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One must not forget to have fun in life, it is what we are here to do.

Abstract

p-adic physics started 35 years ago with a conjecture by I. Volovich on how the description of spacetime below the Planck scale should be using p-adic numbers. The concept was to construct string theory amplitudes in a p-adic worldsheet embedded in a real target spacetime starting from their integral expression. The amplitudes are much simpler to compute and captured many physical properties. The p-adic string worldsheet is described by an infinite graph with no loops known as the Bruhat-Tits tree, whose boundary is identified with the p-adic projective line.

There is a close connection between string amplitudes and local zeta functions. Mathematically there is interest in finding meromorphic continuations of the amplitudes to regularize them. In this thesis, first, we study the bosonic string amplitudes in a background antisymmetric B field. We show the regularization of the amplitudes, the process to obtain them, and analyze the limit $p \to 1$, that interestingly relates p-adic theories with real theories.

Then we present a proposal of an action for the p-adic superstring theory, analogue to the real superstring in the Neveu-Schwarz formalism in conformal gauge. We show that this action has a family of supersymmetry transformations. We also write this action in the superspace formalism. The scattering amplitudes are explicitly computed, recovering previous works.

Recently, the Bruhat-Tits tree has served as the bulk in a p-adic version of the AdS/CFT correspondence. In this context, we show the equivalence and generalization of two alternatives for a general solution of the bulk equations of motion in terms of a bulk-to-boundary propagator. We also lay the ground to implement the process of holographic Wilsonian renormalization. Remarkably, leading behaviors in the real holography are exact behaviors in p-adic holography.

Resumen

La física p-ádica empezó hace 35 años con una conjetura de I. Volovich sobre cómo la descripción del espaciotiempo a la escala de Planck debería ser usando números p-ádicos. El concepto fue construir amplitudes de teoría de cuerdas en una hoja de mundo p-ádica embebida en un espacio real a través de las expresiones integrales. Las amplitudes son mucho más simples de evaluar y se mantienen varias propiedades físicas. La hoja de mundo p-ádica se describe como un árbol infinto sin lazos llamado árbol de Bruhat-Tits cuya frontera es identificada con la línea proyectiva p-ádica.

Existe una conexión cercana entre amplitudes de cuerdas y funciones zeta locales. Matemáticamente interesa regularizar las amplitudes al encontrar sus continuaciones meromorfas. En esta tesis, primero, estudiamos amplitudes de la cuerda bosónica con un campo antisimétrico B de fondo. Mostramos la regularización de las amplitudes, el proceso para obtenerlas y analizamos el límite $p \to 1$ que interesantemente relaciona teorías p-ádicas con teorías reales.

Después presentamos una propuesta de una acción para la supercuerda p-ádica, análoga a la supercuerda real en el formalismo de Neveu-Schwarz en la norma conforme. Mostramos que esta acción tiene una familia de transformaciones supersimétricas. También escribimos la acción usando el formalismo de superespacio. Obtenemos explícitamente las amplitudes recuperando trabajos previos.

Recientemente, el árbol de Bruhat-Tits ha servido como el bulto en una versión p-ádica de la correspondencia AdS/CFT u holografía. En este contexto, mostramos la equivalencia y generalización de dos alternativas para la solución general a las ecuaciones de movimiento del bulto en términos de un propagator bulto-frontera. También sentamos bases para implementar el proceso de renormalización Wilsoniana. Notablemente, los comportamiento dominantes en holografía real son comportamientos exactos en la holografía p-ádica.

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Contents

1 Introduction **1**

I Background 6

2 Mathematical Background **7**

2.1 Introduction 7

2.2 Definitions 8

2.2.1 p-adic Numbers 9

2.2.2 Topology and $\mathbb{Q}_p^n$ 10

2.2.3 The sign function 11

2.2.4 Algebraic extensions 15

2.2.5 Integration 17

2.2.6 Fourier transform 22

2.2.7 Vladimirov Derivative 25

2.3 Local Zeta Functions 27

2.4 Grassmann Variables 31

3 Physics Background **35**

3.1 String Theory Amplitudes 35

3.1.1 String Theory Action 36

3.1.2 Scattering Amplitudes 37

3.1.3 Superstring 42

4 p-adic Physics **46**

4.1 p-adic String Amplitudes 47

4.1.1 Open string 47

4.1.2 Closed String 50

4.1.3 The p-adic string worldsheet 52

4.1.4 p-adic Superstring Amplitudes 55

4.2 Connection between Archimedean and Non-Archime-dean physics 57

4.2.1 Adelic products 57

4.2.2 The limit $p \to 1$ 59

4.3 p-adic strings on a background B field 60

II p-adic String Amplitudes **64**

5 p-Adic String Amplitudes in a B-field **65**
5.1 The limit $p \to 1$ in the deformed effective action . . . 65
5.1.1 Amplitudes from the noncommutative Gerasimov-Shatashvili Lagrangian 66
5.1.2 Four-point amplitudes . . . 67
5.2 Twisted Multivariate Local Zeta Functions . . . 69
5.3 The Ghoshal-Kawano local zeta function . . . 71
5.4 Meromorphic continuation of Ghoshal-Kawano local zeta function . . . 73
5.4.1 Meromorphic continuation of $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ . . . 74
5.5 Explicit computation of $Z^{(4)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ . . . 75
5.6 Explicit computation of $Z^{(5)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ . . . 78
5.7 The limit $p \to 1$ of the Ghoshal-Kawano amplitudes . . . 85

6 Towards Non-Archimedean Superstring Amplitudes **86**
6.1 A p-adic fermionic action . . . 86
6.1.1 Fermionic Green's function . . . 87
6.2 A 'worldsheet' action for the p-adic superstring . . . 88
6.2.1 Equations of motion . . . 89
6.2.2 Supersymmetry transformation . . . 90
6.2.3 Superspace description of the p-adic superstring . . . 91
6.3 Tree-level amplitudes of the p-adic superstring . . . 92
6.3.1 Integral expression . . . 93
6.3.2 Integrating the Grassmann variables . . . 95
6.4 Four-point amplitudes . . . 98
A Vertex operators . . . 99
B Functional derivatives . . . 100

7 Local Zeta Functions and Koba-Nielsen Amplitudes **102**
7.1 Meromorphic Continuation of Koba-Nielsen Amplitudes Defined on Local Fields of Characteristic Zero . . . 102
7.1.1 4-point Koba-Nielsen string amplitude . . . 105
7.1.2 5-point Koba-Nielsen string amplitude . . . 107

III AdS/CFT **110**

8 The Archimedean and non-Archimedean AdS/CFT correspondence **111**
8.1 Essentials of the AdS/CFT correspondence . . . 111
8.1.1 Correlation Functions . . . 113
8.1.2 Holographic Renormalization . . . 116

8.2 The p-adic AdS/CFT correspondence . . . 118
8.2.1 The 2-point function . . . 122
8.3 A coarse-grained realization of p-adic AdS/CFT . . . 124
8.3.1 Removing the cutoffs . . . 128
8.4 p-adic Holographic Renormalization . . . 130
8.4.1 Extracting the counterterms . . . 132
A Explicit proof that both terms in the solution satisfy the equation of motion . . 139
B The case $\Delta = D/2 = D - \Delta$ . . . 142

9 Conclusion **143**

References **146**

List of Figures

4.1 Convergence region for the p-adic Veneziano amplitude . . . 49
4.2 The Bruhat-Tits tree for $p = 2$ . . . 53

8.1 Bruhat-Tits tree of $\mathbb{Q}_{2^2}$ and its boundary as a 2-dimensional space. . . . 119
8.2 Coordinate system used for the Bruhat-Tits tree as the holographic bulk . . . 121
8.3 Summing a laplacian on the Bruhat-Tits tree reduces to boundary terms . . . 123
8.4 Alternative notation for coordinates on the Bruhat-Tits tree . . . 125
8.5 A tree with UV and IR cutoffs is equivalent to infinite disconnected trees . . . 128
8.6 Visualization of the relations between lengths in the tree and the coordinates . . 129
8.7 Visualization of the vertices along which we sum in simplifying boundary terms 135
8.8 Visual aid for the proof of the general solution to the cutoff bulk equations . . . 141

List of Tables

2.1 Examples of quadratic residues for the cases $p = 5, 7, 11, 13$. . . . 13
2.2 Non-trivial possibilities for the p-adic sign of the elements ϵ, p and ϵp. . . . 14
2.3 Expressions for the non-trivial signs in $\mathbb{Q}_p$ . . . 15

8.1 The powers of $|\epsilon|_p$ in the boundary action terms to look for divergencies . . . 137

Chapter 1

Introduction

The 1980's was an interesting decade for theoretical physics, string theory was a very promising theory with high expectations and novel developments kept being made. One of said developments was the introduction of p-adic numbers in the study of string theory. Although it is a little strange, it gave rise to a lot of work at the time, mainly imitations of known results in the usual real theory. Later the amount of work dedicated to this subject faded. It was until some years ago that the interest in the use of p-adic numbers arose back in the newest revolution of theoretical physics, the AdS/CFT correspondence. Once again there are a lot of promising works on this subject that have sprouted new ideas. Time will tell if this interest in p-adic physics will remain and grow or if it will fade and sleep once more. This thesis is a result of some of the latest developments in the use of p-adics numbers and techniques in physical models. We will review here some of the beginnings and foundations of these ideas.

The story begins in 1987 with the reference [1], Volovich's work argued that the Planck scale was a symptom of something much deeper than just an uncertainty relation. It meant that the description of our universe at its most fundamental language: mathematics, needed to be revised and replaced, at least at this scale, for something more natural to the situation. He proposed using p-adic numbers due to their unusual totally disconnected topology. The existence of a Planck length implies that the spacetime considered as a topological space is completely disconnected. The points (which are the connected components) play the role of spacetime quanta. This is precisely the Volovich conjecture on the non-Archimedean nature of the spacetime below the Planck scale, [2, 1, 3], [4, Chapter 6]. There is a natural occurrence of ultrametric spaces in physical models from which one can trace the origin of this idea, see e.g. [5, 2, 1, 6, 7, 8, 9, 4, 10].

The study of a p-adic string theory was introduced many years ago with some periodic fluctuations in their interest, and further developed in [2, 11, 12, 13, 14]. In these references it was studied the proposal of considering the tree-level amplitudes of a bosonic open string theory with the fields on the worldsheet taking values over the p-adic number field $\mathbb{Q}_p$. This is in con-

trast with the conjecture by Volovich where the spacetime is also p-adic. The string amplitudes were introduced by Veneziano in the 60s, [15], further generalizations were obtained by Virasoro [16], Koba and Nielsen [17], among others. In the 80s, Freund, Witten and Volovich, among others, studied string amplitudes at the tree level over different number fields, and suggested the existence of connections between these amplitudes. For some reviews, see [18, 19, 7, 5, 13]. Later these amplitudes were derived from a p-adic worldsheet action defined on the Bruhat-Tits tree [20] or projected out as an effective theory to the boundary of the tree, the p-adic line $\mathbb{Q}_p$, in terms of the Vladimirov derivative [21]. Both methods were found to be equivalent in order to derive the mentioned open string amplitudes. Very recently the procedure followed in [21, 20] has been studied from a rigorous point of view in [22].
In string theory, the scattering amplitudes are obtained integrating over the moduli space of Riemann surfaces. Even for tree-level amplitudes (on the sphere for closed strings and on the disk for open strings) N-point amplitudes are difficult to compute beyond four points. Moreover, the convergence of these integrals is not evident by itself [23, 24]. There is a natural relation of string amplitudes with local zeta functions. In this context the study of the regularization of the amplitudes for non-Archimedean open strings in terms of local zeta functions was discussed in [25]. The techniques developed for the p-adic case can be applied to the Archimedean case as well. Recently, in [26] was established, in a rigorous mathematical way, that Koba-Nielsen amplitudes are bona fide integrals; they admit meromorphic continuations when considered as complex functions of the kinematic parameters. See [27] for a recent survey on the connections between string amplitudes and local zeta functions.

The p-adic strings are surprisingly related to ordinary strings in at least two ways. First, there are several examples that show a relation using what is known as adeles. An adele is a string of numbers with entries in $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p. A mathematical object in a p-adic theory is defined for a given p. An adelic product multiplies the object over all possible p and the result is related to the corresponding object in the real theory. These adelic relations were studied in the past [12, 28]. The second way is through the limit when $p \to 1$, first discussed in [21]. Although this limit remains mostly mysterious, there are more recent examples of this relation. In [29] was showed that the limit $p \to 1$ of the effective action gives rise to a boundary string field theory that was previously proposed by Witten in the context of background independent string theory [30]. A similar relation for another effective Lagrangian was found in [31]. The limit $p \to 1$ in the effective theory can be performed without any problem, since one can consider p as a real parameter and the limit $p \to 1$ makes sense. The resulting theory is related to a field theory describing an open string tachyon [32]. There are also exact noncommutative solitons in this limit, some of these solutions were found in [33]. In [34] a very interesting physical interpretation of this limit was given in terms of a lattice discretization of ordinary string worldsheet. From the perspective of the worldsheet theory, we cannot forget the nature of p as a prime number, thus the analysis of the limit is more subtle. The correct way of taking the limit $p \to 1$ involves the introduction of finite extensions of the p-adic field $\mathbb{Q}_p$ and it was developed in [35] using the theory of topological zeta functions due to Denef and Loeser [36, 37]. The totally ramified extensions gives rise to a finer discretization of the worldsheet following the rules of the renormalization group [34].

String theory is a very strong candidate for a quantum theory of gravity. Its non-perturbative formulation known as AdS/CFT correspondence has been successfully applied to many systems of gravity and field theory [38]. Among its most remarkable applications is that of describing the quantum properties of black holes constructed from D-brane configurations. This feature is achieved by counting the corresponding open-string states of the supersymmetric brane configurations. There are other phenomena also described in terms of brane configurations as the loss of information and the emergence of spacetime itself from entangled states in the dual conformal field theory. Some of these considerations are in an early stage and it is observed that in the standard correspondence is quite difficult to find some progress due to the complicated nature of the calculations. Thus, some simpler models that capture some essential features of the AdS/CFT correspondence are very important to be explored in order to achieve some progress. Recently a model was proposed in [39, 40], regarding a p-adic version of AdS/CFT correspondence. A considerable amount of work has been performed in this context [39, 40, 41, 42, 43]. These p-adic models capture the essential features of the usual correspondence but the computations are often much simpler. This is a common feature of p-adic models. In this p-adic AdS/CFT, the Bruhat-Tits tree plays the role of the bulk, where the spatial dimensions are played by the degree of an unramified extension of $\mathbb{Q}_p$. The boundary field theory is given in terms of a p-adic CFT on the line proposed several years ago [44].

Furthermore, ultrametric spaces have also appeared in models of complex systems. A central paradigm in the physics of certain complex systems (for instance proteins), asserts that the dynamics of such systems can be modeled as a random walk over the leaves of a rooted tree. This tree is a finite ultrametric space constructed out of the energy landscape. Mean-field approximations of these models drive naturally to models involving p-adic numbers, see e.g. [9, 45, 46, 47], and the references therein.

This thesis is based on the following works [48, 49, 50, 27]. The work [27] is a survey of the works done by the authors about the articles that use local zeta function techniques to study string amplitudes. Some further questions are asked as well as stating clearly some conjectures on these topics. The emphasis is made on examples rather than technical results.
In [48] we establish rigorously the regularization of the p-adic open string amplitudes, with Chan-Paton rules and a constant B-field, introduced by Ghoshal and Kawano in [51]. In this study we use techniques of multivariate local zeta functions depending on multiplicative characters and a phase factor which involves an antisymmetric bilinear form. These local zeta functions

are new mathematical objects. In ordinary string theory the effective action for bosonic open strings in gauge field backgrounds was discussed many years ago in [52]. The analysis incorporating a Neveu-Schwarz B field in the target space leads to a noncommutative effective gauge theory on the world-volume of D-branes [53]. The study of the p-adic open string tree amplitudes including Chan-Paton factors was started in [13]. However the incorporation of a B-field in the p-adic context and the computation of the tree level string amplitudes was discussed in [51, 54]. In these works it was reported that the tree-level string amplitudes are affected by a noncommutative factor. In [51] Ghoshal and Kawano introduced new amplitudes involving multiplicative characters and a noncommutative factor. These amplitudes coincide with the ones obtained directly from the noncommutative effective action [33].

In [49] we propose a worldsheet action containing bosons and fermions on the p-adic worldsheet projected on the boundary. We will show that this action is supersymmetric and thus might be considered as a p-adic analogue of the worldsheet superstring action in the superconformal gauge [55]. Moreover we will show that this action can be rewritten as an action in a p-adic version of the ordinary superspace. Furthermore we compute the tree-level N-point open string amplitudes of this superstring action and we obtain the corresponding Koba-Nielsen formula of the well known amplitudes in the NSR formalism [55, 56]. This is carried out explicitly by performing the path integration of this superstring action with N tachyonic vertex operators in the spirit of Refs. [20, 51]. We obtain the amplitudes previously found in Refs. [57, 58].

In the past, direct analogues for the 4-point superstring amplitudes were considered in [28, 18]. We find previous proposals of supersymmetry in the p-adic context in [59, 60]. In order to add fermions in the p-adic string amplitudes we require of extending the non-Archimedean formalism to include Grassmann numbers. Some further developments of the formalism were carried out in [61, 62].

In the context of p-adic AdS/CFT some results concerning the study of non-Archimedean versions of fermionic systems as SYK melonic theories were obtained in [63]. Furthermore, a way of introducing the spin was proposed in Ref. [64]. Motivated in part by these works, in [65] was studied the fermionic field theory on the Bruhat-Tits tree and its effective action on the boundary.

In [50] we establish a non obvious exact connection between two works in the context of the p-adic AdS/CFT. On the one hand in [39] a solution to the bulk equation of motion is found in terms of a p-adic bulk-to-boundary propagator. The bulk action is of a massive scalar field and the infinite Bruhat-Tits tree is built for $\mathbb{Q}_q$, where $q = p^d$ and d is the degree of the unramified extension of $\mathbb{Q}_p$. In the p-adic context, d is interpreted as the bulk dimension. A coordinate system is introduced similar to the Arhchimedean case that allows to write physical quantities in a form that closely resembles the Archimedean results. On the other hand, the work [66] constructs a solution to a bulk field in terms of its value at the boundary of the tree. The tree is considered finite, meaning that there is are both UV and IR cutoffs. The bulk action is of a

massless scalar field and the tree is over $\mathbb{Q}_p$, interpreted as a 2 dimensional bulk.
Our work generalizes both works and shows that they are intimately connected. First we generalize the construction in [66] to the massive case and in d dimensions (with the extended field $\mathbb{Q}_q$). Then we implement the coordinate system used in [39] and remove the cutoffs. We see that the solution constructed is exactly the one using the p-adic bulk-to-boundary propagator. Therefore our generalization with the UV and IR cutoffs is a coarse-grained version of this, including both UV and IR boundary conditions. Using the cutoff version of the action, one can see its behavior as we go to the boundary. Just like in the real case, there are divergencies that need to be renormalized in order to have a well defined boundary action. These divergencies are removed by adding a counterterm action. This process is known as holographic renormalization [67, 68]. We do this process in the p-adic case. Unlike the Archimedean case, the number of divergent terms is fixed and does not depend on the conformal dimension Δ.

This thesis is organized as follows. The chapter 2 reviews the necessary mathematical background. The p-adic numbers and analysis are introduced in a formal way. We present a brief account on the theory of local zeta functions, and a short survey on the use of Grassmann variables is also shown. The next chapter 3 introduces the physical context of string theory focusing on scattering amplitudes. We show the techniques to obtain them in different contexts of string theories. The chapter 4 reviews some of the earlier results on p-adic string theory. Then in 5 we review the results obtained in [48]. In 6 we review the results of [49]. The section 7 reviews a connection between string amplitudes and local zeta functions, and presents the regularization of string amplitudes over local fields from [27]. The chapter 8 contains a short review of the AdS/CFT correspondence in the Archimedean and non-Archimedean setting, then shows the results obtained in [50]. Finally in 9 we give the conclusions of this thesis.

Part I

Background

Chapter 2

Mathematical Background

2.1 Introduction

So you are probably wondering what exactly are p-adic numbers and how can they capture the essence of spacetime in Plank scale. This chapter will introduce them from a more mathematical aspect, as they are best explained in that way. There is an effort to give some intuitive ideas behind the concepts to make it more accessible.

First a small tale about how one must understand the real numbers mathematically. Not that they've been taught wrong to us, but we've gotten so used to them that we probably don't realize just how strange they are. One easily imagines how the rational numbers came to be: the need for an abstract number emerged, and the first step is to count things. Suddenly, the natural numbers arise naturally (obviously). As one gets familiar with counting things, one memorizes certain "patterns", for instance if you count 5 trees, and a little bit later encounter 3 more trees, you would need the number 8 to account for all of them. There must be a relation between 5, 3 and 8. You define a new rule for counting faster, called addition and you conclude that $3 + 5 = 8$. Therefore addition is nothing more than abbreviated counting. An interesting thing is that learning to sum will lead you to questions like "What should I add to 8 to get to 12?", that leads us to the idea of an inverse of addition, subtraction. But this leads to a problem, if we can subtract numbers, what happens if we subtract something from 1? and we discover (yes, discover, I'm a Platonist) the 0 and negative numbers. In the same way we come up with the idea of multiplying as an abbreviated sum, and demanding to have the inverse, we learn how to divide, that leads to the discovery of rational numbers.

As one gets more abstract, algebra emerges and with it, algebraic equations. This leads to the discovery of irrational numbers like $\sqrt{2}$ and all algebraic numbers (solutions to polynomial equations with rational coefficients). Are there other numbers?, yes! transcendental numbers, that are all non algebraic numbers, how convenient. These are numbers like π and e, that have unusual definitions. But, how do we find the rest of them if there are any?, what lies outside algebraic numbers? To answer these questions we need a more systematic definition of

a number, like simply a sequence of digits. This idea is what is behind the definition or the real numbers. We need to understand this in order to understand how p-adic numbers emerge.
There is a lot of literature that introduces p-adic numbers and analysis varying in approaches and the audience they are intended. Personally, I recommend [7] as it is a comprehensive study of the p-adics and its applications to physics with many examples. For a more mathematical approach the reader may see [69, 70, 71], and for an approach friendlier to physicists one may see [18, 13, 39, 40], as well as many modern articles on p-adic physics that contain a short introduction to p-adic numbers, for instance see the appendices of [51, 48, 49].

2.2 Definitions

The field of p-adic numbers is denoted by $\mathbb{Q}_p$, and it is defined as the completion of the rational numbers $\mathbb{Q}$ with respect to the p-adic norm. What is the p-adic norm you ask? well, to answer that consider a rational number $r \in \mathbb{Q}$. From the fundamental theorem of arithmetic we know that it has a unique factorization of integer powers of prime numbers. Therefore we can write

$$r = \prod_{i=1}^{N} p_i^{n_i} = \prod_{i=1}^{\infty} p_i^{n_i}, \tag{2.1}$$

where p_i is the ith prime number and $n_i \in \mathbb{Z}$. It probably puzzles you that the product goes to infinity in the primes. The key is that the exponents are any integers, including 0. Then all of the prime numbers that do not appear in the factorization of r would have an exponent $n_i = 0$ in (2.1). Also notice that n_i can be negative.
To define the p-adic norm, one selects a p_i, and define the following

$$|r|_{p_i} = \begin{cases} {p_i}^{-n_i}, & r \neq 0 \\ 0, & r = 0. \end{cases} \tag{2.2}$$

Technically this is the p_i-adic norm, and I want to emphasize here that the norm depends on the prime that we choose, hence the subscript i. However it is customary to simply use p to stand for any fixed prime number, one doesn't change it in the middle of the discussion. Therefore the p-adic norm will be denoted $|\cdot|_p$ henceforth.
Let's look at some examples, take $r = \frac{45}{64} = \frac{9\cdot5}{8\cdot8} = 2^{-6}\cdot 3^2 \cdot 5$, and let $p = 2$, then $|r|_2 = 2^6 = 64$. Now consider $p = 7$ and $r = 100$, then $|100|_7 = 1$, and finally let $p = 11$ and $r = 11^{10}$, then $|11^{10}|_{11} = 11^{-10}$.
The p-adic norm has a property called *ultrametricity*, a stronger version of the triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}. \tag{2.3}$$

We can see this easily by considering two rational numbers $r_1 = p^{n_1}\bar{r}_1,\ r_2 = p^{n_2}\bar{r}_2$, with $\bar{r}_i$ being coprime to p, without loss of generality consider $n_1 \geq n_2$. Then if $n_1 \neq n_2$ we will have

$|r_1 + r_2|_p = p^{-n_2}|p^{n_1-n_2}\bar{r}_1 + \bar{r}_2|_p = p^{-n_2} = |r_2|_p$. The last equality is because $p^{n_1-n_2}\bar{r}_1 + \bar{r}_2$ cannot be a multiple of p since $\bar{r}_2$ is coprime to p. On the other hand if $n_1 = n_2 = n$, then $|r_1 + r_2|_p = p^{-n}|\bar{r}_1 + \bar{r}_2|_p$. The denominator of $\bar{r}_1 + \bar{r}_2$ cannot be a multiple of p, however the numerator can, but in that case we would get a factor of $|p^k|_p < 1$. This proofs the ultrametric property.

