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Deformations and Extensions of Modified λ-Differential 3-Lie Algebras

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Abstract: In this paper, we propose the representation and cohomology of modified λ-differential 3-Lie algebras. As their applications, the linear deformations, abelian extensions and T^*-extensions of modified λ-differential 3-Lie algebras are also studied.

Keywords: 3-Lie algebras; modified λ-differential operator; representation; cohomology; linear deformation; abelian extension; T^*-extension

MSC: 17A40; 17B10; 17B40; 17B56

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1. Introduction

3-Lie algebra plays an important role in string theory, and it is also used to study supersymmetry and gauge symmetry transformation of the world-volume theory of multiple coincident M2-branes [1,2]. The concept of 3-Lie algebra, general n-Lie algebra, was first introduced by Filippov [3] and can be regarded as a generalization of Lie algebra to higher-order algebra. 3-Lie algebra has attracted the attention of scholars from mathematics and physics [4–6]. Representation theory, cohomology theory, deformations, Nijenhuis operators and extension theory of n-Lie algebras have been widely studied by scholars [7–17].

Derivations play important roles in the study of homotopy Lie algebras [18], differential Galois theory [19], control theory and gauge theories of quantum field theory [20]. Recently, authors have studied algebras with derivations in [21,22] from the operadic point of view. In [23], Tang et al. investigated the deformation and extension of Lie algebras with derivations from the cohomological point of view. The results of [23] have been extended to 3-Lie algebras with derivations [24,25]. More research on algebraic structures with derivations has been developed; see [26–31] and references cited therein.

In recent years, scholars have increasingly focused on structures with arbitrary weights, thanks to the important work of [32–37]. The papers [38–40] established the cohomology, extensions and deformations of Rota–Baxter 3-Lie algebras with any weight λ, as well as the differential 3-Lie algebras with any weight λ. Additionally, the cohomology and deformation of modified Rota–Baxter algebras were studied by Das [41]. The works [42,43] provided insights into the cohomology and deformation of modified Rota–Baxter Leibniz algebras with weight λ. Furthermore, Peng et al. [44] introduced the concept of modified λ-differential Lie algebras.

Motivated by the mentioned work on the modified λ-differential operator on Lie algebras and considering the importance of 3-Lie algebra, representation and cohomology, in this paper, our main objective is to study modified λ-differential 3-Lie algebras. We develop a cohomology theory of modified λ-differential 3-Lie algebras that controls the deformation and extensions of modified λ-differential 3-Lie algebras.

The paper is organized as follows. In Section 2, we introduce the representation and cohomology of modified λ-differential 3-Lie algebras (MD$_\lambda$3-LieAs). In Section 3, we consider linear deformations of MD$_\lambda$3-LieAs. In Section 4, we study abelian extensions of MD$_\lambda$3-LieAs. In Section 5, we study T^*-extensions of MD$_\lambda$3-LieAs.

Throughout this paper, $\mathbf{k}$ denotes a field of characteristic zero. All the vector spaces and linear maps are taken over $\mathbf{k}$.

2. Representations and Cohomologies of MD$_\lambda$3-LieAs

In this section, we introduce the concept of a MD$_\lambda$3-LieA and present some examples. Then, we give representations and cohomologies of MD$_\lambda$3-LieAs.

Definition 1 ([3])**.** *(i) A 3-Lie algebra is a tuple $(\mathfrak{A}, [-,-,-])$ in which $\mathfrak{A}$ is a vector space together with a skew-symmetric ternary operation $[-,-,-] : \wedge^3\mathfrak{A} \to \mathfrak{A}$ such that*

$$[a_1, a_2, [a_3, a_4, a_5]] = [[a_1, a_2, a_3], a_4, a_5] + [a_3, [a_1, a_2, a_4], a_5] + [a_3, a_3, [a_1, a_2, a_5]], \quad (1)$$

for all $a_1, a_2, a_3, a_4, a_5 \in \mathfrak{A}$.
(ii) A homomorphism between two 3-Lie algebras $(\mathfrak{A}_1, [-,-,-]_1)$ and $(\mathfrak{A}_2, [-,-,-]_2)$ is a linear map $\eta : \mathfrak{A}_1 \to \mathfrak{A}_2$ satisfying $\eta([a_1, a_2, a_3]_1) = [\eta(a_1), \eta(a_2), \eta(a_3)]_2, \ \forall\, a_1, a_2, a_3 \in \mathfrak{A}_1$.

Definition 2. *(i) Let $\lambda \in \mathbf{k}$ and $(\mathfrak{A}, [-,-,-])$ be a 3-Lie algebra. A modified λ-differential operator on 3-Lie algebra $\mathfrak{A}$ is a linear map $\mathrm{d} : \mathfrak{A} \to \mathfrak{A}$, such that*

$$\mathrm{d}[a_1, a_2, a_3] = [\mathrm{d}(a_1), a_2, a_3] + [a_1, \mathrm{d}(a_2), a_3] + [a_1, a_2, \mathrm{d}(a_3)] + \lambda[a_1, a_2, a_3], \forall a_1, a_2, a_3 \in \mathfrak{A}. \quad (2)$$

(ii) A modified λ-differential 3-Lie algebra (MD$_\lambda$3-LieA) is a triple $(\mathfrak{A}, [-,-,-], \mathrm{d})$ consisting of a 3-Lie algebra $(\mathfrak{A}, [-,-,-])$ and a modified λ-differential operator d.
(iii) A homomorphism between two MD$_\lambda$3-LieAs $(\mathfrak{A}_1, [-,-,-]_1, \mathrm{d}_1)$ and $(\mathfrak{A}_2, [-,-,-]_2, \mathrm{d}_2)$ is a 3-Lie algebra homomorphism $\eta : (\mathfrak{A}_1, [-,-,-]_1) \to (\mathfrak{A}_2, [-,-,-]_2)$ such that $\eta \circ \mathrm{d}_1 = \mathrm{d}_2 \circ \eta$. Furthermore, if η is nondegenerate, then η is called an isomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$.

Let $(\mathfrak{A}, [-,-,-])$ be a 3-Lie algebra; then, the elements in $\wedge^2\mathfrak{A}$ are called fundamental objects of the 3-Lie algebra $(\mathfrak{A}, [-,-,-])$. There is a bilinear operation $[-,-]$ on $\wedge^2\mathfrak{A}$, which is given by

$$[A, B]_{\mathcal{F}} = [a_1, a_2, b_1] \wedge b_2 + b_1 \wedge [a_1, a_2, b_2], \forall A = a_1 \wedge a_2, B = b_1 \wedge b_2 \in \wedge^2\mathfrak{A}.$$

It is shown in [45] that $(\wedge^2\mathfrak{A}, [-,-]_{\mathcal{F}})$ is a Leibniz algebra. Furthermore, by direct calculation, we have the following result.

Proposition 1. *Let $(\mathfrak{A}, [-,-,-], \mathrm{d})$ be a MD$_\lambda$3-LieA. Then, $(\wedge^2\mathfrak{A}, [-,-]_{\mathcal{F}}, \mathrm{d}_{\mathcal{F}})$ is a Leibniz algebra with a derivation, where $\mathrm{d}_{\mathcal{F}}(a_1 \wedge a_2) = \mathrm{d}(a_1) \wedge a_2 + a_1 \wedge \mathrm{d}(a_2) + \lambda a_1 \wedge a_2$, for all $a_1 \wedge a_2 \in \wedge^2\mathfrak{A}$. See [26] for more details about Leibniz algebras with derivations*

Remark 1. *Let d be a modified λ-differential operator on a 3-Lie algebra $(\mathfrak{A}, [-,-,-])$. If $\lambda = 0$, then d is a derivation on the 3-Lie algebra $\mathfrak{A}$. See [46] for various derivations of 3-Lie algebras.*

Example 1. *Let $(\mathfrak{A}, [-,-,-])$ be a 3-Lie algebra. Then, a linear map $\mathrm{d} : \mathfrak{A} \to \mathfrak{A}$ is a modified λ-differential operator if and only if $\mathrm{d} + \frac{\lambda}{2}\mathrm{id}_{\mathfrak{A}}$ is a derivation on the 3-Lie algebra $\mathfrak{A}$.*

Example 2. *Let $(\mathfrak{A}, [-,-,-], \mathrm{d})$ be a MD$_\lambda$3-LieA. Then, for $k \in \mathbf{k}$, $(\mathfrak{A}, [-,-,-], k\mathrm{d})$ is a MD$_{(k\lambda)}$3-LieA.*

Definition 3. *(i) (see [8]) A representation of a 3-Lie algebra* $(\mathfrak{A}, [-,-,-])$ *on a vector space* $\mathfrak{M}$ *is a skew-symmetric bilinear map* $\rho : \mathfrak{A} \wedge \mathfrak{A} \to \mathrm{End}(\mathfrak{M})$*, such that*

$$\rho([a_1, a_2, a_3], a_4) = \rho(a_2, a_3)\rho(a_1, a_4) + \rho(a_3, a_1)\rho(a_2, a_4) + \rho(a_1, a_2)\rho(a_3, a_4), \tag{3}$$

$$\rho(a_1, a_2)\rho(a_3, a_4) = \rho(a_3, a_4)\rho(a_1, a_2) + \rho([a_1, a_2, a_3], a_4) + \rho(a_3, [a_1, a_2, a_4]), \tag{4}$$

for all $a_1, a_2, a_3, a_4 \in \mathfrak{A}$*. We also denote a representation of* $\mathfrak{A}$ *on* $\mathfrak{M}$ *by* $(\mathfrak{M}; \rho)$*.*
(ii) A representation of a MD_λ*3-LieA* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *is a triple* $(\mathfrak{M}; \rho, \mathrm{d}_\mathfrak{M})$*, where* $(\mathfrak{M}; \rho)$ *is a representation of the 3-Lie algebra* $(\mathfrak{A}, [-,-,-])$ *and* $\mathrm{d}_\mathfrak{M}$ *is a linear operator on* $\mathfrak{M}$*, satisfying the following equation*

$$\mathrm{d}_\mathfrak{M}(\rho(a, b)v) = \rho(\mathrm{d}(a), b)v + \rho(a, \mathrm{d}(b))v + \rho(a, b)\mathrm{d}_\mathfrak{M}(v) + \lambda\rho(a, b)v, \tag{5}$$

for any $a, b \in \mathfrak{A}$ *and* $v \in \mathfrak{M}$*.*

Remark 2. *Let* $(\mathfrak{M}; \rho, \mathrm{d}_\mathfrak{M})$ *be a representation of a* MD_λ*3-LieA* $(\mathfrak{A}, [-,-,-], \mathrm{d})$*. If* $\lambda = 0$*, then* $(\mathfrak{M}; \rho, \mathrm{d}_\mathfrak{M})$ *is a representation of the 3-Lie algebra with a derivation* $(\mathfrak{A}, [-,-,-], \mathrm{d})$*. See [24,25,29] for more details about 3-Lie algebras with derivations.*

