Old Models Never Die: The Revival of the Skyrme Model

The Skyrme–Witten model for baryons is discussed as a form of effective Lagrangian field theory for QCD at small momenta and large N_c (number of colors). The model for two light quark flavors—the original Skyrme model—is reviewed in some detail, leading to predictions of static and dynamic properties of the nucleon and delta. The generalization to three light flavors, which is essential in determining the nucleon quantum numbers, is discussed.

INTRODUCTION

In the fall of 1960, T. H. R. Skyrme wrote a paper on "a non-linear field theory tentatively describing the strongly interacting elementary particles."[1] Strongly interacting particles are now believed to be described by quantum chromodynamics (QCD)—a theory formulated more than a decade after Skyrme's paper—and the concept of an elementary particle has itself undergone great change. Nonetheless, the last five years have seen an upsurge of interest in Skyrme's work, which is now considered to be an effective (Lagrangian) field theory, in some sense modelling QCD in the hitherto inaccessible confinement regime. The desirability of such a model is due to the absence of any calculational scheme, *derived from QCD* (and respecting the symmetries of QCD), which can predict observed hadronic phenomena at low excitation energies, where quarks and gluons are confined in colorless hadrons. (Such a calculational scheme should in principle predict all of nuclear physics.)

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It must be admitted right off that the Skyrme model is as yet far from being such a scheme, and, from a practical point of view, has to be considered principally as a useful alternative to the constituent-quark and bag models. It is conceivable, however, that with time, it may evolve into such a scheme—even at this stage of its development the Skyrme model can be applied to situations where other models cannot.[2] An attractive possibility is that the quark model and the Skyrme model will prove to be for particle physics what the shell model and the collective model are for nuclear physics.

THE SKYRME MODEL AND THE LARGE N_c LIMIT

The degrees of freedom of the Skyrme model are an isospin triplet of pion fields corresponding to the (approximate) Goldstone bosons in the spectrum of QCD. Skyrme made the ingenious guess that the solitons appearing in the classical version of the model would, upon quantization, give rise to fermions (which he identified with nucleons). A demonstration that the Skyrme soliton ("skyrmion") could be a fermion was given by Finkelstein and Rubinstein[3] and by Williams.[3] The recent revival of the Skyrme model was to a large extent due to the work of the Syracuse and Princeton groups.[4] To explore the connection between the Skyrme model and QCD let us review the role of the number of colors.

QCD can be formulated for any number of colors, N_c, independently of the number of quark flavors. The resulting theory is expressed in terms of $N_c^2 - 1$ gauge gluons coupled in an SU(N_c) gauge-invariant way to quarks with N_c colors. It was pointed out by 't Hooft[5] that in the limit $N_c \to \infty$, with $g^2 N_c$ fixed (g being the gauge coupling constant), the theory simplifies considerably: all mesons constructed out of a quark and an antiquark become stable noninteracting particles (lifetimes vary as N_c^{-1} with meson–meson scattering cross sections behaving as N_c^{-2}); meson masses behave as $(N_c)^0$; Zweig's rule becomes exact; and there are no exotic states.

The fact that this hypothetical world with its infinite number of different stable mesons shows some resemblance to the physical world with $N_c = 3$ motivates one to explore the consequences of taking N_c large but not infinite. In the low energy regime it should be possible to describe the large-N_c world as a weakly interacting effective field theory of colorless low lying mesons. (Such a scheme would allow a discussion of glue balls as well.)

Baryons pose a problem for the large-N_c approach. While mesons are simply quark–antiquark for any number of colors, we must construct baryons out of N_c quarks to make a colorless resultant. It is thus probably not too surprising[7] that baryon masses diverge as N_c. In addition Witten[7] showed that baryon–baryon scattering amplitudes diverge as N_c, while meson–baryon scattering amplitudes have an $(N_c)^0$ behavior. In the large-N_c limit, baryons become extended objects with sizes and shapes that behave as N_c^0. Thus in the large-N_c world, mesons and baryons are very different objects. It is not surprising that they are treated differently in the Skyrme model.

Weakly coupled field theories sometimes possess, aside from the usual particles, additional states (solitons) whose masses diverge as the inverse of the coupling, a fact that (among other things) led Pak and Tze[6] to propose that baryons be considered as *chiral* solitons. Noting that $1/N_c$ plays the role of the coupling constant for large-N_c QCD, Witten related the baryon mass divergence, $\sim N_c$, to soliton behavior—as indeed N_c is the inverse of the coupling: $1/(1/N_c)$. Thus Witten[7] was led to suggest that baryons could be solitons in the weakly interacting meson theory describing the large-N_c world.

The simplest model of this kind would incorporate only the lightest bosons, the three pions for two flavors (u and d quarks). It would be nice if one could, starting with QCD for SU(N_c), derive the large-N_c model; however, this has not yet been accomplished.* Lacking a valid derivation, one constructs the effective field theory on the basis of symmetry principles. As a result, the effective theory contains constants which should, in principle, be calculable from large-N_c QCD.

TWO LIGHT QUARK FLAVORS

The phenomenology of the standard model reveals two very light quark flavors, u and d, with masses $m_u \cong 4.5$ MeV and $m_d \cong 8$ MeV. The smallness of these masses on the strong interaction scale

*Although such a procedure can be carried out in 1 + 1 dimensions.[8]

(~300 MeV) makes QCD with $m_u = m_d = 0$ a very good approximate theory. This approximate theory has a global (non-gauge) classical U(2) × U(2) symmetry group broken by the chiral anomaly to [U(2) × U(2)]/U(1). (The axial U(1) invariant subgroup is given by the set of pairs $(F,F^\dagger)$ with F a complex number of modulus 1 multiplying the 2 × 2 unit matrix.) It is believed that QCD with massless u and d quarks has a ground state which is not invariant under the full quantum mechanical [U(2) × U(2)]/U(1) symmetry group but only under its U(2) diagonal subgroup formed by the pairs (M,M), where M is a matrix in U(2). The theory thus has a three-dimensional vacuum-degeneracy manifold {[U(2) × U(2)]/U(1)}/U(2) which is reflected in the spectrum by the appearance of three massless pions—the Goldstone bosons.

