The Theoretical Derivation and Exploration of Gravitational Wave

Xiangyu Wen[1,*]

[1]Tianjin Yinghua experimental school, Tianjin ,301799, China

*Corresponding author's e-mail: 1067592985@qq.com

Abstract. This paper mainly records the theoretical system of gravitational waves and the mathematical derivation process. At the same time, it also mentions the working principle of the laser interference gravitational wave observatory. It belongs to the learning of the gravitational wave entry-level, and it can also help you understand and learn gravity waves faster from a professional point of view. The end of the article contains some of my discussions on the future development prospects of gravitational waves.

1. Introduction

After Newton discovered gravity, the prevailing view at the time was that gravity did not take time to propagate [1]. In other words, gravity travels at an infinite speed. Still, after that, Einstein put forward the theory of relativity. This theory tells us that the speed of light is the fastest speed in the universe. It will not change, so the speed of gravity is infinite. This view is clearly contrary to the theory of relativity [2]. Einstein then suggested that gravity travels at the same speed as light. To explain gravitational waves in another way, when the mass of an object changes, the structure of space-time changes. This change in space-time will propagate in the form of waves. So how people observe and measure gravitational waves becomes a problem. In the beginning, the gravitational wave was only the speculation of general relativity, and there was no direct evidence to prove the existence of the gravitational wave. So people have been trying to observe gravitational waves to prove their existence [3-4].

2. Mathematical derivation process

2.1. Linearization of gravitational field

Because the gravitational wave we can observe is far away from the gravitational source, we must first find the gravitational field under the weak field approximation

The Metric under weak field approximation is:

$$g_{\mu v} = \eta_{\mu v} + h_{\mu v} \qquad \left| h_{\mu v} \right| \ll 1 \tag{1}$$

$n_{\mu v}$ is the metric in Minkowski space-time and $h_{\mu v}$ is the symmetric perturbation tensor field

Its inverse metric:

$$\begin{aligned} g^{\mu v} &= \left(\eta_{\mu v} + h_{\mu v}\right)^{-1} \\ &= \left[\eta_{\mu v}\left(1 + \eta^{\mu v} h_{\mu v}\right)\right]^{-1} \\ &= \eta^{\mu v}(1 + \eta^{\mu v} h_{\mu v})^{-1} \\ &= \eta^{\mu v}(1 - h) \\ &= \eta^{\mu v} - h^{\mu v} \end{aligned} \tag{2}$$

For the Coriolis, it can be approximated to the first order:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$
$$= \frac{1}{2}(\eta^{\lambda\rho} - h^{\lambda\rho})[\partial_\mu(\eta_{\rho\nu} + h_{\rho\nu}) + \partial_\nu(\eta_{\mu\rho} + h_{\mu\rho}) - \partial_\rho(\eta_{\mu\nu} + h_{\mu\nu})] \quad (3)$$
$$\approx \eta^{\lambda\rho}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu})$$

Riemannian tensor[5]:

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda}(\partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \partial_\sigma \Gamma^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\rho\delta}\Gamma^{\delta}_{\sigma\nu} - \Gamma^{\lambda}_{\sigma\delta}\Gamma^{\delta}_{\rho\nu})$$
$$\approx \eta_{\mu\lambda}(\partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \partial_\sigma \Gamma^{\lambda}_{\nu\rho}) \quad (4)$$
$$= \frac{1}{2}(\partial_\rho\partial_\nu h_{\sigma\mu} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\rho\mu})$$

Our definition of Ricci curvature is:

$$R_{\nu\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma} \quad (5)$$

And the Scalar curvature is define as:

$$R = g^{\nu\sigma} R_{\nu\sigma} \quad (6)$$

The Einstein tensor is approximate to :

$$G_{\mu\nu} \approx \frac{1}{2}(\partial_\rho\partial_\nu h^{\rho}_{\sigma} + \partial_\sigma\partial_\mu h^{\mu}_{\nu} - \partial^\mu\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + \eta_{\mu\nu}\partial^\alpha\partial_\alpha h) \quad (7)$$

Bring it into Einstein's field equation($G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$), where $T_{\mu\nu}$= 0 can get the vacuum, weak field and linear gravitational field equation:

$$\partial_\rho\partial_\nu h^{\rho}_{\sigma} + \partial_\sigma\partial_\mu h^{\mu}_{\nu} - \partial^\mu\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + \eta_{\mu\nu}\partial^\alpha\partial_\alpha h = 0 \quad (8)$$

Define the opposite perturbation field:

$$\hbar_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \quad (9)$$
$$\hbar = h - 2h = -h \quad (10)$$

