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# Some topics on Dirac-harmonic maps and the Yang-Mills flow

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# Contents

# Abstract

This thesis contains two parts.

In the first part, we study the Dirac-harmonic map with a Ricci type spinor potential, which is a critical point $(u, \psi)$ of the corresponding functional, where $u$ is a map from a Riemannian spin surface $(M^2, g_{ij})$ to $(N^n, h_{\alpha\beta})$ and $\psi$ is a section of $\Sigma M \otimes u^{-1}TN$. First, we consider the case that $M$ is a compact Riemann surface without boundary. We show that a Dirac-harmonic map with a Ricci type spinor potential is smooth. Furthermore, we obtain a convergence theorem of approximate Dirac-harmonic maps with a Ricci type spinor potential. Then we consider the case that $M$ is a compact Riemannian spin surface with non-empty boundary $\partial M$. We study a free boundary value problem for the Dirac-harmonic maps with a Ricci type spinor potential. For a closed submanifold $\mathcal{S}$ of $N$, we show that a Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S}$ is Hölder continuous. If in addition, we assume that $\mathcal{S}$ is a closed, totally geodesic submanifold of $N$, then we can get a higher regularity of $(u,\, \psi)$.

In the second part, we explore homothetically shrinking solitons of the Yang-Mills flow. We show that the Yang-Mills flow in dimension four cannot develop any type I singularity. Then motivated by the work of Colding-Minicozzi [CM12] on mean curvature flow, we introduce a notion of $\mathcal{F}$-stability for weakly self-similar solutions. We show that the $\mathcal{F}$-stability can be characterized by the semi-positive definiteness of the Jacobi operator acting on a subspace of variation fields.

# Introduction

## 0.1 Motivation to this work

### 0.1.1 Dirac-harmonic maps

Harmonic maps are solutions to a natural geometrical variational problem. Many other canonical or natural maps turn out to be harmonic, such as geodesics, minimal surfaces and harmonic functions. The key questions of harmonic maps between given manifolds are existence, uniqueness and regularity of the harmonic maps.

The harmonic functions on an open subset $\Omega$ of $\mathbb{R}^m$ are solutions of the Laplace equation

$$\Delta f = 0, \qquad \text{where } \Delta := \sum_{i=1}^{m} \frac{\partial^2}{\left(\partial x^i\right)^2}, \quad (x = \left(x^1,\, \cdots,\, x^m\right) \in \Omega),$$

which are critical points of the Dirichlet functional

$$E_\Omega(f) := \frac{1}{2} \int_\Omega |df|^2 \; dx.$$

The harmonic maps between manifolds are defined in a similar way. A smooth map $u$ from $(M^m,\, g_{ij})$ to another compact manifold $(N^n,\, h_{\alpha\beta})$ is called a harmonic map if it is a critical point ( extremal) of the energy functional

$$E(u) := \frac{1}{2} \int_M |du|^2 \; dv_g,$$

which satisfies the following equations in local coordinates

$$\Delta_g u^\alpha + g^{ij}\, \Gamma^\alpha_{\beta\gamma}(u(x))\, \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} \equiv 0,$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of $N$.

By Nash's famous embedding theorem [Na56], $(N,\, h_{\alpha\beta})$ can be isometrically embedded into the Euclidean space $\mathbb{R}^K$. We define the Sobolev space

$$H^1(M,\, N) = \left\{ u \in H^1(M,\, \mathbb{R}^K) \,|\quad u(x) \in N \, a.e. x \in M \right\}.$$

In this Sobolev space, the energy functional $E(u)$ is still well defined and we can look for critical points of $E(u)$ in $H^1(M, N)$, which are called the weakly harmonic maps.

For $m = dim\, M \geq 3$, Rivière [Ri95] constructed everywhere discontinuous weakly Harmonic maps with finite energy, taking values in the sphere $S^2$. One needs to pose a stationary condition to get partial regularity for a harmonic map, for instance [Be93, Ri95, Ev91]. And for $m = dim\, M = 1$, it is easy to show that every weakly harmonic map is smooth.

We focus on $m = dim\, M = 2$, i.e. we assume that $(M, g_{ij})$ is a compact Riemannian surface. Many results were obtained for the regularity of the weakly harmonic maps, for instance [Mo48, Gr81, Sch84]. The general proof is due to Hélein in [He90, He91, He02], a crucial idea of Hélein's proof is to use the enlargement argument, which allows us to replace the target manifold $N$ by another Riemannian manifold $\widetilde{N}$ such that, there exists a global orthonormal frame on $\widetilde{N}$. Furthermore, this global orthonormal frame can be chosen as a Coulomb frame adapted to the map $u$. The regularity to the case of weakly harmonic maps on a surface $M$ with nonempty boundary was proved by Jie Qing [Qi93]. The harmonic maps with a free boundary have also been studied by several authors, for instance [GJ87, Sch06]

Since a compact Riemann surface is a spin manifold, hence there exists a spin structure on $(M^2, g_{ij})$ that allows us to define the associated spinor bundle $\Sigma M$, which is a complex vector bundle with structure group $Spin(2)$. We consider a compact Riemann surface $(M^2, g_{ij})$ with a given spin structure, then the Dirac operator $\partial\!\!\!/ : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$, which is a differential operator that is a formal square root of the Laplacian, can be defined on the spinor bundle $\Sigma M$. More precisely, $\partial\!\!\!/ := \gamma_i \cdot \nabla_{\gamma_i}$, where $\{\gamma_i\}_{i=1,2}$ is a local orthonormal frame on $M$, and $\cdot$ denotes the Clifford multiplication, which maps $TM \otimes \Sigma M$ to $\Sigma M$, and $\nabla$ denotes the connection on $\Sigma M$ which induced from the Livi-Civita connection on $M$. For more information, see [Fr00, LM89].

Let $(M, g)$ be an oriented, compact Riemannian surface with a given spin structure and $(N\, h)$ a compact Riemannian manifold of dimension $n \geq 2$. Let $u$ be a smooth map from $M^m$ to $N^n$ which induces a pull-back vector bundle $u^{-1}TN$. We have a natural connection on the bundle $u^{-1}TN$. A section $\psi$ on $(\Sigma M \otimes u^{-1}TN)$ locally can be expressed as:

$$\psi(x) = \psi^\alpha \otimes \frac{\partial}{\partial y\alpha}(u(x)),$$

where $\psi^\alpha$ is a spinor and $\left\{\frac{\partial}{\partial y\alpha}\right\}_{\alpha=1}^{n}$ is a local basis of $N$. The Dirac operator $\mathcal{D}$

along the map $u$ is defined by

$$\begin{aligned}\mathcal{D}\psi \quad &:= \quad \partial\!\!\!/\psi^\alpha \otimes \frac{\partial}{\partial y^\alpha} + \gamma_i \cdot \psi^\alpha \otimes \nabla_{\gamma_i} \frac{\partial}{\partial y^\alpha} \\ &= \quad \partial\!\!\!/\psi^\alpha(x) \otimes \frac{\partial}{\partial y^\alpha}(u(x)) + \Gamma^\gamma_{\alpha\beta}(u(x))\, \nabla_{\gamma_i} u^\alpha(x)\, \left(\gamma_i \cdot \psi^\beta(x)\right) \otimes \frac{\partial}{\partial y^\gamma}(u(x)),\end{aligned}$$

where $\Gamma^\gamma_{\alpha\beta}$ are Christoffel symbols of $N$.

Chen-Jost-Li-Wang [CJLW05, CJLW06] first introduced the following Dirac-harmonic energy functional

$$L(u,\, \psi) := \frac{1}{2} \int_M \left[|du|^2 + \langle \psi,\, \mathcal{D}\psi \rangle\right] \, dv_g,$$

which is arisen from the supersymmetric nonlinear sigma model [De99]. The critical points of $L(u,\, \psi)$ are called Dirac-harmonic maps.

The Euler-Lagrange equations for $L$ are

$$\begin{cases} \tau(u) &= \mathcal{R}(u, \psi) \\ \mathcal{D}\psi &= 0, \end{cases}$$

where $\tau(u)$ is the tension field and $\mathcal{R}(u, \psi) \in \Gamma(u^{-1}TN)$ defined by

$$\mathcal{R}(u, \psi) = \frac{1}{2} R^\gamma_{\alpha\delta\beta}(u) \, \left\langle \psi^\alpha,\, \nabla u^\beta \cdot \psi^\delta \right\rangle \, \frac{\partial}{\partial y^\gamma}(u).$$

**Definition 0.1.** *For $u \in H^1(M,\, N)$, the set of sections $\psi \in \Gamma(\Sigma M \otimes u^{-1}TN)$ is defined to be all $\psi = \left(\psi^1,\, \psi^2,\, \cdots,\, \psi^K\right) \in (\Gamma(\Sigma M))^K$ with the property that $\psi(x)$ is along the map $u$, namely*

$$\sum_{i=1}^K v_i \psi^i = 0, \qquad \text{for any normal vector } v = (v_1,\, v_2,\, \cdots,\, v_K) \text{ at } u(x).$$

*We say that $\psi = \left(\psi^1,\, \psi^2,\, \cdots,\, \psi^K\right) \in W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$ if $d\psi \in L^{\frac{4}{3}}(M)$ and $\psi^i \in L^4(M)$ for all $1 \le i \le K$.*

A pair of map $(u, \psi) \in H^1(M,\, N) \times W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$ is called a weakly Dirac-harmonic map, if it is a critical point of $L(\cdot,\, \cdot)$ in the Sobolev space $H^1(M,\, N) \times W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$.

There are series of papers studying some analytic aspects of the Dirac-harmonic maps, such as [CJW07, CJW08, WX09, Zh07]. For explicit examples of Dirac-harmonic maps, see [JMZ09]. And for a free boundary value problem for Dirac-harmonic maps, see [CJWZ].

The goal of the first part of this thesis is to study the Dirac-harmonic maps with a Ricci type spinor potential. We consider the following functional $L_\lambda(u,\, \psi)$

$$L_\lambda(u,\,\psi) := \int_M \left\{ \frac{1}{2}\left[ |du|^2 + \langle \psi,\, \mathcal{D}\psi\rangle \right] + \lambda R_{\alpha\beta}(u) \left\langle \psi^\alpha,\, \psi^\beta \right\rangle \right\} dv_g.$$

A pair of map $(u, \psi) \in H^1(M,\,N) \times W^{1,\,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$ is called a weakly Dirac-harmonic map with a Ricci type spinor potential, if it is a critical point of $L_\lambda(\cdot,\,\cdot)$ in the Sobolev space $H^1(M,\,N) \times W^{1,\,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$.

The Euler-Lagrange equation for $L_\lambda(u,\,\psi)$ is

$$\begin{cases} \tau^\gamma(u) = -\lambda\, h^{\gamma\eta}\, R_{\beta\xi}\, \Gamma^\xi_{\alpha\eta} \left( \left\langle \psi^\alpha,\, \psi^\beta \right\rangle + \left\langle \psi^\beta,\, \psi^\alpha \right\rangle \right) \\ \qquad\qquad + \dfrac{1}{2} \left\langle \psi^\alpha,\, \nabla u^\delta \cdot \psi^\beta \right\rangle R^\gamma_{\alpha\beta\delta} + \lambda\, h^{\gamma\eta}\, R_{\alpha\beta,\,\eta} \left\langle \psi^\alpha,\, \psi^\beta \right\rangle & (0.1a) \\ \mathcal{D}\psi^\gamma \;= -2\lambda R^\alpha_\beta(u)\, \psi^\beta, & (0.1b) \end{cases}$$

or

$$\begin{cases} \tau^\gamma(u) \;=\; \frac{1}{2} \left\langle \psi^\alpha,\, \nabla u^\delta \cdot \psi^\beta \right\rangle R^\gamma_{\alpha\beta\delta} + \lambda\, h^{\gamma\eta}\, R_{\alpha\beta;\,\eta} \left\langle \psi^\alpha,\, \psi^\beta \right\rangle \\ \mathcal{D}\psi^\gamma \;=\; -2\lambda R^\alpha_\beta(u)\, \psi^\beta, \end{cases} \tag{0.2}$$

where $R_{\alpha\beta;\,\eta}$ denotes the covariant derivative of the Ricci curvature tensor $R_{\alpha\beta}$ with respect to $\frac{\partial}{\partial y^\eta}$.

We first study the regularity of the Dirac-harmonic maps with a Ricci type spinor potential from a compact Riemann surface $M$; then we consider $M$ as a compact Riemannian spin surface with nonempty boundary $\partial M$, and study a free boundary value problem for the Dirac-harmonic maps with a Ricci type spinor potential.

### 0.1.2 The Yang-Mills flow

The Yang-Mills theory is a fundamental ingredient in particle physics. Mathematically, Yang-Mills theory is a nonlinear generalization of the Hodge theory. In particular, the solutions of the Yang-Mills equations are precisely those connections whose curvature tensors are "harmonic". Such connections and their curvatures are called Yang-Mills connections and Yang-Mills fields respectively.

Let $(M, g)$ be a closed $n$-dimensional Riemannian manifold. Let $G$ be a compact Lie group and $P(M, G)$ a principle bundle over $M$ with the structure group $G$. We now fix a $G$-vector bundle $E = P(M, G) \times_\rho \mathbb{R}^r$, associated with $P(M, G)$ via a faithful representation $\rho : G \to SO(r)$.

Let $\mathfrak{g}$ denote the Lie algebra of $G$. A connection on $E$ is locally a $\mathfrak{g}$-valued 1-form. Using Latin letters for the manifold indices, one may write a connection $A$ in the form of

$$A = A_i dx^i,$$

where $A_i \in so(r)$. Using Greek letters for the bundle indices, one may also write

$$A = A^\alpha_{i\beta} dx^i.$$

The curvature of the connection $A$ is locally a $\mathfrak{g}$-valued 2-form

$$F = \frac{1}{2}F_{ij}dx^i \wedge dx^j = \frac{1}{2}F^\alpha_{ij\beta}dx^i \wedge dx^j.$$

Here

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j],$$

and

$$F^\alpha_{ij\beta} = \partial_i A^\alpha_{j\beta} - \partial_j A^\alpha_{i\beta} + A^\alpha_{i\gamma}A^\gamma_{j\beta} - A^\alpha_{j\gamma}A^\gamma_{i\beta}.$$

Let

$$|F|^2 = g^{ik}g^{jl}F^\alpha_{ij\beta}F^\alpha_{kl\beta}.$$

The Yang-Mills functional, defined on the space of connections, is then given by

$$YM(A) = \frac{1}{2}\int_M |F|^2 d\mu_g. \tag{0.3}$$

Let $\nabla$ denote the covariant differentiation on $\Gamma(E)$ associated with the connection $A$, and also the covariant differentiation on $\mathfrak{g}$-valued $p$-forms induced by $A$ and the Levi-Civita connection of $(M, g)$. A critical point $A$ of the Yang-Mills functional is called a Yang-Mills connection, which satisfies

$$\nabla^p F^\alpha_{pj\beta} = 0.$$

In order to prove the existence of Yang-Mills connections, one can deform the connections by the negative gradient of the Yang-Mills functional, i.e. the Yang-Mills flow.

$$\frac{\partial}{\partial t}A^\alpha_{j\beta} = \nabla^p F^\alpha_{pj\beta}. \tag{0.4}$$

This approach was first suggested for the Yang-Mills functional by Atiyah-Bott [AB83]. Donaldson [Do85, Do87] used the Hermitian Yang-Mills flow to establish the existence of Hermitian-Einstein metrics on stable holomorphic vector bundles over algebraic manifolds. Over a compact Riemannian manifold of dimension two or three, Rade [Ra92] proved that the Yang-Mills flow exists for all time and converges to a Yang-Mills connection. However if the base manifold has dimension five or above, Naito [Na94] showed that there exists a Yang-Mills flow which develops singularity in finite time, see also [Gr01].

**Definition 0.2.** *Assume $A(x,t)$ is a smooth solution to the Yang-Mills flow for $0 \le t < T$ and as $t \to T$ the curvature blows up, i.e. $\limsup_{t\to T}\max_{x\in M}|F(x,t)| = \infty$. One says that the Yang-Mills flow develops a Type-I singularity, or a rapidly forming singularity, if there exists a positive constant $C$ such that*

$$|F(x,t)| \le \frac{C}{T-t}, \tag{0.5}$$

*Otherwise one says that the Yang-Mills flow develops a Type-II singularity. If (0.5) is satisfied and $x_0$ is a point such that $\limsup_{t\to T} |F(x_0,t)| = \infty$, we call $(x_0,T)$ a Type-I singularity.*

It is unknown that whether the Yang-Mills flow over a four-dimensional manifold develops a singularity in finite time. For partial results in this dimension, see for instance [St94, SST98, HT04]. In [We04], by using a monotonicity formula from [Ha93] Weikove shows that rapidly forming singularities in Yang-Mills flow converge, modulo the gauge group, to a nontrivial homothetically shrinking soliton.

**Definition 0.3.** *A solution $A(y,s)$ to the Yang-Mills flow, defined on the trivial bundle over $\mathbb{R}^n \times (-\infty,0)$, is called a homothetically shrinking soliton if it satisfies*

$$A^{\alpha}_{i\beta}(y,s) = \frac{1}{\sqrt{|s|}} A^{\alpha}_{i\beta}(\frac{y}{\sqrt{|s|}}, -1)$$

*for any $y \in \mathbb{R}^n$ and $s < 0$, see [We04].*

The limiting Yang-Mills flow $\widetilde{A}(y,s)$ is actually a homothetically shrinking soliton. For a limiting Yang-Mills flow $\widetilde{A}(y,s)$ which is obtained by Weikove [We04] and any $(x_0,t_0) \in \mathbb{R}^n \times (0,+\infty)$, if we set

$$A(x,t) = A_p(x,t)dx^p = \widetilde{A}_p(\frac{x-x_0}{\sqrt{t_0}}, \frac{t-t_0}{t_0})dx^p, \tag{0.6}$$

then $A(x,t)$ is a solution of the Yang-Mills flow defined on $\mathbb{R}^n \times [0,t_0)$ satisfying

$$\nabla^p F_{pj}(x,t) - \frac{1}{2(t_0-t)}(x-x_0)^p F_{pj}(x,t) = 0. \tag{0.7}$$

The goal of the second part of the thesis is that, by Weinkove's blowup analysis for type I singularity of the Yang-Mills flow [We04] and a simple identity for four dimensional Yang-Mills solitons, the Yang-Mills flow in dimension four cannot develop any type I singularity, see for instance Proposition 4.9. Then motivated by the work of Colding-Minicozzi [CM12] on mean curvature flow, we will introduce the notion of $\mathcal{F}$-stability for weakly self-similar solutions. The $\mathcal{F}$-stability can be characterized by the semi-positive definiteness of the Jacobi operator acting on a subspace of variation fields.

## 0.2 Summary of the thesis

This work is split into two parts. In the following, we will briefly introduce each of these parts.

In chapter 1, we will introduce the Dirac-harmonic maps and the regularity result of the weakly Dirac-harmonic maps.

In chapter 2, we will study the Dirac-harmonic maps with a Ricci type spinor potential. In §2.1, we first calculate the Euler-Lagrange equation for the functional $L_\lambda(u, \psi)$ via moving frames. Second, let $f : (N, h) \to (\widetilde{N}, \widetilde{h})$ be a totally geodesic, isometric embedding map. If the pair $(u, \psi)$ is a Dirac-harmonic map with a Ricci type spinor potential, then for $\widetilde{u} = f(u) : M \to \widetilde{N}$ and $\widetilde{\psi} = f_*(\psi) \in \Gamma(\Sigma M \otimes \widetilde{u}^{-1}T\widetilde{N})$, we verify that the pair $(\widetilde{u}, \widetilde{\psi})$ is also a Dirac-harmonic map with a Ricci type spinor potential. Therefore, we can follow Hélein's enlargement argument, without loss of generality, we can assume that $N$ supports a global orthonormal frame. Furthermore, this global orthonormal frame can be chosen as a Coulomb frame $\{e_\alpha\}_{\alpha=1}^n$ adapted to the map $u$.

In §2.2, we prove the regularity for weakly Dirac-harmonic maps with a Ricci type spinor potential. We show that a weakly Dirac-harmonic maps with a Ricci type spinor potential is smooth. Under the Coulomb frame, we rewrite the equation (0.1a) of Dirac-harmonic maps with a Ricci type spinor potential into the form

$$-d^* \left(\langle du, e_\alpha\rangle_{TN}\right) = \sum_\beta \left(\Theta_{\alpha\beta}\right) \circ \langle du, e_\beta\rangle_{TN} + F_\lambda^\alpha(u, \psi), \tag{0.8}$$

where $\Theta = (\Theta_{\alpha\beta}) \in L^2\left(B_r(x_0),\, so(n) \otimes \Lambda^1\mathbb{R}^2\right)$ satisfies $\left|(\Theta_{\alpha\beta})\right|^2 \le C\left(|\nabla u|^2 + |\psi|^4\right)$; $\circ$ denotes the inner product of the 1-forms; and $F_\lambda(u, \psi)$ denotes the Ricci type spinor potential terms in (0.1a), which satisfies $|F_\lambda(u, \psi)| \le C\,|\psi|^2$. Then we modify the technique by Riviève [Ri07] and Riviève-Struwe [RS08] to obtain a decay estimate in the Morrey space, which implies an $\epsilon$-regularity result by Morrey's decay lemma, namely, $u$ is Hölder continuous. The higher regularity then follows from modifying the proof of Lemma 2.5 and Theorem 2.4 in [CJLW05] by a slight change of analysis caused by the Ricci type spinor potential terms in equations (0.1a) and (0.1b).

In §2.3, we obtain a convergence theorem of weakly convergent sequence of approximate Dirac-harmonic maps with a Ricci type spinor potential. This is a byproduct of the rewriting equation (0.1a) of Dirac-harmonic maps with a Ricci type spinor potential into the form (0.8), we modify the proof of [Ri07, Thm. 1.5] to get a proof.

This part of the work in chapter 2 is a joint work with Deliang Xu [XC13].

In chapter 3, we study a free boundary value problem for the Dirac-harmonic maps with a Ricci type spinor potential. In §3.1.1, we introduce the chirality boundary condition for the usual Dirac operator $\partial\!\!\!/$, which exists on any spin manifold. This type boundary condition has been considered in [GHHP83, HMR02, Bu93, CJWZ] etc. In §3.1.2, we introduce the chirality boundary condition for the Dirac operator $\mathcal{D}$ along a map $u$, which has been considered in [CJWZ] for Dirac-harmonic naps. In §3.1.3, we introduce the free boundary condition for Dirac-harmonic maps with a

Ricci type spinor potential. After a variational calculation, we obtain the equations in Lemma 3.7. In §3.1.4, we first define the weakly Dirac-harmonic maps with a Ricci type spinor potential and free boundary $\mathcal{S}$, where $\mathcal{S}$ is a closed submanifold of the target manifold $N$. Second, if the pair $(u,\,\psi)$ is a Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S}$. Then similar to Chapter 2, for $\widetilde{u}=f(u):M\to\widetilde{N}$, $\widetilde{\psi}=f_*(\psi)\in\Gamma(\Sigma M\otimes\widetilde{u}^{-1}T\widetilde{N})$, and $\widetilde{\mathcal{S}}=f(\mathcal{S})$, we can show that the pair $(\widetilde{u},\,\widetilde{\psi})$ is a weakly Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\widetilde{\mathcal{S}}$. Hence we can follow Hélein's enlargement argument as in chapter 2.

In §3.2, we first prove a lemma , which is analogous to Lemma 3.1 in [CJWZ] and Lemma 3.1 in [Sch06], shows that the image of $u$ over a sufficiently small neighborhood of a boundary point is contained in a tubular neighborhood of the supporting submanifold $\mathcal{S}$. Therefore, if we restrict to a sufficiently small domain, one can use the geodesic reflection $\sigma$ to reflect the pair $(u,\,\psi)$ across $\mathcal{S}$. Then by using the geodesic reflection $\sigma$, we can extend the metric $h$ on the pull-back bundle $u^{-1}TN$ to some metric $\tilde{h}$ on the bundle $u^{-1}TN$(here $u$ is the extended map, which we still denote by $u$) with the extended map $u$, furthermore, we have the extended Riemannian curvature, Ricci curvature, and the extended Christolffel symbols etc. Finally, we will prove that the extended pair $(u,\,\psi)$ satisfies some equations (Theorem 3.15), from which we will prove the continuity of the extended pair $(u,\,\psi)$ in the next section.

In §3.3, we prove the regularity for weakly Dirac-harmonic maps with a Ricci type spinor potential and free boundary on $\mathcal{S}$. In fact, applying Theorem 3.15 and under the Coulomb frame, we can rewrite the equation of Dirac-harmonic maps with a Ricci type spinor potential with free boundary on $\mathcal{S}$ into the form

$$-d^*\left(\langle du,\,e_\alpha\rangle\right)=\sum_\beta\left(\widetilde{\Theta}_{\alpha\beta}\right)\circ\langle du,\,e_\beta\rangle+\widetilde{F}^\alpha_\lambda(u,\,\psi). \tag{0.9}$$

Comparing with (0.8), following the same schedule as the proof in §2.2, we obtain the Hölder continuous of the extended pair $(u,\,\psi)$. The theorem is following:

**Theorem 0.4.** *Let $(M,\,g_{ij})$ be a compact Riemannian spin surface with non-empty boundary $\partial M$, $(N,h_{\alpha\beta})$ any compact Riemannian manifold of dimension $n\ge 2$, and $\mathcal{S}$ a closed submanifold of $N$. Let $(u,\,\psi)$ be a weakly Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S}$. Then for any $\alpha\in(0,\,1)$, we have*

$$u\in C^{0,\,\alpha}(M,\,N).$$

Furthermore, if in addition we assume that $\mathcal{S}$ is a closed, totally geodesic submanifold of $N$, then it follows from [CJWZ, Prop. 3.11] that for any $1\le\beta,\,\delta,\,\eta\le n$ and $\gamma\in(0,\,1)$, the extended Christoffel symbol $\widetilde{\Gamma}^\eta_{\beta\delta}(u)$ is Hölder continuous. Finally, as in §2.2, we can get the following higher regularity:

**Theorem 0.5.** *Let $(M,\, g_{ij})$ be a compact Riemannian spin surface with non-empty boundary $\partial M$, $(N, h_{\alpha\beta})$ any compact Riemannian manifold of dimension $n \geq 2$, and $\mathcal{S}$ a closed, totally geodesic submanifold of $N$. Let $(u,\, \psi)$ be a weakly Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S}$ and suppose that $u \in C^{0,\alpha}(M,\, N)$ for any $\alpha \in (0,\, 1)$. Then there exists some $\gamma \in (0,\, 1)$ such that*

$$u \in C^{1,\gamma}(M,\, N), \quad \psi \in C^{1,\gamma}(\Sigma M \otimes u^{-1}TN).$$

In §4.1, we will give a background and a notation about Yang-Mills flow, which we will use in the following sections.

In §4.2, we define the $\mathcal{F}$-functional with respect to $(x_0,\, t_0)$ on $\mathbb{R}^n \times [0,\, t_0)$ by

$$\mathcal{F}_{x_0,t_0}(A) = t_0^2 \int_{\mathbb{R}^n} |F|^2 (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} dx.$$

We calculate the first variation of the $\mathcal{F}$-functional and after a calculation we show that

**Proposition 0.6.** *In dimension four, the Yang-Mills flow can not develop a singularity of type I.*

The entropy of a connection $A$ on the $G$-vector bundle $E$ over $\mathbb{R}^n$ is defined as follows:

**Definition 0.7.** *Let $A$ be a connection on the $G$-vector bundle $E$ over $\mathbb{R}^n$. The entropy is defined by*

$$\lambda(A) = \sup_{x_0 \in \mathbb{R}^n, t_0 > 0} \mathcal{F}_{x_0,t_0}(A). \tag{0.10}$$

we show that, along a Yang-Mills flow on $E$, the entropy is non-increasing.

In §4.3, we calculate the second variation of the $\mathcal{F}$-functional at a critical point and characterize the $\mathcal{F}$-stability by the semi-positive definiteness of the Jacobi operator acting on a subspace of variation fields. As a byproduct we have a rigidity result as follows:

**Theorem 0.8.** *Let $A$ be a self-similar solution with blows up at $t_0 = 1, x_0 = 0$, If $|F| \leq \frac{1}{2}$, then the $G$-vector bundle $(E,\, A)$ is flat.*

This part of the work in chapter 4 is a joint work with Yongbing Zhang [CZ].

# Part I

# Dirac-harmonic maps with a Ricci type spinor potential

# Chapter 1

# Regularity for Dirac-harmonic maps

## 1.1 Harmonic maps

In this subsection, we introduce the harmonic maps and the tension field.

Let $(M^m, g_{ij})$, $(N^n, h_{\alpha,\beta})$ be Riemannian manifolds, and let $u$ be a smooth map from $M^m$ to $N^n$, which induces a pull-back vector bundle $u^{-1}TN$. We have a natural connection on the bundle $u^{-1}TN$, inherited from the Levi-Civita connection on $TN$.

By using $u$, we have the induced map between tangent bundles $u_* : TM \to TN$, and there is a pull-back 2-tensor $u^*h$, which is symmetric and semi-positive. For any $X, Y \in \Gamma(TM)$, one has

$$u^*h(X,Y) = h(u_*X, u_*Y) = \langle u_*X, u_*Y \rangle_N .$$

Let $du \in \Gamma\left(T^*M \otimes u^{-1}TN\right)$ be defined as

$$du\,(X) \,:=\, u_*X, \quad \forall X \in \Gamma(TM).$$

The energy density of $u$ at a point $x \in M$ is defined as

$$e(u) = \frac{1}{2}\,|du|^2 .$$

We will use Latin letters for the manifold indices of $M$, and Greek letters for the manifold indices of the target manifold $N$. In local coordinates $x^i$ and $y^\alpha$ of $M$ and $N$ at $x$ and $u(x)$, whose tangent vectors of the coordinates are $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^\alpha}$, respectively, where $i = 1, 2 \cdots, m$ and $\alpha = 1, 2 \cdots, n$. Then the energy density is given by

$$e(u) = \frac{1}{2} g^{ij}\,\frac{\partial u^\alpha}{\partial x^i}\,\frac{\partial u^\beta}{\partial x^j}\,h_{\alpha\beta}.$$

In this work we use the Einstein summation.

We define the energy functional $E(u)$ by

$$E(u) = \int_M e(u)\, dv_g.$$

**Definition 1.1.** *The critical points of the energy functional $E$ in the space of maps are called harmonic maps.*

The second fundamental form of $u$ is defined by

$$B_{XY}(u) := (\nabla_X du)(Y) \in \Gamma\left(u^{-1}TN\right), \qquad \forall X, Y \in \Gamma(TM),$$

where $\nabla$ is the induced connection on the vector bundle $T^*M \otimes u^{-1}TN$. For convenience, since there is no confusion, we will use the same notation $\nabla$ to denote different connections on different bundles.

**Definition 1.2.** *A smooth map $u : M \to N$ is called a totally geodesic map if the second fundamental form $B(u) = \nabla\, du \equiv 0$.*

We define the tension field $\tau(u)$ by taking the trace of the second fundamental form $B_{XY}$

$$\tau(u) := g^{ij} B_{\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}}(u).$$

A direct calculation shows that

$$\tau(u) = \left(\Delta_g u^\alpha + g^{ij}\, \Gamma^\alpha_{\beta\gamma}(u(x)) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j}\right) \frac{\partial}{\partial y^\alpha},$$

where $\Gamma^\alpha_{\beta\gamma}$ denote the Christoffel symbols of $N$.

Consider the harmonic maps as extremals of the energy functional $E(u)$, one has the Euler-Lagrange equation of $E$:

$$\tau(u) = 0.$$

Hence we have the following lemma:

**Lemma 1.3.** *A smooth map $u : M \to N$ is a harmonic map if and only if $\tau(u) = 0$.*

For more properties of the harmonic maps, see [He02], [Xi96] and [SY97].

## 1.2 Dirac-harmonic maps

Let $(M, g)$ be an oriented, compact Riemannian surface and $P_{SO(2)} \to M$ its oriented orthonormal frame bundle, which is a $SO(2)$-principal bundle over $M$. A spin structure on $M$ is an equivariant lift of $P_{SO(2)} \to M$ with respect to the double covering $Spin(2) \to SO(2)$, i. e. there exists a Principal $Spin$-bundle $P_{Spin(2)} \to M$ such that there is a bundle map

$$\begin{array}{ccc} P_{Spin(2)} & \longrightarrow & P_{SO(2)} \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

The standard spin representation $\rho : Spin(2) \to U(\mathbb{C}^2)$ allows us to consider the associated complex vector bundle

$$\Sigma M = P_{Spin(2)} \times_\rho \mathbb{C}^2,$$

$\Sigma M$ is a complex vector bundle over $M$ with an Hermitian metric $\langle \cdot, \cdot \rangle$, which is called the spinor bundle for the given spin structure.

There exists a Clifford multiplication $TM \otimes \Sigma M \to \Sigma$, denoted by $v \otimes \psi \mapsto v \cdot \psi$, which satisfies the Clifford relations

$$v \cdot w \cdot \psi + w \cdot v \cdot \psi = -2g(v, w)\psi$$

for all $v, w \in \Gamma(TM)$ and $\psi \in \Gamma(\Sigma M)$. The Clifford multiplication is skew-symmetric in the following way

$$\langle v \cdot \psi, \xi \rangle = -\langle \psi, v \cdot \xi \rangle$$

for any $v \in \Gamma(TM)$ and $\psi, \xi \in \Gamma(\Sigma M)$

Let $\nabla$ be the Levi-Civita connection on $(M, g)$, then there is a connection (still denoted by $\nabla$) on $\Sigma M$ compatible with the Hermitian metric $\langle \cdot, \cdot \rangle$. In local coordinates, let $\{\gamma_1, \gamma_2\}$ be a orthonormal basis of $TM$, then the Dirac operator $\not{\partial} : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ is expressed by $\not{\partial} := \gamma_i \cdot \nabla_{\gamma_i}$.

There is a standard decomposition

$$\Sigma M = \Sigma M^+ \oplus \Sigma M^-,$$

where the two direct summands are respectively the $\pm 1$-eigenspaces of the Clifford multiplication $i\gamma_1 \cdot \gamma_2$.

The Dirac operator is a first order elliptic differential operator, it interchanges the subbundles $\Sigma M^\pm$, and it is a symmetric operator with respect to the $L^2$-product.

**Lemma 1.4.** *Let $\psi, \xi \in \Gamma(\Sigma M)$ be spinor fields. Then*

$$\int_M \langle \partial\!\!\!/ \psi,\, \xi \rangle = \int_M \langle \psi,\, \partial\!\!\!/ \xi \rangle\,.$$

Useful references for spin manifolds, Clifford multiplication and the Dirac operators are [Fr00] and [LM89].

Now Let $u$ be a smooth map from $(M,\, g)$ to another Riemannian manifold $(N\, h)$ of dimension $n \geq 2$. On $\Sigma M \otimes u^{-1}TN$, there is a metric and a natural connection induced from those on $\Sigma M$ and $u^{-1}TN$. In local coordinates, a section $\psi \in \Gamma(\Sigma M \otimes u^{-1}TN)$ can be expressed by

$$\psi(x) = \psi^\alpha(x) \otimes \frac{\partial}{\partial y^\alpha}(u(x)),$$

where $\psi^\alpha$ is a spinor and $\{\frac{\partial}{\partial y^\alpha}\}$ is a local basis of $TN$.

The induced connection $\nabla$ on $\Sigma M \otimes u^{-1}TN$ can be expressed by

$$\nabla\psi(x) = \nabla\psi^\alpha(x) \otimes \frac{\partial}{\partial y^\alpha}(u(x)) + \Gamma^\gamma_{\alpha\beta}(u(x))\, \nabla u^\alpha(x)\, \psi^\beta(x) \otimes \frac{\partial}{\partial y^\gamma}(u(x)),$$

where $\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols of $N$.

We define the Dirac operator $\mathcal{D}$ along the map $u$ by

$$\mathcal{D}\psi(x) = \partial\!\!\!/ \psi^\alpha(x) \otimes \frac{\partial}{\partial y^\alpha}(u(x)) + \Gamma^\gamma_{\alpha\beta}(u(x))\, \nabla_{\gamma_i} u^\alpha(x)\, \left(\gamma_i \cdot \psi^\beta(x)\right) \otimes \frac{\partial}{\partial y^\gamma}(u(x)), \quad (1.1)$$

where $\gamma_1,\, \gamma_2$ is a local orthonormal basis of $M$.