To construct an effective theory for the Goldstone bosons, one defines the degrees of freedom as mapping a point in space-time onto a point in the vacuum-degeneracy manifold. The manifold {[U(2) × U(2)]/U(1)}/U(2) is topologically (but not algebraically) SU(2) $\simeq S^3$ (the unit sphere in Euclidean four-dimensional space). Thus our dynamical degrees of freedom are SU(2) matrices $\mathrm{U}(x,t)$ defined at each space-time point (x,t). Any parametrization of SU(2) will involve three independent parameters which may be related (in a representation-dependent way) to the three pionic fields, $\pi(x,t)$. Thus, for example, using the Cayley representation one has:

$$\mathrm{U}(x,t) = (\mathbf{1} + iF_\pi^{-1}\pi(x,t)\cdot\tau)(\mathbf{1} - iF_\pi^{-1}(x,t)\cdot\tau)^{-1} \qquad (1)$$

where F_π is a constant. (We shall later identify F_π as the pion decay constant, so that experimentally $F_\pi \cong 185$ MeV.) The fact that the Goldstone bosons are pseudo-scalars and not scalars leads to the following behavior of $\mathrm{U}(x,t)$ under parity:

$$\text{Parity: } \mathrm{U}(x,t) \longrightarrow \mathrm{U}(-x,t)^\dagger. \qquad (2)$$

For matrices $\mathrm{U}(x,t)$ that are close to the unit matrix $\mathbf{1}$, one has the expansion:

$$\mathrm{U}(x,t) = \mathbf{1} + 2i\,\pi(x,t)\cdot\tau/F_\pi + \cdots, \qquad (3)$$

which identifies the pion field in the small amplitude regime. There is no such unique identification valid for large pion field amplitudes, but different parametrizations of $U(x,t)$ give rise to identical S-matrix elements. It is desirable to carry out the calculations as far as possible, in a parametrization-independent way.

An element (A_L,A_R) of $U(2) \times U(2)$ operates on $U(x,t)$ by the action:

$$U(x,t) \rightarrow A_L U(x,t) A_R^\dagger. \tag{4}$$

One sees that the vector $U(1)$ subgroup of $U(2) \times U(2)$ formed by the pairs (F,F) does not change $U(x,t)$ at all. This is actually to be expected since in QCD the vector $U(1)$ subgroup is generated by the baryon number carried by the massless quarks, while in the effective theory the pions do not carry baryon number. We shall see in the following that baryon number conservation becomes a topological conservation law in the effective theory. Since the vector $U(1)$ subgroup is ineffective while the axial $U(1)$ subgroup is forbidden (by the anomaly), we may restrict the pair (A_L,A_R) in the transformation law (4) to belong to $SU(2) \times SU(2)$.

To construct a Lagrangian density for the theory, one constructs out of $U(x,t)$ and its first partial derivatives combinations which are both Lorentz invariant and $SU(2) \times SU(2)$ invariant. Since this is to be a low energy theory, one tries to introduce as few derivatives as possible. There is no (nontrivial) term without derivatives since $U(x,t)$ is unimodular ($\det U = 1$); Lorentz invariance eliminates terms with an odd number of derivatives. There is a unique term with two derivatives, three possible terms with four derivatives, etc.

It is convenient to define the following matrix vectors:

$$R_\mu = -i\, U^\dagger(\partial_\mu U) \quad \text{and} \quad L_\mu = -i(\partial_\mu U)U^\dagger. \tag{5}$$

The unique term with two derivatives contributes to the action a term:

$$S = \frac{F_\pi^2}{16} \int d^4x \operatorname{tr}(R_\mu R^\mu) = \frac{F_\pi^2}{16} \int d^4x \operatorname{tr}(L_\mu L^\mu). \tag{6}$$

The constant $F_\pi^2/16$ is chosen so that in the small amplitude approximation the action reduces to that of three free massless pions:

$$S = \frac{1}{2}\int d^4x(\partial_\mu \boldsymbol{\pi}\cdot\partial^\mu \boldsymbol{\pi}) + \cdots . \tag{7}$$

The action (6) is just the action used in the mid 1960's for chiral dynamics. (In fact, substituting (1) in (6) yields Weinberg's action.[9]) At this point the only remnant of the large-N_c argument is the observation that F_π^2 grows linearly with N_c. Since in a functional integral approach to quantization $\hbar$ appears in the combination $\hbar/F_\pi^2$, this remark shows that the large-N_c approximation is formally equivalent to the semiclassical approximation.

The energy momentum tensor corresponding to the action (6) is

$$T_{\mu\nu} = 2\left.\frac{\delta S}{\delta g_{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}} = \frac{F_\pi^2}{16}\,\mathrm{tr}(2R_\mu R_\nu - \eta_{\mu\nu}R_\alpha R^\alpha). \tag{8}$$

In particular the energy integral is given by

$$E[\mathrm{U}] = \frac{F_\pi^2}{16}\int d^3x\ \mathrm{tr}(R_0R_0 + R_iR_i), \tag{9}$$

which is manifestly non-negative.

We now restrict attention to configurations $\mathrm{U}(x,t)$ with finite energy. For the energy functional to be convergent, all four derivatives of U must vanish as x tends to infinity at fixed t. Thus $\mathrm{U}(x,t)$ must tend to a space-time independent matrix U_0. Expressed differently, requiring finite energy implies that the point at spatial infinity maps to a fixed element in the field manifold, U_0, which shows that space ($\mathbf{R}^3$) has been compactified to $\mathbf{R}^3 \cup (\infty)$, which is topologically S^3. This fact allows the use of homotopy theory to classify all finite energy configurations into a countable infinity of homotopy classes,[10] each characterized by an integer:

$$\pi_3(S^3) = Z. \tag{10}$$

Since a Hamiltonian time development of a given initial configuration $U(x,t_0)$ is continuous, the homotopy class does not change with time and is therefore a constant of the motion. This constant is given by

$$B[U] \equiv \frac{-i}{24\pi^2}\epsilon_{ijk}\int d^3x \operatorname{tr}(R_iR_jR_k)$$

$$= \frac{-i}{24\pi^2}\epsilon_{ijk}\int d^3x \operatorname{tr}(L_iL_jL_k). \quad (11)$$

It can be shown that this expression is an integer that measures the number of times the image of space covers SU(2). It is clear that B is the charge of the following current density:

$$B_\mu = \frac{-i}{24\pi^2}\epsilon_{\mu\nu\lambda\sigma}\operatorname{tr}(R^\nu R^\lambda R^\sigma). \quad (12)$$

The current B_μ is conserved irrespective of the equations of motion for $U(x,t)$. Such conservation laws—topological conservation laws—reflect the boundary conditions that the fields have to satisfy rather than the dynamical equations of motion. The constant of motion B was identified by Skyrme as the baryon number. We will show later that B is indeed the same baryon number that is defined in QCD.

The homotopy classification of field configurations gives a certain simplification, as it allows some problems to be treated separately for each connected part of configuration space. An example is the determination of minimum energy configurations in each sector labelled by baryon number B.