2.2.1 p-adic Numbers

To get to $\mathbb{Q}_p$ we start from $\mathbb{Q}$, that is topologically incomplete. Sequences of rational numbers exist that do not converge to a rational number (think of subsequently adding all of the digits of π to 3.14). One then 'fills in the gaps" by inserting the limits of such sequences, calles Cauchy sequences. This process requires the notion of convergence, that requires a norm, any norm. To make it clear we first define what is a norm.

Definition 2.2.1 (Norm)**.** A norm $|\cdot| : K \to \mathbb{R}$ is a mapping from a number field K to a nonnegative real number that obeys the following

- $|x| \geq 0$;
- $|x||y| = |xy|$;
- $|x| = 0 \Leftrightarrow x = 0$.

There are infinite different norms, however most of them are equivalent to each other. There is one important theorem that tells the difference

Theorem 2.2.1 (Ostrowski)**.** *Every possible norm is equivalent to either the absolute value or a p-adic norm.*

The reader may see, for instance, the Theorem 1 in Chapter 1 of [70] for a proof. This theorem tells us that the only possible ways to complete $\mathbb{Q}$ is to either $\mathbb{R}$ or $\mathbb{Q}_p$ for some p. In some way this also serves as justification to investigate physical theories over $\mathbb{Q}_p$. From an abstract perspective one may wonder about exploring all the possible mathematical descriptions of our universe.
Consider a sequence a_n of rational numbers. We will call a sequence Cauchy if for any norm $|\cdot|$ the following is true

$$\lim_{m,n\to\infty} |x_m - x_n| = 0. \tag{2.4}$$

Then we can assert that the sequence has a limit, let's call it x. We also say that a number field is complete if all the limits of Cauchy sequences remain within the field. We point out that the field of rational numbers is not a complete field. One may complete it by adding all the limits x of Cauchy sequences, it is in this sense that one "fills in the gaps" of the field. If we use the absolute value in (2.4) to complete $\mathbb{Q}$, we get $\mathbb{R}$; if instead we use the p-adic norm, we get $\mathbb{Q}_p$. Any p-adic number $x \in \mathbb{Q}_p$ has a unique representation as a series in powers of p as

$$x = p^{v(x)} x_0 + p^{v(x)+1} x_1 + \cdots = p^{v(x)} \sum_{n=0}^{\infty} x_n p^n, \tag{2.5}$$

where $v(x) \in \mathbb{Z}$ is the *order* or *valuation* of x, and $x_n \in \{1, 2, \ldots, p-1\}$ with $x_0 \neq 0$. Now we extend the definition of the p-adic norm to a number $x \in \mathbb{Q}_p$

$$|x|_p = \begin{cases} p^{-v(x)}, & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{2.6}$$

For the unfamiliar reader, the series (2.5) may seem divergent, and indeed would be, were we working with real numbers. However recall that the p-adic norm flips the sign of the valuation in the exponent. Therefore the series is actually convergent p-*adically*. One eventually gets used to this unfamiliar intuitions.

2.2.2 Topology and $\mathbb{Q}_p^n$

There are many unusual properties about $\mathbb{Q}_p$ that we are going to discuss. But first we need to set some common notation for subsets of $\mathbb{Q}_p$. The unit ball and the unit circle centered around zero are respectively denoted by

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p; |x|_p \leq 1\}, \quad \mathbb{Z}_p^\times := \{x \in \mathbb{Q}_p; |x|_p = 1\}. \tag{2.7}$$

$\mathbb{Z}_p$ constitutes an abelian ring, and the unit circle is also called the set of units of $\mathbb{Z}_p$, that is the largest multiplicative group contained within $\mathbb{Z}_p$ (hence the superscript $\times$). This is easily proven because if there existed an element $x \in \mathbb{Z}_p^\times$ with $|x|_p < 1$, then we should also include its multiplicative inverse x^{-1}, that must have $|x^{-1}|_p > 1$ and therefore would be outside $\mathbb{Z}_p$.
A general ball has a center $a \in \mathbb{Q}_p$ and a radius p^r with $r \in \mathbb{Z}$, and is denoted by

$$B_r(a) := \{x \in \mathbb{Q}_p;\ |x-a|_p \leq p^r\} = a + p^{-r}\mathbb{Z}_p. \tag{2.8}$$

An interesting and counterintuitive property of p-adic balls is that all of its points are its center, that is $\forall\, b \in B_r(a)$ we have $B_r(a) = B_r(b)$. Another unusual property is that for any pair of balls $B_{r_1}(a_1)$, $B_{r_2}(a_2)$, either they are disjoint $B_{r_1}(a_1) \cap B_{r_2}(a_2) = \emptyset$, or one is contained inside the other $B_{r_1}(a_1) \subseteq B_{r_2}(a_2)$. Also notice that we defined the ball with the symbol of less or equal '$\leq$' instead of just less than '$<$'. This is because the spectrum of the p-adic norm is discrete and we actually have that

$$\{x \in \mathbb{Q}_p;\ |x-a|_p < p^r\} = \{x \in \mathbb{Q}_p;\ |x-a|_p \leq p^{r-1}\} = B_{r-1}(a).$$

Likewise we can define a general circle with radius p^r centered around a

$$S_r(a) := \{x \in \mathbb{Q}_p;\ |x-a|_p = p^r\} = a + p^{-r}\mathbb{Z}_p^\times. \tag{2.9}$$

We declare the balls as open subsets. Then one shows that the circles are also open subsets. Furthermore it can be proven then that these sets are also closed, that makes the balls and circles *clopen*, as it is sometimes called. This is already very different from the real numbers. A topological space X is called *disconnected* if it can be represented as a union of two disjoint nonempty open subsets, otherwise X is called *connected*. A subset $A \subseteq X$ is called disconnected if it can be represented as

$$A = (Y_1 \cap A) \bigsqcup (Y_2 \cap A),$$

where Y_1, Y_2 are nonempty open subsets of X with $Y_1 \cap A \neq \emptyset$, $Y_2 \cap A \neq \emptyset$, and $\bigsqcup$ denotes the disjoint union. We call X *totally disconnected* if the only connected sets are the empty set and the individual points $a \in X$. One can show that $\mathbb{Q}_p$, as a topological space with balls as open sets, is totally disconnected. It is also locally compact, meaning that every point in $\mathbb{Q}_p$ has a compact neighborhood. A compact subset of $\mathbb{Q}_p$ is compact if and only it is closed and bounded. Furthermore $\mathbb{Q}_p$ is a fractal, meaning that it is homeomorphic to a Cantor-like set of $\mathbb{R}$.

We extend the $p-$adic norm to $\mathbb{Q}_p^n$ by taking

$$||\boldsymbol{x}||_p := \max_{1\le i\le n} |x_i|_p, \text{ for } \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

We define $v(\boldsymbol{x}) = \min_{1\le i\le n}\{v(x_i)\}$, then $||\boldsymbol{x}||_p = p^{-v(\boldsymbol{x})}$. The metric space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a separable complete ultrametric space (here, separable means that $\mathbb{Q}_p^n$ contains a countable dense subset, which is $\mathbb{Q}^n$). p-Adic balls in multiple dimensions are defined as expected

$$B_r(\boldsymbol{a})^n := \{\boldsymbol{x} \in \mathbb{Q}_p^n;\ ||\boldsymbol{x}-\boldsymbol{a}||_p \le p^r\}; \quad S_r(\boldsymbol{a})^n := \{\boldsymbol{x} \in \mathbb{Q}_p^n;\ ||\boldsymbol{x}-\boldsymbol{a}||_p = p^r\}.$$

n-Dimensional balls are the product of one-dimensional balls, $B_r^n(\boldsymbol{a}) = B_r(a_1) \times B_r(a_2) \times \cdots \times B_r(a_n)$ with $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$. However this doesn't happen for circles, $S_r^2(\boldsymbol{a}) \neq S_r(a_1) \times S_r(a_2)$. For example for the 2d unit circle $S_0^2(0) = (\mathbb{Z}_p^2)^\times$, from the definition one can see that in fact $(\mathbb{Z}_p^2)^\times = p\mathbb{Z}_p \times \mathbb{Z}_p^\times \sqcup \mathbb{Z}_p^\times \times p\mathbb{Z}_p \sqcup \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. As a topological space $(\mathbb{Q}_p^n, ||\cdot||_p)$ is totally disconnected. Two balls also are either disjoint or one is inside the other. Subsets are compact if and only if they are closed and bounded. It is also a locally compact topological space.

2.2.3 The sign function

For the usual real case the sign function is very simple and intuitive both because we learn it from an early age, and because indeed it is very simple, maybe the simplest non-trivial case (at least in the sense that we will discuss here). There are analytical ways to define it over $\mathbb{R}$, usually as a limit of a sequence of continuous functions that tend to assign -1 for $\mathbb{R} \ni x < 0$ and 1 if $x > 0$. Here we will define it in a more algebraic way using only two assumptions.

General Sign

Consider a number field K and its corresponding multiplicative group $K^\times$, in which we want to define a sign function. We demand two things of such function

Definition 2.2.2. A sign function (sgn) over a field K is a mapping, $\text{sgn} : K^\times \to \pm 1$, that has the property $\text{sgn}(xy) = \text{sgn}(x)\text{sgn}(y)$, i.e. it is multiplicative.

This simple definition is enough to obtain all the signs that we are going to discuss. There is one more concept to mention, and that is the equivalence of sign functions. We say that two sign functions are equivalent if they assign the same value to each number in the field, that is they are equivalent as functions over $K^\times$ point by point.
The multiplicative property is crucial since it will greatly reduce the possible different signs. Let us deduce a very important consequence of our definition; that all square numbers must have a positive sign.

Proof. Consider an element $x \in K^\times$ and a nontrivial sign function sgn. Then by the multiplicative property we have $\text{sgn}(x^2) = \text{sgn}(x)^2 = (\pm 1)^2 = 1$. □

This tells us that the elements in the field with a negative sign cannot have a square root within the field. It is remarkable that after this we are completely free to assign negative signs to the non quadratic elements of K, as long as we are consistent with the definition. This implies the following, consider a non-quadratic element $a \in K$, and the set $K^{*2} = \{z^2 \; ; \; z \in K\}$. Then all elements in the set aK^{*2} have the same sign. We can easily check this, $\text{sgn}(az^2) = \text{sgn}(a)\text{sgn}(z^2) = \text{sgn}(a)$. A field is partitioned by the sets $a_i K^{*2}$ with a_i a non square. Therefore the set K/K^{*2} will tell us how many different sign functions can be defined.
Let us understand the usual real sign function in this terms. We know that the real numbers smaller than zero do not have a real square root, therefore we can assign a negative sign to them. In this case the set $-1\mathbb{R}^{*2}$ already contains all the numbers smaller than zero. Therefore we are free to choose the signs of the elements in the set. If we choose $\text{sgn}(-1) = 1$ we have the trivial sign in which all the numbers have a positive sign. If we choose $\text{sgn}(-1) = -1$, we get the usual sign that we all know.

The Legendre symbol

For the case $K = \mathbb{F}_p$, the finite field with p elements, there is a special name for the sign function, it is the Legendre symbol, denoted by $\left(\frac{n}{p}\right)$ where $n \in \mathbb{F}_p$. It is defined as

$$\left(\frac{n}{p}\right) = \begin{cases} 1, & \text{if } x^2 \equiv n \bmod p \text{ has a solution;} \\ -1, & \text{otherwise.} \end{cases}$$

This symbol has the multiplicative property of course, it classifies the elements of $\mathbb{F}$ into those with a square root and those without it. This matches exactly our definition of the sign function, so we can truly consider the Legendre symbol as the sign function of $\left(\frac{n}{p}\right)$. It appears frequently in algebraic number theory and several interesting properties are known, like the quadratic reciprocity

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{(p-1)(q-1)/4},$$

for primes p and q. It is very useful in obtaining the Legendre symbol when one of the primes is much larger. A distinguished value is the sign of -1 in this context. It turns out that

$$\left(\frac{-1}{p}\right) = \left(\frac{p-1}{p}\right) = (-1)^{\frac{p-1}{2}} \tag{2.10}$$

This means that in $\mathbb{F}_p$, -1 only has a negative sign if $p \equiv 3 \bmod 4$. Otherwise we would have $\left(\frac{-1}{p}\right) = 1$. This really is counterintuitive, however it is often the case that our intuition betrays us when we dig deep enough. Just to not leave it all very abstract, let us make a couple of examples. We will see $p = 11$ and $p = 13$, and we'll see which numbers are squares. We summarize this in the table 2.1. As we can see, in $\mathbb{F}_{11}$ the numbers with positive sign are

Table 2.1: Examples of quadratic residues for the cases $p = 5, 7, 11, 13$.

x	x^2 mod 5	x^2 mod 7	x^2 mod 11	x^2 mod 13
1	1	1	1	1
2	4	4	4	4
3	4	2	9	9
4	1	2	5	3
5	-	4	3	12
6	-	1	3	10
7	-	-	5	10
8	-	-	9	12
9	-	-	4	3
10	-	-	1	9
11	-	-	-	4
12	-	-	-	1

$1, 3, 4, 5, 9$, and the ones to which we can assign a negative sign are $2, 6, 7, 8, 10$. From the table we can also see that it is enough to look at the first $(p-1)/2$ squares, because if we have $p - n \equiv -n \bmod p$, and $(-n)^2 \equiv n^2 \bmod p$. Also notice that for $\mathbb{F}_{11}$, $-1 \equiv 10$, and has a negative sign, but for $\mathbb{F}_{13}$, $-1 \equiv 12 \equiv 5^2$. So -1 is a square in $\mathbb{F}_{13}$!
Given that all squares in $\mathbb{F}_p^\times$ are determined by half of its elements, there is always $\frac{p-1}{2}$ elements with positive Legendre symbol and $\frac{p-1}{2}$ elements with negative Legendre symbol.

p-adic sign

On to the main point of this section. The sign function in $\mathbb{Q}_p$ is not simple, because there are actually three inequivalent non-trivial sign functions that one can define. Like in the previous cases, we need to classify $\mathbb{Q}_p$ into squares and non squares, but here is the tricky part. The set of p-adic squares $\mathbb{Q}_p^{*2}$ is

$$\mathbb{Q}_p/\mathbb{Q}_p^{*2} = \{1, \epsilon, p, \epsilon p\}. \tag{2.11}$$

where ϵ is an integer that satisfies $\left(\frac{\epsilon}{p}\right) = -1$. This means that any p-adic number can be written uniquely as $x = a^2\tau$ for $a \in \mathbb{Q}_p$ and $\tau \in \{1, \epsilon, p, \epsilon p\}$. This naturally partitions $\mathbb{Q}_p$ into four sectors, three of which can have negative sign. Before designating the signs, let us briefly see why this partitions occurs. It can be shown that any square must have two properties, its norm must be an even power of p, and the first coefficient in the series (2.5) (the x_0) must satisfy $\left(\frac{x_o}{p}\right) = 1$. Clearly $|\epsilon|_p = 1$, then it alone cannot cover all of the nonsquares, then we turn to p, and we must add also their product to cover all $\mathbb{Q}_p$.
As mentioned before, once we have the partition represented in (2.11), we can freely assign negative signs, keeping care of consistency. In words, we can choose to have either $\text{sgn}(\epsilon) = -1$, or $\text{sgn}(p) = -1$, or both, notice that the sign of ϵp is automatically determined since $\text{sgn}(\epsilon p) = \text{sgn}(\epsilon)\text{sgn}(p)$. These choices are presented in the following table

Table 2.2: Non-trivial possibilities for the p-adic sign of the elements ϵ, p and ϵp. The notation is in line with the usual definition of the sign function over fields (See (2.12)).

$\text{sgn}(\epsilon)$	$\text{sgn}(p)$	$\text{sgn}(\epsilon p)$	Notation if $p \equiv 1 \bmod 4$	Notation if $p \equiv 3 \bmod 4$
-1	1	-1	$\text{sgn}_p(\cdot)$	$\text{sgn}_{\epsilon p}(\cdot)$
1	-1	-1	$\text{sgn}_\epsilon(\cdot)$	$\text{sgn}_\epsilon(\cdot)$
-1	-1	1	$\text{sgn}_{\epsilon p}(\cdot)$	$\text{sgn}_p(\cdot)$

The notation for sgn_τ may seem arbitrary, but it comes from the classical definition of the sign function that one finds in the textbooks [7]. For completeness and in benefit of the reader we present it here. The sign function over a field K is defined as

$$\text{sgn}_\tau(x) = \begin{cases} 1, & \text{if } x = a^2 - \tau y^2 \text{ for some } a, b \in K; \\ -1, & \text{otherwise.} \end{cases} \tag{2.12}$$

Here $\tau \in \mathbb{Q}_p$ is a non-square. Personally I find this definition rather hard to grasp, although it is perfectly sensible. In this case, it is straightforward to see that for two different numbers τ, τ', their respective sign will be equivalent if and only if $\tau = x^2\tau'$, for some x. One advantage of this definition is that one may construct a sign function out of any number in the field, but one quickly realizes that the different signs will correspond to the classification presented above regarding the squares of the field.

One important feature is the differentiation between primes congruent to 1 or 3 mod 4. This is due to (2.10), that implies $-1 \equiv x^2$ if $\frac{p-1}{2}$ is even, or equivalently $p \equiv 1 \mod 4$. This impacts the definition in (2.12), since in that case we can ignore the negative sign in the condition.

One important feature of sgn_τ is that it is a locally constant function. In other words that it can be written as a linear combination of characteristic functions. In particular, we can known if a p-adic number is a square based only on the parity of its order and if the first coefficient x_0 is a quadratic residue or not. The sign depends only on the order and the first coefficient. Therefore we have

$$\mathrm{sgn}_\tau(x \pm y) = \mathrm{sgn}_\tau(x) \; \forall \; y; \; |y|_p < |x|_p. \tag{2.13}$$

In the rest of this work we will be using the notation presented in table 2.2, because it is the most common notation and definition. Ultimately we are interested in the cases in which the sign function is antisymmetric, that is when we have $\mathrm{sgn}_\tau(-1) = -1$. This happens when $\tau \neq \epsilon$ and $p \equiv 3 \mod 4$, or equivalently when $\left(\frac{-1}{p}\right) = -1$ and $\mathrm{sgn}(\epsilon) = -1$, because in this case we can choose $\epsilon = -1$. Another very useful result from [64] is the following table that gives explicit expressions for the sign function

Table 2.3: Explicit expressions for evaluating the sign function in different cases [64]. v_x is the valuation of the number x, or its lowest power of p in its series expansion, and x_0 is the coefficient of said power. (See (2.5)).

$p \equiv 1 \mod 4$	$p \equiv 3 \mod 4$
$\mathrm{sgn}_\epsilon(x) = (-1)^{v(x)}$	$\mathrm{sgn}_\epsilon(x) = (-1)^{v(x)}$
$\mathrm{sgn}_p(x) = \left(\frac{x_0}{p}\right)$	$\mathrm{sgn}_p(x) = (-1)^{v(x)} \left(\frac{x_0}{p}\right)$
$\mathrm{sgn}_{\epsilon p}(x) = (-1)^{v(x)} \left(\frac{x_0}{p}\right)$	$\mathrm{sgn}_{\epsilon p}(x) = \left(\frac{x_0}{p}\right)$.

2.2.4 Algebraic extensions

Let us discuss algebraic extensions of number fields. One may see [7, Section 1.4] for a quick introduction to quadratic extensions or [39, Section 2.2] for a more general discussion aimed to physicists. For a more mathematical approach see, for instance [72, Chapters 4-6]. Consider a field $\mathbb{K}$ that is not algebraically closed, meaning there are algebraic equations with coefficients in $\mathbb{K}$ whose solutions are not in $\mathbb{K}$. An algebraic extension of a number field results from adding a *new* element that satisfies an algebraic equation with coefficient in the field. They are added through linear combinations. The most familiar example are complex numbers, where an element of $\mathbb{C}$ is of the form $x + iy$ with $i = \sqrt{-1}$ the imaginary unit that is a solution to the equation $x^2 + 1 = 0$. i is not in $\mathbb{R}$ and since it is a square root, $\mathbb{C}$ is a quadratic extension of $\mathbb{R}$. More generally an algebraic extension of degree n of a field $\mathbb{K}$ is a linear combination $a_0+\tau a_1+\tau^2 a_2+\ldots\tau^{n-1}a_{n-1}$, where $\tau^n\in\mathbb{K}$. It is common to denote the extended field as $\mathbb{K}(\tau)$ and its degree as $n=[\mathbb{K}(\tau):\mathbb{K}]$. Now of course we can add more than one number whose root is not in $\mathbb{K}$. In that case the degree of the extension is the product of the extensions associated to each number. For example consider the field of rational numbers $\mathbb{K}=\mathbb{Q}$, and the extension by adding $\sqrt{2}$ and $\sqrt[3]{3}$. We can denote this field as $\mathbb{Q}(\sqrt{2},\sqrt[3]{3})$, and an element would be of the form

$$\mathbb{Q}(\sqrt{2},\sqrt[3]{3})\ni z=a+b\sqrt{2}+c\sqrt[3]{3}+d\sqrt[3]{3^2}+e\sqrt{6}+f\sqrt{18};\ a,b,c,d,e,f\in\mathbb{Q}\}$$
$$[\mathbb{Q}(\sqrt{2},\sqrt[3]{3}):\mathbb{Q}]=2\cdot 3=6.$$

Another famous example are the Gaussian integers $\mathbb{Z}(i)$, the extension of $\mathbb{Z}$ by the imaginary unit i. A thing to notice is that there are multiple ways to have an extension of a given degree n, it can be made of multiple single number extensions of degree n_i, the only condition is that $\prod_i n_i=n$. We now turn to the specific case of extensions of $\mathbb{Q}_p$. There are basically two places where we can extend $\mathbb{Q}_p$, one is in the residue field $\mathbb{F}_p=\mathbb{Z}_p/p\mathbb{Z}_p$, and the other is by using $p^{1/n}$ instead of p in the power series of a p-adic number. Extending only the residue field $\mathbb{F}_p$ is referred to as an *unramified extension*, and extending only by using $p^{1/n}$ is known as a *totally ramified extension*. A combination of both is simply called a *ramified extension*. We denote by f the degree of extension of the residue field $\mathbb{F}_{p^f}$, and by e the degree of extension by using $p^{1/e}$, e is also known as the ramification index. In a ramified extension the total degree is $e\cdot f=n$, a totally ramified extension has $e=n$ and $f=1$, and an unramified extension is characterized by $e=1$ and $f=n$. The elements of the extended p-adic field K, have an expansion of the form

$$z=p^{v(z)/e}\sum_{i=0}^{\infty}x_i p^{i/e},\quad x_i\in\mathbb{F}_{p^f}. \tag{2.14}$$

One can use something different than $p^{i/e}$, in general it is called a uniformizer and usually denoted by π, the condition is that the norm of the uniformizer is $p^{-1/e}$ and the expansion (2.14) is unique. Notice that now the valuation $v(z)$ is still an integer number. The norm of these numbers is easy to see intuitively from the expansion (2.14), it is now an integer power of $p^{1/e}$ and equal to $p^{-v(z)/e}$. This norm can be defined properly using the associated field norm $N(z)$. Consider an extension K of degree n of $\mathbb{Q}_p$, then K can be represented as an n-dimensional vector space over $\mathbb{Q}_p$. Consider two elements $z,w\in K$, then the map $z\to wz$ can be represented as a linear map over $\mathbb{Q}_p^n$. As such we can obtain its determinant $N(w)$, this is precisely the associated field norm. Then we define the norm over K as

$$|z|_K=|N(z)|_p^{1/n}. \tag{2.15}$$

This is the unique norm on K that satisfies $|z|_K=|z|_p$ for any $x\in\mathbb{Q}_p$[1]. In the recent work [39], the unramified extension of $\mathbb{Q}_p$ is used to construct the Bruhat-Tits tree that is analogue

[1]The power $\frac{1}{n}$ in (2.15) is not standard and one can define the norm without it. In fact, later for the p-adic closed string in section 4.1.2, we will use the norm without this power as it is closer to the Archimedean case.

to AdS_n, and is the type of extension that interest us. In this case $f = n$ and the extension is made completely in the residue field. This means that the spectrum of the norm remains the same. It is usual to denote the unramified extension by $\mathbb{Q}_{p^n} = \mathbb{Q}_q$, with $q = p^n$. We also denote the norm for the unramified extension as $|\cdot|_q$. The relevant sets for $\mathbb{Q}_q$ are denoted in a similar fashion. The unit ball is denoted by $\mathbb{Z}_q$ and the unit circle by $\mathbb{Z}_q^\times$. Multiplying these sets times powers of p gives us the balls and circles of different radii.