Using Fock coordinate conditions ($\partial_\alpha h^{\mu\alpha} - \frac{1}{2}\partial^\mu h = 0$) can be recorded as:

$$\partial_\mu \hbar^{\mu}_{\alpha}=0 \quad (11)$$

And bring it into the vacuum Einstein field equation:

$$\partial^\mu\partial_\mu h_{\nu\sigma} - \frac{1}{2}\eta_{\mu\sigma}\partial^\alpha\partial_\alpha h = \partial_\alpha\partial^\alpha \hbar_{\mu\nu} = 0 \quad (12)$$

Then we can get the wave solution that far away from the gravitational source:

$$\hbar_{\mu\nu} = C_{\mu\nu} e^{ik_\alpha x^\alpha} \qquad C_{\mu\nu} \text{ is a constant tensor.} \quad (13)$$

2.2. Polarization mode of gravitational wave under gauge change

In the second part, we will use the coordinate conditions and gauge conditions to act on the constant tensor $C_{\mu\nu}$, and then obtain the gravitational wave solutions with two polarization modes.

Using the Fock coordinate conditions in the first part, we have obtained that $\partial_\mu \hbar^{\mu}_{\alpha} = 0$ put the plane wave solution into the Fock coordinate conditions to get[5]:

$$\partial_\mu \hbar^{\mu\nu} = iC^{\mu\nu} k_\mu e^{ik_\alpha x^\alpha} = 0 \quad (14)$$

So:

$$k_\mu C^{\mu\nu} = 0 \quad (15)$$

Where μ is the summation index, which means that this constraint has four independent components brought by ν. Now consider the infinitesimal coordinate transformation:

$$x'^\mu = x^\mu + \xi^\mu(x) \quad (16)$$
$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^{\mu}_{\nu} + \partial_\nu \xi^\mu \quad (17)$$

The coordinate transformation of metric is[6]:

$$g'_{\mu\nu} = \frac{\partial x^\alpha \partial x^\beta}{\partial x'^\mu \partial x'^\nu} g_{\alpha\beta}$$
$$= (\delta^{\alpha}_{\mu} - \partial_\mu \xi^\alpha)(\delta^{\beta}_{\nu} - \partial_\nu \xi^\beta) g_{\alpha\beta} \quad (18)$$
$$= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

Under the coordinate transformation, the weak field approximation should still be considered:

$$g'_{\mu\nu}(x') = \eta_{\mu\nu} + h'_{\mu\nu}(x') \tag{19}$$

By comparing the above two equations, we can get the coordinate transformation satisfied by the disturbance field into:

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{20}$$

For the coordinate transformation of the disturbance field with opposite trace:

$$\begin{aligned} \hbar'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h' \\ &= h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu - \tfrac{1}{2}\eta_{\mu\nu}(h - 2\partial_\alpha \xi^\alpha) \\ &= \hbar_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial_\alpha \xi^\alpha \end{aligned} \tag{21}$$

Then use Fock coordinate condition to find the partial derivative on both sides of the above formula:

$$\partial^\mu \hbar'_{\mu\nu} = \partial^\mu \hbar_{\mu\nu} = 0 \tag{22}$$

So:

$$\xi^\mu = B^\mu e^{ik_\alpha x^\alpha} \tag{23}$$

Substitution coordinate transformation of disturbance field with opposite trace($\hbar'_{\mu\nu} = \hbar_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial_\alpha \xi^\alpha$):

$$C'_{\mu\nu} e^{ik_\alpha x^\alpha} = C_{\mu\nu} e^{ik_\alpha x^\alpha} - ik_\mu B_\nu e^{ik_\alpha x^\alpha} - ik_\nu B_\mu e^{ik_\alpha x^\alpha} + i\eta_{\mu\nu} k_\rho B^\rho e^{ik_\alpha x^\alpha} \tag{24}$$

$$C'_{\mu\nu} = C_{\mu\nu} - ik_\mu B_\nu - ik_\nu B_\mu + i\eta_{\mu\nu} k_\rho B^\rho \tag{25}$$