The $B = 0$ Sector

The expression for the energy certainly attains its minimum value (zero) on configurations which are independent of space and time. The demand that $U(x,t)$ tends to a fixed U_0 as $|x| \to \infty$ chooses out of all space-time independent matrices the solution:

$$U(x,t) = U_0. \quad (13)$$

It is also clear that the baryon number integral vanishes for the solution (13); that is:

$$B[\mathrm{U}_0] = 0. \tag{14}$$

We shall refer to U_0 as a "classical-vacuum" solution. Because of the necessity to fulfill the boundary conditions at spatial infinity, this solution is not invariant under the whole of SU(2) × SU(2) but only under an SU(2) subgroup, the isospin symmetry group. Other SU(2) × SU(2) transformations take one classical vacuum to a mathematically different but physically equivalent classical vacuum. Since SU(2) × SU(2) is a symmetry of the action, the choice of a particular classical vacuum is a matter of convenience. We shall choose our standard classical solution vacuum to be the unit matrix:

$$\mathrm{U}_0(x,t) = \mathbf{1}. \tag{15}$$

With this choice of boundary condition, the isospin subgroup is the diagonal SU(2) subgroup.

The $B = 1$ Sector

A solution in the $B = 1$ sector must have nontrivial spatial dependence in order that the baryon number and energy not vanish; a time-independent (static) solution, if it exists, should have minimum energy. Skyrme faced the problem, however, that the energy functional (9) does not have a static solution with $B \neq 0$. To see this, assume $\mathrm{U}_1(x)$ is a static solution and therefore an extremum of $E[\mathrm{U}]$. Calculate $E[\mathrm{U}]$ along the curve $\mathrm{U}_\lambda(x)$ in function space passing through $\mathrm{U}_1(x)$:

$$\mathrm{U}_\lambda(x) = \mathrm{U}_1(\lambda x). \tag{16}$$

This gives

$$E[\mathrm{U}_\lambda] = (1/\lambda)E[\mathrm{U}_1] \tag{17}$$

which for $E[\mathrm{U}_1] > 0$ is not an extremum at $\lambda = 1$.

Thus if we want the effective theory to have a stable classical solution with $B = 1$, we are forced to go beyond the minimal chiral action (6). Of the possible term with four derivatives, Skyrme chose the only one that is at most quadratic in time derivatives. This makes quantization simpler (although there seems to be no compelling reason for this choice).

With this choice the action takes the form:

$$S = \frac{F_\pi^2}{16} \int d^4x \, \mathrm{tr}(R_\mu R^\mu) - \frac{1}{32e^2} \int d^4x \, \mathrm{tr}(R_{\mu\nu} R^{\mu\nu}), \tag{18}$$

where e is a dimensionless constant (Skyrme expected e to be 2π), while the antisymmetric tensor of matrices $R_{\mu\nu}$ is given by

$$R_{\mu\nu} = (-i)[R_\mu, R_\nu] = \partial_\mu R_\nu - \partial_\nu R_\mu. \tag{19}$$

The large-N_c argument shows that $e^2 \propto N_c^{-1}$.

Given the action (18) it is straightforward to show that the Euler–Lagrange equations of motion are conservation equations for the chiral currents, and to obtain the expressions for the energy-momentum tensor and the generators of both the Poincaré group and $SU(2) \times SU(2)$. The expressions for the topological current (12) and the corresponding charge (11) are of course independent of the dynamics. From the energy momentum tensor it follows that the energy functional for a static configuration is

$$E[U(x)] = \frac{F_\pi^2}{16} \int d^3x \, \mathrm{tr}(R_i R_i) + \frac{1}{32e^2} \int d^3x \, \mathrm{tr}(R_{ij} R_{ij}), \tag{20}$$

while the momentum, the Lorentz-boost, and the angular momentum integrals all vanish. (The mixed space-time components T_{0i} vanish for a static configuration.) For configurations with $B \neq 0$, this functional fulfills the following Bogomolnii inequality[10]:

$$E[U] > 3\pi^2 \left(\frac{F_\pi}{e}\right) B[U]. \tag{21}$$

This inequality is interesting because it is compatible with energies linear in the baryon number—the "zeroth order" empirical mass formula for nuclei.

To find an energy minimizing solution in the $B = 1$ sector, one notices that it pays to make the solution as slowly varying as possible. The relevant mathematical concept here is equivariance, which we now define. Given a mapping U from a manifold M to a manifold M', one says that it is equivariant with respect to an abstract group G—which acts on M and M'—if

$$(\mathrm{G} \circ \mathrm{U})\,(x) = \mathrm{U}(\mathrm{G}x), \quad x \in M,\, \mathrm{U}(x) \in M'. \tag{22}$$

To find the equivariance group, consider the degeneracy manifold of the ground state in a $B \neq 0$ sector. A static classical solution at rest with its center of mass at a given point may be acted upon by the Poincaré group, as well as the internal symmetry group $SU(2) \times SU(2)$, producing another solution. For static solutions, time translations make no change, while space translations change the center of mass location. Lorentz boosts change the energy so that only the spatial rotation group SO(3) does not change the energy, the center of mass location, or the (vanishing) momentum. Out of the $SU(2) \times SU(2)$ group only the diagonal subgroup SU(2) does not change the boundary conditions at spatial infinity. We are thus left with the group $SU(2)_I \times SO(3)_J$. An element (A,R) of the group operates on a solution $\mathrm{U}(x)$ as follows:

$$\mathrm{U}(x) \rightarrow A\mathrm{U}\,(Rx)\,A^{\dagger}. \tag{23}$$

In particular it is easy to see that a solution invariant under $SU(2)_I$ must have $B = 0$, and similarly invariance under $SO(3)_J$ forces $B = 0$. Thus the best we can do for $B = 1$ is to demand invariance under the diagonal subgroup $SU(2)_K$ formed by the pairs $(A,R(A))$, where $R(A)$ is the SO(3) matrix representing the SU(2) matrix A. In other words, $\mathrm{U}(x)$ should be equivariant under $\mathrm{G}(=SU(2))$ which acts on space through rotations $x \rightarrow R(A)x$ and simultaneously on the SU(2) isospin manifold by conjugation: $\mathrm{U} \rightarrow A\mathrm{U}A^{\dagger}$.

Denotating the isospin generators by I and rotation generators by J, we see that equivariance implies that the combined generator $K = I + J$ commutes with an equivariant $\mathrm{U}(x)$. This linking together of two very disparate symmetries is curious, but not unexpected. Pauli and Dancoff, in treating pseudoscalar meson field theory by strong coupling techniques,[11] were led to introduce exactly the same composition of spin and isospin. Nuclear physicists are familiar with similar examples of broken symmetry in treating deformed nuclei (rotationally asymmetric intrinsic states). The quantum states are projections from these intrinsic states and do not themselves manifest the broken symmetry.