2.2.5 Integration

In this section I will tell you a little bit about how does one integrate functions over the p-adics. If you know a little bit of measure theory, this should not surprise you. However if you don't, it is going to be quite different from what you are probably used to. You see, thanks to the fundamental theorem of calculus, and because that is how we mostly are taught to deal with it, we see integrating as the inverse operation of differentiating. Although this isn't emphasized enough (in my humble opinion), this is very impressive, since the definition of an integral comes from the idea of Riemann summation. This is very specific to the real numbers, and complex. In physics we need to integrate in different contexts, such as using Grassmann variables in Berezin integration, with its very unusual rules that honestly have little to do with summing of functions. Or the infamous Feynman path integral, in which the only general rule we can give, with some degree of rigor, is Gaussian path integration.
As you will see, integrating over $\mathbb{Q}_p$ boils down to splitting the region of integration into sufficiently small sets, in each of which the function to integrate becomes constant. Doing this, integrating becomes a weighted sum of measures of sets.

Single variable

Let us begin with some formalities for the case of one variable of integration. $\mathbb{Q}_p$ is a field, and as such it is an additive group $(\mathbb{Q}_p, +)$, a locally compact group in fact. This guarantees that it has a measure known as a Haar measure, commonly denoted by dx. The group $(\mathbb{Q}_p, \cdot)$ is also a locally compact group and has a Haar measure $d^*x = dx/|x|_p$. We will use only the Haar measure corresponding to the additive group. Notice that since they have a relation, one can switch between measures as needed. The measure dx is translationally invariant, $d(x+a) = dx$, and must be normalized to make it unique. Conventionally we give the unit ball a measure of 1. Therefore

$$\int_{\mathbb{Z}_p} dx = 1 \tag{2.16}$$

However it is more relevant to us using the definition shown in the main text. This is because it is the one used for AdS/CFT, that represents a larger portion of this thesis.

Now there are two ways in which I can show you the values of the different sets. The first is more direct but the second is honestly funner. Let us first deduce the measure of the unit circle. Relations between sets translate nicely into rules for integrating the sets. For example, integrating over the union of two disjoint sets $A \sqcup B$, is equal to adding the integration over the sets separately

$$\int_{A\sqcup B} dx = \int_A dx + \int_B dx.$$

Similarly, integration over the subtraction of sets $A\backslash B$ is equal to the subtraction of the integrals

$$\int_{A\backslash B} dx = \int_A dx - \int_B dx.$$

We need one more ingredient, the change of the measure under a change of variable. I will present the general rule later, for the moment it is sufficient to know that $d(ax) = |a|_p dx$[2], and $d(ax+b) = |a|_p dx$. Then we can obtain the measure of any ball. Consider the integral

$$\int_{B_r(a)} dx = \int_{a+p^{-r}\mathbb{Z}_p} dx,$$

using the change $x = a + p^{-r}y$ with $y \in \mathbb{Z}_p$ we get

$$\int_{\mathbb{Z}_p} |p^{-r}|_p dy = p^r \int_{\mathbb{Z}_p} dy = p^r.$$

This means that any ball has its radius as its measure. Now we can obtain the measure of the unit circle, that can be written as $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus p\mathbb{Z}_p$. Then

$$\int_{\mathbb{Z}_p^\times} dx = \int_{\mathbb{Z}_p} dx - \int_{p\mathbb{Z}_p} dx = 1 - p^{-1}.$$

More generally for the circle of radius p^r we have

$$\int_{p^{-r}\mathbb{Z}_p^\times} dx = \int_{p^{-r}\mathbb{Z}_p} dx - \int_{p^{-r+1}\mathbb{Z}_p} dx = p^r - p^{(r-1)} = (1-p^{-1})p^r.$$

Let's do a quick consistency check. A ball of radius p^r is made out of all circles with a radius smaller and equal to p^r. This means that $p^{-r}\mathbb{Z}_p = \bigsqcup_{s=0}^\infty p^{-r+s}\mathbb{Z}_p^\times$, and this implies

$$\int_{p^{-r}\mathbb{Z}_p} dx = \sum_{s=0}^\infty \int_{p^{-r+s}\mathbb{Z}_p^\times} dx = \sum_{s=0}^\infty (1-p^{-1})p^{r-s} = (1-p^{-1})p^r \frac{1}{1-p^{-1}} = p^r.$$

This is consistent with the previous result. I will now present the other way of obtaining the measure of a circle and a ball, this method only uses the translational invariance of the measure and the normalization (2.16), and it is taken from [13]. The unit ball may also be decomposed into p balls $\mathbb{Z}_p = \bigsqcup_{a=0}^{p-1} a + p\mathbb{Z}_p$. Then

$$1 = \int_{\mathbb{Z}_p} dx = \sum_{a=0}^{p-1} \int_{a+p\mathbb{Z}_p} dy.$$

[2]This is easily verified using that $d^*(ax) = d^*x \Rightarrow d(ax)/|ax|_p = dx/|x|_p$.

Since $d(a+y) = dy$ every term in the right hand side is equal, there are p of them, therefore

$$1 = p \int_{p\mathbb{Z}_p} dy \Rightarrow \int_{p\mathbb{Z}_p} dy = p^{-1}.$$

And there you have it! It is a matter of applying this subsequently, in fact we can use induction. The induction hypothesis is $\int_{p^n\mathbb{Z}_p} = p^{-n}$. We just showed it for $n = 1$, and now we show it for $n+1$ assuming it is true for n. Consider that $p^n\mathbb{Z}_p = \bigsqcup_{a=0}^{p-1} ap^n + p^{n+1}\mathbb{Z}_p$, and $d(ap^n + y) = dy$. Then

$$p^{-n} = \int_{p^n\mathbb{Z}_p} dx = p \int_{p^{n+1}\mathbb{Z}_p} dx \Rightarrow \int_{p^{n+1}\mathbb{Z}_p} dy = p^{-(n+1)}.$$

This works in the other direction too, by proving it for $n-1$, recovering the entire spectrum. With this we obtain the same result without resorting to measure theory regarding changes of variable. Of course there is no need, but as you may tell, this second method comes from physicists learning new maths, and us physicists like to get the most out of the least possible. I consider this view more intuitive and entertaining, in fact this is how I first learned p-adic integration. I think both sides can benefit from the other.

Let's compute the basic example of integrating a function. This is integrating the p-adic norm to a certain power. The method is the same, there will simply be an additional factor in each term in the sum. Right away we do it generally for any ball of radius p^r.

$$\begin{aligned} \int_{p^{-r}\mathbb{Z}_p} |x|_p^u dx &= \sum_{s=0}^{\infty} \int_{p^{-r+s}\mathbb{Z}_p^\times} |x|_p^u dx = \sum_{s=0}^{\infty} p^{u(r-s)} \int_{p^{-r+s}\mathbb{Z}_p^\times} dx \\ &= \sum_{s=0}^{\infty} p^{u(r-s)}(1-p^{-1})p^{r-s} = (1-p^{-1}) \sum_{s=0}^{\infty} p^{r(u+1)} p^{-s(u+1)} = \frac{(1-p^{-1})p^{r(u+1)}}{1-p^{-(u+1)}}, \end{aligned} \tag{2.17}$$

where $u \in \mathbb{C}$. We can be more general and state this result for a so called radial function $f(|x|_p)$, a function that depends on the norm only. Then

$$\int_{p^{-r}\mathbb{Z}_p} f(|x|_p) dx = (1-p^{-1}) \sum_{s=-\infty}^{r} f(p^s)p^s. \tag{2.18}$$

The sum has changed a little bit, but hopefully you can easily see that it is equivalent.

One more interesting example is integrating the sign function. As we saw in the last section, the p-adic sign function is quite tricky. There are three different possibilities and it possesses no relation with the additive inverse in general, which makes it lack intuition.

Nevertheless, one can integrate this function in the same way relying on (2.13). Consider the following example

$$\begin{aligned} \int_{\mathbb{Z}_p^\times} \operatorname{sgn}_\tau(x) dx &= \sum_{a=1}^{p-1} \int_{a+p\mathbb{Z}_p} \operatorname{sgn}_\tau(x) dx = \sum_{a=1}^{p-1} \operatorname{sgn}_\tau(a) \int_{a+p\mathbb{Z}_p} dx \\ &= p^{-1} \sum_{a=1}^{p-1} \operatorname{sgn}_\tau(a) = \begin{cases} 1-p^{-1}, & \tau = \epsilon \\ 0, & \tau \neq \epsilon. \end{cases} \end{aligned}$$

To see this look at the table 2.2, in the unit circle for $\tau = \epsilon$ the sign is trivial, and for $\tau \neq \epsilon$ the sign is just the Legendre symbol. Recall that in the set $1, \ldots, p-1$ the number of elements with a positive and negative Legendre symbol are equal, hence the result presented. This behavior of the sign function is very typical, usually we have one result for $\tau = \epsilon$ and another for $\tau \neq \epsilon$. There is rarely a distinction between using $\tau = \epsilon p$ and $\tau = p$ for the integrals that we encounter. Now consider the integral

$$\int_{\mathbb{Z}_p} \mathrm{sgn}_\tau(x)dx = \sum_{s=0}^{\infty}\sum_{a=1}^{p-1}\int_{ap^s+p^{s+1}\mathbb{Z}_p} \mathrm{sgn}_\tau(x)dx = \sum_{s=0}^{\infty}\sum_{a=1}^{p-1}\mathrm{sgn}_\tau(p^s a)p^{-(s+1)}$$

$$= p^{-1}\sum_{s=0}^{\infty}(\mathrm{sgn}_\tau(p)p^{-1})^s\sum_{a=1}^{p-1}\mathrm{sgn}_\tau(a) = \begin{cases} \frac{1-p^{-1}}{1+p^{-1}}, & \tau = \epsilon \\ 0, & \tau \neq \epsilon. \end{cases}$$

Again, for the case $\tau \neq \epsilon$ it is the sum over a that is zero, and for the other case we have $\mathrm{sgn}_\epsilon(p) = -1$. Finally we will do one more integral that is very common in the calculations done elsewhere in this thesis. We will integrate $|x|_p^u \mathrm{sgn}_\tau(x) = \pi_{u-1}(x)$, which by the way is a general multiplicative character, an important object mathematically.

$$\int_{\mathbb{Z}_p} |x|_p^u\mathrm{sgn}_\tau(x)dx = \sum_{s=0}^{\infty}\sum_{a=1}^{p-1}\int_{ap^s+p^{s+1}\mathbb{Z}_p} |x|_p^u\mathrm{sgn}_\tau(x)dx = \sum_{s=0}^{\infty}\sum_{a=1}^{p-1}\mathrm{sgn}_\tau(p^s a)p^{-(s+1)-su}$$

$$= p^{-1}\sum_{s=0}^{\infty}(\mathrm{sgn}_\tau(p)p^{-(u+1)})^s\sum_{a=1}^{p-1}\mathrm{sgn}_\tau(a) = \begin{cases} \frac{(1-p^{-1})p^{-u-1}}{1+p^{-u-1}}, & \tau = \epsilon \\ 0, & \tau \neq \epsilon. \end{cases}$$

As we can see, integrating over a single p-adic variable is straight forward, most of the times it reduces to geometric series.

Multivariable

We will now study the procedure done above but for the space $\mathbb{Q}_p^N = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$. Although the idea is the same and many properties remain the same, there is significant change in the process. As mentioned in 2.2.2, the separation of balls into circles is no longer simple.
On to the formalities, we still have a Haar measure that coincides with the product measure. In physics it is very common to use the notation $d^N x := \prod_{i=1}^N dx_i$, so I'll be using them indistinctively, I must say however that the former is more common in this text.
To perform a change of variables we do the usual, change the measure by the Jacobian of the transformation. Consider the change $y_i = f_i(\mathbf{x})$, then the measure $d^N x$ will transform as $d^N x = \left| Jac\left(\frac{\partial \mathbf{x}}{\partial \mathbf{y}}\right)\right|_p d^N y$.
Whenever we have integration over balls, we can easily factorize it into products of balls and hopefully the integrand will factorize as well. It is then a matter of using Fubini's theorem[3]

[3]Fubini's theorem simply states that if a multiple integral is absolutely convergent, then we may integrate in any order. Concretely if $I = \int_{A\times B}|f(x,y)|d(x,y) < \infty$, then $\int_A\left(\int_B f(x,y)dy\right)dx = \int_B\left(\int_A f(x,y)dx\right)dy = I$.

to factorize the entire integral. However, if this is not possible, the strategy is either making a change of variables that allows for factorization, or break down the integration region into balls and circles small enough so that each part can factorize. As a very simple example consider

$$\int_{\mathbb{Z}_p^N} d^N x = \prod_{i=1}^{N} \int_{\mathbb{Z}_p} dx = 1.$$

Since the measure is still a Haar measure invariant under translations, by the reasoning given at the beginning of the previous section, we should have

$$\int_{p^n\mathbb{Z}_p^N} d^N x = \prod_{i=1}^{N} \int_{p^n\mathbb{Z}_p} dx = p^{-Nn}$$

Then we can deduce that the measure of the unit ball should be $d((\mathbb{Z}_p^N)^\times) = 1 - p^{-N}$ and furthermore $d((p^n\mathbb{Z}_p^N)^\times) = (1-p^{-N})p^{-nN}$. This can be recovered by integrating directly the N dimensional circle of radius p^{-n}. We need to consider all of the possible ways to have at least one component in the circle. This is done systematically by considering all proper subsets $I \subset \{1, \ldots, N\}$ and define $\mathbf{S}(I) = \{(x_1, x_2, \ldots, x_N) \in \mathbb{Q}_p^N; |x_i| < p^{-n} \forall i \in I \wedge |x_i| = p^{-n} \forall i \notin I\}$. Then we have

$$\int_{p^n\mathbb{Z}_p^N} d^N x = \sum_I \int_{\mathbf{y}(I)} d^N x = \sum_I \prod_{i\in I} \int_{p^{n+1}\mathbb{Z}_p} dx_i \prod_{j\notin I} \int_{p^n\mathbb{Z}_p^\times} dx_j$$

$$= \sum_I \prod_{i\in I} p^{-n-1} \prod_{j\notin I} (1-p^{-1})p^{-n} = \sum_{k=1}^{N} \binom{N}{k} (p^{-n-1})^{N-k}[(1-p^{-1})p^{-n}]^k$$

$$= \left(p^{-n-1} + (1-p^{-1})p^{-n}\right)^N - p^{-(n+1)N} = (1-p^{-N})p^{-nN}.$$

As you see, integrating over a circle becomes quite challenging compared to a single variable. This becomes more challenging if we want to integrate complicated functions that can include signs. This is exactly the type of integrals that we are going to encounter in the computation of scattering amplitudes in p-adic string theory. We leave the details of the integration to later sections, as it is part of the results obtained for this thesis.

With Algebraic Extensions

Integrating over an extended field is very similar to usual p-adic integration. One only needs to be careful of two things, how the measure changes with the transformation $x \to ax$, and the size of the residue field. We concentrate on the case of unramified extensions because it is the one relevant to us. $\mathbb{Q}_q$ is also a locally compact group and has a Haar measure invariant under translations. The rule for scaling the integration variable is now

$$d(ax) = |a|_q^n dx, \quad a \in \mathbb{Q}_q. \tag{2.19}$$

with the norm $|\cdot|_q$ defined in (2.15), n is the degree of the extension, and $q = p^n$. The unit ball $\mathbb{Z}_q$ is also chosen to have a unit measure, and by the same arguments as shown above in section 2.2.5 we can get the measure of a ball of a different radius, one way is using (2.19) to get

$$\int_{p^{-r}\mathbb{Z}_q} dx = |p^{-r}|_q^n \int_{\mathbb{Z}_q} dy = p^{nr} = q^r. \tag{2.20}$$

Or we use the translation invariance of the measure

$$1 = \int_{\mathbb{Z}_q} dx = \sum_{a=0}^{p^n-1} \int_{a+p\mathbb{Z}_q} dy = p^n \int_{p\mathbb{Z}_q} dy \Rightarrow \int_{p\mathbb{Z}_q} = p^{-n} = q^{-1},$$

then we can iteratively repeat to get the result (2.20). And one can now easily see that

$$\int_{p^{-r}\mathbb{Z}_q^\times} dx = \int_{p^{-r}\mathbb{Z}_q} dx - \int_{p^{-r+1}\mathbb{Z}_q} dx = q^r(1-q^{-1}).$$

Finally we show the integration of $|x|_q^u$ with $u \in \mathbb{C}$, it is

$$\begin{aligned}\int_{p^{-r}\mathbb{Z}_q} |x|_q^u dx &= \sum_{s=0}^{\infty} p^{u(r-s)} \int_{p^{-r+s}\mathbb{Z}_q^\times} dx = \sum_{s=0}^{\infty} p^{u(r-s)}(1-q^{-1})q^{r-s} \\ &= (1-q^{-1}) \sum_{s=0}^{\infty} p^{r(u+n)} p^{-s(u+n)} = \frac{(1-q^{-1})p^{r(u+n)}}{1-p^{-(u+n)}}.\end{aligned} \tag{2.21}$$

It is interesting that for the measure of sets we can replace $p \to q$ going from $\mathbb{Q}_p$ to $\mathbb{Q}_q$, but not for the norm. Had we defined the norm (2.15) without the power $1/n$ we would indeed have that the result (2.21) is the same as the integral over $\mathbb{Q}_p$ but replacing $p \to q$. However we stick to the definition (2.15) as it is the one used in [39] to construct the n-dimensional Bruhat-Tits tree.

2.2.6 Fourier transform

One can do Fourier analysis over $\mathbb{Q}_p^N$. It is completely analogous to the real Fourier analysis. Before defining the p-adic Fourier transform, we define an additive character, i.e., a function $\chi(x) : \mathbb{Q}_p \to \mathbb{C}$, such that $\chi(x+y) = \chi(x)\chi(y)$. This is defined as

$$\chi(x) = \exp(2\pi i \{x\}_p),$$

where i is the usual imaginary unit and $\{x\}_p$ is the fractional part of x, that is, we only keep the coefficients that have a negative power of p in the series expansion of x. This is intuitive given that $\exp(2\pi i n) = 1$ for any integer n.
We define the Fourier transform of a function $f(x)$ as

$$\widetilde{f}(\omega) = \int_{\mathbb{Q}_p} \chi(\omega x) f(x) dx. \tag{2.22}$$

The only difference with the usual case is that the factor of 2π is included in the exponent, this simplifies a little the transform. In fact, one could do it also for the usual real Fourier transform, but for certain conveniences it is usually left as a coefficient of the integral, although this means having a bit of ambiguity in whether the factor $1/2\pi$ is on the direct or inverse transform. Naturally, we expect that the p-adic inverse transform be the same expression but taking the complex conjugate $\chi(x)^* = \chi(-x)$. Indeed this is the case, we have

$$f(x) = \int_{\mathbb{Q}_p} \chi(\omega x)^* \widetilde{f}(\omega) d\omega. \tag{2.23}$$

To show this we need to show that

$$\delta(x) = \int_{\mathbb{Q}_p} \chi(\omega x) d\omega. \tag{2.24}$$

We will do this by showing some integrals of χ over different regions. First, notice that for any number with $|x|_p < 1$ we have $\chi(x) = 1$. This means that integrating on any region inside the ball of radius p^{-1} is equal to the measure of the region. Now consider a circle of radius p^r with $r > 0$ and a number a with $1 \leq |a|_p < p^r$. Then

$$\int_{p^{-r}\mathbb{Z}_p^\times} \chi(x)dx = \int_{a+p^{-r}\mathbb{Z}_p^\times} \chi(y+a)d(y+a)y = \chi(a) \int_{p^r\mathbb{Z}_p^\times} \chi(y)dy$$

Since $\chi(a) \neq 1$ we conclude that

$$\int_{p^{-r}\mathbb{Z}_p^\times} \chi(x)dx = 0; \quad r \geq 1.$$

This leaves us with the case $r = 1$, the unit circle. We do the integral in the usual way by making the change $x = ap^{-1} + y$ with $a \in \{1, \ldots, p-1\}$ and $y \in \mathbb{Z}_p$. Then

$$\begin{aligned}\int_{p^{-1}\mathbb{Z}_p^\times} \chi(x)dx &= \sum_{a=1}^{p-1} \int_{\mathbb{Z}_p} \chi(ap^{-1})\chi(y)dy = \sum_{a=1}^{p-1} \chi(a/p) \int_{\mathbb{Z}_p} dx = \sum_{a=1}^{p-1} (e^{2\pi i/p})^a \\ &= \frac{e^{2\pi i} - e^{2\pi i/p}}{e^{2\pi i/p} - 1} = -1.\end{aligned}$$

From this we can then write the following

$$\int_{p^n\mathbb{Z}_p^\times} \chi(x)dx = \begin{cases} (1-p^{-1})p^{-n}, & n \geq 0 \\ -1, & n = -1 \\ 0, & n \leq -2 \end{cases}; \qquad \int_{p^n\mathbb{Z}_p} \chi(x)dx = \begin{cases} p^{-n}, & n \geq 0 \\ 0, & n < 0. \end{cases} \tag{2.25}$$

Now to integrate $\chi(\omega x)$, it is enough to make the change $\omega x = y$ and $dx = |\omega|_p^{-1} dy$. It is not hard to see that

$$\int_{p^n\mathbb{Z}_p^\times} \chi(\omega x)dx = \begin{cases} (1-p^{-1})p^{-n}, & |\omega|_p \leq p^n \\ -|\omega|_p^{-1}, & |\omega|_p = p^{n+1} \\ 0, & |\omega|_p \geq p^{n+2} \end{cases}; \qquad \int_{p^n\mathbb{Z}_p} \chi(\omega x)dx = \begin{cases} p^{-n}, & |\omega|_p \leq p^n \\ 0, & |\omega|_p > p^n. \end{cases} \tag{2.26}$$

From this result we may take the limit $n \to -\infty$ to retrieve the integral over all of $\mathbb{Q}_p$. Doing so we indeed see that the result would be 0 for every non-zero value of ω, and that if $\omega = 0$, then the result would be ∞. This looks really good for establishing (2.24). To make it more rigorous, we consider a locally constant function f, this means that there exists a ball around x such that $f(x)$ is constant inside the ball. The radius of the ball may be small but it is positive. From this we can see that

$$\lim_{n\to-\infty}\int_{\mathbb{Q}_p}\int_{p^n\mathbb{Z}_p}\chi(\omega x)f(x)d\omega dx = \lim_{n\to-\infty}\int_{|x|_p\leq p^n} p^{-n}f(x)dx$$

$$= \lim_{n\to-\infty} f(0)p^{-n}\int_{|x|\leq p^n} dx = f(0).$$

We have proven the equation (2.23), that is the inverse transform of (2.22). To provide some examples we show first the Fourier transform of the p-adic norm. For a local constant function, the general strategy will be using the first result of (2.26) and sum over the circle's radius.

$$\int_{\mathbb{Q}_p}\chi(\omega x)|x|_p^u dx = \sum_{n=-\infty}^{\infty} p^{-nu}\int_{p^n\mathbb{Z}_p^\times}\chi(\omega x)dx = -p^u|\omega|_p^{-u-1} + \sum_{n=-ord(\omega)}^{\infty}(1-p^{-1})p^{-n(u+1)}$$

$$= -p^u|\omega|_p^{-u-1} + |\omega|_p^{-(u+1)}\frac{1-p^{-1}}{1-p^{-(u+1)}} = \frac{1-p^u}{1-p^{-u-1}}|\omega|_p^{-u-1}. \tag{2.27}$$

As we can see, the transform is another p-adic norm, but the exponent has changed and we got a factor. To check that it is correct, we can take the inverse transform (2.23). Notice that because $|-1|_p = 1$, in this case the inverse transform is the same as the direct transform. Then we can simply take $-(u+1) = u'$ and apply the same result. Doing it we see that $-u'-1 = u$ and $\frac{1-p^{u'}}{1-p^{-u'-1}} = \frac{1-p^{-u-1}}{1-p^u}$, which exactly cancels the factor that we got in the transform. This means that a double Fourier transform of an analytic (admits a series expansion) radial function will result in the identity operation.