Example 3. *Let* $(\mathfrak{M}; \rho)$ *be a representation of a 3-Lie algebra* $(\mathfrak{A}, [-,-,-])$*. Then,* $(\mathfrak{M}; \rho, \mathrm{d}_\mathfrak{M})$ *is a representation of the* MD_λ*3-LieA* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *if and only if* $(\mathfrak{M}; \rho, \mathrm{d}_\mathfrak{M} + \frac{\lambda}{2}\mathrm{id}_\mathfrak{M})$ *is a representation of the 3-Lie algebra with a derivation* $(\mathfrak{A}, [-,-,-], \mathrm{d} + \frac{\lambda}{2}\mathrm{id}_\mathfrak{A})$*.*

Example 4. *Let* $(\mathfrak{M}; \rho)$ *be a representation of a 3-Lie algebra* $(\mathfrak{A}, [-,-,-])$*. Then, for* $k \in \mathbf{k}$*,* $(\mathfrak{M}; \rho, \mathrm{id}_\mathfrak{M})$ *is a representation of the* $MD_{(-2k)}$*3-LieA* $(\mathfrak{A}, [-,-,-], k\mathrm{id}_\mathfrak{A})$*.*

Example 5. *Let* $(\mathfrak{M}; \rho, \mathrm{d}_\mathfrak{M})$ *be a representation of a* MD_λ*3-LieA* $(\mathfrak{A}, [-,-,-], \mathrm{d})$*. Then, for* $k \in \mathbf{k}$*,* $(\mathfrak{M}; \rho, k\mathrm{d}_\mathfrak{M})$ *is a representation of the* $MD_{(k\lambda)}$*3-LieA* $(\mathfrak{A}, [-,-,-], k\mathrm{d})$*.*

Proposition 2. *Let* $(\mathfrak{A}, [-,-,-])$ *be a 3-Lie algebra, and* $(\mathfrak{M}; \rho)$ *be a representation of it. Then,* $(\mathfrak{M}; \rho, \mathrm{d}_\mathfrak{M})$ *is a representation of* MD_λ*3-LieA* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *if and only if* $\mathfrak{A} \oplus \mathfrak{M}$ *is a* MD_λ*3-LieA under the following maps:*

$$\begin{aligned} [a_1 + u_1, a_2 + u_2, a_3 + u_3]_\rho :=& [a_1, a_2, a_3] + \rho(a_1, a_2)u_3 + \rho(a_3, a_1)u_2 + \rho(a_2, a_3)u_1, \\ \mathrm{d} \oplus \mathrm{d}_\mathfrak{M}(a_1 + u_1) :=& \mathrm{d}(a_1) + \mathrm{d}_\mathfrak{M}(u_1), \end{aligned}$$

for all $a_1, a_2, a_3 \in \mathfrak{A}$ *and* $u_1, u_2, u_3 \in \mathfrak{M}$*.*

Proof. Assume that $(\mathfrak{A} \oplus \mathfrak{M}, [-,-,-]_\rho, \mathrm{d} \oplus \mathrm{d}_\mathfrak{M})$ is a MD_λ3-LieA, for any $a_1, a_2 \in \mathfrak{A}$ and $u_3 \in \mathfrak{M}$; we have

$$\begin{aligned} & \mathrm{d} \oplus \mathrm{d}_\mathfrak{M}[a_1 + 0, a_2 + 0, 0 + u_3]_\rho \\ =& [\mathrm{d} \oplus \mathrm{d}_\mathfrak{M}(a_1 + 0), a_2 + 0, 0 + u_3]_\rho + [a_1 + 0, \mathrm{d} \oplus \mathrm{d}_\mathfrak{M}(a_2 + 0), 0 + u_3]_\rho \\ & + [a_1 + 0, a_2 + 0, \mathrm{d} \oplus \mathrm{d}_\mathfrak{M}(0 + u_3)]_\rho + \lambda[a_1 + 0, a_2 + 0, 0 + u_3]_\rho, \end{aligned}$$

which implies that

$$\mathrm{d}_\mathfrak{M}(\rho(a_1, a_2)u_3) = \rho(\mathrm{d}(a_1), a_2)u_3 + \rho(a_1, \mathrm{d}(a_2))u_3 + \rho(a_1, a_2)\mathrm{d}_\mathfrak{M}(u_3) + \lambda\rho(a_1, a_2)u_3.$$

Therefore, $(\mathfrak{M}; \rho, \mathrm{d}_\mathfrak{M})$ is a representation of $(\mathfrak{A}, [-,-,-], \mathrm{d})$.
The converse can be proved similarly. □

Let $(\mathfrak{M};\rho,\mathrm{d}_{\mathfrak{M}})$ be a representation of a MD$_\lambda$3-LieA $(\mathfrak{A},[-,-,-],\mathrm{d})$ and $\mathfrak{M}^* := \mathrm{Hom}(\mathfrak{M},\mathbf{k})$ be a dual space of $\mathfrak{M}$. We define a bilinear map $\rho^* : \wedge^2\mathfrak{A} \to \mathrm{End}(\mathfrak{M}^*)$ and a linear map $\mathrm{d}^*_{\mathfrak{M}} : \mathfrak{M}^* \to \mathfrak{M}^*$ respectively by

$$\langle \rho^*(a_1,a_2)u^*,v\rangle = -\langle u^*,\rho(a_1,a_2)v\rangle \text{ and } \langle \mathrm{d}^*_{\mathfrak{M}}u^*,v\rangle = \langle u^*,\mathrm{d}_{\mathfrak{M}}(v)\rangle, \tag{6}$$

for any $a_1,a_2 \in \mathfrak{A}, v \in \mathfrak{M}$ and $u^* \in \mathfrak{M}^*$.

Proposition 3. *With the above notations, $(\mathfrak{M}^*;\rho^*,-\mathrm{d}^*_{\mathfrak{M}})$ is a representation of the MD$_\lambda$3-LieA $(\mathfrak{A},[-,-,-],\mathrm{d})$.*

Proof. First, It has been proved that [47] $(\mathfrak{M}^*;\rho^*)$ is a representation of the 3-Lie algebra $(\mathfrak{A},[-,-,-])$. Furthermore, for any $a_1,a_2 \in \mathfrak{A}, v \in \mathfrak{M}$ and $u^* \in \mathfrak{M}^*$, by Equations (5) and (6), we have

$$\begin{aligned}
&\langle \rho^*(\mathrm{d}(a_1),a_2)u^*,v\rangle + \langle \rho^*(a_1,\mathrm{d}(a_2))u^*,v\rangle + \langle \rho^*(a_1,a_2)(-\mathrm{d}^*_{\mathfrak{M}})u^*,v\rangle + \langle \lambda\rho^*(a_1,a_2)u^*,v\rangle \\
&\quad - \langle (-\mathrm{d}^*_{\mathfrak{M}})\rho^*(a_1,a_2)u^*,v\rangle \\
=& - \langle u^*,\rho(\mathrm{d}(a_1),a_2)v\rangle - \langle u^*,\rho(a_1,\mathrm{d}(a_2))v\rangle - \langle (-\mathrm{d}^*_{\mathfrak{M}})u^*,\rho(a_1,a_2)v\rangle - \langle u^*,\lambda\rho(a_1,a_2)v\rangle \\
&\quad + \langle \rho^*(a_1,a_2)u^*,\mathrm{d}_{\mathfrak{M}}(v)\rangle \\
=& - \langle u^*,\rho(\mathrm{d}(a_1),a_2)v\rangle - \langle u^*,\rho(a_1,\mathrm{d}(a_2))v\rangle + \langle u^*,\mathrm{d}_{\mathfrak{M}}(\rho(a_1,a_2)v)\rangle - \langle u^*,\lambda\rho(a_1,a_2)v\rangle \\
&\quad - \langle u^*,\rho(a_1,a_2)\mathrm{d}_{\mathfrak{M}}(v)\rangle \\
=& - \langle u^*,\rho(\mathrm{d}(a_1),a_2)v + \rho(a_1,\mathrm{d}(a_2))v - \mathrm{d}_{\mathfrak{M}}(\rho(a_1,a_2)v) + \lambda\rho(a_1,a_2)v + \rho(a_1,a_2)\mathrm{d}_{\mathfrak{M}}(v)\rangle \\
=&0,
\end{aligned}$$

which implies that $\rho^*(\mathrm{d}(a_1),a_2)u^* + \rho^*(a_1,\mathrm{d}(a_2))u^* + \rho^*(a_1,a_2)(-\mathrm{d}^*_{\mathfrak{M}})u^* + \lambda\rho^*(a_1,a_2)u^* - (-\mathrm{d}^*_{\mathfrak{M}})\rho^*(a_1,a_2)u^* = 0$. So, we obtain the result. □

Example 6. *Let $(\mathfrak{A},[-,-,-],\mathrm{d})$ be a MD$_\lambda$3-LieA and define $ad : \mathfrak{A}\wedge\mathfrak{A} \to \mathrm{End}(\mathfrak{A})$ by $ad(a_1,a_2)(a) = [a_1,a_2,a], \forall a_1,a_2,a \in \mathfrak{A}$. Then, $(\mathfrak{A};ad,\mathrm{d})$ is a representation of the MD$_\lambda$3-LieA $(\mathfrak{A},[-,-,-],\mathrm{d})$, which is called the adjoint representation of $(\mathfrak{A},[-,-,-],\mathrm{d})$. Furthermore, $(\mathfrak{A}^*;ad^*,-\mathrm{d}^*)$ is a dual adjoint representation of $(\mathfrak{A},[-,-,-],\mathrm{d})$, which is called the coadjoint representation of $(\mathfrak{A},[-,-,-],\mathrm{d})$.*

Next, we will study the cohomology of a MD$_\lambda$3-LieA with coefficients in its representation.