The equivariant $U(x)$ for the Skyrme model can be written as

$$U(x) = e^{if(r)\hat{r}\cdot\tau}, \tag{24}$$

where the function $f(r)$ is to be determined from the energy minimization condition. The expressions for the energy and the baryon number integrals are now

$$E = \frac{\pi F_\pi}{e}\int_0^\infty d\rho\left[\frac{1}{2}(\rho f')^2 + \sin^2 f + (\sin^2 f)(2f')^2 + 2\left(\frac{\sin^2 f}{\rho}\right)^2\right], \tag{25}$$

$$B = \frac{2}{\pi}\int_0^\infty (\sin^2 f) f'\, d\rho = \frac{1}{\pi}\left[f(0) - \frac{1}{2}\sin 2f(0)\right], \tag{26}$$

where ρ is a dimensionless radial variable given by $\rho = eF_\pi r$, and $f(\infty)$ is set to 0 so that the boundary condition is satisfied.

The variational equation for f is

$$(\rho^2 + 8\sin^2 f)f'' + 2\rho f' + 4\sin 2f(f')^2 - \sin 2f - (8\sin^3 f\cos f)/\rho^2 = 0. \tag{27}$$

The boundary conditions appropriate for a static solution in the $B = 1$ sector are $f(0) = \pi$, $f(\infty) = 0$. From Eq. (25) we see that the energy is linear in F_π/e and therefore behaves as N_c for large N_c. We also see that the equation for $f(\rho)$ does not involve any N_c- dependent quantities so that all shape and length characteristics of the solution are independent of N_c. Solving Eq. (27) numerically, Adkins *et al.*[12] found the energy $E_1 \cong 36.5\ F_\pi/e$ which, factoring out the bound (21), is

$$E_1 \cong 1.23\cdot(3\pi^2 F_\pi/e). \tag{28}$$

To obtain properties of the quantum states, one next needs to quantize the fluctuations about this solution. Before doing this, let us first consider the $B = 2$ sector.

The $B = 2$ Sector

The SU(2) equivariant solution Eq. (24) with $f(0) = 2\pi$ (for $B = 2$) does not give a local minimum of the energy integral in function space. In fact, for such a solution the corresponding energy is 2.98. Obviously the hope in dealing with the $B = 2$ sector is to describe the deuteron as a bound state of two $B = 1$ solutions, so that the total energy must be less than $2E_1$. By substituting in the energy integral a trial configuration of the product form

$$\mathrm{U}_2 \equiv \mathrm{U}_1(r - r_1)A\mathrm{U}_1(r + r_1)A^{\dagger} \tag{29}$$

and treating the matrix A and the distance r_1 as variational parameters, one finds a variational upper bound: $E_2 \leq (2E_1 - F_\pi/e)$ for $A = i\tau_2$ and $|r_1| = (eF_\pi)^{-1}$. It seems that a minimum energy static solution in the $B = 2$ sector does not possess the high symmetry that the $B = 1$ soliton possesses.[13] We will return to this point in the next section.

The problem of determining the nucleon–nucleon interaction from the Skyrme model has been considered in the $B = 2$ sector by Jackson *et al.*[13] From the spherical $B = 2$ solution one obtains a repulsive core of the order of 1 GeV, as noted above. Many other qualitative and even quantitative features of the nucleon–nucleon interaction emerge correctly. But recent work, reported by A. D. Jackson at the 1985 Paris European Physical Society meeting, is not as encouraging; the entire central potential arising in the Skyrme model has positive G-parity and behaves like the exchange of σ-mesons with the "wrong" sign! To overcome this difficulty, rather drastic and disturbing changes in the Lagrangian were proposed.

QUANTIZATION AND THE STATIC PROPERTIES OF THE NUCLEON AND DELTA

It is important to quantize the fluctuations about the classical solutions in order to determine the correct quantum numbers of the states. One starts by considering deviations from the classical solutions in the approximation of small oscillations. For the $B = 0$ sector the equations for small oscillations are just the Klein–Gordon equations for three free massless pions, leading to a quantum description of the pseudoscalar mesons. One may calculate the charged pion life time, thereby identifying the constant F_π as the pion decay constant. (Experimentally, $F_\pi = 186.2 \pm 0.4$ MeV.) One may also expand the Lagrangian in small oscillation eigenmodes and verify that the third-order terms are of order $N_c^{-1/2}$ while the fourth-order terms are of order N_c^{-1} so that indeed the pion–pion scattering cross section is of order N_c^{-2}.

In the $B = 1$ sector the situation is more complicated. Energy stability of the soliton guarantees the reality of the small oscillation frequencies, which are proportional to eF_π and therefore behave as $(N_c)^0$ for large N_c.

However, the nontrivial energy degeneracy manifold signals the existence of zero frequency small oscillation modes. It is important to treat the quantization of the zero modes in any approximation scheme, as they carry the symmetry information and quantum numbers. Starting from the equivariant $B = 1$ static solution (24), we can get any other solution on the energy degeneracy manifold $(\mathrm{SU}(2)_I \times \mathrm{SU}(2)_J)/\mathrm{SU}(2)_K$ by conjugation,

$$\mathrm{U}_1(x) \rightarrow A\ \mathrm{U}_1(x)\ A^\dagger, \tag{30}$$

and the infinitesimal conjugations give rise to zero modes.

A full treatment of the quantization should include both zero and nonzero modes. For the purpose of discussing low-lying excitations, however, it is enough to deal with the lowest lying modes. An extreme approximation, which we now describe, is to quantize only the zero modes. To this end one elevates the matrices $A(t)$ to the status of dynamical variables (collective coordinates familiar in nuclear rotational models). Substituting

$$\mathrm{U}_1(x,t) = A(t)\ \mathrm{U}_1(x)\ A(t)^{-1} \tag{31}$$

in the action (18) and performing the spatial integrations, we get

$$S = \int L\ dt, \tag{32}$$

with L—the effective Lagrangian for the dynamical variables A—given by

$$L = -E_1 + I\,\mathrm{Tr}(\partial_0 A \partial_0 A^{-1}), \tag{33}$$

where the moment of inertia, I, is given by Ref. 12:

$$I = \frac{2\pi}{3e^3 F_\pi} \int_0^\infty (\rho \sin F)^2 \left[1 + 4\left(F'^2 - \left(\frac{\sin F}{\rho}\right)^2 \right) \right] d\rho. \tag{34}$$

A numerical evaluation of the integral gives 50.9. We note in passing that I diverges as N_c for large N_c.

The quantization procedure is immediate, as it is formally equivalent to the problem of a free particle constrained to move on the energy degeneracy manifold S^3. At this point we can make the substitution (31) in all the classical observables and convert them to quantum operators. In particular, the generators of $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_J$ are now operators and, in view of the invariance of the classical soliton under $\mathrm{SU}(2)_K$, fulfill the operator relation $I^2 = J^2$. From the action of the symmetry group $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_J$ on the configuration $\mathrm{U}(x,t)$, it follows that for a matrix B in $\mathrm{SU}(2)_I$:

$$A \to BA, \tag{35}$$

while under a matrix C in $\mathrm{SU}(2)_J$:

$$A \to AC. \tag{36}$$

The quantum problem thus has $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_J$ invariance and one can diagonalize the effective Hamiltonian simultaneously with the three commuting operators $I^2 = J^2$, I_3, J_3. The effective Hamiltonian is

$$H = E_1 + \frac{1}{2I} J^2. \tag{37}$$

This Hamiltonian has a discrete spectrum with eigenvalues $E_1 + J(J+1)/2I$, where the allowed values of J are $0, 1/2, 1, 3/2, 2, \ldots$

and the degeneracies are $(2J + 1)^2$. The mass splittings in this rotational band diminish as $1/N_c$ for large-N_c.

