Reestablishment of SO(4) symmetry in the relativistic Coulomb problem from $\mathcal{N}=4$ SYM

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Reestablishment of SO(4) symmetry in the relativistic Coulomb problem from $\mathcal{N} = 4$ SYM

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Abstract. We obtain a scalar field theory by applying a specific Higgs mechanism on $\mathcal{N} = 4$ SYM and imposing a constraint. We see that the resulting relativistic equation describes a charged particle non-minimally coupled to the Coulomb potential. The resulting non-minimal coupling allows us to build a relativistic Runge-Lenz vector that generates, together with the angular momentum, the SO(4) algebra. We also construct a Kustaanheimo-Stiefel duality between this modified relativistic Coulomb problem and the modified relativistic harmonic oscillator and see how the integrals of motion of both problems are mapped.

1. Introduction

The study of $\mathcal{N} = 4$ SYM is an active area of research. In particular, its integrability accounts for important results in the context of scattering amplitudes. For instance, by using a duality symmetry in momentum space the complete calculation of all tree level amplitudes as well as up to four loops in perturbation theory was accomplished [1]. The authors of [2] considered the possibility of constructing a consistent quantum field theory that preserves the analog of the Runge-Lenz (RL) vector of the Kepler potential in classical mechanics. It turns out that this theory can be obtained through a particular spontaneous symmetry breaking of $\mathcal{N} = 4$ SYM. The existence of the RL vector is in contrast to the fact that a relativistic particle minimally coupled to a Coulomb potential breaks the symmetry associated to the non relativistic problem. However, we will see that it is possible to restore the SO(4) symmetry in the relativistic context if we introduce a non-minimal coupling of the particle with the scalar Coulomb field and choose a particular reference frame that breaks the explicit Lorentz covariance. This non-minimal coupling is widely used in the theory of nuclear spectra to give rise to the so-called Klein-Gordon and Dirac equations with Scalar and Vector Potentials of Equal Magnitude (SVPEM) which possess an enhanced symmetry algebra [3, 4, 5, 6].

In Section 2, we present the simplest implementation of a Higgs mechanism to extract the modified Klein-Gordon equation. Then, in Section 3, we see the deduction of this equation from a modified action principle for a relativistic particle. In Section 4, we obtain the RL vector and show that it generates, together with the angular momentum, the SO(4) algebra, yielding the relativistic spectrum of the modified Klein-Gordon equation. The results confirm previous analyses using different approaches [6, 7]. Next, in Section 5, we make use of the Kustaanheimo-Stiefel (KS) transformation and show the relation of the spectra of the relativistic spectrum of

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the hydrogen-like atom in two dimensions and the relativistic harmonic oscillator in 2d. The results again confirm the analysis where the spectrum of the relativistic harmonic oscillator was constructed by directly solving the modified Klein-Gordon equation [5]. In addition to this, we present the mapping of the integrals of motion under the KS duality. Finally, in Section 6, we give some conclusions and discuss possible directions for future work. The results presented here have been published in JHEP [8].

2. Modified Klein-Gordon equation from $\mathcal{N} = 4$ SYM

The Klein-Gordon equation describing a charged spinless particle in a Coulomb field is obtained by the minimal coupling prescription $\partial_\mu \to D_\mu = \partial_\mu + \frac{i}{\hbar c} A_\mu$, where a particular reference frame is chosen in order to have $A^\mu = (-\alpha/r, \mathbf{0})$, $\alpha > 0$:

$$-\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{i}{\hbar c}\frac{\alpha}{r}\right)^2 \phi + \nabla^2 \phi - \left(\frac{mc}{\hbar}\right)^2 \phi = 0. \tag{1}$$

Expanding, we get

$$-\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi + \frac{2i\alpha}{\hbar c^2 r}\frac{\partial \phi}{\partial t} + \frac{\alpha^2}{\hbar^2 c^2 r^2}\phi - \left(\frac{mc}{\hbar}\right)^2 \phi = 0. \tag{2}$$

Separation of time leads to the stationary Klein-Gordon equation with Coulomb potential that has the well known spectrum [7, 9]

$$E_{n\ell} = m \left[1 + \left(\frac{\gamma}{n - (\ell + 1/2) + \sqrt{(\ell + 1/2)^2 - \gamma^2}}\right)^2\right]^{-1/2}, \quad n = 0, 1, 2, \ldots, \tag{3}$$

where $\gamma = \alpha/\hbar c$. We can see the breaking of the n^2 degeneracy of the hydrogen atom due to the appearance of the orbital quantum number ℓ in the spectrum.

