chosen, where α_1, α_i are arbitrary real numbers and j is any disjoint decomposition of the set of indices $\{1, 2, \cdots, n\}$. The related restricted function is

$$\Psi = \alpha_1 X + \sum_j \alpha_j x_j - \text{Const.} \tag{22}$$

where $\sum_i \alpha_i^2 \neq 0$. The normal covector on this hypersurface is

$$n_\alpha = N\left(0, \alpha_1, \cdots, \alpha_{j_p}, \cdots, \alpha_{j_k}, \cdots\right), \tag{23}$$

where $N = \left|g^{x_jx_{j'}}\Psi_{,x_j}\Psi_{,x_{j'}}\right|^{-\frac{1}{2}}$. Then the normal vector can be obtained as

$$n^\alpha = N\begin{bmatrix} 0 \\ \alpha_1 g^{XX} + \sum_j \alpha_j g^{Xx_j} \\ \alpha_1 g^{x_1X} + \sum_j \alpha_j g^{x_1x_j} \\ \vdots \\ \alpha_1 g^{x_iX} + \sum_j \alpha_j g^{x_ix_j} \end{bmatrix}. \tag{24}$$

Following the similar procedure, one can obtain the extrinsic curvature

$$K = \frac{N}{\sqrt{|g|}}\partial_X\left[\sqrt{|g|} \times \left(\alpha_1 g^{XX} + \sum_j \alpha_j g^{Xx_j}\right)\right]$$

$$+\frac{N}{\sqrt{|g|}}\sum_i \partial_{x_i}\left[\sqrt{|g|}\times\left(\alpha_1 g^{x_i X}+\sum_j \alpha_j g^{x_i x_j}\right)\right]$$
$$\backsim \frac{N\partial_X g}{2g}\left(\alpha_1 g^{XX}+\sum_j \alpha_j g^{X x_j}\right)$$
$$+\frac{N\partial_X g}{2g}\sum_i\left(\alpha_1 g^{x_i X}+\sum_j \alpha_j g^{x_i x_j}\right)$$
$$\propto M_{SS}^{-1}. \quad (25)$$

It is interesting that the information of phase structure contained in the specific heat at constant X, x_i ensemble can be reflected by $\alpha_1 X+\sum_j \alpha_j x_j=$ Const. hypersurfaces.

4. Extrinsic curvature and thermodynamic stability

We have generally proved that extrinsic curvature can reflect the critical point of phase transition for black holes. Here we will show that extrinsic curvature can reflect the thermodynamic stability information. General prove of it has been tried and we conclude that it seems impossible as concrete quantities are needed in some deductions or some assumptions must be taken. Here RN–AdS and KN–AdS black holes are used as toy models to exhibit our ideas.

4.1. The consideration of thermodynamic fluctuation

The mass of RN–AdS black holes is

$$M=\frac{l^2\pi^2 Q^2+l^2\pi S+S^2}{2l^2\pi^{3/2}\sqrt{S}}, \quad (26)$$

where l, S, Q are AdS radius, Bekenstein–Hawking entropy and electric charge, respectively. The first law of the black hole is

$$dM=TdS+\Phi dQ, \quad (27)$$

where Φ is the electric potential. The extrinsic curvature at $Q=$const. hypersurface can be calculated as

$$K_{RN-AdS}=\frac{36\pi^{5/2}l^2 Q\left(\pi^2 l^2 Q^2+S^2\right)}{\left(3\pi^2 l^2 Q^2-\pi l^2 S+3S^2\right)\left(3\pi^2 l^2 Q^2+\pi l^2 S+3S^2\right)}\times\sqrt{\frac{l^2 S}{\pi l^2\left(3\pi Q^2+S\right)+3S^2}}. \quad (28)$$