Now we obtain the Fourier transform of $|x|_p^u\text{sgn}_\tau(x)$. It is convenient to take the change $x = \omega^{-1}y$, then

$$\int_{\mathbb{Q}_p}\chi(\omega x)|x|_p^u\text{sgn}_\tau(x)dx = |\omega|_p^{-u-1}\text{sgn}_\tau(\omega)\int_{\mathbb{Q}_p}\chi(y)|y|_p^u\text{sgn}_\tau(y)dy.$$

Let us focus on the remaining integral

$$\int_{\mathbb{Q}_p}\chi(y)|y|_p^u\text{sgn}_\tau(y)dy = \sum_{n=-\infty}^{\infty}p^{-nu}\sum_{a=1}^{p-1}\text{sgn}_\tau(ap^n)\int_{ap^n+p^{n+1}\mathbb{Z}_p}\chi(y)dy$$

$$= \sum_{n=-\infty}^{\infty}[\text{sgn}_\tau(p)p^{-u}]^n\left(\sum_{a=1}^{p-1}\text{sgn}_\tau(a)\chi(ap^n)\right)\int_{p^{n+1}\mathbb{Z}_p}\chi(y)dy$$

$$= \sum_{n=-1}^{\infty}[\text{sgn}_\tau(p)p^{-u}]^n\left(\sum_{a=1}^{p-1}\text{sgn}_\tau(a)\chi(ap^n)\right)p^{-n-1}$$

$$= \text{sgn}_\tau(p)p^u\mathcal{C}(p) + p^{-1}\sum_{n=0}^{\infty}[\text{sgn}_\tau(p)p^{-u-1}]^n\sum_{a=1}^{p-1}\text{sgn}_\tau(a) = \begin{cases} \frac{1+p^u}{1+p^{-u-1}}, & \tau = \epsilon \\ \text{sgn}_\tau(p)p^u\mathcal{C}(p), & \tau \neq \epsilon \end{cases},$$

where $\mathcal{C}(p) = \sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\chi(ap/n) = \sqrt{p(-1)^{\frac{p-1}{2}}}$, meaning that $\mathcal{C}(p) = \sqrt{p}$ for $p \equiv 1 \mod 4$, and $\mathcal{C}(p) = i\sqrt{p}$ for $p \equiv 3 \mod 4$. This surprising result comes from number theory and it is an example of a Gauss sum associated to the Legendre symbol, that is a primitive Dirichlet character.

The final result took into account that $\text{sgn}_\epsilon(p) = -1$, $\text{sgn}_\epsilon(a) = 1$ and $\sum_{a=1}^{p-1}\text{sgn}_{\tau\neq\epsilon}(a) = 0$. Therefore the desired transform is

$$\int_{\mathbb{Q}_p}\chi(\omega x)|x|_p^u\text{sgn}_\tau(x)dx = \begin{cases} \frac{1+p^u}{1+p^{-u-1}}|\omega|_p^{-u-1}\text{sgn}_\tau(\omega), & \tau = \epsilon \\ \text{sgn}_\tau(p)p^u\mathcal{C}(p)|\omega|_p^{-u-1}\text{sgn}_\tau(\omega), & \tau \neq \epsilon \end{cases}. \tag{2.28}$$

It is important to note here that in this case, the inverse transform is slightly different than a second transform. This comes from the sign function, for which $\text{sgn}_\tau(-x) = \text{sgn}_\tau(-1)\text{sgn}_\tau(x)$. We may check the result (2.28) by taking the inverse transform. In this case it is a second Fourier transform but multiplied times $\text{sgn}_\tau(-1)$, but since the function to transform is real, it is better to simply take the complex conjugate of (2.28) with $u \to -u-1$. The reader may do so to find that we return to our original function.

2.2.7 Vladimirov Derivative

Since we have been talking a lot about integrating over $\mathbb{Q}_p$, maybe you're wondering about differentiation. As exciting as the title of this subsection sounds, there are deadly technical difficulties in properly defining a derivative on the type of functions that we are working with. The issue is in the difference in nature between the argument and the function.

We mentioned that for this work, a p-adic function is a mapping $f : \mathbb{Q}_p \to \mathbb{C}$. Then if we naively try to define a derivative as usual, we would have

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}. \tag{2.29}$$

But that means dividing a complex number by a p-adic number, and that doesn't make sense at all!, at least not in our context. There is a way of defining a proper derivative, or differential operator for functions that go from $\mathbb{Q}_p$ to $\mathbb{Q}_p$, but that goes beyond our topics of interest.[4]

There is still hope, we can work around the problem of defining the limit (2.29) through the use of Fourier transforms. Remember that over $\mathbb{R}$, in Fourier space, the derivative of a function $d^n f/dx^n$ is $\omega^n\tilde{f}(\omega)$ (assuming a vanishing of f at infinity). Therefore a way to get the derivative of a function is by taking the inverse Fourier transform of a power of the Fourier variable times

[4]For the interested reader, one way to do it is for analytic functions through its series expansion, see [7, Chapter II]. Another way is with a limit more in the classical sense, see [73, Chapter 4] but beware, the treatment is quite mathematical.

the transform of the function.
We know how to do Fourier transforms over $\mathbb{Q}_p$, so why not define derivatives through it? This is precisely what is known as a Vladimirov derivative, or operator. It is defined as

$$D_s f(x) = \int_{\mathbb{Q}_p} \chi(\omega x)^* |\omega|_p^s \tilde{f}(\omega) d\omega, \tag{2.30}$$

where s is the order of the derivative and $\tilde{f}$ is the Fourier transform of f. It is called a pseudo-differential operator because it is not defined as a limit. The Vladimirov derivative can also be defined by the expression

$$D_s f(x) = \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{|x-y|^{s+1}} dy. \tag{2.31}$$

Of course both definitions are equivalent, but it as a little hard to show it. As we can see, D_s is a non local operator, that is, it depends on the value of the function everywhere, not just where one wants to obtain its derivative. The operator admits any order $s \in \mathbb{C}$, although one may need to take an analytic continuation.
Let's do one example, as usual, we will do it for $|x|_p^u$. Using the definition (2.30) and the result (2.27), we immediately see that

$$D_s |x|_p^u = \int_{\mathbb{Q}_p} \chi(\omega x)^* \frac{1-p^u}{1-p^{-u-1}} |\omega|_p^{s-u-1} d\omega = \frac{1-p^u}{1-p^{-u-1}} \frac{1-p^{s-u-1}}{1-p^{u-s}} |x|_p^{u-s}.$$

There is a variation on the Vladimirov derivative that includes a multiplicative character in the definition (2.31). In general it could be any character, but here we specialize it to the p-adic sign function. We define then the twisted Vladimirov derivative (for the purposes of this thesis, we may also call this operator a fermionic Vladimirov derivative), and it is defined as

$$D_s f(x) = \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{|x-y|^{s+1}} \operatorname{sgn}_\tau(x-y) dy. \tag{2.32}$$

Notice that it is a matter of using the more general multiplicative character $|\cdot|_p \operatorname{sgn}_\tau(\cdot)$, instead of just the norm. The term 'twisted' refers to modifying the operator by a character. We leave a more in depth treatment of this type of Vladimirov operator to chapter 6, as we use it there for the fermionic term in the p-adic superstring action.

2.3 Local Zeta Functions

In this section we will talk about the essentials of the theory of local zeta functions and quickly get to the computations and results relevant to this thesis. A little heads up, there is an unfortunate disconnection between the language used in pure mathematics and in pure physics. Therefore, in benefit for the reader who comes from only one background, I will try my best to make the distinction clear, although one quickly gets used to it.

So let's begin, first, what do we mean by *local*? It means that the domain of integration of the zeta function is a subset of a local field. A number field is called local if every point has a compact neighborhood. We can also say that the field is locally compact if it has a locally compact topology. We also demand the field to be Hausdorff[5]. Examples of local fields include the reals $\mathbb{R}$, complex $\mathbb{C}$, p-adics $\mathbb{Q}_p$ and the formal Laurent series field with coefficients on a finite field with q elements $\mathbb{F}_q((x))$.

We are not going to need all the theory of local zeta functions, its definition and a theorem is enough for us. This section is mostly based on the work [74], the reader is referred to this article and the references therein for further details. We want to study local zeta functions because it turns out that the string amplitudes defined over different number fields, as mathematical objects can be mapped to local zeta functions. Local zeta functions and their analytic properties are not a new subject in mathematics, in fact their study began in the 60's. However their connection with physics is a recent finding. Furthermore, there is interest from both disciplines, as physics learns about the mathematical properties of physical objects that they then interpret appropriately. And mathematics is inspired to study new mathematical objects that come from the connection with physics. For a recent survey on this topic see [27].

A local zeta function is a type of integral functional over a given polynomial over a given local field $\mathbb{K}$. The ingredients to build it are: a polynomial function on several variables $f \in \mathbb{K}[\vec{x}]$, a complex number s, and a complex valued locally constant function with compact support ϕ, also known as a test function. Then the local zeta function attached to this data is

$$Z_\phi(s, f) = \int_{\mathbb{K}^n \setminus \cup_i f_i^{-1}(0)} \phi(x) \prod_{i=1}^{m} |f_i(x)|_{\mathbb{K}}^{s_i} \prod_{i=1}^{n} dx_i, \quad Re(s) > 0. \tag{2.33}$$

For technical issues one should remove the zeros of the polynomial f from the integration region. The set $D_{\mathbb{K}} := \cup_{i=1}^{m} f_i^{-1}(0)$ is called the divisor, because f_i is a polynomial, its divisor is a set of points, that has measure zero, so removing this doesn't affect the integration. The local zeta function (also known as Igusa's local zeta function) is holomorphic (no poles) for $Re(s) > 0$, and furthermore if K has characteristic zero, it admits a meromorphic continuation to the whole complex plane on s.

Multivariate local zeta functions over local fields of zero characteristic were studied by Loeser [75]. In the case of zero characteristic, the main tool to show the existence of a meromorphic

[5]A topological space X is Hausdorff is for any two points $x, y \in X$, $x \neq y$, there exist open subsets U, V such that $x \in U$, $y \in V$ and $U \bigcap V = \emptyset$

continuation of the multivariate local function is the Hironaka's resolution of singularities theorem. By applying this theorem to the divisor, the mutivariate local zeta function is reduced to the case of monomial integrals [76], and [75]. Currently, the methods used by Igusa are not available in positive characteristic, so the problem of the meromorphic continuations in this setting it is still an open problem.
We will state now the theorem by Hironaka to resolve singularities, it is a technical theorem and is here for completeness. One thing is needed to understand it better, for a general local field $\mathbb{K}$, a $\mathbb{K}$-analytic manifold is defined very similarly as for a usual real or complex manifold. One needs a set of analytic mappings between charts of an atlas that cover the manifold. These mapping must be convergent power series. The following theorem is as stated in section 8 of [27], the reader is referred there for more details.

Theorem 2.3.1 (Hironaka)**.** *Let $\mathbb{K}$ be a local field of characteristic zero. There exists an embedded resolution $\sigma : X \to \mathbb{K}^n$ of the divisor $D_{\mathbb{K}}$, that is,*
(i) X is an n-dimensional $\mathbb{K}$-analytic manifold, σ is a proper $\mathbb{K}$-analytic map which is a composition of a finite number of blow-ups at closed submanifolds, and which is an isomorphism outside of $\sigma^{-1}(D_{\mathbb{K}})$;
(ii) $\sigma^{-1}(D_{\mathbb{K}})$ is a normal crossings divisor, meaning that $\sigma^{-1}(D_{\mathbb{K}}) = \cup_{i\in T} E_i$, where the E_i are closed submanifolds of X of codimension one, each equipped with an m-tuple of nonnegative integers $(N_{f_1,i}, \dots, N_{f_m,i})$ and a positive integer v_i, satisfying the following. At every point b of X there exist local coordinates $(y_1, \dots, y_n)$ on X around b such that, if $E_1, \dots, E_r$ are the E_i containing b, we have on some open neighborhood V of b that E_i is given by $y_i = 0$ for $i \in \{1, \dots, r\}$,

$$\sigma^* (dx_1 \wedge \ldots \wedge dx_n) = \eta (y) \left(\prod_{i=1}^{r} y_i^{v_i - 1} \right) dy_1 \wedge \ldots \wedge dy_n, \tag{2.34}$$

and

$$f_j^*(y) := (f_j \circ \sigma)(y) = \varepsilon_{f_j}(y) \prod_{i=1}^{r} y_i^{N_{f_j,i}}, \qquad \text{for } j = 1, \dots, m, \tag{2.35}$$

where $\eta(y)$ and the $\varepsilon_{f_j}(y)$ belong to $\mathcal{O}_{X,b}^{\times}$, the group of units of the local ring of X at b.

The Hironaka resolution theorem allows expressing a multivariate local zeta function as a linear combination of monomial integrals, through a finite sequence of changes of variables. This is very useful in determining whether or not the string amplitudes have a region of convergence, and stating that they are indeed regular expressions. As we will see, the scattering amplitudes from p-adic string theory can be put in a form so that they become sums of local zeta functions and apply the theorems. It is in this way that mathematics meets physics yet again. However, as a preview, we will see that the string amplitudes are not quite of the form (2.33), the ingredient missing is the compact support, however it has been shown that nevertheless they have a connected region of convergence [25].

As a trivial check of this, look at the following integral done already in the previous section. We denote by $\mathbf{1}(x)$ the characteristic ball of the argument, i.e.

$$\mathbf{1}(x) = \begin{cases} 1, & |x|_p \le 1; \\ 0, & |x|_p > 1. \end{cases}$$

Then

$$Z(s,n) = \int_{\mathbb{Q}_p} \mathbf{1}(p^{-n}x)|x|_p^s dx = \int_{p^{-n}\mathbb{Z}_p} |x|_p^s dx = \frac{(1-p^{-1})p^{n(s+1)}}{1-p^{-n(s+1)}}, \tag{2.36}$$

is a local zeta function, and it is a rational function in the variable $p^{n(s+1)}$. As a more interesting and relevant example, we work out the following

$$Z^{(4)}(a,b) = \int_{\mathbb{Q}_p} |x|_p^a |1-x|_p^b dx \tag{2.37}$$

If you notice, the compact support is missing, however, we can show that this integral does converge in a connected region, and therefore admits a meromorphic continuation on the variables a, b. To do the integral, we divide the integration region into 3 parts: (i) $p\mathbb{Z}_p$, (ii) $\mathbb{Z}_p^\times$ and (iii) $\mathbb{Q}_p \setminus \mathbb{Z}_p$. In (i) we have

$$Z^{(4)}_{(i)}(a,b) = \int_{p\mathbb{Z}_p} |x|_p^a dx = \frac{(1-p^{-1})p^{-(a+1)}}{1-p^{-(a+1)}}, \ \mathrm{Re}(a) < -1.$$

For region (ii)

$$Z^{(4)}_{(ii)}(a,b) = \int_{\mathbb{Z}_p^\times} |x|_p^a |1-x|_p^b dx = \sum_{j=2}^{p-1} \int_{j+p\mathbb{Z}_p} dx + \int_{1+p\mathbb{Z}_p} |1-x|_p^b dx$$

$$= (p-2)p^{-1} + \int_{p\mathbb{Z}_p} |y|_p dy = (p-2)p^{-1} + \frac{(1-p^{-1})p^{-1(b+1)}}{1-p^{-(b+1)}}, \ \mathrm{Re}(b) < -1.$$

For region (iii) we must do the change of variables $x = 1/y$, and realize that in this case $|1-x|_p = |x|_p$. Then

$$Z^{(4)}_{(iii)}(a,b) = \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} |x|_p^{a+b} dx = \int_{p\mathbb{Z}_p} |y|_p^{-a-b-2} dy = \frac{(1-p^{-1})p^{a+b+1}}{1-p^{a+b+1}}, \ \mathrm{Re}(a+b) > 1.$$

Therefore the zeta function does exist and it has the rational form

$$Z^{(4)}(a,b) = Z^{(4)}_{(i)}(a,b) + Z^{(4)}_{(ii)}(a,b) + Z^{(4)}_{(iii)}(a,b)$$

$$= (p-2)p^{-1} + \frac{(1-p^{-1})p^{-(a+1)}}{1-p^{-(a+1)}} + \frac{(1-p^{-1})p^{-(b+1)}}{1-p^{-(b+1)}} + \frac{(1-p^{-1})p^{a+b+1}}{1-p^{a+b+1}}. \tag{2.38}$$

It converges in the set given by the conditions $\mathrm{Re}(a) < -1, \ \mathrm{Re}(a) < -1, \ \mathrm{Re}(a+b) > 1$, that is a connected set in the plane (a, b), it is a small triangle in fact. This is shown later in the Fig. 4.1.

The expression (2.37) is actually the 4-point tachyon scattering amplitude for the p-adic bosonic string studied back in the 80's. As a quick note, the general Koba-Nielsen N-point function has been proven to be a bona fide integral in a connected region, this was done recently in the work [25], showing a connection of interests from mathematicians and physicists. This amplitude is

$$Z^{(N)}(\mathbf{s}) = \int_{\mathbb{Q}_p^N} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1-x_i|_p^{s_{i(N-1)}} \prod_{2\leq i<j\leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i. \tag{2.39}$$

Once again, notice that we do not have a compact support, however, just like in the $N = 4$ case, it can be integrated and shown to be a rational function. This is a non-trivial example where the compact support is not necessary in order to ensure the convergence of the integral. Furthermore, this has later been extended to the case of the usual real bosonic string, where the integral is almost the same as (2.39), but over the fields $\mathbb{R}$ and $\mathbb{C}$ for the open and closed strings, respectively.

2.4 Grassmann Variables

This is a review of some well known results in the use of Grassmann variables. In physics they serve as classical representations of fermion fields, the main property is that these variables anticommute, endowing them with Fermi-Dirac statistics, and also making operations like integration seem very strange at first glance. This section will be an overview of results and techniques, we will omit more formal definitions and constructions, for more details one can check [77]. Or for a more path integral oriented introduction see [78, Appendix 2].
A Grassmann algebra is defined as a set of variables with the following property

$$\Lambda := \{\theta_i,\ i \in \mathcal{I}; \{\theta_i, \theta_j\} := \theta_i\theta_j + \theta_j\theta_i = 0\}, \tag{2.40}$$

where $\mathcal{I}$ is an index set. Notice that this implies $\theta_i^2 = 0$ for every i. From this we deduce that all analytic functions of Grassmann variables are at most linear. Then in particular we have

$$\exp\{\theta_i\} = 1 + \theta; \quad \log(1 + \theta_i) = \theta_i. \tag{2.41}$$

A product of two Grassmann variables is commutative, i.e. $[\theta_i\theta_j, \theta_k\theta_l] = 0$. These properties later become particularly interesting when exponentiating a bilinear form acting on Grassmann variables.
Since ultimately we are interested in working with both Grassmann and usual (commutative) complex variables, it is said that Grassmann variables have an odd parity, and normal variables have an even parity, and then the product of two objects with different parity is

$$x\theta = (-1)^{\hat{x}+\hat{X}}\theta x, \tag{2.42}$$

where the hat $\hat{\cdot}$ indicates the parity of the variable, it is either 0 or 1. Throughout this section θ will be a variable of odd parity or Grassmann. First, we need to know that for N complex Grassmann variables $\theta_i,\ 1 \le i \le N$, we have the identities

$$\begin{aligned} \theta_{i_1}\theta_{i_2}\cdots\theta_{i_N} &= (-)^P\theta_1\theta_2\cdots\theta_N; && (2.43)\\ \bar{\theta}_{i_1}\theta_{i_1}\bar{\theta}_{i_2}\theta_{i_2}\cdots\bar{\theta}_{i_N}\theta_{i_N} &= (-1)^{\frac{N(N-1)}{2}}\bar{\theta}_{i_1}\cdots\bar{\theta}_{i_N}\theta_{i_1}\cdots\theta_{i_N}; && (2.44)\\ &= \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2\cdots\bar{\theta}_N\theta_N, && (2.45) \end{aligned}$$

here $(-)^P$ stands for the sign of the permutation of the set $\{i_1 i_2 \cdots {}_N\}$ and $\bar{\cdot}$ stands for the complex conjugate. The second line is achieved simply by repeatedly applying the anticommutative property. The third line follows from the fact that products of pairs of Grassmann variables commute with each other, or in other words, a product of two odd variables is an even variable. We want to be adding variables with the same parity.

Now consider a symmetric matrix M^{ij} of rank N. Then

$$\left(\sum_{i,j=1}^{N}\bar{\theta}_i M^{ij}\theta_j\right)^k = \left(\sum_{i=1}^{N}\bar{\theta}_i\Theta^i\right)^k = \sum_{\{i_1\dots i_k\}\in\{1\dots N\}}\sum_{P_k}\bar{\theta}_{P_k(1)}\Theta^{P_k(1)}\bar{\theta}_{P_k(2)}\Theta^{P_k(2)}\cdots\bar{\theta}_{P_k(k)}\Theta^{P_k(k)}$$
$$= \sum_{\{i_1\dots i_k\}\in\{1\dots N\}}\sum_{P_k}\sum_{\{j_1\dots j_k\}\in\{1\dots N\}}\sum_{P_{k'}}\bar{\theta}_{P_k(i_1)}M^{P_k(i_1)P_{k'}(j_1)}\theta_{P_{k'}(j_1)}\cdots\bar{\theta}_{P_k(j_n)}M^{P_k(i_n)P_{k'}(j_N)}\theta_{P_{k'}(j_N)}$$
$$= \sum_{\substack{\{i_1\dots i_k\}\subseteq\{1\dots N\}\\ \{j_1\dots j_k\}\subseteq\{1\dots N\}}}\bar{\theta}_{i_1}\theta_{j_1}\cdots\bar{\theta}_{i_k}\theta_{j_k}\sum_{P_k}(-)^{P_k}\left[\sum_{P_{k'}}(-)^{P_{k'}}M^{P_k(i_1)P_{k'}(j_1)}\dots M^{P_k(i_n)P_{k'}(j_N)}\right] \tag{2.46}$$
$$= \sum_{\substack{I_k=\{i_1\dots i_k\}\subseteq\{1\dots N\}\\ J_k=\{j_1\dots j_k\}\subseteq\{1\dots N\}}}\bar{\theta}_{i_1}\theta_{j_1}\cdots\bar{\theta}_{i_k}\theta_{j_k}\sum_{P_k}(-)^{P_k}\left[(-)^{P_k}\det(M^{I_kJ_k})\right]$$
$$= k!\sum_{\substack{I_k,J_k\subseteq\{1\dots N\}\\ |I_k|=|J_k|=k}}\bar{\theta}_{i_1}\theta_{j_1}\cdots\bar{\theta}_{i_k}\theta_{j_k}\det(M^{I_kJ_k}).$$

Some notation explanation is in order. We made the substitution $\Theta^i = \sum_{j=i}^{N} M^{ij}\theta_j$. Sets I_k, J_k run through all $\binom{N}{k}$ possible subsets of $\{1\dots N\}$ with k elements. P_k is the permutations of k elements. $M^{I_kJ_k}$ is the $k\times k$ matrix constructed with rows I_k and columns J_k of M. Notice that we have in total $\left[\binom{N}{k}k!\right]^2 < N^{2k}$ terms, this is due to $\bar{\theta}_i^2 = \theta_j^2 = 0$.
In words, (2.46) is the sum of all possible determinants of $k\times k$ submatrices of M.
Therefore we can write it as

$$\exp\left\{\sum_{i,j=1}^{N}\bar{\theta}_i M^{ij}\theta_j\right\} = \sum_{k=0}^{\infty}\frac{\left(\sum_{i,j=1}^{N}\bar{\theta}_i M^{ij}\theta_j\right)^k}{k!} = \sum_{k=0}^{N}\sum_{\substack{I_k,J_k\subseteq\{1\dots N\}\\ |I_k|=|J_k|=k}}\bar{\theta}_{i_1}\theta_{j_1}\cdots\bar{\theta}_{i_k}\theta_{j_k}\ \det(M^{I_kJ_k}) \tag{2.47}$$

The sum in k is truncated to N because all the following terms have at least one Grassmann variable squared. This expression will be very useful when doing integration of Gaussian integrands.
First, we must define the operation of integration, known as the Berezin integral, and it is defined as

$$\int d\theta_i\theta_i = 1; \quad \int d\theta_i = 0. \tag{2.48}$$

If this seems odd to you, indeed it is. This has nothing really to do with normal integration, it is best if you think about it as a definition rather than an integral in the usual sense. This definition leads to analogous results when doing calculations in fermionic theories, and it is specially suited to use in supersymmetric theories in the superspace approach. The integral is defined to be linear, as one would expect. Then for a general function $f(\theta) = a + b\theta$, where $a, b \in K$ are in some number field, we have

$$\int d\theta f(\theta) = \int d\theta(a+b\theta) = 0 + b\int d\theta\theta = b. \tag{2.49}$$

The integral (2.48) can be extended to several variables in an intuitive way, however one must take into account that the differentials $d\theta_i$ will also anticommute, that is $d\theta_i d\theta_j = -d\theta_j d\theta_i$. It is important in (2.48) to notice that the order of the integrand and differential matters, it is very common in physics to write the differential or measure right after the integral sign. As long as the convention is clear it won't be a problem. Here we put the differential right after the integral sign.
For multiple variables, we make sure that the proper order is used, for example

$$\int d\theta_i d\theta_j \theta_i \theta_j = -\int d\theta_i \left(\int d\theta_j \theta_j \right) \theta_i = -\int d\theta_i \theta_i = -1. \tag{2.50}$$

and more generally

$$\int d\theta_N \dots d\theta_1 \theta_{i_1} \dots \theta_{i_N} = (-)^{P(i)}, \tag{2.51}$$

where $P(i)$ is the sign of the permutation of the is. Finally it is now straightforward to see the very nice result for a Gaussian integral in Grassmann variables

$$\int d\theta_N d\bar{\theta}_N \dots d\theta_1 d\bar{\theta}_1 \exp\left\{ \sum_{i,j=1}^{N} \bar{\theta}_i M^{ij} \theta_j \right\} = \det(M). \tag{2.52}$$