As the usual Dirac operator $\partial\!\!\!/$, the Dirac operator $\mathcal{D}$ along the map $u$ is a symmetric operator with respect to the $L^2$-product.

**Lemma 1.5.** *Let $\psi, \xi \in \Gamma(\Sigma M \otimes u^{-1}TN)$, then*

$$\int_M \langle \mathcal{D}\psi,\, \xi \rangle = \int_M \langle \psi,\, \mathcal{D}\xi \rangle\,.$$

The Dirac-harmonic energy functional was first introduced by Chen-Jost-Li-Wang in [CJLW05] and [CJLW06]:

$$L(u,\, \psi) := \frac{1}{2} \int_M \left[ |du|^2 + \langle \psi,\, \mathcal{D}\psi \rangle \right]\, dv_g. \quad (1.2)$$

The critical points of $L(u,\, \psi)$ are called Dirac-harmonic maps.

The Dirac-harmonic maps are natural extensions of harmonic maps and harmonic spinors. In fact, when $\psi = 0$, $L(u,\, 0)$ is the energy functional $E(u)$, and its critical points are harmonic maps. On the other hand, when $u$ is a constant map, then $L(constant, \psi)$ is the Dirac functional, and its critical points are harmonic spinors. Harmonic spinors also have been well understood (see [LM89], [Hi74], [Ba96] and [BS92] for relevant references).

Calculating the Euler-Lagrange equations of $L$, one has:

**Proposition 1.6.** *[CJLW06] The Euler-Lagrange equations for L are*

$$\tau(u) = \mathcal{R}(u, \psi); \tag{1.3}$$
$$\mathcal{D}\psi = 0, \tag{1.4}$$

*where $\tau(u)$ is the tension field and $\mathcal{R}(u, \psi) \in \Gamma(u^{-1}TN)$ defined by*

$$\mathcal{R}(u, \psi) = \frac{1}{2} R^{\gamma}_{\alpha\delta\beta}(u) \left\langle \psi^{\alpha}, \nabla u^{\beta} \cdot \psi^{\delta} \right\rangle \frac{\partial}{\partial y^{\gamma}}(u). \tag{1.5}$$

*Here $R^{\gamma}_{\alpha\delta\beta}(u) = h^{\gamma\eta} R_{\alpha\delta\beta\eta}(u)$ are the components of the Riemannian curvature tensor of $h$.*

Throughout this thesis we use the following notations:

$$R_{\alpha\beta\delta\gamma} := \left\langle R^N \left( \frac{\partial}{\partial y^{\alpha}}, \frac{\partial}{\partial y^{\beta}} \right) \frac{\partial}{\partial y^{\delta}}, \frac{\partial}{\partial y^{\gamma}} \right\rangle_{TN},$$

for a local basis $\left\{ \frac{\partial}{\partial y^{\alpha}} \right\}$ of $TN$.

## 1.3 weakly Dirac-harmonic maps and the regularity

Let $(M, g)$ be an oriented, compact Riemannian surface with a given spin structure and $(N, h)$ a Riemannian manifold of dimension $n \geq 2$.

First, we will define the natural Sobolev space in which the functional $L(\cdot, \cdot)$ is well defined. By Nash's embedding theorem, we may assume that $(N, h)$ is isometrically embedded into the Euclidean space $\mathbb{R}^K$ for sufficiently large $K$.

**Definition 1.7.** *The Sobolev space $H^1(M, N)$ is defined by*

$$H^1(M, N) = \left\{ u \in H^1(M, \mathbb{R}^K) \mid \quad u(x) \in N \, a.e. x \in M \right\}.$$

**Definition 1.8.** *For $u \in H^1(M, N)$, the set of sections $\psi \in \Gamma(\Sigma M \otimes u^{-1}TN)$ is defined to be all $\psi = \left(\psi^1, \psi^2, \cdots, \psi^K\right) \in \left(\Gamma(\Sigma M)\right)^K$ with the property that $\psi(x)$ is along the map $u$, namely*

$$\sum_{i=1}^{K} v_i \psi^i = 0, \quad \text{for any normal vector } v = (v_1, v_2, \cdots, v_K) \text{ at } u(x).$$

*We say that* $\psi = \left(\psi^1, \psi^2, \cdots, \psi^K\right) \in W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$ *if* $d\psi \in L^{\frac{4}{3}}(M)$ *and* $\psi^i \in L^4(M)$ *for all* $1 \le i \le K$.

**Definition 1.9.** *A pair of map* $(u, \psi) \in H^1(M, N) \times W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$ *is called a weakly Dirac-harmonic map, if it is a critical point of* $L(\cdot, \cdot)$ *over the Sobolev space* $H^1(M, N) \times W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$.

Since $M$ is a compact Riemannian surface, we denote by $i_M > 0$ the injectivity radius of $M$. For $r \in (0, i_M)$, we denote by $B_r(x)$ the geodesic ball in $M$ with center $x \in M$ and radius $r$. Then we have the following regularity of the Dirac-harmonic maps.

**Theorem 1.10.** *($\epsilon$-regularity)* *There exists* $\epsilon_0 > 0$ *depending only on* $(M, g)$ *and* $(N, h)$*such that if* $(u, \psi) \in H^1(M, N) \times W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$ *is a weakly Dirac-harmonic map, for some* $x_0 \in M$ *and* $0 < r_0 \le i_M$ *such that*

$$\int_{B_{r_0}(x_0)} \left[|du|^2 + |\psi|^4\right] \le \epsilon_0^2,$$

*then*

$$\|u\|_{C^k} + \|\psi\|_{C^k} \le C\left(\int_{B_{r_0}(x_0)} \left[|du|^2 + |\psi|^4\right]\right),$$

*where the norm on the left hand is on* $B_{\frac{r_0}{2}}(x_0)$ *and the constant* $C$ *depends only on* $k$ *and the geometry of* $N$.

**Theorem 1.11.** *Let* $(u, \psi) \in H^1(M, N) \times W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$ *be a weakly Dirac-harmonic map. Then* $(u, \psi) \in C^\infty(M, N) \times C^\infty\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$

Theorem 1.11 was first proved by Chen-Jost-Li-Wang in [CJLW05] for the target manifold $N = S^{K-1} \subset \mathbb{R}^K$ and then by Zhu in [Zh09] for hypersurfaces $N \subset \mathbb{R}^K$, the general result was independently proved by Wang-Xu [WX09] and Chen-Jost-Wang-Zhu [CJWZ].

In [WX09], Wang-Xu also proved a convergence theorem of weakly convergent sequences of approximate Dirac-harmonic maps, which extends a corresponding convergence of approximate harmonic maps from surfaces due to Bethuel [Be93]. The theorem is following:

**Theorem 1.12.** *[WX09]* *Let* $(u_p, \psi_p) \in H^1(M, N) \times W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$ *be a sequence of weakly solutions to the approximate Dirac-harmonic map equation*

$$\tau(u) = \mathcal{R}(u, \psi) + S_p\,; \tag{1.6}$$

$$\mathcal{D}\psi = B_p. \tag{1.7}$$

*Assume that* $S_p \to 0$ *strongly in* $H^{-1}(M)$ *and* $B_p \rightharpoonup 0$ *weakly in* $L^{\frac{4}{3}}(\Sigma M \otimes u^{-1}TN)$. *If* $u_p \rightharpoonup u$ *in* $H^1(M, N)$ *and* $\psi_p \rightharpoonup \psi$ *in* $W^{1,\frac{4}{3}}(\Gamma(\Sigma M \otimes u^{-1}TN))$, *then* $(u, \psi) \in H^1(M, N) \times W^{1,\frac{4}{3}}(\Gamma(\Sigma M \otimes u^{-1}TN))$ *is a weakly Dirac-harmonic map.*

# Chapter 2

# Regularity for Dirac-harmonic map with a Ricci type spinor potential

## 2.1 Euler-Lagrange equation for $L_\lambda(u,\,\psi)$ via moving frames

Let $(M,g)$ be an oriented, compact Riemannian surface with a given spin structure and $(N,\,h)$ a Riemannian manifold of dimension $n \geq 2$. For a pair of map $(u,\psi) \in H^1(M,\,N) \times W^{1,\,\frac{4}{3}}(\Gamma(\Sigma M \otimes u^{-1}TN))$, we consider the following functional

$$L_\lambda(u,\,\psi) := \int_M \left\{ \frac{1}{2}\left[ |du|^2 + \langle \psi,\, \mathcal{D}\psi \rangle \right] + \lambda R_{\alpha\beta}(u) \left\langle \psi^\alpha,\, \psi^\beta \right\rangle \right\} dv_g.$$

**Remark 2.1.** $(i)$ *Since $\dim M = 2$, it follows from the Sobolev's embedding theorem that $\psi \in L^4(\Gamma(\Sigma M \otimes u^{-1}TN))$ provide $\psi \in W^{1,\,\frac{4}{3}}(\Gamma(\Sigma M \otimes u^{-1}TN))$. Thus, by Hölder's inequality one has*

$$\left| \int_M \langle \psi,\, \mathcal{D}\psi \rangle \, dv_g \right| \leq \|\psi\|_{L^4} \, \|\nabla\psi\|_{L^{\frac{4}{3}}} \leq C \, \|\psi\|_{L^4} \left[ \|d\psi\|_{L^{\frac{4}{3}}} + \|du\|_{L^2}^{\frac{3}{4}} \, \|\psi\|_{L^2}^{\frac{3}{4}} \right] < \infty.$$

*Hence $L_\lambda(u,\,\psi)$ is well defined on $H^1(M,\,N) \times W^{1,\,\frac{4}{3}}(\Gamma(\Sigma M \otimes u^{-1}TN))$.*

$(ii)$ *It follows from Lemma 1.5 that*

$$\int_M \overline{\langle \psi,\, \mathcal{D}\psi \rangle} = \int_M \langle \mathcal{D}\psi,\, \psi \rangle = \int_M \langle \psi,\, \mathcal{D}\psi \rangle\,,$$

*which implies*

$$\int_M \langle \psi, \mathcal{D}\psi \rangle \in \mathbb{R}. \tag{2.1}$$

*(iii)* $\int_M R_{\alpha\beta}(u) \left\langle \psi^\alpha, \psi^\beta \right\rangle dv_g \in \mathbb{R}.$
*In fact, we have*

$$\int_M R_{\alpha\beta}(u) \left\langle \psi^\alpha, \psi^\beta \right\rangle dv_g = \int_M R_{\beta\alpha}(u) \left\langle \psi^\beta, \psi^\alpha \right\rangle dv_g = \int_M R_{\alpha\beta}(u) \overline{\langle \psi^\alpha, \psi^\beta \rangle} \, dv_g.$$

First, we will deduce the Dirac-harmonic equations with a Ricci type potential in local coordinates, which will be essentially depended on the choice of local coordinates of the target manifold $N$.

The Euler-Lagrange equation for $L_\lambda(u, \psi)$ can be calculated in a similar way as in [CJLW06] and [CJW07].

In local coordinates, one can write the metric $h$ as follows

$$h = h_{\alpha\beta} dy^\alpha \otimes dy^\beta,$$

then $\psi$ can be expressed by

$$\psi(x) = \psi^\alpha(x) \otimes \frac{\partial}{\partial y^\alpha}(u(x)),$$

and $\mathcal{D}\psi$ can be expressed by

$$\mathcal{D}\psi = (\mathcal{D}\psi)^\alpha (x) \otimes \frac{\partial}{\partial y^\alpha}(u(x)) := \mathcal{D}\psi^\alpha(x) \otimes \frac{\partial}{\partial y^\alpha}(u(x)).$$

Define

$$A := \frac{1}{2} |du|^2, \quad B := \frac{1}{2} h_{\alpha\beta}(u) \left\langle \psi^\alpha, \mathcal{D}\psi^\beta \right\rangle, \quad C := \lambda R_{\alpha\beta}(u) \left\langle \psi^\alpha, \psi^\beta \right\rangle. \tag{2.2}$$

Then

$$L_\lambda(u, \psi) := \int_M (A + B + C) \, dv_g.$$

Similar to in [CJLW06], we have

$$\begin{aligned} \delta_\psi \int_M B &= \frac{1}{2} \int_M [\langle \delta_\psi \psi, \mathcal{D}\psi \rangle + \langle \psi, \mathcal{D}\delta_\psi \psi \rangle] \\ &= \frac{1}{2} \int_M [\langle \delta_\psi \psi, \mathcal{D}\psi \rangle + \langle \mathcal{D}\psi, \delta_\psi \psi \rangle] \\ &= \int_M Re \, \langle \delta_\psi \psi, \mathcal{D}\psi \rangle \\ &= \int_M Re \left[ h_{\alpha\beta}(u) \left\langle \delta_\psi \psi^\alpha, \mathcal{D}\psi^\beta \right\rangle \right], \end{aligned} \tag{2.3}$$

where $Re(z)$ denotes the real part of $z \in \mathbb{C}$. And we calculate $\delta_\psi C$ as

$$\begin{aligned} \delta_\psi C &= \lambda R_{\alpha\beta}(u) \left( \left\langle \delta_\psi \psi^\alpha, \, \psi^\beta \right\rangle + \left\langle \psi^\alpha, \, \delta_\psi \psi^\beta \right\rangle \right) \\ &= 2\lambda \, Re \left[ R_{\alpha\beta}(u) \left\langle \delta_\psi \psi^\alpha, \, \psi^\beta \right\rangle \right]. \end{aligned} \tag{2.4}$$

Combining (2.3) and ( 2.4), we have

$$\delta_\psi L_\lambda(u, \, \psi) = \int_M Re \left[ h_{\alpha\beta}(u) \left\langle \delta_\psi \psi^\alpha, \, \mathcal{D}\psi^\beta \right\rangle + 2\lambda R_{\alpha\beta}(u) \left\langle \delta_\psi \psi^\alpha, \, \psi^\beta \right\rangle \right] \, dv_g,$$

hence

$$h_{\alpha\beta}(u) \; \mathcal{D}\psi^\beta + 2\lambda R_{\alpha\beta}(u) \, \psi^\beta = 0,$$

which implies that

$$\mathcal{D}\psi^\alpha = -2\lambda R^\alpha_\beta(u) \, \psi^\beta, \tag{2.5}$$

where we have used the notation

$$R^\alpha_\beta = h^{\alpha\gamma} \, R_{\beta\gamma}.$$

Now we deduce the $u$-variation $\{u_t\}$ in the direction $\delta u = \frac{du_t}{dt}\Big|_{t=0} = v$ and $u_0 = u$, we have

$$\begin{aligned} \delta_u L_\lambda(u, \, \psi) &= \left. \frac{dL_\lambda(u_t, \, \psi)}{dt} \right|_{t=0} \\ &= \int_M \left( \frac{\partial}{\partial t} A + \frac{\partial}{\partial t} B + \frac{\partial}{\partial t} C \right) \, dv_g \Big|_{t=0}. \end{aligned} \tag{2.6}$$

It was known that (see e.g., [Jo02], [CJLW06] or [CJW07])

$$\int_M \frac{\partial}{\partial t} A \, dv_g \Big|_{t=0} = - \int_M h_{\alpha\eta} \, \tau^\alpha(u) \, v^\eta, \tag{2.7}$$

and

$$\begin{aligned} \int_M \frac{\partial}{\partial t} B \, dv_g \Big|_{t=0} &= \frac{1}{2} \int_M \left( \left\langle \psi^\alpha, \, \mathcal{D}\psi^\beta \right\rangle + \left\langle \mathcal{D}\psi^\beta, \, \psi^\alpha \right\rangle \right) \, v^\eta \, \Gamma^\delta_{\alpha\eta} \, h_{\delta\beta} \\ &\quad + \frac{1}{2} \int_M \left\langle \psi^\alpha, \, \nabla u^\delta \cdot \psi^\beta \right\rangle \, R_{\eta\delta\beta\alpha} v^\eta, \end{aligned} \tag{2.8}$$

where we denote by $\tau^\alpha(u)$ the tension field of the map $u$, and $\Gamma^\delta_{\alpha\eta}$ the Christoffel symbols of $N$.

By an easy computation for the Ricci type spinor potential term, we have

$$\int_M \frac{\partial}{\partial t} C\, dv_g \Big|_{t=0} = \lambda \int_M R_{\alpha\beta,\,\eta} \left\langle \psi^\alpha,\, \psi^\beta \right\rangle v^\eta. \tag{2.9}$$

Using (2.5), we have

$$\begin{aligned} \left\langle \psi^\alpha,\, \mathcal{D}\psi^\beta \right\rangle v^\eta\, \Gamma^\delta_{\alpha\eta}\, h_{\delta\beta} &= -2\lambda R^\gamma_\delta \left\langle \psi^\alpha,\, \psi^\delta \right\rangle v^\eta\, \Gamma^\xi_{\alpha\eta}\, h_{\xi\gamma} \\ &= -2\lambda R_{\delta\xi} \left\langle \psi^\alpha,\, \psi^\delta \right\rangle v^\eta\, \Gamma^\xi_{\alpha\eta}. \end{aligned} \tag{2.10}$$

Combining (2.6)-(2.10), we obtain

$$\begin{aligned} & \frac{dL_\lambda(u_t,\, \psi)}{dt}\Big|_{t=0} \\ = & \int_M \left\{ -h_{\alpha\eta}\, \tau^\alpha(u) - \lambda R_{\delta\xi}\, \Gamma^\xi_{\alpha\eta} \left( \left\langle \psi^\alpha,\, \psi^\delta \right\rangle + \left\langle \psi^\delta,\, \psi^\alpha \right\rangle \right) \right. \\ & \left. + \frac{1}{2} \left\langle \psi^\alpha,\, \nabla u^\delta \cdot \psi^\beta \right\rangle R_{\eta\delta\beta\alpha} + \lambda R_{\alpha\beta,\,\eta} \left\langle \psi^\alpha,\, \psi^\beta \right\rangle \right\} v^\eta\, dv_g. \end{aligned} \tag{2.11}$$

Therefore, we get the Euler-Lagrange equation for $L_\lambda(u,\, \psi)$

$$\left\{ \begin{array}{l} \tau^\gamma(u) = -\lambda\, h^{\gamma\eta}\, R_{\beta\xi}\, \Gamma^\xi_{\alpha\eta} \left( \left\langle \psi^\alpha,\, \psi^\beta \right\rangle + \left\langle \psi^\beta,\, \psi^\alpha \right\rangle \right) \\ +\frac{1}{2} \left\langle \psi^\alpha,\, \nabla u^\delta \cdot \psi^\beta \right\rangle R^\gamma_{\alpha\beta\delta} + \lambda\, h^{\gamma\eta}\, R_{\alpha\beta,\,\eta} \left\langle \psi^\alpha,\, \psi^\beta \right\rangle \\ \mathcal{D}\psi^\gamma = -2\lambda R^\alpha_\beta(u)\, \psi^\beta. \end{array} \right. \tag{2.12}$$

Note that

$$\begin{aligned} & -h^{\gamma\eta}\, R_{\beta\xi}\, \Gamma^\xi_{\alpha\eta} \left( \left\langle \psi^\alpha,\, \psi^\beta \right\rangle + \left\langle \psi^\beta,\, \psi^\alpha \right\rangle \right) + h^{\gamma\eta}\, R_{\alpha\beta,\,\eta} \left\langle \psi^\alpha,\, \psi^\beta \right\rangle \\ = & \; h^{\gamma\eta} \left( R_{\alpha\beta,\,\eta} - \Gamma^\xi_{\alpha\eta} R_{\beta\xi} - \Gamma^\xi_{\beta\eta} R_{\alpha\xi} \right) \left\langle \psi^\alpha,\, \psi^\beta \right\rangle \\ = & \; h^{\gamma\eta}\, R_{\alpha\beta;\,\eta} \left\langle \psi^\alpha,\, \psi^\beta \right\rangle, \end{aligned}$$

where $R_{\alpha\beta;\,\eta}$ denotes the covariant derivative of the Ricci curvature tensor $R_{\alpha\beta}$ with respect to $\frac{\partial}{\partial y^\eta}$.

Hence, we can write the Euler-Lagrange equation for $L_\lambda(u,\, \psi)$ as

$$\left\{ \begin{array}{lcl} \tau^\gamma(u) &=& \frac{1}{2} \left\langle \psi^\alpha,\, \nabla u^\delta \cdot \psi^\beta \right\rangle R^\gamma_{\alpha\beta\delta} + \lambda\, h^{\gamma\eta}\, R_{\alpha\beta;\,\eta} \left\langle \psi^\alpha,\, \psi^\beta \right\rangle \\ \mathcal{D}\psi^\gamma &=& -2\lambda R^\alpha_\beta(u)\, \psi^\beta. \end{array} \right. \tag{2.13}$$

Unfortunately, the above extrinsic version of the Euler-Lagrange equation for $L_\lambda(u,\, \psi)$ is not so convenient for us to handle the right hand term globally, because we could not directly extend the local basis $\left\{ \frac{\partial}{\partial y^\alpha} \right\}$ to $\mathbb{R}^K$. In the sequel of this section, we will prefer to use the moving frame, which was initially used by Hélein [He91] to prove his famous regularity result for harmonic maps from a surface and then used by many other authors [Be93, Wa05] for the study of regularities of elliptic and parabolic systems.

To derive the Euler-Lagrange equation for $L_\lambda(u,\,\psi)$ which we call Dirac-harmonic maps with a Ricci type spinor potential, we first assume that $N$ is parallizable. In this case, there exists a global orthonormal frame $\{\hat{e}_\alpha\}_{\alpha=1}^n$, $n = dim\, N$. Hence

$$e_\alpha = \hat{e}_\alpha(u(x)), \;\; 1 \le \alpha \le n,$$

forms an orthonormal frame on $u^{-1}(TN)$. Using this moving frame, we can write the spinor field $\psi$ along the map $u$ as follows:

$$\psi = \psi^\alpha \otimes e_\alpha,$$

where $\psi^\alpha \in \Gamma(\Sigma M)$, $\alpha = 1,\,2,\,\cdots,\,n$.

For simplicity, we also assume $\frac{\partial}{\partial x_i}$, $1 \le i \le 2$, is a local orthonormal coordinate frame on $M$.

Recall that the tension field $\tau$ of $u : M \to N$ is defined by

$$\tau(u) = g^{ij}\left(\nabla_{\frac{\partial}{\partial x_i}} du\right)(\frac{\partial}{\partial x_j}),$$

and

$$\tau^\alpha(u) = \langle \tau(u),\, e_\alpha\rangle_{u^{-1}(TN)}\,, \;\; \text{for } 1 \le \alpha \le n,$$

is the $\alpha$-component of $\tau(u)$ with respect to the frame $\{e_\alpha\}$.

The Dirac operator along the map $u$ with respect to the frame $\{e_\alpha\}$ is given by

$$\begin{aligned} \mathcal{D}\psi &= \frac{\partial}{\partial x_i}\cdot \nabla_{\frac{\partial}{\partial x_i}}(\psi^\alpha \otimes e_\alpha) \\ &= \partial\!\!\!/ \psi^\alpha \otimes e_\alpha + \Gamma^\gamma_{\alpha\beta}\left\langle \nabla_{\frac{\partial}{\partial x_i}} u,\, e_\beta\right\rangle_{u^{-1}(TN)}\left(\frac{\partial}{\partial x_i}\cdot\psi^\alpha\right)\otimes e_\gamma, \end{aligned}$$

where $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of $N$ with respect to the frame $\{e_\alpha\}$.

Denoting

$$\mathcal{D}\psi^\gamma := (\mathcal{D}\psi)^\gamma = \langle \mathcal{D}\psi,\, e_\gamma\rangle_{u^{-1}(TN)}\,, \;\; \text{for } 1 \le \gamma \le n.$$

Then we have

$$\mathcal{D}\psi^\gamma := (\mathcal{D}\psi)^\gamma = \partial\!\!\!/ \psi^\gamma + \Gamma^\gamma_{\alpha\beta}\left\langle \nabla_{\frac{\partial}{\partial x_i}} u,\, e_\beta\right\rangle_{u^{-1}(TN)}\left(\frac{\partial}{\partial x_i}\cdot\psi^\alpha\right). \tag{2.14}$$

The Ricci type spinor potential term can be expressed by

$$R_{\alpha\beta}(u)\left\langle \psi^\alpha,\, \psi^\beta\right\rangle = Ric(u)(e_\alpha,\, e_\beta)\left\langle \psi^\alpha,\, \psi^\beta\right\rangle.$$

Under these notations, we have:

**Lemma 2.2.** *Suppose that* $u : M \to N$ *and* $\psi \in \Gamma\left(\Sigma M \otimes u^{-1}TN\right)$ *is a pair of weakly Dirac-harmonic map with a Ricci type spinor potential. Then the Euler-Lagrange equation can be expressed under the frame* $\{e_\alpha\}$ *by*

$$\mathcal{D}\psi^\alpha = -2\lambda\, Ric(u)(e_\alpha,\, e_\beta)\, \psi^\beta, \tag{2.15}$$

$$\begin{aligned} \tau^\gamma(u) \;\; &= \;\; \frac{1}{2} R^N(e_\gamma,\, e_\delta,\, e_\alpha,\, e_\beta) \left\langle \nabla_{\frac{\partial}{\partial x_i}} u,\, e_\delta \right\rangle_{u^{-1}(TN)} \left\langle \psi^\beta,\, \frac{\partial}{\partial x_i} \cdot \psi^\alpha \right\rangle_{\Sigma M} \\ &\quad + \lambda\, R_{\alpha\beta,\,\gamma}(u) \left\langle \psi^\alpha,\, \psi^\beta \right\rangle - 2\lambda\, Ric(u)(e_\beta,\, e_\delta)\, \Gamma^\beta_{\alpha\gamma} \left( Re \left\langle \psi^\alpha,\, \psi^\delta \right\rangle \right), \end{aligned} \tag{2.16}$$

*where* $\{\frac{\partial}{\partial x_1},\, \frac{\partial}{\partial x_2}\}$ *is a local orthonormal coordinate frame on* $M$ *and* $Re(z)$ *denotes the real part of* $z \in \mathbb{C}$.

*Proof.* We just center on calculating the second equation (2.16) and the curvature term variational equation. We consider a variation $\{u_t\}$ of $u$ such that

$$\left.\frac{du_t}{dt}\right|_{t=0} = \eta = \eta^\gamma e_\gamma, \;\; \text{and } u_0 = u.$$

$$\begin{aligned} \frac{d}{dt}\mathcal{D}\psi \;\; &= \;\; \nabla_{\frac{d}{dt}} \left( \frac{\partial}{\partial x_i} \cdot \nabla_{\frac{\partial}{\partial x_i}} \left(\psi^\alpha \otimes e_\alpha\right) \right) \\ &= \;\; \frac{\partial}{\partial x_i} \cdot \nabla_{\frac{\partial}{\partial x_i}} \psi^\alpha \otimes \nabla_{\frac{d}{dt}} e_\alpha + \frac{\partial}{\partial x_i} \cdot \psi^\alpha \otimes \nabla_{\frac{d}{dt}} \nabla_{\frac{\partial}{\partial x_i}} e_\alpha \\ &= \;\; \frac{\partial}{\partial x_i} \cdot \nabla_{\frac{\partial}{\partial x_i}} \psi^\alpha \otimes \nabla_{\frac{d}{dt}} e_\alpha \\ &\quad + \frac{\partial}{\partial x_i} \cdot \psi^\alpha \otimes \left( \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{d}{dt}} e_\alpha + R^{u^{-1}} \left( \frac{d}{dt},\, \frac{\partial}{\partial x_i} \right) e_\alpha \right) \\ &= \;\; \frac{\partial}{\partial x_i} \cdot \nabla_{\frac{\partial}{\partial x_i}} \left( \psi^\alpha \otimes \nabla_{\frac{d}{dt}} e_\alpha \right) + \frac{\partial}{\partial x_i} \cdot \psi^\alpha \otimes R^{u^{-1}} \left( \frac{d}{dt},\, \frac{\partial}{\partial x_i} \right) e_\alpha \\ &= \;\; \mathcal{D}\nabla_{\frac{d}{dt}} \psi + \frac{\partial}{\partial x_i} \cdot \psi^\alpha \otimes R^N \left( u_* \left( \frac{d}{dt} \right),\, u_* \left( \frac{\partial}{\partial x_i} \right) \right) e_\alpha. \end{aligned} \tag{2.17}$$

Hence

$$\begin{aligned}
\frac{d}{dt}\int_M \langle\psi,\mathcal{D}\psi\rangle &= \int_M \left[\left\langle\nabla_{\frac{d}{dt}}\psi,\mathcal{D}\psi\right\rangle + \left\langle\psi,\mathcal{D}\nabla_{\frac{d}{dt}}\psi\right\rangle\right] \\
&\quad + \int_M \left\langle\psi^\beta,\, \frac{\partial}{\partial x_i}\cdot\psi^\alpha\right\rangle \left\langle e_\beta,\, R^N\left(u_*\left(\frac{d}{dt}\right),\, u_*\left(\frac{\partial}{\partial x_i}\right)\right) e_\alpha\right\rangle \\
&= \int_M \left[\left\langle\nabla_{\frac{d}{dt}}\psi,\mathcal{D}\psi\right\rangle + \left\langle\mathcal{D}\psi,\nabla_{\frac{d}{dt}}\psi\right\rangle\right] \\
&\quad + \int_M \left\langle\psi^\beta,\, \frac{\partial}{\partial x_i}\cdot\psi^\alpha\right\rangle \left\langle e_\beta,\, R^N\left(u_*\left(\frac{d}{dt}\right),\, u_*\left(\frac{\partial}{\partial x_i}\right)\right) e_\alpha\right\rangle .
\end{aligned} \tag{2.18}$$

Noting that

$$\begin{aligned}
u_*\left(\frac{d}{dt}\right) &= \left.\frac{du_t}{dt}\right|_{t=0} = \eta^\gamma e_\gamma, \\
u_*\left(\frac{\partial}{\partial x_i}\right) &= \left\langle\nabla_{\frac{\partial}{\partial x_i}}u,\, e_\delta\right\rangle e_\delta,
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{\frac{d}{dt}}\psi &= \nabla_{\frac{d}{dt}}\left(\psi^\alpha\otimes e_\alpha\right) \\
&= \left(\psi^\alpha\otimes\nabla_{\frac{d}{dt}}e_\alpha\right) \\
&= \eta^\gamma\,\Gamma^\beta_{\alpha\gamma}\,\psi^\alpha\otimes e_\beta .
\end{aligned}$$

Hence finally we obtain

$$\begin{aligned}
\frac{d}{dt}\int_M \langle\psi,\mathcal{D}\psi\rangle &= \int_M \left(\langle\psi^\alpha,\mathcal{D}\psi^\beta\rangle + \langle\mathcal{D}\psi^\beta,\,\psi^\alpha\rangle\right)\,\eta^\gamma\,\Gamma^\beta_{\alpha\gamma} \\
&\quad + \int_M \left\langle\psi^\beta,\,\frac{\partial}{\partial x_i}\cdot\psi^\alpha\right\rangle \left\langle\nabla_{\frac{\partial}{\partial x_i}}u,\, e_\delta\right\rangle \left\langle R^N\left(e_\gamma,\, e_\delta\right)e_\alpha,\, e_\beta\right\rangle\,\eta^\gamma .
\end{aligned} \tag{2.19}$$

On the other hand, similar to in [CJW07], we have

$$\begin{aligned}
\delta_\psi\left[\int_M \langle\psi,\mathcal{D}\psi\rangle\right] &= \int_M \langle\delta_\psi\psi,\mathcal{D}\psi\rangle + \int_M \langle\psi,\mathcal{D}\delta_\psi\psi\rangle \\
&= \int_M \left[\langle\delta_\psi\psi^\alpha,\mathcal{D}\psi^\alpha\rangle + \langle\mathcal{D}\psi^\alpha,\,\delta_\psi\psi^\alpha\rangle\right],
\end{aligned} \tag{2.20}$$

and $\delta_\psi C$ can be calculated as

$$\begin{aligned} & \delta_\psi \left[ \int_M Ric(u)(e_\alpha,\, e_\beta)\, \left\langle \psi^\alpha, \psi^\beta \right\rangle \right] \\ = & \int_M Ric(u)(e_\alpha,\, e_\beta)\, \left[ \left\langle \delta_\psi \psi^\alpha, \psi^\beta \right\rangle + \left\langle \psi^\alpha, \delta_\psi \psi^\beta \right\rangle \right]. \end{aligned} \tag{2.21}$$

Combining (2.20) and (2.21), we have

$$\delta_\psi L_\lambda(u, \psi) = \int_M Re\left( \langle \delta_\psi \psi^\alpha, \mathcal{D}\psi^\alpha \rangle + 2\lambda\, Ric(u)(e_\alpha,\, e_\beta)\, \left\langle \delta_\psi \psi^\alpha, \psi^\beta \right\rangle \right),$$

where $Re(z)$ denotes the real part for $z \in \mathbb{C}$. Hence

$$\mathcal{D}\psi^\alpha + 2\lambda\, Ric(u)(e_\alpha,\, e_\beta)\, \psi^\beta = 0, \tag{2.22}$$

which implies (2.15).

Plugging (2.22) into (2.19), we obtain

$$\begin{aligned} \delta_u L_\lambda(u, \psi) \;=\; & \int_M \left[ -\tau^\gamma(u) - \lambda\, Ric(u)(e_\beta,\, e_\delta)\, \Gamma^\beta_{\alpha\gamma}\, \left[ \left\langle \psi^\alpha,\, \psi^\delta \right\rangle + \left\langle \psi^\delta,\, \psi^\alpha \right\rangle \right] \right] \eta^\gamma \\ & + \frac{1}{2} \int_M \left\langle \psi^\beta,\, \frac{\partial}{\partial x_i} \cdot \psi^\alpha \right\rangle \left\langle \frac{\partial u}{\partial x_i},\, e_\delta \right\rangle \left\langle R^N\left(e_\gamma,\, e_\delta\right) e_\alpha,\, e_\beta \right\rangle \eta^\gamma \\ & + \int_M \lambda\, R_{\alpha\beta,\,\gamma}(u) \left\langle \psi^\alpha,\, \psi^\beta \right\rangle \eta^\gamma, \end{aligned}$$

and this implies (2.16)

□

Now we modify Hélein's enlargement argument to $N$ so that the assumption on the existence of global orthonormal frame can be achieved. To start it, we assume that $(\tilde{N},\, \tilde{h})$ is another Riemannian manifold without boundary such that there exists a totally geodesic, isometric embedding map $f\, :\, (N,\, h) \to (\tilde{N},\, \tilde{h})$. Given a pair $(u,\, \psi)$ with $u\, :\, (M,\, g) \to (N,\, h)$ and $\psi \in \Gamma\left(\Sigma M \otimes u^{-1}TN\right)$, we can consider the following map:

$$\begin{aligned} \tilde{u} \;&=\; f \circ u\, :\, (M,\, g) \to (\tilde{N},\, \tilde{h}), \\ \tilde{\psi} \;&=\; f_*\psi = \psi^\alpha \otimes f_*(e_\alpha)\, \in\, \Gamma\left( \Sigma M \otimes \tilde{u}^{-1}TN \right). \end{aligned}$$

Denote by $\tau(\tilde{u})$ and $\tilde{\mathcal{D}}$ the tension field of $\tilde{u}\, :\, (M,\, g) \to (\tilde{N},\, \tilde{h})$ and the Dirac operator along the map $\tilde{u}$ on $\Sigma M \otimes \tilde{u}^{-1}TN$ respectively. Then we have the following proposition.