The corresponding classical stationary problem with conserved angular momentum L and energy $E = cp^0$ is given by

$$-\frac{(E + \alpha/r)^2}{c^2} + (p_r^2 + L^2/r^2) + m^2 c^2 = 0. \tag{4}$$

The solutions of this equation for bounded orbits are rosettes [10], in contrast with the non-relativistic problem where the orbits are ellipses [11]. As a consequence, the RL vector is not conserved.

On the other hand, the modified Klein-Gordon equation includes a non-minimal coupling through the mass,

$$-\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{i}{\hbar c}\frac{\alpha}{r}\right)^2 \phi + \nabla^2 \phi - \left(\frac{mc}{\hbar} - \frac{\alpha}{\hbar c r}\right)^2 \phi = 0. \tag{5}$$

Expanding this equation, it can be seen that the modification of the mass results in the cancellation of the quadratic potential term,

$$-\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi + \frac{2i\alpha}{\hbar c^2 r}\frac{\partial \phi}{\partial t} + \frac{2\alpha m}{\hbar^2 r}\phi - \left(\frac{mc}{\hbar}\right)^2 \phi = 0. \tag{6}$$

This equation can be readily cast in relativistic form,

$$\partial_\mu \partial^\mu \phi - 2iA_\mu \partial^\mu \phi - A_\mu A^\mu \phi - \left(m - \frac{\alpha}{r}\right)^2 \phi = 0, \tag{7}$$

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where $\hbar = c = 1$ and the metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

To undertake the spontaneous breaking of symmetry that will allow us to extract the above equation, we consider only the bosonic sector of $\mathcal{N} = 4$ SYM which has the following Lagrangian density [12]

$$\mathcal{L} = \text{Tr}\left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\sum_{i=1}^{6} D_\mu\Phi_i D^\mu\Phi_i + \frac{g^2}{4}\sum_{i,j=1}^{6}[\Phi_i, \Phi_j]^2 \right\}. \tag{8}$$

Here, the six scalar fields are $N \times N$ traceless Hermitian matrices in the adjoint representation of SU(N). The action of the covariant derivative on a generic field W is given by

$$D_\mu W = \partial_\mu W - ig[A_\mu, W], \tag{9}$$

where under a gauge transformation U, the gauge field A_μ and the scalar fields Φ_i transform as

$$\Phi_i \to U\Phi_i U^\dagger, \qquad A_\mu \to UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger, \tag{10}$$

and as usual, the field strength $F_{\mu\nu}$ is defined by the commutator of the covariant derivatives

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{11}$$

The resulting equations of motion are [13]

$$D_\mu F^{\nu\mu} = ig\sum_{i=1}^{6}[\Phi_i, D^\nu\Phi_i], \tag{12}$$

$$D_\mu D^\mu \Phi_i = g^2\sum_{j=1}^{6}[\Phi_j, [\Phi_j, \Phi_i]]. \tag{13}$$

We choose to work with the group SU(2) for simplicity. Therefore, the fields will be expressible in terms of the Pauli matrices τ^a as

$$\Phi_i = \Phi_i^a\frac{\tau^a}{2} = \frac{1}{2}\begin{pmatrix} \Phi_i^0 & \Phi_i^- \\ \Phi_i^+ & -\Phi_i^0 \end{pmatrix}, \qquad A_\mu = A_\mu^a\frac{\tau^a}{2} = \frac{1}{2}\begin{pmatrix} A_\mu^3 & A_\mu^1 - iA_\mu^2 \\ A_\mu^1 + iA_\mu^2 & -A_\mu^3 \end{pmatrix}, \tag{14}$$

where $\Phi_i^\pm = \Phi_i^1 \pm i\Phi_i^2$.

