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A Dual Formulation of Non-Abelian Gauge Theories

Yoichi Kazama†
Fermi National Accelerator Laboratory
Batavia, Illinois 60510

and

Robert Savit*
Physics Department
University of Michigan
Ann Arbor, Michigan 48104

ABSTRACT

Motivated by an analogy with Abelian theories, we construct a new duality transformation for non-Abelian gauge theories. For large coupling the gauge theory vacuum should have its most important contributions from confugurations of the dual fields without large fluctuations. The generating functional when expressed in terms of the dual variables has a rather simple and suggestive structure which manifests some features of a gauge theory, but now a gauge theory of the dual variables with the coupling constant inverted. We discuss several aspects of this representation including the possibility of using it as the basis for a strong coupling expansion for the field theory. We also investigate a systematic, formal (perturbative) solution to a constraint condition among our dual variables which has the form of a Bianchi identity.

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I. Introduction

Duality transformations have proven to be extremely useful for theories with an Abelian symmetry.[1] These transformations are generalizations of the work of Kramers and Wannier on the two dimensional Ising model.[2] Among their virtues is the property that the strong coupling region of a theory is mapped into the weak coupling region of another (dual) theory, and vice versa. (In statistical mechanics, a statistical system at high temperatures is mapped into a dual statistical system at low temperatures.) Applying a duality transformation to a strong coupling theory one may thereby achieve two ends: First, since the dual theory is a weak coupling theory, the picture of the vacuum (or ground state) should be much simpler in terms of the dual variables. For small values of the dual coupling constant the fluctuations of the dual variables should not be too large. Second, one may achieve a calculational advantage since it may be possible to do perturbation theory in the dual coupling constant. In addition, it is sometimes possible to derive a third benefit from using duality transformations, in that the rôle of the topological excitations of the theory may become transparent.

Unfortunately, the generalization of duality transformations to theories with a general non-Abelian symmetry is not so straightforward. A number of different approaches to this problem have been tried,[3] yielding varying amounts of insight, but so far none has provided a form which is as elegant as that obtained for

Abelian theories. This is particularly frustrating in view of the fact that continuum QCD cries out for a tractable strong coupling calculational scheme to verify (or disprove) confinement and compute the hadronic spectrum. In this paper we shall describe another attempt to construct a dual form of the non-Abelian gauge theory. Our approach leads us to a relatively simple and suggestive dual representation of the theory with a number of intriguing features. Unfortunately, as with other investigations of non-Abelian duality, we have not yet been able to establish a well-defined strong coupling expansion using our representation. Nevertheless, we feel our representation is sufficiently attractive to merit further study and discussion.

To motivate our approach, it is convenient to first review the derivation of the dual form for an Abelian theory. For our purposes the most appropriate theory to consider is a four dimensional Abelian gauge theory. There are two important ingredients in the Abelian duality transformation which we wish to use in our recipe for non-Abelian theories. To illustrate these we first describe the duality transformation for the trivial theory of free photons in the continuum in four Euclidean dimensions. Following this exercise, we briefly review how the same approach generates a simple dual form for non-trivial Abelian theories, using as our example compact QED on a four dimensional lattice. To finish setting the stage and motivating out work, we then briefly describe the problems one encounters if one tries to construct a non-Abelian duality transformation by the most straightforward generalization of the Abelian case.[4]

The generating functional for free photons is

$$Z = \int DA_\mu \; e^{-\frac{k}{2}(F_{\mu\nu},F_{\mu\nu})} \tag{1.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and

$$(F_{\mu\nu},F_{\mu\nu}) \equiv \int d^4x \; F_{\mu\nu}(x)F_{\mu\nu}(x)$$

The constant k has no physical significance in (1.1), but is the analogue of the inverse of a coupling constant in interacting theories and is useful for seeing the effect of the transformation. We now introduce a field, $w_{\mu\nu}$ which is Fourier conjugate to $F_{\mu\nu}$, and write (up to overall constants)

$$Z = \int DA_\mu \; Dw_{\mu\nu} \; e^{-\frac{1}{2k}(w_{\mu\nu},w_{\mu\nu})+i(\tilde{w}_{\mu\nu},F_{\mu\nu})} \tag{1.2}$$

where $\tilde{w}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} w_{\rho\sigma}$. The last term can be integrated by parts. Ignoring the surface term (or, choosing boundary conditions so that it is zero) we have

$$Z = \int Da_\mu D\, w_{\tilde{\mu}\nu} \; e^{-\frac{1}{2k}(w_{\mu\nu},w_{\mu\nu})-i(A_\mu,\partial_\nu\tilde{w}_{\mu\nu})} \; . \tag{1.3}$$

Integrating over A_μ, (1.3) becomes

$$Z = \int Dw_{\mu\nu} e^{-\frac{1}{2k}(w_{\mu\nu}, w_{\mu\nu})} \prod_x \delta(\partial_\nu \tilde{w}_{\mu\nu}(x)) \tag{1.4}$$

The delta functions in (1.4) will be satisfied if and only if we write

$$\tilde{w}_{\mu\nu} = \epsilon_{\mu\nu\lambda\sigma} \, \partial_\lambda B_\sigma \tag{1.5}$$

Using (1.5) in (1.4) yields (up to overall constants)

$$Z = \int DB_\mu \; e^{-\frac{1}{2k}(\partial_\lambda B_\sigma - \partial_\sigma B_\lambda, \; \partial_\lambda B_\sigma - \partial_\sigma B_\lambda)} \tag{1.6}$$

which is the same as (1.1), but with $k \to 1/k$.

There are several points to note about this result. First, while it is perfectly true that (1.1) and (1.6) are trivial the process by which we went from (1.1) to (1.6) is not. In particular, there are two important steps. The first is the Fourier transform (1.2) which has the effect of inverting the coupling constant (or what would be the coupling constant were the theory not trivial). Second is the appearance in (1.4) of a Bianchi-identity-like delta function which is generated by an integration over the original degrees of freedom of the system, in this case the A_μ's. We shall incorporate these two elements in our treatment of non-Abelian theories. Notice also that the integration over A_μ was performed without fixing a gauge.