**Proposition 2.3.** *Let $f\,:\,(N,\,h)\to(\tilde N,\,\tilde h)$ be a totally geodesic, isometric embedding map. Suppose $(u,\psi)\in H^1(M,\,N)\times W^{1,\,\frac{4}{3}}\left(\Gamma(\Sigma M\otimes u^{-1}TN)\right)$ is a pair of weakly Dirac-harmonic map with a Ricci type spinor potential. Then $(\tilde u,\,\tilde\psi)\in H^1(M,\,\tilde N)\times W^{1,\,\frac{4}{3}}\left(\Gamma(\Sigma M\otimes\tilde u^{-1}T\tilde N)\right)$ is also a pair of weakly Dirac-harmonic map with a Ricci type spinor potential from $(M,\,g)$ to $(\tilde N,\,\tilde h)$.*

*Proof.* By the chain rule of the tension field (See [Xi96, Chapter 1, (1.4.1)]), one has

$$\begin{aligned}
\tau(\tilde u) &= tr\left[(\nabla df)(\nabla u,\,\nabla u)\right]+f_*(\tau(u))\\
&= f_*(\tau(u))\\
&= \frac{1}{2}\,f_*\left[R^N(e_\gamma,\,e_\delta,\,e_\alpha,\,e_\beta)\left\langle\nabla_{\frac{\partial}{\partial x_i}}u,\,e_\delta\right\rangle_{u^{-1}(TN)}\left\langle\psi^\beta,\,\frac{\partial}{\partial x_i}\cdot\psi^\alpha\right\rangle_{\Sigma M}e_\gamma\right]\\
&\quad+\lambda\,f_*\left\{R_{\alpha\beta,\,\gamma}(u)\left\langle\psi^\alpha,\,\psi^\beta\right\rangle e_\gamma-2Ric(u)(e_\beta,\,e_\delta)\left[Re\left\langle\psi^\alpha,\,\psi^\delta\right\rangle\right]\Gamma^\beta_{\alpha\gamma}\,e_\gamma\right\},
\end{aligned}$$

where we have used the fact $\nabla df=0$ (see Definition 1.2).

We set

$$e^*_\alpha(\tilde u)=f_*(u)(e_\alpha),\quad\text{for } 1\le\alpha\le n.$$

Since $f$ is an isometric embedding, we conclude that $\{e^*_\alpha(\tilde u)\}_{1\le\alpha\le n}$ forms an orthonormal frame on $\tilde u^{-1}T\hat N$, where $\hat N:=f(N)$ is a totally geodesic submanifold of $(\tilde N,\,\tilde h)$.

Denote by $\tilde\Gamma^\beta_{\alpha\gamma}$ the Christoffel symbols of $(\tilde N,\,\tilde h)$, then $\Gamma^\beta_{\alpha\gamma}(x)=\tilde\Gamma^\beta_{\alpha\gamma}(\tilde u(x))$ for $x\in M$. Hence we have

$$\begin{aligned}
&f_*\left[R^N(e_\gamma,\,e_\delta,\,e_\alpha,\,e_\beta)\left\langle\nabla_{\frac{\partial}{\partial x_i}}u,\,e_\delta\right\rangle_{u^{-1}(TN)}\left\langle\psi^\beta,\,\frac{\partial}{\partial x_i}\cdot\psi^\alpha\right\rangle_{\Sigma M}e_\gamma\right]\\
=\;&R^N(e_\gamma,\,e_\delta,\,e_\alpha,\,e_\beta)\left\langle\nabla_{\frac{\partial}{\partial x_i}}u,\,e_\delta\right\rangle_{u^{-1}(TN)}\left\langle\psi^\beta,\,\frac{\partial}{\partial x_i}\cdot\psi^\alpha\right\rangle_{\Sigma M}f_*(e_\gamma)\\
=\;&R^{\tilde N}(e^*_\gamma,\,e^*_\delta,\,e^*_\alpha,\,e^*_\beta)\left\langle\nabla_{\frac{\partial}{\partial x_i}}\tilde u,\,e^*_\delta\right\rangle_{\tilde u^{-1}(T\tilde N)}\left\langle\psi^\beta,\,\frac{\partial}{\partial x_i}\cdot\psi^\alpha\right\rangle_{\Sigma M}e^*_\gamma
\end{aligned}$$

and

$$\begin{aligned}
&f_*\left\{R_{\alpha\beta,\,\gamma}(u)\left\langle\psi^\alpha,\,\psi^\beta\right\rangle e_\gamma-2Ric(u)(e_\beta,\,e_\delta)\left[Re\left\langle\psi^\alpha,\,\psi^\delta\right\rangle\right]\Gamma^\beta_{\alpha\gamma}\,e_\gamma\right\}\\
&=R^{\tilde N}_{\alpha\beta,\,\gamma}(\tilde u)\left\langle\psi^\alpha,\,\psi^\beta\right\rangle e^*_\gamma-2Ric^{\tilde N}(\tilde u)(e^*_\beta,\,e^*_\delta)\left[Re\left\langle\psi^\alpha,\,\psi^\delta\right\rangle\right]\tilde\Gamma^\beta_{\alpha\gamma}\,e^*_\gamma,
\end{aligned}$$

where we have used the Gauss-Codazzi equation and the the totally geodesic isometric embedding of $f$ to guarantee the validity of the above two equations.

Hence we have

$$\tau(\tilde{u}) = \frac{1}{2} R^{\tilde{N}}(e^*_\gamma,\, e^*_\delta,\, e^*_\alpha,\, e^*_\beta) \left\langle \nabla_{\frac{\partial}{\partial x_i}} \tilde{u},\, e^*_\delta \right\rangle_{\tilde{u}^{-1}(T\tilde{N})} \left\langle \psi^\beta,\, \frac{\partial}{\partial x_i} \cdot \psi^\alpha \right\rangle_{\Sigma M} e^*_\gamma$$
$$+\lambda \left( R^{\tilde{N}}_{\alpha\beta,\,\gamma}(\tilde{u}) \left\langle \psi^\alpha,\, \psi^\beta \right\rangle e^*_\gamma - 2Ric^{\tilde{N}}(\tilde{u})(e^*_\beta,\, e^*_\delta) \left[ Re \left\langle \psi^\alpha,\, \psi^\delta \right\rangle \right] \tilde{\Gamma}^\beta_{\alpha\gamma}\, e^*_\gamma \right).$$

This implies that $\tilde{u}$ satisfies the second equation (2.16) of the Dirac-harmonic map with a Ricci type spinor potential.

To verify that $\tilde{\psi}$ satisfies the first equation (2.15), we compute, as in Chen-Jost-Li-Wang [CJLW05, (2.6)], as follows

$$\begin{aligned} \tilde{\mathcal{D}}\tilde{\psi} &= f_*(\mathcal{D}\psi) + \left\langle \nabla_{\frac{\partial}{\partial x_i}} u,\, e_\delta \right\rangle_{u^{-1}(TN)} \frac{\partial}{\partial x_i} \cdot \psi^\gamma \otimes \nabla(df)(e_\beta,\, e_\gamma) \\ &= -2\lambda\, f_* \left( Ric(u)(e_\alpha,\, e_\beta)\, \psi^\beta \otimes e_\alpha \right) \\ &= -2\lambda\, Ric^{\tilde{N}}(\tilde{u})(e^*_\alpha,\, e^*_\beta)\, \psi^\beta \otimes e^*_\alpha, \end{aligned}$$

where we also have used the fact $\nabla(df) = 0$.

This proves the assertion.

□

With the help of the above proposition, now we can follow the enlargement construction to replace $N$ by another Riemannian manifold $\tilde{N}$ such that on the neighborhood of the image of the map $\tilde{u}$ there is a global orthonormal frame. Thus, without loss of generality, we assume that $N$ supports a global orthonormal frame $\{e_\alpha\}_{\alpha=1}^n$. Moreover, as in Hélein [He91, He02], we may assume that $\{e_\alpha\}_{\alpha=1}^n$ is a Coulomb frame with the following properties:

$$d^* \left( \langle d\, e_\alpha,\, e_\beta \rangle \right) = 0,\ 1 \le \alpha \le n, \tag{2.23}$$

$$\sum_{\alpha=1}^n \int_M \left| \nabla^{u^{-1}TN}\, e_\alpha \right|^2 dv_g \le C \int_M |\nabla u|^2\, dv_g \tag{2.24}$$

where $d^*$ is the conjugate operator of $d$.

## 2.2 Regularity for Dirac-harmonic map with a Ricci type spinor potential

In this section we will prove the regularity for Dirac-harmonic map with a Ricci type spinor potential. The theorem is following:

**Theorem 2.4.** *Let $(M,\, g)$ be an oriented, compact Riemannian surface with a given spin structure, and $(N,\, h)$ another Riemannian manifold of dimension $n \ge 2$. Suppose that $(u, \psi) \in H^1(M,\, N) \times W^{1,\, \frac{4}{3}} \left( \Gamma(\Sigma M \otimes u^{-1}TN) \right)$ is a pair of weakly Dirac-harmonic map with a Ricci type spinor potential, then $(u, \psi)$ must be smooth.*

To prove theorem 2.4, we will first prove an $\epsilon$-regularity theorem. We define an $n \times n$ 1-form valued matrix $\Omega$ as follows:

$$\Omega_{\gamma\delta} = \sum_{i=1}^{2} \left[ \frac{1}{2} R^N(e_\gamma,\, e_\delta,\, e_\alpha,\, e_\beta) \left\langle \psi^\beta,\, \frac{\partial}{\partial x_i} \cdot \psi^\alpha \right\rangle_{\Sigma M} \right] dx^i, \text{ for } 1 \leq \gamma, \delta \leq n.$$

Then we have the following fact.

**Proposition 2.5.** *For the above defined matrix $\Omega = \{\Omega_{\gamma\delta}\}$, the following properties hold:*

(1), $\Omega_{\gamma\delta}$ *is real* 1*-form valued for any* $1 \leq \gamma, \delta \leq n$*,*

(2), $\Omega$ *is skew-symmetric*

$$\Omega_{\gamma\delta} = -\Omega_{\delta\gamma}.$$

*Proof.* (1). We just need to observe that the skew-symmetry of the Clifford multiplication and the properties of the Hermitian metric $\langle \cdot,\, \cdot \rangle_{\Sigma M}$. We have

$$\begin{aligned} \overline{\left\langle \psi^\beta,\, \frac{\partial}{\partial x_i} \cdot \psi^\alpha \right\rangle_{\Sigma M}} &= \left\langle \frac{\partial}{\partial x_i} \cdot \psi^\alpha,\, \psi^\beta \right\rangle_{\Sigma M} \\ &= -\left\langle \psi^\alpha,\, \frac{\partial}{\partial x_i} \cdot \psi^\beta \right\rangle_{\Sigma M}. \end{aligned}$$

Therefore, using the anti-symmetric property of the Riemannian curvature $R^N$

$$R^N(e_\gamma,\, e_\delta,\, e_\beta,\, e_\alpha) = -R^N(e_\gamma,\, e_\delta,\, e_\alpha,\, e_\beta),$$

we can easily conclude that

$$\overline{\Omega_{\gamma\delta}} = \Omega_{\gamma\delta}.$$

This proves (1).

(2) follows directly from the anti-symmetric property of the Riemannian curvature $R^N$ with respect to its first two-components.

□

Under these notations, it is not hard to see that the second equation (2.16) of the Dirac-harmonic map with a Ricci type spinor potential can be written as

$$\begin{aligned} \tau^\gamma(u) &= \sum_{\delta=1}^{n} (\Omega_{\gamma\delta}) \circ \langle du,\, e_\delta \rangle_{TN} + \lambda\, R_{\alpha\beta,\,\gamma}(u) \left\langle \psi^\alpha,\, \psi^\beta \right\rangle \\ &\quad -\lambda\, Ric(u)(e_\beta,\, e_\delta)\, \Gamma^\beta_{\alpha\gamma} \left( \left\langle \psi^\alpha,\, \psi^\delta \right\rangle + \left\langle \psi^\delta,\, \psi^\alpha \right\rangle \right), \end{aligned} \tag{2.25}$$

where $\circ$ denotes the inner product of the 1-forms for distinguishing the Clifford multiplication, and

$$\langle du,\, e_\delta\rangle_{TN} = \left\langle \nabla_{\frac{\partial}{\partial x^i}} u,\, e_\delta \right\rangle_{TN} dx^i.$$

Note that for any given $1 \le \alpha \le n$, this equation yields another form of the second equation of the Dirac-harmonic map with a Ricci type spinor potential:

$$\begin{aligned}
-d^*\left(\langle du,\, e_\alpha\rangle_{TN}\right) &= \langle \tau(u),\, e_\alpha\rangle_{TN} + \langle du,\, de_\alpha\rangle_{TN} \\
&= \tau^\alpha(u) + \left\langle du,\, \sum_\delta \langle e_\delta,\, de_\alpha\rangle_{TN}\, e_\delta \right\rangle_{TN} \\
&= \sum_\delta \left[\Omega_{\alpha\delta} + \langle e_\delta,\, de_\alpha\rangle_{TN}\right] \circ \langle du,\, e_\delta\rangle_{TN} \\
&\quad + F_\lambda^\alpha(u,\, \psi).
\end{aligned} \tag{2.26}$$

For simplicity, here and in the sequel, we will denote

$$F_\lambda^\alpha(u,\, \psi) := \lambda\, R_{\gamma\beta,\,\alpha}(u) \left\langle \psi^\gamma,\, \psi^\beta \right\rangle - \lambda\, Ric(u)(e_\beta,\, e_\delta)\, \Gamma^\beta_{\alpha\gamma} \left( \left\langle \psi^\gamma,\, \psi^\delta \right\rangle + \left\langle \psi^\delta,\, \psi^\gamma \right\rangle \right). \tag{2.27}$$

To prove the theorems, we need a useful lemma of Rivière [Ri07, Lemma A.3].

**Lemma 2.6.** *There exist $\epsilon(n) > 0$ and $C(n)$ such that, for every $\Omega = \{\Omega_{\gamma\delta}\}_{1\le\gamma,\le\delta\le n}$ in $L^2\left(B_1,\, so(n) \otimes \Lambda^1\mathbb{R}^2\right)$ satisfying*

$$\int_{B_1} |\Omega|^2\, dv_g \le \epsilon(n),$$

*there exist $\xi \in W^{1,2}\left(B_1,\, so(n) \otimes \Lambda^2\mathbb{R}^2\right)$ and $P \in W^{1,2}\left(B_1,\, SO(n)\right)$ such that :*

*i)*

$$P^{-1}dP + P^{-1}\Omega P = d^*\xi \quad \text{in } B_1, \tag{2.28}$$

*ii)*

$$\xi = 0 \quad \text{on } \partial B_1, \tag{2.29}$$

*iii)*

$$\|\xi\|_{W^{1,2}(B_1)} + \|P\|_{W^{1,2}(B_1)} \le C(n)\, \|\Omega\|_{L^2(B_1)}\,. \tag{2.30}$$

*where $so(n)$ denotes the Lie algebra of $SO(n)$, or the space of all the skew-symmetric matrices, and $B_1 \subset \mathbb{R}^2$ is the unit ball.*

Now we are going to prove the regularity of the weakly Dirac-harmonic map with a Ricci type spinor potential.

Since the regularity is a local property, without loss of generality, we may assume for simplicity that $(M,\, g) = (B_1, g_0)$, where $g_0 = dx^2 + dy^2$ is the standard metric on $\mathbb{R}^2$. Note that in this case, the spinor bundle $\Sigma M$ is trivial and hence we can choose a coordinate system such that $\Sigma M = \mathbb{C}^2$ and the Dirac operator on $\Sigma M$ can be identified by the $\overline{\partial}$ operator as follows. The Clifford multiplication of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ on spinor fields can be identified by the multiplication with matrices

$$v_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \, v_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

If $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : B_1 \to \mathbb{C}^2$ is a spinor field, then the Dirac operator $\mathcal{D}$ acts on $\psi$ is given by

$$\mathcal{D}\psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x}\psi_1 \\ \frac{\partial}{\partial x}\psi_2 \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y}\psi_1 \\ \frac{\partial}{\partial y}\psi_2 \end{pmatrix} = 2 \begin{pmatrix} \frac{\partial}{\partial \overline{z}}\psi_2 \\ -\frac{\partial}{\partial z}\psi_1 \end{pmatrix}, \tag{2.31}$$

where

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \, \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

To prove Theorem 2.4, we first prove the following $\epsilon$-regularity theorem.

**Theorem 2.7.** *($\epsilon$-regularity) There exists $\epsilon_0 > 0$ such that if*

$$(u, \, \psi) \in H^1(B_1, \, N) \times W^{1, \frac{4}{3}}(B_1, \, \mathbb{C}^2 \otimes u^{-1}TN)$$

*is a pair of weakly Dirac-harmonic map with a Ricci type spinor potential satisfying*

$$\int_{B_1} \left[|du|^2 + |\psi|^4\right] \leq \epsilon_0^2, \tag{2.32}$$

*then*

$$\|u\|_{C^k} + \|\psi\|_{C^k} \leq C \left( \int_{B_1} \left[|du|^2 + |\psi|^4\right] \right),$$

*where the norm on the left hand is on $B_{\frac{1}{2}}$ and the constant $C$ depends only on $k$ and the geometry of $N$.*

*Proof.* Denoting

$$\Theta = (\Theta_{\alpha\beta}) := (\Omega_{\alpha\beta} + \langle e_\alpha, \, de_\beta \rangle),$$

then it follows from Proposition 2.5 and the fact $(\langle e_\alpha, \, de_\beta \rangle)$ is skew-symmetric, we know that $\Theta = (\Theta_{\alpha\beta})$ is skew-symmetric, or in other words, $\Theta = (\Theta_{\alpha\beta}) \in L^2 \left(B_1, \, so(n) \otimes \Lambda^1\mathbb{R}^2\right)$.

On the other hand, we rewrite the second equation (2.16) of the Dirac-harmonic map with a Ricci type spinor potential (also (2.26)) as:

$$-d^*\left(\langle du,\, e_\alpha\rangle_{TN}\right) = \sum_\beta (\Theta_{\alpha\beta}) \circ \langle du,\, e_\beta\rangle_{TN} + F^\alpha_\lambda(u,\,\psi), \tag{2.33}$$

where $F^\alpha_\lambda(u,\,\psi)$ is defined in (2.27).

It is obvious from the definition of $(\Theta_{\alpha\beta})$ that

$$\int_{B_1} |(\Theta_{\alpha\beta})|^2 \le C_1 \int_{B_1} |\nabla u|^2 + C_2 \int_{B_1} |\psi|^4, \tag{2.34}$$

where we have used the property (2.24) of the Coulomb frame.

First from (2.34), we know that there exists $r_0$, $0 < r_0 \le 1$ such that

$$\int_{B_{r_0}} |(\Theta_{\alpha\beta})|^2 \le \epsilon(n),$$

then we can handle the regularity problem in $B_{r_0}$ with $r_0$ fixed, (or rescaling by $x \mapsto \frac{x}{r_0}$, and let $U(x) = u(\frac{x}{r_0})$, $\Psi(x) = \frac{1}{\sqrt{r_0}}\psi(\frac{x}{r_0})$, then (2.33) could be reduced to a similar manner but with a varied term $r_0^2\, F^\alpha_\lambda(U, \Psi)$).

Hence without loss of generality, by (2.34) we can always assume that $\int_{B_1} |\Theta|^2 = \int_{B_1} |(\Theta_{\alpha\beta})|^2 \le \epsilon(n)$, where $\epsilon(n)$ is the same constant as in Lemma 2.6. Applying Lemma 2.6, we can find reversible matrix valued map $P \in W^{1,2}\left(B_1,\, SO(n)\right)$ and $\xi \in W^{1,2}\left(B_1,\, so(n) \otimes \Lambda^2\mathbb{R}^2\right)$ related to $(\Theta_{\alpha\beta})$ satisfying the three properties as in Lemma 2.6.

Applying the gauge transformation $P^{-1}$ to $(\langle du,\, e_\alpha\rangle_{TN})$ and observing the identity $dP^{-1} = -P^{-1}dPP^{-1}$, from (2.33) we obtain the following equation

$$
\begin{aligned}
& -div\left(P^{-1}\begin{pmatrix}\langle du,\, e_1\rangle\\ \cdot\\ \cdot\\ \cdot\\ \langle du,\, e_n\rangle\end{pmatrix}\right)\\
= \; & P^{-1}dPP^{-1}\circ\begin{pmatrix}\langle du,\, e_1\rangle\\ \cdot\\ \cdot\\ \cdot\\ \langle du,\, e_n\rangle\end{pmatrix} + P^{-1}\left(\Theta_{\alpha\beta}\right)\circ\begin{pmatrix}\langle du,\, e_1\rangle\\ \cdot\\ \cdot\\ \cdot\\ \langle du,\, e_n\rangle\end{pmatrix}\\
& +P^{-1}\begin{pmatrix}F_\lambda^1(u,\, \psi)\\ \cdot\\ \cdot\\ \cdot\\ F_\lambda^n(u,\, \psi)\end{pmatrix}\\
= \; & \left(P^{-1}dP + P^{-1}\,\Theta\, P\right)\circ P^{-1}\begin{pmatrix}\langle du,\, e_1\rangle\\ \cdot\\ \cdot\\ \cdot\\ \langle du,\, e_n\rangle\end{pmatrix}\\
& +P^{-1}\begin{pmatrix}F_\lambda^1(u,\, \psi)\\ \cdot\\ \cdot\\ \cdot\\ F_\lambda^n(u,\, \psi)\end{pmatrix}\\
= \; & d^*\xi\circ P^{-1}\begin{pmatrix}\langle du,\, e_1\rangle\\ \cdot\\ \cdot\\ \cdot\\ \langle du,\, e_n\rangle\end{pmatrix} + P^{-1}\begin{pmatrix}F_\lambda^1(u,\, \psi)\\ \cdot\\ \cdot\\ \cdot\\ F_\lambda^n(u,\, \psi)\end{pmatrix}. \qquad (2.35)
\end{aligned}
$$

The components of (2.35) can be written as

$$-div\left(p_{\alpha\beta}^{-1}\,\langle du,\, e_\beta\rangle\right) = d^*\xi_{\alpha\beta}\circ p_{\beta\gamma}^{-1}\,\langle du,\, e_\gamma\rangle + p_{\alpha\beta}^{-1}\,F_\lambda^\beta(u,\, \psi), \quad \alpha = 1,\, 2,\, \cdots,\, n \text{ in } B_1, \qquad (2.36)$$

where $P^{-1} = \left(p_{\alpha\beta}^{-1}\right)$.

For clarifying the estimates, we can assume that $u \in H_0^1(B_1,\, N)$. Otherwise, we can modify $u$ by multiplying a cut-off function and reduce (2.35) in a similar manner but with extra term which will not cause difficulties for the following estimates. For the detail technique we refer to Rivière-Struwe [RS08].

Now for $0 < R < 1$, by the Hodge decomposition, we have

$$p_{\alpha\beta}^{-1}\ \langle du,\, e_\beta\rangle = df_\alpha + d^* g_\alpha + h_\alpha, \quad \alpha = 1,\, 2,\, \cdots,\, n,$$

where $f_\alpha \in H_0^1(B_R)$ and $g_\alpha \in H^1(B_R,\, \Lambda^2\mathbb{R}^2)$ with vanishing boundary value on $\partial B_R$, and $h_\alpha \in L^2(B_R,\, \Lambda^1\mathbb{R}^2)$ is a harmonic 1-form. For the Hodge decomposition of forms in Sobolev spaces, we refer to Iwaniec-Martin [IM01, Cor. 10.5.1, p.236].

It follows from (2.36) that

$$-\Delta f_\alpha = -div\left(p_{\alpha\beta}^{-1}\ \langle du,\, e_\beta\rangle\right) = d^*\xi_{\alpha\beta} \circ p_{\beta\gamma}^{-1}\ \langle du,\, e_\gamma\rangle + p_{\alpha\beta}^{-1}\ F_\lambda^\beta(u,\,\psi), \tag{2.37}$$

and

$$\begin{aligned} -\Delta g_\alpha &= -d\left(p_{\alpha\beta}^{-1}\ \langle du,\, e_\beta\rangle\right) \\ &= -dp_{\alpha\beta}^{-1} \wedge (\langle du,\, e_\beta\rangle) - p_{\alpha\gamma}^{-1}\ (\langle de_\gamma,\, e_\beta\rangle) \wedge (\langle du,\, e_\beta\rangle)\,, \end{aligned} \tag{2.38}$$

where $\alpha = 1,\, 2,\, \cdots,\, n$.

First, we estimate $g_a$. In this case, we need to interpret that the target manifold $N$ is isometrically embedded into $\mathbb{R}^K$, so we can write equation (2.38) as

$$\begin{aligned} -\Delta g_a &= -dp_{\alpha\beta}^{-1} \wedge (\langle du,\, e_\beta\rangle) - p_{\alpha\gamma}^{-1}\ (\langle de_\gamma,\, e_\beta\rangle) \wedge (\langle du,\, e_\beta\rangle) \\ &= -\left(\frac{\partial p_{\alpha\beta}^{-1}}{\partial x}\, dx + \frac{\partial p_{\alpha\beta}^{-1}}{\partial y}\, dy\right) \wedge \left(\left\langle \nabla_{\frac{\partial}{\partial x}} u,\, e_\beta\right\rangle dx + \left\langle \nabla_{\frac{\partial}{\partial y}} u,\, e_\beta\right\rangle dy\right) \\ &\quad -p_{\alpha\gamma}^{-1}\ \left(\left\langle \nabla_{\frac{\partial}{\partial x}} e_\gamma,\, e_\beta\right\rangle dx + \left\langle \nabla_{\frac{\partial}{\partial y}} e_\gamma,\, e_\beta\right\rangle dy\right) \wedge \\ &\quad \wedge \left(\left\langle \nabla_{\frac{\partial}{\partial x}} u,\, e_\beta\right\rangle dx + \left\langle \nabla_{\frac{\partial}{\partial y}} u,\, e_\beta\right\rangle dy\right) \\ &= -\left(\frac{\partial p_{\alpha\beta}^{-1}}{\partial x}\, \left\langle \nabla_{\frac{\partial}{\partial y}} u,\, e_\beta\right\rangle - \frac{\partial p_{\alpha\beta}^{-1}}{\partial y}\, \left\langle \nabla_{\frac{\partial}{\partial x}} u,\, e_\beta\right\rangle\right) dx \wedge dy \\ &\quad -p_{\alpha\gamma}^{-1}\ \left(\left\langle \nabla_{\frac{\partial}{\partial x}} e_\gamma,\, e_\beta\right\rangle \left\langle \nabla_{\frac{\partial}{\partial y}} u,\, e_\beta\right\rangle - \left\langle \nabla_{\frac{\partial}{\partial y}} e_\gamma,\, e_\beta\right\rangle \left\langle \nabla_{\frac{\partial}{\partial x}} u,\, e_\beta\right\rangle\right) dx \wedge dy \\ &= -\left(\frac{\partial p_{\alpha\beta}^{-1}}{\partial x}\, u_y^i\, e_\beta^i - \frac{\partial p_{\alpha\beta}^{-1}}{\partial y}\, u_x^i\, e_\beta^i\right) dx \wedge dy \\ &\quad -p_{\alpha\gamma}^{-1}\ \left((e_\gamma)_x^i\ e_\beta^i u_y^j\, e_\beta^j - (e_\gamma)_y^i\ e_\beta^i u_x^j\, e_\beta^j\right) dx \wedge dy, \end{aligned} \tag{2.39}$$

where $P^{-1} = \left(p_{\alpha\gamma}^{-1}\right)$, $(e_\gamma)_x^i = \left(\nabla_{\frac{\partial}{\partial x}} e_\gamma\right)^i$ and $(\cdot)^i$ denote realizing the embedding into $\mathbb{R}^K$.

Fix a number $1 < p < 2$ and let $q > 2$ be the conjugate exponent of $p$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$). Since $g_\alpha \in H^1(B_R,\, \Lambda^2\mathbb{R}^2)$ with vanishing boundary value on $\partial B_R$, by duality we have

$$\|dg_\alpha\|_{L^p} \le C \sup_{\substack{\|d\varphi\|_{L^q}\le 1\\ \varphi\in W_0^{1,q}(B_R)}} \int_{B_R} \langle dg_\alpha,\, d\varphi\rangle\,. \tag{2.40}$$

For $R < 1$, the Sobolev embedding theorem implies $W_0^{1,q}(B_R) \hookrightarrow C^{1-\frac{2}{q}}(B_R)$ and for $\varphi \in W_0^{1,q}(B_R)$, there holds

$$\|\varphi\|_{L^\infty(B_R)} \le CR^{1-\frac{2}{q}}\ \|\nabla\varphi\|_{L^q(B_R)}\,,$$

and by Hölder's inequality, there holds

$$\|\nabla\varphi\|_{L^2(B_R)} \le C\,R^{1-\frac{2}{q}}\ \|\nabla\varphi\|_{L^q(B_R)}\,.$$

For such $\varphi$ then we can estimate

$$\begin{aligned}
\int_{B_R} \langle dg_\alpha,\, d\varphi\rangle &= = -\int_{B_R} \varphi\Delta\, g_\alpha \\
&= -\int_{B_R} \left(\frac{\partial p^{-1}_{\alpha\beta}}{\partial x}\, u^i_y - \frac{\partial p^{-1}_{\alpha\beta}}{\partial y}\, u^i_x\right) e^i_\beta\,\varphi \\
&\quad -\int_{B_R} p^{-1}_{\alpha\gamma}\ \left((e_\gamma)^i_x\ e^i_\beta u^j_y\, e^j_\beta - (e_\gamma)^i_y\ e^i_\beta u^j_x\, e^j_\beta\right)\ \varphi \\
&:= I + II.
\end{aligned} \tag{2.41}$$

For $v \in L^1(B_R)$, we denote by $\overline{v}_{B_R}$ the average value of $v$ on $B_R$, that is,

$$\overline{v}_{B_R} := \frac{1}{\pi R^2}\int_{B_R} v.$$

Integrating by parts, we have

$$\begin{aligned}
I &= -\int_{B_R} \left(\frac{\partial p^{-1}_{\alpha\beta}}{\partial x}\, u^i_y - \frac{\partial p^{-1}_{\alpha\beta}}{\partial y}\, u^i_x\right) e^i_\beta\,\varphi \\
&= \int_{B_R} \left(\frac{\partial p^{-1}_{\alpha\beta}}{\partial x}\,\frac{\partial(e^i_\beta\,\varphi)}{\partial y} - \frac{\partial p^{-1}_{\alpha\beta}}{\partial y}\,\frac{\partial(e^i_\beta\,\varphi)}{\partial x}\right)\ \left(u^i - \overline{u^i}_{B_R}\right) \\
&\le C\left\|\left(\frac{\partial p^{-1}_{\alpha\beta}}{\partial x}\,\frac{\partial(e^i_\beta\,\varphi)}{\partial y} - \frac{\partial p^{-1}_{\alpha\beta}}{\partial y}\,\frac{\partial(e^i_\beta\,\varphi)}{\partial x}\right)\right\|_{\mathcal{H}(B_R)} \|u\|_{BMO(B_R)} \\
&\le C\ \|dP\|_{L^2(B_R)}\ \left(\|\varphi\|_{L^\infty(B_R)}\ \|\nabla e_\beta\|_{L^2(B_R)} + \|d\varphi\|_{L^2(B_R)}\right)\ \|u\|_{BMO(B_R)} \\
&\le C\,\epsilon_0\,R^{\frac{2}{p}-1}\ \|u\|_{BMO(B_R)}\,,
\end{aligned} \tag{2.42}$$

where we have used the duality of the Hardy space $\mathcal{H}$ and the $BMO$ space, the fact $|e_b\|_{L^\infty} \le 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In a similar way, we can estimate

$$
\begin{aligned}
II &= -\int_{B_R} p_{\alpha\gamma}^{-1} \left( (e_\gamma)^i_x \, e^i_\beta u^j_y \, e^j_\beta - (e_\gamma)^i_y \, e^i_\beta u^j_x \, e^j_\beta \right) \varphi \\
&= -\int_{B_R} \left( \frac{\partial e^i_\gamma}{\partial x} \, u^j_y - \frac{\partial e^i_\gamma}{\partial y} \, u^j_x \right) p_{\alpha\gamma}^{-1} \, e^i_\beta \, e^j_\beta \, \varphi \\
&= \int_{B_R} \left( \frac{\partial e^i_\gamma}{\partial x} \, \frac{\partial \left( p_{\alpha\gamma}^{-1} \, e^i_\beta \, e^j_\beta \, \varphi \right)}{\partial y} - \frac{\partial e^i_\gamma}{\partial y} \, \frac{\partial \left( p_{\alpha\gamma}^{-1} \, e^i_\beta \, e^j_\beta \, \varphi \right)}{\partial x} \right) \left( u^i - \overline{u^i}_{B_R} \right) \\
&\le \left\| \frac{\partial e^i_\gamma}{\partial x} \, \frac{\partial \left( p_{\alpha\gamma}^{-1} \, e^i_\beta \, e^j_\beta \, \varphi \right)}{\partial y} - \frac{\partial e^i_\gamma}{\partial y} \, \frac{\partial \left( p_{\alpha\gamma}^{-1} \, e^i_\beta \, e^j_\beta \, \varphi \right)}{\partial x} \right\|_{\mathcal{H}(B_R)} \|u\|_{BMO(B_R)} \\
&\le C \, \|\nabla e_g\|_{L^2} \left( \|\varphi\|_{L^\infty(B_R)} \, \|\nabla e_\beta\|_{L^2(B_R)} + \|\varphi\|_{L^\infty(B_R)} \, \|dP\|_{L^2} + \|d\varphi\|_{L^2(B_R)} \right) \\
&\quad \times \|u\|_{BMO(B_R)} \\
&\le C \, \epsilon_0 \, R^{\frac{2}{p}-1} \, \|u\|_{BMO(B_R)} \, ,
\end{aligned} \tag{2.43}
$$

where we have used the fact $\left\| p_{\alpha\gamma}^{-1} \right\|_{L^\infty} \le C$, since $\left( p_{\alpha\gamma}^{-1} \right) \in SO(n)$. And we also have used that $\|\nabla e_\gamma\|_{L^2} \le C \, \|du\|_{L^2}$ (see ( 2.24)).

Hence we conclude

$$
\|g_\alpha\|_{W^{1,\,p}(B_R)} \le C \, \epsilon_0 \, R^{\frac{2}{p}-1} \, \|u\|_{BMO(B_R)} \, . \tag{2.44}
$$

Now we proceed to estimate $\|f_\alpha\|_{W^{1,\,p}(B_R)}$ in a similar way.
It follows from equation (2.37) that

$$
\begin{aligned}
-\Delta f_\alpha &= d^* \xi_{\alpha\beta} \circ p_{\beta\gamma}^{-1} \, \langle du, \, e_\gamma \rangle + p_{\alpha\beta}^{-1} \;\; F^\beta_\lambda(u, \, \psi) \\
&= \left( -\frac{\partial \xi_{\alpha\beta}}{\partial y}, \, \frac{\partial \xi_{\alpha\beta}}{\partial x} \right) \circ p_{\beta\gamma}^{-1} \, \left( \left\langle \nabla_{\frac{\partial}{\partial x}} u, \, e_\gamma \right\rangle , \, \left\langle \nabla_{\frac{\partial}{\partial y}} u, \, e_\gamma \right\rangle \right) \\
&\quad + p_{\alpha\gamma}^{-1} \, F^\gamma_\lambda(u, \, \psi) \\
&= \frac{\partial \xi_{\alpha\beta}}{\partial x} \, p_{\beta\gamma}^{-1} \left\langle \nabla_{\frac{\partial}{\partial y}} u, \, e_\gamma \right\rangle - \frac{\partial \xi_{\alpha\beta}}{\partial y} \, p_{\beta\gamma}^{-1} \left\langle \nabla_{\frac{\partial}{\partial x}} u, \, e_\gamma \right\rangle + p_{\alpha\gamma}^{-1} \, F^\gamma_\lambda(u, \, \psi) \\
&= \frac{\partial \xi_{\alpha\beta}}{\partial x} \, p_{\beta\gamma}^{-1} u^i_y e^i_\gamma - \frac{\partial \xi_{\alpha\beta}}{\partial y} \, p_{\beta\gamma}^{-1} u^i_x e^i_\gamma + p_{\alpha\gamma}^{-1} \, F^\gamma_\lambda(u, \, \psi),
\end{aligned} \tag{2.45}
$$

where we have used the same notations as in (2.39).