Recall from [14] that let $(\mathfrak{M};\rho)$ be a representation of a 3-Lie algebra $(\mathfrak{A},[-,-,-])$. Denote the $n-$cochains of $\mathfrak{A}$ with coefficients in representation $(\mathfrak{M};\rho)$ by

$$C^n_{\mathrm{3Lie}}(\mathfrak{A},\mathfrak{M}) := \mathrm{Hom}((\wedge^2\mathfrak{A})^{\otimes n-1}\wedge\mathfrak{A},\mathfrak{M}), n \geq 1.$$

The coboundary operator $\delta : C^n_{\mathrm{3Lie}}(\mathfrak{A},\mathfrak{M}) \to C^{n+1}_{\mathrm{3Lie}}(\mathfrak{A},\mathfrak{M})$, for $A_i = a_i \wedge b_i \in \wedge^2\mathfrak{A}, a_{n+1} \in \mathfrak{A}$ and $f \in C^n_{\mathrm{3Lie}}(\mathfrak{A},\mathfrak{M})$, as

$$\begin{aligned}
&\delta f(A_1,\cdots,A_n,a_{n+1}) \\
=&(-1)^{n+1}\big(\rho(b_n,a_{n+1})f(A_1,\cdots,A_{n-1},a_n) + \rho(a_{n+1},a_n)f(A_1,\cdots,A_{n-1},b_n)\big) \\
&+ \sum_{i=1}^{n}(-1)^{i+1}\rho(a_i,b_i)f(A_1,\cdots,A_{i-1},A_{i+1},\cdots,A_n,a_{n+1}) \\
&+ \sum_{i=1}^{n}(-1)^{i}f(A_1,\cdots,A_{i-1},A_{i+1},\cdots,A_n,[a_i,b_i,a_{n+1}]) \\
&+ \sum_{1\leq i<k\leq n}(-1)^{i}f(A_1,\cdots,A_{i-1},A_{i+1},\cdots,A_{k-1},[a_i,b_i,a_k]\wedge b_k + a_k\wedge[a_i,b_i,b_k],\cdots,A_n,a_{n+1}),
\end{aligned}$$

it was proved that $\delta\circ\delta = 0$.

Lemma 1. *Let* $(\mathfrak{M};\rho,\mathrm{d}_{\mathfrak{M}})$ *be a representation of a* MD_λ*3-LieA* $(\mathfrak{A},[-,-,-],\mathrm{d})$. *For any* $n\geq 1$, *we define a linear map* $\Phi:\mathcal{C}^n_{\mathrm{3Lie}}(\mathfrak{A},\mathfrak{M})\to\mathcal{C}^n_{\mathrm{3Lie}}(\mathfrak{A},\mathfrak{M})$ *by*

$$\begin{aligned}\Phi f(A_1,\cdots,A_{n-1},a_n)=&\sum_{i=1}^{n-1}f(A_1,\cdots,A_{i-1},\mathrm{d}(a_i)\wedge b_i+a_i\wedge\mathrm{d}(b_i),A_{i+1},\cdots,A_{n-1},a_n)\\&+f(A_1,\cdots,A_{n-1},\mathrm{d}(a_n))+(n-1)\lambda f(A_1,\cdots,A_{n-1},a_n)\\&-\mathrm{d}_{\mathfrak{M}}(f(A_1,\cdots,A_{n-1},a_n)),\end{aligned}$$

for any $f\in\mathcal{C}^n_{\mathrm{3Lie}}(\mathfrak{A},\mathfrak{M})$ *and* $A_i=a_i\wedge b_i\in\wedge^2\mathfrak{A}, i=1,\cdots,n-1, a_n\in\mathfrak{A}$. *Then,* Φ *is a cochain map; i.e.,* $\Phi\circ\delta=\delta\circ\Phi$.

Proof. It follows by a straightforward tedious calculations. □

Let $(\mathfrak{M};\rho,\mathrm{d}_{\mathfrak{M}})$ be a representation of a MD_λ3-LieA $(\mathfrak{A},[-,-,-],\mathrm{d})$; we define n-cochains for MD_λ3-LieA as follows:

$$\mathcal{C}^n_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M}):=\begin{cases}\mathcal{C}^n_{\mathrm{3Lie}}(\mathfrak{A},\mathfrak{M})\oplus\mathcal{C}^{n-1}_{\mathrm{3Lie}}(\mathfrak{A},\mathfrak{M}), & n\geq 2,\\ \mathrm{Hom}(\mathfrak{A},\mathfrak{M}), & n=1.\end{cases}$$

We define a linear map $\partial:\mathcal{C}^n_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})\to\mathcal{C}^{n+1}_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})$ by

$$\begin{aligned}\partial(f)&=(\delta f,-\Phi f), && \text{if } f\in\mathcal{C}^1_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M});\\ \partial(f,g)&=(\delta f,\delta g+(-1)^n\Phi f), && \text{if } (f,g)\in\mathcal{C}^n_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M}), n\geq 2.\end{aligned}$$

In view of Lemma 1, we have the following theorem.

Theorem 1. *The linear map* ∂ *is a coboundary operator; that is,* $\partial\circ\partial=0$.

Therefore, we obtain a cochain complex $(\mathcal{C}^*_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M}),\partial)$, for $n\geq 2$, and we denote the set of n-cocycles by $\mathcal{Z}^n_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})=\{(f,g)\in\mathcal{C}^n_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})\mid\partial(f,g)=0\}$, the set of n-coboundaries by $\mathcal{B}^n_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})=\{\partial(f,g)\mid(f,g)\in\mathcal{C}^{n-1}_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})\}$ and the n-th cohomology group of the MD_λ3-LieA $(\mathfrak{A},[-,-,-],\mathrm{d})$ with coefficients in the representation $(\mathfrak{M};\rho,\mathrm{d}_{\mathfrak{M}})$ by $\mathcal{H}^n_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})=\mathcal{Z}^n_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})/\mathcal{B}^n_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})$.

Lastly, we calculate the 1-cocycle and 2-cocycle.

For $f\in\mathcal{C}^1_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})$, f is a 1-cocycle if $\partial(f)=(\delta f,-\Phi f)=0$, i.e.,

$$\rho(b_1,a_2)f(a_1)+\rho(a_2,a_1)f(b_1)+\rho(a_1,b_1)f(a_2)-f([a_1,b_1,a_2])=0$$

and

$$\mathrm{d}_{\mathfrak{M}}(f(a_1))-f(\mathrm{d}(a_1))=0.$$

For $(f,g)\in\mathcal{C}^2_{\mathrm{md3Lie}^\lambda}(\mathfrak{A},\mathfrak{M})$, (f,g) is a 2-cocycle if $\partial(f,g)=(\delta f,\delta g+\Phi f)=0$, i.e.,

$$\begin{aligned}&-\rho(b_2,a_3)f(a_1,b_1,a_2)-\rho(a_3,a_2)f(a_1,b_1,b_2)+\rho(a_1,b_1)f(a_2,b_2,a_3)-\rho(a_2,b_2)f(a_1,b_1,a_3)\\&-f(a_2,b_2,[a_1,b_1,a_3])+f(a_1,b_1,[a_2,b_2,a_3])-f([a_1,b_1,a_2],b_2,a_3)-f(a_2,[a_1,b_1,b_2],a_3)=0\end{aligned}$$

and

$$\begin{aligned}&\rho(b_1,a_2)f(a_1)+\rho(a_2,a_1)f(b_1)+\rho(a_1,b_1)f(a_2)-f([a_1,b_1,a_2])\\&+f(\mathrm{d}(a_1),b_1,a_2)+f(a_1,\mathrm{d}(b_1),a_2)+f(a_1,b_1,d(a_2))+\lambda f(a_1,b_1,a_2)-\mathrm{d}_{\mathfrak{M}}(f(a_1,b_1,a_2))=0.\end{aligned}$$

3. Linear Deformations of MD_λ3-LieAs

In this section, we study linear deformations of MD_λ3-LieAs.

Let $(\mathfrak{A}, [-,-,-], \mathrm{d})$ be a MD$_\lambda$3-LieA. Denote $\nu_0 = [-,-,-]$ and $\mathrm{d}_0 = \mathrm{d}$. Consider a family of linear maps:

$$\nu_t = \nu_0 + t\nu_1 + t^2\nu_2,\ \ \nu_1, \nu_2 \in \mathcal{C}^2_{\mathrm{3Lie}}(\mathfrak{A}, \mathfrak{A}), \quad \mathrm{d}_t = \mathrm{d}_0 + t\mathrm{d}_1,\ \ \mathrm{d}_1 \in \mathcal{C}^1_{\mathrm{3Lie}}(\mathfrak{A}, \mathfrak{A}).$$

Definition 4. *A linear deformation of the MD$_\lambda$3-LieA $(\mathfrak{A}, [-,-,-], \mathrm{d})$ is a pair (ν_t, d_t) which endows $(\mathfrak{A}[[t]], \nu_t, \mathrm{d}_t)$ with the MD$_\lambda$3-LieA.*

Proposition 4. *The pair (ν_t, d_t) generates a linear deformation of the MD$_\lambda$3-LieA $(\mathfrak{A}, [-,-,-], \mathrm{d})$ if and only if the following equations hold:*

$$\sum_{i+j=n} \nu_i(a_1, a_2, \nu_j(a_3, a_4, a_5)) = \sum_{i+j=n} \nu_i(\nu_j(a_1, a_2, a_3), a_4, a_5) + \sum_{i+j=n} \nu_i(a_3, \nu_j(a_1, a_2, a_4), a_5) + \sum_{i+j=n} \nu_i(a_3, a_4, \nu_j(a_1, a_2, a_5)), \tag{7}$$

$$\sum_{i+l=n} \mathrm{d}_l(\nu_i(a_1, a_2, a_3)) = \sum_{i+l=n} \nu_i(\mathrm{d}_l(a_1), a_2, a_3) + \sum_{i+l=n} \nu_i(a_1, \mathrm{d}_l(a_2), a_3) + \sum_{i+l=n} \nu_i(a_1, a_2, \mathrm{d}_l(a_3)) + \lambda\nu_n(a_1, a_2, a_3), \tag{8}$$

for any $a_1, a_2, a_3, a_4, a_5 \in \mathfrak{A}$ and $i, j = 0, 1, 2, l = 0, 1$.