The infinities that result are contained in the over complete set of delta functions. Nevertheless, it is important to recognize that one can formally carry out the A_μ integrations in (1.3) without choosing a gauge. Finally, we remark that the fact that (1.6) is a gauge theory is peculiar to four dimensions, and is because the dual of a two-form (i.e. $F_{\mu\nu}$) in four dimensions is also a two-form. Had we carried out the duality transformation in three dimensions, the delta functions in (1.4) would have been satisfied by $\tilde{w}_{\mu\nu} = \epsilon_{\mu\nu\lambda}\partial_\lambda\phi$, and (1.6) would have become a theory of a free massless scalar field.

We next briefly describe the effect of the duality transformation on a non-trivial theory, the U(1) lattice gauge theory. A discussion of the U(1) lattice theory will give us the background to see where the most straightforward generalization of duality to a non-Abelian theory ceases to be simple. We recall that the generating functional for the U(1) lattice gauge theory is

$$Z = \int_{-\pi}^{\pi} D\theta_\mu \, e^L = \int_{-\pi}^{\pi} \prod_{j,\mu} d\theta_\mu(j) \prod_p e^{\beta\cos[f_{\mu\nu}(j)]} \qquad (1.7)$$

where

$$L = \beta\sum_p \cos[\Delta_\mu\theta_\nu(j) - \Delta_\nu\theta_\mu(j)] \qquad (1.8)$$

and $f_{\mu\nu}(j)$ is the argument of the cosine in (1.8). Associated with each link of a four-dimensional hypercubic lattice is a phase, $e^{i\theta_\mu(j)}$, where j is a vector which labels the lattice site (for simplicity we drop the vector notation on j) and μ is a direction index. Δ_ν is a discrete difference operator: i.e. $\Delta_\nu\theta_\mu(j) \equiv \theta_\mu(j)-\theta_\mu(j-\hat{\nu})$, and the sum (product) over p in (1.8) ((1.7)) is a sum (product) over all plaquettes of the lattice.

Following our discussion of the free photon case, we Fourier expand each factor in the argument of (1.7) writing

$$e^{\beta\cos[f_{\mu\nu}(j)]} = \sum_{n_{\mu\nu}} I_{n_{\mu\nu}(j)}(\beta)\, e^{i\tilde{n}_{\mu\nu}(j)f_{\mu\nu}(j)}$$
$$\simeq e^{\beta} \sum_{n} e^{-\frac{1}{2\beta}n_{\mu\nu}^2+i\tilde{n}_{\mu\nu}f_{\mu\nu}} \qquad (1.9)$$

where for simplicity we have introduced a large β approximation for the modified Bessel function, $I_n(\beta)$. Using (1.9) in (1.7) and ignoring some overall factors, we find after a little algebra

$$Z = \int_{-\pi}^{\pi} D\theta_\mu \sum_{\{n\}} e^{\sum -\frac{1}{2\beta} n_{\mu\nu}^2 - i\theta_\mu\Delta_\mu\tilde{n}_{\mu\nu}} \qquad (1.10a)$$

$$= \sum_{\{n\}} e^{\sum -\frac{1}{2\beta} n_{\mu\nu}^2} \prod_{j,\mu} \delta(\Delta_\mu\tilde{n}_{\mu\nu}) \qquad (1.10b)$$

The gaggle of Kronecker δ-functions in (1.10b) are generated by integrating over the original gauge fields, θ_μ, and the sum in the exponent of (1.10b) is over all plaquettes of the lattice (more properly, over all plaquettes of the dual lattice). (1.10b) is clearly of the same form as (1.4). To complete the transformation for the present case we note that the delta functions will be satisfied if and only if we write

$$\tilde{n}_{\mu\nu}(j) = \varepsilon_{\mu\nu\lambda\sigma}\, \Delta_\lambda\, \phi_\sigma(j) \qquad (1.11)$$

where the ϕ_μ's are integer-valued fields associated with the links of the (dual) lattice. (1.10) then becomes

$$Z = \sum_{\{\phi\}} e^{\sum_P -\frac{1}{2\beta} (\Delta_\mu\phi_\nu - \Delta_\nu\phi_\mu)^2} \qquad (1.12)$$

where we have ignored the harmless overall infinities associated with not choosing a gauge.

In the dual transformation of this non-trivial Abelian theory we again see the two important ingredients emphasized earlier: First, as a result of the Fourier transform (or character expansion) the coupling constant, β^{-1}, is inverted in the dual theory and second, as a result of integrating over the original field variables which now appear linearly in the exponent (see e.g. (1.10a)) we produce a set of Bianchi-identity-like delta functions which force us to a certain representation of the new field variables, $n_{\mu\nu}$.

In the next section we will present our non-Abelian duality transformation based on these ingredients, but first we want to address the following question which may have arisen in the reader's mind: The starting point for the Abelian duality transformation is the character expansion for the interactions (e.g. (1.9)). Such an expansion is also possible for non-Abelian interactions. What happens if one slavishly imitates the procedure described here for Abelian theories? For example, one can expand the interactions of an O(3) symmetric theory in spherical harmonics, use the addition formula to factor out the dependence on the fields at different lattice points, and integrate over these fields as we did in, say, (1.10). One is then left with a theory in which the indices of the spherical harmonics, ℓ and m appear as fields which must be summed over. In analogy with (1.10b) there is a "Lagrangian" which depends on $\{\ell,m\}$ and a set of constraints which are produced by the integration over the original field variables. Unfortunately, these constraints are not just simple delta functions which can be completely and identically satisfied by a clever representation for the ℓ's and m's. Thus the non-Abelian dual theory generated in this way is much more complex than in the Abelian case, although in certain limits it may be possible to construct a tractable approximation using this representation.[4] Noting the failure of this most obvious generalization of Abelian duality, we turn now to describe another approach.

II. The Duality Transformation

As was mentioned in the introduction, the basic ingredients for the success of the duality transformation for Abelian gauge theories are the Fourier transformation and the Bianchi constraint which emerges upon integration over the original vector potential. In this section we shall extend this principle and try to formulate duality transformations for non-Abelian gauge theories. Specifically we shall consider a non-Abelian gauge theory (with a simple structural Lie group, G) coupled to an external source in Euclidean four dimensional space, E^4. Although the manipulations will be somewhat formal, at the end an intriguing form of a dual theory emerges. Various aspects of our dual formulations will be elaborated in Section III.