For clarifying the estimates for $f_\alpha$, we can decompose $f_\alpha$ as $f_\alpha = f_{\alpha 1} + f_{\alpha 2}$, such that

$$\begin{cases} -\Delta f_{\alpha 1} = \frac{\partial \xi_{\alpha\beta}}{\partial x} \, p^{-1}_{\beta\gamma} u^i_y e^i_\gamma - \frac{\partial \xi_{\alpha\beta}}{\partial y} \, p^{-1}_{\beta\gamma} u^i_x e^i_\gamma \\ f_{\alpha 1}|_{\partial B_R} = 0, \end{cases}$$

and

$$\begin{cases} -\Delta f_{\alpha 2} = p^{-1}_{\alpha\gamma} \, F^\gamma_\lambda(u,\, \psi) \\ f_{\alpha 2}|_{\partial B_R} = 0. \end{cases}$$

By duality we have

$$\|f_{\alpha 1}\|_{W^{1,\,p}(B_R)} \le C \sup_{\substack{\|d\varphi\|_{L^q} \le 1 \\ \varphi \in W^{1,\,q}_0(B_R)}} \int_{B_R} \langle df_{\alpha 1},\, d\varphi \rangle \,, \tag{2.46}$$

and we estimate

$$\begin{aligned} \int_{B_R} \langle df_{\alpha 1},\, d\varphi \rangle &= -\int_{B_R} \varphi \, \Delta f_{\alpha 1} \\ &= \int_{B_R} \varphi \left( \frac{\partial \xi_{\alpha\beta}}{\partial x} \, p^{-1}_{\beta\gamma} u^i_y e^i_\gamma - \frac{\partial \xi_{\alpha\beta}}{\partial y} \, p^{-1}_{\beta\gamma} u^i_x e^i_\gamma \right) \\ &= -\int_{B_R} \left( \frac{\partial \xi_{\alpha\beta}}{\partial x} \frac{\partial \left(p^{-1}_{\beta\gamma} \, \varphi \, e^i_\gamma\right)}{\partial y} - \frac{\partial \xi_{\alpha\beta}}{\partial y} \frac{\partial \left(p^{-1}_{\beta\gamma} \, \varphi \, e^i_\gamma\right)}{\partial x} \right) \left( u^i - \overline{u^i}_{B_R} \right) \\ &\le \left\| \frac{\partial \xi_{\alpha\beta}}{\partial x} \frac{\partial \left(p^{-1}_{\beta\gamma} \, \varphi \, e^i_\gamma\right)}{\partial y} - \frac{\partial \xi_{\alpha\beta}}{\partial y} \frac{\partial \left(p^{-1}_{\beta\gamma} \, \varphi \, e^i_\gamma\right)}{\partial x} \right\|_{\mathcal{H}(B_R)} \|u\|_{BMO(B_R)} \\ &\le C \, |\nabla \, \xi\|_{L^2} \, \left| \nabla \, \left(p^{-1}_{\beta\gamma} \, \varphi \, e^i_\gamma\right) \right\|_{L^2} \, \|u\|_{BMO(B_R)} \\ &\le C \, \epsilon_0 \, R^{\frac{2}{p}-1} \, \|u\|_{BMO(B_R)} \,. \end{aligned} \tag{2.47}$$

Hence, it follows from (2.47) that

$$\|f_{\alpha 1}\|_{W^{1,\,p}(B_R)} \le C \, |\nabla \, \xi\|_{L^2} \; R^{\frac{2}{p}-1} \; \|u\|_{BMO(B_R)} \le C \, \epsilon_0 \, R^{\frac{2}{p}-1} \; \|u\|_{BMO(B_R)} \,. \tag{2.48}$$

To estimate $f_{\alpha 2}$, we just use the standard estimates for elliptic equation

$$\|f_{\alpha 2}\|_{W^{2,\,2}(B_r)} \le C \; \|F_\lambda(u,\, \psi)\|_{L^2(B_R)} \,,$$

which means

$$\|f_{\alpha 2}\|_{W^{1,\,p}(B_r)} \le C \, r^{\frac{2}{p}} \; \|F_\lambda(u,\, \psi)\|_{L^2(B_R)} \,. \tag{2.49}$$

From the classic Campanato estimates for harmonic functions, as in Giaguinta [Gi83, Proof of the Theorem 2.2, p.84], which yields for any $0 \le r \le R$ and the harmonic function $h_\alpha$

$$\int_{B_r} |h_\alpha|^p \leq C \left(\frac{r}{R}\right)^2 \int_{B_R} |h_\alpha|^p . \tag{2.50}$$

Thus, from (2.44), (2.48), (2.49) and (2.50), we can proceed the estimate for $u$ as follows:

$$\begin{aligned} \int_{B_r} |\nabla u|^p &\leq C \sup_\alpha \int_{B_r} |h_\alpha|^p + C \sup_\alpha \int_{B_r} (|df_\alpha|^p + |dg_\alpha|^p) \\ &\leq C \sup_\alpha \left(\frac{r}{R}\right)^2 \int_{B_R} |h_\alpha|^p + C \sup_\alpha \int_{B_r} (|df_\alpha|^p + |dg_\alpha|^p) \\ &\leq C \left(\frac{r}{R}\right)^2 \int_{B_R} |\nabla u|^p + C\, \epsilon_0^p\, R^{2-p}\, \|u\|_{BMO(B_R)} \\ &\quad + C\, r^2\, \|F_\lambda(u,\, \psi)\|^p_{L^2(B_R)} . \end{aligned} \tag{2.51}$$

For $1 \leq p \leq 2$, the Morrey space $M^{p,p}(B_R)$ is defined by

$$M^{p,p}(B_R) := \left\{ v \in L^p_{loc}(B_R) \,:\, \|v\|_{M^{p,p}(B_R)} < \infty \right\},$$

where

$$\|v\|_{M^{p,p}(B_R)} = \left( \sup_{r \leq R} \left\{ r^{p-2} \int_{B_R} |v|^p \right\} \right)^{\frac{1}{p}} .$$

It easy to see that $M^{2,2}(B_R) = L^2\,(B_R)$ and $M^{p,p}(B_R)$ behaves like $L^2$ from the view of scallions. We also note that

$$\|u\|_{BMO(B_R)} \leq C\, \|\nabla u\|_{M^{p,p}(B_R)} . \tag{2.52}$$

Now as in [RS08], for $x_0 \in B_{\frac{1}{2}}$, and $0 < r \leq \frac{1}{2}$, we set

$$\Phi(x_0,\, r) \; = r^{p-2} \int_{B_r(x_0)} |\nabla u|^p ,$$

and for $0 < R \leq \frac{1}{2}$, we define

$$\Psi(R) \; = \sup_{\substack{x_0 \in B_1 \\ 0 < r \leq R}} \Phi(x_0,\, r).$$

Then by (2.52), we have

$$\sup_{x_0 \in B_1} \|u\|^p_{BMO(B_R(x_0))} \leq C\Psi(R).$$

Thus, it follows from (2.51) that, there is a universal constant $C_1 > 0$ such that for any $x_0 \in B_{\frac{1}{2}}$ and $0 < r < R \leq \frac{1}{2}$, we have

$$\begin{aligned} \Phi(x_0,\, r) &\leq C\left(\frac{r}{R}\right)^p \Phi(x_0,\, R) + C\,\epsilon_0^p \left(\frac{r}{R}\right)^{p-2} \Phi(x_0,\, R) \\ &\quad + C\, r^p\, \|F_\lambda\|^p_{L^2(B_R)} \\ &\leq C_1 \left(\frac{r}{R}\right)^p \left(1 + \left(\frac{r}{R}\right)^{-2} \epsilon_0^p\right) \Psi(R) + C_1\, r^p\, \|F_\lambda\|^p_{L^2(B_R)} \\ &\leq C_1 \left(\frac{r}{R}\right)^p \left(1 + \left(\frac{r}{R}\right)^{-2} \epsilon_0^p\right) \Psi(R) + C_1\, r^p\, \epsilon_0^p, \end{aligned} \tag{2.53}$$

where we have used the fact $\|F_\lambda\|^2_{L^2(B_R)} \leq C\, \|\psi\|^4_{L^4(B_R)}$ (see (2.27)).

Now for any given $\alpha \in (0,1)$, fixing $\frac{r}{R} = \theta_0$ such that $2C_1\, \theta_0^{p(1-\alpha)} \leq 1$ (without loss of generality, we assume $2C_1 > 1$) , and choose $\epsilon_0 > 0$ such that $\epsilon_0^p = \theta_0^2$, then for any $R_0 < \frac{1}{2}$ and $0 < R < R_0$, we have

$$\begin{aligned} \Phi(x_0,\, \theta_0 R) &\leq C_1\, \theta_0^p \left(1 + \theta_0^{-2}\, \epsilon_0^p\right) \Psi(R) + C_1\, (\theta_0 R)^p\, \epsilon_0^p \\ &\leq \theta_0^{p\alpha}\, \Psi(R) + C_1\, \theta_0^{p+2}\, R^p \\ &\leq \theta_0^{p\alpha}\, \Psi(R_0) + C_1\, \theta_0^{p+2}\, R_0^p, \end{aligned} \tag{2.54}$$

Taking supremum with respect to $x_0$ and $R < R_0$, we have

$$\Psi(\theta_0\, R_0) \leq \theta_0^{p\alpha}\, \Psi(R_0) + C_1\, \theta_0^{p+2}\, R_0^p,\ \forall\, 0 < R_0 < \frac{1}{2}. \tag{2.55}$$

Then for any $r \in (0,\, \theta_0]$, choosing $k \in \mathbb{N}$ such that $\theta_0^{k+1} < r \leq \theta_0^k$ and iterating (2.55), we obtain

$$\begin{aligned} \Psi(r) &\leq \Psi(\theta_0^k) \leq \theta_0^{p\alpha}\, \Psi(\theta_0^{k-1}) + C_1\, \theta_0^{p+2}\, \theta_0^{p(k-1)} \\ &\leq \theta_0^{pk\alpha}\, \Psi(1) + C_1\, \theta_0^{p+2}\, \theta_0^{p(k-1)} \sum_{j=0}^{k-1} \theta_0^{pj(\alpha-1)} \\ &= \theta_0^{pk\alpha}\, \Psi(1) + C_1\, \theta_0^{pk+2}\, \frac{\theta_0^{pk(\alpha-1)} - 1}{\theta_0^{p(\alpha-1)} - 1} \\ &\leq \theta_0^{pk\alpha}\, \Psi(1) + \frac{C_1}{2C_1 - 1}\, \theta_0^{pk\alpha+2} \\ &\leq \theta_0^{pk\alpha}\, \left(\Psi(1) + C\, \epsilon_0^p\right). \end{aligned} \tag{2.56}$$

Using the fact that $\frac{1}{2} \leq \frac{k \log \theta_0}{\log r} < 1 < \frac{(k+1)\log \theta_0}{\log r}$, we have

$$\theta_0^{pk\alpha} < r^{\frac{p\alpha}{2}}.$$

It follows from ( 2.56) that

$$\Psi(r) \leq r^{\frac{p\alpha}{2}} \left(\Psi(1) + C\,\epsilon_0^p\right), \forall r \in (0,\, \theta_0]. \tag{2.57}$$

This implies that $u$ is Hölder continuous in $B_{\frac{1}{2}}$ by using the well-known Morrey's decay Lemma, c.f. [Gi83].

By this $C^{0,\alpha}$ continuous in hand, the rest of the proof of the regularity of $(u, \psi)$ is standard by using a similar argument of Chen-Jost-Li-Wang [CJLW05], practically we just need to modify the proof of Lemma 2.5 and Theorem 2.4 in [CJLW05] by a slight change of analysis caused by the Ricci type spinor potential terms in equations (2.15) and (2.16) in Lemma 2.2. The higher order estimates of $(u,\, \psi)$ follows from a bootstrap argument.

□

**Proof of Theorem** 2.4

Theorem 2.4 directly follows from the $\epsilon$- regularity theorem, since the regularity is a local property.

## 2.3 Approximate compactness

In this section we study the approximate compactness of Dirac-harmonic maps with a Ricci type spinor potential. The theorem is as follows:

**Theorem 2.8.** *Let* $(u_p, \psi_p) \in H^1(M,\, N) \times W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$ *be a sequence of weak solutions to the approximate Dirac-harmonic map equation with a Ricci type spinor potential*

$$\begin{cases} \tau^\gamma(u_p) &= \frac{1}{2}\left\langle \psi_p^\alpha,\, \nabla u_p^\delta \cdot \psi_p^\beta \right\rangle R^\gamma_{\alpha\beta\delta} + \lambda\, h^{\gamma\eta}\, R_{\alpha\beta;\,\eta}\left\langle \psi_p^\alpha,\, \psi_p^\beta\right\rangle + S_p^\gamma \\ \mathcal{D}\psi_p^\gamma &= -2\lambda R^\alpha_\beta(u_p)\,\psi_p^\beta + B_p^\gamma \end{cases} \tag{2.58}$$

*from a compact Riemannian surface* $(M,\, g_{ij})$ *to another compact Riemannian manifold* $(N^n,\, h_{\alpha\beta})$ *of dimension* $n \geq 2$*. Assume that* $\|u_p\|_{H^1}$*,* $\|\psi_p\|_{W^{1,\frac{4}{3}}}$ *are uniformly bounded,* $S_p = \left(S_p^\alpha\right)$ *strongly converges to* 0 *in* $H^{-1}\left(M,\, \mathbb{R}^K\right)$ *and* $B_p$ *strongly converges to* 0 *in* $L^{\frac{4}{3}}(\Sigma M \otimes u_p^{-1}TN)$*. Then there exists a subsequence* $(u_{p_k}, \psi_{p_k})$ *of* $(u_p, \psi_p)$*, such that* $(u_{p_k}, \psi_{p_k})$ *weakly converges in* $H^1(M,\, N)\times W^{1,\frac{4}{3}}\left(\Gamma(\Sigma M \otimes u^{-1}TN)\right)$ *to a pair* $(u, \psi)$*, which is a weakly Dirac-harmonic map with a Ricci type spinor potential. i.e.* $(u, \psi)$ *satisfies the equations*

$$\begin{cases} \tau^\gamma(u) &= \frac{1}{2}\left\langle \psi^\alpha,\, \nabla u^\delta \cdot \psi^\beta \right\rangle R^\gamma_{\alpha\beta\delta} + \lambda\, h^{\gamma\eta}\, R_{\alpha\beta;\,\eta}\left\langle \psi^\alpha,\, \psi^\beta\right\rangle \\ \mathcal{D}\psi^\gamma &= -2\lambda R^\alpha_\beta(u)\,\psi^\beta. \end{cases} \tag{2.59}$$

*Proof.* Since $(u_p, \psi_p)$ are uniformly bounded in $H^1(M, N) \times W^{1,\frac{4}{3}}$, we can always assume that the sequence pair $(u_p, \psi_p)$ weakly converges to some pair $(u, \psi)$ in $H^1(M, N) \times W^{1,\frac{4}{3}}(\Gamma(\Sigma M \otimes u^{-1}TN))$.

Observing that we can modify the argument of Proposition 2.3, we may assume that there exists a global orthonormal frame $\{(e_a)_p\}$ . As in [He91, He02], we may assume that $\{(e_a)_p\}_{\alpha=1}^n$ is a Coulomb moving frame with properties (2.23) and (2.24). Then the equation (2.58) under the orthonormal frame $\{(e_a)_p\}$ can be written as

$$\mathcal{D}\psi_p^\alpha = -2\lambda\, Ric(u_p)((e_\alpha)_p, (e_\beta)_p)\, \psi_p^\beta + B_p^\alpha, \tag{2.60}$$

$$\begin{aligned}
\tau^\gamma(u_p) &= \frac{1}{2} R^N((e_\gamma)_p, (e_\delta)_p, (e_\alpha)_p, (e_\beta)_p) \left\langle \nabla_{\frac{\partial}{\partial x_i}} u_p, (e_\delta)_p \right\rangle \left\langle \psi_p^\beta, \frac{\partial}{\partial x_i} \cdot \psi_p^\alpha \right\rangle_{\Sigma M} \\
&\quad + \lambda\, R_{\alpha\beta,\gamma}(u_p) \left\langle \psi_p^\alpha, \psi_p^\beta \right\rangle - 2\lambda\, Ric(u)((e_\beta)_p, (e_\delta)_p)\, \Gamma_{\alpha\gamma}^\beta \left( Re \left\langle \psi_p^\alpha, \psi_p^\delta \right\rangle \right) \\
&\quad + S_p^\gamma \\
&= \frac{1}{2} R^N((e_\gamma)_p, (e_\delta)_p, (e_\alpha)_p, (e_\beta)_p) \left\langle \nabla_{\frac{\partial}{\partial x_i}} u_p, (e_\delta)_p \right\rangle \left\langle \psi_p^\beta, \frac{\partial}{\partial x_i} \cdot \psi_p^\alpha \right\rangle_{\Sigma M} \\
&\quad + F_\lambda^\gamma(u_p, \psi_p) + S_p^\gamma,
\end{aligned} \tag{2.61}$$

where $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ is a local orthonormal coordinate frame on $M$ and $Re(z)$ denotes the real part of $z \in \mathbb{C}$, and we have used the notations $B_p^\alpha = \langle B_p, (e_\alpha)_p \rangle$ , $S_p^\gamma = \langle S_p, (e_\gamma)_p \rangle$.

For every $x \in M$ we assign $r_{x,p}$ such that $\int_{B_{r_{x,p}}(x)} |\Theta_p|^2 = \epsilon(n)$ or $r_{x,p} = \infty$ in case $\int_M |\Theta_p|^2 < \epsilon(n)$, where $\epsilon(n)$ is the constant as in Lemma 2.6 and $\Theta_p$ are indicated in (2.33) related to $(u_p, \psi_p)$. Hence $\{B_{r_{x,p}}(x)\}$ realizes a covering of $M$. By the compactness of $M$, we extract a Vitali covering from $\{B_{r_{x,p}}(x)\}$ which ensures that every point of $M$ is covered by a number of balls bounded by a universal number. Since $\int_M |\Theta_p|^2$ is uniformly bounded, the number of balls in such Vitali covering for each $p$, and after extraction of a subsequence (we still denote by $(u_p, \psi_p)$), we can assume that it is fixed and equals to $J$ independent of $p$. Let $\{B_{r_{i,p}}(x_{i,p})\}_{i=1,2,\cdots,J}$ be this covering. After extracting a subsequence, we still denote it by $(u_p, \psi_p)$, we can always assume that each sequence $\{x_{i,p}\}$ converges to a limit $x_i$ in $M$, and that each sequence $\{r_{i,p}\}$ converges to a non-negative number $r_i$ ( of course $r_i$ could be 0), $i = 1, 2, \cdots, J$.

We claim that $(u, \psi)$ satisfies the equations (2.15) and (2.16) on each ball $B_{r_i}(x_i)$, hence satisfies (2.59) on each ball $B_{r_i}(x_i)$.

Let $P_{i,p}$ and $\xi_{i,p}$ be given by Lemma 2.6 in Ball $B_{r_i}(x_i)$ for $\Theta_{i,p}$. Then we have

$$
\begin{aligned}
& -div\left( P_{i,p}^{-1} \begin{pmatrix} \left\langle du_p,\, (e_1)_p \right\rangle \\ \cdot \\ \cdot \\ \cdot \\ \left\langle du_p,\, (e_n)_p \right\rangle \end{pmatrix} \right) \\
= \;& d^*\xi_{i,p} \circ P_{i,p}^{-1} \begin{pmatrix} \left\langle du_p,\, (e_1)_p \right\rangle \\ \cdot \\ \cdot \\ \cdot \\ \left\langle du,\, (e_n)_p \right\rangle \end{pmatrix} + P_{i,p}^{-1} \begin{pmatrix} F_\lambda^1(u_p,\, \psi_p) \\ \cdot \\ \cdot \\ \cdot \\ F_\lambda^n(u_p,\, \psi_p) \end{pmatrix} + P_{i,p}^{-1}\, S_p,
\end{aligned} \tag{2.62}
$$

or equivalently

$$
\begin{aligned}
& -div\left( \left(p_{i,q}^{-1}\right)_{\alpha\beta} \left\langle du_q,\, (e_\beta)_q \right\rangle \right) \\
= \;& d^*\,(\xi_{i,q})_{\alpha\beta} \circ \left(p_{i,q}^{-1}\right)_{\beta\gamma} \left\langle du_q,\, (e_\gamma)_q \right\rangle + \left(p_{i,q}^{-1}\right)_{\alpha\beta} F_\lambda^\beta(u_q,\, \psi_q) + \left(p_{i,q}^{-1}\right)_{\alpha\beta} S_q^\beta
\end{aligned} \tag{2.63}
$$

for $\alpha = 1, 2, \cdots, n$, and

$$
\mathcal{D}\psi_p^\alpha = -2\lambda\, Ric(u_p)((e_\alpha)_p,\, (e_\beta)_p)\, \psi_p^\beta + B_p^\alpha \tag{2.64}
$$

in $B_{r_i}(x_i)$. Where $P_{i,p}$ and $\xi_{i,p}$ satisfy

$$
P_{i,p}^{-1} dP_{i,p} + P_{i,p}^{-1}\, \Theta_{i,p}\, P_{i,p} = d^*\xi_{i,p}.
$$

We can extract a subsequence which still denoted by $(u_p,\, \psi_p)$ such that each of its related moving frame$\{(e_\alpha)_p\}$ and the couples $P_{i,p}$ , $\xi_{i,p}$ weakly converge in $W^{1,2}$ to some limit $\{e_\alpha\}$ and $P_i$, $\xi_i$ respectively and strongly in $L^2$ in every ball $B_{r_i}(x_i)$; here the moving frame $\{(e_\alpha)_p\} \rightharpoonup \{e_\alpha\}$ in $W^{1,2}$, hence we have $\{e_\alpha\}$ is an orthonormal frame along the map $u$.

First we note that since $B_p \to 0$ strongly in $L^4$, it is easy to see that

$$
\begin{aligned}
& \mathcal{D}\psi_p^\gamma + 2\lambda\, Ric(u_p)((e_\gamma)_p,\, (e_\beta)_p)\, \psi_p^\beta \\
& \to \partial\!\!\!/ \psi^\gamma + \Gamma_{\alpha\beta}^\gamma \left\langle \nabla_{\frac{\partial}{\partial x_i}} u,\, e_\beta \right\rangle_{u^{-1}(TN)} \left( \tfrac{\partial}{\partial x_i} \cdot \psi^\alpha \right) + 2\lambda\, Ric(u)(e_\alpha,\, e_\beta)\, \psi^\beta
\end{aligned} \tag{2.65}
$$

in $\mathcal{D}'(B_{r_i}(x_i))$. Hence we have

$$\mathcal{D}\psi^\gamma + 2\lambda\, Ric(u)(e_\gamma,\, e_\beta)\,\psi^\beta = 0.$$

On the other hand, in every ball $B_{r_i}(x_i)$ we have

$$P_{i,p}^{-1}\begin{pmatrix} \left\langle du_p,\,(e_1)_p\right\rangle \\ \cdot \\ \cdot \\ \cdot \\ \left\langle du_p,\,(e_n)_p\right\rangle \end{pmatrix} \to P_i^{-1}\begin{pmatrix} \langle du,\, e_1\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle du,\, e_n\rangle \end{pmatrix} \quad \text{in } \mathcal{D}'\left(B_{r_i}(x_i)\right), \tag{2.66}$$

and

$$P_{i,p}^{-1}dP_{i,p} + P_{i,p}^{-1}\,\Theta_{i,p}\,P_{i,p} - d^*\xi_{i,p} \;\to\; P_i^{-1}dP_i + P_i^{-1}\,\Theta_i\,P_i - d^*\xi_i \tag{2.67}$$

in $\mathcal{D}'\left(B_{r_i}(x_i)\right)$. Obviously we have

$$\left|P_{i,p}^{-1}\,F_\lambda(u_p,\,\psi_p)\right| \le C\,\left|\psi_p\right|^2.$$

Hence

$$P_{i,p}^{-1}\begin{pmatrix} F_\lambda^1(u_p,\,\psi_p) \\ \cdot \\ \cdot \\ \cdot \\ F_\lambda^n(u_p,\,\psi_p) \end{pmatrix} \to\; P_i^{-1}\begin{pmatrix} F_\lambda^1(u,\,\psi) \\ \cdot \\ \cdot \\ \cdot \\ F_\lambda^n(u,\,\psi) \end{pmatrix} \quad \text{in } \mathcal{D}'\left(B_{r_i}(x_i)\right). \tag{2.68}$$

We also need to get the weak convergence of the term

$$d^*\xi_{i,p}\circ P_{i,p}^{-1}\begin{pmatrix} \left\langle du_p,\,(e_1)_p\right\rangle \\ \cdot \\ \cdot \\ \cdot \\ \left\langle du,\,(e_n)_p\right\rangle, \end{pmatrix},$$

or its components

$$d^*\left(\xi_{i,q}\right)_{\alpha\beta}\circ\left(p_{i,q}^{-1}\right)_{\beta\gamma}\left\langle du_q,\,(e_\gamma)_q\right\rangle \quad \alpha = 1,\,2,\,\cdots,\,n.$$

For any $\varphi \in C_0^\infty\left(B_{r_i}(x_i)\right)$, we have

$$\begin{aligned}
& \int_{B_{r_i}(x_i)} d^*\left(\xi_{i,q}\right)_{\alpha\beta} \circ \left(p_{i,q}^{-1}\right)_{\beta\gamma} \left(\left\langle du_q,\, \left(e_\gamma\right)_q\right\rangle\right) \varphi \\
= & \int_{B_{r_i}(x_i)} d^*\left(\xi_{i,q}\right)_{\alpha\beta} \circ \left(p_i^{-1}\right)_{\beta\gamma} \left(\langle du_q,\, e_\gamma\rangle\right) \varphi \\
& + \int_{B_{r_i}(x_i)} d^*\left(\xi_{i,q}\right)_{\alpha\beta} \circ \left(p_{i,q}^{-1} - p_i^{-1}\right)_{\beta\gamma} \left(\langle du_q,\, e_\gamma\rangle\right) \varphi \\
& + \int_{B_{r_i}(x_i)} d^*\left(\xi_{i,q}\right)_{\alpha\beta} \circ \left(p_{i,q}^{-1}\right)_{\beta\gamma} \left(\left\langle du_q,\, \left(e_\gamma\right)_q\right\rangle - \langle du_q,\, e_\gamma\rangle\right) \varphi \\
& := I + II + III. \qquad (2.69)
\end{aligned}$$

As a same calculation of (2.41) and (2.45), we can write down the three terms in the following:

$$\begin{aligned}
I &= \int_{B_{r_i}(x_i)} d^*\left(\xi_{i,q}\right)_{\alpha\beta} \circ \left(p_i^{-1}\right)_{\beta\gamma} \left(\langle du_q,\, e_\gamma\rangle\right) \varphi \\
&= \int_{B_{r_i}(x_i)} \left(-\frac{\partial\left(\xi_{i,q}\right)_{\alpha\beta}}{\partial y},\, \frac{\partial\left(\xi_{i,q}\right)_{\alpha\beta}}{\partial x}\right) \left(p_i^{-1}\right)_{\beta\gamma} \left(\left\langle \nabla_{\frac{\partial}{\partial x}} u_q,\, e_\gamma\right\rangle,\, \left\langle \nabla_{\frac{\partial}{\partial y}} u_q,\, e_\gamma\right\rangle\right) \varphi \\
&= \int_{B_{r_i}(x_i)} \left(\frac{\partial\left(\xi_{i,q}\right)_{\alpha\beta}}{\partial x} \nabla_{\frac{\partial}{\partial y}} u_q^\gamma - \frac{\partial\left(\xi_{i,q}\right)_{\alpha\beta}}{\partial y} \nabla_{\frac{\partial}{\partial x}} u_q^\gamma\right) \left(p_i^{-1}\right)_{\beta\gamma} \varphi \\
&\to \int_{B_{r_i}(x_i)} \left(\frac{\partial\left(\xi_i\right)_{\alpha\beta}}{\partial x} \nabla_{\frac{\partial}{\partial y}} u^\gamma - \frac{\partial\left(\xi_i\right)_{\alpha\beta}}{\partial y} \nabla_{\frac{\partial}{\partial x}} u^\gamma\right) \left(p_i^{-1}\right)_{\beta\gamma} \varphi \\
&= \int_{B_{r_i}(x_i)} d^*\xi_i \circ P_i^{-1}\left(\langle du,\, e_\alpha\rangle\right) \varphi, \qquad (2.70)
\end{aligned}$$

where we have used the fact that

$$\left(\frac{\partial\left(\xi_{i,q}\right)_{\alpha\beta}}{\partial x} \nabla_{\frac{\partial}{\partial y}} u_q^\gamma - \frac{\partial\left(\xi_{i,q}\right)_{\alpha\beta}}{\partial y} \nabla_{\frac{\partial}{\partial x}} u_q^\gamma\right) \to \left(\frac{\partial\left(\xi_i\right)_{\alpha\beta}}{\partial x} \nabla_{\frac{\partial}{\partial y}} u^\gamma - \frac{\partial\left(\xi_i\right)_{\alpha\beta}}{\partial y} \nabla_{\frac{\partial}{\partial x}} u^\gamma\right)$$

in $\mathcal{D}'\left(B_{r_i}(x_i)\right)$, which could be proved by the so called Div-Curl Lemma. In fact,

$$\left(\frac{\partial\left(\xi_{i,q}\right)_{\alpha\beta}}{\partial x} \nabla_{\frac{\partial}{\partial y}} u_q^\gamma - \frac{\partial\left(\xi_{i,q}\right)_{\alpha\beta}}{\partial y} \nabla_{\frac{\partial}{\partial x}} u_q^\gamma\right) \in \mathcal{H}\left(B_{r_i}(x_i)\right).$$

Thus we have on $\left(B_{r_i}(x_i)\right)$

$$\left(\frac{\partial\left(\xi_{i,q}\right)_{\alpha\beta}}{\partial x}\nabla_{\frac{\partial}{\partial y}}u_q^{\gamma}-\frac{\partial\left(\xi_{i,q}\right)_{\alpha\beta}}{\partial y}\nabla_{\frac{\partial}{\partial x}}u_q^{\gamma}\right)\to\left(\frac{\partial\left(\xi_{i}\right)_{\alpha\beta}}{\partial x}\nabla_{\frac{\partial}{\partial y}}u^{\gamma}-\frac{\partial\left(\xi_{i}\right)_{\alpha\beta}}{\partial y}\nabla_{\frac{\partial}{\partial x}}u^{\gamma}\right)$$

weakly in $\mathcal{H}\left(B_{r_i}(x_i)\right)$.

Combing this with the fact that $\left(p_i^{-1}\right)_{\beta\gamma}\varphi\in BMO\left(B_{r_i}(x_i)\right)$, we get (2.70).

For the convergence of $II$, we use the Hodge decomposition theorem again( cf. Iwaniec-Martin [IM01]), such that

$$\left(p_{i,q}^{-1}\right)_{\beta\gamma}\left(\langle du_q,\,e_\gamma\rangle\right)=df_{i,p}^{\beta}+d^*g_{i,p}^{\beta},\quad\beta=1,\,2,\,\cdots,\,n,$$

and $f_{i,p}$, $g_{i,p}$ satisfy

$$\|f_{i,p}\|_{L^2}+\|g_{i,p}\|_{L^2}\le C\,\|\nabla u_p\|_{L^2}\,,$$

where the norm is on the ball $B_{r_i}(x_i)$.

We may assume that $f_{i,p}\to f_i$ and $g_{i,p}\to g_i$ weakly in $W^{1,2}$. It follows from (2.66) that

$$\left(p_i^{-1}\right)_{\beta\gamma}\left(\langle du,\,e_\gamma\rangle\right)=df_i^{\beta}+d^*g_i^{\beta},\quad\beta=1,\,2,\,\cdots,\,n.$$

Therefore we have

$$\begin{aligned}
II &= \int_{B_{r_i}(x_i)} d^*\left(\xi_{i,q}\right)_{\alpha\beta}\circ\left(p_{i,q}^{-1}-p_i^{-1}\right)_{\beta\gamma}\left(\langle du_q,\,e_\gamma\rangle\right)\varphi\,dv_g\\
&\to \int_{B_{r_i}(x_i)} d^*\left(\xi_{i,q}\right)_{\alpha\beta}\circ\left(df_{i,p}^{\beta}-df_i^{\beta}+d^*g_{i,p}^{\beta}-d^*g_i^{\beta}\right)\varphi\,dv_g\\
&= \int_{B_{r_i}(x_i)} d^*\left(\xi_{i,q}\right)_{\alpha\beta}\circ\left(d^*g_{i,p}^{\beta}-d^*g_i^{\beta}\right)\varphi\,dv_g\\
&\quad -\int_{B_{r_i}(x_i)} d^*\left(\xi_{i,q}\right)_{\alpha\beta}\circ\left(f_{i,p}^{\beta}-f_i^{\beta}\right)d\varphi\,dv_g\\
&\to \int_{B_{r_i}(x_i)} d^*\left(\xi_{i,q}\right)_{\alpha\beta}\circ\left(d^*g_{i,p}^{\beta}-d^*g_i^{\beta}\right)\varphi\,dv_g\\
&\to \int_{B_{r_i}(x_i)}\varphi\,dv\quad\text{ as } p\to\infty,
\end{aligned}\tag{2.71}$$

where $dv=\sum_{j\in\Lambda}a_j\,\delta_{\bar{x}_j}$, $\Lambda$ is an at most countable set, $a_j\in\mathbb{R}$, $\bar{x_j}\in B_{r_i}(x_i)$, and $\sum_{j\in\Lambda}|a_j|<\infty$. Here we have used a compensated compactness result [Wa05, Lemma 3.4], which was first developed by Freire-Müller-Struwe [FMS97] in the context of wave maps research.

The convergence of $III$ can be treated in a similar way as above, we can calculate

$$III \to \sum_{j\in\Lambda} b_j\, \delta_{\bar{x}_j}, \tag{2.72}$$

where we use the same index set $\Lambda$, since it is also at most countable, and $\sum_{j\in\Lambda} |b_j| < \infty$.

Combining (2.66)-(2.72) and the fact

$$P_{i,p}^{-1} S_p \to 0 \qquad \text{in } \mathcal{D}'(B_{r_i}(x_i)),$$

we then have

$$\begin{aligned} -div\left(\left(p_i^{-1}\right)_{\alpha\beta} \langle du,\, e_\beta\rangle\right) &= d^*\,(\xi_i)_{\alpha\beta} \circ \left(p_i^{-1}\right)_{\beta\gamma} \langle du,\, e_\gamma\rangle + \left(p_i^{-1}\right)_{\alpha\beta} F_\lambda^\beta(u,\, \psi) \\ &\quad + \sum_{j\in\Lambda} (a_j + b_j)\; \delta_{\bar{x}_j}, \end{aligned}$$

and

$$\mathcal{D}\psi^\alpha = -2\lambda\, Ric(u)(e_\alpha,\, e_\beta)\, \psi^\beta$$

in $B_{r_i}(x_i)$. On the other hand, we know that $\delta_x \notin W^{-1,2} + L^1$, so we conclude that $a_j + b_j = 0$, for $j \in \Lambda$. Which implies that the pair $(u,\, \psi)$ satisfies the equations (2.15) and (2.16) in the ball $B_{r_i}(x_i)$.

The claim is proved.

Now we complete the proof of the theorem by modifying an argument arising by Rivière in [Ri07] for proving approximate compactness for nonlinear systems which share an anti-symmetric structure. It is clearly that every point of $M$ is in the closure of the union of the balls $B_{r_i}(x_i)$. Let $x \in M$ be a point which belongs to none of the balls $B_{r_i}(x_i)$. Then it must be seat on the circle, the boundary of one of the balls $B_{r_i}(x_i)$. Because of the convexity of the circle and the balls , it has to seat at the boundary of at least two different circles. Two different circles can intersect at only finitely many points( 0, 1 or 2 points), since there are finitely many circles, only finitely many points in $M$ can be outside of the union of the balls $B_{r_i}(x_i)$. Thus the distribution $-div\left(\left(p_i^{-1}\right)_{\alpha\beta} \langle du,\, e_\beta\rangle\right) - d^*\,(\xi_i)_{\alpha\beta} \circ \left(p_i^{-1}\right)_{\beta\gamma} \langle du,\, e_\gamma\rangle - \left(p_i^{-1}\right)_{\alpha\beta} F_\lambda^\beta(u,\, \psi)$ is supported at most finitely many points. Since $-div\left(\left(p_i^{-1}\right)_{\alpha\beta} \langle du,\, e_\beta\rangle\right) - d^*\,(\xi_i)_{\alpha\beta} \circ \left(p_i^{-1}\right)_{\beta\gamma} \langle du,\, e_\gamma\rangle - \left(p_i^{-1}\right)_{\alpha\beta} F_\lambda^\beta(u,\, \psi) \in W^{-1,2} + L^1$, so it is identically zero on $M$. This completes the proof.