The following two criteria are chosen to get two gauge freedom degrees

(1)

$$C'^\mu_\mu = 0 \tag{26}$$

$$C'^\mu_\mu = C^\mu_\mu - ik_\mu B^\mu - ik_\nu B^\nu + i\delta^\mu_\mu k_\rho B^\rho = C^\mu_\mu + 2ik_\mu B^\mu = 0 \tag{27}$$

$$k_\mu B^\mu = \tfrac{1}{2} i C^\mu_\mu \tag{28}$$

(2)

$$C'_{0\mu} = 0 \tag{29}$$

When:$\mu = 0$:

$$\begin{aligned} C'_{00} &= C_{00} - ik_0 B_0 - ik_0 B_0 + i\eta_{00} k_\rho B^\rho \\ &= C_{00} - 2ik_0 B_0 - ik_\rho B^\rho \\ &= C_{00} - 2ik_0 B_0 + \frac{1}{2} C^\mu_\mu = 0 \end{aligned} \tag{30}$$

$$B_0 = \tfrac{1}{2ik_0}(C_{00} + \tfrac{1}{2} C^\mu_\mu) \tag{31}$$

When $\mu = w$(w=1,2,3……),same as above:

$$B_w = \tfrac{i}{2k_0^2}[-2k_0 C_{oj} + k_j(C_{00} + \tfrac{1}{2} C^\mu_\mu)] \tag{32}$$

Combining all the above gauge conditions, the $k_\mu C^{\mu\nu} = 0$ obtained by Fock coordinate condition and the $C^\mu_\mu = 0$, $C_{0\mu} = 0$ obtained by gauge condition (1) (2), we call these conditions the transverse traceless gauge condition of gravitational wave, that is (transverse traceless gauge). When we got the coefficient tensor before, we found that $C_{\mu\nu}$ has ten independent components. Now using TT gauge, we can reduce $9 - 1 = 8$ degrees of freedom, that is,$C_{\mu\nu}$ has only two independent components.

The disturbance field under TT specification is:

$$h^{TT} = 0 \tag{33}$$

$$\hbar^{TT}_{\mu\nu} = h^{TT}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h^{TT} = h^{TT}_{\mu\nu} \tag{34}$$

We consider the propagation of gravitational waves in a particular direction, that is the propagation in the Z direction $k^\mu = (\omega, 0,0, k^3) = (\omega, 0,0, \omega)$, then we use TT gauge:

$$k^\mu C_{\mu\nu} = \omega C_{0\nu} + \omega C_{3\nu} = 0 \tag{35}$$

So:

$$C_{3\nu} = 0 \tag{36}$$

The coefficient tensor is represented by a matrix[6]:

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{37}$$

Let's make $h_+ = C_{11}$,$h_* = C_{12}$, the coefficient matrix is substituted into the plane wave solution:

$$h_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_* & 0 \\ 0 & h_* & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\alpha x^\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_* & 0 \\ 0 & h_* & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-ct)/c} \tag{38}$$

Among them, the two polarization states (+ state and * state) are:

$$\epsilon_{\mu\nu}^{+} = h_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \epsilon_{\mu\nu}^{*} = h_* \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{39}$$

The two polarization states of gravitational waves can be represented by the following figure 1:

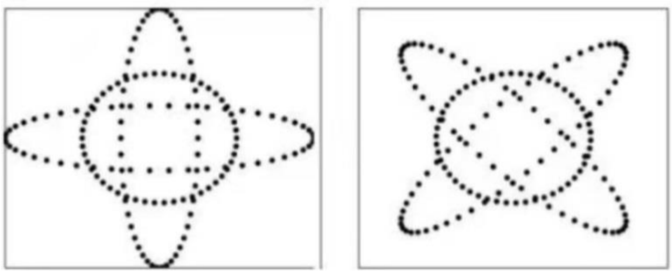

Figure 1. The two polarization states of gravitational waves can be represented in the figure above, the left one is + states, right one is * states [7]

3. Observation of gravitational waves

Generally speaking, due to conservation of energy when an object emits electromagnetic waves, it causes its own energy to drop, so physicists suspect. In the process of the propagation of gravity, does it also produce radiation outward and reduce its own energy. For example, in a binary star structure, the process of transmitting gravitational waves outward, their energy will be reduced, and their gravitational radius will continue to shrink. In 1974 The Arecibo radio telescope observed a pair of pulsating binary stars (P1913+16)and provides evidence for the idea of gravitational waves. In 1974 The Arecibo radio telescope received a weak pulse signal from the universe It means we found a pulsar, generally speaking, the frequency of the pulsar is extremely stable, but according to the data the period of this pulsar is deviated by 58 / 1 billion seconds so the scientists speculated that there is a companion star, which affects the propagation of electromagnetic waves so according to the relativistic calculation of gravitational waves The scientists calculated that, apart from due to gravitational radiation, it will cause their energy to decrease, the pulsing binary pair's revolution period would be reduced by 75 parts per million second per year. Eventually, through years of observation Its revolution period will be reduced by 76±2 millionths of a second a year, this result is in good agreement with the predictions made by general relativity.