We now introduce the Higgs mechanism by giving a vacuum expectation value v to Φ_1 [7, 14, 15]

$$\Phi_1 = \frac{1}{2}\begin{pmatrix} \Phi_i^0 + v & 0 \\ 0 & -\Phi_i^0 - v \end{pmatrix}, \tag{15}$$

and taking the other fields as

$$\Phi_2 = \frac{1}{2}\begin{pmatrix} 0 & \Phi_2^- \\ \Phi_2^+ & 0 \end{pmatrix}, \quad \Phi_i = 0,\ i = 3, 4, 5, 6; \quad A_\mu = A_\mu^a\frac{\tau^a}{2} = \frac{1}{2}\begin{pmatrix} A_\mu^3 & 0 \\ 0 & -A_\mu^3 \end{pmatrix}. \tag{16}$$

Now, the Lagrangian (8) reads

$$\begin{aligned}\mathcal{L} = &-\frac{1}{8}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}\partial_\mu\Phi_2^-\partial^\mu\Phi_2^+ - \frac{1}{4}\partial_\mu\Phi_1^0\partial^\mu\Phi_1^0 + \frac{ig}{4}A^\mu(\Phi_2^-\partial_\mu\Phi_2^+ - \Phi_2^+\partial_\mu\Phi_2^-) \\ &- \frac{g^2}{4}\Phi_2^-\Phi_2^+A_\mu A^\mu - \frac{g^2}{4}(\Phi_1^0 + v)^2\Phi_2^-\Phi_2^+.\end{aligned} \tag{17}$$

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Here, we have defined $A_\mu = A^3_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. A crucial step is to implement the constraint

$$\Phi^0_1 + \frac{\alpha}{r} = 0 \tag{18}$$

in the previous Lagrangian. After the strong implementation of this second class constraint, we obtain

$$\begin{aligned}\mathcal{L} = &-\frac{1}{8}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}\partial_\mu\Phi^-_2\partial^\mu\Phi^+_2 + \frac{ig}{4}A^\mu(\Phi^-_2\partial_\mu\Phi^+_2 - \Phi^+_2\partial_\mu\Phi^-_2)\\ &-\frac{g^2}{4}\Phi^-_2\Phi^+_2A_\mu A^\mu - \frac{g^2}{4}\left(v - \frac{\alpha}{r}\right)^2\Phi^-_2\Phi^+_2\end{aligned} \tag{19}$$

up to a boundary term. Thus, the field content of our theory has been reduced to one vector field A_μ and one complex scalar Φ^-_2.

The equation of motion for the scalar field Φ^-_2 is

$$\partial_\mu\partial^\mu\Phi^-_2 - ig(\partial_\mu A^\mu)\Phi^-_2 - 2igA_\mu\partial^\mu\Phi^-_2 - g^2A_\mu A^\mu\Phi^-_2 - g^2(v+\Phi^0_1)^2\Phi^-_2 = 0. \tag{20}$$

Of course, the equation for Φ^+_2 will be the complex conjugate of this one. Denoting $\Phi^-_2 = \phi$ and taking the Coulomb potential $A^\mu = (-\alpha/r, \mathbf{0})$ with $g = 1$ (which is equivalent to absorbing the coupling constant into α), this equations becomes

$$\partial_\mu\partial^\mu\phi - 2iA_\mu\partial^\mu\phi - A_\mu A^\mu\phi - \left(m - \frac{\alpha}{r}\right)^2\phi = 0, \tag{21}$$

which is clearly (7). We see that the vacuum expectation value of Φ_1 is the mass m of the scalar field ϕ, and that the imposed constraint provides the non-minimal coupling $m \to m - \alpha/r$ necessary to enhance the symmetry of the field theory from SO(3) to SO(4).