□

**Remark 2.9.** *From the proof of the main theorems, it is easy to see that one can also study and prove the same kind of results for nonlinear coupled elliptic system in two dimension, such as Dirac-harmonic maps with other kind of nonlinear potential,when* *one assumes that the potential forcing term satisfies:* $|F_\lambda(u, \psi)| \leq C\,|\psi|^p$, *for* $0 \leq p < 4$. *However, we could not prove the regularity result for solution of Euler-Lagrange systems of the functional* $L_c$, *which was proposed in [CJW07] related with a general curvature potential term.*

# Chapter 3

# The boundary value problem for Dirac-harmonic maps with a Ricci type spinor potential

In Chen-Jost-Wang-Zhu [CJWZ], boundary conditions that of the type of the D-branes of superstring theory have been studied. After a geometric derivation of the boundary conditions, Chen-Jost-Wang-Zhu provided analytic regularity theory for the Dirac-harmonic maps at such a boundary. In this chapter, we will study the boundary value problem for Dirac-harmonic maps with a Ricci type spinor potential.

## 3.1 Free boundary problem for Dirac-harmonic maps

In this section, we will introduce the free boundary problem for Dirac-harmonic maps. For more details, see [CJWZ].

### 3.1.1 Chirality boundary conditions for the Dirac operator $\partial\!\!\!/$

In this subsection, we will recall the chirality boundary conditions for the usual Dirac operator $\partial\!\!\!/$ (see [HMR02]).

Let $M$ be a compact Riemannian spin surface $M$ with non-empty boundary $\partial M$. Then $M$ admits a chirality operator, the Clifford multiplication $G = i\gamma_1 \cdot \gamma_2$, where $\{\gamma_1, \gamma_2\}$ is an orthonormal frame on $M$. $G : \Sigma M \to \Sigma M$ is an endomorphism of the spinor bundle $\Sigma M$ satisfying:

$$G^2 = I, \quad \langle G\,\psi,\, G\,\varphi\rangle = \langle \psi,\, \varphi\rangle\,, \tag{3.1}$$
$$\nabla_X(G\,\psi) = G\,\nabla_X\psi, \quad X\cdot G\,\psi = -G\,(X\cdot\psi) \tag{3.2}$$

for all $X \in \Gamma(TM)$ and $\psi, \varphi \in \Gamma(\Sigma M)$. Here $I$ denotes the identity endomorphism of $\Sigma M$, and $\nabla$ denotes the induced connection by the Levi-Civita connection $\nabla$ on $TM$.

Denote by

$$\mathbf{S}(\partial M) := \Sigma M|_{\partial M}$$

the restricted spinor bundle with induced Hermitian product.

Let $\vec{n}$ be the outward unit normal vector field on $\partial M$, then the fibre preserving endomorphism $\vec{n}G : \Gamma\left(\mathbf{S}(\partial M)\right) \to \Gamma\left(\mathbf{S}(\partial M)\right)$ is self-adjoint with respect to the pointwise Hermitian product, whose square is the identity $I$. That is,

$$\langle \vec{n}G\psi, \varphi\rangle = \langle \psi, \vec{n}G\varphi\rangle \tag{3.3}$$
$$\left(\vec{n}G\right)^2 = I. \tag{3.4}$$

Hence it has two eigenvalues $+1$ and $-1$, and we have the decomposition

$$\mathbf{S}(\partial M) = V^+ \oplus V^-,$$

where $V^\pm$ is the eigensubbundle corresponding to the eigenvalue $\pm 1$.

Now we define the boundary condition as follows:

$$\mathbf{B}^\pm : L^2\left(\mathbf{S}(\partial M)\right) \to L^2(V^\pm) \tag{3.5}$$
$$\psi \mapsto \frac{1}{2}\left(I \pm \vec{n}G\right)\psi, \tag{3.6}$$

that is, the orthonormal projection onto the eigensubbundle $V^\pm$. $\mathbf{B}^\pm$ is indeed a local elliptic boundary condition for the Dirac operator $\partial\!\!\!/$ ( see [HMR02]).

We say that a spinor $\psi \in W^{1,\frac{4}{3}}$ satisfies the boundary condition $\mathbf{B}^\pm$ if

$$\mathbf{B}^\pm\psi\big|_{\partial M} = 0. \tag{3.7}$$

This type of (local) boundary condition has already been considered by many authors, for instance, [GHHP83, HMR02, Bu93, CJWZ] etc..

We have the following proposition, see [HMR02, p. 384] or [CJWZ, Prop. 3.1] for a proof.

**Proposition 3.1.** *If $\psi,\,\varphi \in W^{1,\frac{4}{3}}$ satisfy the boundary condition $\mathbf{B}^{\pm}$, then*

$$\langle \vec{n}\cdot\psi,\,\varphi\rangle = 0, \qquad on\ \partial M. \tag{3.8}$$

*In particular, one has*

$$\int_{\partial M} \langle \vec{n}\cdot\psi,\,\varphi\rangle = 0. \tag{3.9}$$

### 3.1.2 Chirality condition for the Dirac operator $\mathcal{D}$ along a map $u$

For a Riemannian spin surface with non-empty boundary, the Dirac operator $\mathcal{D}$ along a map is in general not formally self-adjoint. We have the following proposition.

**Proposition 3.2.**

$$\int_M \langle \mathcal{D}\,\psi,\,\varphi\rangle = \int_M \langle \psi,\,\mathcal{D}\,\varphi\rangle + \int_{\partial M} \langle \vec{n}\cdot\psi,\,\varphi\rangle \tag{3.10}$$

*for all $\psi, \varphi \in \Gamma\,(\Sigma M \otimes u^{-1}TN)$.*

Now we extend the chirality boundary condition to the Dirac operator $\mathcal{D}$ along a map $u : M \to N$.

**Definition 3.3.** *Given a submanifold $\mathcal{S}$ of $N$, we say a $(1,\,1)$ tensor $R$ on $\mathcal{S}$ is compatible, if the following properties hold*

$$\langle R(y)V,\,R(y)W\rangle = \langle R(y)V,\,R(y)W\rangle \quad \forall V,\,W \in T_yN,\,y \in \mathcal{S},$$

*and*

$$R(y)R(y)\,V = V \quad \forall V \in T_yN,\,y \in \mathcal{S}.$$

Such a compatible $(1,\,1)$ tensor on $\mathcal{S}$ always exists. For example, the identity map $id$ of $TN$, i.e.

$$id\,: T_yN \to T_yN, \quad \forall y \in \mathcal{S}.$$

Let $\mathcal{S}$ be a closed submanifold of $N$ with a compatible $(1,\,1)$ tensor $R$ and we consider a map $u \in C^\infty(M,\,N)$ such that $u(\partial M) \subset \mathcal{S}$.

We denote by

$$\mathbf{S}(\partial M)_u := \left(\Sigma M \otimes u^{-1}TN\right)\big|_{\partial M}$$

the restricted spinor bundle with the induced metric.

Let $\{e_\alpha\}$ be a local orthonormal frame of $N$, then a section $\psi \in \Gamma(\mathbf{S}(\partial M)_u)$ can be expressed as

$$\psi = \psi^\alpha \otimes e_\alpha.$$

Denote by $Id$ the identity map acting on $\Gamma\left(u^{-1}TN|_{\partial M}\right)$, and we define the endomorphism $\vec{n}G \otimes R : \Gamma(\mathbf{S}(\partial M)_u) \to \Gamma(\mathbf{S}(\partial M)_u)$ by

$$(\vec{n}G \otimes R)\,\psi := \vec{n}G\,\psi^\alpha \otimes R\,e_\alpha, \quad \forall \psi = \psi^\alpha \otimes e_\alpha \in \Gamma(\mathbf{S}(\partial M)_u). \tag{3.11}$$

Then one has

$$\begin{aligned} \langle(\vec{n}G \otimes R)\,\psi,\, \varphi\rangle &= \langle\psi,\, (\vec{n}G \otimes R)\,\varphi\rangle\,, &(3.12)\\ (\vec{n}G \otimes R)^2 &= I \otimes Id, &(3.13)\end{aligned}$$

that is, $\vec{n}G \otimes R$ is self-adjoint and its square is the identity map. Hence we have the decomposition

$$\mathbf{S}(\partial M)_u = V_u^+ \oplus V_u^-,$$

where $V_u^\pm$ is the eigensubbundle corresponding to the eigenvalue $\pm 1$ of $\vec{n}G \otimes R$. As in Section 3.1.1, the orthonormal projection onto the eigensubbundle $V_u^\pm$:

$$\begin{aligned} \mathbf{B}_u^\pm : \Gamma(\mathbf{S}(\partial M)_u) &\to \Gamma(\mathbf{S}(\partial M)_u)\\ \psi &\mapsto \frac{1}{2}\left(I \otimes Id \pm \vec{n}G \otimes R\right)\psi \end{aligned}$$

defines an elliptic boundary condition for the Dirac operator $\mathcal{D}$ along the map $u$.

We say that a spinor field $\psi \in \Gamma\left(\Sigma M \otimes u^{-1}TN\right)$ along a map $u$ satisfies the boundary condition $\mathbf{B}_u^\pm$ if

$$\mathbf{B}_u^\pm\ \psi|_{\partial M} = 0. \tag{3.14}$$

Furthermore, we have the following proposition [CJWZ, Prop. 3.3]:

**Proposition 3.4.** *If $\psi,\, \varphi \in \Gamma\left(\Sigma M \otimes u^{-1}TN\right)$ satisfy the chirality boundary condition $\mathbf{B}_u^\pm$, then*

$$\langle\vec{n}\cdot\psi,\, \varphi\rangle = 0, \quad \text{on } \partial M. \tag{3.15}$$

*In particular, one has*

$$\int_{\partial M} \langle\vec{n}\cdot\psi,\, \varphi\rangle = 0. \tag{3.16}$$

### 3.1.3 Free boundary condition for Dirac-harmonic maps with a Ricci type spinor potential

Let $\mathcal{S}$ be a closed $s$-dimensional submanifold of $(N^n, h)$. First, we will introduce a natural (1, 1) tensor $R$ which is compatible with respect to $\mathcal{S}$.

We consider a tubular neighborhood $U_\delta := \{y \in N \mid d_h(y, \mathcal{S}) < \delta\}$ of $\mathcal{S}$ in $N$, where $\delta$ is a small enough constant such that for any $y \in U_\delta$, there exists only one minimal geodesic connecting $y$ and some $y^{'} \in \mathcal{S}$ which attains the distance from $y$ to the closed submanifold $\mathcal{S}$. Then on $U_\delta$, we can define the geodesic reflection $\sigma$ as follows:

$$\begin{aligned} \sigma : U_\delta &\to U_\delta \\ y := \exp_{y'} v &\mapsto \sigma(y) := \exp_{y'}(-v), \end{aligned}$$

where $v \in T_{y'}N$ is uniquely determined by $y$. It is clear that $\sigma^2 = id : U_\delta \to U_\delta$, hence for $\delta$ small enough, $\sigma$ is a diffeomorphism. Thus, we can define a (1, 1) tensor $R$ on $\mathcal{S}$ by

$$R(y) := d\sigma(y), \quad \forall y \in \mathcal{S},$$

and since $\sigma|_{\mathcal{S}} = id$, hence $R(y) \in T_y^*N \otimes T_yN$ is indeed a (1, 1) tensor on $\mathcal{S}$. $R$ satisfies the following properties:

*i)* $R$ is compatible;

*ii)* $R(z)|_{T_y\mathcal{S}} = id, \; R(z)|_{T_y^\perp\mathcal{S}} = -id, \quad \forall y \in \mathcal{S},$

where $id$ denotes the identity endomorphism and $T_y^\perp\mathcal{S}$ denotes the subspace of $T_yN$ that is normal to $T_y\mathcal{S}$.

For a given $s$-dimensional closed submanifold $\mathcal{S}$ of $N$, in the sequel, we will always associate with it the compatible (1, 1) tensor $R$ constructed via the reflection $\sigma$ for $\mathcal{S}$.

Assume $u \in C^\infty(M, N)$ satisfies the boundary condition that $u(\partial M) \subset \mathcal{S}$ and let $\psi \in \Gamma(\Sigma M \otimes u^{-1}TN)$. We impose the free boundary condition for $\psi$ as the chirality boundary condition corresponding to $\mathcal{S}$, that is,

$$\mathbf{B}_u^{\pm}\, \psi|_{\partial M} = 0.$$

For more details and an example for $M = \mathbb{R}^2_+$, see [CJWZ].

We define

$$\begin{aligned} \chi(M, N; \mathcal{S}) \;:=\; & \{(u, \psi) |\, u \in C^\infty(M, N),\, u(\partial M) \subset \mathcal{S}; \\ & \psi \in \Gamma\left(\Sigma M \otimes u^{-1}TN\right), \mathbf{B}_u^{\pm}\, \psi|_{\partial M} = 0\}. \end{aligned}$$

**Definition 3.5.** *A pair $(u,\psi)\in\chi(M,N;\mathcal{S})$ is called a Dirac-harmonic map from $M$ to $N$ with a Ricci type spinor potential and free boundary on $\mathcal{S}$ if it is a critical point of $L_\lambda(\cdot,\cdot)$ in $\chi(M,N;\mathcal{S})$.*

Recall the definition of $L_\lambda(\cdot,\cdot)$ in section 2.1,

$$L_\lambda(u,\psi):=\int_M\left\{\frac{1}{2}\left[|du|^2+\langle\psi,\mathcal{D}\psi\rangle\right]+\lambda R_{\alpha\beta}(u)\left\langle\psi^\alpha,\psi^\beta\right\rangle\right\}dv_g.$$

Now Let $(u,\psi)\in\chi(M,N;\mathcal{S})$ be a pair of Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S}$, we will calculate the Euler-lagrange equation of the the functional $L_\lambda(u,\psi)$.

First, we consider a family of $(u,\psi_t)$ with $\frac{d\psi_t}{dt}\big|_{t=0}=\eta$ and $\psi_0=\psi$. Then we calculate

$$\begin{aligned}\left.\frac{dL_\lambda(u,\psi_t)}{dt}\right|_{t=0} &= \frac{1}{2}\int_M\left.\frac{d}{dt}\langle\psi_t,\mathcal{D}\psi_t\rangle\right|_{t=0}+\lambda\int_M R_{\alpha\beta}(u)\left.\frac{d}{dt}\left\langle\psi_t^\alpha,\psi_t^\beta\right\rangle\right|_{t=0}\\ &= \frac{1}{2}\int_M\left[\langle\eta,\mathcal{D}\psi\rangle+\langle\psi,\mathcal{D}\eta\rangle\right]\\ &\quad+\lambda\int_M R_{\alpha\beta}(u)\left[\left\langle\eta^\alpha,\psi^\beta\right\rangle+\left\langle\psi^\alpha,\eta^\beta\right\rangle\right]\\ &= \frac{1}{2}\int_M\left[\langle\eta,\mathcal{D}\psi\rangle+\langle\mathcal{D}\psi,\eta\rangle\right]-\frac{1}{2}\int_{\partial M}\langle\vec{n}\cdot\psi,\eta\rangle\\ &\quad+\lambda\int_M R_{\alpha\beta}(u)\left[\left\langle\eta^\alpha,\psi^\beta\right\rangle+\left\langle\psi^\alpha,\eta^\beta\right\rangle\right]\\ &= \int_M Re\left(h_{\alpha\beta}\left\langle\eta^\alpha,\mathcal{D}\psi^\beta\right\rangle+2\,\lambda R_{\alpha\beta}(u)\left\langle\eta^\alpha,\psi^\beta\right\rangle\right)\\ &\quad-\frac{1}{2}\int_{\partial M}\langle\vec{n}\cdot\psi,\eta\rangle,\end{aligned}\tag{3.17}$$

where we have used Proposition 3.2, and $Re(z)$ denotes the real part of $z\in\mathbb{C}$.

Since $\psi$, $\eta$ satisfy the boundary condition $\mathbf{B}_u^\pm$, hence it follows from Proposition 3.4 that $\int_{\partial M}\langle\vec{n}\cdot\psi,\eta\rangle=0$.

Thus, it follows from (3.17) that

$$\mathcal{D}\psi^\alpha=-2\,\lambda R_\beta^\alpha(u)\,\psi^\beta.\tag{3.18}$$

Now we deduce the $u$-variation $\{u_t\}$ in the direction $\frac{du_t}{dt}\big|_{t=0}=v$ with $u_0=u$. We choose a local basis $\left\{\frac{\partial}{\partial y^\alpha}\right\}$ of $N$, then $v=v^\alpha\frac{\partial}{\partial y^\alpha}$ and $\psi=\psi^\alpha\otimes\frac{\partial}{\partial y^\alpha}$, we compute

$$
\begin{aligned}
\left.\frac{dL_\lambda(u_t,\,\psi)}{dt}\right|_{t=0} &= \frac{1}{2}\int_M \left.\frac{d}{dt}\,|du_t|^2\right|_{t=0} + \frac{1}{2}\int_M \left.\frac{d}{dt}\,\langle\psi,\,\mathcal{D}\psi\rangle\right|_{t=0} \\
&\quad \left. +\lambda\,\frac{d}{dt}\int_M R_{\alpha\beta}(u)\,\left\langle\psi^\alpha,\,\psi^\beta\right\rangle\right|_{t=0} \\
&= \int_M \langle du,\,dv\rangle + \frac{1}{2}\int_M \left.\left[\left\langle\frac{d}{dt}\psi,\,\mathcal{D}\psi\right\rangle + \left\langle\psi,\,\frac{d}{dt}\mathcal{D}\psi\right\rangle\right]\right|_{t=0} \\
&\quad +\lambda\int_M R_{\alpha\beta,\,\gamma}(u)\,\left\langle\psi^\alpha,\,\psi^\beta\right\rangle v^\gamma \\
&= -\int_M \langle\tau(u),\,v\rangle + \frac{1}{2}\int_M \left\langle\psi^\alpha,\,\mathcal{D}\psi^\beta\right\rangle\,v^\eta\,\Gamma^\delta_{\alpha\,\eta}h_{\delta\beta} \\
&\quad +\frac{1}{2}\int_M \left\langle\psi,\,\mathcal{D}\left(\psi^\alpha\otimes\nabla_{\frac{d}{dt}}\frac{\partial}{\partial y^\alpha}\right)\right\rangle \\
&\quad +\frac{1}{2}\int_M \left\langle\psi^\alpha,\,\nabla u^\delta\cdot\psi^\beta\right\rangle\,R_{\eta\delta\beta\alpha}v^\eta + \lambda\int_M R_{\alpha\beta,\,\eta}(u)\,\left\langle\psi^\alpha,\,\psi^\beta\right\rangle v^\eta \\
&\quad +\int_{\partial M}\langle u_{\vec{n}},\,v\rangle \\
&= -\int_M \langle\tau(u),\,v\rangle + \frac{1}{2}\int_M \left[\left\langle\psi^\alpha,\,\mathcal{D}\psi^\beta\right\rangle + \left\langle\mathcal{D}\psi^\beta,\,\psi^\alpha\right\rangle\right]\,v^\eta\,\Gamma^\delta_{\alpha\,\eta}h_{\delta\beta} \\
&\quad +\frac{1}{2}\int_M \left\langle\psi^\alpha,\,\nabla u^\delta\cdot\psi^\beta\right\rangle\,R_{\eta\delta\beta\alpha}v^\eta + \lambda\int_M R_{\alpha\beta,\,\eta}(u)\,\left\langle\psi^\alpha,\,\psi^\beta\right\rangle v^\eta \\
&\quad +\int_{\partial M}\langle u_{\vec{n}},\,v\rangle - \int_{\partial M}\left\langle\vec{n}\cdot\psi,\,\psi^\alpha\otimes\nabla_{\frac{d}{dt}}\frac{\partial}{\partial y^\alpha}\right\rangle \\
&= \int_M \left\{-h_{\alpha\eta}\,\tau^\alpha(u) - \lambda R_{\delta\xi}\,\Gamma^\xi_{\alpha\eta}\,\left(\left\langle\psi^\alpha,\,\psi^\delta\right\rangle + \left\langle\psi^\delta,\,\psi^\alpha\right\rangle\right)\right. \\
&\quad \left. +\frac{1}{2}\,\left\langle\psi^\alpha,\,\nabla u^\delta\cdot\psi^\beta\right\rangle\,R_{\eta\delta\beta\alpha} + \lambda R_{\alpha\beta,\,\eta}\,\left\langle\psi^\alpha,\,\psi^\beta\right\rangle\right\}\,v^\eta \\
&\quad +\int_{\partial M} h_{\gamma\eta}\,\left(2u^\gamma_{\vec{n}} - h^{\gamma\xi}\,\left\langle\vec{n}\cdot\psi^\alpha,\,\psi^\beta\right\rangle\Gamma^\delta_{\beta\xi}\,h_{\alpha\delta}\right)\,v^\eta \\
&= \int_M \left\{-h_{\alpha\eta}\,\tau^\alpha(u) + \frac{1}{2}\,\left\langle\psi^\alpha,\,\nabla u^\delta\cdot\psi^\beta\right\rangle\,R_{\eta\delta\beta\alpha}\right. \\
&\quad \left. +\,\lambda R_{\alpha\beta;\,\eta}\,\left\langle\psi^\alpha,\,\psi^\beta\right\rangle\right\}\,v^\eta \\
&\quad +\int_{\partial M} h_{\gamma\eta}\,\left(2u^\gamma_{\vec{n}} - h^{\gamma\xi}\,\left\langle\vec{n}\cdot\psi^\alpha,\,\psi^\beta\right\rangle\Gamma^\delta_{\beta\xi}\,h_{\alpha\delta}\right)\,v^\eta,
\end{aligned} \tag{3.19}
$$

where $\Gamma^\xi_{\alpha\eta}$ are the Christoffel symbols of $N$, $u_{\vec{n}} = \frac{\partial u}{\partial \vec{n}}$, and $R_{\alpha\beta;\,\eta}$ denotes the covariant derivative of the Ricci curvature tensor $R_{\alpha\beta}$ with respect to $\frac{\partial}{\partial y^\eta}$; and we have used the fact $\mathcal{D}\psi^\alpha = -2\,\lambda R^\alpha_\beta(u)\,\psi^\beta$.

Since $v = \frac{du_t}{dt}\Big|_{t=0}$ is arbitrary, hence we have

$$\tau^{\gamma}(u) = \frac{1}{2}\left\langle \psi^{\alpha},\, \nabla u^{\delta}\cdot\psi^{\beta}\right\rangle R^{\gamma}_{\alpha\beta\delta} + \lambda\, h^{\gamma\eta}\, R_{\alpha\beta;\,\eta}\left\langle \psi^{\alpha},\, \psi^{\beta}\right\rangle \tag{3.20}$$

and

$$\left(2u^{\gamma}_{\vec{n}} - h^{\gamma\xi}\left\langle \vec{n}\cdot\psi^{\alpha},\, \psi^{\beta}\right\rangle \Gamma^{\delta}_{\beta\xi}\, h_{\alpha\delta}\right)\frac{\partial}{\partial y^{\gamma}} \perp \mathcal{S}. \tag{3.21}$$

Let $A(\cdot,\,\cdot)$ be the second fundamental form of the closed submanifold $\mathcal{S}$ in $N$. And for $\xi \in T^{\perp}N$, let $P_{\mathcal{S}}(\xi;\,\cdot)$ be the shape operator of $\mathcal{S}$ in $N$, then one has $\langle P_{\mathcal{S}}(\xi;\,X),\,Y\rangle = \langle A(X,\,Y),\,\xi\rangle$. After a calculation, we have the following proposition [CJWZ, Prop. 3.4]:

**Proposition 3.6.** *The condition (3.21) is equivalent to*

$$\left(\frac{\partial u}{\partial\vec{n}}\right)^{\top} = P_{\mathcal{S}}(\vec{n}\cdot\psi^{\perp};\,\psi^{\top}).$$

*In particular, if $\mathcal{S}$ is a totally geodesic submanifold of $N$, this reads*

$$\frac{\partial u}{\partial\vec{n}} \perp \mathcal{S}.$$

Therefore, from the above calculations and Proposition 3.6 , we obtain the following equivalent definition of Dirac-harmonic maps with a Ricci type spinor potential and free boundary on $\mathcal{S}$.

**Lemma 3.7.** *A pair $(u,\,\psi) \in \chi\,(M,\,N;\,\mathcal{S})$ is a Dirac-harmonic map with a Ricci type spinor potential from $M$ to $N$ and free boundary on $\mathcal{S}$ if and only if $(u,\,\psi)$ satisfies the equations*

$$\mathcal{D}\psi^{\alpha} = -2\,\lambda R^{\alpha}_{\beta}(u)\,\psi^{\beta}, \tag{3.22}$$

$$\tau^{\gamma}(u) = \frac{1}{2}\left\langle \psi^{\alpha},\, \nabla u^{\delta}\cdot\psi^{\beta}\right\rangle R^{\gamma}_{\alpha\beta\delta} + \lambda\, h^{\gamma\eta}\, R_{\alpha\beta;\,\eta}\left\langle \psi^{\alpha},\, \psi^{\beta}\right\rangle, \tag{3.23}$$

*and satisfies the following free boundary conditions:*

*i)*

$$\left(\frac{\partial u}{\partial\vec{n}}\right)^{\top} = P_{\mathcal{S}}(\vec{n}\cdot\psi^{\perp};\,\psi^{\top}); \tag{3.24}$$

*ii)*

$$\mathbf{B}^{\pm}_{u}\ \psi|_{\partial M} = 0. \tag{3.25}$$

### 3.1.4 Weakly Dirac-harmonic maps with a Ricci type spinor potential and a free boundary on $\mathcal{S}$

In this section, we will define the free boundary conditions for weakly Dirac-harmonic maps with a Ricci type spinor potential. By Nash's embedding theorem, we may assume that $(N, h)$ is isometrically embedded into the Euclidean space $\mathbb{R}^K$. By using the orthonormal decomposition $\mathbb{R}^K_y = T_yN \oplus T_y^{\perp}N$, for any $y \in N$, we can consider the bundles $\Sigma M \otimes u^{-1}TN$ and $\mathbf{S}(\partial M)_u = \Sigma M \otimes u^{-1}TN|_{\partial M}$ as subbundles of $\Sigma M \otimes u^{-1}\mathbb{R}^K$ and $\Sigma M \otimes u^{-1}\mathbb{R}^K\big|_{\partial M}$ respectively.

Let $V_\delta N$ be a tubular neighborhood of $N$ in $\mathbb{R}^K$ with the projection $P : V_\delta \to N$ (see [He02]). We define

$$\widetilde{R}(y) := d(\sigma \circ P)(y), \quad \forall y \in \mathcal{S}.$$

Since $(\sigma \circ P)|_{\mathcal{S}} = id$, hence $\widetilde{R}(y) \in T_y^*\mathbb{R}^k \otimes T_y\mathbb{R}^K$ is a (1, 1) tensor on $\mathcal{S}$. One can verify that $\widetilde{R}$ is compatible (for more details see [CJWZ]).

Denote

$$L^2\left(\mathbf{S}(\partial M)_u\right) := \left\{ \psi|_{\partial M} \,\middle|\, \psi \in W^{1,\frac{4}{3}}\left(\Sigma M \otimes u^{-1}TN\right) \right\}.$$

Then we can define an endomorphism

$$\vec{n}G \otimes \widetilde{R} : L^2\left(\mathbf{S}(\partial M)_u\right) \to L^2\left(\mathbf{S}(\partial M)_u\right),$$

which is self-adjoint and its square is the identity map. Moreover, we have the decomposition $\mathbf{S}(\partial M)_u = V_u^+ \oplus V_u^-$ and we can define an elliptic boundary condition

$$\widetilde{\mathbf{B}}_u^{\pm} : L^2\left(\mathbf{S}(\partial M)_u\right) \to L^2\left(V_u^{\pm}\right)$$

for the Dirac operator $\mathcal{D}$ along the map $u$.

For convenience, we will still denote $\widetilde{\mathbf{B}}_u^{\pm}$ by $\mathbf{B}_u^{\pm}$.

In analogy to the case of smooth sections in Proposition 3.4, we have:

**Proposition 3.8.** *For $u \in H^1(M, N)$, if $\psi, \varphi \in W^{1,\frac{4}{3}}(\Sigma M \otimes u^{-1}TN)$ satisfy the chirality boundary condition $\mathbf{B}_u^{\pm}$, then*

$$\langle \vec{n} \cdot \psi, \varphi \rangle = 0, \quad \text{on } \partial M, \tag{3.26}$$

*in particular, one has*

$$\int_{\partial M} \langle \vec{n} \cdot \psi, \varphi \rangle = 0. \tag{3.27}$$

Now we define the class $\chi^{1,\,2}_{1,\,\frac{4}{3}}(M,\,N;\,\mathcal{S})$ of the pair $(u,\psi)$ with free boundary as follows:

$$\begin{aligned}\chi^{1,\,2}_{1,\,\frac{4}{3}}(M,\,N;\,\mathcal{S}) \;&:=\; \Big\{(u,\psi)\,\big|u\in H^1(M,N)\,,u(\partial M)\subset\mathcal{S};\\ &\qquad \psi\in W^{1,\,\frac{4}{3}}\left(\Sigma M\otimes u^{-1}TN\right),\,\mathbf{B}^{\pm}_u\,\psi|_{\partial M}=0\Big\},\end{aligned}$$

where $u(\partial M)\subset\mathcal{S}$ means that $u|_{\partial M}$ maps almost all of $\partial M$ into $\mathcal{S}$ and $\mathbf{B}^{\pm}_u\,\psi|_{\partial M}=0$ means that $\mathbf{B}^{\pm}_u\,\psi|_{\partial M}$ vanish on almost all of $\partial M$.

**Definition 3.9.** *A pair $(u,\,\psi)\in\chi^{1,\,2}_{1,\,\frac{4}{3}}(M,\,N;\,\mathcal{S})$ is called a weakly Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S}$ if it is a critical point of the functional $L_\lambda(\cdot,\,\cdot)$ in $\chi^{1,\,2}_{1,\,\frac{4}{3}}(M,\,N;\,\mathcal{S})$.*

Similar to Proposition 2.3, we have the following proposition:

**Proposition 3.10.** *Let $f\,:\,(N,\,h)\to(\widetilde{N},\,\widetilde{h})$ be a totally geodesic, isometric embedding map. Suppose $(u,\psi)\in\chi^{1,\,2}_{1,\,\frac{4}{3}}(M,\,N;\,\mathcal{S})$ is a pair of weakly Dirac-harmonic map with a Ricci type spinor potential with free boundary on $\mathcal{S}$. Then $(\tilde{u},\,\tilde{\psi})\in\chi^{1\,2}_{1,\frac{4}{3}}\left(M,\,\widetilde{N};\,\widetilde{\mathcal{S}}\right)$ is also a pair of weakly Dirac-harmonic map with a Ricci type spinor potential from $(M,\,g)$ to $(\widetilde{N},\,\widetilde{h})$ with free boundary on $\widetilde{\mathcal{S}}$, where $\tilde{u}=f\circ u$, $\tilde{\psi}=u_*\psi$ and $\widetilde{\mathcal{S}}=f(\mathcal{S})$.*

With the help of the above proposition, now we can follow Hélein's enlargement construction to replace $N$ by another Riemannian manifold $\widetilde{N}$ such that there is a global orthonormal frame on the neighborhood of the image of the map $\tilde{u}$. Thus, without loss of generality, we assume that $N$ supports a global orthonormal frame $\{e_\alpha\}^n_{\alpha=1}$, which is a Coulomb frame with the properties (2.23) and (2.24). Under this orthonormal frame $\{e_\alpha\}^n_{\alpha=1}$, a spinor field $\psi$ along the map $u$ can be expressed as $\psi=\psi^\alpha\otimes e_\alpha$.

Now we can rewrite Lemma 2.2 as follows:

**Proposition 3.11.** *Suppose that $(u,\psi)\in H^1(M,\,N)\otimes W^{1,\,\frac{4}{3}}(\Sigma M\otimes u^{-1}TN)$ is a weakly Dirac-harmonic map with a Ricci type spinor potential. Then*

$$\int_M\langle\psi,\,\mathcal{D}\xi\rangle+2\lambda\int_M Ric(u)(e_\alpha,\,e_\beta)\left\langle\psi^\alpha,\,\xi^\beta\right\rangle=0,$$

$$\begin{aligned}-\int_M du\cdot\nabla V \;&=\; \int_M\Big\{\frac{1}{2}R^N(e_\eta,\,e_\delta,\,e_\alpha,\,e_\beta)\,\langle u_*(\gamma_i),\,e_\delta\rangle_{u^{-1}(TN)}\left\langle\psi^\beta,\,\gamma_i\cdot\psi^\alpha\right\rangle_{\Sigma M}\\ &\quad+\lambda R_{\alpha\beta,\,\eta}\left\langle\psi^\alpha,\,\psi^\beta\right\rangle-\lambda R_{\delta\xi}\,\Gamma^\xi_{\alpha\eta}\left(\left\langle\psi^\alpha,\,\psi^\delta\right\rangle+\left\langle\psi^\delta,\,\psi^\alpha\right\rangle\right)\Big\}\,V^\eta\end{aligned}$$

*for all compactly supported $\xi \in W^{1,\frac{4}{3}} \cap L^\infty(\Sigma M \otimes u^{-1}TN)$ and for all compactly supported $V \in H^1 \cap L^\infty(M, u^{-1}TN)$. Here $\{\gamma_1, \gamma_2\}$ is a local orthonormal coordinate frame on $M$.*

Similarly, we have the following proposition for Dirac-harmonic maps with a Ricci type spinor potential and free boundary on $\mathcal{S}$.

**Proposition 3.12.** *Suppose that $(u, \psi) \in \chi^{1\,2}_{1,\frac{4}{3}}(M, N; \mathcal{S})$ is a pair of weakly Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S}$. Then*

$$\int_M \langle \psi, \mathcal{D}\xi\rangle + 2\lambda \int_M Ric(u)(e_\alpha, e_\beta)\left\langle \psi^\alpha, \xi^\beta\right\rangle = 0,$$

$$\begin{aligned} -\int_M du \cdot \nabla V &= \int_M \left\{\frac{1}{2} R^N(e_\eta, e_\delta, e_\alpha, e_\beta)\, \langle u_*(\gamma_i), e_\delta\rangle_{u^{-1}(TN)} \left\langle \psi^\beta, \gamma_i \cdot \psi^\alpha\right\rangle_{\Sigma M} \right.\\ &\quad \left. + \lambda R_{\alpha\beta,\eta}\left\langle \psi^\alpha, \psi^\beta\right\rangle - \lambda R_{\delta\xi}\,\Gamma^\xi_{\alpha\eta}\left(\left\langle \psi^\alpha, \psi^\delta\right\rangle + \left\langle \psi^\delta, \psi^\alpha\right\rangle\right)\right\} V^\eta \end{aligned}$$

*for all $\xi \in W^{1,\frac{4}{3}} \cap L^\infty(\Sigma M \otimes u^{-1}TN)$ such that $B_u^{\pm}\,\xi|_{\partial M} = 0$ and for all $V \in H^1 \cap L^\infty(M, u^{-1}TN)$ such that $V(x) \in T_{u(x)}\mathcal{S}$ for a.e. $x \in \partial M$. Here $\{\gamma_1, \gamma_2\}$ is a local orthonormal coordinate frame on $M$.*

## 3.2 The reflection construction

Since the goal of this chapter is to study the regularity of the weakly Dirac-harmonic maps with a Ricci type spinor potential and free boundary on $\mathcal{S}$, and since the regularity is a local property, without loss of generality, we may assume for simplicity that $(M, g) = (B_1^+, g_0)$, where $B_1^+ := \{(x_1, x_2) \in \mathbb{R}^2\,|x_1^2 + x_2^2 \le 1,\, x_2 \ge 0\}$ and $g_0 = dx_1^2 + dx_2^2$ is the standard metric on $\mathbb{R}^2$, and the free boundary portion $\mathbf{I} := \{(x_1, 0)\,|-1 \le x_1 \le 1\}$. Moreover, we identify $\frac{\partial}{\partial x_1}$ with $\gamma_i$, $i = 1, 2$, then the outward unit normal vector field $\vec{n} = -\gamma_2$, hence $\vec{n}G = -i\,\gamma_1$.