To avoid proliferation of indices, we will use the following inner product notations throughout, except when a more explicit notation is helpful:

For tensors $F^i_{\mu\nu}$ and $G^i_{\mu\nu}$, (i=group index)

$$(F,G) \equiv \int d^4x \; F^i_{\mu\nu} \; G^i_{\mu\nu}$$

and for vectors A^i_μ and B^i_μ,

$$(A,B) \equiv \int d^4x \; A^i_\mu \; B^i_\mu, \text{ etc.,}$$

where summation over the repeated indices is understood.

Our starting point is the partition function, Z, of the theory given, in the notation above, by

$$Z = \int DA \exp\{-\tfrac{1}{4}(F(A;g),F(A;g)) - ig(\xi,A)\} \qquad (2.1)$$

where

$$F^i_{\mu\nu}(A;g) = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g c_{ijk} A^j_\mu A^k_\nu \qquad (2.2)$$

and ξ^i_μ is a fixed, external source while c_{ijk} are the structure constants of the group, G. Gauge fixing (e.g. axial gauges) may be done without difficulty, but since, as we shall see, it does not play any essential role in the transformation, we shall formally proceed without explicitly fixing a gauge. As is usual for duality transformations[1] the infinities generated by such a procedure will appear in our dual form as a set of redundant delta-functions.

It is convenient to scale A^i_μ and $F^i_{\mu\nu}$ defining

$$a = gA,\ f(a) = gF(A;g) \qquad (2.3)$$

Then Z becomes

$$Z = \int Da \exp\{-\frac{1}{4g^2}(f(a),f(a)) - i(\xi,a)\} \qquad (2.4)$$

We now introduce an antisymmetric tensor field $w^i_{\mu\nu}(x)$ and Fourier transform (2.4) into the so called first order form, viz.,

$$Z = \int DaDw \exp\{-\frac{g^2}{4}(w,w) - \tfrac{i}{2}(\tilde{w},f(a)) - i(\xi,a)\} \ (2.5)$$

where

$$\tilde{w}^i_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta} w^i_{\alpha\beta} \qquad (2.6)$$

is the dual of $w^i_{\mu\nu}$. Notice that by this transformation, the coupling constant g is effectively inverted.

Let us examine the second term in the exponent of (2.5) more closely. Explicitly,

$$(\tilde{w},f(a)) = \int d^4x\ \tilde{w}^i_{\mu\nu}\ (\partial_\mu a^i_\nu - \partial_\nu a^i_\mu + c_{ijk} a^j_\mu a^k_\nu)$$
$$= \int d^4x\ \tilde{w}^i_{\mu\nu}(2\partial_\mu a^i_\nu + c_{ijk} a^j_\mu a^k_\nu) \qquad (2.7)$$

It is convenient to introduce a symmetric, in general non-degenerate, matrix $T^{jk}_{\mu\nu}(\tilde{w})$ defined by

$$T^{jk}_{\mu\nu}(\tilde{w}) \equiv \tilde{w}^i_{\mu\nu}\ c_{ijk} \qquad (2.8)$$

This is to be regarded as a matrix in the pairs of indices (j,μ) and (k,ν). For G=SU(2), for example, it is a 12x12 matrix. Then by integration by parts, (2.7) becomes

$$(\tilde{w},f(a)) = 2\int d^4x\partial_\mu(\tilde{w}^i_{\mu\nu}a^i_\nu)$$
$$- 2(\partial\tilde{w},a)+(a,Ta) \qquad (2.9)$$

where $\partial\tilde{w}$ stands for $\partial_\mu\tilde{w}^i_{\mu\nu}$. The first term is a surface term. By Stokes' theorem,

$$\int d^4x\ \partial_\mu(\tilde{w}^i_{\mu\nu}a^i_\nu) = \int_\sigma ds^\mu\tilde{w}^i_{\mu\nu}a^i_\nu \qquad (2.10)$$

where σ is a closed surface at infinity. This may be further rewritten by introducing a surface current, $\xi_{[\sigma]}$, defined by

$$\xi_{[\sigma]}{}^i_\nu \equiv \hat{s}_\mu\ \tilde{w}^i_{\mu\nu}\ \delta_{[\sigma]} \qquad (2.11)$$

where $\delta_{[\sigma]}$ is a delta-function with its support on the surface σ, and $\hat{s}_\mu$ is the unit vector outwardly normal to σ. Then (2.10) takes the form

$$\int d^4x\partial_\mu(\tilde{w}^i_{\mu\nu}a^i_\nu) = \int d^4x\ \xi^i_{[\sigma]\mu}a^i_\mu = (\xi_{[\sigma]},a). \qquad (2.12)$$

Thus, the surface term in (2.9) has the same form as the external current term in (2.4). (2.9) now becomes

$$(\tilde{w}, f(a)) = (a, Ta) - 2(\partial\tilde{w}, a) + 2(\xi_{[\sigma]}, a) \qquad (2.13)$$

This form will be useful later. Using (2.13) in (2.5), we obtain

$$Z = \int DaDw \exp\{-\frac{g^2}{4}(w,w) - \frac{i}{2}(a,Ta)$$
$$-i(\xi+\xi_{[\sigma]}, a) + i(\partial\tilde{w}, a)\} \qquad (2.14)$$