Unlike with usual variables, this Gaussian integral gives the determinant of the matrix, instead of being the reciprocal with a factor of π. These type of integrals can be used to write determinants in a neat way as Gaussian integrals using Grassmann variables, and are an elegant tool to write results in a supersymmetric way. There is more to say, like for example how does the change of variables work, however we do not need it for this thesis, therefore we omit it.
Now we define the differentiation. This definition is more intuitive

$$\partial_{\theta_i} \theta_i = 1; \quad \partial_{\theta_i} 1 = 0. \tag{2.53}$$

In case you missed it, this is actually the same definition as (2.48). In this context we can make the formal replacement $\partial_{\theta_i} = \int d\theta_i$. Therefore you already know how to differentiate, just do the same as integration and you are done, or viceversa, whatever feels more natural.
Grassmann variables are used in physics to both describe fermion particles through Grassmann valued fermion fields, and as auxiliary variables to manipulate expression in a formal way so that we have more elegant and easier to handle expressions. Here we see one of those examples that is really important when handling the superstring amplitudes later in this work.
Consider a Grassmann valued field $\psi(y)$, and an auxiliary Grassmann variable θ, then we can write as

$$\psi(y) = \int d\theta \, (1 + \theta\psi(y)) = \int d\theta e^{\theta\psi(y)}. \tag{2.54}$$

From this we can better express things like $\psi(x)e^{k\cdot X}$, with k, X even variables, as

$$\psi(y)e^{k\cdot X} = \int d\theta e^{\theta\psi(y) + k\cdot X} \tag{2.55}$$

Because $\theta\psi(x)$ together form an even variable, there are no issues in the product of the exponentials. Something similar can be done with an expression having a logarithm,

$$\log(y_i - y_j) + \frac{\theta_i\theta_j}{y_i - y_j} = \log(y_i - y_j) + \log\left(1 + \frac{\theta_i\theta_j}{|y_i - y_j|}\right) \tag{2.56}$$

$$= \log(y_i - y_j + \theta_i\theta_j), \tag{2.57}$$

where we used (2.41). This appears in the simplest case for Archimedean bosonic superstring amplitudes [55]. This identity can be generalized for constants A, B and w to

$$A\log|y_i - y_j| + B\frac{\theta_i\theta_j}{|y_i - y_j|^w} = \frac{A}{w}\left(\log|y_i - y_j|^w + \frac{Bw}{A}\frac{\theta_i\theta_j}{|y_i - y_j|^w}\right) \tag{2.58}$$

$$= \frac{A}{w}\log\left(|y_i - y_j|^w + \frac{Bw}{A}\theta_i\theta_j\right).$$

In fact we can take the exponential of the previous expression to get

$$\exp\left\{\frac{A}{w}\log\left(|y_i - y_j|^w + \frac{Bw}{A}\theta_i\theta_j\right)\right\} = \left(|y_i - y_j|^w + \frac{Bw}{A}\theta_i\theta_j\right)^{A/w}. \tag{2.59}$$

Granted, this is a little bit formal, but it is compatible with the rules of commutativity given, however it can be a little bit misleading. What one really has is

$$\left(|y_i - y_j|^w + \frac{Bw}{A}\theta_i\theta_j\right)^{A/w} = |y_i - y_j|^A\left(1 + B\frac{\theta_i\theta_j}{|y_i - y_j|^w}\right), \tag{2.60}$$

But, let's face it, that does not look as elegant.

Chapter 3

Physics Background

3.1 String Theory Amplitudes

Well what can I tell you, if you are reading this thesis then you probably know some of the basics of string theory, but more than likely you know more than me, I must humbly admit. Nevertheless let us talk a little bit about what string theory is. In my opinion it is one of the most notorious theories in theoretical physics. It has been around for about 60 years now, starting from an effective description of high energy physics for hadrons with the Veneziano Amplitude in an attempt to describe the Regge slope observed in the experiments. Then it was realized that the amplitude came from a deeply fundamental theory of one dimensional objects named strings. This theory has provided great richness in both physics and mathematics. On the one hand it is still a strong candidate for the unification of forces in physics. On the other hand it has inspired many mathematics, in fact some mathematics has been developed thanks to string theory, that particular direction of information flow was really unexpected. To this day string theory remains strong and it keeps bringing connections between physics and mathematics, the very proof of it is, of course, this work; although in this thesis the place for string theory is still pretty abstract.

This thesis is not about strings, but rather it is in the context of strings, and for that reason we will not make a revision of all of string theory, that would be extremely long, out of the scope of my work and this has been extensively covered the literature. We will see some basics of the bosonic string, in particular the symmetries of the theory and how we can gauge fix to simplify the calculations, this makes the theory ready for "p-adization". We will also talk a little about scattering amplitudes and focus on tachyon scattering for the bosonic string, as well as physical properties we want in our amplitudes. Then we will comment about superstrings and the superspace with their (super)amplitudes.

3.1.1 String Theory Action

This section is largely based on [55], see also [78, 79, 80, 81, 82]. The action of string theory is the Polyakov action, although originally the proposed action was the Nambu-Goto action, proportional to the string worldvolume, it is not convenient since it is not polynomial (it is proportional to $\sim \sqrt{\det(\partial X/\partial\sigma)}$). The Polyakov action is equivalent at the cost of having an extra degree of freedom, the intrinsic metric of the string worldsheet h_{ab}. The Polyakov action is

$$S_{Pol}[X] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \tag{3.1}$$

here $1/4\pi\alpha'$ is the string tension, α' is the Regge slope, σ are the worldsheet coordinates, h is the intrinsic worldsheet metric, X is the embedding field interpreted as spacetime coordinates and η is the spacetime Minkowski metric. By equivalence of the actions I mean that classically they posees the same equation of motion. This happens when we take the metric h_{ab} on shell and substitute it in (3.1), we recover the Nambu-Goto action. In other words, using the Nambu-Goto action is equivalent to using the Polyakov action but having the constrain that h_{ab} must be on shell. This constrain turns out to be the requirement of having a vanishing energy-momentum tensor $T_{ab} = 0$.

There are a couple of symmetries that (3.1) has, one of them reparametrization invariance under worldsheet diffeomorphisms. If we take a general analytic reparametrization $\sigma^a \to \bar{\sigma}^a(\sigma)$, the action keeps its form in terms of the new parameters $\bar{\sigma}$. It also has conformal invariance, meaning that under the change of the metric $h_{ab} \to \bar{h}_{ab} = e^{\phi(x)}h_{ab}$, the action is also invariant. Both of these can be verified directly in the Polyakov action. A symmetry means that we have redundancies in our description of the system, and this gives us the freedom to choose any reparametrization that suits us. Of course we want to do our lives easier and choose one that simplifies the action, after all, since it is a redundancy, we would not be changing any of the physics of the system.

Let us do a quick check of the degrees of freedom, the metric h_{ab} is symmetric in the worldsheet, so it has 3 degrees of freedom, and a reparametrization consists of two functions, that is, two degrees of freedom. Then we can make h_{ab} of the form $e^{\phi(\sigma)}\eta_{ab}$, where η_{ab} is the worldsheet flat metric, and e^ϕ is a conformal factor. When one inserts this choice into (3.1), the conformal factor disappears for free! this is because the factors coming from $\sqrt{-h}$ and h^{ab} cancel each other. Then the action is simply

$$S_{Pol}[X] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{3.2}$$

This choice of parametrization is known as *conformal gauge*. Further choices of gauge (reparametrization) fixing can be made, such as *lightcone gauge*, however we will not see them here. In this coordinates (with a flat metric), the equation of motion is a 1+1 dimensional wave equation. Things get even nicer if we go to complex coordinates defined as $z = \sigma + i\tau$, $\bar{z} = \sigma - i\tau$, for example the equation of motion is now $\partial\bar{\partial}X = 0$. One later can quantize the theory using canonical quantization introducing equal time commutators for the (now operators) $\hat{X}$, and proceed to find things like the spectra of the theory and anomalies. We will not see all of that here because the interest is just to have an idea of what string theory is, and go straight to the scattering amplitudes. The interested reader can easily find lots of information of this vast subject (because things get very tricky for this theory, physically and mathematically) in the references given at the beginning of this section, or simply find literature that suits your needs, honestly, there is now maybe too many references one may consult.

3.1.2 Scattering Amplitudes

One of the most interesting computations one can make on a quantum theory, if not the most important, is the computation of the transition or scattering amplitudes. These quantify the probability that on a given interaction (or more generically, a process), we start with a given set of initial or incoming states and end up with another given set of final or outgoing states (by initial it is assumed that these states are infinitely far away and in the far past as a free particle, and the same for final states, infinitely away and in the far future as free particles). In fact there are formalisms that focus solely on what is known as the S matrix, whose components in the state basis are precisely the transition amplitudes.

In particular for string theory, and in a bigger class of theories with conformal symmetry known as Conformal Field Theories (CFT), the way to obtain the scattering amplitudes is through the use of vertex operators. As we mentioned, the scattering amplitudes are for a given set of asymptotically initial and final states, in string theory this would mean certain vibration modes or curves at the infinite edges of the worldsheet. In practice what one does is to compactify the string worldsheet to a disk or a sphere. This can be achieved we still have the conformal factor from the metric that vanished for free, therefore we still have that coin to buy some further simplifications. With a conformal mapping, that is a rescaling of the metric, we can bring the asymptotic states to a finite space, in fact, the curves assumed to be the states of the string can be mapped to points on the sphere or disk. It is precisely in those points where one inserts the vertex operator that carry the memory of the state. This means that we have a vertex operator for every possible excitation state of the string.

The actual formula to compute the amplitude is the correlation function of the vertex operators

$$A(\Lambda_1, k_1; \ldots; \Lambda_N, k_N) = \kappa^{N-2} \int DXDhe^{-S_{Pol}[X,h]} \prod_{i=1}^{N} V_{\Lambda_i}(k_i), \tag{3.3}$$

where Λ_i is a symbol for all the quantum numbers that specify a state, k_i is its momentum and κ is a coupling constant. The vertex operator is of the form

$$V_\Lambda(k) =: \int d^2\sigma \sqrt{h} W_\Lambda(X, \partial X) e^{ik\cdot x} :,$$

where $W_\Lambda(X, \partial X)$ is a polynomial in the fields X and its derivatives and the : mean normal ordering. This form is because the vertex operators should reflect the symmetries of our theory,

i.e. they should be reparametrization invariant, that is why we integrate over the worldsheet, another way to understand this is that there is no special point in the worldsheet, therefore we should be democratic and consider all points evenly. The momentum of the state is given by the factor $e^{ik\cdot X}$, and the type of particle is determined by W_Λ, for instance for tachyons, the lowest energy state, we have the simplest possibility, $W_\Lambda = 1$. for a vector state with a polarization ζ_μ we have $W_\Lambda = \zeta_\mu k^\mu$, and so on.
If you notice in the path integral of (3.3) we are integrating over all the possible metrics h, as we should, however, there is a mathematical theorem that allows us to save the trouble of doing that integral. Remember that when doing the path integral, we should remove all the symmetries which are just redundancies and lead to overcounting. Again, using the reparametrization invariance and conformal invariance of the metric, we know that we can choose to have the flat metric η^{ab}, the theorem by Riemann ensures us that this can be done globally for the sphere. Therefore we can simply drop the integral over h and consider it flat inside the action in (3.3). It is very important to mention that we are assuming the worldsheet to be a sphere, this only applies to closed strings at tree level, the full amplitude should be summing over the all the possible topologies of a closed surface, this basically means having handles. For example the one loop amplitudes would be over the torus, that has only one handle. And in the case of open strings, the tree level worldsheet is a disk, and we add boundary components to add loops. As a fundamental example we sketch the computation of tachyon scattering. In this case the vertex operator is simply

$$V(k_i) =: \int_\Sigma d^2\sigma e^{ik_i\cdot X(\sigma_i,\tau_i)} : . \tag{3.4}$$

The fact that it is only an exponential with an argument linear in the field X is very convenient, because then its correlator can be seen as a source for the generating function of the theory. Remember that the generating function is the path integral of $e^{-S[X]+\int J\cdot X}$, and because $S[X]$ is quadratic in X, we can complete the square in the argument of the exponential using the Green's function of the operator in $S[X]$. Assuming $S[X] = \frac{1}{2}\int X\cdot \mathcal{A}X$ we have the identity

$$-\frac{1}{2}X\cdot\mathcal{A}X + X\cdot J = -\frac{1}{2}\left(X - \int GJ\right)\cdot\mathcal{A}\left(X - \int GJ\right) + \frac{1}{2}\int JGJ, \tag{3.5}$$

where G is the Green's function of the operator $\mathcal{A}$. Depending on the nature or $\mathcal{A}$ one may need to choose appropriate boundary conditions in order for (3.5) to hold; for example for the closed string we integrate over a closed surface with no boundary, or for the open string we can take Neumann boundary conditions. Using that $D(X - \int GJ) = DX$ for the functional measure, we can write the generating function as

$$Z[J] = \int DXe^{-S[X]+\int J\cdot X} = Ze^{-\frac{1}{2}\int JGJ}.$$

Now we go back to our case of strings. $S[X]$ is the Polyakov action in the conformal gauge, then the operator $\mathcal{A}$ would be of the form $\partial\bar{\partial}$, and if we want to insert the tachyon vertex operators, it is equivalent to the generating function with the sources

$$J^\mu(z) = \sum_{i=1}^N k_i^\mu \delta^2(z - z_i).$$

All that is left now is to get the Green's function of the operator $\mathcal{A}$. Denoting the complex Laplacian with respect to z, our Green's function should satisfy

$$\triangle_z G(z, z') = 2\pi\delta^2(z - z').$$

The solution to this is given by

$$G(z, z') = -\frac{1}{2\pi}\int d^2q \frac{e^{iq\cdot(z-z')}}{q^2} = \ln(\mu|z - z'|),$$

where μ is an infrared cutoff needed to handle the divergence at $q = 0$. Substituting this into the expression $\int JGJ$ we get

$$\exp\left\{\int d^2z d^2z' \sum_{i\neq j}^N k_{i,\mu} k_j^\mu \delta^2(z - z_i) ln(\mu|z - z'|)\delta^2(z' - z_j)\right\} = \exp\{k_i \cdot k_j \sum_{i\neq j} ln(\mu|z_i - z_j|)|\}$$
$$= K(\mu)\prod_{i\neq j}|z_i - z_j|^{k_j\cdot k_j}.$$

When we substitute this into the amplitudes we get

$$A(k_1, \ldots, k_N) \sim \kappa^{N-2}\int \prod_{i=1}^N d^2z_i \prod_{i\neq j}|z_i - z_j|^{k_i\cdot k_j}. \tag{3.6}$$

where we have omitted both $K(\mu)$ and the partition function, as they are just constants (I know, they may be infinite, but stick with it, we can regularize them, plus there is a formal step already when considering the Feynman integral). This is nice and symmetric, however, there is still work to do, the amplitudes (3.6), as they are, are infinite. There is still a remnant symmetry that is giving us redundancies. If you payed attention, we didn't actually used the conformal symmetry. Since the action is conformally invariant, the conformal factor gets cancelled without a choice of one. Now we want to fix it. For our purposes it is enough to know that the conformal invariance can be set using three parameters, and the corresponding transformations are of the form

$$z' = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}. \tag{3.7}$$

Notice that although we have four parameters, we only have three degrees of freedom. This is simply because of the linear fraction form, one can always scale all of the parameters by a factor and the transformation is the same. The form (3.7) is also highly suggestive of the group of transformations it represents, known as Möbius transformations. They are isomorphic to the group $SL(2, \mathbb{C})$, the special linear group of dimension 2. The usual representation is with 2x2

matrices that have a unit determinant. The product of these matrices is homomorphic to the composition of the transformation (3.7).
This transformation has the property of being 3-transitive. This means that one single transformation can take any three points to wherever you wish. In particular this fact is used to *fix* three points in (3.6) and finally have a finite answer. This gauge fixing is achieved by choosing appropriately the parameters of the transformation. It is customary to fix three points to 0, 1 and ∞ (remember that we are in the projective line). To make this happen we need to take the following transformation

$$z_i' = \frac{(z_i - z_A)(z_B - z_C)}{(z_i - z_C)(z_B - z_A)}, \quad i \neq A, B, C.$$

This transformation sends $z_A \to 0$, $z_B \to 1$ and $z_C \to \infty$. This is easily seen when trying to obtain z_A', z_B' and z_C', I invite the reader to do it and see it for him/herself. What this transformation does is to factorize the integrand into two parts, one dependent on the variables z_i' and the other on z_A, z_B, and z_C. The details will be done for the p-adic case later in this thesis.
We still need to decide which points we are going to fix. The choice is arbitrary, but it is usual to choose the points $z_1 = z_A$, $z_{N-1} = z_B$, and $z_N = z_C$. After doing the transformation and normalizing the gauge factor that comes out, we are left with the following amplitudes

$$A(k_1, \ldots, k_n) \sim \kappa^{N-2} \int \prod_{i=2}^{N-2} d^2 z_i \prod_{i=2}^{N-2} |z_i|^{k_1 \cdot k_i} |1 - z_i|^{k_i \cdot k_{N-1}} \prod_{2 \leq i < j \leq N-2} |z_i - z_j|^{k_i \cdot k_j} \tag{3.8}$$

These are the Koba-Nielsen type amplitudes for the closed string, that is the N-point tachyon scattering amplitudes. The case $N = 4$ is known as the Virasoro-Shapiro Amplitude and it looks like

$$A(k_1, k_2, k_3, k_4) \sim \int d^2 z |z|^{k_1 \cdot k_2} |1 - z|^{k_2 \cdot k_3} = \frac{\Gamma(1 + \alpha(s))\Gamma(1 + \alpha(t))\Gamma(1 + \alpha(u))}{\Gamma(1 + \alpha(s) + \alpha(t) + \alpha(u))}.$$

The form on the right hand side in terms of gamma functions is very important because it shows explicitly the crossing symmetry over the channels s, t and u. It is also very relevant to this thesis because in the p-adic string theory, this type of expressions are the ones that led to the description of p-adic strings, there are analogues over $\mathbb{Q}_p$ of the gamma functions. But that is a story for another chapter, see 4.
We will now comment on the open strings amplitudes. The story is really the same, there are only a couple of changes. Firstly, the worldsheet is now a disk instead of a sphere. This means that it has a boundary where the vertex operators are now inserted. They are inserted there because remember that the disk is actually a conformal map of the original worldsheet, that is a 2d surface with asymptotic curves at infinity that are mapped to points in a compact space. These points must lie on the boundary, otherwise we would be changing the topology of the worldsheet by adding boundary components inside the disk. This would mean that we are no

longer at tree level.
The vertex operators are almost the same as for the closed string (3.4), except that the integral is over the boundary of the disk. The disk is mapped conformally to the upper half plane, and the boundary becomes the projective real line. This makes a very important distinction, the variables of integration are no longer complex, they are real, and furthermore they are ordered. The cyclic order in which the vertex are inserted must always be preserved. This is because there is no reparametrization that will change this, hence it is something fundamental. The Green's function in the upper half plane is the same $\ln|z-z'|$ but now subject to the boundary condition

$$\frac{\partial G(z,z')}{dy} = 0\Big|_{y=0}.$$

By this we mean the normal derivative (in the y direction) at the boundary must vanish. This can be achieved through the method of images. We do not present the details here but the end result is that the propagator is

$$G(x,x') = 2\ln|x-x'|.$$

Then the only difference is the factor of 2 and that the variables are real now. The rest proceeds in the same way, we evaluate the generating function with the appropriately chosen sources to get the correlation of the vertex operators. The end result is almost the same, as for the closed strings; after gauge fixing the 3 points with the same conventions we end up with the following amplitudes

$$A(k_1,\dots,k_N) = g^{N-2}\int_{0<x_2<x_3<\cdots<x_{N-2}} \prod_{i=2}^{N-2} |x_i|^{k_1\cdot k_i}|1-x_i|^{k_i\cdot k_{N-2}} \prod_{2\le i<j\le N-2} |x_i-x_j|^{k_i\cdot k_j} \prod_{i=2}^{N-2} dx_i. \tag{3.9}$$

Notice that in this case the integration region where the variables are ordered, and they all lie between 0 and 1 because of our gauge fixing to the points 0, 1 and ∞. These are known as the Koba-Nielsen amplitudes, they are the generalization of the famous Veneziano amplitude (that is for 4 points) to N points. Just to show it, we present the case $N=4$ that is the Veneziano amplitude

$$A(k_1,k_2,k_3,k_4) = g^2\int_0^1 (x)^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}dx = g^2 B(-\frac{s}{2}-2,-\frac{t}{2}-2).$$

Here $B(\cdot,\cdot)$ is the Beta function, and notice that the absolute values are not necessary since the factors are always positive in the integration region. As a final remark, we mention that it is also possible to construct amplitudes that have both open and closed string states. Basically they will have the same form but with real ordered variables for the open string states, and complex variables for the closed string states. Again later in this work we will see the p-adic analogs of the Koba-Nielsen amplitudes.
Finally we want to simply mention the possibility of having background fields in the space-time where the string propagates. These background fields may be interpreted as coupling the

fundamental string to the massless closed string excitations. The first and most obvious is generalizing the spacetime metric to a general Riemannian metric $g_{\mu\nu}(X)$, giving us the action

$$S[X] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X).$$

This is an example of a non linear sigma model. It represents the string propagating in a curved spacetime, and it is associated with the coupling of gravitons. There are two other fields that can be coupled by adding terms with background fields. One of them is the Kalb-Ramond field $B_{\mu\nu}$ that is antisymmetric in its indices, and enters as the term

$$S_B[X] = frac14\pi\alpha' \int_\Sigma d^2\sigma \varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X), \tag{3.10}$$

where ε^{ab} is worldsheet antisymmetric tensor. The other associated closed string field is the dilaton $\Phi(X)$, that has the action term

$$S_\Phi[X] = \frac{1}{4\pi}\int d^2\sigma\sqrt{-h}\Phi(X)R^{(2)} = \xi,$$

where $R^{(2)}$ is the two dimensional Ricci scalar. In 2 dimensions it turns out that ξ is actually a topological invariant, known as the Euler characteristic of the 2d manifold. We include this brief mention of the background fields because later on we will study the p-adic analog of (3.10) along with its consequences in the scattering amplitudes.

3.1.3 Superstring

The Action

The previous section dealt with what is known as the bosonic string theory. It is the most basic string theory, however, as you have seen (a little) it can become quite complicated. Unfortunately, it is not a satisfactory theory, mainly for two reasons. One, it has a tachyon in its spectrum, a non-physical particle because it has a negative mas squared. This is obviously nonsense and cannot occur, although some people like to think of them as particles that travel faster than the speed of light and therefore travel backwards in time. Another interpretation is that in fact we have not found the true vacuum of the theory, however to this day no one has been able to find it if there is one. Nevertheless that would also mean not having a satisfactory theory yet. The second reason is that it does not include fermions, a very fundamental and important aspect of modern physics.

It turns out that in order to incorporate fermions in string theory, we need to include supersymmetry as well. Supersymmetry is a symmetry between bosons and fermions, each particle has a supersymmetric partner. Superstrings also lack the problem of the tachyon, for physical reasons we can remove it from the spectrum of the theory. Therefore superstrings are a much stronger candidate for a physical theory. In this brief section we will only focus on some aspects of the superstring action, and we will then go straight to the amplitudes. The amplitudes keep their essence but they become more complicated, as we will see. The idea to obtain them is the same, however things are more involved thanks to the incorporation of fermions.

The action of the superstrings is a generalization of the bosonic action in conformal gauge. We add a term involving 2-dimensional Majorana spinors Ψ:

$$S[X,\psi] = -\frac{1}{2\pi}\int d^2\sigma \eta^{ab}[\partial_a X^\mu \partial_b X^\nu - i\bar{\Psi}^\mu \rho_a \partial_b \Psi^\nu]\eta_{\mu\nu}, \tag{3.11}$$

where ρ^a are 2d Dirac matrices given by

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and satisfy the relation $\{\rho^a, \rho^b\} = -2\eta^{ab}$. The fermionic part is simpler than it seems. It is customary to denote the components of the spinor as $\Psi = (\psi_- \ \psi_+)^T$. Expanding the term we get

$$S_f = \frac{i}{\pi}\int d^2\sigma \eta_{\mu\nu}[\psi_-^\mu \partial_+ \psi_-^\nu + \psi_+^\mu \partial_- \psi_+^\nu],$$

where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. The equations of motion are easily shown to be $\partial_+\psi_-^\nu = \partial_-\psi_+^\nu = 0$. The solution to these equations and the corresponding canonical quantization will not concerns us here, we will show a few more properties of the superstring action and then proceed to calculate the tachyon scattering amplitude.