The Lagrangian (19) can be rewritten as

$$\mathcal{L} = \frac{1}{2}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - \frac{1}{2}\left(m - \frac{\alpha}{r}\right)^2\phi^*\phi\right], \tag{22}$$

where the covariant derivative is given by $D_\mu = \partial_\mu - iA_\mu$. We can recognize this as one half the Lagrangian for scalar electrodynamics with modified mass.

Now, we obtain the equation of motion for the gauge field (12),

$$-\partial_\mu\partial^\mu A^\nu + \partial^\nu(\partial\cdot A) = \frac{i}{2}(\phi\,\partial^\nu\phi^* - \phi^*\partial^\nu\phi) - A^\nu|\phi|^2. \tag{23}$$

The application of the divergence to (23) reveals that the conserved current is

$$J^\nu = \frac{i}{2}(\phi\,\partial^\nu\phi^* - \phi^*\partial^\nu\phi) - A^\nu|\phi|^2, \tag{24}$$

which means that the density ρ is

$$\rho = \frac{i}{2}(\phi^*\partial_t\phi - \phi\,\partial_t\phi^*) + \frac{\alpha}{r}|\phi|^2. \tag{25}$$

On the other hand, if we set $A^\mu = (-\alpha/r, \mathbf{0})$ in the equation of motion (23), we find

$$\nabla^2\left(\frac{\alpha}{r}\right) = \frac{i}{2}(\phi^*\partial_t\phi - \phi\,\partial_t\phi^*) + \frac{\alpha}{r}|\phi|^2, \tag{26}$$

which implies

$$\rho = \nabla^2\left(\frac{\alpha}{r}\right), \tag{27}$$

just as expected, or, upon integration,

$$\int \mathrm{d}^3r\,\rho = -4\pi\alpha. \tag{28}$$

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3. Modified relativistic particle

A different approach to the modified Klein-Gordon equation is to analyze the action principle that gives rise to it. From (5), it is clear that the mass should be affected by the addition of the potential, so that the action reads

$$S = \int \mathrm{d}\tau \, L = \int \mathrm{d}\tau \, \left[-mc\sqrt{-\left(1 - \frac{\alpha}{mc^2 r}\right)^2 \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + \frac{1}{c} A_\mu \dot{x}^\mu \right]. \tag{29}$$

This corresponds to a relativistic particle interacting with a Coulomb potential in a curved spacetime described by the conformally flat metric

$$g_{\mu\nu} = \left(1 - \frac{\alpha}{mc^2 r}\right)^2 \eta_{\mu\nu}. \tag{30}$$

From here we get

$$\left(p_\mu - \frac{1}{c}A_\mu\right)\left(p^\mu - \frac{1}{c}A^\mu\right) + m^2c^2\left(1 - \frac{\alpha}{mc^2 r}\right)^2 = 0. \tag{31}$$

Taking $A^\mu = (-\alpha/r, \mathbf{0})$, this constraint is reduced to

$$(p^0)^2 + \frac{2\alpha}{cr}p^0 + \frac{2\alpha m}{r} - \boldsymbol{p}^2 - m^2c^2 = 0, \tag{32}$$

which reproduces the modified Klein-Gordon equation (6) after the application of the quantization prescription $p_\mu = -i\hbar\partial_\mu$. Separating the time as $\phi(\boldsymbol{r}, t) = \exp(-iEt/\hbar)\varphi(\boldsymbol{r})$ in the resultant equation (6) we find

$$\frac{E^2}{\hbar^2 c^2}\varphi + \nabla^2\varphi + \frac{2\alpha E}{\hbar^2 c^2 r}\varphi + \frac{2\alpha m}{\hbar^2 r}\varphi - \left(\frac{mc}{\hbar}\right)^2\varphi = 0. \tag{33}$$

This can be cast in the form of a Schrödinger-like equation

$$-\frac{\hbar^2}{(E/c^2 + m)}\nabla^2\varphi - \frac{2\alpha}{r}\varphi = (E - mc^2)\varphi \tag{34}$$

with the spectrum [7]

$$E_n = m\left(1 - \frac{2\gamma^2}{n^2 + \gamma^2}\right), \quad n = 1, 2, 3, \ldots. \tag{35}$$