4. LIGO and the principle of observing gravitational waves

Since the transmission speed of gravitational waves is not infinite So at two points in different locations, the time we receive gravitational waves is different. This is the most basic principle for us to observe gravitational waves. But the reality is the displacement caused by gravitational waves may be only the size of a thousandth of a proton, this change is very difficult to observe. Here is a brief introduction to the laser interference gravitational wave observatory. In fact, its principle of observation is very simple. First, we emit two lasers at a vertical angle, and a reflecting mirror is placed in front of the two laser beams. When the light comes back along the mirror, interference fringes are formed, since there is a positional difference between the two mirrors, gravitational waves will then pass through two mirrors and will have an impact on both mirrors, the two mirrors are not affected at the same time. So, we can observe gravitational waves by observing changes in interference fringes.

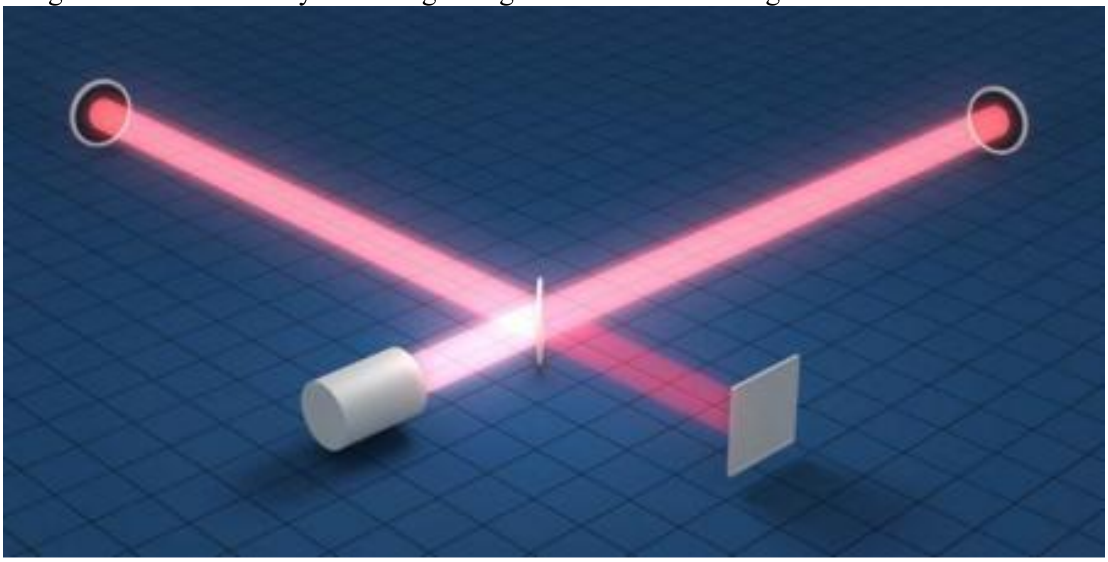

Figure 2. This is the theory graph of the laser interference gravitational wave observatory [8]

But will not rule out a special case that gravitational waves travel in the direction of an angle of 45 degrees and affect both mirrors at the same time, but I think, the probability of producing such a perfect angle is very small. There are a lot of errors in real life. Observing this very small change is not a very easy thing for example, we want to make sure that the deflection of the mirror does not come from the vibration of the earth or by human-made so the United States built two laser interference gravitational wave observatories in two places that very far away from each other. Thereby reducing the error.

So after years of waiting, LIGO observed gravitational waves from the merge of two black holes(GW150914) more than a billion light-years away. And get such a set of data at 2015.

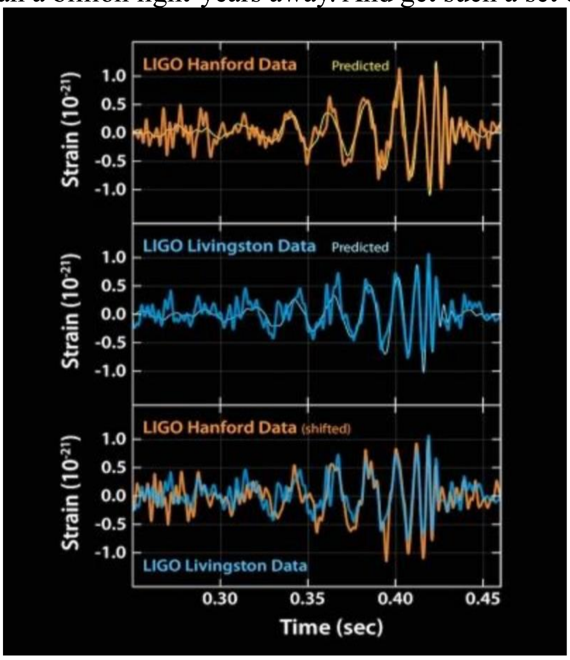

Figure 3. The above picture shows the data of the collision of two black holes observed by the Laser Interference Gravitational Wave Observatory [9]