Proof. $(\mathfrak{A}[[t]], \nu_t, \mathrm{d}_t)$ is a MD$_\lambda$3-LieA if and only if

$$\begin{aligned} &\nu_t(a_1, a_2, \nu_t(a_3, a_4, a_5)) \\ &= \nu_t(\nu_t(a_1, a_2, a_3), a_4, a_5) + \nu_t(a_3, \nu_t(a_1, a_2, a_4), a_5) + \nu_t(a_3, a_4, \nu_t(a_1, a_2, a_5)), \end{aligned} \tag{9}$$

$$\begin{aligned} &\mathrm{d}_t(\nu_t(a_1, a_2, a_3)) \\ &= \nu_t(\mathrm{d}_t(a_1), a_2, a_3) + \nu_t(a_1, \mathrm{d}_t(a_2), a_3) + \nu_t(a_1, a_2, \mathrm{d}_t(a_3)) + \lambda\nu_t(a_1, a_2, a_3). \end{aligned} \tag{10}$$

Comparing the coefficients of t^n on both sides of the above equations, Equations (9) and (10) are equivalent to Equations (7) and (8), respectively. □

Corollary 1. *Let $(\mathfrak{A}[[t]], \nu_t, \mathrm{d}_t)$ be a linear deformation of a MD$_\lambda$3-LieA $(\mathfrak{A}, \nu_0, \mathrm{d})$. Then, (ν_1, d_1) is a 2-cocycle of $(\mathfrak{A}, [-,-,-], \mathrm{d})$ with the coefficient in the adjoint representation $(\mathfrak{A}; ad, \mathrm{d})$.*

Proof. For $n = 1$, Equations (7) and (8) are equivalent to

$$\begin{aligned} &\nu_1(a_1, a_2, [a_3, a_4, a_5]) + [a_1, a_2, \nu_1(a_3, a_4, a_5)] \\ =&\nu_1([a_1, a_2, a_3], a_4, a_5) + [\nu_1(a_1, a_2, a_3), a_4, a_5] + \nu_1(a_3, [a_1, a_2, a_4], a_5) + [a_3, \nu_1(a_1, a_2, a_4), a_5] \\ &+ \nu_1(a_3, a_4, [a_1, a_2, a_5]) + [a_3, a_4, \nu_1(a_1, a_2, a_5)], \\ &\mathrm{d}_1([a_1, a_2, a_3]) + \mathrm{d}(\nu_1(a_1, a_2, a_3)) \\ =&[\mathrm{d}_1(a_1), a_2, a_3] + \nu_1(\mathrm{d}(a_1), a_2, a_3) + [a_1, \mathrm{d}_1(a_2), a_3] + \nu_1(a_1, \mathrm{d}(a_2), a_3) + [a_1, a_2, \mathrm{d}_1(a_3)] \\ &+ \nu_1(a_1, a_2, \mathrm{d}(a_3)) + \lambda\nu_1(a_1, a_2, a_3), \end{aligned}$$

which imply that $\delta\nu_1 = 0, \delta\mathrm{d}_1 + \Phi\nu_1 = 0$, respectively. Hence, (ν_1, d_1) is a 2-cocycle of $(\mathfrak{A}, [-,-,-], \mathrm{d})$ with the coefficient in the adjoint representation $(\mathfrak{A}; ad, \mathrm{d})$. □

Definition 5. *The* 2*-cocycle (ν_1, d_1) is called the infinitesimal of the linear deformation $(\mathfrak{A}[[t]], \nu_t, \mathrm{d}_t)$ of $(\mathfrak{A}, [-,-,-], \mathrm{d})$.*

Definition 6. *(i) Two linear deformations* $(\mathfrak{A}[[t]], \nu_t, \mathrm{d}_t)$ *and* $(\mathfrak{A}[[t]], \nu'_t, \mathrm{d}'_t)$ *of the* MD_λ*3-LieA* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *are said to be equivalent if there exists a linear map* $N : \mathfrak{A} \to \mathfrak{A}$*, such that* $N_t = \mathrm{id}_{\mathfrak{A}} + tN$ *satisfying*

$$N_t(\mathrm{d}_t(a_1)) = \mathrm{d}'_t(N_t(a_1)), \tag{11}$$
$$N_t \nu_t(a_1, a_2, a_3) = \nu'_t(N_t(a_1), N_t(a_2), N_t(a_3)), \tag{12}$$

for any $a_1, a_2, a_3 \in \mathfrak{A}$*.*
(ii) A linear deformation $(\mathfrak{A}[[t]], \nu_t, \mathrm{d}_t)$ *of the* MD_λ*3-LieA* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *is said to be trivial if* $(\mathfrak{A}[[t]], \nu_t, \mathrm{d}_t)$ *is equivalent to* $(\mathfrak{A}, [-,-,-], \mathrm{d})$*.*

Comparing the coefficients of t on both sides of the above Equations (11) and (12), we have

$$\nu_1(a_1, a_2, a_3) - \nu'_1(a_1, a_2, a_3) = [Na_1, a_2, a_3] + [a_1, Na_2, a_3] + [a_1, a_2, Na_3] - N[a_1, a_2, a_3],$$
$$\mathrm{d}_1(a) - \mathrm{d}'_1(a) = \mathrm{d}(N_1 a) - N_1 \mathrm{d}(a).$$

Thus, we have the following theorem.

Theorem 2. *The infinitesimals of two equivalent linear deformations of* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *are in the same cohomological class in* $\mathcal{H}^2_{\mathrm{md3Lie}^\lambda}(\mathfrak{A}, \mathfrak{A})$*.*

Let $(\mathfrak{A}[[t]], \nu_t, \mathrm{d}_t)$ be a trivial deformation of $(\mathfrak{A}, [-,-,-], \mathrm{d})$. Then, there exists a linear map $N : \mathfrak{A} \to \mathfrak{A}$, such that $N_t = \mathrm{id}_{\mathfrak{A}} + tN$ satisfying

$$N_t(\mathrm{d}_t(a_1)) = \mathrm{d}(N_t(a_1)), \tag{13}$$
$$N_t \nu_t(a_1, a_2, a_3) = [N_t(a_1), N_t(a_2), N_t(a_3)]. \tag{14}$$

Compare the coefficients of $t^i (1 \le i \le 3)$ on both sides of Equations (13) and (14), and we can obtain

$$N\mathrm{d}(a_1) = \mathrm{d}(Na_1), \tag{15}$$
$$\nu_1(a_1, a_2, a_3) + N[a_1, a_2, a_3] = [Na_1, a_2, a_3] + [a_1, Na_2, a_3] + [a_1, a_2, Na_3], \tag{16}$$
$$\nu_2(a_1, a_2, a_3) + N\nu_1(a_1, a_2, a_3) = [Na_1, Na_2, a_3] + [a_1, Na_2, Na_3] + [Na_1, a_2, Na_3], \tag{17}$$
$$N\nu_2(a_1, a_2, a_3) = [Na_1, Na_2, Na_3]. \tag{18}$$

Thus, from a trivial deformation, we can obtain the following definition of Nijenhuis operator.

Definition 7. *Let* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *be a* MD_λ*3-LieA. A linear map* $N : \mathfrak{A} \to \mathfrak{A}$ *is called a Nijenhuis operator if the following equations hold:*

$$N \circ \mathrm{d} = \mathrm{d} \circ N, \tag{19}$$
$$\begin{aligned}[Na_1, Na_2, Na_3] =& N([a_1, Na_2, Na_3] + [Na_1, a_2, Na_3] + [Na_1, Na_2, a_3]) \\ & - N^2([Na_1, a_2, a_3] + [a_1, Na_2, a_3] + [a_1, a_2, Na_3]) + N^3[a_1, a_2, a_3],\end{aligned} \tag{20}$$

for any $a_1, a_2, a_3 \in \mathfrak{A}$*.*

Proposition 5. *Let* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *be a* MD_λ*3-LieA, and* $N : \mathfrak{A} \to \mathfrak{A}$ *a Nijenhuis operator. Then,* $(\mathfrak{A}, [-,-,-]_N, \mathrm{d})$ *is a* MD_λ*3-LieA, where*

$$\begin{aligned}[a_1, a_2, a_3]_N =& [a_1, Na_2, Na_3] + [Na_1, a_2, Na_3] + [Na_1, Na_2, a_3] \\ & - N([Na_1, a_2, a_3] + [a_1, Na_2, a_3] + [a_1, a_2, Na_3]) + N^2[a_1, a_2, a_3].\end{aligned}$$

Proof. In the light of [9], $(\mathfrak{A}, [-,-,-]_N)$ is a 3-Lie algebra. Next, we prove that d is a modified λ-differential operator of $(\mathfrak{A}, [-,-,-]_N)$, for any $a_1, a_2, a_3 \in \mathfrak{A}$, by Equations (2) and (19), and we have

$$\begin{aligned}
&\mathrm{d}[a_1, a_2, a_3]_N \\
=&\mathrm{d}[a_1, Na_2, Na_3] + \mathrm{d}[Na_1, a_2, Na_3] + \mathrm{d}[Na_1, Na_2, a_3] \\
&- N(\mathrm{d}[Na_1, a_2, a_3] + \mathrm{d}[a_1, Na_2, a_3] + \mathrm{d}[a_1, a_2, Na_3]) + N^2\mathrm{d}[a_1, a_2, a_3] \\
=&[\mathrm{d}(a_1), Na_2, Na_3] + [a_1, N\mathrm{d}(a_2), Na_3] + [a_1, Na_2, N\mathrm{d}(a_3)] + \lambda[a_1, Na_2, Na_3] \\
&+ [N\mathrm{d}(a_1), a_2, Na_3] + [Na_1, \mathrm{d}(a_2), Na_3] + [Na_1, a_2, N\mathrm{d}(a_3)] + \lambda[Na_1, a_2, Na_3] \\
&+ [N\mathrm{d}(a_1), Na_2, a_3] + [Na_1, N\mathrm{d}(a_2), a_3] + [Na_1, Na_2, \mathrm{d}(a_3)] + \lambda[Na_1, Na_2, a_3] \\
&- N([N\mathrm{d}(a_1), a_2, a_3] + [Na_1, \mathrm{d}(a_2), a_3] + [Na_1, a_2, \mathrm{d}(a_3)] + \lambda[Na_1, a_2, a_3]) \\
&- N([\mathrm{d}(a_1), Na_2, a_3] + [a_1, N\mathrm{d}(a_2), a_3] + [a_1, Na_2, \mathrm{d}(a_3)] + \lambda[a_1, Na_2, a_3]) \\
&- N([\mathrm{d}(a_1), a_2, Na_3] + [a_1, \mathrm{d}(a_2), Na_3] + [a_1, a_2, N\mathrm{d}(a_3)] + \lambda[a_1, a_2, Na_3]) \\
&+ N^2([\mathrm{d}(a_1), a_2, a_3] + [a_1, \mathrm{d}(a_2), a_3] + [a_1, a_2, \mathrm{d}(a_3)] + \lambda[a_1, a_2, a_3]) \\
=&[\mathrm{d}(a_1), a_2, a_3]_N + [a_1, \mathrm{d}(a_2), a_3]_N + [a_1, a_2, \mathrm{d}(a_3)]_N + \lambda[a_1, a_2, a_3]_N.
\end{aligned}$$

So, we obtain the conclusion. □

Definition 8. *A linear map $R : \mathfrak{M} \to \mathfrak{A}$ is called an $\mathcal{O}$-operator on the MD$_\lambda$3-LieA $(\mathfrak{A}, [-,-,-], \mathrm{d})$ with respect to the representation $(\mathfrak{M}; \rho, \mathrm{d}_{\mathfrak{M}})$ if the following equations hold:*

$$\begin{aligned}
R \circ \mathrm{d}_{\mathfrak{M}} =& \mathrm{d} \circ R, \\
[Rv_1, Rv_2, Rv_3] =& R(\rho(Rv_1, Rv_2)v_3 + \rho(Rv_2, Rv_3)v_1 + \rho(Rv_3, Rv_1)v_2),
\end{aligned}$$

for any $v_1, v_2, v_3 \in \mathfrak{M}$.