At this point one may perform the Gaussian integration over the vector potential, a^i_μ, and obtain the so called field strength form[5] in which the theory is written entirely in terms of $w^i_{\mu\nu}$. But we shall resist the temptation to do the Gaussian integral, and instead follow a different route in our search for a dual theory. We recall from our study of Abelian theories that the second important feature of duality transformations is the existence of a Bianchi-identity-like delta function constraint through which one makes a transition into the space of dual variables. To make the scheme clear, we shall achieve this in two steps. First, we perform a second Fourier transform on the quadratic part (in a) of the integrand in (2.14) by introducing a current $j^i_\mu(x)$. Viz.,

$$\exp(-\frac{i}{2}(a,Ta)) = \int Dj\ (\det T)^{-1/2} \exp\{\frac{i}{2}(j,T^{-1}j) - i(j,a)\} \qquad (2.15)$$

where $(\det T)^{-1/2}$ is necessary to cancel the functional determinant associated with the Gaussian integration. (2.14) then becomes

$$Z = \int DaDwDj\,(\det T)^{-1/2} \exp\{-\frac{g^2}{4}(w,w)+\frac{i}{2}(j,T^{-1}j) + i(\partial\tilde{w}-j-\xi-\xi_{[\sigma]},a)\} \tag{2.16}$$

for which the integration over the a^i_μ is now trivial to perform. Making the shift of the variable $j \to j+\xi+\xi_{[\sigma]}$, the result is

$$Z = \int DwDj\,(\det T)^{-1/2}\delta(\partial\tilde{w}-j) \times \exp\{-\frac{g^2}{4}(w,w) + \frac{i}{2}(j+\xi+\xi_{[\sigma]},T^{-1}(j+\xi+\xi_{[\sigma]}))\} \tag{2.18}$$

As in the Abelian case, the delta function that appears in (2.18) comes precisely from the integration over the original degrees of freedom, i.e. the A^i_μ. To see that it is a kind of Bianchi constraint, we make a change of variables from j^i_μ to a dual vector potential b^i_μ defined by

$$\begin{aligned} j^i_\nu &\equiv c_{ijk}\tilde{w}^j_{\mu\nu}\, b^k_\mu \\ &= T^{ik}_{\nu\mu} b^k_\mu \end{aligned} \tag{2.19}$$

Then (2.18) becomes

$$Z = \int DwDb\,\sqrt{\det T}\,\delta(\partial\tilde{w}-Tb) \times \exp\{-\frac{g^2}{4}(w,w)+\frac{i}{2}(b,Tb)-i(b,\xi_{[\sigma]}) -i(b,\xi)+\frac{i}{2}(\xi+\xi_{[\sigma]},T^{-1}(\xi+\xi_{[\sigma]}))\} \tag{2.20}$$

Now with the use of this delta function, the second and the third term in the exponent of (2.20) may be written as

$$\begin{aligned} \frac{i}{2}(b,Tb)-i(b,\xi_{[\sigma]}) &= -\frac{i}{2}(b,Tb)+i(b,Tb)-i(b,\xi_{[\sigma]}) \\ &= -\frac{i}{2}\{(b,Tb)-2(\partial\tilde{w},b)+2(\xi_{[\sigma]},b)\} \end{aligned} \tag{2.21}$$

Eq. (2.13), however, tells us that this is nothing but

$$-\frac{i}{2}\,(\tilde{w},\ f(b))$$

Thus Eq. (2.20) simplifies to

$$\begin{aligned} Z = \int & Dw Db\ \sqrt{\det T}\ \delta(\partial\tilde{w}-Tb) \\ & \times \exp\{-\frac{g^2}{4}(w,w)-\frac{i}{2}(\tilde{w},f(b)) \\ & - i(b,\xi)+\frac{i}{2}\ (\xi+\xi_{[\sigma]},T^{-1}(\xi+\xi_{[\sigma]}))\} \end{aligned} \tag{2.22}$$

To see the effect of the duality transformation more clearly, let us scale back the variables in the following manner. Define W and B by

$$\begin{cases} w = \dfrac{1}{g}\,W \\ b = \dfrac{1}{g}\,B \end{cases} \tag{2.23}$$

$\xi_{[\sigma]}(\tilde{w})$ and $T(\tilde{w})$ in terms of W are given by

$$\begin{cases} \xi_{[\sigma]}(\tilde{w}) = \dfrac{1}{g}\ \xi_{[\sigma]}(\tilde{W}) \equiv \dfrac{1}{g}\ \bar{\xi}_{[\sigma]} \\ T(\tilde{w}) = \dfrac{1}{g}\ T(\tilde{W}) \equiv \dfrac{1}{g}\ \bar{T} \end{cases} \tag{2.24}$$

Thus (2.22) becomes

$$\begin{aligned} Z = \int & DWDB\ \sqrt{\det\ \bar{T}}\ \delta(\partial\tilde{W}-\frac{1}{g}\bar{T}B) \\ & \times \exp\ \{-\frac{1}{4}(W,W)-\frac{i}{2g^2}\ (\tilde{W},\ F(B;\frac{1}{g})) \\ & -\frac{i}{g}(B,\xi)+\frac{i}{2g}\ (g\xi+\bar{\xi}_{[\sigma]},\bar{T}^{-1}(g\xi+\bar{\xi}_{[\sigma]}))\} \end{aligned} \tag{2.25}$$

This is the direct generalization to the non-Abelian case of the dual form for Abelian theories, e.g. Eqs. (1.4) and (1.10). This form has a number of intriguing features as we shall discuss in detail in the next section.

Before doing so, however, let us briefly comment on a mathematical ambiguity apparent in (2.25). There appears, in the last scalar product in the exponent, a term $(\xi_{[\sigma]}, T^{-1}\xi_{[\sigma]})$. This contains two surface delta-functions and hence is difficult to interpret. Such a term would have a well-defined meaning had we worked on a lattice. This suggests that a further study on regularization and renormalization of the theory is needed to clarify its meaning. Not having done this, we shall hereafter work on a closed manifold, so that we may drop the surface terms and study the corresponding expression for Z given by

$$Z = \int DWDB \sqrt{\det \overline{T}}\, \delta(\partial \tilde{W} - \frac{1}{g} \overline{T}B)$$
$$\exp\{-\frac{1}{4}(W,W) + \frac{i}{2g^2} (B,\overline{T}B) - \frac{i}{g}(B,\xi)$$
$$+ \frac{i}{2}g(\xi,\overline{T}^{-1}\xi)\}. \qquad (2.26)$$

We shall now turn to a discussion of various features of this form.