It is noteworthy that we have recovered the degeneracy since the orbital quantum number ℓ does not appear in this formula. This points to the existence of an additional integral of motion, i.e., the relativistic generalization of the RL vector which will enable us to recover the SO(4) symmetry. The constraint (32) leads to the classical problem (cf. eq. 4)

$$\frac{p_r^2 + L^2/r^2}{E/c^2 + m} - \frac{2\alpha}{r} = E - mc^2. \tag{36}$$

The quantum and classical equations suggest that we should make the identifications

$$E/c^2 + m \leftrightarrow 2m_s, \quad E - mc^2 \leftrightarrow E_s, \quad 2\alpha \leftrightarrow \alpha_s \tag{37}$$

to recover a Schrödinger equation with mass m_s, energy E_s, and coupling constant α_s. Notice that under the previous substitutions, the spectrum (35) can be easily obtained from that of the hydrogen atom.

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4. The relativistic SO(4) algebra

In this section, we shall take units such that $\hbar = c = 1$. Using as a model the non-relativistic construction [11], we easily find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{p}\times\boldsymbol{L}-(E+m)\,\frac{\alpha\boldsymbol{r}}{r}\right]=0, \tag{38}$$

where $\boldsymbol{L}$ is the angular momentum that generates the SO(3) algebra. This indicates that the relativistic generalization of the RL vector is

$$\boldsymbol{A}=\boldsymbol{p}\times\boldsymbol{L}-(E+m)\,\frac{\alpha\boldsymbol{r}}{r}. \tag{39}$$

Therefore, the non-minimal coupling $m \to m - \alpha/r$, or equivalently, the transformation to a conformally flat space with metric (30), allows us to introduce a RL vector and restore the SO(4) symmetry in the relativistic case.

The complete spectrum can be constructed from the relativistic SO(4) algebra, generalizing the non relativistic result as presented in [16]. We first introduce a redefinition of the RL vector (39):

$$\boldsymbol{A}'=\frac{2}{E+m}\boldsymbol{A}=\frac{1}{E+m}(\boldsymbol{p}\times\boldsymbol{L}-\boldsymbol{L}\times\boldsymbol{p})-2\alpha\frac{\boldsymbol{r}}{r}.$$

This vector $\boldsymbol{A}'$ satisfies

$$[\boldsymbol{A}',H]=0,\qquad \boldsymbol{L}\cdot\boldsymbol{A}'=\boldsymbol{A}'\cdot\boldsymbol{L},$$

and

$$\boldsymbol{A}'^2=4\left[\alpha^2+\frac{E-m}{E+m}(1+L^2)\right]. \tag{40}$$

We can see that the corresponding relativistic algebra closes as

$$[L_i,L_j]=i\varepsilon_{ijk}L_k,$$
$$[A'_i,J_j]=i\varepsilon_{ijk}A'_k,$$
$$[A'_i,A'_j]=-4i\left(\frac{E-m}{E+m}\right)\varepsilon_{ijk}L_k.$$

Defining

$$\boldsymbol{D}=\sqrt{-\frac{E+m}{4(E-m)}}\boldsymbol{A}'$$

and

$$\boldsymbol{M}=\frac{1}{2}(\boldsymbol{L}-\boldsymbol{D}),\quad \boldsymbol{N}=\frac{1}{2}(\boldsymbol{L}+\boldsymbol{D}),$$

it is easy to show the the original algebra splits into the product of two SO(3) algebras

$$[M_i,M_j]=i\varepsilon_{ijk}M_k,\quad [N_i,N_j]=i\varepsilon_{ijk}N_k$$

with the constraint

$$\boldsymbol{M}^2=\boldsymbol{N}^2. \tag{41}$$

The operator $\boldsymbol{M}^2+\boldsymbol{N}^2$ will have the eigenvalues $2\ell(\ell+1)$ with $\ell = 0, 1, 2, ...$ because of the constraint (41). On the other hand, we can find that