The action (3.11) is invariant under the transformation

$$\delta X^\mu = \bar{\epsilon}\psi^\mu; \quad \delta\psi^\mu = -i\rho^a \partial_a X^\mu \epsilon.$$

where ϵ is a constant anticommuting spinor. Since these variations mix the bosonic and fermionic fields, they are known as supersymmetric transformations, and is the reason why we call the action (3.11) supersymmetric.

Superspace

The action (3.11) can be written using the so-called superspace formalism. In this formalism we add a set of Grassmann coordinates to write things in a manifestly supersymmetric way using superfields and superoperators. The prefix super refers to the use of both usual and Grassmann or anticommuting coordinates. Since we have two worldsheet coordinates, we will add two Grasmann coordinates in the form of one complex anticommuting variable θ. A general superfield may be expanded in powers of θ as

$$Y^\mu(\sigma,\theta) = X^\mu(\sigma) + \bar{\theta}\Psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma). \tag{3.12}$$

This is because remember that $(\theta_a)^2 = 0$. In the above expression θ without an index is actually a 2-component spinor

$$\theta = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}; \qquad \bar{\theta} = \theta^T\rho^0 = \begin{pmatrix} i\theta^2 & -i\theta^1 \end{pmatrix}.$$

With this definitions one can see that $\int d^2\theta\ \bar{\theta}\theta = -2i$. In the expansion (3.12) the real field B is included for generality but it is not necessary to write the action (3.11) in a superspace formalism, therefore we will set it to zero from now on. There is one more ingredient, a superderivative D, or more precisely, a superspace covariant derivative; it is defined by

$$D = \frac{\partial}{\partial\bar{\theta}} - i\rho^a\theta\partial_a.$$

The superstring action (3.11) now takes the form

$$S = \frac{i}{4\pi}\int d^2\sigma d^2\theta \bar{D}Y^\mu DY_\mu. \tag{3.13}$$

This action is manifestly supersymmetric and there is a lot more that can be said about it, but we would diverge from our main purpose. We refer to [55] for further reading, this is also contained in other literature on string theory.

Scattering Amplitudes

Now that we have a better understanding of the superstring action, we will proceed to obtain the most basic amplitudes. These will also be the tachyon amplitudes, although, as previously mentioned, they do not form a part of the physical spectrum of the superstring, they do appear in the calculations and one may obtain them as a warmup in preparation for other amplitudes. The vertex operator is now

$$V_T(k,y) = k\cdot\psi(y) : e^{ik\cdot X(y)} :$$

As you can see the fermionic field appears as a coefficient of the bosonic tachyon vertex operator. This may pose a problem since it means we cannot use the techniques used before. But there is in fact a way to do it, using the power of the Grassmann variables. The trick is to write the coefficient and the vertex operator in the following way

$$k\cdot\psi = \int d\theta e^{\theta k\cdot\psi} \Rightarrow \frac{1}{\sqrt{y}}V_T(k,y) = \int \exp\{ik\cdot X(y) + \theta k\cdot\psi(y)/\sqrt{y}\}.$$

The division by $\sqrt{y}$ is convenient because of the propagators

$$\left\langle \frac{\psi^\mu(y_i)}{\sqrt{y_i}}\frac{\psi^\nu(y_j)}{\sqrt{y_j}} \right\rangle = \frac{\eta^{\mu\nu}}{y_i - y_j}; \quad \langle X^\mu(y_i)X^\nu(y_j)\rangle = -\eta^{\mu\nu}\ln(y_i - y_j).$$

From this we can compute the following

$$\begin{aligned} \left\langle \frac{V_T(y_i)}{\sqrt{y_i}}\frac{V_T(y_j)}{\sqrt{y_j}} \right\rangle &= \int d\theta_i d\theta_j \exp\left\{k_i\cdot k_j\left(\ln(y_i-y_j) - \frac{\theta_i\theta_j}{y_i-y_j}\right)\right\} \\ = \int d\theta_i d\theta_j \exp\left\{k_i\cdot k_j \ln(y_i - y_j - \theta_i\theta_j)\right\} &= \int d\theta_i d\theta_j (y_i - y_j - \theta_i\theta_j)^{k_i\cdot k_j} \end{aligned} \tag{3.14}$$

Such manipulations are possible due to the unusual properties of the Grassmann variables. From this it is straightforward to compute the scattering amplitudes, we simply take the correlation of a product of N vertex operators and integrate over the insertion points y_i. Then

$$A_N(\mathbf{k}) = \int \prod_{i=1}^{N} dy_i \prod_{i=1}^{N} d\theta_i \prod_{1\leq i<j\leq N} (y_i - y_j - \theta_i\theta_j)^{k_i\cdot k_j}. \tag{3.15}$$

The last expression is a bit formal since we do not have a well defined notion of exponentiating a Grassmann variable to a complex power. However it should be understood as in the first line of (3.14), the exponential of two separate terms. In that form it is clear that we should expand the fermionic part using the algebraic rules of the Grassmann variables, and then using the integration over the θs to eliminate them completely, since they were introduced only as auxiliary variables. Then one can proceed with the gauge fixing of three points. The resulting expression is a sum of integrals of the Koba-Nielsen type with some factors involving the kinematic variables k, called appropriately kinematic factors. This will be done in more detail in the p-adic case, where we follow very closely the steps presented here.

Just as a last comment, we mention that other vertex operators can be put in a similar form using auxiliary Grassmann variables. For example a vector state vertex operator with polarization ζ can be written as

$$\frac{1}{y}V(\zeta, k, y) = \int d\phi d\theta \exp(ik\cdot X + \theta\phi\zeta\cdot \dot{X}/y + \theta k\cdot\psi/\sqrt{y} - \phi\zeta\cdot\psi/\sqrt{y}).$$

Again using an appropriately chosen (much more complicated) current, we can compute the scattering amplitudes of these states. We recommend seeing [56] for further details on this technique. This was a very quick and selective overview of some aspects of superstring theory, we reiterate that there is a lot of literature that the reader may consult to get much more information, for example [55, Section 7.3].

Chapter 4

p-adic Physics

p-Adic physics began in the 1980s with an article by Volovich [1] where he made a bold conjecture motivated by the existence of the Planck scale. He argued that at such scale the structure of the spacetime is not understood and therefore cannot be assumed to be well described by real numbers. Because of the unfathomable difference between cosmological lengths and the Planck scale, he argued that it is no longer fruitful to take the Archimedean property for granted. Therefore he proposed using p-adic numbers instead, that have a topology that is totally disconnected (meaning the only connected components are the individual points) and this fits better the ideas about elementary particles. He applied it to the bosonic string theory, specifically he constructed a p-adic analog of the Veneziano amplitude and proposed further developments.

Over the coming years, the idea of constructing p-adic analogs of string amplitudes and other systems flourished giving numerous articles on the subject. The main feature was that the expressions were usually simpler or easier to handle than the real or Archimedean counterparts. Many different amplitudes and systems were turned p-adic in some sense, although mostly by analogy with real physics.

In recent years, p-adic physics has seen a resurgence in the context of the AdS/CFT correspondence or holography. A p-adic analog of holography has been proposed [39, 40] with various following articles [83, 84, 41, 64, 85, 42, 86, 87, 88, 89, 90] touching on various subjects. Because the p-adics are not known in most of the physical community, many of these works are made in collaboration with mathematicians [91, 92, 93, 94]. Naturally this has led some more formal aspects of theoretical physics to be explored with rigor [35, 25, 27]. Some examples of purely non-Archimedean physical theories attempting to build completely rigorous computations [95, 22, 96]. Some ideas developed in the p-adic context have proven to be useful to solve issues in the ardchimedean physics [97, 26]. And in some cases both Archimedean and non-Archimedean cases are treated simultaneously [98, 99, 63, 100, 101, 102].

In this chapter we will talk about the origins and foundations for p-adic string theory, where p-adic physics originated. We will also mention some interesting and surprising relations between Archimedean and Non-Archimedean physics as well as a brief mention of other physical systems for which p-adic analogs have been constructed. One highly relevant reference is [18] for the initial stages of p-adic physics, one can see [5] for a more modern but brief review.

4.1 p-adic String Amplitudes

As was mentioned in the introduction, in [1] Volovich made a bold conjecture that at the Planck scale physics should be described using the field of p-adic numbers $\mathbb{Q}_p$. This proposal is quite radical as it changes completely the nature of spacetime, it even messes with causality because of the unordered nature of $\mathbb{Q}_p$. So where can we use $\mathbb{Q}_p$ other than in spacetime coordinates? Remember that in string theory we handle two spaces, the target space, that is the spacetime, and the worldsheet, usually taken to be a 2-dimensional Riemannian surface. So here we have two different spaces that are used for a single theory. If we do not want our spacetime to be p-adic we then need to make the worldsheet p-adic. Additionally, this choice makes the least deviation from usual physics because we really do not expect to see and find the actual nature of strings anyway. Some authors have also mentioned that going to $\mathbb{Q}_p$ is a natural generalization, given that in experiments we cannot measure other numbers than rationals, because we do not have infinite precision. This justifies exploring all the possible fields that contain the rationals, and because of Ostrowski's theorem, we know that the only possibilities are the reals and the p-adics.
Then we will be working with a p-adic valued worldsheet (more light on what this is in the next section) and complex valued amplitudes. The way we are going to work with string theory is focusing on the scattering amplitudes.

4.1.1 Open string

Historically, the p-adic string worlsheet was introduced through the integration variables of the scattering amplitudes. It was done in that way because one can easily write down p-adic analogs of the expressions for the amplitudes. In [103] it was shown that the sum of the channels of the Veneziano amplitude is

$$A(s,t,u) = A(st) + A(su) + A(ut); \quad A(st) = \int_0^1 x^{-\alpha(s)}(1-x)^{-\alpha(t)}dx, \tag{4.1}$$

provided that $\alpha(s) + \alpha(t) + \alpha(u) = 2$. We recall here the conditions on the momenta variables for this theory, the Mandelstam variables and the definition of $\alpha(s)$:

$$k_i^2 = 2 \qquad \sum_i k_i = 0. \tag{4.2}$$

$$s = -(k_1+k_2)^2; \quad t = -(k_1+k_3)^2; \quad u = -(k_2+k_3)^2; \quad \alpha(s) = \frac{s}{2} + 2 = -k_1 \cdot k_2.$$

The expression (4.1) can be written as a single integral using the absolute value

$$A(s,t,u) = \int_{-\infty}^{\infty} |x|^{-\alpha(s)}|1-x|^{-\alpha(t)}dx. \tag{4.3}$$

It is in this form that one can intuitively guess the p-adic analog. We replace the integration region, that is over the entire real line, with the p-adic line, that is, $\mathbb{Q}_p$. Also the absolute value is the natural norm on $\mathbb{R}$ so one is quick to suggest the change $|\cdot| \to |\cdot|_p$. Remember that the p-adic norm is a real number, so the amplitudes remain Archimedean while the integration is over $\mathbb{Q}_p$. Then the p-adic 4-point tree level open string tachyon amplitude is

$$A_p(s,t,u) = \int_{\mathbb{Q}_p} |x|_p^{-\alpha(s)}|1-x|_p^{-\alpha(t)}dx. \tag{4.4}$$

As you can see, compared to (4.3) we merely replaced the integration domain from $\mathbb{R} \to \mathbb{Q}_p$ and the absolute value for the p-adic norm. We will now integrate the expression (4.4) to get an explicit result. First we divide $\mathbb{Q}_p$ into three regions $D_1 := \{x \in \mathbb{Q}_p; |x|_p < 1\} = p\mathbb{Z}_p$, $D_2 := \{x \in \mathbb{Q}_p; |x|_p = 1\} = \mathbb{Z}_p^{\times}$, and $D_3 := \{x \in \mathbb{Q}_p; |x|_p > 1\} = \mathbb{Q}_p \setminus \mathbb{Z}_p$. Then for the first region we see that the result is

$$\int_{D_1} |x|_p^{-\alpha(s)}|1-x|_p^{-\alpha(t)}dx = \int_{p\mathbb{Z}_p} |x|_p^{-\alpha(s)}dx = \frac{(1-p^{-1})p^{\alpha(s)-1}}{1-p^{\alpha(s)-1}}.$$

We used the results and techniques of section 2.2.5. We can see that we have a term that depends only on the s channel. We comment on this later, let us do the rest of the regions. For the region D_2 we have

$$\int_{D_2} |x|_p^{-\alpha(s)}|1-x|_p^{-\alpha(t)}dx = \int_{1+p\mathbb{Z}_p} |1-x|_p^{-\alpha(t)}dx + \sum_{j=2}^{p-1}\int_{j+p\mathbb{Z}_p} dx$$

$$= \int_{p\mathbb{Z}_p} |y|_p^{-\alpha(t)}dy + p^{-1}(p-2) = \frac{(1-p^{-1})p^{\alpha(t)-1}}{1-p^{\alpha(t)-1}} + 1 - 2p^{-1}.$$

Again, we have an equal term dependent on the kinematic variables, but now it depends only on the channel t. In this case, we also got a constant term. On to the third region, for this case we must perform the change of variables $x = 1/y$. Then we will have the following changes

$$|x|_p^{-s} = |y|_p^{s}; \quad dx = |y|_p^{-2}dy; \quad x \in \mathbb{Q}_p \setminus \mathbb{Z}_p \Rightarrow y \in p\mathbb{Z}_p.$$

Then we get

$$\int_{D_3} |x|_p^{-\alpha(s)}|1-x|_p^{-\alpha(t)}dx = \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |x|_p^{-\alpha(s)-\alpha(t)}dx = \int_{p\mathbb{Z}_p} |y|_p^{\alpha(s)+\alpha(t)-2}dy = \frac{(1-p^{-1})p^{\alpha(u)-1}}{1-p^{\alpha(u)-1}}.$$

It is a little tricky to see that we get in fact the u channel, I invite the reader to check it, is a fun little exercise, one just needs to use the conditions (4.2) in the form $k_j \cdot (\sum_i k_i) = 0$. Then the full amplitude is

$$A_p(s,t,u) = (1-2p^{-1}) + \frac{(1-p^{-1})p^{\alpha(s)-1}}{1-p^{\alpha(s)-1}} + \frac{(1-p^{-1})p^{\alpha(t)-1}}{1-p^{\alpha(t)-1}} + \frac{(1-p^{-1})p^{\alpha(u)-1}}{1-p^{\alpha(u)-1}}. \tag{4.5}$$

We notice that, like in the Archimedean case, we have a term per channel, and each term has the same poles for the corresponding channel, from which one can say something about the p-adic string spectrum. However we refer to the references [18, 13] for that discussion. Here, we will look at something much closer to the spirit of this thesis. Since we are interested in the mathematical aspects of the amplitudes, we care about whether or not they are well defined, that is if there exists a connected region of convergence for this integral. Back in the section 2.2.5 where we learned p-adic integration, we saw that each of the integrals done here do converge but not for all $\alpha(\cdot) \in \mathbb{C}$. Then one wonders if there is a common region of convergence. The answer is yes! there is a common region of convergence that we will illustrate right now. In each term that involves a channel, one can see that the region of convergence is when $\mathrm{Re}(\alpha(\cdot)) < 1$ for s, t and u. And we have the relation $\alpha(u) = 2 - \alpha(s) - \alpha(t)$. Then we have the set of conditions

$$\begin{aligned} \mathrm{Re}(\alpha(s)) &< 1; \\ \mathrm{Re}(\alpha(t)) &< 1; \\ \mathrm{Re}(\alpha(s) + \alpha(t)) &> 1. \end{aligned} \tag{4.6}$$

After a careful examination one can see that there is a in fact a region of convergence for the real parts of $\alpha(s)$ and $\alpha(t)$, this region is illustrated in Figure 4.1. Having found it, we can say

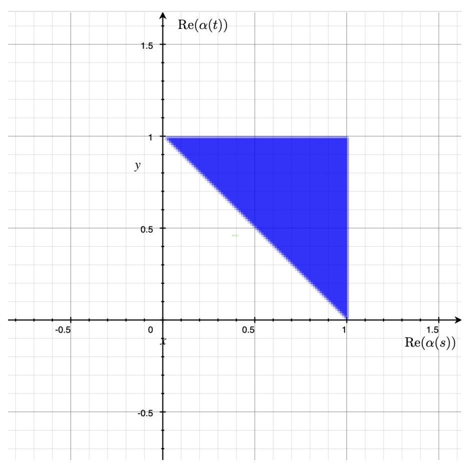

Figure 4.1: The region of convergence for the p-adic open string tree level tachyon 4-point amplitude. From the conditions (4.6), one can deduce that the region of convergence is the triangle shown, that is a connected region and therefore can be used to say that the amplitude is meromorphic on the plane and admits an analytic continuation.

that the amplitude admits a meromorphic continuation to the whole complex plane in each of the kinematical parameters, i.e. the momenta k_i. This is very important, having a region of convergence gives mathematical rigor to the amplitudes, which in turn endows physical sense to the interpretation we give them.

We now proceed to the full generalization of (4.4) to the case of N points, also known as the

Koba-Nielsen amplitudes. It turns out that the "trick" of summing over all channels into a single integral using absolute values holds for general N. Then we present the Koba-Nielsen amplitudes, that are the scattering amplitudes for N tachyons at tree level

$$A^{(N)}(\mathbf{k}) = \int_{\mathbb{R}^{N-3}} \prod_{i=2}^{N-2} |x_i|^{k_1\cdot k_i}|1-x_i|^{k_i\cdot k_{N-1}} \prod_{2\leq i<j\leq N-2} |x_i - x_j|^{k_i\cdot k_j} \prod_{i=2}^{N-2} dx_i. \tag{4.7}$$

where $\mathbf{k} = (k_1, \ldots, k_N) \in \mathbb{C}^N$. Of course you can quickly guess the p-adic analog of this expression, the same recipe applies. Without further ado, we write the p-adic Koba-Nielsen type string amplitudes.

$$A_p^{(N)}(\mathbf{k}) = \int_{\mathbb{Q}_p{}^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{k_1\cdot k_i}|1-x_i|_p^{k_i\cdot k_{N-1}} \prod_{2\leq i<j\leq N-2} |x_i - x_j|_p^{k_i\cdot k_j} \prod_{i=2}^{N-2} dx_i. \tag{4.8}$$

There are mathematical theorems that allow us to say that this integral is a rational function in the variables $p^{k_i\cdot k_j}$. The theorem mentioned is due to Igusa and is discussed further in the section on Local Zeta Functions 2.33. We know that it is a rational function, however we do not know its region of convergence. Recently, in [25] it was shown explicitly how to construct such region for arbitrary N. Of course it is a very complicated region, but as in the 4-point case, it is enclosed by a set of hyperplanes. Once again this allows for a regularization of the amplitudes and in principle one can get an explicit expression with a recursive formula presented in [25], although it is very lengthy and tedious.

4.1.2 Closed String

The previous amplitudes were for the open string, this can all be made for the closed string as well. As reference we have the Virasoro-Shapiro amplitude, that mathematically is basically the same as Veneziano, but the integration variable is complex and we integrate them over the entire complex plane. All of the gauge fixing has been done in the same way as for the open string.

$$A_{\mathbb{C}}(s,t,u) = \int_{\mathbb{C}} |z|^{k_1\cdot k_2/4}|1-z|^{k_1\cdot k_3/4} d^2z = \frac{\Gamma_{\mathbb{C}}(-\alpha(s)/2)\Gamma_{\mathbb{C}}(-\alpha(t)/2)}{\Gamma_{\mathbb{C}}(-\alpha(s)/2-\alpha(t)/2)}. \tag{4.9}$$

There are slight differences with some constants because we have a different mass relation, for closed strings we have $k^2 = 8$; of course we are using the standard norm on complex variables $|z|^2 = z\bar{z}$. Now $\alpha(x) = x/4 + 2$ and

$$\Gamma_{\mathbb{C}}(s) := \int_{\mathbb{C}} \exp[2\pi i(z+\bar{z})]|z|_{\mathbb{C}}^{s-1} dz,$$

is the Gelfand-Graev gamma function over $\mathbb{C}$ [18]. This is generalized to N-points in the same way by the expression

$$A_{\mathbb{C}}^{(N)}(\mathbf{k}) = \int_{\mathbb{C}^{N-3}} \prod_{i=2}^{N-2} |z_i|^{k_1\cdot k_i/4}|1-z_i|^{k_i\cdot k_{N-1}/4} \prod_{2\leq i<j\leq N-2} |z_i - z_j|^{k_i\cdot k_j/4} \prod_{i=2}^{N-2} d^2z_i. \tag{4.10}$$

Now the question is how to generalize this to the p-adic case. And it is here that there is a very interesting and I dare saying open problem. What is the p-adic analog of $\mathbb{C}$? Let us consider for a moment the mathematical properties of $\mathbb{C}$. It is an algebraically closed field, which makes it ideal to use to solve algebraic equations because there is always an answer. Another property is that it is a quadratic extension of $\mathbb{R}$, that is that it is made up of a linear combination of 2 real numbers, using $\sqrt{-1}$, that is the only number needed to make $\mathbb{R}$ complete. And finally it is a complete field, in the sense that every Cauchy sequence of complex numbers converges to a complex number. This is rare, it is unusual that we get such a nice field when extending the reals just by $\sqrt{-1}$. I mention this because it is a completely different story for $\mathbb{Q}_p$. First of all, there are more possible algebraic extensions than quadratic, in fact, there are an infinite number of possible extensions, with different degrees of roots added to the field. By adding them, we would get an algebraically closed field, great. But there is one problem, it is not a complete field, so it cannot still be a true analog of $\mathbb{C}$. If one goes to complete the algebraically closed p-adic field, one would get finally to what are called the complex p-adic numbers $\mathbb{C}_p$, that are both algebraically closed and complete. However, as you may have guessed, it is a much more complicated field, or rather, a very different field that $\mathbb{Q}_p$.

Physicists encountered this problem many years ago, and the route they took was to simply take a quadratic extension of $\mathbb{Q}_p$ as the analog field to build the closed string amplitudes. Beware that this does not mean that they thought it was the analog for the complex numbers, rather it is the best analog for coordinates of a 2 dimensional string worldsheet, after all, complex numbers were used because of the convenience, not because they were fundamental to the theory. So, we want to work on $\mathbb{Q}_p(\sqrt{\tau})$ with τ a number that does not have a square root on $\mathbb{Q}_p$. $\mathbb{Q}_p(\sqrt{\tau})$ is the quadratic extension of $\mathbb{Q}_p$ (see the section 2.2.4) and an element of this field has the form $z = x + \sqrt{\tau}y$ with $x, y \in \mathbb{Q}_p$. Then the p-adic closed string amplitudes analog to (4.9),(4.10) are

$$A_{\mathbb{Q}_p(\sqrt{\tau})}(s,t,u) = \int_{\mathbb{Q}_p(\sqrt{\tau})} |z|^{k_1\cdot k_2/4}|1-z|^{k_1\cdot k_3/4}dz; \tag{4.11}$$

$$A^{(N)}_{\mathbb{Q}_p(\sqrt{\tau})}(\mathbf{k}) = \int_{\mathbb{Q}_p(\sqrt{\tau})^{N-3}} \prod_{i=2}^{N-2} |z_i|^{k_1\cdot k_i/4}|1-z_i|^{k_i\cdot k_{N-1}/4} \prod_{2\leq i<j\leq N-2} |z_i - z_j|^{k_i\cdot k_j/4} \prod_{i=2}^{N-2} dz_i, \tag{4.12}$$

where $|z| = |z\bar{z}|_p = |x^2 - \tau y^2|_p$ [1]. The techniques to integrate this are really the same as for $\mathbb{Q}_p$, one only needs to adjust the measure change when changing variables, and the size of the residue field (see the section 2.2.5). Another issue arises (Yet another one!), there is more than one way to extend quadratically the p-adics. Recall the section 2.2.3, where we saw that there are three distinct quadratic extensions of $\mathbb{Q}_p$. The interesting thing is that we can easily check each of them for the case of 4-points, and see that the answers are very similar to the open

[1] Notice that this norm is different from the one defined in section 2.2.4, here it does not have the power 1/2. Like for the complex case, depends on your background you may be more familiar with the complex norm defined as $|z| = z\bar{z}$ or $|z| = (z\bar{z})^{1/2}$. The same applies to the p-adic case, here for the closed string it is better to not take the square root as it is completely analogous to the Archimedean case.

string case. In fact, for $\tau \neq \varepsilon$, the result is exactly the same as for the open string case, that is

$$A^{(4)}_{\mathbb{Q}_p(\sqrt{\tau})}(\mathbf{k}) = A^{(4)}_{\mathbb{Q}_p}(\mathbf{k}); \quad \tau \neq \varepsilon.$$

And when $\tau = \varepsilon$ we just need to make the replacement $p \to p^2$,

$$A^{(4)}_{\mathbb{Q}_p(\sqrt{\varepsilon})}(\mathbf{k}) = A^{(4)}_{\mathbb{Q}_p}(\mathbf{k})|_{p\to p^2}.$$

This can be extended to the general case of N-points. So as you can see we can obtain p-adic closed string amplitudes very easily provided we have the open string case. This is extremely useful as we can immediately conclude that the p-adic closed string amplitudes also admit a meromorphic continuation to the whole plane $\mathbb{C}$ in the form of rational functions in the variables $q^{-k_i k_j}$. One can go further and define amplitudes with both open and closed strings, for this type of things the reader may consult [18]. I would like to remind you that these amplitudes are for tachyons and are at tree level, there has been some discussion about loop amplitudes [104] and vector states [58], but the topic has been little discussed since.