The corresponding eigenfunctions are the rotation matrices $D^{(I)*}_{I_3,J_3}(A)$. Since by (31) the action of $A(t)$ on the classical solution is the adjoint action, the two matrices $A(t)$ and $-A(t)$ lead to the same solution and cannot be distinguished classically. Quantum mechanically this means that the wave functions must be related by a phase that can be made real. We thus face a choice: we can either quantize the system, demanding that all wave functions be the same for the two cases, or that the wave functions change sign. The first choice leads to bosonic states with $I = J = 0,1,2,\ldots$ and the second to fermionic states with $I = J = 1/2,3/2,5/2,\ldots$. To describe low lying baryons, one makes the second choice.

It is interesting to note that the spectrum (37) shows that we have recovered another old model: the strong coupling model of the 1950's.

Let us remark that since quantization about an SU(2) equivariant solution leads only to states with $I = J$, the deuteron ($I = 0, J = 1$) must necessarily be the result of quantizing about a different class of classical solutions.

We now identify the $I = J = 1/2$ states with the nucleon at 939 MeV and the $I = J = 3/2$ states with the Δ resonance at 1232 MeV. In the model there is an infinite rotational band of states, but these are not observed experimentally. The lowest lying spurious state is $I = J = 5/2$, for which the model predicts the mass relation:

$$M_{5/2} - M_{3/2} = 5/3(M_{3/2} - M_{1/2}). \tag{38}$$

Taking this formula at face value and substituting the experimental numbers for the right-hand side, one finds $M_{5/2} = 1720$ MeV. One could of course claim that such states are artifacts of the large-N_c approximation as a quark model calculation shows. One could also say that this provides an *a posteriori* limit on the validity of the model—that for an excitation energy of 780 MeV the model certainly fails. The fact that this energy is similar to the masses of the vector mesons ρ and ω may tell us that to improve the model we have to go beyond the initial minimal model and introduce the vector mesons explicitly.

Adkins *et al.*[12] studied in detail the predictions of the model for the static properties of the nucleon. We reproduce their results in Table I.

In their work the nucleon and delta masses are treated as input data to fix F_π and e. Generally, theory and experiment differ by about 30% (with some notable exceptions, F_π and g_A). This success is very encouraging, since the model had only two free parameters. But experience with chiral models would lead us to expect somewhat better agreement between theory and experiment.

There are two relations[12] predicted by the model which are independent of the values of e and F_π:

$$\text{isoscalar } g\text{-factor} \equiv g_{I=0} = (4/9)\langle r^2\rangle_{I=0}\cdot(M_{1/2})(M_{3/2}-M_{1/2}), \quad (39)$$

$$\text{isovector } g\text{-factor} \equiv g_{I=1} = 2M_{1/2}(M_{3/2}-M_{1/2})^{-1}. \quad (40)$$

The first relation agrees with the experimental data at the 7% level, while the second is off by about 47%. Mattis and Karliner[14] noted that the right-hand side of (39) behaves as $(N_c)^0$, while in

TABLE I

Quantity	Prediction	Experiment
M_N	input	939 MeV
M_Δ	input	1232 MeV
F_π	129 MeV	186 MeV
$\langle r^2\rangle^{1/2}_{I=0}$	0.59 fm	0.72 fm
$\langle r\rangle^{1/2}_{M,I=0}$	0.92 fm	0.81 fm
μ_p	1.87	2.79
μ_n	−1.31	−1.91
$\left\lvert\frac{\mu_p}{\mu_n}\right\rvert$	1.43	1.46
g_A	0.61	1.23
$g_{\pi NN}$	8.9	13.5
$g_{\pi N\Delta}$	13.2	20.3
$\mu_{N\Delta}$	2.3	3.3

(40) the right-hand side behaves as N_c^2. This empirical result suggests that it is best to fix the model parameters from observables which are asymptotically independent of N_c. Such observables are the mean square radii, $\langle r^2 \rangle$. One such observable is enough to fix eF_π, but this argument does not give a reliable method for either e or F_π separately.

There have been several attempts to improve the Skyrme model. A quark mass term can be added to the action, explicitly breaking chiral symmetry. A term that breaks chiral $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry to the diagonal subgroup $\mathrm{SU}(2)_I$ has the form:

$$S_{\text{mass}} = (m_\pi^2 F_\pi^2/8) \int \mathrm{tr}(\mathrm{U} - \mathbf{1})\, d^4x, \tag{41}$$

where the normalization is such that for matrices U near the unit matrix, (41) reduces to a mass term giving all three pions equal masses. (This term also respects the restriction imposed by Dashen's theorem.[15])

Weinberg[16] showed that if the effective theory is to include unequal masses for the u quark and d quark, then one must introduce terms that are at least of order $(m_u - m_d)^2$. Thus corrections to $\mathrm{SU}(2)_I$ are indeed small and may be safely ignored as long as electromagnetic corrections are not included. The changes that the mass term (41) introduces are not a significant improvement except for observables that diverge in the chiral limit.[17]

Another variant of the model introduces the vector meson ω[18] and replaces the *ad hoc* Skyrme term by an ω-π interaction term. Once again there is no marked improvement. (We discuss below more elaborate schemes for the introduction of vector mesons into the model.)

DYNAMIC PROPERTIES OF THE NUCLEON AND DELTA

Let us consider pion–nucleon scattering. One notices that since the classical soliton decays to the classical vacuum once we get away from the center of the soliton, it follows that small oscillations about the soliton become small oscillations about the vacuum when not in the neighborhood of the soliton. Since small oscillations about the vacuum are to be interpreted as pions, we thus get a description of the scattering process: pion + soliton $\rightarrow$ pion + soliton. This approach was initiated by Hayashi *et al.* at Siegen[19] and elaborated and extended by Peskin, Karliner and Mattis at SLAC.[20,21]