$$\boldsymbol{M}^2+\boldsymbol{N}^2=\frac{1}{2}\left[L^2-\frac{E+m}{4(E-m)}\boldsymbol{A}'^2\right]=-\frac{1}{2}\left(\frac{E+m}{E-m}\alpha^2+1\right),$$

where we used (40). With this at hand, we obtain the energy eigenvalues

$$E_n = m\left(1 - \frac{2\alpha^2}{n^2+\alpha^2}\right). \tag{42}$$

This is the same spectrum that we got through the identifications (37) and that is reported in [7], reproduced here with $n = 2\ell+1$. Due to the hidden symmetry lying under this non-minimal coupling which reveals the existence of the relativistic RL vector, $\mathcal{N}=4$ SYM is dubbed as the "hydrogen atom quantum field theory" [17].

5. Relativistic Kustaanheimo-Stiefel duality

Now, we relate the wave functions and the spectra of the modified Coulomb problem and the modified relativistic harmonic oscillator. We will consider the illustrative case of two dimensions (for the treatment of arbitrary dimensions see [8]). At the end, we see how the integrals of motion of both problems are related by the KS duality.

The KS transformation relates the variables of the Coulomb problem (x, y, t) with the harmonic oscillator (u, v, s) as [18]

$$x = u^2 - v^2, \quad y = 2uv, \quad \frac{\mathrm{d}t}{\mathrm{d}s} = u^2 + v^2 = r. \tag{43}$$

We denote by $\bar{\phi}(u, v, s)$ the field that results from evaluating the original field in terms of the new variables,

$$\bar{\phi}(u, v, s) = \phi(x(u, v), y(u, v), t(s)). \tag{44}$$

Then, the transformation of the differential operators is

$$\frac{\partial\phi}{\partial t} = \frac{\partial\bar{\phi}}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{u^2+v^2}\frac{\partial\bar{\phi}}{\partial s}, \tag{45}$$

$$\frac{\partial^2\phi}{\partial t^2} = \frac{1}{(u^2+v^2)^2}\frac{\partial^2\bar{\phi}}{\partial s^2}, \tag{46}$$

$$\nabla^2\phi = \frac{1}{4(u^2+v^2)}\nabla_u^2\bar{\phi}, \tag{47}$$

where

$$\nabla_u^2\bar{\phi} = \frac{\partial^2\bar{\phi}}{\partial u^2} + \frac{\partial^2\bar{\phi}}{\partial v^2}. \tag{48}$$

Thus, the modified Klein-Gordon equation (6) becomes

$$-\frac{1}{c^2(u^2+v^2)^2}\frac{\partial^2\bar{\phi}}{\partial s^2} + \frac{1}{4(u^2+v^2)}\nabla_u^2\bar{\phi} + \frac{2i\alpha}{\hbar c^2(u^2+v^2)}\frac{\partial\bar{\phi}}{\partial s} + \frac{2\alpha m}{\hbar^2(u^2+v^2)}\bar{\phi} - \left(\frac{mc}{\hbar}\right)^2\bar{\phi} = 0. \tag{49}$$

The separation of time in terms of new variables is

$$\bar{\phi}(u, v, s) = \exp\left[-\frac{i}{\hbar}E\int^s \mathrm{d}s'\,(u^2+v^2)\right]\bar{\varphi}(u, v). \tag{50}$$

Substituting it in (49) we get

$$-\frac{\hbar^2}{4\,(E/c^2+m)}\nabla_u^2\bar{\varphi} - (E - mc^2)(u^2+v^2)\bar{\varphi} = 2\alpha\bar{\varphi}. \tag{51}$$

This is the stationary Schrödinger equation for a two-dimensional harmonic oscillator with mass $2(E/c^2+m)$, energy 2α, and frequency

$$\omega = c\sqrt{-\frac{E-mc^2}{E+mc^2}}. \tag{52}$$

Now, the time-dependent Schrödinger equation for the harmonic oscillator is

$$-\frac{\hbar^2}{4\mathcal{M}}\nabla_u^2\bar{\xi} + \frac{\mathcal{M}\omega^2}{2}(u^2+v^2)\bar{\xi} = i\hbar\frac{\partial\bar{\xi}}{\partial s}, \tag{53}$$

where

$$\bar{\xi}(u,v,s) = \exp\left(-\frac{i}{\hbar}\mathcal{E}s\right)\bar{\varphi}(u,v) = \exp\left(-\frac{i}{\hbar}2\alpha s\right)\bar{\varphi}(u,v). \tag{54}$$