4.1.3 The p-adic string worldsheet

The amplitudes (4.8),(4.12) are called p-adic string amplitudes, however, is this named justified? Is there a p-adic string, or are the expressions merely interesting mathematical objects inspired by string theory? In other words, should physicists be interested in this topic, or just be amused and satisfied that we gave mathematicians something interesting that they can publish? This type of questions are plagued throughout string theory history and is still relevant today in my opinion (and it is a topic that fascinates me if I am honest).

In [20], A. V. Zabrodin shed some light into this questions. He proved that the amplitudes built as direct p-adic analogs of the mathematical expressions, do in fact come from a more fundamental approach, i.e. a field theory defined on a suitable 2d space. This space is to be interpreted as the p-adic string worldsheet. Using standard techniques from field theory he was able to derive the tree level amplitudes starting from an action analog to the Polyakov action in the p-adic worldsheet.

Enough suspense, the space he used is actually a discrete space, a tree known as the Bruhat-Tits tree. It is a uniform infinite graph with no loops and valence $p+1$, that means that from each vertex there are $p+1$ edges. There is a natural interpretation of the boundary of the tree, it is identified with the projective line over $\mathbb{Q}_p$, See Fig. 4.2. This surprising result comes from a more mathematical translation of what the string worldsheet is. By that I mean that the usual string worldsheet is mapped to the upper half plane $\mathbb{H}$. This admits a homogenous space description as $SL(2,\mathbb{R})/SO(2)$. It is at this description that we switch to the p-adic world, the natural analog of $SL(2,\mathbb{R})$ is $PGL(2,\mathbb{Q}_p) = GL(2,\mathbb{Q}_p)/{\mathbb{Q}_p}^{\times}$ the group of fractional linear transformations of the projective line $P(\mathbb{Q}_p)$. Its maximal compact subgroup $PGL(2,\mathbb{Z}_p)$, is the analog of $SO(2)$. Therefore the p-adic analog of the string worldsheet turns out to be

$$\mathbb{H} = SL(2,\mathbb{R})/SO(2) \qquad \to \qquad PGL(2,\mathbb{Q}_p)/PGL(2,\mathbb{Z}_p) = T_p.$$

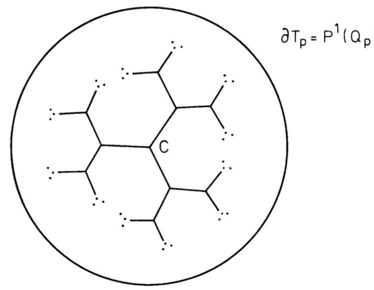

Figure 4.2: The Bruhat-Tits tree for the case $p = 2$. From an arbitrary chosen center point C, $p+1$ edges come out and continue infinitely (represented by the dots) towards the boundary. The circle represents the boundary and it can be identified with the projective p-adic line.[Image taken from [20]]

T_p is the Bruhat-Tits tree, and it is in this tree where we define the field theory. There is plenty to be said about how one works mathematically on the tree, but those details are left in [20] to be explored by the reader. Here we highlight the definition and derivation of the amplitudes. We start by defining the following action on the tree

$$S_p[X] = \frac{\beta_p}{2}\sum_e (X^\mu(z'_e) - X^\mu(z_e))(X_\nu(z'_e) - X_\nu(z_e)), \tag{4.13}$$

where $X^\mu(z_e)$ is the embedding function of the string worldsheet into spacetime, it is defined over the vertices of the tree; the sum is over all of the edges e of the tree and z_e, z'_e are the endpoints of the edge e. We assume a flat metric for the target space. This action is meant to be an analog of the Polyakov action in the conformal gauge, as you can see it is made of the square of the difference between neighboring points, that is the discrete analog of a derivative. To obtain the amplitudes we do the usual thing, they are the expectation values of vertex operators inserted on the boundary of the worldsheet, in this case the tree. The boundary is regularized to be of a finite radius R from a chosen center vertex, the distance is the number of edges from the center to the boundary. The we can define the amplitudes as

$$A_p^{(N)}(\mathbf{k}) = \lim_{R\to\infty}\sum_{\{z_j\}\in\partial T_p(R)} \frac{\int DX \exp\left(-S_p[X] + i\sum_{j=1}^N k_j \cdot X(z_j)\right)}{\int DX \exp\left(-S_p[X]\right)} \tag{4.14}$$

The z_j are the insertion points of the operators at the regulated boundary of the tree $\partial T_p(R)$; the sum is then over all possible insertion vertices at the boundary of the worldsheet. From here the usual techniques are applied, one finds a solution to the equation of motion of the argument in the exponential of the numerator. This is

$$\beta_p \Box X^\mu_{cl}(z) = i\sum_{j=1}^N k_j^\mu \delta_{z,z_j}, \quad \text{where} \quad \Box X(z) = \sum_{z'\sim z}(X(z) - X(z')). \tag{4.15}$$

The operation $\Box$ is the standard laplacian on a lattice, it is the difference between a central point and its closest neighbors, $z' \sim z$ means that z' is a neighbor of the vertex z. A slight clarification if you read [20], there the laplacian is defined as the negative of the definition in (4.15), the reason we stick to our choice is because it is used more in recent articles. In the classical solution suitable coundary conditions analog to Neumann boundary conditions are taken (See [20] for details). Then this solution is substituted in (4.14). One can define a measure on the tree for branches, that is for subtrees the go from a selected vertex outwardly to the boundary. This definition is

$$\mu_0(z) = p^{-d(C,z)},$$

where $d(\cdot,\cdot)$ is the number of edges between the center C of the tree and the vertex z. With the definition of R, in the limit $R \to \infty$ the sum over the regulated boundary can be changed to an integration measure as

$$\sum_{\partial T_p(R)} \to p^R \int_{\partial T_p} d\mu_0(y).$$

It turns out that the limit only exists if we choose $k_i^2 = 2\beta_p \ln(p)$, and with the condition $k_i^2 = 2$ we obtain that $\beta_p = 1/\ln(p)$. Finally after substituting everything and rearranging we get

$$A_p^{(N)}(\mathbf{k}) = \int_{\partial T_p} \prod_{i<j} |y_i, y_j|_p^{k_i \cdot k_j} \prod_{j=i}^{N} d\mu_0(y_j). \tag{4.16}$$

The measure is slightly different than the usual Haar measure on $\mathbb{Q}_p$, but it can be written in terms of the p-adic measure. After gauge fixing the usual three points and adjusting the measure, one can see that (4.16) is in fact the same as (4.8) proposed earlier. This derivation put the more heuristic previous derivations on a much stronger foundation, the p-adic amplitudes indeed come from a field theory defined on a suitable analog of the string worldsheet. Many articles since have used the Bruhat-Tits as the usual playground to define and study p-adic physics. This has acquired great relevance recently in the context of the AdS/CFT correspondence, although there the role of the Bruhat-Tits tree is not quite the same, it plays the role of the bulk, and the boundary, identified with $P(\mathbb{Q}_p)$, is where one expects to have p-adic CFTs. We talk more about this later in chapter 8.

There is another interesting and important aspect found in the appendix C of [20]. This is the fact that the action on the tree (4.13) can be integrated out to give an effective action on the boundary, from which one can obtain the amplitudes (4.16) in an alternative way. I also recommend the more recent article [65] that uses a similar technique to get the effective action on the boundary but in the case when the action has a mass term. I will not reproduce any details here, I just want to present the effective action

$$\tilde{S}_p[X] = \frac{p(p-1)}{4(p+1)\ln(p)} \int_{\mathbb{Q}_p{}^2} \frac{(X^\mu(y) - X^\mu(x))(X_\mu(y) - X_\mu(x))}{|y-x|_p^2} dydx. \tag{4.17}$$

This action had already been used in other works [21, 105, 106], and also in more recent works [51, 40, 49]. The reason is that it is easier to work with, because one does not have to deal with the discrete geometry of the tree and it is easier to generalize.

4.1.4 p-adic Superstring Amplitudes

In this section we are going to mention the previous work done on p-adic superstring amplitudes. The work done particularly on amplitudes was done at the end of the 1980s. This means that the general strategy was to take the mathematical expressions for the amplitudes (usually given in terms of gamma functions) and construct p-adic analogs of them, see [18, 13, 28]. Some works constructed fermionic actions over $\mathbb{Q}_p$ using Vladimirov derivatives and proposed supersymmetric formulations [59, 60]. Another approach is to take p-adic analogs of propagators to construct the amplitudes [58, 57]. There are more modern works that try a more fundamental approach by introducing fermions and spinors in the Bruhat-Tits tree [64, 65]. A common technique used recently is to start from Archimedean actions in momentum space, where operators take an algebraic form, and then turning them p-adic in an intuitive way, this allows to describe in a simultaneous way bosons and fermions [63].

Here I will show some of the p-adic superstring amplitudes that have been built. The integral representation of the amplitudes usually can be written in terms of gamma functions, this applies to the previous two cases seen before of the open and closed bosonic string. We'll come back to those in the next section. For this section we simply show that the 4-point amplitude of the superstring in the Neveu-Schwarz-Ramond formalism can be written as

$$K(\mathbf{k},\zeta)\left(\frac{\Gamma(-\frac{1}{2}s)\Gamma(-\frac{1}{2}s)}{\Gamma(-\frac{1}{2}(s+t))}+\frac{\Gamma(-\frac{1}{2}s)\Gamma(-\frac{1}{2}u)}{\Gamma(-\frac{1}{2}(s+u))}+\frac{\Gamma(-\frac{1}{2}u)\Gamma(-\frac{1}{2}u)}{\Gamma(-\frac{1}{2}(u+t))}\right), \tag{4.18}$$

where $K(\mathbf{k},\zeta)$ is a kinematical factor, a polynomial on the momenta variables and polarization vectors of the states ζ. For this case we have the mass condition $s+t+u=0$. The common expression (4.18) admits another form as [18]

$$-iK(\mathbf{k},\zeta)\hat{\Gamma}(-\frac{1}{2}s)\hat{\Gamma}(-\frac{1}{2}t)\hat{\Gamma}(-\frac{1}{2}t),$$

where

$$\hat{\Gamma}(x):=\int_{\mathbb{R}}e^{2\pi iw}|w|^{x-1}\mathrm{sgn}(w)dw.$$

It is in this form that one can easily write a p-adic analog of this amplitude. We take the definition

$$\hat{\Gamma}_p^{(\tau)}(x):=\int_{\mathbb{Q}_p}e^{2\pi iw}|w|_p^{x-1}\mathrm{sgn}(w)dw=\begin{cases}(1+p^{s-1})/(1+sp^{-x}), & \tau=\varepsilon\\ \pm\sqrt{\mathrm{sgn}(-1)}p^{x-1/2}, & \tau\neq\varepsilon\end{cases}. \tag{4.19}$$

The sign shown as $\pm$ depends on the prime p and on τ, but it is not relevant here. Then it is simply proposed the following p-adic 4-point superstring amplitude

$$A_p^{(4)}(\mathbf{k},\zeta)=-iK(\mathbf{k},\zeta)\hat{\Gamma}_p^{(\tau)}(-\frac{1}{2}s)\hat{\Gamma}_p^{(\tau)}(-\frac{1}{2}t)\hat{\Gamma}_p^{(\tau)}(-\frac{1}{2}t). \tag{4.20}$$

Another example is in [107], where the amplitude for 4 external massless scalars, is written in the following form ready to turn p-adic

$$A = -\frac{64}{3}C(s^2+t^2+u^2)\left(\frac{\Gamma(2-\frac{1}{2}\alpha_s)\Gamma(1-\frac{1}{2}\alpha_t)\Gamma(1-\frac{1}{2}\alpha_u)}{\Gamma(3-\frac{1}{2}\alpha_s-\frac{1}{2}\alpha_t)\Gamma(2-\frac{1}{2}\alpha_t-\frac{1}{2}\alpha_u)\Gamma(3-\frac{1}{2}\alpha_u-\frac{1}{2}\alpha_t s)}+2\text{perms.}\right),$$

where now $\alpha_x = 2+\frac{1}{4}x$. And now, in the usual analogy from the time, it is known that the expression in terms of gamma functions comes from the integral

$$\int d^2z|z|^{2A}|1-z|^{2B} = \pi\frac{\Gamma(A+1)\Gamma(B+1)\Gamma(-1-A-B)}{\Gamma(A+B+2)\Gamma(-A)\Gamma(-B)}.$$

This is easily turned p-adic as

$$\int_{\mathbb{Q}_p(\sqrt{p})} dz|z|_p^{2A}|z|_p^{2B} = \Gamma_p(A+1)\Gamma_p(B+1)\Gamma_p(-1-A-B),$$

with the Gel'fand-Graev p-adic gamma function [7, 18]

$$\Gamma_p(s) = \int_{\mathbb{Q}_p} e^{2\pi ix}|x|_p^{s-1}dx = \frac{1-p^{s-1}}{1-p^{-s}}.$$

One can then turn the amplitude (4.20) p-adic. We do not show the explicit expression, we care here more about the method. We point out that the choice $\tau = p$ was taken because the other two cases will give the same result but replacing $p \to p^2$.
Now, we mention also the work [57], where a similar approach to the previous two was referred to as the simplest approach and another route is proposed. There the amplitudes for the superstring are taken from the analogs of the integral expressions. Recall that this was the initial recipe for the open strings, however, there it was done because the amplitude summing over all channels, was written as a single integral over the real line. This doesn't happen for the superstring, so one faces the problem of finding the p-adic version of an ordered integral. The solution presented is to turn the amplitude for the emission of 4 fermions given by

$$A_{4F}(\mathbf{k},\mathbf{u}) = u^\alpha u^\beta u^\gamma u^\delta(-g^2/2)\int_0^1 d^2z|z|^{-1+k_2k_3}|1-z|^{-1-k_3k_4}[|1-z|\gamma^\mu_{\alpha\beta}\gamma^\mu_{\gamma\delta} - |z|\gamma^\mu_{\alpha\delta}\gamma^\mu_{\beta\gamma}],$$

into a p-adic analog. The main problem is the analog of the interval $[0,1]$. The solution they proposed is to use an analog of the characteristic function $\theta_{\tau,[0,1]}(z)$. There is more than one way to define it and it depends on τ. For example for $\tau = \varepsilon$ we can have

$$\theta_{\varepsilon,[0,1]}(z) = \frac{1}{2}[\text{sgn}_\varepsilon(z) - \text{sgn}_\varepsilon(-1)\text{sgn}_\varepsilon(1-z)]$$

Then the amplitude may be written in terms of p-adic Beta functions with different multiplicative characters, some just including the norm and some having also the sign function. We omit details, the reader may consult the references cited.
Another more fundamental approach is to go to the expressions for the amplitudes in terms of

propagators, and use an appropriate p-adic fermion propagator. In [58] a p-adic analog of the fermionic action is proposed

$$S_F^{(p,\tau)} \sim \int_{\mathbb{Q}_p} dx dy \psi(x) \frac{\text{sgn}(x-y)}{|x-y|_p} \psi(y).$$

To keep the statistics one must choose to have $\text{sgn}(-1) = -1$. This restricts us to $\tau \neq \varepsilon$ and $p \equiv 3 \mod 4$. [2] Then the expression used to obtain the tachyon amplitudes for the superstring is

$$A_N^{(p)} = V^{-1} \int_{\mathbb{Q}_p^N} \prod_{i=1}^{N} dx_i \prod_{i<j} \text{sgn}(x_I - x_j) \left\langle \prod_{i=1}^{N} k_i \psi(x_i) \right\rangle_F \left\langle \prod_{j=1}^{N} \exp\{ik_j \phi(x_j)\} \right\rangle_B \tag{4.21}$$

This comes from an almost identical expression in the Archimedean case, just making the obvious replacements between number fields. Then, using the propagator

$$\langle \psi(x) \psi(x') \rangle_F = \frac{\text{sgn}(x-x')}{|x-x'|_p},$$

the amplitude for 4 points becomes

$$\begin{aligned} \int_{\mathbb{Q}_p} dx |x|_p^{-\alpha(s)} |1-x|_p^{-\alpha(t)} \left\{ \alpha^2(s) \frac{\text{sgn}(1-x)}{|x|_p} + \alpha^2(t) \frac{\text{sgn}(x)}{|1-x|_p} - \alpha^2(u) \text{sgn}(x) \text{sgn}(1-x) \right\} \\ = -p^{-1} \sum_{x=s,t,u} \alpha^2(x) \frac{1-p^{\alpha(x)+1}}{1-p^{\alpha(x)}}. \end{aligned} \tag{4.22}$$

This explicitly shows that the amplitude is symmetric in the channels.

4.2 Connection between Archimedean and Non-Archimedean physics

After all of this "p-adization" you might be wondering if it is at all useful for the Archimedean world, or is it just an interesting academic exercise. Well, this questions has been asked before and some answers have been given. We present two ways in which the Non-Archimedean and Archimedean worlds connect.

4.2.1 Adelic products

One very straight forward way is through what is known as adelic products. Now what are adeles? We take this section from [18], see the references therein for further reading. An adele x is an infinite string of numbers over the fields $\mathbb{R}$, $\mathbb{Q}_p$ for every p.

$$x = (x_\infty, x_2, x_3, \ldots, x_p, \ldots), \quad x_\infty \in \mathbb{Q}_\infty = \mathbb{R}, \quad x_p \in \mathbb{Q}_p.$$

[2] It is worth mentioning that a very similar fermionic term for the action was also proposed in the works [59, 60], the only difference being the exponent in the norm on the denominator, they proposed using 3/2 on grounds of being a sort of square root of the bosonic term.

It is common to use the notation $\mathbb{Q}_\infty$ for the real numbers in this context. The real and p-adic fields will be given the same treatment so it is convenient to have a common index that includes both. The set of adeles is denoted by $\mathbb{A}$ and forms a ring under addition and multiplication componentwise. It is common to demand that the adeles have only a finite number of components outside the p-adic ball. This ensures that $\mathbb{A}$ is locally compact and thus have a Haar measure, that would be the product of the measures of each field. An adele product is a product over the fields of objects defined for a general field. The basic adele product is using the norm. It is not hard to show that

$$1 = \prod_{v=2,3,5,\ldots}^{\infty} |r|_v = |r|_{\mathbb{A}}, \quad r \in \mathbb{Q}.$$

It is basically a consequence of the fundamental theorem of arithmetic, the p-adic measures extract the reciprocal prime factors of the rational r. On the process we defined the adelic norm as the product over the norms. These types of products over the fields are of special interest when they converge to a simple amount, like a constant. One less trivial example is with gamma functions, we can define the adelic gamma function

$$\Gamma_{\mathbb{A}}(s) = \int_{\mathbb{A}} \chi_{\mathbb{A}}(x)|x|_{\mathbb{A}}^{s-1} dx = \prod_v \Gamma_v(s), \quad \chi_{\mathbb{A}}(x) = \prod_v \chi_v(x_v).$$

It can be shown that the product is in fact identically 1, meaning $\Gamma_{\mathbb{A}}(s) = 1$. So as a function in the adeles it is trivial, but it presents a nontrivial relation between Archimedean and non-Archimedean places.

One of the most amazing results using these ideas is regarding the 4 point string amplitude. It is very satisfying that there is an adelic product for this amplitude that relates the Archimedean and Non-Archimedean theories [108, 12]. Concretely we have the relation

$$\prod_{v=2,3,5,\ldots}^{\infty} B^{(v)}(-\alpha(s), -\alpha(t)) = 1 \tag{4.23}$$

where

$$B^{(v)}(-\alpha(s), -\alpha(t)) = \int_{\mathbb{Q}_v} dy |y|_v^{-\alpha(s)-1} |1-y|_v^{-\alpha(t)-1}.$$

Since we know that the four point amplitudes are proportional to a beta function as defined above, we can conclude from (4.23) that the four point amplitudes for open string have an adelic relation. This is also true for closed strings with the Virasoro-Shapiro amplitude and the p-adic analogues. It could be said that the p-adic amplitudes are just parts of the real amplitude. Sometimes mathematicians talk about this using the word local to refer to the p-adic part of an expression for a specific prime p.

Following these ideas, in [28] it is shown another proposal for a superstring 4-point amplitude that satisfies an adelic relation.

4.2.2 The limit $p \to 1$

There is another interesting and surprising link between the theories. In [29] it was shown that the effective action for the p-adic open string amplitudes, when taking the formal limit $p \to 1$, one actually ends up with an action that is of the form of a string field theory. Let us see this in a little more detail. The effective action that reproduces the scattering amplitudes of the p-adic string tachyon is [13]

$$S(\phi) = \frac{1}{g^2}\frac{p^2}{p-1}\int d\sigma \left(-\frac{1}{2}\phi p^{-\frac{1}{2}\Delta}\phi + \frac{1}{p+1}\phi^{p+1}\right). \tag{4.24}$$

This is a non local action as portrayed by the infinite derivative operator $p^{-\frac{1}{2}\Delta}$, where Δ is the laplacian. The equations of motion for this action are

$$p^{-\frac{1}{2}\Delta}\phi = \phi^p. \tag{4.25}$$

It is here where we take the limit $p \to 1$. This limit is also discussed in [21]. We expand the equations of motion around $p = 1$ using the expansions

$$p^{-\frac{1}{2}\Delta} = 1 - \frac{(p-1)}{2}\Delta + \frac{1}{2}\frac{(p-1)^2}{2^2}\Delta^2 - \dots$$

$$\phi^p = \phi + (p-1)\phi\ln\phi + \frac{1}{2}(p-1)^2\phi^2(\ln\phi)^2 + \dots$$

Substituting in the equations (4.25), to order $(p-1)$ we get

$$\Delta\phi = -2\phi\ln\phi.$$

This equation can be obtained from a more traditional lagrangian for the tachyon field ϕ with a usual kinetic term and the potential

$$V(\phi) = -\phi^2\ln\frac{\phi^2}{e}.$$

But this is remarkable, because the same potential appears, after a field redefinition, in the off-shell effective Lagrangian for tachyons on the string field theory. More recently in [31], the $\lambda\phi^4$ scalar field theory is studied over $\mathbb{Q}_p$. There the limit $p \to 1$ is analysed and found to be also connected to real string field theory by means of exact relations between the effective potentials of real and p-adic theories.

The limit is formal because remember that p should be a prime, and this is essential for its mathematical construction, so technically, the limit cannot be taken. However in [35] a rigorous way to take this limit was established. The first step is to generalize the Koba-Nielsen amplitudes (4.7) to the case in which the integration variables are now on an algebraic extension of $\mathbb{Q}_p$ of degree e, denoted by $\mathbb{K}_e$. There is more than one way to extend the field given a degree, for more details see the section 2.2.4 but only the degree of the extension is relevant here. Then we have the generalized p-adic amplitudes

$$A^{(N)}(\mathbf{k},\mathbb{K}_e)=\int_{\mathbb{K}_e{}^{N-3}}\prod_{i=2}^{N-2}|x_i|_{\mathbb{K}_e}^{k_1\cdot k_i}|1-x_i|_{\mathbb{K}_e}^{k_i\cdot k_{N-1}}\prod_{2\leq i<j\leq N-2}|x_i-x_j|_{\mathbb{K}_e}^{k_i\cdot k_j}\prod_{i=2}^{N-2}dx_i.$$

Basically all of the previous results for the normal p-adic amplitudes hold, it is just a matter of adjusting certain powers of p that come from the extension. And as usual we attach a local zeta function to the amplitudes for their mathematical analysis. It is through this variable that we will take the limit $p\to 1$, not by actually taking said limit, but by taking the equivalent limit $e\to 0$. This makes sense because one can interpolate appropriately between different values of e, and in particular obtain an explicit expression including the parameter e from which to take the limit $e\to 0$. In this limit the zeta functions are called topological because their formulas are independent of the resolution of singularities chosen.

$$Z_{top}(\mathbf{s})=\lim_{e\to 0}Z(\mathbf{s},\mathbb{K}_e).$$

Then we simply take the definition of topological amplitudes as the ones attached to the topological zeta functions

$$A^{(N)}_{top}(\mathbf{k})=Z_{top}(\mathbf{s})|_{s_{ij}=k_i\cdot k_j}.$$

The crucial thing is that this rigorous limit in fact coincides with the naive limit of simply making $p\to 1$ in the explicit expressions for the amplitudes. Thus recovering the previous results setting in a stronger foundation.

