Geometric Quantization and Graded Lie Algebras

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A SURVEY OF THE APPLICATIONS OF GEOMETRIC QUANTI ATION.

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One of the principal themes of this conference is geometric quantisation. This is a convenient term for a concept which, to a greater or lesser extent, has been found to be relevant to such topics as

(i) the passage from a classical dynamical system with Poisson bracket (symplectic manifold) to a quantum system (Hilbert space), see Segal (12), Kostant (5), Souriau (15), and Sniatycki (this conference),

(ii) the construction of irreducible representations of Lie groups, see Kirillov (4), Borel-Weil (13), Kostant-Auslander (6), Blattner (1),

(iii) the construction of primitive ideals in the enveloping algebra of a Lie algebra see Dixmier (2) and Joseph (this conference),

(iv) the construction of physical systems which are elementary with respect to a prescribed symmetry group, see Souriau (15), Renouard (11), Rawnsley (10),

(v) the study of dynamical groups, see Onofri (8), Sternberg-Wolf (in progress),

(vi) coherent states for a Lie group, Onofri, Bacry (this conference),

(vii) twistor theory of general relativity, see Penrose (9), and Woodhouse (this conference),

(viii) partial differential equations, see Maslov (7), Hormander-Duistermaat (3).

The basic ingredient in geometric quantisation is the notion of a <u>polarised symplectic manifold</u> (M, w, F). Here M denotes a real differentiable manifold, w a non-degenerate closed differential 2-form (<u>symplectic form</u>) on M, and F an involutive sub-bundle of the complexified tangent bundle of M which is maximally isotropic with respect to w (<u>polarisation of w</u>). The quantisation process may then be described as the construction of a complex hermitian line bundle L over M with connection form α having w as curvature form, and the construction of a Hilbert space from sections of the line bundle whose covariant derivatives in the directions of F vanish. Such a line bundle can be constructed if and only if w has integral periods (quantisation condition) and is unique if, for example, M is simply connected. In this process certain functions on M have a natural representation as operators on the Hilbert space, in such a way that the Poisson bracket of functions corresponds to the commutator of operators. In this scheme a function ϕ is represented formally by the operator

$$i\hbar \nabla_{\xi_\phi} + \phi$$

where ∇_{ξ_ϕ} denotes covariant differentiation along the Hamiltonian vector field ξ_ϕ generated by ϕ.

We now describe how these ideas arise naturally in the topics listed above.

(i) Let M be the classical phase space of a dynamical system based on a configuration space X. Let w be the 2-form which equals Planck's constant times $\Sigma dp_j \wedge dq^j$ in local canonical coordinates $p_1, \ldots, p_n$, $q^1, \ldots q^n$. Let F be the polarisation spanned by the vector fields $\frac{\partial}{\partial p_j}$. The line bundle L in this case is a product $M \times \mathbb{C}$ and the connection form α equals $\Sigma p_j dq^j + \frac{1}{i}\frac{dz}{z}$ where z is the coordinate on $\mathbb{C}$. The Hilbert space is $L^2(X)$ considered as the completion of a space of functions on M. For $X = \mathbb{R}^n$ this yields the usual Schrödinger representation $p_j \to i\hbar\frac{\partial}{\partial q^j}$ and $q_j \to$ multiplication by q_j. For X an arbitrary Riemannian manifold, the Hamiltonian of a free particle is quantised as the Laplace-Beltrami operator.

(ii) Let G be a connected Lie group and let $\underline{G}^*$ be the algebraic dual of its Lie algebra $\underline{G}$. Let M be any orbit of the coadjoint action of G on $\underline{G}^*$.

Each element X of $\underline{G}$ is a linear function on $\underline{G}^*$ which restricts to a function ϕ^X on M. There is a unique symplectic structure w on M such that the Poisson bracket of ϕ^X and ϕ^Y equals $\phi^{[X,Y]}$ for all X,Y in $\underline{G}$. Let F be a polarisation of w which is invariant under G. Then G acts on the Hilbert space of the quantisation process by unitary operators. For compact groups, and for solvable groups of type I, all irreducible unitary representations can be obtained in this way. For semi-simple Lie groups the irreducible representations occurring in the Plancherelmeasure can be obtained in this way.