In this section, we first prove a lemma , which is analogous to Lemma 3.1 in [CJWZ] and Lemma 3.1 in [Sch06]. We show that the image of $u$ over a sufficiently small neighborhood of a boundary point is contained in a tubular neighborhood of the supporting submanifold $\mathcal{S}$. Hence if we restrict to a sufficiently small domain, one can use the geodesic reflection $\sigma$ to reflect the pair $(u, \psi)$ across $\mathcal{S}$. Then by using the geodesic reflection $\sigma$, we can extend the metric $h$ on the pull-back bundle $u^{-1}TN \to B_1^+$ to some metric $\tilde{h}$ on the bundle $u^{-1}TN \to B_1$ with the extended map $u$. Finally, we will prove that the extended pair $(u, \psi)$ satisfies some equations which are similar to the equations in Proposition 3.11 and Proposition 3.12. These equations will be used to show the regularity of the weakly Dirac-harmonic maps with a Ricci type spinor potential and free boundary on $\mathcal{S}$ in the next section.

First, we have the following lemma.

**Lemma 3.13.** *Let $(N,\, h)$ be a compact Riemannian manifold of dimension $n \geq 2$, isometrically embedded in $\mathbb{R}^K$ and $\mathcal{S}$ a closed submanifold of $N$. Then there is an $\epsilon_0 = \epsilon(N)$ such that for all weakly Dirac-harmonic maps $(u, \psi) \in \chi^{1\,2}_{1,\frac{4}{3}}(M,\, N;\, \mathcal{S})$ with a Ricci type spinor potential and free boundary on $\mathcal{S}$ satisfying*

$$\int_{B_1^+} \left(|du|^2 + |\psi|^4\right) \leq \epsilon_0^2.$$

*Then it holds $d_h\left(u(x),\, \mathcal{S}\right) \leq C\,\epsilon_0$ for all $x \in B^+_{1/4}$ with a constant $C = C(N)$. Moreover, there is a $Q \in \mathcal{S}$ such that $u(x) \in B_{C\,\epsilon_0}(Q)$ for all $x \in B^+_{1/4}$ with a constant $C = C(N)$.*

*Proof.* Since we have the $\epsilon$-regularity for Dirac-harmonic maps with a Ricci type spinor potential ( Theorem 2.7 ), the proof of Lemma 3.13 is quite similar to the proof of [CJWZ, Lemma 3.1], we omit it. □

The above lemma shows that if the energy $\int_{B_1^+}\left(|du|^2 + |\psi|^4\right)$ is small enough, then

$$u\left(B^+_{1/4}\right) \subset \mathbf{U}_\delta := \{y \in N \;: d_h\left(y,\, \mathcal{S}\right) < \delta\}$$

for some $\delta > 0$.

Denote

$$\Sigma(x) := d\sigma\left(u(x)\right), \quad \forall x \in B^+_{1/4}.$$

and we define a morphism

$\mathbf{T}^{\pm}_u : W^{1,\frac{4}{3}}\left(\Sigma B^+_{1/4} \otimes u^{-1}TN\right) \to W^{1,\frac{4}{3}}\left(\Sigma B^+_{1/4} \otimes (\sigma \circ u)^{-1}\, TN\right)$ by

$$\mathbf{T}^{\pm}_u := \pm i\, \gamma_1 \otimes \Sigma.$$

By definition, we have the equivalent boundary condition

$$\mathcal{B}^{\pm}_u\, \psi|_{\mathbf{I}} = 0 \;\text{ if and only if } \mathbf{T}^{\pm}_u\, \psi|_{\mathbf{I}} = \pm\, \psi|_{\mathbf{I}} \tag{3.28}$$

for all $\psi \in W^{1,\frac{4}{3}}\left(\Sigma B^+_{1/4} \otimes u^{-1}TN\right)$, where $\mathbf{I} := \partial B^+_{1/4} = \left\{(x_1,\, 0)\,\middle|\, -\frac{1}{4} \leq x_1 \leq \frac{1}{4}\right\}$

In the following, we will only consider the case of $(\mathcal{B}^+_u,\, \mathbf{T}^+_u)$, since the case of $(\mathcal{B}^+_u,\, \mathbf{T}^+_u)$ is similar. For simplicity, we will denote $(\mathcal{B}^+_u,\, \mathbf{T}^+_u)$ by $(\mathcal{B}_u,\, \mathbf{T}_u)$

Now we extend the pair $(u,\, \psi)$ to the lower half disk

$$B^-_{1/4} := \left\{(x_1,\, x_2) \in \mathbb{R}^2 \,\middle|\, x_1^2 + x_2^2 \leq \frac{1}{4},\, x_2 \leq 0\right\}.$$

For $x = (x_1,\, x_2)$, denote $x^* = (x_1,\, -x_2)$. We define

$$\begin{aligned} u(x^*) &:= \sigma(u(x)), \quad x^* \in B^-_{1/4}, \\ \psi(x^*) &:= \mathbf{T}_u\, \psi(x), \quad x^* \in B^-_{1/4}. \end{aligned}$$

It follows from (3.28) that the extension for $(u, \psi)$ is well defined.

Now using the extended map $u$, one can extend $\Sigma(x)$ to $B_{1/4}$ (still denote by $\Sigma(x)$), which satisfies $\Sigma^{-1}(x) = \Sigma(x^*)$. Moreover, one can extend $\mathbf{T}_u$ to some morphism (still denote by $\mathbf{T}_u$)

$$\mathbf{T}_u : W^{1,\frac{4}{3}}\left(\Sigma B_{1/4} \otimes u^{-1}TN\right) \to W^{1,\frac{4}{3}}\left(\Sigma B_{1/4} \otimes (\sigma \circ u)^{-1} TN\right).$$

For any $\psi \in W^{1,\frac{4}{3}}\left(\Sigma B_{1/4} \otimes u^{-1}TN\right)$, it can be expressed as

$$\psi(x) = \psi^\alpha(x) \otimes e_\alpha(x), \quad x \in B_{1/4}.$$

By definition, one has

$$\psi(x^*) = \mathbf{T}_u(x)\psi(x) = i\gamma_1 \cdot \psi^\alpha(x) \otimes \Sigma(x) e_\alpha(x), \quad x^* \in B_{1/4},$$

Since $\{e_\alpha\}$ is an orthonormal basis of $N$, so is $\Sigma(x)e_\alpha(x)$. Hence for $x \in B_{1/4}$, there exists $A(x^*) \in O(n)$ such that

$$\begin{pmatrix} \Sigma(x)e_1(x) \\ \cdots \\ \Sigma(x)e_n(x) \end{pmatrix} = A(x^*) \begin{pmatrix} e_1(x^*) \\ \cdots \\ e_n(x^*) \end{pmatrix},$$

or

$$\Sigma(x)e_\alpha(x) = A^\beta_\alpha(x^*)e_\beta(x^*), \quad \alpha = 1, \cdots, n.$$

This implies that $\psi^\alpha(x^*) = A^\beta_\alpha(x)\left(i\gamma_1 \cdot \psi^\beta(x)\right)$, $x \in B_{1/4}$. Similarly, we have $i\gamma_1 \cdot \psi^\alpha(x) = A^\beta_\alpha(x^*)\psi^\beta(x^*)$, $x \in B_{1/4}$.

Furthermore, one can calculate that $\mathbf{T}_u(x)\,\mathbf{T}_u(x^*)\,\psi(x^*) = \psi(x^*)$. For more details, see [CJWZ].

Now using the geodesic reflection $\sigma$, we are able to extend the metric $h$ on the bundle $u^{-1}TN \to B^+_{1/4}$ to some metric $\tilde{h}$ on the bundle $u^{-1}TN \to B_{1/4}$ with the extended map $u$ as follows:

$$\langle V(x), W(x)\rangle_{\tilde{h}} := \begin{cases} \langle V(x), W(x)\rangle_h, & x \in B^+_{1/4} \\ \langle \Sigma(x)\,V(x), \Sigma(x)\,W(x)\rangle_h, & x \in B^-_{1/4} \end{cases}$$

for all $V(x), W(x) \in \Gamma\left(B_{1/4}, u^{-1}TN\right)$. Similarly, we have the extended metrics (with respect to $h$) on $\Sigma B_{1/4} \otimes u^{-1}TN$, $TB_{1/4} \otimes u^{-1}TN$ and $T^*\Sigma B_{1/4} \otimes u^{-1}TN$ respectively.

Moreover, we have the extended covariant $\widetilde{\nabla}$ with respect to $\tilde{h}$ defined as follows

$$\begin{aligned}&\widetilde{\nabla}_{X(x)}\,V(x)\\ :=\ &\begin{cases} \nabla_{u_*(X(x))}\,V(x) & x \in B^+_{1/4} \\ \Sigma(x^*)\nabla_{(\sigma\circ u)_*(X(x))}\,(\Sigma(x)V(x)) = \Sigma(x^*)\nabla_{\Sigma(x)u_*(X(x))}\,(\Sigma(x)V(x)) & x \in B^-_{1/4}, \end{cases}\end{aligned}$$

where $X \in \Gamma(TB_{1/4})$ and $V \in \Gamma(B_{1/4},\, u^{-1}TN)$.
And the extended Riemannian curvature tensor $\widetilde{R}$ is defined by

$$\begin{aligned} &\widetilde{R}(u)\ (V(x),\, W(x))\, U(x) \\ := &\begin{cases} R(u)\ (V(x),\, W(x))\, U(x) & x \in B^+_{1/4} \\ \Sigma(x^*)R(u)\ (\Sigma(x)V(x),\, \Sigma(x)W(x))\, (\Sigma(x)U(x)) & x \in B^-_{1/4}, \end{cases} \end{aligned}$$

where $U, V, W \in \Gamma(B_{1/4},\, u^{-1}TN)$.

We define

$$\widetilde{R}_{\alpha\beta\gamma\delta}(u) := \left\langle \widetilde{R}(u)\ (e_\alpha,\, e_\beta)\, e_\gamma,\, e_\delta \right\rangle_{\tilde{h}},$$

then the extended Ricci curvature $\widetilde{Ric}_{\alpha\beta}(u)$ can be expressed as follows:

$$\begin{aligned} &\widetilde{Ric}_{\alpha\beta}(u(x)) \\ = &\ \widetilde{R}_{\alpha\delta\delta\beta}(u(x)) = \left\langle \widetilde{R}(u)\ (e_\alpha,\, e_\delta)\, e_\delta,\, e_\beta \right\rangle_{\tilde{h}} \\ = &\begin{cases} Ric_{\alpha\beta}(u(x)) & x \in B^+_{1/4} \\ \langle\, R(u)\ (\Sigma(x)e_\alpha,\, \Sigma(x)e_\delta)\, (\Sigma(x)e_\delta)\,,\, \Sigma(x)e_\beta\, \rangle_h & x \in B^-_{1/4} \end{cases} \\ = &\begin{cases} Ric_{\alpha\beta}(u(x)) & x \in B^+_{1/4} \\ A^\gamma_\alpha(x^*)A^\eta_\beta(x^*)\, \langle\, R(u)\ (e_\gamma(x^*),\, e_\delta(x^*))\, (e_\delta(x^*))\,,\, e_\eta(x^*)\, \rangle_h & x \in B^-_{1/4} \end{cases} \\ = &\begin{cases} Ric_{\alpha\beta}(u(x)) & x \in B^+_{1/4} \\ A^\gamma_\alpha(x^*)A^\eta_\beta(x^*)Ric_{\gamma\eta}(u(x^*)) & x \in B^-_{1/4}. \end{cases} \end{aligned}$$

The extended Christoffel symbol $\widetilde{\Gamma}^\gamma_{\alpha\beta}(u)$ can be expressed as

$$\begin{aligned} \widetilde{\Gamma}^\gamma_{\alpha\beta}(u(x)) &= \begin{cases} \Gamma^\gamma_{\alpha\beta}(u(x)) & x \in B^+_{1/4} \\ \left\langle \widetilde{\nabla}_{e_\alpha} e_\beta,\, e_\gamma \right\rangle_{\tilde{h}} (u(x)) & x \in B^-_{1/4} \end{cases} \\ &= \begin{cases} \Gamma^\gamma_{\alpha\beta}(u(x)) & x \in B^+_{1/4} \\ \left\langle\, \Sigma(x)\Sigma(x^*)\nabla_{\Sigma(x)e_\alpha} (\Sigma(x)e_\beta)\,,\, \Sigma(x)e_\gamma \right\rangle_h & x \in B^-_{1/4} \end{cases} \\ &= \begin{cases} \Gamma^\gamma_{\alpha\beta}(u(x)) & x \in B^+_{1/4} \\ A^\delta_\alpha(x^*)A^\xi_\beta(x^*)A^\eta_\gamma(x^*) \left\langle\, \nabla_{e_\delta(x^*)} (e_\xi(x^*))\,,\, e_\eta(x^*) \right\rangle_h & x \in B^-_{1/4} \end{cases} \\ &= \begin{cases} \Gamma^\gamma_{\alpha\beta}(u(x)) & x \in B^+_{1/4} \\ A^\delta_\alpha(x^*)A^\xi_\beta(x^*)A^\eta_\gamma(x^*)\Gamma^\eta_{\delta\xi}(u(x^*)) & x \in B^-_{1/4}. \end{cases} \end{aligned}$$

Finally, recall that the Dirac operator along the map $u$ can be expressed as:

$$\mathcal{D} = \partial\!\!\!/ \otimes Id + \gamma_i \otimes \nabla_{u_*\gamma_i},$$

and we can define the Dirac operator along the extended map $u$ with respect to the extended metric $\tilde{h}$ by

$$\widetilde{\mathcal{D}} := \not\partial \otimes Id + \gamma_i \otimes \widetilde{\nabla}_{u_*\gamma_i}.$$

We have the following relation between $\widetilde{\mathcal{D}}$ and $\mathcal{D}$:

**Lemma 3.14.** *i) For $\psi,\, \varphi \in W^{1,\frac{4}{3}}\left(\Sigma B_{1/4} \otimes u^{-1}TN\right)$, we have*

$$\langle \psi(x)(x),\, \varphi(x)\rangle_{\tilde{h}} = \langle \mathbf{T}_u(x)\psi(x)(x),\, \mathbf{T}_u(x)\varphi(x)\rangle\,, \quad \forall x \in B^-_{1/4}.$$

*ii) For any $\xi \in W^{1,\frac{4}{3}}\left(\Sigma B_{1/4} \otimes u^{-1}TN\right)$, denote $\xi^*(x) := \mathbf{T}_u(x^*)\,\xi(x^*)$, $\forall x \in B_{1/4}$, then we have*

$$\widetilde{\mathcal{D}}_{x^*}\,\xi(x^*) = \mathbf{T}_u(x)\,\mathcal{D}_x\xi^*(x), \quad \forall x \in B_{1/4}.$$

*Proof.* See [CJWZ, Lemma 3.2] and [CJWZ, Lemma 3.3]. □

For the equations of the extended pair $(u,\, \psi)$, in analogy to Proposition 3.11 and Proposition 3.12, we have the following theorem.

**Theorem 3.15.** *Suppose $(u,\, \psi) \in \chi^{1\,2}_{1,\frac{4}{3}}\left(B^+_1,\, N;\, \mathcal{S}\right)$ is a weakly Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S}$. We extend the pair $(u\,\psi)$ to the whole disk $B_{1/4}$ as before. Then*

$$\begin{aligned}-\int_{B_{1/4}} du \cdot_{\tilde{h}} \widetilde{\nabla} V \;=\; &\int_{B_{1/4}} \Big\{\frac{1}{2}\widetilde{R}(e_\eta,\, e_\delta,\, e_\alpha,\, e_\beta)\;\langle u_*(\gamma_i),\, e_\delta\rangle_{\tilde{h}}\;\left\langle \psi^\beta,\, \gamma_i\cdot\psi^\alpha\right\rangle \\ &+ \lambda \widetilde{R}_{\alpha\beta,\,\eta}\;\left\langle \psi^\alpha,\, \psi^\beta\right\rangle - \lambda\widetilde{R}_{\delta\xi}\,\widetilde{\Gamma}^\xi_{\alpha\eta}\;\left(\left\langle\psi^\alpha,\, \psi^\delta\right\rangle + \left\langle\psi^\delta,\, \psi^\alpha\right\rangle\right)\Big\}\; V^\eta,\end{aligned}$$

$$\int_{B_{1/4}} \left\langle \psi,\, \widetilde{\mathcal{D}}\xi\right\rangle_{\tilde{h}} + 2\lambda\int_{B_{1/4}} \widetilde{Ric}(u)(e_\alpha,\, e_\beta)\;\left\langle\psi^\alpha,\, \xi^\beta\right\rangle = 0$$

*for all compactly supported $V \in H^1 \cap L^\infty\left(B_{1/4},\, u^{-1}TN\right)$ and all compactly supported $\xi \in W^{1,\frac{4}{3}} \cap L^\infty\left(\Sigma B_{1/4} \otimes u^{-1}TN\right)$.*

*Proof.* we modify the proof of Theorem 3.1 in [CJWZ]. For a compactly supported $V \in H^1 \cap L^\infty\left(B_{1/4},\, u^{-1}TN\right)$, we decompose the vector field as in [Sch06] into the equivariant and the antiequivariant part with respect to the geodesic reflection $\sigma$,i.e. $V = V_e + V_a$. Then for $x \in B_{1/4}$, we have

$$V_e(x) := \frac{1}{2}\left[V(x) + \Sigma(x^*)V(x^*)\right], \quad V_a(x) := \frac{1}{2}\left[V(x) - \Sigma(x^*)V(x^*)\right].$$

It follows from $\Sigma(x)\,\Sigma(x^*) = Id(u(x))\,: T_{u(x)}N \to T_{u(x)}N$ that

$$V_e(x^*) \,=\Sigma(x)\,V_e(x), \quad V_a(x^*) \,= -\Sigma(x)\,V_a(x).$$

By the definition of $\Sigma(x)$ and $\sigma|_{\mathcal{S}} = id$, we have for $x_0 \in \mathbf{I}$

$$V_e(x_0) \,= \frac{1}{2}\,[V(x_0) + \Sigma(x_0)V(x_0)] \in T_{u(x_0)}\mathcal{S}.$$

Hence it follows from proposition 3.12 that

$$\begin{aligned} -\int_{B^+_{1/4}} du \cdot \widetilde{\nabla} V_e \;=\; & \int_{B^+_{1/4}} \Big\{ \frac{1}{2}\widetilde{R}(e_\eta,\, e_\delta,\, e_\alpha,\, e_\beta)\, \langle u_*(\gamma_i),\, e_\delta\rangle_{u^{-1}(TN)}\, \left\langle \psi^\beta,\, \gamma_i\cdot\psi^\alpha\right\rangle \\ & + \lambda \widetilde{R}_{\alpha\beta,\,\eta}\, \left\langle\psi^\alpha,\, \psi^\beta\right\rangle - \lambda \widetilde{R}_{\delta\xi}\, \Gamma^\xi_{\alpha\eta}\, \left(\left\langle\psi^\alpha,\, \psi^\delta\right\rangle + \left\langle\psi^\delta,\, \psi^\alpha\right\rangle\right)\Big\}\, V_e^\eta. \end{aligned} \tag{3.29}$$

As in [CJWZ, Thm. 3.1] or [Sch06, Lemma 3.2], we have

$$\int_{B^-_{1/4}} du \cdot \widetilde{\nabla} V_e = \int_{B^+_{1/4}} du \cdot \nabla V_e, \tag{3.30}$$

$$\int_{B^-_{1/4}} du \cdot \widetilde{\nabla} V_a = -\int_{B^+_{1/4}} du \cdot \nabla V_a, \tag{3.31}$$

$$\begin{aligned} & \frac{1}{2}\int_{B^-_{1/4}} \widetilde{R}(V_e,\, e_\delta,\, e_\alpha,\, e_\beta)\, \langle u_*(\gamma_i),\, e_\delta\rangle_{\widetilde{h}}\, \left\langle\psi^\beta,\, \gamma_i\cdot\psi^\alpha\right\rangle \\ = \; & \frac{1}{2}\int_{B^+_{1/4}} R(V_e,\, e_\delta,\, e_\alpha,\, e_\beta)\, \langle u_*(\gamma_i),\, e_\delta\rangle\, \left\langle\psi^\beta,\, \gamma_i\cdot\psi^\alpha\right\rangle, \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} & \frac{1}{2}\int_{B^-_{1/4}} \widetilde{R}(V_a,\, e_\delta,\, e_\alpha,\, e_\beta)\, \langle u_*(\gamma_i),\, e_\delta\rangle_{\widetilde{h}}\, \left\langle\psi^\beta,\, \gamma_i\cdot\psi^\alpha\right\rangle \\ = \; & -\frac{1}{2}\int_{B^+_{1/4}} R(V_a,\, e_\delta,\, e_\alpha,\, e_\beta)\, \langle u_*(\gamma_i),\, e_\delta\rangle\, \left\langle\psi^\beta,\, \gamma_i\cdot\psi^\alpha\right\rangle. \end{aligned} \tag{3.33}$$

Moreover, we claim that the following four identities hold:

$$\int_{B^-_{1/4}} \widetilde{R}_{\alpha\beta,\,\eta}\, \left\langle\psi^\alpha,\, \psi^\beta\right\rangle V_e^\eta = \int_{B^+_{1/4}} R_{\alpha\beta,\,\eta}\, \left\langle\psi^\alpha,\, \psi^\beta\right\rangle V_e^\eta, \tag{3.34}$$

$$\int_{B^-_{1/4}} \widetilde{R}_{\alpha\beta,\,\eta}\,\left\langle \psi^\alpha,\,\psi^\beta\right\rangle V^\eta_a = -\int_{B^+_{1/4}} R_{\alpha\beta,\,\eta}\,\left\langle \psi^\alpha,\,\psi^\beta\right\rangle V^\eta_a, \tag{3.35}$$

$$\int_{B^-_{1/4}} \widetilde{R}_{\delta\xi}\,\widetilde{\Gamma}^\xi_{\alpha\eta}\left(\left\langle \psi^\alpha,\,\psi^\delta\right\rangle + \left\langle \psi^\delta,\,\psi^\alpha\right\rangle\right) V^\eta_e = \int_{B^+_{1/4}} R_{\delta\xi}\,\Gamma^\xi_{\alpha\eta}\,\left(\left\langle \psi^\alpha,\,\psi^\delta\right\rangle + \left\langle \psi^\delta,\,\psi^\alpha\right\rangle\right) V^\eta_e, \tag{3.36}$$

and

$$\int_{B^-_{1/4}} \widetilde{R}_{\delta\xi}\widetilde{\Gamma}^\xi_{\alpha\eta}\left(\left\langle \psi^\alpha,\,\psi^\delta\right\rangle + \left\langle \psi^\delta,\,\psi^\alpha\right\rangle\right) V^\eta_a = -\int_{B^+_{1/4}} R_{\delta\xi}\,\Gamma^\xi_{\alpha\eta}\,\left(\left\langle \psi^\alpha,\psi^\delta\right\rangle + \left\langle \psi^\delta,\psi^\alpha\right\rangle\right) V^\eta_a. \tag{3.37}$$

If the claim is true, then from (3.29)-(3.37), we have

$$\begin{aligned} -\int_{B_{1/4}} du\cdot\widetilde{\nabla}V \;&=\; \int_{B_{1/4}} \Big\{\frac{1}{2}\widetilde{R}(e_\eta,\,e_\delta,\,e_\alpha,\,e_\beta)\,\langle u_*(\gamma_i),\,e_\delta\rangle_{\widetilde{h}}\,\left\langle \psi^\beta,\,\gamma_i\cdot\psi^\alpha\right\rangle \\ &\quad + \lambda\widetilde{R}_{\alpha\beta,\,\eta}\,\left\langle \psi^\alpha,\,\psi^\beta\right\rangle - \lambda\widetilde{R}_{\delta\xi}\,\Gamma^\xi_{\alpha\eta}\,\left(\left\langle \psi^\alpha,\,\psi^\delta\right\rangle + \left\langle \psi^\delta,\,\psi^\alpha\right\rangle\right)\Big\}\;V^\eta, \end{aligned}$$

which proves the first equation of Theorem 3.15.

Now we are going to prove the claim.

The proof of (3.34):

Let $x\in B^+_{1/4}$, then $x^*\in B^-_{1/4}$. Recall that $\psi(x^*) = i\gamma_1\cdot\psi^\alpha(x)\otimes\Sigma(x)e_\alpha(x)$ with $\Sigma(x)e_\alpha(x) = A^\beta_\alpha(x^*)e_\beta(x^*)$ and $\psi^\alpha(x^*) = A^\beta_\alpha(x)\left(i\gamma_1\cdot\psi^\beta(x)\right)$, where $\left(A^\alpha_\beta(x)\right)$, $\left(A^\alpha_\beta(x^*)\right)\in O(n)$. We calculate

$$\begin{aligned} & \widetilde{R}_{\alpha\beta,\,\eta}(u(x^*))\,\left\langle \psi^\alpha,\,\psi^\beta\right\rangle(x^*)\,V^\eta_e(x^*) \\ =\;& V_e(x^*)\left(\widetilde{Ric}\,(e_\alpha,\,e_\beta)\circ u\right)\,\left\langle \psi^\alpha(x^*),\,\psi^\beta(x^*)\right\rangle \\ =\;& A^\gamma_\alpha(x)A^\delta_\beta(x)\,(\Sigma(x)V_e(x))\left(\widetilde{Ric}\,(e_\alpha,\,e_\beta)\circ u\right)\,\left\langle i\gamma_1\cdot\psi^\gamma(x),\,i\gamma_1\cdot\psi^\delta(x))\right\rangle \\ =\;& A^\gamma_\alpha(x)A^\delta_\beta(x)\,(V_e(x))\left(\widetilde{Ric}\,(e_\alpha,\,e_\beta)\circ u\circ\sigma\right)\,\left\langle \psi^\gamma(x),\,\psi^\delta(x))\right\rangle \\ =\;& A^\gamma_\alpha(x)A^\delta_\beta(x)A^\xi_\alpha(x)A^\eta_\beta(x)\,(V_e(x))\,(Ric\,(e_\xi,\,e_\eta)\circ u)\,\left\langle \psi^\gamma(x),\,\psi^\delta(x))\right\rangle \\ =\;& (V_e(x))\,(Ric\,(e_\gamma,\,e_\delta)\circ u)\,\left\langle \psi^\gamma(x),\,\psi^\delta(x))\right\rangle \\ =\;& R_{\alpha\beta,\eta}(u(x))\,\left\langle \psi^\alpha(x),\,\psi^\beta(x))\right\rangle\,V^\eta_e(x). \end{aligned}$$

Similarly, by using the fact $V_a(x^*) = -\Sigma(x)\,V_a(x)$, one checks (2.35).

The proof of (3.36):

We calculate

$$\begin{aligned}
& \widetilde{R}_{\delta\xi}\left(u(x^*)\right) \widetilde{\Gamma}^{\xi}_{\alpha\eta}\left(u(x^*)\right) \left(\left\langle \psi^{\alpha},\, \psi^{\delta}\right\rangle (x^*) + \left\langle \psi^{\delta},\, \psi^{\alpha}\right\rangle (x^*)\right) V^{\eta}_{e}(x^*) \\
= & \left(A^{\beta}_{\delta} A^{\gamma}_{\xi} A^{\mu}_{\alpha} A^{\nu}_{\eta} A^{\zeta}_{\xi} A^{\theta}_{\alpha} A^{\tau}_{\delta}\right)(x) R_{\beta\gamma}\left(u(x)\right) \Gamma^{\zeta}_{\mu\nu}\left(u(x)\right) \left\{\left\langle i\gamma_1 \cdot \psi^{\theta}(x),\, i\gamma_1 \cdot \psi^{\tau}(x)\right)\right\rangle \\
& + \left\langle i\gamma_1 \cdot \psi^{\tau}(x),\, i\gamma_1 \cdot \psi^{\theta}(x)\right)\right\rangle\Big\} \left\langle V_e(x^*),\, e_{\eta}\right\rangle_{\widetilde{h}} \\
= & A^{\nu}_{\eta}(x) R_{\delta\xi}\left(u(x)\right) \Gamma^{\xi}_{\alpha\nu}\left(u(x)\right) \left\{\left\langle \psi^{\alpha}(x),\, \psi^{\delta}(x)\right)\right\rangle + \left\langle \psi^{\delta}(x),\, \psi^{\alpha}(x)\right)\right\rangle\Big\} \\
& \times \left\langle \Sigma(x^*)\Sigma(x) V_e(x),\, \Sigma(x^*) e_{\eta}(x^*)\right\rangle_{h} \\
= & R_{\delta\xi}\left(u(x)\right) \Gamma^{\xi}_{\alpha\eta}\left(u(x)\right) \left\{\left\langle \psi^{\alpha}(x),\, \psi^{\delta}(x)\right)\right\rangle + \left\langle \psi^{\delta}(x),\, \psi^{\alpha}(x)\right)\right\rangle\Big\} V^{\eta}_{e}(x).
\end{aligned}$$

Similarly, by using the fact $V_a(x^*) = -\Sigma(x)\, V_a(x)$, one checks (2.38).
This completes the proof of the claim.
Now we are going to prove the second equation of Theorem 3.15.
For any compactly supported $\xi \in W^{1,\frac{4}{3}} \cap L^{\infty}\left(\Sigma B_{1/4} \otimes u^{-1}TN\right)$, we have

$$\int_{B^{+}_{1/4}} \left\langle \psi,\, \widetilde{\mathcal{D}}\xi \right\rangle_{\widetilde{h}} = \int_{B^{+}_{1/4}} \left\langle \mathcal{D}\psi,\, \xi \right\rangle - \int_{\mathbf{I}} \left\langle (-\gamma_2)\cdot \psi,\, \xi \right\rangle,$$

where we have used the fact $\vec{n} = -\gamma_2$ and Proposition 3.2.

By using Lemma 3.14, we calculate

$$\begin{aligned}
\int_{x^* \in B^{-}_{1/4}} \left\langle \psi(x^*),\, \widetilde{\mathcal{D}}_{x^*}\xi(x^*) \right\rangle_{\widetilde{h}} &= \int_{x^* \in B^{-}_{1/4}} \left\langle \mathbf{T}_u(x^*)\psi(x^*),\, \mathbf{T}_u(x^*)\, \widetilde{\mathcal{D}}_{x^*}\xi(x^*) \right\rangle \\
&= \int_{x \in B^{+}_{1/4}} \left\langle \psi(x),\, \mathcal{D}_x \xi^*(x) \right\rangle \\
&= \int_{x \in B^{+}_{1/4}} \left\langle \mathcal{D}\psi(x),\, \xi^*(x) \right\rangle - \int_{\mathbf{I}} \left\langle (-\gamma_2)\cdot \psi(x),\, \xi^*(x) \right\rangle.
\end{aligned}$$

Hence we obtain

$$\int_{B_{1/4}} \left\langle \psi,\, \widetilde{\mathcal{D}}\xi \right\rangle_{\widetilde{h}} = \int_{B^{+}_{1/4}} \left\langle \mathcal{D}\psi,\, \xi + \xi^* \right\rangle - \int_{\mathbf{I}} \left\langle (-\gamma_2)\cdot \psi,\, \xi + \xi^* \right\rangle. \tag{3.38}$$

As in [CJWZ, Thm. 3.1], we have

$$\int_{\mathbf{I}} \left\langle (-\gamma_2)\cdot \psi,\, \xi + \xi^* \right\rangle = 0. \tag{3.39}$$

Finally, recall that $\mathcal{D}\psi^{\alpha} = -2\lambda\, Ric(u)\left(e_{\alpha},\, e_{\beta}\right) \psi^{\beta}$, we have

$$\int_{B^+_{1/4}} \langle \mathcal{D}\psi,\, \xi\rangle \;= -2\lambda \int_{B^+_{1/4}} Ric(u)\,(e_\alpha,\, e_\beta)\,\left\langle \psi^\beta,\, \xi^\beta\right\rangle, \tag{3.40}$$

and

$$\begin{aligned}
&\int_{x\in B^+_{1/4}} \langle \mathcal{D}\psi(x),\, \xi^*(x)\rangle \\
=\;& \int_{x^*\in B^-_{1/4}} \langle\, \mathcal{D}\psi(x),\, \mathbf{T}_u(x^*)\xi(x^*)\,\rangle \\
=\;& \int_{x^*\in B^-_{1/4}} \langle\, \mathbf{T}_u(x^*)\mathbf{T}_u(x)\mathcal{D}\psi(x),\, \mathbf{T}_u(x^*)\xi(x^*)\,\rangle \\
=\;& \int_{x^*\in B^-_{1/4}} \langle\, \mathbf{T}_u(x)\mathcal{D}\psi(x),\, \xi(x^*)\,\rangle_{\widetilde{h}} \\
=\;& -2\lambda \int_{x^*\in B^-_{1/4}} \left\langle\, \mathbf{T}_u(x)\, \left(Ric(u)\,(e_\alpha,\, e_\beta)\;\psi^\beta\otimes e_\alpha\right)(x),\, \xi(x^*)\,\right\rangle_{\widetilde{h}} \\
=\;& -2\lambda \int_{x^*\in B^-_{1/4}} R_{\alpha\beta}\,(u(x))\left\langle\, (i\gamma_1)\cdot\psi^\beta(x)\otimes\Sigma(x)e_\alpha(x),\, \xi(x^*)\,\right\rangle_{\widetilde{h}} \\
=\;& -2\lambda \int_{x^*\in B^-_{1/4}} \left(A^\delta_\alpha A^\gamma_\beta A^\mu_\beta A^\nu_\alpha\right)(x^*)\widetilde{R}_{\delta\gamma}\,(u(x^*))\,\langle\, \psi^\mu(x^*)\otimes e_\nu(x^*),\, \xi(x^*)\,\rangle_{\widetilde{h}} \\
=\;& -2\lambda \int_{x^*\in B^-_{1/4}} \widetilde{R}_{\alpha\beta}\,(u(x^*))\left\langle\, \psi^\beta(x^*)\otimes e_\alpha(x^*),\, \xi(x^*)\,\right\rangle_{\widetilde{h}} \\
=\;& -2\lambda \int_{B^-_{1/4}} \widetilde{R}_{\alpha\beta}\,(u)\left\langle\, \psi^\beta,\, \xi^\alpha\right\rangle,
\end{aligned} \tag{3.41}$$

where we have used the fact that $i\gamma_1\cdot\psi^\beta(x) = A^\mu_\beta(x^*)\psi^\mu(x^*)$.

Hence, combining (3.38)-(3.41), one gets

$$\int_{B_{1/4}} \left\langle \psi,\, \widetilde{\mathcal{D}}\xi\right\rangle_{\widetilde{h}} + 2\lambda \int_{B_{1/4}} \widetilde{Ric}(u)(e_\alpha,\, e_\beta)\,\left\langle \psi^\alpha,\, \xi^\beta\right\rangle \,= 0.$$

This completes the proof.

□

From Theorem 3.15, we have the following corollary.

**Corollary 3.16.** *Assumption and notations as above, then the extended fields* $(u,\, \psi)$ *satisfy in* $B_{1/4}$

$$\mathcal{D}\psi^\alpha = \partial\!\!\!/\psi^\alpha + \widetilde{\Gamma}^\alpha_{\beta\gamma}(u)\left(\partial_i u^\beta\right)(\gamma_i\cdot\psi^\gamma) = -2\lambda\,\widetilde{Ric}(u)(e_\alpha,\, e_\beta)\,\psi^\beta \quad \alpha = 1,\, 2,\, \cdots,\, n,$$

$$\begin{aligned}\Delta u^{\gamma} &= -\widetilde{\Gamma}^{\gamma}_{\alpha\beta}(u)\,(\partial_i u^{\alpha})\left(\partial_i u^{\beta}\right) + \frac{1}{2}\,\widetilde{R}(e_{\gamma},\,e_{\delta},\,e_{\alpha},\,e_{\beta})\,\langle\nabla_{\gamma_i}u,\,e_{\delta}\rangle_{\widetilde{h}}\,\left\langle\psi^{\beta},\,\gamma_i\cdot\psi^{\alpha}\right\rangle\\ &\quad +\lambda\,\widetilde{R}_{\alpha\beta,\gamma}(u)\left\langle\psi^{\alpha},\,\psi^{\beta}\right\rangle - 2\lambda\,\widetilde{Ric}(u)(e_{\beta},\,e_{\delta})\,\widetilde{\Gamma}^{\beta}_{\alpha\gamma}\,\left(Re\left\langle\psi^{\alpha},\,\psi^{\delta}\right\rangle\right),\end{aligned}$$

*where* $\gamma = 1,\,2,\,\cdots,\,n$ *and* $\partial_i := \frac{\partial}{\partial x^i}$, $i = 1,\,2$.