Technically one expands the action about the classical soliton using eigenmodes of the small oscillations. Since the static soliton is invariant under $SU(2)_K$, it is useful to expand in small oscillation eigenmodes that are irreducible representations of $SU(2)_K$. This implies that the partial-wave S-matrix elements have the form

$$S(L',L_3',I_3';L,L_3,I_3) = \sum_{K,K_3} \langle L'I_3'1I_3'|KK_3\rangle\, T(K,L,L')\, \langle LL_3 1I_3|KK_3\rangle, \quad (42)$$

where L and L' are the orbital angular momenta of the incoming and outgoing pions, respectively, and I_3 and I_3' serve to define their respective charges. In an approximation where the pion momenta are large enough so that the soliton rotates through a negligible angle during the scattering process, one needs only to project the incoming state on the $|K,K_3\ L,L_3\rangle$ states and sum the scattering contributions of the relevant processes. Such an approximation is expected to break down near threshold—although this is where the old current algebra results are supposed to hold. The resulting S-matrix elements in this approximation are

$$S(L'R'LRIJ) = \sum_K (-1)^{(R-R')}\sqrt{(2R'+1)(2R'+1)}\,(2K+1) \times \begin{Bmatrix} K & I & J \\ R' & L' & 1 \end{Bmatrix} \times \mathring{S}(KLL') \begin{Bmatrix} KIJ \\ RL1 \end{Bmatrix}, \quad (43)$$

where R and R' are the values of $I = J$ for the incoming and outgoing baryons, respectively, and

$$\begin{Bmatrix} J1 & J2 & J3 \\ J4 & J5 & J6 \end{Bmatrix}$$

are Wigner 6-J symbols. The SLAC group made a detailed comparison of the model with the data, carefully distinguishing between those aspects that need only the existence of a static classical solution and those aspects that depend on the detailed behavior of the solution. Just assuming a classical solution leads to linear relations among the reduced partial-wave amplitudes $S(I,J)$ in pion–nucleon scattering:

$$S(3/2,L-1/2) = ((L-1)S(1/2,L-1/2)+(3L+3)S(1/2,L+1/2))/(4L+2)$$

and (44)

$$S(3/2,L+1/2) = ((3L)S(1/2,L-1/2)+(L+2)S(1/2,L+1/2))/(4L+2).$$

In particular one gets $S(1/2,3/2) = S(3/2,1/2)$—a relation which stems from yet another old model: the Chew–Low static model.

The linear relations (44) do not work very well for S, P, or D waves for which the (ignored) coupling effects between rotational and the vibrational small oscillation modes seem to be important. An interesting observation made by the SLAC group is that these relations—with the added assumption that for a given L the important reduced T-matrix elements are the ones with lowest value K—lead to the pattern

$$|T(1/2,L-1/2)| > |T(3/2,L+1/2)| >> |T(1/2,L+1/2)| \sim |T(3/2,L-1/2)|. \quad (45)$$

A detailed construction of the phase shifts gives a reasonable agreement with the data for F and higher waves.

It was remarked by Schnitzer[22] that to get the current algebra results, one should take into account the zero modes to all orders and the vibrations to lowest nonvanishing order.

THE INCORPORATION OF VECTOR MESONS

The starting point for the construction of the effective theory was the assumed $SU(2)_L \times SU(2)_R$ vacuum degeneracy and the related Goldstone phenomenon. This puts the pions in a distinguished position, which from a quark model point of view is problematic. In the quark model the pseudoscalar mesons and the vector mesons are distinguished only by spin-spin interactions between the quark and the antiquark. It would therefore be more in keeping with the quark model to enlarge the effective theory by including vector mesons.

An interesting way to do this is based on the equivalence of nonlinear sigma-type models using a homogeneous manifold G/H (as in the Skyrme model) and a model with G(global) $\times$ H(gauge) symmetry broken by a Higgs mechanism (so that all vector mesons in the adjoint representation of H become massive and the vacuum is invariant only under the diagonal subgroup of G(global) $\times$ H(rigid)). When this approach is implemented for $[SU(2)_L \times SU(2)_R]/SU(2)_I$, the ρ mesons are identified as the gauge mesons of $SU(2)_I$ that became massive by the Higgs mechanism. Bando *et al.*[23] showed that in such a scheme some features of low energy π and ρ dynamics are correctly reproduced. To recover Skyrme's Lagrangian, one considers the limit $m_\rho \to \infty$. This results in an expression for e in terms of the π and ρ dynamics.

THREE LIGHT FLAVORS

The effective theories presented in the previous sections may be generalized to the phenomenologically interesting case of three light flavors. The mass of the strange quark, about 160 MeV, is not very small on the strong interaction scale. However, we know that the flavor symmetry $SU(3)_f$ and the chiral symmetry $SU(3)_L \times SU(3)_R$ do give a reasonable approximate description of the real world. As elucidated by Witten,[24] there are two qualitatively new elements in the generalization from two light flavors to three.

The first is the need to add an anomaly ("Wess–Zumino term"[25]) to the action, that is, a term which cannot be written as a space-time integral over a local Lagrangian. (To preserve the locality of the equations of motion, the first variation of the anomaly term must be local.)

The second new element is that this additional term—call it Γ—when properly normalized, has an integer coefficient equal to N_c, the number of quark colors in the underlying theory.[24] The physical origin of the anomaly is a discrete symmetry of the naive effective chiral Lagrangian which is not a symmetry of QCD.[24]

This additional term in the action—denoted by $N_c\Gamma$—has interesting and far-reaching consequences[26]:

(i) It forces a unique fermion/boson character (unlike the Skyrme model where one could choose either alternative). In fact, for the quantum states of the $B = 1$ sector, *even* N_c results in bosons, *odd* N_c in fermions.

(ii) It validates yet another old result: the Michel–Lurcat rule relating baryon number B and spin S: $(-1)^{B+2S} = 1$.

(iii) For $N_c = 3$ it implies zero triality for physical states.

(iv) It validates that the topological baryon number B is, as conjectured by Skyrme, exactly the same quantum number B entering in the QCD definition of hypercharge (strangeness plus baryon number B).

(v) Using the experimental fact that the nucleon lies in a $SU(3)_f$ octet, it implies that $N_c = 3$.

Consider, then, QCD with three light flavors in the approximation where the quark masses are all zero. The vacuum degeneracy manifold is now $[SU(3)_L \times SU(3)_R]/SU(3)_f$ which is topologically the group manifold SU(3). The Skyrme–Witten effective action is

$$S = \int \left[\frac{F_\pi^2}{16} \operatorname{Tr}(R_\alpha R^\alpha) - \frac{1}{32e^2} \operatorname{Tr}(R_{\alpha\beta} R^{\alpha\beta}) \right] d^4x + N_c\Gamma, \quad (46)$$

where the matrices R_α and $R_{\alpha\beta}$ are defined as in (5) and (19) with the difference that U is now a 3×3 matrix in SU(3). The anomaly term Γ is defined by:

$$\Gamma \equiv i/(240\pi^2) \int_{M_5} \operatorname{Tr}(\omega^5). \quad (47)$$

Here the integration region M^5 is a five-dimensional manifold with compactified space-time as boundary. To define ω we note that since the homotopy group $\pi_4(SU(3))$ is trivial, it follows that for any given configuration $U(x,t)$ with $U(\infty,0) = \mathbf{1}$, there is a (homotopically) unique extension, $U(x,t,s)$ to the whole of M^5 such that $U(x,t,1) = \mathbf{1}$; $U(x,t,0) = U(x,t)$ and $U(\infty,0,s) = \mathbf{1}$. Picking any one of these homotopically equivalent extensions, the matrix of differential one-forms ω is defined by

$$\omega = R_\mu \, dx^\mu - iU^\dagger(\partial U/\partial s)ds. \tag{48}$$

It should be noted that Γ—as an integral over differential forms—is independent of the metric on space-time, $g_{\mu\nu}$, and therefore, according to (8), does not contribute to the energy momentum tensor $T_{\mu\nu}$. It does, however, modify the equations of motion and therefore gives a contribution to the chiral currents.