III. Aspects of the Dual Representation

Let us recall that in Abelian theories, the final step in the duality transformation is to express the theory in terms of the dual vector potential by integrating out the dual field strength variable with the help of the Bianchi-identity-like delta function. To proceed along the same lines with the form (2.25) obtained in the previous section we need to solve the corresponding delta function constraint, i.e.,

$$\partial\tilde{W}-\frac{1}{g}\bar{T}B = 0. \tag{3.1}$$

An obvious solution is the usual field strength form:

$$W^i_{\mu\nu} = F^i_{\mu\nu}(B;\frac{1}{g}) = \partial_\mu B^i_\nu-\partial_\nu B^i_\mu+\frac{1}{g}c^{ijk}B^j_\mu B^k_\nu. \tag{3.2}$$

Let us suppose for the moment that this is the only solution and perform the integration over W. Noting that in such a case

$$\begin{aligned}\frac{i}{2g^2}(B,\bar{T}B) = -\frac{i}{2g^2}(\tilde{F}(B;\frac{1}{g}),F(B;\frac{1}{g}))\\ +\frac{i}{g^2}\int d^4x\partial_\mu(\tilde{W}^i_{\mu\nu}B^i_\nu) = \text{surface term} = 0\end{aligned} \tag{3.3}$$

the partition function takes the form

$$\begin{aligned}Z = \int DB\sqrt{\det\bar{T}(\tilde{F}(B;\frac{1}{g}))}\exp(-\frac{1}{4}(F(B;\frac{1}{g}),F(B;\frac{1}{g}))\\ -\frac{i}{g}(B,\xi)+\frac{i}{2}g(\xi,\bar{T}^{-1}(\tilde{F}(B;\frac{1}{g}))\xi))\end{aligned} \tag{3.4}$$

So we obtain the intriguing result that the dual theory is again a non-Abelian gauge theory with the coupling constant inverted, albeit with more complicated self-interactions and more complicated interactions with the external source.[6]

Before continuing, we wish to point out a peculiar property of (3.4) under space inversion. Unlike the other terms, the last term in the exponent is apparently odd under this operation. This is most likely an indication that the solution (3.2) is not unique, and that this peculiarity will disappear after the full set of solutions of the Bianchi constraint are taken into account.

Although the form (3.4) is attractive and suggestive, we must go back and ask whether the Bianchi-like constraint does admit solutions other than the field strength form (3.2). One first notes that, due to the linear nature of the constraint for W,

$$W = \lambda\ F(B;\tfrac{1}{g}) \tag{3.5}$$

is also an admissible solution for any constant, λ. Beyond this class of solutions, the question becomes a very complicated one, in contrast to the Abelian case. Although the constriant equation is linear in W, it becomes non-linear when W is expressed in terms of B. This prevents one from writing down any simple solution besides (3.2) and (3.5).

Obviously, one needs a more systematic approach to the problem. One such possibility, which we shall explore, is a perturbative solution based on an expansion in powers of 1/g. Afterall, the solutions (3.2) and (3.5) do have such a form. Further, in the limit of large g, such an expansion may be useful in trying to formulate a strong coupling approximation to Z. We shall first try to construct a formal perturbative solution and then later discuss its use in a strong coupling approximation. As it turns out, even a perturbative analysis of (3.1) is not so straightforward and requires some general results of harmonic analysis which are best described in the language of differential forms. We shall therefore only discuss the results of our analysis and relegate the details to the appendix.

The results were obtained in a recursive form (see (A.20) or (A.22) in the Appendix), and when translated into a more familiar language are as follows: The general solution of (3.1) may be written in an expansion in g^{-1} as

$$W^i_{\mu\nu} = \sum_{n=0}^{\infty} \left(\frac{1}{g}\right)^n W^i_{(n)\mu\nu} \tag{3.6}$$

$$\begin{aligned} W^i_{(n)\mu\nu}(x) = {} & \partial_\mu \alpha^i_{(n)\nu}(x) - \partial_\nu \alpha^i_{(n)\mu}(x) + \gamma^i_{(n)\mu\nu}(x) \\ & - \frac{1}{2} c^{ijk} \varepsilon_{\mu\nu\alpha_1\alpha_2} \varepsilon_{\beta_1\beta_2\beta_3\alpha_2} \\ & \times \int d^4y\, G(x-y) \frac{\partial}{\partial y_{\alpha_1}} [W^j_{(n-1)\beta_1\beta_2}(y) B^k_{\beta_3}(y)] \end{aligned} \tag{3.7}$$

where $G(x-y)$ is the Green's function defined by

$$\partial^2 G(x-y) = \delta^{(4)}(x-y), \tag{3.8}$$

$a^i_{(n)\mu}(x)$ is a vector field, and $\gamma^i_{(n)\mu\nu}(x)$ is a tensor field satisfying

$$\partial^2 \gamma^i_{(n)\mu\nu}(x) = 0.$$

Furthermore, the fields in (3.7) must satisfy the constraint equation

$$\begin{aligned} c^{ijk} W^j_{(n)\mu\nu} B^k_\rho(x) = -c^{ijk}\varepsilon_{\mu\nu\alpha_1\alpha_2} \varepsilon_{\beta_1\beta_2\beta_3\alpha_2} \\ \times \int d^4y \, G(x-y) \frac{\partial}{\partial y_\rho} \frac{\partial}{\partial y_{\alpha_1}} [W^j_{(n)\beta_1\beta_2}(y) B^k_{\beta_3}(y)] \end{aligned} \tag{3.9}$$

It is not difficult to check that Eqs. (3.6)-(3.9) do generate the solution to (3.1) of the form (3.5) (see appendix). However, the complicated integrability condition (3.9) has so far prevented us from using the recursion relation (3.7) to explicitly write down the general solution to (3.1). (See the appendix for further discussion.)

But let us suppose that we were able to overcome this technical difficulty and explicitly develop W in a power series in g^{-1}. We now ask whether such an expansion in g^{-1} can be used as the basis for a systematic approximation procedure for quantities of physical interest in the theory. Of particular interest, of course, are quantities which are sensitive to the large distance structure of the theory such as the asymptotic behavior of the Wilson loop. For such quantities is there any intuitive reason to suppose that such an expansion might be sensible?