For KN–AdS black holes, the mass can be expressed as

$$M^2=\frac{J^2}{l^2}+\frac{\pi\left(4J^2+Q^4\right)}{4S}+\frac{S\left(\frac{S^2}{2\pi^2 l^2}+Q^2+\frac{S}{\pi}\right)}{2\pi l^2}+\frac{Q^2}{2}+\frac{S}{4\pi}, \quad (29)$$

where J is the angular momentum. The first law of the black hole is

$$dM=TdS+\Phi dQ+\Omega dJ. \quad (30)$$

We first calculate the extrinsic curvature of the black hole at $Q=$Const. hypersurface and then let the angular momentum decrease to zero. At first sight, one expects that K_{RN-AdS} will be identical to $K_{KN-AdS}^{J\to 0}$, as KN–AdS geometry will degenerate to RN–AdS geometry under the condition $J\to 0$. However, we obtain

$$K_{KN-AdS}^{J\to 0}=\frac{\mathcal{C}_0}{S\left(\pi l^2\left(3\pi Q^2-S\right)+3S^2\right)\left(\pi l^2\left(\pi Q^2+S\right)+S^2\right)}, \quad (31)$$

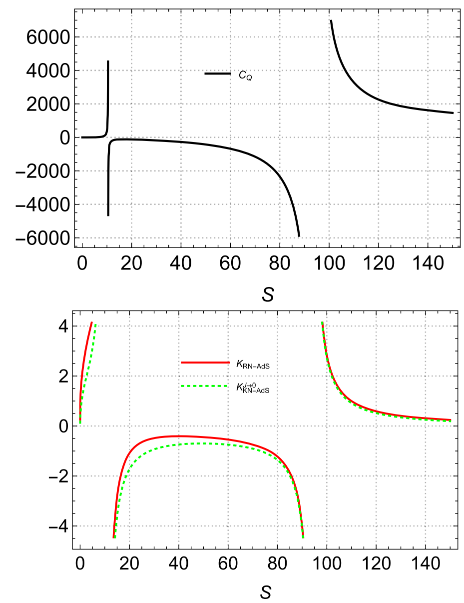

Fig. 1. Variations of C_Q, K_{RN-AdS}, $K_{KN-AdS}^{J\to 0}$ with S for $Q=1, l=10$.

where

$$\mathcal{C}_0=8\pi^{5/2}Q\left(\frac{l^2 S}{\pi l^2\left(3\pi Q^2+S\right)+3S^2}\right)^{3/2}\times\left[\pi^2 l^4\left(9\pi^2 Q^4+6\pi Q^2 S-S^2\right)+6\pi l^2 S^2\left(3\pi Q^2+S\right)+9S^4\right]. \quad (32)$$

It is clearly that $K_{RN-AdS}\neq K_{KN-AdS}^{J\to 0}$. How to explain this?

As is well known, the Riemannian thermodynamic curvature is used as a method of considering the fluctuation phenomena and it is suggested that there may be a relationship between the curvature and the interactions of the underlying statistical system [24,25]. When we calculate extrinsic curvature of the RN–AdS black hole, there are two fluctuating parameters S and Q. However, considering extrinsic curvature of the KN–AdS black hole, another fluctuating parameter J is included. So it is not strange that $K_{KN-AdS}^{J\to 0}$ being different from K_{RN-AdS}. Existence of difference between K_{RN-AdS} and $K_{KN-AdS}^{J\to 0}$ exhibits the non-trivial fluctuated effects of the angular momentum on the extrinsic curvature of the black holes.

It is shown that extrinsic curvature has the same sign as the heat capacity around the phase transition point for an RN–(A)dS black hole so that information of thermodynamic stability can be reflected [20]. One would like to ask that weather this fine property of extrinsic curvature can endure the test of consideration of thermodynamic fluctuations. The answer is affirmative. To show it explicitly, we plot the diagrams of K_{RN-AdS} and $K_{KN-AdS}^{J\to 0}$ in Fig. 1. One can see from the diagram that even if the thermodynamic fluctuation of one more parameter is considered, extrinsic curvature can still tell us stability information.