It is also apparent from (47)—since SU(2) has only three dimensions—that the anomaly Γ vanishes for two light flavors.

The term Γ is dimensionless and must be multiplied by $\hbar$ to serve properly as a term in the action. (This $\hbar$ is not explicit because of the convention $\hbar = 1$.) Thus the anomaly disappears in the classical limit $\hbar \to 0$ but not in the limit $N_c \to \infty$. (It is this latter limit that is of interest in the Skyrme–Witten model, and one should be aware of the distinction to avoid ambiguity in the (standard) usage whereby a skyrmion is denoted as a "classical" soliton.)

One may add a quark mass term to break $SU(3)_L \times SU(3)_R$ to its diagonal subgroup $SU(3)_f$. The simplest term is of the form:

$$S'_{\text{mass}} \equiv (m^2 F_\pi^2/16) \int \text{tr}[\mu(U^\dagger + U - 2\,\mathbf{1})]d^4x, \tag{49}$$

where m^2 is the average mass squared in the 0^- octet while μ is a Hermitian matrix proportional to the (diagonal) quark mass matrix ($m_u = m_d \neq m_s$) and thus splits the masses of the π, κ, and η.[27]

The discussion of the $B = 0$ sector with three light flavors is very similar to the previous discussion for two light flavors. In the absence of a mass term, the zero energy solutions are degenerate. Any particular solution U_0 = constant breaks the symmetry (to $SU(3)_f$) and one makes the conventional choice $U_0 = \mathbf{1}$. Small oscillations about this solution when quantized describe an octet of massless pseudoscalar bosons.

The $B \neq 0$ sectors are more interesting and involve new aspects. There are now two kinds of equivariant solutions, depending on the way the abstract group SU(2) is represented as a subgroup of SU(3). If we choose the conventional isospin subgroup (namely SU(3) matrices with 1 in the lower right corner), we get, for the equivariant static solution, the form

$$\mathrm{U}(x) = \begin{pmatrix} \mathrm{U}_1(x) & 0 \\ 0 & 1 \end{pmatrix} \tag{50}$$

where $\mathrm{U}_1(x)$ is the same matrix used for the two-flavor solution.

One may, however, choose to represent SU(2) by real orthogonal 3×3 matrices. The resulting solutions have the property that their baryon number is necessarily an even integer. Balachandran *et al.*[28] suggested identifying the lowest lying $B = 2$ solutions of this kind with the quark model dibaryon H. (We shall return to dibaryons below.)

A solution of the form (50) has a very important symmetry: the SU(3) generator λ_8 commutes with any $\mathrm{U}(x)$ of the form in Eq. (50). As a result, the degeneracy manifold of the solutions is not SU(3) but the seven-dimensional submanifold SU(3)/U(1)$_Y$ (where U(1)$_Y$ is generated by λ_8). Quantization of the zero-modes proceeds similarly to the two-flavor case.

The anomaly in the action now makes the dynamics (quantization of the zero modes) analogous to the dynamics of a charged particle moving in the field of a magnetic monopole (where the angular manifold is SU(2)/U(1) $\sim S^2$). The corresponding generalization of wave functions (eigensections or monopolar harmonics) is based on the representation matrices of SU(3): $D^{[M]^*}(\mathrm{SU}(3))$, where $[M] = [m_{13}, m_{23}, m_{33} = 0]$ labels an irreducible representation (irrep) in SU(3). (Matrix elements of $D^{[M]^*}$ form a complete set of functions on the SU(3) group manifold.)

Going from SU(3) to SU(3)/U(1)$_Y$ forces a restriction on one of two sets of Y, I, I_3 quantum numbers that characterize a state (matrix element of $D^{[M]^*}$): namely the "right hypercharge" must be equal to the $Y_R = (1/3)N_cB$. This specific value is a result of evaluating the anomaly; the "baryon number" B is the Skyrme expression (11), numerically equal to 1. It follows that if $N_c = 3$, the right hypercharge is integral and the baryons must have triality zero ($m_{13} + m_{23} \equiv 0 \bmod 3$). Alternatively, if we use the empirical result that the nucleon has integral hypercharge and belongs to an octet, then the number of colors must be three. (Witten showed that gauging the anomaly to include electromagnetic interactions and studying the decay $\pi_0 \rightarrow \gamma + \gamma$ recovered Gell-Mann's result that $N_c = 3$). Equivariance, (23), implies that the "right isospin" quantum numbers, I' and I'_3, must be the spin quantum numbers J and J_3, respectively.

The dynamics of the zero modes leads to the mass formula[29,30]:

$$E(m_{13},m_{23};JB) = 1.23(3\pi^2)F_\pi/e + [m_{13}(m_{13} + 3) - m_{23}(m_{13} - m_{23}) - B^2/4]/2I_3 + J(J + 1)/2I_2, \quad (51)$$

where the two moments of inertia, I_3 and I_2 are integrals involving the radial function $f(r)$ and its derivative, and m_{13}, m_{23} label the triality zero $SU(3)_f$ irrep $[m_{13}, m_{23}, 0]$ with B = baryon number. The expression $m_{13}(m_{13} + 3) - m_{23}(m_{13} - m_{23})$ is the eigenvalue of the Casimir operator for SU(3).

The lowest lying states—as determined by the mass formula (51)—can be identified with the baryon octet ([210]) and decimet ([300]). This identification not only determines $N_c = 3$ and triality = 0, as mentioned, but (in the intrinsic frame) verifies that Skyrme's quantum number (11) is actually *the* baryon number. The equivariance condition determines the spin to be 1/2 for the baryon octet and 3/2 for the decimet.

Once again we have an infinite band of states with unbounded $SU(3)_f$ and spin quantum numbers. As in the two-flavor cases we may consider the other states as artifacts of the model and/or as indications for its limits of validity.

It is possible to add the meson mass term (49) to the action. The component of the μ-matrix proportional to the 3×3 unit matrix does not break SU(3) and is expected only to change the numerical values of the constants in Eq. (51).