Remark 3. *Obviously, an invertible linear map $R : \mathfrak{M} \to \mathfrak{A}$ is an $\mathcal{O}$-operator if and only if R^{-1} is a 1-cocycle of the MD$_\lambda$3-LieA $(\mathfrak{A}, [-,-,-], \mathrm{d})$ with coefficients in the representation $(\mathfrak{M}; \rho, \mathrm{d}_{\mathfrak{M}})$.*

Proposition 6. *Let $(\mathfrak{M}; \rho, \mathrm{d}_{\mathfrak{M}})$ be a representation of a MD$_\lambda$3-LieA $(\mathfrak{A}, [-,-,-], \mathrm{d})$. Then, $R : \mathfrak{M} \to \mathfrak{A}$ is an $\mathcal{O}$-operator if and only if $\overline{R} = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} : \mathfrak{A} \oplus \mathfrak{M} \to \mathfrak{A} \oplus \mathfrak{M}$ is a Nijenhuis operator on semidirect product MD$_\lambda$3-LieA $(\mathfrak{A} \oplus \mathfrak{M}, [-,-,-]_\rho, \mathrm{d} \oplus \mathrm{d}_{\mathfrak{M}})$.*

Proof. For any $a_1, a_2, a_3 \in \mathfrak{A}$ and $u_1, u_2, u_3 \in \mathfrak{M}$, by $\overline{R}^2 = 0$, we have

$$\begin{aligned}
&(\mathrm{d} \oplus \mathrm{d}_{\mathfrak{M}})\overline{R}(a_1 + u_1) = (\mathrm{d} \oplus \mathrm{d}_{\mathfrak{M}})(Ru_1 + 0) = \mathrm{d}(Ru_1) + 0, \\
&\overline{R}(\mathrm{d} \oplus \mathrm{d}_{\mathfrak{M}})(a_1 + u_1) = \overline{R}(\mathrm{d}(a_1) + \mathrm{d}_{\mathfrak{M}}(u_1)) = R\mathrm{d}_{\mathfrak{M}}(u_1) + 0, \\
&\overline{R}([a_1 + u_1, \overline{R}(a_2 + u_2), \overline{R}(a_3 + u_3)]_\rho + [\overline{R}(a_1 + u_1), a_2 + u_2, \overline{R}(a_3 + u_3)]_\rho \\
&+ [\overline{R}(a_1 + u_1), \overline{R}(a_2 + u_2), a_3 + u_3]_\rho) - [\overline{R}(a_1 + u_1), \overline{R}(a_2 + u_2), \overline{R}(a_3 + u_3)]_\rho \\
=&\overline{R}([a_1 + u_1, Ru_2 + 0, Ru_3 + 0]_\rho + [Ru_1 + 0, a_2 + u_2, Ru_3 + 0]_\rho + [Ru_1 + 0, Ru_2 + 0, a_3 + u_3]_\rho) \\
&- [Ru_1 + 0, Ru_2 + 0, Ru_3 + 0]_\rho \\
=&\overline{R}([a_1, Ru_2, Ru_3] + \rho(Ru_2, Ru_3)u_1 + [Ru_1, a_2, Ru_3] + \rho(Ru_3, Ru_1)u_2 + [Ru_1, Ru_2 a_3] + \rho(Ru_1, Ru_2)u_3) \\
&- [Ru_1, Ru_2, Ru_3] + 0 \\
=&R(\rho(Ru_2, Ru_3)u_1 + \rho(Ru_3, Ru_1)u_2 + \rho(Ru_1, Ru_2)u_3) - [Ru_1, Ru_2, Ru_3] + 0,
\end{aligned}$$

which implies that R is an $\mathcal{O}$-operator if and only if $\overline{R}$ is a Nijenhuis operator. □

4. Abelian Extensions of MD$_\lambda$3-LieAs

In this section, we study abelian extensions of MD$_\lambda$3-LieAs and show that they are classified by the second cohomology groups.

Notice that a vector space $\mathfrak{M}$ together with a linear map $\mathrm{d}_{\mathfrak{M}}$ is naturally a MD$_\lambda$3-LieA where the bracket on $\mathfrak{M}$ is defined to be $[-,-,-]_{\mathfrak{M}} = 0$.

Definition 9. *An abelian extension of* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *by* $(\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}})$ *is a short exact sequence of homomorphisms of MD*$_\lambda$*3-LieAs*

$$0 \to (\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}}) \xrightarrow{i} (\widehat{\mathfrak{A}}, [-,-,-]_{\widehat{\mathfrak{A}}}, \widehat{\mathrm{d}}) \xrightarrow{p} (\mathfrak{A}, [-,-,-], \mathrm{d}) \to 0,$$

i.e., there exists a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M} & \xrightarrow{i} & \widehat{\mathfrak{A}} & \xrightarrow{p} & \mathfrak{A} & \longrightarrow & 0 \\ & & \mathrm{d}_{\mathfrak{M}} \big\downarrow & & \widehat{\mathrm{d}} \big\downarrow & & \mathrm{d} \big\downarrow & & \\ 0 & \longrightarrow & \mathfrak{M} & \xrightarrow{i} & \widehat{\mathfrak{A}} & \xrightarrow{p} & \mathfrak{A} & \longrightarrow & 0, \end{array}$$

where the MD$_\lambda$*3-LieA* $(\widehat{\mathfrak{A}}, [-,-,-]_{\widehat{\mathfrak{A}}}, \widehat{\mathrm{d}})$ *satisfies* $[-,u,v]_{\widehat{\mathfrak{A}}} = 0$, *for all* $u, v \in \mathfrak{M}$.

We will call $(\widehat{\mathfrak{A}}, [-,-,-]_{\widehat{\mathfrak{A}}}, \widehat{\mathrm{d}})$ *an abelian extension of* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *by* $(\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}})$.

A section of an abelian extension $(\widehat{\mathfrak{A}}, [-,-,-]_{\widehat{\mathfrak{A}}}, \widehat{\mathrm{d}})$ of $(\mathfrak{A}, [-,-,-], \mathrm{d})$ by $(\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}})$ is a linear map $s : \mathfrak{A} \to \widehat{\mathfrak{A}}$ such that $p \circ s = \mathrm{id}_{\mathfrak{A}}$.

Let $(\mathfrak{M}; \rho, \mathrm{d}_{\mathfrak{M}})$ be a representation of a MD$_\lambda$3-LieA $(\mathfrak{A}, [-,-,-], \mathrm{d})$. Assume that $(f, g) \in \mathcal{C}^2_{\mathrm{md3Lie}^{\sim}}(\mathfrak{A}, \mathfrak{M})$. Define $[-,-,-]_{\rho f} : \wedge^3(\mathfrak{A} \oplus \mathfrak{M}) \to \mathfrak{A} \oplus \mathfrak{M}$ and $\mathrm{d}_g : \mathfrak{A} \oplus \mathfrak{M} \to \mathfrak{A} \oplus \mathfrak{M}$, respectively, by

$$\begin{aligned} &[a_1 + u_1, a_2 + u_2, a_3 + u_3]_{\rho f} \\ &= [a_1, a_2, a_3] + \rho(a_2, a_3)u_1 + \rho(a_3, a_1)u_2 + \rho(a_1, a_2)u_3 + f(a_1, a_2, a_3), \quad (21) \\ &\mathrm{d}_g(a_1 + u_1) = \mathrm{d}(a_1) + \mathrm{d}_{\mathfrak{M}}(u_1) + g(a_1), \ \forall a_1, a_2, a_3 \in \mathfrak{A}, \ u_1, u_2, u_3 \in \mathfrak{M}. \quad (22) \end{aligned}$$

Proposition 7. *The triple* $(\mathfrak{A} \oplus \mathfrak{M}, [-,-,-]_{\rho f}, \mathrm{d}_g)$ *is a MD*$_\lambda$*3-LieA if and only if* (f, g) *is a 2-cocycle in the cohomology of the MD*$_\lambda$*3-LieA* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *with the coefficient in* $(\mathfrak{M}; \rho, \mathrm{d}_{\mathfrak{M}})$. *In this case,*

$$0 \to (\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}}) \hookrightarrow (\mathfrak{A} \oplus \mathfrak{M}, [-,-,-]_{\rho f}, \mathrm{d}_g) \xrightarrow{p} (\mathfrak{A}, [-,-,-], \mathrm{d}) \to 0$$

is an abelian extension.