## 3.3 Continuity and higher regularity

In this section we will use Theorem 3.15 to modify the proof of the regularity theorems in Chapter 2 to show the Hölder continuity for weakly Dirac-harmonic maps with a Ricci type spinor potential and free boundary on $\mathcal{S}$. Furthermore, we show that if $\mathcal{S}$ is a closed, totally geodesic submanifold of $N$, then we have higher regularity for the pair $(u,\,\psi)$.

We define an $n\times n$ 1-form valued matrix $\widetilde{\Omega}$ as follows:

$$\widetilde{\Omega}_{\gamma\delta} = \sum_{i=1}^{2}\left[\frac{1}{2}\,\widetilde{R}(e_{\gamma},\,e_{\delta},\,e_{\alpha},\,e_{\beta})\,\left\langle\psi^{\beta},\,\gamma_i\cdot\psi^{\alpha}\right\rangle\right]\,dx^i,\ \text{for } 1\le\gamma,\delta\le n.$$

Similar to Proposition 2.5, we have the following proposition:

**Proposition 3.17.** *For the above defined matrix* $\widetilde{\Omega} = \left\{\widetilde{\Omega}_{\gamma\delta}\right\}$, *the following properties hold:*

(1). $\widetilde{\Omega}_{\gamma\delta}$ *is real* 1*-form valued for any* $1\le\gamma,\delta\le n$,

(2). $\widetilde{\Omega}$ *is skew-symmetric*

$$\widetilde{\Omega}_{\gamma\delta} = -\widetilde{\Omega}_{\delta\gamma}.$$

Under this notation, the first equation in Theorem 3.15 can be written as

$$\begin{aligned}-\int_{B_{1/4}} du\cdot_{\widetilde{h}}\widetilde{\nabla}V &= \int_{B_{1/4}}\left\{\frac{1}{2}\widetilde{R}(e_{\eta},\,e_{\delta},\,e_{\alpha},\,e_{\beta})\,\langle u_*(\gamma_i),\,e_{\delta}\rangle_{\widetilde{h}}\,\left\langle\psi^{\beta},\,\gamma_i\cdot\psi^{\alpha}\right\rangle\right.\\ &\quad\left.+\,\lambda\widetilde{R}_{\alpha\beta,\eta}\left\langle\psi^{\alpha},\,\psi^{\beta}\right\rangle - \lambda\widetilde{R}_{\delta\xi}\,\widetilde{\Gamma}^{\xi}_{\alpha\eta}\,\left(\left\langle\psi^{\alpha},\,\psi^{\delta}\right\rangle + \left\langle\psi^{\delta},\,\psi^{\alpha}\right\rangle\right)\right\}V^{\eta}\\ &= \int_{B_{1/4}}\left\{\sum_{\delta=1}^{n}\left(\widetilde{\Omega}_{\eta\delta}\right)\circ\langle du,\,e_{\delta}\rangle_{\widetilde{h}} + \widetilde{F_{\lambda}}^{\eta}(u,\,\psi)\right\}V^{\eta},\end{aligned}$$

where $\widetilde{F_{\lambda}}^{\eta}(u,\,\psi) := \lambda\widetilde{R}_{\alpha\beta,\eta}\left\langle\psi^{\alpha},\,\psi^{\beta}\right\rangle - \lambda\widetilde{R}_{\delta\xi}\,\widetilde{\Gamma}^{\xi}_{\alpha\eta}\,\left(\left\langle\psi^{\alpha},\,\psi^{\delta}\right\rangle + \left\langle\psi^{\delta},\,\psi^{\alpha}\right\rangle\right)$ and $\circ$ denotes the inner product of the 1-forms.

For any $\varphi \in C_0^\infty(B_{1/4})$, and for any fixed $1 \le \alpha \le n$, we take $V = \varphi\, e_\alpha$ in Theorem 3.15, then we have

$$\begin{aligned} & \int_{B_{1/4}} \left\{ \sum_{\delta=1}^{n} \left(\widetilde{\Omega}_{\alpha\delta}\right) \circ \langle du,\, e_\delta\rangle_{\widetilde{h}} + \widetilde{F_\lambda}^{\alpha}(u,\, \psi) \right\} \varphi \\ = & -\int_{B_{1/4}} du \cdot_{\widetilde{h}} \widetilde{\nabla}\,(\varphi\, e_\alpha) \\ = & -\int_{B_{1/4}} \langle du,\, e_\alpha\rangle_{\widetilde{h}} \circ d\varphi - \int_{B_{1/4}} \left\langle e_\delta,\, \widetilde{\nabla} e_\alpha \right\rangle_{\widetilde{h}} \circ \langle du,\, e_\delta\rangle_{\widetilde{h}}\, \varphi. \end{aligned}$$

Since $\varphi \in C_0^\infty(B_{1/4})$ is arbitrary, we obtain

$$-d^* \langle du,\, e_\alpha\rangle_{\widetilde{h}} = \left(\widetilde{\Omega}_{\alpha\delta} + \left\langle \widetilde{\nabla} e_\alpha,\, e_\delta \right\rangle_{\widetilde{h}}\right) \circ \langle du,\, e_\delta\rangle_{\widetilde{h}} + \widetilde{F_\lambda}^{\alpha}(u,\, \psi). \tag{3.42}$$

Define

$$\widetilde{\Theta} = \left(\widetilde{\Theta}_{\alpha\beta}\right) := \left(\widetilde{\Omega}_{\alpha\beta} + \left\langle \widetilde{\nabla} e_\alpha,\, e_\beta \right\rangle_{\widetilde{h}}\right).$$

then one has

$$-d^* \langle du,\, e_\alpha\rangle_{\widetilde{h}} = \sum_{\beta=1}^{n} \left(\widetilde{\Theta}_{\alpha\beta}\right) \circ \langle du,\, e_\beta\rangle_{\widetilde{h}} + \widetilde{F_\lambda}^{\alpha}(u,\, \psi). \tag{3.43}$$

Comparing with (2.33), and following the same schedule as the proof of Theorem 2.7, we obtain the Hölder continuous of the extended pair $(u,\, \psi)$. More precisely, we have

**Theorem 3.18.** *Let $(M,\, g_{ij})$ be a compact Riemannian spin surface with non-empty boundary $\partial M$, $(N, h_{\alpha\beta})$ any compact Riemannian manifold of dimension $n \ge 2$, and $\mathcal{S}$ a closed submanifold of $N$. Let $(u,\, \psi)$ be a weakly Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S}$. Then for any $\alpha \in (0,\, 1)$, we have*

$$u \in C^{0,\,\alpha}(M,\, N).$$

Now let $(u,\, \psi)$ be a weakly Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S} \subset N$, and we assume that $u \in C^{0,\,\alpha}(B_1^+,\, N)$ for any $\alpha \in (0,\, 1)$. If in addition we assume that $\mathcal{S}$ is a closed, totally geodesic submanifold of $N$, then it follows from [CJWZ, Prop. 3.11] that for any $1 \le \beta,\, \delta,\, \eta \le n$ and $\gamma \in (0,\, 1)$, the extended Christoffel symbol $\widetilde{\Gamma}^{\eta}_{\beta\delta}(u)$ is Hölder continuous, namely

$$\widetilde{\Gamma}^{\eta}_{\beta\delta}(u) \in C^{0,\,\gamma}\left(B_{1/4}\right). \tag{3.44}$$

With (3.44) and Theorem 3.15, one can choose the adapted coordinates in some neighborhood of the point $u(0) \in \mathcal{S}$ as in [CJWZ], these coordinates are also called Fermi coordinates(see [Gr04] for more details). Then by using a similar argument as in Chen-Jost-Li-Wang [CJLW05], practically we just need to modify the proof of Lemma 2.5 and Theorem 2.4 in [CJLW05] by a slight change of the analysis caused by the Ricci type spinor potential terms in the equations in Theorem 3.15. Then we get the following higher regularity:

**Theorem 3.19.** *Let $(M,\, g_{ij})$ be a compact Riemannian spin surface with non-empty boundary $\partial M$, $(N, h_{\alpha\beta})$ any compact Riemannian manifold of dimension $n \geq 2$, and $\mathcal{S}$ a closed, totally geodesic submanifold of $N$. Let $(u,\, \psi)$ be a weakly Dirac-harmonic map with a Ricci type spinor potential and free boundary on $\mathcal{S}$ and suppose that $u \in C^{0,\alpha}(M,\, N)$ for any $\alpha \in (0,\, 1)$. Then there exists some $\gamma \in (0,\, 1)$ such that*

$$u \in C^{1,\gamma}(M,\, N), \quad \psi \in C^{1,\gamma}(\Sigma M \otimes u^{-1}TN).$$

**Remark 3.20.** *From the proof, one can see that we can also study and prove the same kind of results for nonlinear coupled elliptic system in two dimension, such as Dirac-harmonic maps with other kind of nonlinear potential, when one assumes that the potential forcing term satisfies: $|F_\lambda(u,\, \psi)| \leq C\,|\psi|^p$, for $0 \leq p < 4$.*

# Part II

# An entropy formula and a stability notion in the Yang-Mills flow

# Chapter 4

# An entropy formula and a stability notion in the Yang-Mills flow

In this chapter we shall explore homothetically shrinking solitons of the Yang-Mills flow.

## 4.1 Preliminaries

Let $(M, g)$ be a closed $n$-dimensional Riemannian manifold. Let $G$ be a compact Lie group and $P(M, G)$ a principle bundle over $M$ with the structure group $G$. We now fix a $G$-vector bundle $E = P(M, G) \times_\rho \mathbb{R}^r$, which is associated with $P(M, G)$ via a faithful representation $\rho : G \to SO(r)$.

Let $\mathfrak{g}$ denote the Lie algebra of $G$. A connection on $E$ is locally a $\mathfrak{g}$-valued 1-form.

In this chapter, we will use Latin letters for the manifold indices. One may write a connection $A$ in the form of

$$A = A_i dx^i,$$

where $A_i \in so(r)$. And we will use Greek letters for the bundle indices. One may also write

$$A = A^\alpha_{i\beta} dx^i.$$

The curvature of the connection $A$ is locally a $\mathfrak{g}$-valued 2-form

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j = \frac{1}{2} F^\alpha_{ij\beta} dx^i \wedge dx^j.$$

Here

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

and

$$F^\alpha_{ij\beta} = \partial_i A^\alpha_{j\beta} - \partial_j A^\alpha_{i\beta} + A^\alpha_{i\gamma} A^\gamma_{j\beta} - A^\alpha_{j\gamma} A^\gamma_{i\beta}.$$

Let

$$|F|^2 = g^{ik}g^{jl}F^{\alpha}_{ij\beta}F^{\alpha}_{kl\beta}.$$

The Yang-Mills functional, defined on the space of connections, is then given by

$$YM(A) = \frac{1}{2}\int_M |F|^2 d\mu_g. \tag{4.1}$$

Let $\nabla$ denote the covariant differentiation on $\Gamma(E)$ associated with the connection $A$, and also the covariant differentiation on $\mathfrak{g}$-valued $p$-forms induced by $A$ and the Levi-Civita connection of $(M, g)$. A critical point $A$ of the Yang-Mills functional is called a Yang-Mills connection, which satisfies

$$\nabla^p F^{\alpha}_{pj\beta} = 0.$$

In normal coordinates of $(M, g)$, we have

$$\nabla^p F^{\alpha}_{pj\beta} = \partial_p F^{\alpha}_{pj\beta} + A^{\alpha}_{p\gamma}F^{\gamma}_{pj\beta} - F^{\alpha}_{pj\gamma}A^{\gamma}_{p\beta}.$$

As the $L^2$-gradient flow of the Yang-Mills functional, the Yang-Mills flow is defined by

$$\frac{\partial}{\partial t}A^{\alpha}_{j\beta} = \nabla^p F^{\alpha}_{pj\beta}. \tag{4.2}$$

Assume $A(x, t)$ is a smooth solution to the Yang-Mills flow for $0 \le t < T$ and as $t \to T$ the curvature blows up, i.e. $\limsup_{t\to T} \max_{x\in M} |F(x, t)| = \infty$. If there exists a positive constant $C$ such that

$$|F(x, t)| \le \frac{C}{T - t}, \tag{4.3}$$

one says that the Yang-Mills flow develops a Type-I singularity, or a rapidly forming singularity. Otherwise one says that the Yang-Mills flow develops a Type-II singularity. We call $(x_0, T)$ a Type-I singularity, if (4.3) is satisfied and $x_0$ is a point such that $\limsup_{t\to T} |F(x_0, t)| = \infty$.

Let $A(x, t) = A_p(x, t)dx^p$ be a smooth solution to the Yang-Mills flow and $(x_0, T)$ be a Type-I singularity. We now follow [We04] introducing the blowup procedure around $(x_0, T)$. Let $B_r(x_0)$ be a small geodesic ball of radius $r$ centered at $x_0$ such that over which $E$ is trivial. For simplicity one can identify $B_r(x_0)$ with the ball $B_r(0)$ in $\mathbb{R}^n$. Let $\lambda_i$ be a sequence of positive numbers tending to zero. For each $i$, one gets a Yang-Mills flow $A^{(i)}(y, s)$ by setting

$$A^{(i)}(y, s) = A^{(i)}_p(y, s)dy^p = \lambda_i A_p(\lambda_i y, T + \lambda_i^2 s)dy^p, \quad y \in B_{r/\lambda_i}(0), s \in [-\lambda_i^{-2}T, 0). \tag{4.4}$$

Note that by the assumption (4.3), the curvature of $A^{(i)}$ satisfies

$$|F^{(i)}(y, s)| = \lambda_i^2|F(x, t)| = |s|^{-1}(T - t)|F(x, t)| \le C|s|^{-1}.$$

Let $h = h^\alpha_\beta$ be a gauge transformation which acts on connections by

$$h^*\nabla = h^{-1} \circ \nabla \circ h,$$

or equivalently,

$$h^*A = h^{-1}dh + h^{-1}Ah.$$

Note that gauge transformations preserve Yang-Mills flows. Hence $h^*A^{(i)}(y,s)$ defines a solution to the Yang-Mills flow.

Using a monotonicity formula for the Yang-Mills flow [CS94, Ha93, Na94] and a theorem of Uhlenbeck [Uh82] to improve the regularity of the connections, Weinkove [We04] proved the following theorem.

**Theorem 4.1.** *Let $A(x,t)$ be a smooth solution of the Yang-Mills flow on $M$ for $0 \le t < T$ with a Type-I singularity $(x_0, T)$. Then there exists a sequence of blowups $A^{(i)}(y,s)$ defined by (4.4) and a sequence of gauge transformations $h_i$ such that $h_i^*A^{(i)}(y,s)$ converges smoothly on any compact set to a flow $\widetilde{A}(y,s)$. Here $\widetilde{A}(y,s)$, defined on $\mathbb{R}^n \times (-\infty, 0)$, is a solution of the Yang-Mills flow, which has non-zero curvature and satisfies*

$$\widetilde{\nabla}^p \widetilde{F}_{pj} - \frac{1}{2|s|} y^p \widetilde{F}_{pj} = 0. \tag{4.5}$$

In Theorem 4.1, $h_i$ are chosen as suitable Coulomb gauge transformations so that for any $s < 0$ and $k \ge 1$, $||h_i^*A^{(i)}(y,s)||_{C^k}$ is uniformly bounded. The bounds do not depend on $i$. Hence for any $s < 0$ and $k \ge 1$, $||\widetilde{A}(y,s)||_{C^k}$ is uniformly bounded.

A solution $A(y,s)$ to the Yang-Mills flow, defined on the trivial bundle over $\mathbb{R}^n \times (-\infty, 0)$, is called a homothetically shrinking soliton if it satisfies

$$A^\alpha_{i\beta}(y,s) = \frac{1}{\sqrt{|s|}} A^\alpha_{i\beta}\left(\frac{y}{\sqrt{|s|}}, -1\right)$$

for any $y \in \mathbb{R}^n$ and $s < 0$, see [We04]. The limiting Yang-Mills flow $\widetilde{A}(y,s)$ is actually a homothetically shrinking soliton. Choosing an exponential gauge for $\widetilde{A}(y,s)$, in which $y^p \widetilde{A}^\alpha_{p\beta} = 0$, it was proved in [We04] that

$$\widetilde{A}^\alpha_{i\beta}(y,s) = \frac{1}{\sqrt{|s|}} \widetilde{A}^\alpha_{i\beta}\left(\frac{y}{\sqrt{|s|}}, -1\right).$$

For a limiting Yang-Mills flow $\widetilde{A}(y,s)$ obtained in Theorem 4.1 and any $(x_0, t_0) \in \mathbb{R}^n \times (0, +\infty)$, if we set

$$A(x,t) = A_p(x,t)dx^p = \widetilde{A}_p\left(\frac{x - x_0}{\sqrt{t_0}}, \frac{t - t_0}{t_0}\right)dx^p, \tag{4.6}$$

then $A(x,t)$ is a solution of the Yang-Mills flow defined on $\mathbb{R}^n \times [0, t_0)$ satisfying

$$\nabla^p F_{pj}(x,t) - \frac{1}{2(t_0 - t)}(x - x_0)^p F_{pj}(x,t) = 0. \tag{4.7}$$

## 4.2 $\mathcal{F}$-functional and an entropy

In this section we define the $\mathcal{F}$-functional and entropy for connections on a $G$-vector bundle $E$ over $\mathbb{R}^n$. The definitions of $\mathcal{F}$-functional and entropy are closely related to the monotonicity formula [CS94, Ha93, Na94].

Let $A(x,t)$ be a solution to the Yang-Mills flow defined on $\mathbb{R}^n \times [0,T)$. For any $t_0 > 0$ and $t \in [0, t_0)$, define

$$\Phi_{x_0,t_0}(t) = (t_0 - t)^2 \int_{\mathbb{R}^n} |F(x,t)|^2 G_{x_0,t_0}(x,t)dx,$$

here

$$G_{x_0,t_0}(x,t) = [4\pi(t_0 - t)]^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}.$$

The monotonicity formula of the Yang-Mills flow is

$$\frac{d}{dt}\Phi_{x_0,t_0}(t) = -4(t_0-t)^2 \int_{\mathbb{R}^n} |\nabla^p F_{pj} - \frac{1}{2(t_0-t)}(x-x_0)^p F_{pj}|^2 G_{x_0,t_0}(x,t)\, dx$$

Denote two $\mathfrak{g}$-valued 1-forms $J$ and $K$ respectively by

$$J_j dx^j = \nabla^p F_{pj} dx^j, \quad K_j dx^j = (x - x_0)^p F_{pj} dx^j.$$

Here, and in the sequel, $(x - x_0)^p$ denotes the $p$-th component of $(x - x_0) \in \mathbb{R}^n$.

**Definition 4.2.** *A Yang-Mills flow of connections $A(x,t)$ on the bundle $E$ is called a self-similar solution blows up at $(x_0, t_0)$ if it satisfies*

$$J(x,t) - \frac{1}{2(t_0-t)}K(x,t) = 0. \tag{4.8}$$

**Definition 4.3.** *A smooth connection $A(x)$ on a $G$-vector bundle $E$ over $\mathbb{R}^n$ is called a (Yang-Mills) soliton blows up at $(x_0, t_0)$ if it satisfies*

$$J(x) - \frac{1}{2t_0}K(x) = 0. \tag{4.9}$$

Let

$$YM_w(A) = \int_{\mathbb{R}^n} |F|^2 e^{-\frac{|x-x_0|^2}{4t_0}} dx.$$

$A(x)$ is a soliton blows up at $(x_0, t_0)$ if and only if it is a critical point of $YM_w$.

**Definition 4.4.** *The $\mathcal{F}$-functional with respect to $(x_0, t_0)$ is defined by*

$$\mathcal{F}_{x_0,t_0}(A) = t_0^2 \int_{\mathbb{R}^n} |F|^2 (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} dx.$$

From now on for simplicity, we assume all the $C^k$-norms of the connection $A(x)$ under our consideration are bounded by polynomials in $|x|$. We now compute the first variation of the $\mathcal{F}$-functional. Consider a differentiable family $(x_s, t_s, A_s)$. Let

$$G_s(x) = (4\pi t_s)^{-\frac{n}{2}} e^{-\frac{|x-x_s|^2}{4t_s}},$$

and

$$\dot{t}_s = \frac{d}{ds}t_s, \quad \dot{x}_s = \frac{d}{ds}x_s, \quad \theta_s = \frac{d}{ds}A_s.$$

**Proposition 4.5.** *The first variation of the $\mathcal{F}$-functional is given by*

$$\begin{aligned}
\frac{d}{ds}\mathcal{F}_{x_s,t_s}(A_s) &= \int_{\mathbb{R}^n} \dot{t}_s(\frac{4-n}{2}t_s + \frac{1}{4}|x-x_s|^2)|F_s|^2 G_s(x)dx \\
&\quad + \int_{\mathbb{R}^n} \frac{1}{2}t_s < \dot{x}_s, x - x_s > |F_s|^2 G_s(x)dx \\
&\quad - \int_{\mathbb{R}^n} 4t_s^2 < J - \frac{K}{2t_s}, \theta > G_s(x)dx.
\end{aligned}$$

*Proof.* Note that

$$\frac{\partial}{\partial s}G_s(x) = (-\frac{n}{2}\frac{\dot{t}_s}{t_s} + \frac{\dot{t}_s|x-x_s|^2}{4t_s^2} + \frac{< \dot{x}_s, x-x_s >}{2t_s})G_s(x),$$

and

$$\frac{\partial}{\partial s}|F_s|^2 = 2F^\alpha_{ij\beta}(\nabla_i\theta^\alpha_{j\beta} - \nabla_j\theta^\alpha_{i\beta}),$$

we then have

$$\begin{aligned}
\frac{d}{ds}\mathcal{F}_{x_s,t_s}(A_s) &= \int_{\mathbb{R}^n} 2t_s\dot{t}_s|F_s|^2 G_s(x)dx \\
&\quad + \int_{\mathbb{R}^n} 2t_s^2 F^\alpha_{ij\beta}(\nabla_i\theta^\alpha_{j\beta} - \nabla_j\theta^\alpha_{i\beta})G_s(x)dx \\
&\quad + \int_{\mathbb{R}^n} t_s^2|F_s|^2(-\frac{n}{2}\frac{\dot{t}_s}{t_s} + \frac{\dot{t}_s|x-x_s|^2}{4t_s^2} + \frac{< \dot{x}_s, x-x_s >}{2t_s})G_s(x)dx \\
&= \int_{\mathbb{R}^n} 2t_s\dot{t}_s|F_s|^2 G_s(x)dx + \int_{\mathbb{R}^n} 4t_s^2 F^\alpha_{ij\beta}\nabla_i\theta^\alpha_{j\beta}G_s(x)dx \\
&\quad + \int_{\mathbb{R}^n} t_s^2|F_s|^2(-\frac{n}{2}\frac{\dot{t}_s}{t_s} + \frac{\dot{t}_s|x-x_s|^2}{4t_s^2} + \frac{< \dot{x}_s, x-x_s >}{2t_s})G_s(x)dx.
\end{aligned}$$

By integration by parts, we get

$$\begin{aligned}
\frac{d}{ds}\mathcal{F}_{x_s,t_s}(A_s) &= \int_{\mathbb{R}^n} 2t_s\dot{t}_s|F_s|^2G_s(x)dx \\
&+ \int_{\mathbb{R}^n} [-4t_s^2\nabla_i F^{\alpha}_{ij\beta}\theta^{\alpha}_{j\beta} + 2t_s(x-x_s)^i F^{\alpha}_{ij\beta}\theta^{\alpha}_{j\beta}]G_s(x)dx \\
&+ \int_{\mathbb{R}^n} t_s^2|F_s|^2(-\frac{n}{2}\frac{\dot{t}_s}{t_s} + \frac{\dot{t}_s|x-x_s|^2}{4t_s^2} + \frac{< \dot{x}_s, x-x_s >}{2t_s})G_s(x)dx. \\
&= \int_{\mathbb{R}^n} \dot{t}_s(\frac{4-n}{2}t_s + \frac{1}{4}|x-x_s|^2)|F_s|^2G_s(x)dx \\
&+ \int_{\mathbb{R}^n} \frac{1}{2}t_s < \dot{x}_s, x-x_s > |F_s|^2G_s(x)dx \\
&- \int_{\mathbb{R}^n} 4t_s^2 < J - \frac{K}{2t_s}, \theta > G_s(x)dx.
\end{aligned}$$

□

From Proposition 4.5, we see the following:

**Proposition 4.6.** *A connection $A(x)$ is a critical point of $\mathcal{F}_{x_0,t_0}$ if and only if $A(x)$ is a soliton blows up at $(x_0, t_0)$.*

We shall check that $(A(x), x_0, t_0)$ is a critical point of the $\mathcal{F}$-functional if and only if $A(x)$ is a soliton blows up at $(x_0, t_0)$. To check this we will need some identities on solitons; we also need such identities to compute the second variation of the $\mathcal{F}$-functional in next section. Denote

$$G(x) = (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}}.$$

**Lemma 4.7.** *Let $A(x)$ be a soliton blows up at $(x_0, t_0)$ and $\varphi = \varphi^p\partial_p$ a vector field on $\mathbb{R}^n$ such that $|\varphi|$ is a polynomial in $|x - x_0|$. Then we have*

$$\int_{\mathbb{R}^n} \varphi^p(x-x_0)^p|F|^2G(x)dx = \int_{\mathbb{R}^n} [2t_0\partial_p(\varphi^p)|F|^2 - 8t_0\partial_i\varphi^p F^{\alpha}_{pj\beta}F^{\alpha}_{ij\beta}]G(x)dx.$$

*In particular,*

*(a)* $\int_{\mathbb{R}^n} |x-x_0|^2|F|^2G(x)dx = \int_{\mathbb{R}^n} 2(n-4)t_0|F|^2G(x)dx;$

*(b)* $\int_{\mathbb{R}^n}(x-x_0)^k|F|^2G(x)dx = 0;$

*(c)* $\int_{\mathbb{R}^n} |x-x_0|^4|F|^2G(x)dx = \int_{\mathbb{R}^n}[4(n-2)(n-4)t_0^2|F|^2 - 64t_0^3|J|^2]Gdx;$

*(d)* $\int_{\mathbb{R}^n} |x-x_0|^2 < V, x-x_0 > |F|^2G(x)dx = 0;$

(e) $\int_{\mathbb{R}^n} < x - x_0, V >^2 |F|^2 G dx = \int_{\mathbb{R}^n} (2t_0 |V|^2 |F|^2 - 8t_0 < V^i F_{ij}, V^p F_{pj} >) G dx.$

*Proof.* By the assumption on the growth rate of $|A(x)|_{C^k(B_r)}$, we are free in doing integration by parts for those vector fields $\varphi$ under consideration.

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi^p (x - x_0)^p |F|^2 G(x) dx &= \int_{\mathbb{R}^n} -2t_0 \varphi^p |F|^2 \partial_p G(x) dx \\ &= \int_{\mathbb{R}^n} 2t_0 [\partial_p(\varphi^p) |F|^2 + \varphi^p \partial_p(|F|^2)] G(x) dx.\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{\mathbb{R}^n} 8t_0 \varphi^p F^\alpha_{pj\beta} J^\alpha_{j\beta} G dx &= \int_{\mathbb{R}^n} 8t_0 \varphi^p [\partial_i (F^\alpha_{pj\beta} F^\alpha_{ij\beta}) - \nabla_i F^\alpha_{pj\beta} F^\alpha_{ij\beta}] G dx \\ &= \int_{\mathbb{R}^n} -8t_0 F^\alpha_{pj\beta} F^\alpha_{ij\beta} [\partial_i \varphi^p - \frac{(x - x_0)^i}{2t_0} \varphi^p] G dx \\ &\quad + \int_{\mathbb{R}^n} -4t_0 \varphi^p (\nabla_i F^\alpha_{pj\beta} F^\alpha_{ij\beta} + \nabla_j F^\alpha_{ip\beta} F^\alpha_{ij\beta}) G dx.\end{aligned}$$

It then follows from the Bianchi identity that

$$\begin{aligned}\int_{\mathbb{R}^n} 8t_0 \varphi^p F^\alpha_{pj\beta} J^\alpha_{j\beta} G dx &= \int_{\mathbb{R}^n} -8t_0 F^\alpha_{pj\beta} F^\alpha_{ij\beta} [\partial_i \varphi^p - \frac{(x - x_0)^i}{2t_0} \varphi^p] G dx \\ &\quad + \int_{\mathbb{R}^n} -4t_0 \varphi^p \nabla_p F^\alpha_{ij\beta} F^\alpha_{ij\beta} G dx \\ &= \int_{\mathbb{R}^n} -8t_0 F^\alpha_{pj\beta} F^\alpha_{ij\beta} [\partial_i \varphi^p - \frac{(x - x_0)^i}{2t_0} \varphi^p] G dx \\ &\quad + \int_{\mathbb{R}^n} -2t_0 \varphi^p \partial_p (|F|^2) G dx.\end{aligned}$$

Thus we get

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi^p (x - x_0)^p |F|^2 G(x) dx &= \int_{\mathbb{R}^n} 2t_0 [\partial_p(\varphi^p) |F|^2 + \varphi^p \partial_p(|F|^2)] G(x) dx \\ &= \int_{\mathbb{R}^n} 2t_0 \partial_p(\varphi^p) |F|^2 G(x) dx - \int_{\mathbb{R}^n} 8t_0 \varphi^p F^\alpha_{pj\beta} J^\alpha_{j\beta} G dx \\ &\quad + \int_{\mathbb{R}^n} -8t_0 F^\alpha_{pj\beta} F^\alpha_{ij\beta} [\partial_i \varphi^p - \frac{(x - x_0)^i}{2t_0} \varphi^p] G dx \\ &= \int_{\mathbb{R}^n} [2t_0 \partial_p(\varphi^p) |F|^2 - 8t_0 \partial_i \varphi^p F^\alpha_{pj\beta} F^\alpha_{ij\beta}] G(x) dx \\ &\quad - \int_{\mathbb{R}^n} 8t_0 \varphi^p F^\alpha_{pj\beta} (J^\alpha_{j\beta} - \frac{1}{2t_0} K^\alpha_{j\beta}) G dx.\end{aligned}$$

Therefore, for a soliton blows up at $(x_0, t_0)$

$$\int_{\mathbb{R}^n} \varphi^p (x-x_0)^p |F|^2 G(x)dx = \int_{\mathbb{R}^n} [2t_0 \partial_p(\varphi^p)|F|^2 - 8t_0 \partial_i \varphi^p F^\alpha_{pj\beta} F^\alpha_{ij\beta}] G(x)dx. \quad (4.10)$$

Taking $\varphi^p = (x-x_0)^p$, by (4.10) we get

$$\int_{\mathbb{R}^n} |x-x_0|^2 |F|^2 G(x)dx = \int_{\mathbb{R}^n} 2(n-4)t_0 |F|^2 G(x)dx.$$

Taking $\varphi^p = \delta^p_k$, by (4.10) we get for any $k$

$$\int_{\mathbb{R}^n} (x-x_0)^k |F|^2 G(x)dx = 0.$$

Taking $\varphi^p = |x-x_0|^2 (x-x_0)^p$, using (4.10) we compute that

$$\begin{aligned} & \int_{\mathbb{R}^n} |x-x_0|^4 |F|^2 G(x)dx \\ = & \int_{\mathbb{R}^n} [2t_0(n+2)|x-x_0|^2|F|^2 - 8t_0|x-x_0|^2|F|^2 - 16t_0|K|^2] G dx \\ = & \int_{\mathbb{R}^n} [4(n-2)(n-4)t_0^2 |F|^2 - 64 t_0^3 |J|^2] G dx. \end{aligned}$$

Taking $\varphi^p = |x-x_0|^2 V^p$, by (4.10) we get

$$\begin{aligned} & \int_{\mathbb{R}^n} |x-x_0|^2 < V, x-x_0 > |F|^2 G(x)dx \\ = & \int_{\mathbb{R}^n} -32 t_0^2 < J_j, V^p F_{pj} > G dx. \end{aligned}$$

On the other hand, if we take $\varphi^p =< V, x-x_0 > (x-x_0)^p$ and by (b) we get

$$\begin{aligned} & \int_{\mathbb{R}^n} |x-x_0|^2 < V, x-x_0 > |F|^2 G(x)dx \\ = & \int_{\mathbb{R}^n} -16 t_0^2 < J_j, V^i F_{ij} > G dx. \end{aligned}$$

Thus

$$\int_{\mathbb{R}^n} |x-x_0|^2 < V, x-x_0 > |F|^2 G(x)dx = \int_{\mathbb{R}^n} -32 t_0^2 < J_j, V^p F_{pj} > G dx = 0.$$

Taking $\varphi^p =< V, x-x_0 > V^p$, by (4.10) we get

$$\int_{\mathbb{R}^n} < x-x_0, V >^2 |F|^2 G dx = \int_{\mathbb{R}^n} (2t_0 |V|^2 |F|^2 - 8t_0 < V^i F_{ij}, V^p F_{pj} >) G dx.$$

□

By the first variation formula and $(a), (b)$ in Lemma 4.7 we get the following proposition.

**Proposition 4.8.** *$(A(x), x_0, t_0)$ is a critical point of the $\mathcal{F}$-functional if and only if $A(x)$ is a soliton blows up at $(x_0, t_0)$.*

By Weinkove's result (Theorem 4.1), at a type I singularity of a Yang-Mills flow one obtains a soliton on the trivial bundle over $\mathbb{R}^n$, i.e. $A(x, 0)$ in (4.6), whose curvature is uniformly bounded on $\mathbb{R}^n$ and non-zero. However in dimension four, by Lemma 4.7 (a), such a soliton must have vanishing curvature.

**Proposition 4.9.** *In dimension four, the Yang-Mills flow cannot develop a singularity of type I.*

We end this section by introducing the entropy of a connection $A(x)$ on the $G$-vector bundle $E$ over $\mathbb{R}^n$. Along a Yang-Mills flow on $E$, the entropy is non-increasing.