Guadagnini[29] calculated the contribution of a λ_8 component in the quark mass matrix μ (49) to the baryon masses. The agreement with experiment was rather poor, and he suggested that the effective theory should contain an additional term that breaks SU(3). Once again the missing term is suspected to be related to the vector mesons, namely a term representing φ-ω mixing. Adding such a term we find that the octet and decimet masses split in a way characteristic of another old model: static SU(6). We find also that the Skyrme–Witten model provides one relation that goes beyond static SU(6):

$$M(2695,8) = (-\sqrt{6}/27)M(405,8) \cong -0.091M(405,8), \quad (52)$$

where $M(2695,8)$ and $M(405,8)$ denote the octet component of the SU(6) tensor operators 2695 and 405, respectively. This result explains the smallness of the experimental number found for $M(2695,8)$.

It has always been a puzzle that empirically the meson mass splittings involved squared masses (m^2) while baryon splittings involved the masses themselves (m). Let us note explicitly that the Skyrme–Witten model, augmented by quark mass splittings, automatically involves exactly this feature, thus nicely resolving the puzzle.

Adkins and Nappi[31] calculated the magnetic moments of the octet and decimet. With no SU(3) breaking term in the Lagrangian, the results just reproduce the old Coleman–Glashow results but now with a definite value for the F/D ratio. This allows all magnetic moments to be expressed in terms of one of them, say, the proton magnetic moment. (We saw in Table I that the Skyrme model predicts this value as well.)

One finds that the ratio μ_N/μ_P is $-3/4$ rather than the quark model prediction $-2/3$. Even if one adds SU(3) breaking terms, making the moments depend on five free parameters, agreement with experiment is not very good. Adkins and Nappi then added terms linear in the time derivatives of the zero-mode coordinates (to be interpreted as $1/N_c$ corrections.) This leaves two linear relations. One says that the magnetic moment matrix has no SU(3) singlet component

$$\mu(P) + \mu(N) + 3/2[\mu(\Sigma^+) + \mu(\Sigma^-)] + \mu(\Xi^0) + \mu(\Xi^-) + \mu(\Lambda) = 0. \quad (53)$$

(Here the old Marshak–Okubo–Sudarshan sum rule $\mu(\Sigma^0) =$

$1/2[\mu(\Sigma^+) + \mu(\Sigma^-)]$ was used to express the unmeasured Σ^0 magnetic moment.)

The other linear relation is

$$\frac{3\sqrt{3}}{3}\mu(\Sigma - \Lambda) - \mu(\Sigma^+) - \mu(\Sigma^-) - \mu(\Xi^0)$$
$$- \mu(\Xi^-) + 3\mu(P) + \mu(N) + 2\mu(\Lambda) = 0. \quad (54)$$

For these two relations agreement with experiment is reasonable.

To sum up: For the $SU(3)_f$ Skyrme–Witten model, one may say that qualitatively the determination of the quantum numbers is remarkable, but that quantitatively the model needs improvements rather more urgently than the two-flavor case.

Let us remark on the question as to whether the Skyrme–Witten model yields static SU(6) in effect. As noted above (Eq. (52)) the Skyrme–Witten model actually goes beyond static SU(6). Similarly, the μ_N/μ_P ratio differs from the prediction of static SU(6). The quantization of the zero modes in the $B = 1$ sector of the Skyrme–Witten model actually gives a tower of SU(3) × SU(2) irreps: (8,1/2), (10,3/2), (10,1/2), (27,3/2), . . . of which only the lowest two fit with static SU(6).

Nonetheless, SU(6) does provide a convenient language for the lowest states.

DIBARYONS

We conclude by discussing dibaryons in the three-flavor model. The classical solution for the dibaryon arises by realizing SU(2) via its three-dimensional irrep with generators Λ; the corresponding equivariant solution has the form

$$U_2(x) = e^{i\psi(r)}\mathbf{1} + i\sin\chi(r)\,\Lambda\cdot x\, e^{-i\psi(r)/2}$$
$$+ [\cos\chi(r)e^{-i\psi(r)/2} - e^{i\psi(r)}](\Lambda\cdot x)^2. \quad (55)$$

Here, for $B = 2$ the radial functions $\chi(r)$ and $\psi(r)$ must satisfy the boundary conditions:

$$\begin{aligned} \chi(0) &= \pi, \quad \chi(\infty) = 0, \\ \psi(0) &= 2\pi/3, \quad \psi(\infty) = 0. \end{aligned} \tag{56}$$

The corresponding classical energy is about $2.36[3\pi^2 F_\pi/e]$ which is less than twice the classical energy for a single nucleon at rest. We thus get (classically) a bound state in the chiral limit of the theory.

Quantization of the zero modes proceeds as before with two important differences: (i) the contribution of the anomaly term to the zero-mode Lagrangian vanishes, and (ii) the geometry of the energy degeneracy manifold is quite different. Indeed the energy degeneracy manifold is eight dimensional, $[\mathrm{SU}(3)_f \times \mathrm{SU}(2)_J]/\mathrm{SU}(2)_K$, since there is no continuous group which leaves the classical solution invariant. Moreover, the energy degeneracy manifold has *two disconnected pieces*, as can be seen by considering the parity transform of the solution: $\mathrm{U}(x) \rightarrow \mathrm{U}^\dagger(-x)$. If the energy degeneracy manifold had had only one connected piece, there would have existed a matrix in SU(3), say M, such that

$$M\, \mathrm{U}_2(x)\, M^\dagger = \mathrm{U}_2^\dagger(-x) \tag{57}$$

from which it would follow that $\mathrm{Tr}(\mathrm{U}_2(x)) = \mathrm{Tr}(\mathrm{U}_2^\dagger(-x))$, which contradicts the boundary conditions. Therefore the quantization of the zero modes must take into account the tunnelling between the two parts of the energy degeneracy manifold, giving rise to a parity doubling of the states. As far as the continuous symmetries are concerned, the eigenstates are characterized by a triality zero SU(3) representation $[2k\ k\ 0]$ with quantum numbers Y, I, I_3 and SO(3) quantum numbers S, S_3 with S a non-negative integer. (There is no additional quantum number (analogous to hypercharge) commuting with S and S_3, but for the low lying irreps, any S appears at most once.)

The ground state is expected to be an SU(3) singlet with spin-parity 0^+, presumably with a (parity doublet) 0^- nearby. The lowest excited states are octets with 1^+ and 1^- and 2^+ and 2^-. Balachandran *et al.*[28] interpreted the ground state as the dibaryon H of the quark model, but to determine in detail the energy and width of such a state (and the rest of the band), one needs to take very accurately into account SU(3) symmetry breaking in order to determine the stability against decay into two Λ baryons. Experience with SU(3) symmetry breaking does not encourage a belief in the reliability of the model to such an extent. One can only hope for more experimental interest in this mass region to help resolve the problem.

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