Proof. $(\mathfrak{A} \oplus \mathfrak{M}, [-,-,-]_{\rho f}, \mathrm{d}_g)$ is a MD$_\lambda$3-LieA if and only if

$$\begin{aligned} &[[a_1 + u_1, a_2 + u_2, a_3 + u_3]_{\rho f}, a_4 + u_4, a_5 + u_5]_{\rho f} + [a_3 + u_3, [a_1 + u_1, a_2 + u_2, a_4 + u_4]_{\rho f}, a_5 + u_5]_{\rho f} \\ &+ [a_3 + u_3, a_4 + u_4, [a_1 + u_1, a_2 + u_2, a_5 + u_5]_{\rho f}]_{\rho f} - [a_1 + u_1, a_2 + u_2, [a_3 + u_3, a_4 + u_4, a_5 + u_5]_{\rho f}]_{\rho f} \\ &= 0, \quad (23) \\ &[\mathrm{d}_g(a_1 + u_1), a_2 + u_2, a_3 + u_3]_{\rho f} + [a_1 + u_1, \mathrm{d}_g(a_2 + u_2), a_3 + u_3]_{\rho f} + [a_1 + u_1, a_2 + u_2, \mathrm{d}_g(a_3 + u_3)]_{\rho f} \\ &+ \lambda [a_1 + u_1, a_2 + u_2, a_3 + u_3]_{\rho f} - \mathrm{d}_g [a_1 + u_1, a_2 + u_2, a_3 + u_3]_{\rho f} \\ &= 0, \quad (24) \end{aligned}$$

for any $a_1, a_2, a_3, a_4, a_5 \in \mathfrak{A}$, $u_1, u_2, u_3, u_4, u_5 \in \mathfrak{M}$. Furthermore, Equations (23) and (24) are equivalent to

$$\begin{aligned}&\rho(a_4,a_5)f(a_1,a_2,a_3)+f([a_1,a_2,a_3],a_4,a_5)+\rho(a_5,a_3)f(a_1,a_2,a_4)+f(a_3,[a_1,a_2,a_4],a_5)\\&+\rho(a_3,a_4)f(a_1,a_2,a_5)+f(a_3,a_4,[a_1,a_2,a_5])-\rho(a_1,a_2)f(a_3,a_4,a_5)-f(a_1,a_2,[a_3,a_4,a_5])\\&=0,\end{aligned}\quad (25)$$

$$\begin{aligned}&\rho(a_2,a_3)g(a_1)+f(\mathrm{d}(a_1),a_2,a_3)+\rho(a_3,a_1)g(a_2)+f(a_1,\mathrm{d}(a_2),a_3)+\rho(a_1,a_2)g(a_3)\\&+f(a_1,a_2,\mathrm{d}(a_3))+\lambda f(a_1,a_2,a_3)-\mathrm{d}_{\mathfrak{M}}(f(a_1,a_2,a_3))-g([a_1,a_2,a_3])=0,\end{aligned}\quad (26)$$

using Equations (25) and (26), we obtain $\delta f=0$ and $\delta g+\Phi f=0$, respectively. Therefore, $\partial(f,g)=(\delta f,\delta g+\Phi f)=0$, which implies that (f,g) is a 2-cocycle.

Conversely, if (f,g) satisfying Equations (25) and (26), then $(\mathfrak{A}\oplus\mathfrak{M},[-,-,-]_{\rho f},\mathrm{d}_g)$ is a MD$_\lambda$3-LieA. □

Let $(\widehat{\mathfrak{A}},[-,-,-]_{\widehat{\mathfrak{A}}},\widehat{\mathrm{d}})$ be an abelian extension of $(\mathfrak{A},[-,-,-],\mathrm{d})$ by $(\mathfrak{M},[-,-,-]_{\mathfrak{M}},\mathrm{d}_{\mathfrak{M}})$ and $s:\mathfrak{A}\to\widehat{\mathfrak{A}}$ a section. Define $\varrho:\wedge^2\mathfrak{A}\to\mathrm{End}(\mathfrak{M})$, $\upsilon:\wedge^3\mathfrak{A}\to\mathfrak{M}$ and $\mu:\mathfrak{A}\to\mathfrak{M}$, respectively, by

$$\begin{aligned}\varrho(a_1,a_2)u:&=[s(a_1),s(a_2),u]_{\widehat{\mathfrak{A}}},\\ \upsilon(a_1,a_2,a_3):&=[s(a_1),s(a_2),s(a_3)]_{\widehat{\mathfrak{A}}}-s([a_1,a_2,a_3]),\\ \mu(a_1):&=\widehat{\mathrm{d}}(s(a_1))-s(\mathrm{d}(a_1)),\quad\forall a_1,a_2,a_3\in\mathfrak{A},u\in\mathfrak{M}.\end{aligned}$$

Note that ϱ is independent on the choice of s.

Proposition 8. *With the above notations, $(\mathfrak{M},\varrho,\mathrm{d}_{\mathfrak{M}})$ is a representation of the MD$_\lambda$3-LieA $(\mathfrak{A},[-,-,-],\mathrm{d})$ and (υ,μ) is a 2-cocycle in the cohomology of the MD$_\lambda$3-LieA $(\mathfrak{A},[-,-,-],\mathrm{d})$ with the coefficient in $(\mathfrak{M};\varrho,\mathrm{d}_{\mathfrak{M}})$. Furthermore, the cohomological class of the 2-cocycle $[(\upsilon,\mu)]\in\mathcal{H}^2_{\mathrm{md3Lie}^{\sim}}(\mathfrak{A},\mathfrak{M})$ is independent of the choice of sections of p.*

Proof. First, for any $a_1,a_2,a_3,a_4\in\mathfrak{A},u\in\mathfrak{M}$, by Equation (1), we obtain

$$\begin{aligned}&\varrho(a_2,a_3)\varrho(a_1,a_4)u+\varrho(a_3,a_1)\varrho(a_2,a_4)u+\varrho(a_1,a_2)\varrho(a_3,a_4)u-\varrho([a_1,a_2,a_3],a_4)u\\=&[s(a_2),s(a_3),[s(a_1),s(a_4),u]_{\widehat{\mathfrak{A}}}]_{\widehat{\mathfrak{A}}}+[s(a_3),s(a_1),[s(a_2),s(a_4),u]_{\widehat{\mathfrak{A}}}]_{\widehat{\mathfrak{A}}}\\&+[s(a_1),s(a_2),[s(a_3),s(a_4),u]_{\widehat{\mathfrak{A}}}]_{\widehat{\mathfrak{A}}}-[s([a_1,a_2,a_3]),s(a_4),u]_{\widehat{\mathfrak{A}}}\\=&[s(a_2),s(a_3),[s(a_1),s(a_4),u]_{\widehat{\mathfrak{A}}}]_{\widehat{\mathfrak{A}}}+[s(a_3),s(a_1),[s(a_2),s(a_4),u]_{\widehat{\mathfrak{A}}}]_{\widehat{\mathfrak{A}}}\\&+[s(a_1),s(a_2),[s(a_3),s(a_4),u]_{\widehat{\mathfrak{A}}}]_{\widehat{\mathfrak{A}}}-[[s(a_1),s(a_2),s(a_3)]_{\widehat{\mathfrak{A}}}-\upsilon(a_1,a_2,a_3),s(a_4),u]_{\widehat{\mathfrak{A}}}\\=&0,\\&\varrho(a_3,a_4)\varrho(a_1,a_2)u+\varrho([a_1,a_2,a_3],a_4)u+\varrho(a_3,[a_1,a_2,a_4])u-\varrho(a_1,a_2)\varrho(a_3,a_4)u\\=&[s(a_3),s(a_4),[s(a_1),s(a_2),u]_{\widehat{\mathfrak{A}}}]_{\widehat{\mathfrak{A}}}+[s([a_1,a_2,a_3]),s(a_4),u]_{\widehat{\mathfrak{A}}}+[s(a_3),s([a_1,a_2,a_4]),u]_{\widehat{\mathfrak{A}}}\\&-[s(a_1),s(a_2),[s(a_3),s(a_4),u]_{\widehat{\mathfrak{A}}}]_{\widehat{\mathfrak{A}}}\\=&[s(a_3),s(a_4),[s(a_1),s(a_2),u]_{\widehat{\mathfrak{A}}}]_{\widehat{\mathfrak{A}}}+[[s(a_1),s(a_2),s(a_3)]-\upsilon(a_1,a_2,a_3),s(a_4),u]_{\widehat{\mathfrak{A}}}\\&+[s(a_3),[s(a_1),s(a_2),s(a_4)]-\upsilon(a_1,a_2,a_4),u]_{\widehat{\mathfrak{A}}}-[s(a_1),s(a_2),[s(a_3),s(a_4),u]_{\widehat{\mathfrak{A}}}]_{\widehat{\mathfrak{A}}}\\=&0.\end{aligned}$$

In addition, by Equation (2), we have

$$\begin{aligned}&\mathrm{d}_{\mathfrak{M}}(\varrho(a_1,a_2)u)=\mathrm{d}_{\mathfrak{M}}([s(a_1),s(a_2),u]_{\widehat{\mathfrak{A}}})\\=&[\widehat{\mathrm{d}}(s(a_1)),s(a_2),u]_{\widehat{\mathfrak{A}}}+[s(a_1),\widehat{\mathrm{d}}(s(a_2)),u]_{\widehat{\mathfrak{A}}}+[s(a_1),s(a_2),\mathrm{d}_{\mathfrak{M}}(u)]_{\widehat{\mathfrak{A}}}+\lambda[s(a_1),s(a_2),u]_{\widehat{\mathfrak{A}}}\\=&[s(\mathrm{d}(a_1))+\mu(a_1),s(a_2),u]_{\widehat{\mathfrak{A}}}+[s(a_1),s(\mathrm{d}(a_2))+\mu(a_2),u]_{\widehat{\mathfrak{A}}}+[s(a_1),s(a_2),\mathrm{d}_{\mathfrak{M}}(u)]_{\widehat{\mathfrak{A}}}\\&+\lambda[s(a_1),s(a_2),u]_{\widehat{\mathfrak{A}}}\\=&\varrho(\mathrm{d}(a_1),a_2)u+\varrho(a_1,\mathrm{d}(a_2))u+\varrho(a_1,a_2)\mathrm{d}_{\mathfrak{M}}(u)+\lambda\varrho(a_1,a_2)u.\end{aligned}$$

Hence, $(\mathfrak{M}, \varrho, \mathrm{d}_{\mathfrak{M}})$ is a representation over $(\mathfrak{A}, [-,-,-], \mathrm{d})$.