Using (50) to substitute $\bar{\varphi}$ in terms of $\bar{\phi}$ we get

$$\begin{aligned}\bar{\xi}(u,v,s) =& \exp\left[-\frac{i}{\hbar}\left(2\alpha s - E\int^s \mathrm{d}s'\,(u^2+v^2)\right)\right]\bar{\phi}(u,v,s)\\ =& \exp\left[-\frac{i}{\hbar}F(s)\right]\bar{\phi}(u,v,s).\end{aligned} \tag{55}$$

Here, $F(s)$ is the generating function of the canonical (and non-holonomic) KS transformation on the extended phase space [19]. Hence, we have related the wave functions $\bar{\phi}$ of the Coulomb problem (written in terms of new variables) with the wave functions $\bar{\xi}$ of the harmonic oscillator. It is also seen that the wave function φ solves both stationary equations.

Now, we establish the relation between the spectra of the modified relativistic harmonic oscillator and the modified Coulomb problem. In the Schrödinger case, the relation between the harmonic oscillator mass, coupling constant, energy, and angular momentum $(\mathcal{M}_s, k_s, \mathcal{E}_s, \mathcal{L}_s)$ and the same variables of the Coulomb problem $(m_s, \alpha_s, E_s, L_s)$ is

$$\mathcal{M}_s = 4m_s, \quad k_s = -2E_s, \quad \mathcal{E}_s = \alpha_s, \quad \mathcal{L}_s = 2L_s. \tag{56}$$

We know that the identification (37) allows us to map the relativistic problem onto the non relativistic one, so this suggests that we should then apply the KS duality (56) and finally make again the identification (37) in the opposite sense to recover the spectrum of the relativistic hydrogen-like problem. Explicitly, we begin with the spectrum of the modified relativistic harmonic oscillator in two dimensions [5]

$$\frac{(\mathcal{E}/c^2+\mathcal{M})(\mathcal{E}-\mathcal{M}c^2)^2}{\hbar^2 k} = (4q+2|\mathcal{L}|+2)^2, \quad q = 0,1,2,\ldots. \tag{57}$$

Applying (37) we get the Schrödinger spectrum of the two-dimensional harmonic oscillator

$$\mathcal{E}_s = \hbar\sqrt{\frac{k_s}{\mathcal{M}_s}}(2q+|\mathcal{L}|+1). \tag{58}$$

The KS duality gives

$$E_s = -\frac{m_s\alpha_s^2}{2\hbar^2(q+|L|+1/2)^2}, \tag{59}$$

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which is the spectrum of the non relativistic two-dimensional hydrogen atom [20]. Then, again from (37) we find

$$E = mc^2 \left[1 - \frac{2\gamma^2}{(q + |L| + 1/2)^2 + \gamma^2} \right], \tag{60}$$

where $\gamma = \alpha/\hbar c$. By making $n = q + |L|$, we readily recognize this as the spectrum of the modified relativistic Coulomb problem in two dimensions (35). With these results at hand, we are able to build the relativistic KS transformation duality (cf. eq. 56):

$$\begin{aligned} E/c^2 + m =& \frac{1}{4}(\mathcal{E}/c^2 + \mathcal{M}), \\ E - mc^2 =& -k, \\ \alpha =& \frac{1}{2}(\mathcal{E} - \mathcal{M}c^2), \end{aligned} \tag{61}$$

which performs the mapping

$$\frac{\boldsymbol{\mathcal{P}}^2}{\mathcal{E}/c^2 + \mathcal{M}} + k(u^2 + v^2) = \mathcal{E} - \mathcal{M}c^2 \implies \frac{\boldsymbol{p}^2}{E/c^2 + m} - \frac{2\alpha}{r} = E - mc^2. \tag{62}$$