4.2. The selection of hypersurfaces

The specific heat with fixed Q, J for KN–AdS black hole can be calculated as

$$C_{QJ} = \frac{\mathcal{C}_1 \mathcal{C}_2}{\mathcal{C}_3}, \tag{33}$$

where

$$\mathcal{C}_1 = -8S\pi^4 J^2 l^4 + 2Sl^4 \left(\pi^2 S^2 - \pi^4 Q^4\right) + 4S\pi l^2 S^2 \left(\pi Q^2 + 2S\right) + 6SS^4,$$

$$\mathcal{C}_2 = 4\pi^3 J^2 l^2 \left(\pi l^2 + S\right) + \left(\pi l^2 \left(\pi Q^2 + S\right) + S^2\right)^2,$$

and

$$\begin{aligned}\mathcal{C}_3 =& 16\pi^7 J^4 l^6 \left(3\pi l^2 + 4S\right) \\ &+ \left[\pi l^2 \left(3\pi Q^2 - S\right) + 3S^2\right] \left[\pi l^2 \left(\pi Q^2 + S\right) + S^2\right]^3 \\ &+ 8\pi^3 J^2 l^2 \Big[\pi^3 l^6 \left(3\pi^2 Q^4 + 4\pi Q^2 S + 3S^2\right) + 15\pi l^2 S^4 \\ &\qquad + 6S^5\Big] \\ &+ 16\pi^5 J^2 l^6 S \left(\pi^2 Q^4 + 3\pi Q^2 S + 6S^2\right).\end{aligned} \tag{34}$$

Now we consider the choice of different hypersurfaces. First, let us choose the hypersurface $Q = \text{Const.}$ and calculate the extrinsic curvature

$$K_Q = \frac{\mathcal{C}_4}{\mathcal{C}_3}, \tag{35}$$

then the hypersurface $J = \text{Const.}$ is chosen and its corresponding extrinsic curvature is

$$K_J = \frac{\mathcal{C}_5}{\mathcal{C}_3}. \tag{36}$$

Besides, we choose a hypersurface $\alpha Q + \beta J = \text{Const.}$, where $\alpha > 0$, $\beta > 0$, and calculate the extrinsic curvature as

$$K_{QJ} = \frac{\mathcal{C}_6}{\mathcal{C}_3}. \tag{37}$$

$\mathcal{C}_4, \mathcal{C}_5$ and $\mathcal{C}_6$ are too lengthy to show here, instead, diagrams of C_{QJ}, K_Q, K_J and K_{JQ} vs. S are shown in Fig. 2.

On the one hand, as has been proved in the above section, one can see from the figure that all of K_Q, K_J, K_{QJ} diverge where the specific heat C_{QJ} does at constant Q and J ensemble.

On the other hand, what one can see is all of them have the same signs as the specific heat. This is fascinating and we suspect that this is a general property which can be applied to other black hole models.

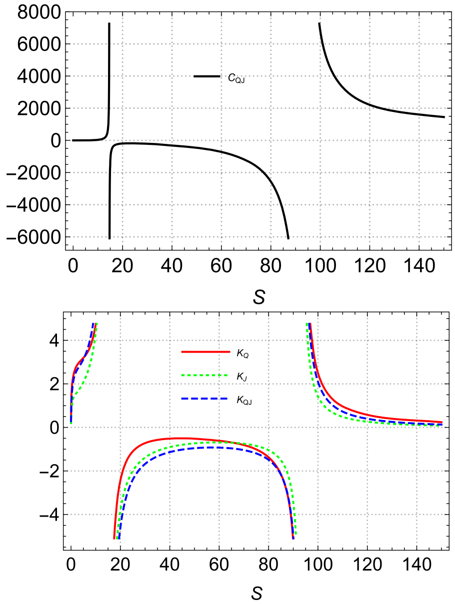

Fig. 2. Variations of C_{QJ}, K_Q, K_J, K_{QJ} with S for $Q = 1, l = 10, J = \frac{1}{2}, \alpha = 4, \beta = 3$.

5. Conclusion

Based on the first law of black hole thermodynamics, we investigate extrinsic curvature scalars in Legendre invariant thermodynamic metric space, both in the constant extensive variable ensemble and the constant intensive variable ensemble. It is verified that the thermodynamic extrinsic curvature scalar can provide information of phase transition structures for black holes, i.e., there exists correspondence of singularity between thermodynamic extrinsic curvature scalar and specific heat. Besides, we have shown that extrinsic curvature scalar can reflect the thermodynamic stability of the black holes, although thermodynamic fluctuations of more parameters are considered, or different hypersurfaces are chosen. Considering this, we argue that extrinsic curvature scalar of the Legendre transformation invariant space is a better quantity than intrinsic curvature scalar to reflect the phase transition and thermodynamic stability information for black holes. Our work hence extends previous studies on geometrothermodynamics.

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