5. Gravitational radiation

In the previous article, we have mentioned the theory of gravitational radiation, because we also confirmed the existence of gravitational waves by observing the kinetic energy loss of the binary system. In the almost vacuum of the universe, the conversion of energy is mainly kinetic energy and gravitational potential energy. When a star moves, it produces gravitational radiation then the kinetic energy of the stars will be lost, this is called gravitational damping. Of course, the mass of a star can also be converted into gravitational radiation. When some intense astral changes occur, such as the fusion of black holes and the explosion of supernovae, all of these conditions produce gravitational radiation. This energy conversion process is very complicated, but the source of energy for gravitational radiation is the mass of a star. Having said that, l have to mention this equation: $\mathrm{E} = \mathrm{mc}^2$. For ease of expression, we often say that quality is lost. In fact, there is no "loss" in quality. The rigorous natural language expression for this equation is that mass-energy equivalence. That is to say, mass and energy are one thing. This concept is very basic, but for those who don't do physical research, they may feel counterintuitive and prefer the notion of loss of quality. In fact, the change of gravitational potential energy will also cause the change of system mass. For example, on the earth, when you parachute from high altitude, you will consume gravitational potential energy and convert it into kinetic energy during the process of falling. This part of kinetic energy will generate heat energy with air friction. This heat energy will most likely radiate to a part of the space. The energy released into space also takes away the mass of the system. In other words, the mass of the earth's system decreases during this process. At the same time, when you parachute, your fall also radiates gravitational waves, radiating into space and taking away the system's mass. In fact, the departure of energy is also the departure of mass. This is because gravitational waves, electromagnetic waves, have no static mass but have mass [10].

But in practice, there is also a difference between the black hole and an ordinary star. Black holes differ from ordinary stars in that they are dense and have strong gravitational effects. If it is not a black hole, then neutron stars are dense and have strong gravitational effects. For example, PSR1913+16 is a binary neutron star system. In the course of surround, gravitational waves are radiated, and kinetic energy is consumed. In fact, the two stars radiate gravitational waves all the time, and getting closer and closer slowly. And like the two black holes spinning around, they will eventually merge together. Intense merging will also radiate a large number of gravitational and electromagnetic waves, and will cause a "loss" of quality after merging. However, unlike neutron stars, black holes do not radiate electromagnetic waves, only gravitational waves.

6. Conclusion

There are still many problems to be solved about gravitational waves. Still, the exploration and observation of gravitational waves can help us better understand the universe, and we may also find the source of the big bang through gravitational waves. Not only that, I would like to quote a sentence once said by Ferrari: Although this kind of thing is of little use at present, no one knows what will happen in a few centuries.

References

[1] Hustad, D. . (2016). Isaac newton discovers gravity.
[2] Pauli, W. , Field, G. , & Lenzen, V. F. . (1958). Theory of relativity. PERGAMON PR.
[3] None. (1947). Introduction to the theory of relativity. The Australasian Journal of Optometry, 1(11), 15-16.
[4] Rosser, & Gv, W. . (1900). An introduction to the theory of relativity. *s.n.* DOI: 10.1049/ep.1964.0362
[5] Ohanian, H., & Ruffini, R. (2013). Gravitation and Spacetime (3rd ed.). Cambridge: Cambridge University Press. doi:10.1017/CBO9781139003391
[6] Carroll, S. (2019). Spacetime and Geometry: An Introduction to General Relativity. Cambridge: Cambridge University Press. doi:10.1017/9781108770385
[7] FangYu Li, Hao Wen (2019). Deep analysis of several typical gravity detection devices

http://m.xueshutang.com/dzwuli/141950.html
[8] TengXunKeJi (2016) Humans have once again detected gravitational waves from the merger of double black holes https://www.nxpic.org.cn/article/id-328203 Photo copyright：B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration)
[9] LIGO Collaboration (2015) https://gracedb.ligo.org/superevents/public/O3/
[10] Romero, G. E. , & Vila, G. S. . (2013). Introduction to black hole astrophysics. *Lecture Notes in Physics*, 876. DOI: 10.1007/978-3-642-39596-3