Since $(\widehat{\mathfrak{A}}, [-,-,-]_{\widehat{\mathfrak{A}}}, \widehat{\mathrm{d}})$ is an abelian extension of $(\mathfrak{A}, [-,-,-], \mathrm{d})$ by $(\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}})$, by Proposition 7, (υ, μ) is a 2-cocycle. Moreover, let $s_1, s_2 : \mathfrak{A} \to \widehat{\mathfrak{M}}$ be two distinct sections providing 2-cocycles (υ_1, μ_1) and (υ_2, μ_2), respectively. Define linear map $\iota : \mathfrak{A} \to \mathfrak{M}$ by $\iota(a_1) = s_1(a_1) - s_2(a_1)$. Then,

$$
\begin{aligned}
&\upsilon_1(a_1, a_2, a_3)\\
=&[s_1(a_1), s_1(a_2), s_1(a_3)]_{\widehat{\mathfrak{A}}_1} - s_1([a_1, a_2, a_3])\\
=&[s_2(a_1) + \iota(a_1), s_2(a_2) + \iota(a_2), s_2(a_3) + \iota(a_3)]_{\widehat{\mathfrak{A}}_1} - (s_2([a_1, a_2, a_3]) + \iota([a_1, a_2, a_3]))\\
=&[s_2(a_1), s_2(a_2), s_2(a_3)]_{\widehat{\mathfrak{A}}_2} + \varrho(a_2, a_3)\iota(a_1) + \varrho(a_3, a_1)\iota(a_2) + \varrho(a_1, a_2)\iota(a_3)\\
&- s_2([a_1, a_2, a_3]) - \iota([a_1, a_2, a_3])\\
=&[s_2(a_1), s_2(a_2), s_2(a_3)]_{\widehat{\mathfrak{A}}_2} - s_2([a_1, a_2, a_3])\\
&+ \varrho(a_2, a_3)\iota(a_1) + \varrho(a_3, a_1)\iota(a_2) + \varrho(a_1, a_2)\iota(a_3) - \iota([a_1, a_2, a_3])\\
=&\upsilon_2(a_1, a_2, a_3) + \delta\iota(a_1, a_2, a_3)
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_1(a_1) &= \widehat{d}(s_1(a_1)) - s_1(\mathrm{d}(a_1))\\
&= \widehat{\mathrm{d}}(s_2(a_1) + \iota(a_1)) - \big(s_2(\mathrm{d}(a_1)) + \iota(\mathrm{d}(a_1))\big)\\
&= \big(\widehat{\mathrm{d}}(s_2(a_1)) - s_2(\mathrm{d}(a_1))\big) + \widehat{\mathrm{d}}(\iota(a_1)) - \iota(\mathrm{d}(a_1))\\
&= \mu_2(a_1) + \mathrm{d}_{\mathfrak{M}}(\iota(a_1)) - \iota(\mathrm{d}(a_1))\\
&= \mu_2(a_1) - \Phi\iota(a_1),
\end{aligned}
$$

which implies that $(\upsilon_1, \mu_1) - (\upsilon_2, \mu_2) = (\delta\iota, -\Phi\iota) = \partial(\iota) \in \mathcal{C}^2_{\mathrm{md3Lie}^{\lambda}}(\mathfrak{A}, \mathfrak{M})$. So $[(\upsilon_1, \mu_1)] = [(\upsilon_2, \mu_2)] \in \mathcal{H}^2_{\mathrm{md3Lie}^{\lambda}}(\mathfrak{A}, \mathfrak{M})$. □

Definition 10. *Let* $(\widehat{\mathfrak{A}}_1, [-,-,-]_{\widehat{\mathfrak{A}}_1}, \widehat{\mathrm{d}}_1)$ *and* $(\widehat{\mathfrak{A}}_2, [-,-,-]_{\widehat{\mathfrak{A}}_2}, \widehat{\mathrm{d}}_2)$ *be two abelian extensions of* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *by* $(\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}})$. *They are said to be equivalent if there is an isomorphism of MD*$_\lambda$*3-LieA* $\eta : (\widehat{\mathfrak{A}}_1, [-,-,-]_{\widehat{\mathfrak{A}}_1}, \widehat{\mathrm{d}}_1) \to (\widehat{\mathfrak{A}}_2, [-,-,-]_{\widehat{\mathfrak{A}}_2}, \widehat{\mathrm{d}}_2)$ *such that the following diagram is commutative:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathfrak{M}, \mathrm{d}_{\mathfrak{M}}) & \xrightarrow{i_1} & (\widehat{\mathfrak{A}}_1, \widehat{\mathrm{d}}_1) & \xrightarrow{p_1} & (\mathfrak{A}, \mathrm{d}) & \longrightarrow & 0\\
 & & \| & & \eta\big\downarrow & & \| & & \\
0 & \longrightarrow & (\mathfrak{M}, \mathrm{d}_{\mathfrak{M}}) & \xrightarrow{i_2} & (\widehat{\mathfrak{A}}_2, \widehat{\mathrm{d}}_2) & \xrightarrow{p_2} & (\mathfrak{A}, \mathrm{d}) & \longrightarrow & 0.
\end{array}
$$

Now, we are ready to classify abelian extensions of a MD$_\lambda$3-LieA.

Theorem 3. *There is a one-to-one correspondence between equivalence classes of abelian extensions of a MD*$_\lambda$*3-LieA* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *by* $(\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}})$ *and the second cohomology group* $\mathcal{H}^2_{\mathrm{md3Lie}^{\lambda}}(\mathfrak{A}, \mathfrak{M})$ *of* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *with coefficients in the representation* $(\mathfrak{M}, \varrho, \mathrm{d}_{\mathfrak{M}})$.

Proof. Assume that $(\widehat{\mathfrak{A}}_1, [-,-,-]_{\widehat{\mathfrak{A}}_1}, \widehat{\mathrm{d}}_1)$ and $(\widehat{\mathfrak{A}}_2, [-,-,-]_{\widehat{\mathfrak{A}}_2}, \widehat{\mathrm{d}}_2)$ are two equivalent abelian extensions of $(\mathfrak{A}, [-,-,-], \mathrm{d})$ by $(\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}})$ with the associated isomorphism $\eta : (\widehat{\mathfrak{A}}_1, [-,-,-]_{\widehat{\mathfrak{A}}_1}, \widehat{\mathrm{d}}_1) \to (\widehat{\mathfrak{A}}_2, [-,-,-]_{\widehat{\mathfrak{A}}_2}, \widehat{\mathrm{d}}_2)$. Let s_1 be a section of $(\widehat{\mathfrak{A}}_1, [-,-,-]_{\widehat{\mathfrak{A}}_1}, \widehat{\mathrm{d}}_1)$. As $p_2 \circ \eta = p_1$, we have

$$p_2 \circ (\eta \circ s_1) = p_1 \circ s_1 = \mathrm{id}_{\mathfrak{A}}.$$

That is, $\eta \circ s_1$ is a section of $(\widehat{\mathfrak{A}}_2, [-,-,-]_{\widehat{\mathfrak{A}}_2}, \widehat{\mathrm{d}}_2)$. Denote $s_2 := \eta \circ s_1$. Since η is an isomorphism of MD$_\lambda$3-LieAs such that $\eta|_{\mathfrak{M}} = \mathrm{id}_{\mathfrak{M}}$, we obtain

$$\begin{aligned} v_2(a_1, a_2, a_3) &= [s_2(a_1), s_2(a_2), s_2(a_3)]_{\widehat{\mathfrak{A}}_2} - s_2([a_1, a_2, a_3]) \\ &= [\eta(s_1(a_1)), \eta(s_1(a_2)), \eta(s_1(a_3))]_{\widehat{\mathfrak{A}}_2} - \eta(s_1([a_1, a_2, a_3])) \\ &= \eta\big([s_1(a_1), s_1(a_2), s_1(a_3)]_{\widehat{\mathfrak{A}}_1} - s_1([a_1, a_2, a_3])\big) \\ &= \eta(v_1(a_1, a_2, a_3)) \\ &= v_1(a_1, a_2, a_3) \end{aligned}$$

and

$$\begin{aligned} \mu_2(a_1) &= \widehat{\mathrm{d}}_2(s_2(a_1)) - s_2(\mathrm{d}(a_1)) = \widehat{\mathrm{d}}_2\big(\eta(s_1(a_1))\big) - \eta\big(s_1(\mathrm{d}(a_1))\big) \\ &= \eta\big(\widehat{\mathrm{d}}_1(s_1(a_1)) - s_1(\mathrm{d}(a_1))\big) \\ &= \eta(\mu_1(a_1)) \\ &= \mu_1(a_1). \end{aligned}$$

Hence, all equivalent abelian extensions give rise to the same element in $\mathcal{H}^2_{\mathrm{md3Lie}^\lambda}(\mathfrak{A}, \mathfrak{M})$.

Conversely, suppose that $[(f_1, g_1)] = [(f_2, g_2)] \in \mathcal{H}^2_{\mathrm{md3Lie}^\lambda}(\mathfrak{A}, \mathfrak{M})$, and we can construct two abelian extensions $0 \to (\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}}) \hookrightarrow (\mathfrak{A} \oplus \mathfrak{M}, [-,-,-]_{\rho f_1}, \mathrm{d}_{g_1}) \xrightarrow{p_1} (\mathfrak{A}, [-,-,-], \mathrm{d}) \to 0$ and $0 \to (\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}}) \hookrightarrow (\mathfrak{A} \oplus \mathfrak{M}, [-,-,-]_{\rho f_2}, \mathrm{d}_{g_2}) \xrightarrow{p_2} (\mathfrak{A}, [-,-,-], \mathrm{d}) \to 0$ via Equations (21) and (22). Then, there exists a linear map $\iota : \mathfrak{A} \to \mathfrak{M}$ such that

$$(f_2, g_2) = (f_1, g_1) + \partial(\iota).$$

Define linear map $\eta_\iota : \mathfrak{A} \oplus \mathfrak{M} \to \mathfrak{A} \oplus \mathfrak{M}$ by $\eta_\iota(a_1 + u_1) := a_1 + \iota(a_1) + u_1$, $a_1 \in \mathfrak{A}, u_1 \in \mathfrak{M}$. Then, η_ι is an isomorphism of these two abelian extensions $(\mathfrak{A} \oplus \mathfrak{M}, [-,-,-]_{\rho f_1}, \mathrm{d}_{g_1})$ and $(\mathfrak{A} \oplus \mathfrak{M}, [-,-,-]_{\rho f_2}, \mathrm{d}_{g_2})$. □

Remark 4. *In particular, any vector space* $\mathfrak{M}$ *with linear transformation* $\mathrm{d}_{\mathfrak{M}}$ *can serve as a trivial representation of* $(\mathfrak{A}, [-,-,-], \mathrm{d})$. *In this situation, central extensions of* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *by* $(\mathfrak{M}, [-,-,-]_{\mathfrak{M}}, \mathrm{d}_{\mathfrak{M}})$ *are classified by the second cohomology group* $\mathcal{H}^2_{\mathrm{md3Lie}^\lambda}(\mathfrak{A}, \mathfrak{M})$ *of* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *with the coefficient in the trivial representation* $(\mathfrak{M}, \rho = 0, \mathrm{d}_{\mathfrak{M}})$.

5. T^*-Extensions of MD$_\lambda$3-LieAs

The T^*-extension of a 3-Lie algebra was studied in [11]. In this section, we consider T^*-extensions of MD$_\lambda$3-LieAs by the second cohomology groups with the coefficient in a coadjoint representation.