**Definition 4.10.** *Let $A(x)$ be a connection on the $G$-vector bundle $E$ over $\mathbb{R}^n$. The entropy is defined by*

$$\lambda(A) = \sup_{x_0 \in \mathbb{R}^n, t_0 > 0} \mathcal{F}_{x_0,t_0}(A). \tag{4.11}$$

**Proposition 4.11.** *Let $A(x,t)$ be a solution on $\mathbb{R}^n \times [0, T)$ to the Yang-Mills flow with uniformly bounded Yang-Mills functional. Then the entropy $\lambda(A(x,t))$ is non-increasing in $t$.*

*Proof.* Let $t_1 < t_2 < T$. For any given positive number $\epsilon$, there exists $(x_0, t_0)$ such that

$$\mathcal{F}_{x_0,t_0}(A(x,t_2)) \geq \lambda(A(x,t_2)) - \epsilon.$$

It follows from the monotonicity formula that for any $\delta \in (0, t_2)$

$$\Phi_{x_0,t_0+t_2}(A(x,t_2)) \leq \Phi_{x_0,t_0+t_2}(A(x,t_2-\delta)).$$

Note that

$$\Phi_{x_0,t_0+t_2}(A(x,t_2)) = \mathcal{F}_{x_0,t_0}(A(x,t_2)), \quad \Phi_{x_0,t_0+t_2}(A(x,t_2-\delta)) = \mathcal{F}_{x_0,t_0+\delta}(A(x,t_2-\delta)).$$

Hence we get

$$\mathcal{F}_{x_0,t_0}(A(x,t_2)) \leq \mathcal{F}_{x_0,t_0+\delta}(A(x,t_2-\delta)).$$

Taking $\delta = t_2 - t_1$, it implies that

$$\mathcal{F}_{x_0,t_0}(A(x,t_2)) \leq \mathcal{F}_{x_0,t_0+t_2-t_1}(A(x,t_1)) \leq \lambda(A(x,t_1)).$$

□

## 4.3 A stability for solitons

We now compute the second variation of the $\mathcal{F}$-functional at a soliton $A(x)$ on the $G$-vector bundle $E$ over $\mathbb{R}^n$. Let

$$\dot{t}_s|_{s=0} = q, \quad \dot{x}_s|_{s=0} = V, \quad \theta_s = \frac{d}{ds}A_s,$$

and

$$\mathcal{F}''_{x_0,t_0}(q,V,\theta) = \frac{d^2}{ds^2}|_{s=0}\mathcal{F}_{x_s,t_s}(A_s).$$

**Proposition 4.12.** *Let $A(x)$ be a soliton on the $G$-vector bundle $E$ over $\mathbb{R}^n$. Then*

$$\frac{1}{4t_0}\mathcal{F}''_{x_0,t_0}(q,V,\theta) = \int_{\mathbb{R}^n} < L\theta - 2qJ - V\cdot F, \theta > Gdx - \int_{\mathbb{R}^n}(q^2|J|^2 + \frac{1}{2}|V\cdot F|^2)Gdx, \tag{4.12}$$

*here*

$$\mathcal{R}(\theta)(X) = [F(\partial_i, X), \theta_i], \quad (V\cdot F)(\partial_j) = V^i F_{ij},$$

*and*

$$L = t_0[D^*D + \mathcal{R} + i_{\frac{1}{2t_0}(x-x_0)}D].$$

*Proof.* Note that

$$\begin{aligned}
\frac{d}{ds}\mathcal{F}_{x_s,t_s}(A_s) &= \int_{\mathbb{R}^n} \dot{t}_s(\frac{4-n}{2}t_s + \frac{1}{4}|x-x_s|^2)|F_s|^2 G_s(x)dx \\
&\quad + \int_{\mathbb{R}^n} \frac{1}{2}t_s < \dot{x}_s, x - x_s > |F_s|^2 G_s(x)dx \\
&\quad - \int_{\mathbb{R}^n} 4t_s^2 < J - \frac{K}{2t_s}, \theta > G_s(x)dx.
\end{aligned}$$

By using the assumption that $A(x)$ is a soliton blows up at $(x_0,t_0)$ and $(a)$, $(b)$ in Lemma 4.7 ,

$$\begin{aligned}
\mathcal{F}''_{x_0,t_0}(q,V,\theta) &= \int_{\mathbb{R}^n}[q(\frac{4-n}{2}q - \frac{1}{2} < x-x_0, V >) + \frac{1}{2}t_0 < V, -V >]|F|^2 Gdx \\
&\quad + \int_{\mathbb{R}^n}[q(\frac{4-n}{2}t_0 + \frac{1}{4}|x-x_0|^2) + \frac{1}{2}t_0 < V, x-x_0 >]\frac{\partial}{\partial s}|F_s|^2 Gdx \\
&\quad + \int_{\mathbb{R}^n}[q(\frac{4-n}{2}t_0 + \frac{1}{4}|x-x_0|^2) + \frac{1}{2}t_0 < V, x-x_0 >]|F|^2\frac{\partial}{\partial s}G_s dx \\
&\quad - \int_{\mathbb{R}^n} 4t_0^2 < \frac{\partial}{\partial s}(J - \frac{K}{2t_s}), \theta > Gdx.
\end{aligned}$$

Note that

$$\frac{\partial}{\partial s}|F_s|^2 = 2F^{\alpha}_{ij\beta}(\nabla_i\theta^{\alpha}_{j\beta} - \nabla_j\theta^{\alpha}_{i\beta}) = 4F^{\alpha}_{ij\beta}\nabla_i\theta^{\alpha}_{j\beta},$$

$$\frac{\partial}{\partial s}G_s(x) = (-\frac{n}{2}\frac{\dot{t}_s}{t_s} + \frac{\dot{t}_s|x-x_s|^2}{4t_s^2} + \frac{< \dot{x}_s, x-x_s >}{2t_s})G_s(x),$$

$$\frac{\partial}{\partial s}J^\alpha_{j\beta} = \nabla_p\nabla_p\theta^\alpha_{j\beta} - \nabla_p\nabla_j\theta^\alpha_{p\beta} + \theta^\alpha_{p\gamma}F^\gamma_{pj\beta} - F^\alpha_{pj\gamma}\theta^\gamma_{p\beta},$$

$$\frac{\partial}{\partial s}(-\frac{1}{2t_s}K^\alpha_{j\beta}) \;\; = \;\; \frac{q}{2t_0^2}(x-x_0)^k F^\alpha_{kj\beta} + \frac{1}{2t_0}V^k F^\alpha_{kj\beta} - \frac{1}{2t_0}(x-x_0)^k(\nabla_k\theta^\alpha_{j\beta} - \nabla_j\theta^\alpha_{k\beta}).$$

Thus we get

$$\begin{aligned}
\mathcal{F}''_{x_0,t_0}(q,V,\theta) \;\; = \;\; & \int_{\mathbb{R}^n}[q(\frac{4-n}{2}q - \frac{1}{2} < x-x_0, V >) + \frac{1}{2}t_0 < V, -V >]|F|^2 G dx \\
& + \int_{\mathbb{R}^n}\left\{q(\frac{4-n}{2}t_0 + \frac{1}{4}|x-x_0|^2)\right. \\
& \left. + \frac{1}{2}t_0 < V, x-x_0 >\right\} 4F^\alpha_{ij\beta}\nabla_i\theta^\alpha_{j\beta}G dx \\
& - \int_{\mathbb{R}^n} 4t_0^2[\nabla_p(\nabla_p\theta^\alpha_{j\beta} - \nabla_j\theta^\alpha_{p\beta}) + \theta^\alpha_{p\gamma}F^\gamma_{pj\beta} - F^\alpha_{pj\gamma}\theta^\gamma_{p\beta}]\theta^\alpha_{j\beta}G dx \\
& - \int_{\mathbb{R}^n} 4t_0^2\left\{\frac{q}{2t_0^2}(x-x_0)^k F^\alpha_{kj\beta} + \frac{1}{2t_0}V^k F^\alpha_{kj\beta}\right. \\
& \left. - \frac{1}{2t_0}(x-x_0)^k(\nabla_k\theta^\alpha_{j\beta} - \nabla_j\theta^\alpha_{k\beta})\right\}\theta^\alpha_{j\beta}G dx \\
& + \int_{\mathbb{R}^n}[q(\frac{4-n}{2}t_0 + \frac{1}{4}|x-x_0|^2) + \frac{1}{2}t_0 < V, x-x_0 >]|F|^2 \\
& \quad \times(-\frac{n}{2}\frac{q}{t_0} + \frac{q|x-x_0|^2}{4t_0^2} + \frac{< V, x-x_0 >}{2t_0})G dx.
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
& \int_{\mathbb{R}^n}[q(\frac{4-n}{2}t_0 + \frac{1}{4}|x-x_0|^2) + \frac{1}{2}t_0 < V, x-x_0 >]4F^\alpha_{ij\beta}\nabla_i\theta^\alpha_{j\beta}G dx \\
= \;\; & \int_{\mathbb{R}^n} -4[q(\frac{4-n}{2}t_0 + \frac{1}{4}|x-x_0|^2) + \frac{1}{2}t_0 < V, x-x_0 >] < J - \frac{1}{2t_0}K, \theta > G dx \\
& - \int_{\mathbb{R}^n} 4[\frac{1}{2}q(x-x_0)^i + \frac{1}{2}t_0 V^i]F^\alpha_{ij\beta}\theta^\alpha_{j\beta}G dx \\
= \;\; & \int_{\mathbb{R}^n}[-2q(x-x_0)^i - 2t_0 V^i]F^\alpha_{ij\beta}\theta^\alpha_{j\beta}G dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{F}''_{x_0,t_0}(q,V,\theta) &= \int_{\mathbb{R}^n}[q(\frac{4-n}{2}q-\frac{1}{2}<x-x_0,V>)+\frac{1}{2}t_0<V,-V>]|F|^2Gdx \\
&+\int_{\mathbb{R}^n}[-2q(x-x_0)^i-2t_0V^i]F^{\alpha}_{ij\beta}\theta^{\alpha}_{j\beta}Gdx \\
&-\int_{\mathbb{R}^n}4t_0^2[\nabla_p(\nabla_p\theta^{\alpha}_{j\beta}-\nabla_j\theta^{\alpha}_{p\beta})+\theta^{\alpha}_{p\gamma}F^{\gamma}_{pj\beta}-F^{\alpha}_{pj\gamma}\theta^{\gamma}_{p\beta}]\theta^{\alpha}_{j\beta}Gdx \\
&-\int_{\mathbb{R}^n}[2q(x-x_0)^i+2t_0V^i]F^{\alpha}_{ij\beta}\theta^{\alpha}_{j\beta}Gdx \\
&+\int_{\mathbb{R}^n}2t_0(x-x_0)^k(\nabla_k\theta^{\alpha}_{j\beta}-\nabla_j\theta^{\alpha}_{k\beta})\theta^{\alpha}_{j\beta}Gdx \\
&+\int_{\mathbb{R}^n}[q(\frac{4-n}{2}t_0+\frac{1}{4}|x-x_0|^2)+\frac{1}{2}t_0<V,x-x_0>]|F|^2 \\
&\quad\times(-\frac{n}{2}\frac{q}{t_0}+\frac{q|x-x_0|^2}{4t_0^2}+\frac{<V,x-x_0>}{2t_0})Gdx.
\end{aligned}$$

So we get

$$\begin{aligned}
\mathcal{F}''_{x_0,t_0}(q,V,\theta) &= -\int_{\mathbb{R}^n}4t_0^2[\nabla_p(\nabla_p\theta^{\alpha}_{j\beta}-\nabla_j\theta^{\alpha}_{p\beta})+\theta^{\alpha}_{p\gamma}F^{\gamma}_{pj\beta}-F^{\alpha}_{pj\gamma}\theta^{\gamma}_{p\beta}]\theta^{\alpha}_{j\beta}Gdx \\
&+\int_{\mathbb{R}^n}2t_0(x-x_0)^k(\nabla_k\theta^{\alpha}_{j\beta}-\nabla_j\theta^{\alpha}_{k\beta})\theta^{\alpha}_{j\beta}Gdx \\
&-\int_{\mathbb{R}^n}[4q(x-x_0)^i+4t_0V^i]F^{\alpha}_{ij\beta}\theta^{\alpha}_{j\beta}Gdx \\
&+\int_{\mathbb{R}^n}[q(\frac{4-n}{2}q-\frac{1}{2}<x-x_0,V>)+\frac{1}{2}t_0<V,-V>]|F|^2Gdx \\
&+\int_{\mathbb{R}^n}[q(\frac{4-n}{2}t_0+\frac{1}{4}|x-x_0|^2)+\frac{1}{2}t_0<V,x-x_0>]|F|^2 \\
&\quad\times(-\frac{n}{2}\frac{q}{t_0}+\frac{q|x-x_0|^2}{4t_0^2}+\frac{<V,x-x_0>}{2t_0})Gdx.
\end{aligned}$$

Now let $D$ denote the covariant exterior differentiation on $\mathfrak{g}$-valued 1-form and

$D^*$ the adjoint operator of $D$. Then

$$\begin{aligned}
\mathcal{F}''_{x_0,t_0}(q,V,\theta) &= \int_{\mathbb{R}^n} 4t_0^2 < D^*D\theta_j + [F_{ij},\theta_i],\theta_j > Gdx \\
&+ \int_{\mathbb{R}^n} 4t_0^2 < i_{\frac{1}{2t_0}(x-x_0)} D\theta,\theta > Gdx \\
&- \int_{\mathbb{R}^n} 4t_0 < 2qJ_j + V^i F_{ij},\theta_j > Gdx \\
&+ \int_{\mathbb{R}^n} [q(\frac{4-n}{2}q - \frac{1}{2} < x-x_0, V >) + \frac{1}{2}t_0 < V, -V >]|F|^2 Gdx \\
&+ \int_{\mathbb{R}^n} [q(\frac{4-n}{2}t_0 + \frac{1}{4}|x-x_0|^2) + \frac{1}{2}t_0 < V, x-x_0 >]|F|^2 \\
&\quad \times(-\frac{n}{2}\frac{q}{t_0} + \frac{q|x-x_0|^2}{4t_0^2} + \frac{< V, x-x_0 >}{2t_0})Gdx. \qquad (4.13)
\end{aligned}$$

By using Lemma 4.7, one can compute

$$\begin{aligned}
& \int_{\mathbb{R}^n} [q(\frac{4-n}{2}t_0 + \frac{1}{4}|x-x_0|^2) + \frac{1}{2}t_0 < V, x-x_0 >]|F|^2 \\
&\quad \times(-\frac{n}{2}\frac{q}{t_0} + \frac{q|x-x_0|^2}{4t_0^2} + \frac{< V, x-x_0 >}{2t_0})Gdx. \\
= & -\int_{\mathbb{R}^n} \frac{n(4-n)}{4} q^2\,|F|^2 Gdx + \int_{\mathbb{R}^n} \frac{q^2(2-n)}{4t_0}\,|x-x_0|^2\,|F|^2 Gdx \\
& + \int_{\mathbb{R}^n} \frac{q(2-n)}{2}\,\langle V, x-x_0\rangle\,|F|^2 Gdx + \int_{\mathbb{R}^n} \frac{q^2}{16t_0^2}\,|x-x_0|^4\,|F|^2 Gdx \\
& + \int_{\mathbb{R}^n} \frac{q}{4t_0}\,|x-x_0|^2\,\langle V, x-x_0\rangle\,|F|^2 Gdx + \int_{\mathbb{R}^n} \frac{\langle V, x-x_0\rangle^2}{4}\,|F|^2 Gdx \\
= & -\int_{\mathbb{R}^n} \frac{n(4-n)}{4} q^2\,|F|^2 Gdx + \int_{\mathbb{R}^n} \frac{q^2(2-n)}{4t_0}\,2(n-4)t_0\,|F|^2 Gdx \\
& + \int_{\mathbb{R}^n} \frac{q^2}{16t_0^2}\,\left\{4(n-2)(n-4)t_0^2|F|^2 - 64t_0^3|J|^2\right\} Gdx \\
& + \int_{\mathbb{R}^n} \frac{1}{2}\left(t_0|V|^2|F|^2 - 4t_0 < V^i F_{ij}, V^p F_{pj} >\right) Gdx \\
= & \int_{\mathbb{R}^n} \frac{q^2(n-4)}{2}|F|^2 Gdx + \int_{\mathbb{R}^n} \frac{t_0}{2}|V|^2|F|^2 G\,dx \\
& -4t_0 \int_{\mathbb{R}^n} \left(q^2|J|^2 + \frac{1}{2}\,|V^p\,F_{pj}|^2\right). \qquad (4.14)
\end{aligned}$$

Combining (4.13) and (4.14), we obtain

$$\begin{aligned}\mathcal{F}''_{x_0,t_0}(q,V,\theta) &= \int_{\mathbb{R}^n} 4t_0^2 < D^*D\theta_j + [F_{ij},\theta_i],\theta_j > Gdx \\ &+\int_{\mathbb{R}^n} 4t_0^2 < i_{\frac{1}{2t_0}(x-x_0)}D\theta,\theta > Gdx \\ &-\int_{\mathbb{R}^n} 4t_0 < 2qJ_j + V^iF_{ij},\theta_j > Gdx \\ &-4t_0\int_{\mathbb{R}^n}(q^2|J|^2+\frac{1}{2}|V^pF_{pj}|^2)Gdx.\end{aligned}$$

Now let

$$\mathcal{R}(\theta)(X) = [F(\partial_i,X),\theta_i], \quad (V\cdot F)(\partial_j) = V^iF_{ij},$$
$$L = t_0[D^*D+\mathcal{R}+i_{\frac{1}{2t_0}(x-x_0)}D],$$

then we have

$$\begin{aligned}\frac{1}{4t_0}\mathcal{F}''_{x_0,t_0}(q,V,\theta) &= \int_{\mathbb{R}^n} < L\theta - 2qJ - V\cdot F,\theta > Gdx \\ &-\int_{\mathbb{R}^n}(q^2|J|^2+\frac{1}{2}|V\cdot F|^2)Gdx.\end{aligned}$$

□

Now we are going to consider the $\mathcal{F}$-stability for weakly self-similar solutions which blows up at $(x_0,\, t_0)$. Without loss of generality, we may assume $t_0 = 1, x_0 = 0$, then the operator $L$ in Proposition 4.12 corresponds to

$$L = D^*D + \mathcal{R} + i_{\frac{x}{2}}D.$$

**Definition 4.13.** *A self-similar solution $A$ with blow up at $t_0 = 1, x_0 = 0$ is called $\mathcal{F}$-stable if for any $X \in W^{2,2}_\omega$ there exist a real number $q$ and a constant vector field $V$ on $\mathbb{R}^n$ such that $\mathcal{F}''(q,V,X) \geq 0$, where*

$$W^{2,2}_\omega := \left\{ X \in \Gamma\left(\mathfrak{g}\otimes\Lambda^1\mathbb{R}^n\right) \middle| \int_{\mathbb{R}^n}\left(|X|^2+|DX|^2+|LX|^2\right)G\,dx < \infty \right\}.$$

**Proposition 4.14.** *Let $A$ be a self-similar solution with blow up at $(x_0 = 0, t_0 = 1)$. That is $J = \frac{1}{2}K$. Then we have*

$$-LJ = J, \quad -L(V\cdot F) = \frac{1}{2}V\cdot F.$$

*Proof.* Note that

$$J_j = \nabla_p F_{pj} = \frac{1}{2} x^p F_{pj}$$

and

$$-L = -D^* D - \mathcal{R} - i_{\frac{x}{2}} D.$$

In fact, we have

$$\begin{aligned} -LJ_j &= \nabla_p \nabla_p J_j - \nabla_p \nabla_j J_p - [F_{pj}, J_p] - \frac{1}{2}(DJ)(x^p \partial_p, \partial_j) \\ &= \nabla_p \nabla_p J_j - \nabla_p \nabla_j J_p - [F_{pj}, J_p] - \frac{1}{2} x^p (\nabla_p J_j - \nabla_j J_p). \end{aligned}$$

Now

$$\nabla_p J_j = \nabla_p (\frac{1}{2} x^q F_{qj}) = \frac{1}{2} F_{pj} + \frac{1}{2} x^q \nabla_p F_{qj}.$$

Then

$$\begin{aligned} \nabla_p \nabla_p J_j &= \nabla_p F_{pj} + \frac{1}{2} x^q \nabla_p \nabla_p F_{qj} \\ &= \nabla_p F_{pj} + \frac{1}{2} x^q \nabla_p (-\nabla_q F_{jp} - \nabla_j F_{pq}) \\ &= \nabla_p F_{pj} - \frac{1}{2} x^q (\nabla_q \nabla_p F_{jp} + F_{pq} F_{jp} - F_{jp} F_{pq}) - \frac{1}{2} x^q \nabla_p \nabla_j F_{pq} \\ &= J_j + \frac{1}{2} x^q \nabla_q J_j + [J_p, F_{jp}] - \frac{1}{2} x^q \nabla_p \nabla_j F_{pq} \end{aligned}$$

and

$$\begin{aligned} \nabla_p \nabla_j J_p &= \nabla_p (-\frac{1}{2} F_{pj} + \frac{1}{2} x^q \nabla_j F_{qp}) \\ &= -\frac{1}{2} \nabla_p F_{pj} + \frac{1}{2} x^q \nabla_p \nabla_j F_{qp} \end{aligned}$$

We now compute that

$$\begin{aligned} \frac{1}{2} x^p \nabla_j J_p &= \frac{1}{2} \nabla_j (x^p J_p) - \frac{1}{2} J_j \\ &= \frac{1}{2} \nabla_j (x^p \frac{1}{2} x^q F_{qp}) - \frac{1}{2} J_j \\ &= -\frac{1}{2} J_j. \end{aligned}$$

Hence we get

$$-LJ = J.$$

$$
\begin{aligned}
-L(V^q F_{qj}) &= \nabla_p \nabla_p (V^q F_{qj}) - \nabla_p \nabla_j (V^q F_{qp}) \\
&\quad -[F_{pj}, V^q F_{qp}] - \frac{1}{2} x^p [\nabla_p (V^q F_{qj}) - \nabla_j (V^q F_{qp})].
\end{aligned}
$$

Now

$$
\begin{aligned}
\nabla_p \nabla_p (V^q F_{qj}) &= V^q \nabla_p \nabla_p F_{qj} \\
&= V^q (-\nabla_q F_{jp} - \nabla_j F_{pq}) \\
&= -V^q (\nabla_q \nabla_p F_{jp} + F_{pq} F_{jp} - F_{jp} F_{pq}) - V^q \nabla_p \nabla_j F_{pq} \\
&= V^q \nabla_q (\frac{1}{2} x^p F_{pj}) + [V^q F_{qp}, F_{jp}] + V^q \nabla_p \nabla_j F_{qp},
\end{aligned}
$$

hence

$$
\begin{aligned}
-L(V^q F_{qj}) &= V^q \nabla_q (\frac{1}{2} x^p F_{pj}) - \frac{1}{2} x^p [\nabla_p (V^q F_{qj}) - \nabla_j (V^q F_{qp})] \\
&= \frac{1}{2} V^q F_{qj} + \frac{1}{2} x^p V^q (\nabla_q F_{pj} + \nabla_p F_{jq}) + \nabla_j F_{qp}) \\
&= \frac{1}{2} V^q F_{qj}.
\end{aligned}
$$

□

From Proposition 4.14, one knows that if $J$ and $V \cdot F$ are in the space $W^{2,2}_{\omega}$, then $J$ and $V \cdot F$ are eigenvector fields of $L$ with respect to the eigenvalues $-1$ and $-\frac{1}{2}$ respectively( if they are not identically zero). The $\mathcal{F}$-stability can be characterized by the eigenvector fields space as follows:

**Theorem 4.15.** *Let $A$ be a self-similar solution with blow up at $t_0 = 1, x_0 = 0$ such that $J$ and $V \cdot F$ are in the space $W^{2,2}_{\omega}$. Then $A$ is $\mathcal{F}$-stable if and only if the eigenvector field space satisfies the following properties:*

(1) $\mathbb{X}_{-1} = \{\mathbb{R}J\}$ ,

(2) $\mathbb{X}_{-\frac{1}{2}} = \{V \cdot F, \qquad V \in \mathbb{R}^n\}$ ,

(3) $\mathbb{X}_{\lambda} = \{0\}$, *for each* $\lambda < 0$ *and* $\lambda \neq -1, -\frac{1}{2}$.

*Proof.* Let $X$ be a $\mathfrak{g}$-value 1-form in $W^{2,2}_{\omega}$ and of the form

$$
X = q_0 \, J + V_0 \cdot F + X_1 \quad q_0 \in \mathbb{R} \text{ and } V_0 \in \mathbb{R}^n
$$

such that

$$
\int_{\mathbb{R}^n} \langle J, \ X_1 \rangle \, G \, dx = \int_{\mathbb{R}^n} \langle V \cdot F, \ X_1 \rangle \, G \, dx = 0 \quad \forall V \in \mathbb{R}^n.
$$

Then it follows from Proposition 4.12 and Proposition 4.14 that

$$\begin{aligned}
&\frac{1}{4}\mathcal{F}''_{(0,1)}(q, V, X) \\
= &\int_{\mathbb{R}^n} \langle LX - 2qJ - V\cdot F,\ X\rangle\, G\, dx - \int_{\mathbb{R}^n}\left(q^2\,|J|^2 + \frac{1}{2}\,|V\cdot F|^2\right) G\, dx \\
= &\int_{\mathbb{R}^n}\left\langle -q_0 J - \frac{1}{2}V_0\cdot F + LX_1,\ q_0\, J + V_0\cdot F + X_1\right\rangle G\, dx \\
&+ \int_{\mathbb{R}^n}\langle -2qJ - V\cdot F,\ q_0\, J + V_0\cdot F + X_1\rangle\, G\, dx \\
&- \int_{\mathbb{R}^n}\left(q^2\,|J|^2 + \frac{1}{2}\,|V\cdot F|^2\right) G\, dx \\
= &\int_{\mathbb{R}^n}\langle LX_1,\ X_1\rangle\, G\, dx - (q_0+q)^2\int_{\mathbb{R}^n}|J|^2\, G\, dx - \frac{1}{2}\int_{\mathbb{R}^n}|V\cdot F|^2\, G\, dx.
\end{aligned}$$

Let $q = -q_0$, $v = -v_0$, one has the equivalence. □

As a byproduct, we also have the following rigidity theorem.

**Theorem 4.16.** *Let $A$ be a self-similar solution with blows up at $t_0 = 1, x_0 = 0$, If $|F| \le \frac{1}{2}$, then the $G$-vector bundle $(E,\, A)$ is flat.*

*Proof.* Since $|F|$ is bounded, and $J_j = \frac{1}{2}x^p F_{pj} = \nabla_p F_{pj}$, one can integrate by parts

$$\int_{\mathbb{R}^n}\left\langle D^*DJ + i_{\frac{x}{2}}\, DJ,\ J\right\rangle G\, dx = \int_{\mathbb{R}^n}|D\, J|^2\, G\, dx.$$

On the other hand, one computes

$$\begin{aligned}
\int_{\mathbb{R}^n}\left\langle D^*DJ + i_{\frac{x}{2}}\, DJ,\ J\right\rangle G\, dx &= \int_{\mathbb{R}^n}\langle LJ - \mathcal{R}J,\ J\rangle\, G\, dx \\
&= -\int_{\mathbb{R}^n}|J|^2\, G\, dx - \int_{\mathbb{R}^n}\langle [F_{ij},\, J_i]\,,\ J_j\rangle\, G\, dx \\
&\le \int_{\mathbb{R}^n}(2\,|F| - 1)\,|J|^2\, G\, dx,
\end{aligned}$$

where we have used the fact that $|\langle [F_{ij},\, J_i]\,,\ J_j\rangle| \le 2\,|F|\,|J|^2$.

Hence we have

$$\int_{\mathbb{R}^n}|D\, J|^2\, G\, dx \le \int_{\mathbb{R}^n}(2\,|F| - 1)\,|J|^2\, G\, dx.$$

.

Using the assumption that $|F| \leq \frac{1}{2}$, one has

$$DJ \equiv 0.$$

Therefore, $J$ is parallel and $|J|$ is a constant.

Notice that $J_i dx^i = \frac{1}{2} x^p F_{pi} dx^i$, we have $J(x) = j(0) = 0$. Hence we finally obtain that, the curvature $F$ vanishes and the vector bundle is flat. □

# Bibliography


[AB83] M. F. Atiyah, R. Bott, The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A* **308** (1983), no. 1505, 523-615.

[Ba96] Christian Bär, Metrics with harmonic spinors. *Geom. Funct. Anal.* **6** (1996), no. 6, 899-942

[BS92] Christian Bär, Paul Schmutz, Harmonic spinors on Riemann surfaces. *Ann. Global Anal. Geom.* **10** (1992), no. 3, 263-273

[Be93] Fabrice Bethuel, On the singular set of stationary harmonic maps . *Manuscripta Math.* **78** (1993), no. 4, 417-443

[Bu93] Ulrich Bunke, Comparison of Dirac operators on manifolds with boundary . (English summary) Proceedings of the Winter School "Geometry and Physics" (Srn, 1991). *Rend. Circ. Mat. Palermo (2) Suppl.* No. 30 (1993), 133-141

[CJLW05] Qun Chen, Jürgen Jost, Jiayu Li, Guofang Wang, Regularity theorems and energy identities for Dirac-harmonic maps , *Math. Z.* **251** (2005), 61-84.

[CJLW06] Qun Chen, Jürgen Jost, Jiayu Li, Guofang Wang, Dirac-harmonic maps, *Math. Z.* **254** (2006), 409-432.

[CJW07] Qun Chen, Jürgen Jost, Guofang Wang, Liouville theorems for Dirac-harmonic maps. *J. Math. Phys.* **48** (2007), no. 11, 113-517.

[CJW08] Qun Chen, Jürgen Jost, Guofang Wang, Nonlinear Dirac equations on Riemann surfaces. *Ann. Global Anal. Geom.* **33** (2008), no. 3, 253-270.

[CJWZ] Qun Chen, Jürgen Jost, Guofang Wang, Miaomiao Zhu, The boundary value problem for Dirac-harmonic maps *J. Eur. Math. Soc.* **15** (2013), 997-1031.

[CS94] Yunmei Chen, Chun-Li Shen, Monotonicity formula and small action regularity for Yang-Mills flows in higher dimensions. *Calc. Var. Partial Differential Equations 2* (1994), no. 4, 389-403.

[CZ] Zhengxiang Chen, Yongbing Zhang, Stabilities of homothetically shrinking Yang-Mills soliton . In preparation.

[CM12] Tobias H. Colding, William P. Minicozzi II, Generic mean curvature flow I: generic singularities. (English summary) *Ann. of Math.* (2) **175** (2012), no. 2, 755-833.

[De99] Pierre Deligne, Note on quantization. Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), 367-375, Amer. Math. Soc., Providence, RI, 1999.

[Do85] S. K. Donaldson, Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. *Proc. London Math. Soc.* (3) **50** (1985), no. 1, 1-26.

[Do87] S. K. Donaldson, Infinite determinants, stable bundles and curvature. *Duke Math. J.* **54** (1987), no. 1, 231-247.

[Ev91] Lawrence C. Evans, Partial regularity for stationary harmonic maps into spheres. *Arch. Rational Mech. Anal.* **116** (1991), 101-113.

[FMS97] A. Freire, S. Müller, M. Struwe, Weak convergence of wave maps from (1+2)-dimensional Minkowski space to Riemannian manifolds. (English summary) *Invent. Math.* **130** (1997), no. 3, 589-617.

[Fr00] Thomas Friedrich, Dirac operators in Riemannian geometry. Translated from the 1997 German original by Andreas Nestke. Graduate Studies in Mathematics, 25. American Mathematical Society, Providence, RI, 2000.

[GHHP83] G. W. Gibbons, S. W. Hawking, Gary T. Horowitz, Malcolm J. Perry, Positive mass theorems for black holes . *Comm. Math. Phys.* **88** (1983), no. 3, 295-308.

[Gi83] Mariano Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems. in Annals of Mathematics Studies, 105. Princeton University Press, Princeton, NJ, 1983.

[GJ87] Robert Gulliver, Jürgen Jost, Harmonic maps which solve a free-boundary problem . *J. Reine Angew. Math.* **381** (1987), 61-89.

[Gr04] Alfred Gray, Tubes. Second edition. With a preface by Vicente Miquel. Progress in Mathematics, 221. Birkhäuser Verlag, Basel, 2004.

[Gr01] Joseph F. Grotowski, Finite time blow-up for the Yang-Mills heat flow in higher dimensions. *Math. Z.* **237** (2001), no. 2, 321-333.

[Gr81] M. Grüter, Regularity of weak $H$-surfaces. *J. Reine Angew. Math.* **329** (1981), 1-15.

[Ha93] Richard S Hamilton, Monotonicity formulas for parabolic flows on manifolds. *Comm. Anal. Geom.* **1** (1993), no. 1, 127-137.

[He90] Frédéric Hélein, Régularite des applications faiblement harmoniques entre une surface et une sphere. *C. R. Acad. Sci. Paris Ser. I Math.* **311** (1990), 519-524.

[He91] Frédéric Hélein, Régularite des applications faiblement harmoniques entre une surface et une variété riemannienne . (French) [Regularity of weakly harmonic maps between a surface and a Riemannian manifold] *C. R. Acad. Sci. Paris Sér. I Math.* **312** (1991), no. 8, 591-596.

[He02] Frédéric Hélein, Harmonic maps, conservation laws and moving frames, Translated from the 1996 French original. With a foreword by James Eells. Second edition. Cambridge Tracts in Mathematics, 150. Cambridge University Press, Cambridge, 2002.

[HMR02] Oussama Hijazi, Sebastián Montiel, Antonio Roldán, Eigenvalue boundary problems for the Dirac operator , *Comm. Math. Phys.* **231** (2002), no. 3, 375-390.

[Hi74] Nigel Hitchin, Harmonic spinors , *Advances in Math.* **14** (1974), 1-55.

[HT04] Min-Chun Hong, Gang Tian, Global existence of the $m$-equivariant Yang-Mills flow in four dimensional spaces. *Comm. Anal. Geom.* **12** (2004), no. 1-2, 183-211.

[IM01] Tadeusz Iwaniec, Gaven Martin, Geometric function theory and nonlinear analysis, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2001.

[Jo02] Jürgen Jost, Riemannian geometry and geometric analysis . Third edition. Springer-Verlag, Berlin, 2002.

[JMZ09] Jürgen Jost, Xiaohuan Mo, Miaomiao Zhu, Some explicit constructions of Dirac-harmonic maps . *J. Geom. Physics* **59** (2009), no. 11, 1512-1527.

[LM89] H. Blaine Lawson, Marie-Louise Michelsohn Spin Geometry,Princeton University Press, 1989.

[Mo48] C. Morrey, The problem of Plateau on a Riemannian manifold. *Ann. of Math.* **49** (1948), 807-951.

[Na94] Hisashi Naito, Finite time blowing-up for the Yang-Mills gradient flow in higher dimensions. *Hokkaido Math. J.* **23** (1994), no. 3, 451-464.

[Na56] John Nash, The imbedding problem for Riemannian manifolds. *Ann. of Math. (2)* **63** (1956), 20-63.

[Qi93] Jie Qing, Boundary regularity of weakly harmonic maps from surfaces. (English summary) *J. Funct. Anal.* **114** (1993), no. 2, 458-466.

[Ra92] Johan Rade, On the Yang-Mills heat equation in two and three dimensions. *J. Reine Angew. Math.* **431** (1992), 123-163.

[Ri95] Tristan Rivière, Everywhere discontinuous Harmonic Maps into Spheres. *Acta Matematica.* **175** (1995),197-226.

[Ri07] Tristan Rivière, Conservation laws for conformally invariant variational problems . *Invent. Math.* **168** (2007), no. 1, 1-22.

[RS08] Tristan Rivière, Michael Struwe, Partial regularity for harmonic maps, and related problems. *Comm. Pure and Applied Math.* **61** (2008), no. 4, 451-463.

[Sch06] Christoph Scheven, Partial regularity for stationary harmonic maps at a free boundary. *Math. Z.* **253** (2006), no. 1, 135-157.

[Sch84] Richard M. Schoen, Analytic Aspects of the Harmonic Map Problem. Seminar on Nonlinear Partial Differential Equations(S. S. Chern editor), MSRI Publications Volume 2, Springer-Verlag New York, 1984, pp 321-358.

[SST98] Andreas E. Schlatter, Michael Struwe, A. Shadi Tahvildar-Zadeh, Global existence of the equivariant Yang-Mills heat flow in four space dimensions. *Amer. J. Math.* **120** (1998), no. 1, 117-128.

[SY97] R. Schoen, S. T. Yau, Lectures on harmonic maps, Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.

[St94] Michael Struwe, The Yang-Mills flow in four dimensions. *Calc. Var. Partial Differential Equations* 2 (1994), no. 2, 123-150.

[Uh82] Karen K. Uhlenbeck, Connections with Lp bounds on curvature. *Comm. Math. Phys.* **83** (1982), no. 1, 31-42.

[Wa05] Changyou Wang, A compactness theorem of $n$-harmonic maps . *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005), no. 4, 509-519.

[WX09] Changyou Wang, Deliang Xu, Regularity of Dirac-harmonic maps. *Int. Math. Res. Not. IMRN* (2009), no. 20, 3759-3792.

[We04] Ben Weinkove, Singularity formation in the Yang-Mills flow. *Calc. Var. Partial Differ. Equat.* **19** (2004), no. 2, 211-220.

[Xi96] Yuanlong Xin, Geometry of harmonic maps, Progress in Nonlinear Differential Equations and their Applications, **23** , Birkhäuser Boston, Inc., Boston, MA, 1996.

[XC13] Deliang Xu und Zhengxiang Chen, Regularity for Dirac-harmonic map with Ricci type spinor potential *Calc. Var. Partial Differ. Equat.* **46** (2013), no. 3-4, 571-590.

[Zh07] Liang Zhao, Energy identities for Dirac-harmonic maps *Calc. Var. Partial Differential Equations* **28** (2007), no. 1, 121-138.

[Zh09] Miaomiao Zhu, Regularity for weakly Dirac-harmonic maps to hypersurfaces . *Ann. Global Anal. Geom.* **35** (2009), no. 4, 405-412.