We finally analyze how the integrals of motion of both problems are related. We start with the non-relativistic case and then, through the replacements (37), we move to the modified relativistic problem. Since the energy of a system becomes the coupling constant of the other, we only analyze the remaining integrals of motion. For the two-dimensional Coulomb problem, they are the Runge-Lenz vector $\boldsymbol{A}_s$ and the angular momentum L. The Runge-Lenz vector is

$$\begin{aligned} A_{s\,x} =& x p_y^2 - y p_x p_y - \frac{m_s \alpha_s x}{\sqrt{x^2 + y^2}}, \\ A_{s\,y} =& y p_x^2 - x p_x p_y - \frac{m_s \alpha_s y}{\sqrt{x^2 + y^2}}, \end{aligned} \tag{63}$$

and the angular momentum reads $L_s = x p_y - y p_x$. Remember that the subindex s refers to the Schrödinger (non-relativistic) case.

On the other hand, for the bidimensional harmonic oscillator we can find a conserved tensor $C_{s\,ij}$ whose components are [11]

$$\begin{aligned} C_{s\,11} =& \frac{\mathcal{P}_u^2}{2\mathcal{M}_s} + \frac{k}{2}u^2, \\ C_{s\,12} =& \frac{\mathcal{P}_u \mathcal{P}_v}{2\mathcal{M}_s} + \frac{k}{2}uv, \\ C_{s\,22} =& \frac{\mathcal{P}_v^2}{2\mathcal{M}_s} + \frac{k}{2}v^2, \end{aligned} \tag{64}$$

and the angular momentum $\mathcal{L}_s = u p_v - v p_u$.

A straightforward calculation employing the non-relativistic KS duality (56) shows that the mapping is

$$\begin{aligned} C_{s\,11} =& \frac{\mathcal{P}_u^2}{2\mathcal{M}_s} + \frac{k}{2}u^2 = -\frac{A_{s\,x}}{2m_s} + \frac{\alpha_s}{2}, \\ C_{s\,12} =& \frac{\mathcal{P}_u \mathcal{P}_v}{2\mathcal{M}_s} + \frac{k}{2}uv = -\frac{A_{s\,y}}{2m_s}, \\ C_{s\,22} =& \frac{\mathcal{P}_v^2}{2\mathcal{M}_s} + \frac{k}{2}v^2 = \frac{A_{s\,x}}{2m_s} + \frac{\alpha_s}{2}, \\ \mathcal{L}_s =& 2L_s. \end{aligned} \tag{65}$$

Finally, we apply (37) to find the relativistic mapping

$$\begin{aligned} \frac{\mathcal{P}_u^2}{\mathcal{E}/c^2+\mathcal{M}} + \frac{k}{2}u^2 =& -\frac{A_x}{E/c^2+m} + \alpha, \\ \frac{\mathcal{P}_u\mathcal{P}_u}{\mathcal{E}/c^2+\mathcal{M}} + \frac{k}{2}uv =& -\frac{A_y}{E/c^2+m}, \\ \frac{\mathcal{P}_v^2}{\mathcal{E}/c^2+\mathcal{M}} + \frac{k}{2}v^2 =& \frac{A_x}{E/c^2+m} + \alpha, \\ \mathcal{L} =& 2L. \end{aligned} \tag{66}$$

6. Conclusions

We have studied the features of a hydrogen-like relativistic theory emerging from $\mathcal{N}=4$ SYM through of a particular Higgs mechanism, and we have found that it is possible to restore the hidden SO(4) symmetry through the introduction of a relativistic RL vector. We were able to reconstruct the spectrum of the system by means of a natural identification of relativistic and non-relativistic quantities. Furthermore, the KS transformation allowed us to relate the spectra of the modified relativistic harmonic oscillator and the modified relativistic Coulomb problem, thus enlarging this KS duality to the relativistic realm. It is left for future work the implementation of a Higgs mechanism that takes into account the charged components of the vector field A_μ as well as an analysis of the fermionic sector of $\mathcal{N}=4$ SYM.

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