Let $(\mathfrak{A}, [-,-,-], \mathrm{d})$ be a MD$_\lambda$3-LieA and $\mathfrak{A}^*$ be the dual space of $\mathfrak{A}$. By Example 6, $(\mathfrak{A}^*; ad^*, -\mathrm{d}^*)$ is a coadjoint representation of $(\mathfrak{A}, [-,-,-], \mathrm{d})$. Suppose that $(f, g) \in \mathcal{C}^2_{\mathrm{md3Lie}^\lambda}(\mathfrak{A}, \mathfrak{A}^*)$. Define a trilinear map $[-,-,-]_f : \wedge^3(\mathfrak{A} \oplus \mathfrak{A}^*) \to \mathfrak{A} \oplus \mathfrak{A}^*$ and a linear map $\mathrm{d}_g : \mathfrak{A} \oplus \mathfrak{A}^* \to \mathfrak{A} \oplus \mathfrak{A}^*$, respectively, by

$$\begin{aligned} &[a_1 + \alpha_1, a_2 + \alpha_2, a_3 + \alpha_3]_f \\ &= [a_1, a_2, a_3] + ad^*(a_2, a_3)\alpha_1 + ad^*(a_3, a_1)\alpha_2 + ad^*(a_1, a_2)\alpha_3 + f(a_1, a_2, a_3), \quad (27) \\ &\mathrm{d}_g(a_1 + \alpha_1) = \mathrm{d}(a_1) - \mathrm{d}^*(\alpha_1) + g(a_1), \ \forall a_1, a_2, a_3 \in \mathfrak{A}, \ \alpha_1, \alpha_2, \alpha_3 \in \mathfrak{A}^*. \quad (28) \end{aligned}$$

Similar to Proposition 7, we have the following result.

Proposition 9. *With the above notations,* $(\mathfrak{A} \oplus \mathfrak{A}^*, [-,-,-]_f, \mathrm{d}_g)$ *is a MD$_\lambda$3-LieA if and only if* (f, g) *is a 2-cocycle in the cohomology of the MD$_\lambda$3-LieA* $(\mathfrak{A}, [-,-,-], \mathrm{d})$ *with the coefficient in the representation* $(\mathfrak{A}^*; ad^*, -\mathrm{d}^*)$.

Definition 11. *The MD$_\lambda$3-LieA $(\mathfrak{A}\oplus\mathfrak{A}^*, [-,-,-]_f, \mathrm{d}_g)$ is called the T^*-extension of the MD$_\lambda$3-LieA $(\mathfrak{A}, [-,-,-], \mathrm{d})$. Denote the T^*-extension by $T^*_{(f,g)}(\mathfrak{A}) = (T^*(\mathfrak{A}) = \mathfrak{A}\oplus\mathfrak{A}^*, [-,-,-]_f, \mathrm{d}_g)$.*

Definition 12. *Let $(\mathfrak{A}, [-,-,-], \mathrm{d})$ be a MD$_\lambda$3-LieA. $(\mathfrak{A}, [-,-,-], \mathrm{d})$ is said to be metrised if it has a non-degenerate symmetric bilinear form $\varpi_{\mathfrak{A}} \in \otimes^2\mathfrak{A}^*$ which satisfies*

$$\varpi_{\mathfrak{A}}([a_1,a_2,a_3],a_4) + \varpi_{\mathfrak{A}}(a_3,[a_1,a_2,a_4]) = 0, \tag{29}$$

$$\varpi_{\mathfrak{A}}(\mathrm{d}(a_1),a_2) + \varpi_{\mathfrak{A}}(a_1,\mathrm{d}(a_2)) = 0, \ \ \forall a_1,a_2,a_3,a_4 \in \mathfrak{A}. \tag{30}$$

We may also say that $(\mathfrak{A}, [-,-,-], \mathrm{d}, \varpi_{\mathfrak{A}})$ is a metric MD$_\lambda$3-LieA.

Define a bilinear map $\varpi : \wedge^2 T^*(\mathfrak{A}) \to \mathfrak{A}$ by

$$\varpi(a_1+\alpha_1, a_2+\alpha_2) = \alpha_1(a_2) + \alpha_2(a_1), \ \ \forall a_1,a_2 \in \mathfrak{A}, \alpha_1,\alpha_2 \in \mathfrak{A}^* \tag{31}$$

Proposition 10. *With the above notations, $(T^*_{(f,g)}(\mathfrak{A}), \varpi)$ is a metric MD$_\lambda$3-LieA if and only if*

$$f(a_1,a_2,a_3)(a_4) + f(a_1,a_2,a_4)(a_3) = 0, \ \ g(a_1)(a_2) + g(a_2)(a_1) = 0, \ \ \forall a_1,a_2,a_3,a_4 \in \mathfrak{A}.$$

Proof. For any $a_1,a_2,a_3,a_4 \in \mathfrak{A}$, $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \mathfrak{A}^*$, using Equations (6), (27)–(31) we have

$$\begin{aligned}
&\varpi([a_1+\alpha_1, a_2+\alpha_2, a_3+\alpha_3]_f, a_4+\alpha_4) + \varpi(a_3+\alpha_3, [a_1+\alpha_1, a_2+\alpha_2, a_4+\alpha_4]_f) \\
=&\varpi([a_1,a_2,a_3] + ad^*(a_2,a_3)\alpha_1 + ad^*(a_3,a_1)\alpha_2 + ad^*(a_1,a_2)\alpha_3 + f(a_1,a_2,a_3), a_4+\alpha_4) \\
&+ \varpi(a_3+\alpha_3, [a_1,a_2,a_4] + ad^*(a_2,a_4)\alpha_1 + ad^*(a_4,a_1)\alpha_2 + ad^*(a_1,a_2)\alpha_4 + f(a_1,a_2,a_4)) \\
=&\alpha_4([a_1,a_2,a_3]) + ad^*(a_2,a_3)\alpha_1(a_4) + ad^*(a_3,a_1)\alpha_2(a_4) + ad^*(a_1,a_2)\alpha_3(a_4) + f(a_1,a_2,a_3)(a_4) \\
&+ \alpha_3([a_1,a_2,a_4]) + ad^*(a_2,a_4)\alpha_1(a_3) + ad^*(a_4,a_1)\alpha_2(a_3) + ad^*(a_1,a_2)\alpha_4(a_3) + f(a_1,a_2,a_4)(a_3) \\
=&\alpha_4([a_1,a_2,a_3]) - \alpha_1([a_2,a_3,a_4] - \alpha_2([a_3,a_1,a_4]) - \alpha_3([a_1,a_2,a_4]) + f(a_1,a_2,a_3)(a_4) \\
&+ \alpha_3([a_1,a_2,a_4]) - \alpha_1([a_2,a_4,a_3]) - \alpha_2([a_4,a_1,a_3]) - \alpha_4(a_1,a_2,a_3) + f(a_1,a_2,a_4)(a_3) \\
=&f(a_1,a_2,a_3)(a_4) + f(a_1,a_2,a_4)(a_3) \\
=&0, \\
&\varpi(\mathrm{d}_g(a_1+\alpha_1), a_2+\alpha_2) + \varpi(a_1+\alpha_1, \mathrm{d}_g(a_2+\alpha_2)) \\
=&\varpi(\mathrm{d}(a_1) - \mathrm{d}^*(\alpha_1) + g(a_1), a_2+\alpha_2) + \varpi(a_1+\alpha_1, \mathrm{d}(a_2) - \mathrm{d}^*(\alpha_2) + g(a_2)) \\
=&-\mathrm{d}^*(\alpha_1)(a_2) + g(a_1)(a_2) + \alpha_2(\mathrm{d}(a_1)) + \alpha_1(\mathrm{d}(a_2)) - \mathrm{d}^*(\alpha_2)(a_1) + g(a_2)(a_1) \\
=&-\alpha_1(\mathrm{d}(a_2)) + g(a_1)(a_2) + \alpha_2(\mathrm{d}(a_1)) + \alpha_1(\mathrm{d}(a_2)) - \alpha_2(\mathrm{d}(a_1)) + g(a_2)(a_1) \\
=&g(a_1)(a_2) + g(a_2)(a_1) \\
=&0.
\end{aligned}$$

Thus, we obtain the result. □

Let $(\mathfrak{A}, [-,-,-], \mathrm{d}, \varpi_{\mathfrak{A}})$ be a metric MD$_\lambda$3-LieA, then $\varpi_{\mathfrak{A}}$ induces an isomorphism $\varpi^\natural_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}^*$ defined by

$$\langle \varpi^\natural_{\mathfrak{A}}(a_1), a_2\rangle = \varpi_{\mathfrak{A}}(a_1,a_2), \forall a_1,a_2 \in \mathfrak{A}.$$

Proposition 11. *With the above notations, $\varpi^\natural_{\mathfrak{A}}$ is an isomorphism from the adjoint representation $(\mathfrak{A}; ad, \mathrm{d})$ to the coadjoint representation $(\mathfrak{A}^*; ad^*, -\mathrm{d}^*)$.*

Proof. For any $a_1,a_2,a_3,a_4 \in \mathfrak{A}$, by Equations (6) and (29), we have

$$\begin{aligned}
\langle \varpi^\natural_{\mathfrak{A}}(ad(a_1,a_2)a_3), a_4\rangle &= \varpi_{\mathfrak{A}}([a_1,a_2,a_3],a_4) = -\varpi_{\mathfrak{A}}(a_3,[a_1,a_2,a_4]) \\
&= -\langle \varpi^\natural_{\mathfrak{A}}(a_3), [a_1,a_2,a_4]\rangle = -\langle \varpi^\natural_{\mathfrak{A}}(a_3), ad(a_1,a_2)a_4\rangle \\
&= \langle (ad(a_1,a_2))^*\varpi^\natural_{\mathfrak{A}}(a_3), a_4\rangle,
\end{aligned}$$

which implies that $\varpi_{\mathfrak{A}}^{\natural}(ad(a_1,a_2)a_3) = (ad(a_1,a_2))^*\varpi_{\mathfrak{A}}^{\natural}(a_3)$.

In addition, for any $a_1, a_2 \in \mathfrak{A}$, by Equations (6) and (30), we have

$$\begin{aligned}\langle\varpi_{\mathfrak{A}}^{\natural}(\mathrm{d}(a_1)), a_2\rangle &= \varpi_{\mathfrak{A}}(\mathrm{d}(a_1), a_2) = -\varpi_{\mathfrak{A}}(a_1, \mathrm{d}(a_2)) = -\langle\varpi_{\mathfrak{A}}^{\natural}(a_1), \mathrm{d}(a_2)\rangle \\ &= -\langle \mathrm{d}^*\varpi_{\mathfrak{A}}^{\natural}(a_1), a_2\rangle,\end{aligned}$$

which implies that $\varpi_{\mathfrak{A}}^{\natural}(\mathrm{d}(a_1)) = -\mathrm{d}^*\varpi_{\mathfrak{A}}^{\natural}(a_1)$. Therefore, $\varpi_{\mathfrak{A}}^{\natural}$ is an isomorphism from $(\mathfrak{A}; ad, \mathrm{d})$ to $(\mathfrak{A}^*; ad^*, -\mathrm{d}^*)$. □

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