The almost universal expectation is that QCD in one way or another confines quarks. If so, then the large distance structure of the QCD vacuum has significant contributions from field configurations with very large values of A_μ and $F_{\mu\nu}$. Now, in the sense of functional Fourier transforms, $w_{\mu\nu}$ is conjugate to $f_{\mu\nu}$ and j_μ is conjugate to a_μ, so we might suppose that the dominant contributions to the large distance structure of the theory come primarily from configurations of $w_{\mu\nu}$ and j_μ which do not have very large amplitude fluctuations. To see this more clearly we note in (2.4) that we expect significant contributions to Z from $f \lesssim \mathcal{O}(g)$. Thus we expect that $w \lesssim \mathcal{O}(\frac{1}{g})$. Now, at long distances, (or large coupling) we anticipate that the dominant contributions to f comes from the term $c_{ijk}a^j_\mu a^k_\nu$. In the first place, if the dominant contribution to f came from the term linear in a^i_μ, then the strong coupling theory would look much like an Abelian theory <u>in the original variables</u>, a^i_μ, plus, perhaps, perturbative corrections, which is almost certainly not true. Second, since we expect large coupling to be associated with long distances (or low momenta) the term linear in a^i_μ may well be additionally suppressed by the fact that the term involves a derivative (or extra momentum factor.) Thus, we expect that $[a,a] \lesssim \mathcal{O}(g)$. Ignoring for the moment the fact that the commutator is not just a simple product, we are led to suppose that $a_\mu^i < \mathcal{O}(\sqrt{g})$. Thus, from (2.15) we expect that $j \lesssim \mathcal{O}(\frac{1}{\sqrt{g}})$. So, indeed, Z should be dominated by configurations of w and j which do not have large fluctuations. Continuing with our heuristic arguments, we see from (2.19) that since $j \lesssim \mathcal{O}(\frac{1}{\sqrt{g}})$ and $w \lesssim \mathcal{O}(\frac{1}{g})$, then $b_\mu^k \lesssim \mathcal{O}(\sqrt{g})$. Thus, superficially, fluctuations in b can grow as g grows. These considerations imply that the most naive approach to constructing a strong coupling perturbation expansion for Z may not be sensible. To see this lets look at the exponent of, say (2.26). Ignoring terms involving ξ, we note that the remaining two terms are both $\lesssim \mathcal{O}(1)$ so it is apparently not fruitful to try to perturb in either one of them. (Note that a similar analysis can be made using (2.25)). On the other hand, while this may be a correct conclusion, it is important to remember that the arguments from which it was drawn were not completely airtight. In particular, in deducing that $a \lesssim \mathcal{O}(\sqrt{g})$ and that $b \lesssim \mathcal{O}(\sqrt{g})$ we treated cross products and commutators as if they were simple multiplication. It is certainly possible for example, to have $[a,a] \lesssim \mathcal{O}(g)$ but still have $a \gtrsim \mathcal{O}(\sqrt{g})$. Unfortunately, it is not simple to rigorously determine whether or not there really are significant contributions to Z from configurations with $b \sim \mathcal{O}(\sqrt{g})$, or whether the important contributions come only from smaller values of b. Thus without further study we can draw no firm conclusions about how one might realize a sensible large coupling perturbation theory starting from the form (2.26).

Before leaving this topic it is worthwhile emphasizing that even if the fluctuations in b can be as large as $\mathcal{O}(\sqrt{g})$, the fluctuations in the dual field strength, w, are still small (see e.g. (2.22)). This is to be contrasted with the fact that when g is large and $a \sim \mathcal{O}(\sqrt{g})$ the fluctuations in the original field strength, f, are large. Furthermore we note that Z has a representation in terms of the dual variables $w_{\mu\nu}$ and j_μ (see e.g. (2.18)) both of which have small fluctuations for large g. Thus, the strong coupling vacuum defined in terms of the dual variables w and j should be "simple" (i.e. not contain large fluctuations) while it is an open question whether it is "simple" in terms of b defined through (2.19).

There are two other observations we wish to make about our result, one heuristic and one technical. The first concerns the size of the term $\propto\xi^2$ in the exponent of, say (2.26). Recalling that $w \lesssim \mathcal{O}(\frac{1}{g})$ we see that the coefficient of the ξ^2 term in (2.25) is a number of order g. Now ξ represents the original external current which could be a quark current, so the ξ^2 term in (2.25) can be thought of as representing a kind of induced potential between quarks. But the coefficient of this term is $\propto g$ and so this effective quark interaction is very strong for large g. Qualitatively, one expects a strong force to be generated between quarks for large coupling, and it is suggestive that such an effect is readily apparent in our dual form. The question of whether or not this is significant for confinement requires further study.

Cur final comment is a technical one. If one chooses to boldly pursue the analysis of (2.26) and tries to develop a computationally tractable strong coupling scheme, one must decide how to handle the factor $[\det T]^{1/2}$ in the functional integral. At least for the case of SU(2), an explicit expression for this determinant exists.[8] Unfortunately, such an expression is not necessarily computationally useful. An alternative method of dealing with this determinant is to introduce ghost fields and write an exponentiated representation for it in the usual way. But from the point of view of perturbation theory this is not much better since there are no quadratic damping factors for the ghost field integrals. A possible solution to this problem is to couple massive Higgs fields to the original gauge fields, a_μ^i, and perform the duality transformations on this coupled theory. The exponentiated representation of the determinant will then have a well-defined perturbation expansion for any negative non-zero Higgs mass term.[9] Now the vacuum of the theory will certainly be different with the Higgs field than without. However, it may be that for sufficiently small negative

Higgs mass term we would be able to see the effects of the unbroken, symmetric theory over distances $r \lesssim \frac{1}{m}$. If so, it might be possible to deduce some of the features of the symmetric, strong coupling vacuum from such a calculation. Such a result would be analogous to the behavior expected near a second order phase transition.

IV. Conclusions and Summary

Motivated by an analogy with Abelian theories, we have constructed a duality transformation for non-Abelian theories which has a number of interesting properties. First, under our transformation the coupling constant gets inverted in a certain sense, so that our dual variables are expected to have only small fluctuations when the coupling constant is large. This is precisely the kind of behavior we encounter when applying duality transformations to Abelian theories.

Second, our dual form has, loosely speaking, the structure of a gauge theory. Recall, for instance, that the contribution to the dual form of Z from $W_{\mu\nu}(B_\nu)$ chosen to be the usual field strength tensor (which therefore satisfied the Bianchi-identity like constraint in (2.26)) had a form quite close to that of the usual generating functional for the non-Abelian gauge theory, but with $g \to g^{-1}$. Finally, the dual form of the generating functional, e.g. (2.26), is rather elegant.