(iii) Let $\underline{G}$ be a Lie algebra and $\underline{G}^*$ be its algebraic dual. Each $f \varepsilon \underline{G}^*$ defines a bilinear form B_f on $\underline{G}$ by $B_f(X,Y) = f([X,Y])$. A subalgebra F of $\underline{G}$ is called a polarisation at f if F is maximal among vector subspaces of $\underline{G}$ on which B_f vanishes. The restriction of f to F is a 1-dimensional representation of F and we can induce to a representation of $\underline{G}$. In this way we can obtain, when $\underline{G}$ is solvable and defined over an algebraically closed field, all ideals in the enveloping algebra which are kernels of irreducible representations (primitive ideals). If we denote by M the orbit of f under the coadjoint representation of G, then B_f induces a non-degenerate skew-symmetric form on the tangent space to M at f, and this gives the symplectic structure w on M. The subalgebra F projects to a polarisation F of w. Thus we have a polarised symplectic manifold (M, w, F). When w satisfies the quantisation condition, the resulting unitary representation of G corresponds to the induced representation of $\underline{G}$.

(iv) Let G be the connected component of the Poincaré group and $\underline{G}$ its Lie algebra. The usual generators P^α, $M^{\alpha\beta}$ can be considered as linear functions on the dual $\underline{G}^*$. For fixed real s, the submanifold M_s of $\underline{G}^*$ given by

$$\tfrac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} P^\beta M^{\gamma\delta} = s P_\alpha , \quad P_o > 0$$

is a 6-dimensional orbit of G in $\underset{\sim}{G}^*$ and carries a natural G-invariant symplectic form w_s. The space-time translations in G generate a polarisation F_s of w_s and (M_s, w_s, F_s) is then a polarised symplectic manifold. This has been proposed by Souriau as the phase space of a classical free elementary relativistic particle of rest mass zero and spin s. Quantisation yields the usual unitary representation of G associated with such a particle in quantum mechanics.

(v) The classical phase space M of the Kepler problem is a 6-dimensional symplectic manifold on which SO(4,2) acts leaving the symplectic form w invariant. Geometric quantisation has been applied in this context using a number of different polarisations.

(vi) Let G be a compact semi-simple Lie group acting irreducibly on a finite dimensional complex vector space H. If ψ is a ray in H with isotropy group a Cartan subgroup, then the G-orbit M of ψ carries a G-invariant symplectic form w and a complex structure F. Geometric quantisation of the polarised symplectic manifold (M, w, F) gives the coherent states associated with G.

(vii) The twistor space M associated with a curved space-time by Penrose is an 8-dimensional symplectic manifold and is also a 4-dimensional complex manifold. If w is the symplectic form and if F is generated by the anti-holomorphic directions, then (M, w, F) is a polarised symplectic manifold. For flat space-time, M is complex vector space $\mathbb{C}^4$ with coordinates (Z^o, Z^1, Z^2, Z^3) and symplectic form

$$w = i(dZ^o \wedge d\bar{Z}^2 + dZ^1 \wedge d\bar{Z}^3 + dZ^2 \wedge d\bar{Z}^o + dZ^3 \wedge d\bar{Z}^1).$$

The 4-fold cover SU(2,2) of the conformal group acts on M leaving w and

F invariant. For fixed real s the set of U(1) orbits in the hypersurface

$$Z^o \bar{Z}^2 + Z^1 \bar{Z}^3 + Z^2 \bar{Z}^o + Z^3 \bar{Z}^1 = s$$

has the induced structure of a 6-dimensional polarised symplectic manifold (M_s, w_s, F_s). Geometric quantisation gives the holomorphic twistor functions used by Penrose to represent a mass-zero spin s particle.

(viii) In the applications to partial differential equations on a manifold X, the basic symplectic manifold is the cotangent bundle $M = T^*(X)$. For example, Rockland (this conference) considers a pseudo differential operator P such that the subset Σ of M where the principal symbol vanishes (the characteristic variety) has an induced symplectic structure. He also uses polarisations of each fibre of the conormal bundle of Σ

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