One of the factors that appears in the dual form of Z is a functional delta function enforcing a Bianchi-identity like constraint on the fields $W_{\mu\nu}$ and B_μ. It is therefore of interest to ask what W's are allowed by this Bianchi-like constraint for a given B. This problem was discussed in Section III and a systematic formal, perturbative (in powers of g^{-1}) solution was studied. Having understood, at least formally, what W's and B's are allowed by the Bianchi constraint, we investigated the possibility of using our dual form of the generating functional to construct a systematic strong coupling approximation scheme. Unfortunately such a scheme proved to be very elusive, and we failed to satisfactorily formulate one. Nevertheless, in analogy to what happens in Abelian theories, it is quite possible that one or the other of our dual forms may be able to provide us with a starting point for such an expansion. In view of this possibility and in view of the suggestive structure and intriguing properties of our dual form we feel that this approach merits further study.

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Appendix. Perturbative Solution to the Bianchi Constraint.

In this appendix, we shall discuss the perturbative solution to the Bianchi constraint

$$\partial_\mu \tilde{W}^i_{\mu\nu} - \frac{1}{g} c_{ijk} \tilde{W}^j_{\mu\nu} B^k_\mu = 0 \tag{A.1}$$

defined over a closed, compact, but not necessarily simply connected manifold, M. To facilitate our manipulations and to utilize some powerful results of harmonic analysis, we employ the elegant formalism of differential forms. We shall give only the minimum of definitions and notations which are necessary for the analysis and quote theorems and propositions without proof. The interested reader can find more details in, for example, Ref. 10.

We begin by defining the Lie algebra valued 1-form B and 2-form W by

$$\begin{aligned} B &\equiv B^i_\mu e_i \, dx^\mu \equiv B^i e_i \\ W &\equiv \frac{1}{2} W^i_{\mu\nu} e_i \, dx^\mu \wedge dx^\nu \equiv W^i e_i \end{aligned} \tag{A.2}$$

where $\{e_i\}$ are the generators of the Lie algebra of the structure group G satisfying

$$[e_i, e_j] = c_{ijk} e_k \tag{A.3}$$

and the symbol $\wedge$ denotes the antisymmetric exterior product. The bracket product such as $[W,B]$ will often be used, which is a shorthand notation for $W^i \wedge B^j [e_i, e_j]$. The linear operator * (the Hodge star operator) produces from a p-form its dual (4-p)-form (in four dimensions). In particular *B (a 3-form) formally, what W's and B's are allowed by the Bianchi constraint, we investigated the possibility of using our dual form of the generating functional to construct a systematic strong coupling approximation scheme. Unfortunately such a scheme proved to be very elusive, and we failed to satisfactorily formulate one. Nevertheless, in analogy to what happens in Abelian theories, it is quite possible that one or the other of our dual forms may be able to provide us with a starting point for such an expansion. In view of this possibility and in view of the suggestive structure and intriguing properties of our dual form we feel that this approach merits further study.

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and the symbol $\wedge$ denotes the antisymmetric exterior product. The bracket product such as $[W,B]$ will often be used, which is a shorthand notation for $W^i \wedge B^j [e_i, e_j]$. The linear operator $*$ (the Hodge star operator) produces from a p-form its dual (4-p)-form (in four dimensions). In particular $*B$ (a 3-form) and $^*W$ (a 2-form) are defined by

$$^*B = \frac{1}{3!}\,\varepsilon_{\mu\nu\rho\sigma}\; B^{i}_{\sigma} e_i dx^\mu \wedge dx^\nu \wedge dx^\sigma$$

$$^*W = \frac{1}{2}\,\tilde{W}^{i}_{\mu\nu} e_i dx^\mu \wedge dx^\nu \qquad (A.4)$$

$$^{**}B = -B, \quad ^{**}W = W$$

(The * operation depends on the metric of the manifold. Throughout we consider a Euclidean metric.)

The differential operator d which sends a p-form into a p+1-form is defined by

$$d^2 = 0$$

$$d(B_\nu dx^\nu) = B_{\nu,\mu} dx^\mu \wedge dx^\nu = \frac{1}{2}(B_{\nu,\mu} - B_{\mu,\nu}) dx^\mu \wedge dx^\nu$$

$$d(\tfrac{1}{2} W_{\mu\nu} dx^\mu \wedge dx^\nu) = \tfrac{1}{2} W_{\mu\nu,\sigma}\; dx^\sigma \wedge dx^\mu \wedge dx^\nu \qquad (A.5)$$

where $,\sigma$ denotes differentiation with respect to x^σ. The co-differential operator δ and the Laplace-Beltrami operator Δ are defined by

$$\delta = -{}^*d^* \qquad \text{(Euclidean metric)}$$

$$\Delta = d\delta + \delta d$$

δ and Δ map a p-form into a p-1 form and a p form, respectively. A differential form γ is said to be harmonic if $\Delta\gamma = 0$. We use $\mathcal{K}$ to denote the space of harmonic forms and the projection operator which projects out the harmonic part of a differential form will be denoted by H. Δ in general does not have an inverse. Rather, there exists a unique integral operator, G, (Green's operator) such that

$$\Delta G = G\Delta = 1-H, \quad GH = HG = 0 \tag{A.6}$$

so that Δ has a unique inverse if and only if it is restricted to operate on the space of forms without a harmonic part. Some useful properties of these differential operators are

$$\begin{array}{ll} \text{(i)} & d\Delta = \Delta d, \quad \delta\Delta = \Delta\delta \\ \text{(ii)} & Gd = dG, \quad G\delta = \delta G \\ \text{(iii)} & \delta\gamma = 0 = d\gamma \text{ if } \gamma\epsilon\mathcal{K} \\ \text{(iv)} & \delta^2 = 0 \\ \text{(v)} & H\delta = \delta H = 0, \quad Hd = dH = 0 \end{array} \tag{A.7}$$

Finally, between two p-forms an inner product $\langle\alpha,\beta\rangle$ is defined by

$$\langle\alpha,\beta\rangle \equiv \int_M \alpha\wedge{*\beta},$$

which has the following properties:

$$\begin{array}{ll} \text{(i)} & \langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle \\ \text{(ii)} & \langle\alpha,\alpha\rangle = 0 \Rightarrow \alpha = 0 \\ \text{(iii)} & \langle\delta\alpha,\beta\rangle = \langle\alpha,d\beta\rangle \\ \text{(iv)} & \langle\alpha,\Delta\alpha\rangle = 0 \Rightarrow \Delta\alpha = 0 \end{array} \tag{A.8}$$

With these notations and definitions we may now start our analysis. The Bianchi constraint takes the simple form

$$dW = \frac{1}{g}[W,B] \tag{A.9}$$

Applying the operator d to both sides of (A.9) and recalling that $d^2=0$, we have the integrability condition,

$$d[W,B] = 0. \tag{A.10}$$

To solve (A.9) perturbatively in powers of $1/g$, it is best to first transform it into an integral equation. For this purpose we can use the Hodge decomposition theorem.[10] It states that any differential form W may be uniquely decomposed in the following form:

$$W = d\alpha + \delta\beta + \gamma$$

$$\text{with } \gamma \in \mathcal{H} \tag{A.11}$$

Applying this to (A.9), we get

$$d\delta\beta = \frac{1}{g}[W,B] \tag{A.12}$$

Applying the theorem once again to β itself, viz.,

$$\beta = d\alpha' + \delta\beta' + \gamma', \ \gamma' \in \mathcal{H} \tag{A.13}$$

we obtain $\delta\beta = \delta d\alpha'$. That is, the $\delta\beta' + \gamma'$ terms do not contribute to W. We may therefore always choose β such that

$$d\beta = 0, \quad H\beta = 0 \tag{A.14}$$

Using this gauge condition, we have $d\delta\beta = (d\delta + \delta d)\beta = \Delta\beta$ and Eq. (A.12) becomes

$$\Delta\beta = \frac{1}{g}[W,B] \tag{A.15}$$

Now in order that this has a solution, the RHS of (A.15) must be in $\mathcal{H}$. Thus we must demand

$$H[W,B] = 0 \tag{A.16}$$

It is then not difficult to show that (A.10) and (A.16) may be replaced by a single condition

$$(1 - dG\delta)[W,B] = 0 \tag{A.17}$$

Then the solution to (A.15) is

$$\beta = \frac{1}{g} G[W,B] \tag{A.18}$$

Notice that due to the integrability condition this properly satisfies the gauge condition (A.14). Putting this back into (A.11), we find

$$W = d\alpha + \gamma + \frac{1}{g}\delta G[W,B] \tag{A.19}$$

This, together with (A.17), is the desired integral equation.

To solve (A.19) we may not simply iterate it since $d\alpha + \gamma$ is not necessarily of order 1. However, we can in general expand W, α and γ in powers of $1/g$, viz.,

$$W = \sum_{n=0}^{\infty} \left(\frac{1}{g}\right)^n W_{(n)}$$

$$\alpha = \sum_{n=0}^{\infty} \left(\frac{1}{g}\right)^n \alpha_{(n)}$$

$$\gamma = \sum_{n=0}^{\infty} \left(\frac{1}{g}\right)^n \gamma_{(n)} \quad . \tag{A.20}$$

Substitution of (A.20) into (A.17) and (A.19) yields

$$(1-dG\delta)[W_{(n)},B] = 0 \tag{A.21}$$

with

$$W_{(n)} = d\alpha_{(n)} + \gamma_{(n)} + \delta G[W_{(n-1)},B] \tag{A.22}$$

$$(W_{(-1)} \equiv 0)$$

If it were not for the integrability condition (A.21), the recursive formula (A.22) would immediately give the desired solution. But because of (A.21), $d\alpha_{(n)}$ and $\gamma_{(n)}$ are constrained in a complicated manner for each n making it difficult to explicitly write down the most general solution.

It is however instructive to show how the particular solutions of the form $W = \lambda F(B;\frac{1}{g})$ are generated using (A.21) and (A.22). First, we take $d\alpha_{(0)} = \lambda dB$ and $\gamma_{(0)} = 0$, i.e.

$$W_{(0)} = \lambda dB \tag{A.23}$$

This satisfies (A.21). We then choose $d\alpha_{(1)} = \frac{\lambda}{2} dG\delta[B,B]$ and $\gamma_{(1)} = \frac{\lambda}{2} H[B,B]$ in (A.22). This gives

$$\begin{aligned} W_{(1)} &= \frac{\lambda}{2} dG\delta[B,B] + \frac{\lambda}{2} H[B,B] + \delta G[dB,B] \\ &= \frac{\lambda}{2} (dG\delta + \delta Gd + H)[B,B] \\ &= \frac{\lambda}{2} (G\Delta + H)[B,B] \\ &= \frac{\lambda}{2} [B,B] \end{aligned} \tag{A.24}$$

Now (A.21) is again satisfied due to the Jacobi identity

$$[[B,B],B] = 0 \tag{A.25}$$

Choosing $d\alpha_{(n)} = \gamma_{(n)} = 0$ for $n \geq 2$, we then find $W_{(n)}=0$ for $n \geq 2$. Therefore the series terminates and we obtain an exact solution

$$\begin{aligned} W = W_{(0)} + \frac{1}{g} W_{(1)} &= \lambda(dB + \frac{1}{2g} [B,B]) \\ &= \lambda F(B;\frac{1}{g}) \end{aligned} \tag{A.26}$$

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6. The fact that our dual form for Z is gauge-theory-like follows from the fact that in four dimensions the dual of a two-form is a two-form. Had we performed our transformation in three dimensions our dual thoery would have had the structure of a scalar field theory. In this respect our dual forms have a geometric structure similar to that found in Abelian theories (see Ref. 1) and also reminiscent of the structure presented by 't Hooft (Ref. 3) in his discussion of duality. It is also worth pointing out that our procedure can be applied in a quite similar way to non-Abelian scalar theories as well.
7. This apparently space inversion violating term should not, of course, be confused with the term of the form $i\tilde{w}F$ which appears in, for example (1.2) and (2.5). At the level of (1.2) or (2.5) there is no bonafide space inversion violation.

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