4.3 p-adic strings on a background B field

In [51] a p-adic action including a term incorporating the background antisymmetric B- field was proposed. From this action the scattering amplitudes were derived using the path integral. Several issues were discussed in comparison with the Archimedean case. The action is assumed to come from a theory in the bulk in the same sense as in [20], however such a theory in the bulk is not described. After the derivation of the amplitudes some issues are discussed and a final form of the amplitudes for the open string is presented, the 4 point amplitude is obtained explicitly and the expression for the general N-point case is given. We will briefly summarize their procedure as it is the starting point for one of the articles that form the main body of this thesis.

As mentioned in a previous section, one can add terms to the string action that incorporate background fields, in particular a fixed antisymmetric matrix field usually denoted by $B_{\mu\nu}$. In the Archimedean case the added term is equivalent to a boundary term

$$\frac{1}{2}\int_{\Sigma}d^z\varepsilon^{ab}\partial_aX^\mu(z)\partial_bX^\nu(z)B_{\mu\nu}=\int_{\partial\Sigma}d\xi B_{\mu\nu}X^\mu(\xi)\partial_tX^\nu(\xi),$$

where Σ is the string worldsheet and ∂_t is the tangential derivative along $\partial\Sigma$. Now this form is very useful because we already know the effective action on the boundary of the p-adic worldsheet, it is (4.17). But one is faced with the question of what is the p-adic analog of ∂_t, the tangential derivative. Well in [58] they encountered the same question in a different context and proposed it to be

$$\partial_t^{(p)} X^\mu(\xi) = \int_{\mathbb{Q}_p} d\xi' \frac{\text{sgn}(\xi - \xi')}{|\xi - \xi'|_p^2} X^\mu(\xi').$$

Using it now it is possible to write a p-adic analog of the string action incorporating the B field. The proposal is the following

$$\begin{aligned} S_p[X,B] = \frac{T_0}{2} \Bigg[&\int_{\mathbb{Q}_p^2} \eta_{\mu\nu} \frac{(X^\mu(\xi) - X^\mu(\xi'))(X^\nu(\xi) - X^\nu(\xi'))}{|\xi - \xi'|_p^2} d\xi d\xi' \\ &+ i \frac{p+1}{p^2 \hat{\Gamma}^{(\tau)}(-1)} \int_{\mathbb{Q}_p^2} B_{\mu\nu} X^\mu(\xi) \frac{\text{sgn}(\xi - \xi')}{|\xi - \xi'|_p^2} X^\nu(\xi') d\xi d\xi' \Bigg], \end{aligned} \tag{4.26}$$

with the string tension $T_0 = \frac{p(p-1)}{2(p+1)\ln p} \frac{1}{\alpha'}$, and the generalized p-adic gamma function defined previously in (4.19). One often writes the first term as the analog to the kinetic term because it can be written in terms of a derivative, in some literature is known as a normal derivative because it can be seen to be the boundary limit of an outwardly radial difference in the tree (see for instance [20, 65]), but it also coincides with a Vladimirov derivative [7] of first order. Since we are talking about [51], we will use their definition, that is the following

$$\partial_n^{(p)} f(\xi) = \int_{\mathbb{Q}_p} \frac{f(\xi') - f(\xi)}{|\xi' - \xi|_p^2} d\xi'.$$

Using this definition the action can be written as

$$S_p[X,B] = T_0 \left[\int_{\mathbb{Q}_p} X^\mu(\xi) \left(-\eta_{\mu\nu} \partial_n^{(p)} + i \frac{p+1}{2p^2 \hat{\Gamma}^{(\tau)}(-1)} B_{\mu\nu} \partial_t^{(p)} \right) X^\nu(\xi) d\xi \right] = \int_{\mathbb{Q}_p} X^\mu(\xi) \Delta_{\mu\nu} X^\nu(\xi) d\xi.$$

Where we have defined the operator $\Delta_{\mu\nu}$ as the operator in parenthesis times the tension on the first equality. As usual, we want to obtain the scattering amplitudes by averaging the insertion of the tachyon vertex operators $e^{ik\cdot X(\xi)}$

$$A_B^{(p)}(\mathbf{k}) = \frac{\int DX \exp\left(-S_p[X,B] + i \sum_{I=1}^N k_I \cdot X(\xi_I) \right)}{\int DX \exp\left(-S_p[X,B] \right)}.$$

This is the same as a generating function $\mathcal{Z}[J]$ with the source $J_\mu(\xi) = \sum_{I=1}^N k_\mu^I \delta(\xi - \xi_I)$. The generating function satisfies the Dyson-Schwinger equation

$$\int_{\mathbb{Q}_p} d\xi' \Delta_{\mu\nu}(\xi - \xi') \frac{\delta \ln \mathcal{Z}[J]}{J_\nu(\xi')} = -J_\mu(\xi).$$

But with the source given, it is equivalent to obtaining the inverse operator or Green's function of $\Delta_{\mu\nu}$, that satisfies

$$\int_{\mathbb{Q}_p} \xi'' \Delta_{\mu\lambda}(\xi - \xi'')\mathcal{G}^{\lambda\nu}(\xi'' - \xi') = \delta_\mu^\nu \delta(\xi - \xi').$$

The strategy to solve this equation is using Fourier analysis to solve it algebraically and then transform back. We do not reproduce the details here but they are in [51]. The result is that the Green's function is

$$\mathcal{G}^{\mu\nu}(\xi - \xi') = -\alpha' G^{\mu\nu} \ln |\xi - \xi'|_p + \frac{i}{2}\theta^{\mu\nu}\text{sgn}(\xi - \xi'),$$

where the open string metric G and the antisymmetric parameter θ are defined by the relation

$$G^{-1} + \frac{i}{2}\frac{p-1}{\alpha' p \ln p \hat{\Gamma}^{(\tau)}(0)}\theta = \frac{1}{\eta - iB}.$$

Now finally the amplitudes will be given by

$$\begin{aligned} \left\langle e^{ik^1 \cdot X(\xi_1)} \dots e^{ik^N \cdot X(\xi_N)} \right\rangle_B &= \exp\left(-\frac{1}{2}\sum_{I,J=1}^N k_\mu^i k_\nu^J \mathcal{G}^{\mu\nu}(\xi_I - \xi_J)\right) \\ &= \prod_{\substack{I,J=1 \\ I<J}}^N \frac{\exp\left(-\frac{i}{2}\theta^{\mu\nu}k_\mu^i k_\nu^J \text{sgn}(\xi_I - \xi_J)\right)}{|\xi_I - \xi_J|_p^{-\alpha' G^{\mu\nu}k_\mu^I k_\nu^J}}. \end{aligned} \tag{4.27}$$

All that is left is to integrate over the insertion points to obtain the amplitudes. Two remarks before that. In the Archimedean case the incorporation of the B field in the quantized theory causes the spacetime coordinates to acquire a noncommutative product, making the spacetime noncommutative. This product is known as a Moyal product. Formally, one can argue the same happens in the p-adic case, one needs to look at the limit $\xi \to 0$ for the spacetime fields $X^\mu(\xi)$. Now this is a little complicated because there is no notion of ordering over $\mathbb{Q}_p$, however one may adopt a convention to do so, taking into account the values of sgn. When one does this one can see that indeed the spacetime coordinates do not commute in the target space because

$$[X^\mu(0), X^\nu(0)] = i\theta^{\mu\nu}.$$

And one can argue that the spacetime fields now have a noncommutative Moyal product $\Phi(X) \star \Psi(X)$ as a consequence of having the B-field. The authors in [51] warn us that the discussion is rather formal and should be taken cautiously.
Another important fact is the amplitudes lack a conformal invariance, or invariance under the group of Möbius transformations $GL(2, \mathbb{Q}_p)$. This is because of the sign function, that stops the symmetry from happening. In principle this does not allow the gauge fixing of 3 insertion points. However, to match the usual results, the authors took the gauge fixing by hand anyway. Now, there is some disagreement between the amplitudes just obtained in (4.27) and the amplitudes coming from a non-commutative deformation of the effective Lagrangian for the p-adic

string amplitudes (see [33])

$$\mathcal{L}^p_{NC}[\phi] = \frac{p}{p-1}\left[-\frac{1}{2}\phi\star p^{-\frac{1}{2}\Box-1}\phi + \sum_{n=3}^{p+1}\frac{(p-1)!}{n!(p-n+1)!}\left(\frac{g}{p}\right)^n(\star\phi)^n\right], \tag{4.28}$$

where ϕ is the tachyon field on the D-brane and g is the open string coupling. To remedy this, there is a variant on the proposal for the p-adic amplitudes incorporating a B-field in order to reproduce the deformed action (4.28). This approach takes the Archimedean amplitudes and constructs a direct analog from the 4-point amplitude and is then generalized to the N-point case. If you recall, the Archimedean amplitudes for the open string are really integrals of variables in the range $(0,1)$, only after summing over all channels were we able to write it as a single integral over the entire real domain. This doesn't always happen, and when we incorporate a B-field, it cannot be made any longer, so we go back to integrating channel by channel in the range $(0,1)$. For example the 4-point amplitude in this case is

$$\mathcal{A}^{(4)}_{B,tu} = \int_0^1 d\xi|\xi|^{k_1\cdot k_3}|1-\xi|^{k_2\cdot k_3}\cos\frac{1}{2}\left(k_1\theta k_2 + \text{sgn}(\xi)k_1\theta k_3 + \text{sgn}(1-\xi)k_2\theta k_3\right).$$

The main issue is how to implement the interval $(0,1)$ in $\mathbb{Q}_p$. The strategy is to write the integral above as an integral over $\mathbb{R}$ with a characteristic function $\Xi(\xi)$, of the desired interval, using for instance step functions in terms of sign functions. There are several alternatives to do it that are equivalent in the reals but not in the p-adics (we simply use the p-adic sign instead). One chooses the one that matches best the deformed effective action (4.28), by analysing the results. The chosen characteristic function for $\mathbb{Q}_p$ is

$$\Xi(\xi) = \frac{1}{2}[1+\text{sgn}(\xi)]\frac{1}{2}[1+\text{sgn}(1-\xi)] =: H_\tau(\xi)H_\tau(1-\xi),$$

where we defined the p-adic step functions $H_\tau(\xi) = \frac{1}{2}[1+\text{sgn}(\xi)]$. Using this, the p-adic analog for the 4-point amplitude in the tu channel of the open string in a background B-field is

$$\begin{aligned}\mathcal{A}^{(4)}_{p,tu}(\mathbf{k},\theta) &= \int_{\mathbb{Q}_p} d\xi\Xi(\xi)|\xi|_p^{k_1\cdot k_3}|1-\xi|_p^{k_2\cdot k_3}\\ &\times\cos\frac{1}{2}\left(\text{sgn}(-1)k_1\theta k_2 + \text{sgn}(\xi)k_1\theta k_3 + \text{sgn}(1-\xi)k_2\theta k_3\right).\end{aligned} \tag{4.29}$$

Upon summing the remaining channels st and ut the result agrees qualitatively with the deformed effective action (4.28). This has, formally, a straightforward generalization to the N-point case as

$$\begin{aligned}&\int_{\mathbb{Q}_p^{N-3}}\prod_{i=1}^{N-2}|x_i|_p^{k_1\cdot k_i}|1-x_i|_p^{k_i\cdot k_{N-1}}H_\tau(x_i)H_\tau(1-x_i)\\ \times&\prod_{2\le i<j\le N-2}|x_i-x_j|_p^{k_i\cdot k_j}H_\tau(x_i-x_j)\prod_{i=1}^{N-2}\exp\left(-\frac{i}{2}\sum_{2\le i<j\le N-2}k_i\theta k_j\text{sgn}(k_i-k_j)\right)dx_i.\end{aligned} \tag{4.30}$$

To get the full amplitude we should sum over all possible permutations of the momenta variables k_i. We baptize these amplitudes as Ghoshal-Kawano amplitudes, in honor of their authors, and will be studied in detail as the main part of [48] in an upcoming chapter.

Part II

p-adic String Amplitudes

Chapter 5

p-Adic String Amplitudes in a B-field

In this chapter we are going to study in detail the p-adic version of string amplitudes coupled to a constant background B-field. We already saw them in the previous chapter and named them Ghoshal-Kawano amplitudes in honor of their authors. This chapter is based on the work [48]. We attach to each amplitude a multivariate local zeta function depending on the kinematic parameters, the B-field and the Chan-Paton factors. We show that these integrals admit meromorphic continuations in the kinematic parameters. This result allows us to regularize the Ghoshal-Kawano amplitudes. The regularized amplitudes do not have ultraviolet divergencies. Due to the need for a certain symmetry, the theory works only for prime numbers which are congruent to 3 modulo 4. We also discuss the limit p→1in the noncommutative effective field theory and in the Ghoshal-Kawano amplitudes. We show that in the case of four points, the limit p→1 of the regularized Ghoshal-Kawano amplitudes coincides with the Feynman amplitudes attached to the limit p→1 of the noncommutative Gerasimov-Shatashvili Lagrangian.

5.1 The limit $p \to 1$ in the deformed effective action

Remember from the section 4.3 that the effective action for the p-adic tachyon string amplitudes was deformed by a noncommutative product to account for the effects of incorporating a constant background B-field, we write here the action after a suitable field redefinition

$$S[\phi] = \frac{1}{g^2}\frac{p^2}{p-1}\int d^D x \left(-\frac{1}{2}\phi \star p^{-\frac{1}{2}\Delta}\phi + \frac{1}{p+1}(\star\phi)^{p+1}\right). \tag{5.1}$$

where g is the coupling constant and Δ is the Laplacian operator. By $(\star\phi)^n$ we mean $\phi\star\phi\star\cdots\star\phi$, n times. The product $\star$ is called a Moyal product and it is associative, consider two smooth functions f and g, then their Moyal product is defined as

$$(f \star g)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\mu}\right) f(x+y)g(x+z)\Big|_{y=z=0}. \tag{5.2}$$

We want to take the limit $p \to 1$, to do it, we need to expand the equations of motion around $p = 1$ and keep only the linear term. First, the equations of motion for this action are

$$p^{-\frac{1}{2}\Delta}\phi = (\star\phi)^p. \tag{5.3}$$

Notice that p in this context is just a parameter, and does not carry the fundamental importance of being a prime number. This was noticeable from the action because it is defined for real variables, this means that the limit $p \to 1$ can be taken without trouble. Following [29] we take the Taylor expansion of both sides of the equation (5.3)

$$p^{-\frac{1}{2}\Delta}\phi = \exp\left(-\frac{1}{2}\ln p\Delta\right)\phi = \phi - \frac{1}{2}\Delta\phi(p-1) + \cdots$$

$$(\star\phi)^p = \exp\left(p\ln(\star\phi)\right) = \phi + \phi\star\ln(\star\phi)(p-1) + \cdots$$

Keeping only the linear terms we have that the equality becomes

$$\Delta\phi = -2\phi\star\ln(\star\phi).$$

Now we can construct an action whose equation of motion is given by the equation above, this action is

$$S[\phi] = \int d^Dx\left((\partial\phi)^2 - (\star\phi)^2\star\ln\left[\frac{(\star\phi)^2}{e}\right]\right). \tag{5.4}$$

In noncommutative field theory, it is well known that the nontrivial noncommutative effect comes from the potential energy of the Lagrangian. The propagators associated with the kinetic energy of the Lagrangian are the same as the ones of the commutative theory. Thus the free Lagrangian with an external source $J(x)$ is

$$S_0[\phi] = \int d^Dx\left((\partial\phi)^2 + \phi(x)^2 + J(x)\phi(x)\right).$$

The propagators are given by $x_{ij} = \frac{1}{k_i\cdot k_j+1}$, where k_i, with $i = 1, ..., N$, are the external momenta of the particles. The Feynman rule for the interaction vertex can be obtained in the noncommutative theory by considering the cubic, quartic, etc. interaction terms and computing the correlation functions, see for instance, [109, 110].

5.1.1 Amplitudes from the noncommutative Gerasimov-Shatashvili Lagrangian

In this subsection we show how to extract the 4-point amplitudes from the noncommutative Gerasimov-Shatashvili Lagrangian (5.4). First we need to look closer at the interactions of the theory, that is the potential term. The generating functional of the free theory is

$$\mathcal{Z}_0[J] = \mathcal{N}[\det(\Delta - 1)]^{-1/2}\exp\left\{-\frac{i}{2\hbar}\int d^Dx\int d^Dx' J(x)G_F(x-x')J(x')\right\},$$

where $G_F(x-x')$ is the Green function of time-ordered product of two fields of the theory, $\mathcal{N}$ is a normalization constant, $[\det(\Delta-1)]^{-1/2}$ is a suitable regularization of the divergent determinant bosonic operator. Then the noncommutative action is

$$S(\phi)=\int d^Dx\big[(\partial\phi)^2+\phi^2-U(\star\phi)\big], \tag{5.5}$$

where $U(\star\phi)=2(\star\phi)^2\star\log(\star\phi)$. For the perturbative treatment we need the Taylor expansion of $U(\star\phi)$ to get the powers of ϕ, this is

$$U(\star\phi)=-\frac{25}{6}\phi\star\phi+8\phi\star\phi\star\phi-6\phi\star\phi\star\phi\star\phi+\cdots. \tag{5.6}$$

With this the generating $Z[J]$ functional incorporating the interaction is

$$\begin{aligned}\mathcal{Z}[J]=\exp\bigg\{&\frac{25i}{6\hbar}\int d^Dx\Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)\star\Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)\\ &-\frac{8i}{\hbar}\int d^Dx\Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)\star\Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)\star\Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)\\ &+\frac{6i}{\hbar}\int d^Dx\Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)\star\Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)\star\Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)\star\Big(-i\hbar\frac{\delta}{\delta J(x)}\Big)+\cdots\bigg\}\mathcal{Z}_0[J].\end{aligned} \tag{5.7}$$

In order to obtain the N-point amplitudes we apply N functional derivatives with respect to J to $\mathcal{Z}[J]$. We are interested in checking whether the connected tree-level scattering amplitudes of this theory match exactly with the corresponding p-adic amplitudes in the limit when p tends to one. The computation of the field theory performed here will be compared to the computation of the p-adic string amplitudes later at section 5.7.

5.1.2 Four-point amplitudes

In this case the connected amplitudes come from two terms of the potential, the cubic and the quartic terms. First we consider the quartic term from (5.6). After expanding the exponential function in the interacting generating functional, for this term we get

$$\begin{aligned}\mathcal{Z}[J]&=\cdots+6i\hbar^3\int d^Dx\bigg\{\Big(\frac{\delta}{\delta J(x)}\Big)\star\Big(\frac{\delta}{\delta J(x)}\Big)\star\Big(\frac{\delta}{\delta J(x)}\Big)\star\Big(\frac{\delta}{\delta J(x)}\Big)\bigg\}\mathcal{Z}_0[J]+\cdots\\ &=\cdots+6i\hbar^3\lim_{x=y_1=y_2=y_3=y_4}\ \lim_{w_1=w_2=w_3=w_4=0}\int d^Dy_1d^Dy_2d^Dy_3d^Dy_4\\ &\quad\times\exp\Big\{\frac{i}{2}\theta^{\mu_1\nu_1}\frac{\partial}{\partial w_1^{\mu_1}}\frac{\partial}{\partial w_2^{\nu_1}}\Big\}\exp\Big\{\frac{i}{2}\theta^{\mu_2\nu_2}\frac{\partial}{\partial w_3^{\mu_2}}\frac{\partial}{\partial w_4^{\nu_2}}\Big\}\\ &\quad\times\bigg\{\Big(\frac{\delta}{\delta J(y_1+w_1)}\Big)\Big(\frac{\delta}{\delta J(y_2+w_2)}\Big)\Big(\frac{\delta}{\delta J(y_3+w_3)}\Big)\Big(\frac{\delta}{\delta J(y_4+w_4)}\Big)\bigg\}\mathcal{Z}_0[J]+\cdots.\end{aligned} \tag{5.8}$$

A straightforward computation of the 4-point vertex gives

$$\frac{\delta^4\mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0}$$

$$= \cdots + 768 i \hbar^3 \int d^D x \Bigg\{ \Bigg\{ \cos\left(\frac{\partial_1 \theta \partial_2}{2}\right) \cos\left(\frac{\partial_3 \theta \partial_4}{2}\right) + \cos\left(\frac{\partial_1 \theta \partial_3}{2}\right) \cos\left(\frac{\partial_2 \theta \partial_4}{2}\right)$$

$$+ \cos\left(\frac{\partial_1 \theta \partial_4}{2}\right) \cos\left(\frac{\partial_2 \theta \partial_3}{2}\right) \Bigg\} \left[-\frac{i}{2\hbar} G_F(x - x_1) \right] \left[-\frac{i}{2\hbar} G_F(x - x_2) \right]$$

$$\times \left[-\frac{i}{2\hbar} G_F(x - x_3) \right] \left[-\frac{i}{2\hbar} G_F(x - x_4) \right] + \cdots , \tag{5.9}$$

where $G_F(x - y)$ is the propagator and $\partial_{1,2,3,4}$ are the partial derivative with respect to the coordinates x_1, x_2, x_3 and x_4, respectively.

The interaction term $8(\star\phi)^3$ from the potential contributes to the 4-point tree amplitudes at the second order in perturbation theory. They are described by Feynman diagrams with two vertices located at points y and z connected by a propagator $G_F(y - z)$ and with two external legs attached to each vertex. The relevant part of the generating functional is

$$\mathcal{Z}[J] = \cdots + 64 \hbar^4 \int d^D y \int d^D z \, \left(\star \frac{\delta}{\delta J(y)} \right)^3 \left(\star \frac{\delta}{\delta J(z)} \right)^3 \mathcal{Z}_0[J] + \cdots . \tag{5.10}$$

We write the Moyal product explicitly as

$$\mathcal{Z}[J] = \cdots + 64 \hbar^4 \lim_{y = y_1 = y_2 = y_3} \lim_{z = z_1 = z_2 = z_3} \lim_{w_1 = w_2 = w_3 = w_4 = 0} \int d^D y_1 d^D y_2 d^D y_3 d^D z_1 d^D z_2 d^D z_3$$

$$\times \exp\left\{ \frac{i}{2} \theta^{\mu_1 \nu_1} \frac{\partial}{\partial w_1^{\mu_1}} \frac{\partial}{\partial w_2^{\nu_1}} \right\} \exp\left\{ \frac{i}{2} \theta^{\mu_2 \nu_2} \frac{\partial}{\partial w_3^{\mu_2}} \frac{\partial}{\partial w_4^{\nu_2}} \right\}$$

$$\times \left\{ \left(\frac{\delta}{\delta J(y_1 + w_1)} \right) \left(\frac{\delta}{\delta J(y_2 + w_2)} \right) \left(\frac{\delta}{\delta J(y_3)} \right) \right\}$$

$$\times \left\{ \left(\frac{\delta}{\delta J(z_1 + w_3)} \right) \left(\frac{\delta}{\delta J(z_2 + w_4)} \right) \left(\frac{\delta}{\delta J(z_3)} \right) \right\} \mathcal{Z}_0[J] + \cdots .$$

Then the connected 4-point amplitudes from the cubic interaction yields to

$$\left. \frac{\delta^4 \mathcal{Z}[J]}{\delta J(x_1) \delta J(x_2) \delta J(x_3) \delta J(x_4)} \right|_{J=0} = \cdots + 8192 \hbar^4 \int d^D y \int d^D z \, \left[-\frac{i}{2\hbar} G_F(y - z) \right]$$

$$\times \Bigg\{ \cos\left(\frac{\partial_1 \theta \partial_2}{2}\right) \cos\left(\frac{\partial_3 \theta \partial_4}{2}\right)$$

$$\times \Bigg\{ \left[-\frac{i}{2\hbar} G_F(y - x_1) \right] \left[-\frac{i}{2\hbar} G_F(y - x_2) \right] \left[-\frac{i}{2\hbar} G_F(z - x_3) \right] \left[-\frac{i}{2\hbar} G_F(z - x_4) \right]$$

$$+ \left[-\frac{i}{2\hbar} G_F(z - x_1) \right] \left[-\frac{i}{2\hbar} G_F(z - x_2) \right] \left[-\frac{i}{2\hbar} G_F(y - x_3) \right] \left[-\frac{i}{2\hbar} G_F(y - x_4) \right] \Bigg\}$$

$$+ \cos\left(\frac{\partial_1 \theta \partial_3}{2}\right) \cos\left(\frac{\partial_2 \theta \partial_4}{2}\right)$$

$$\times \Bigg\{ \left[-\frac{i}{2\hbar} G_F(y - x_1) \right] \left[-\frac{i}{2\hbar} G_F(z - x_2) \right] \left[-\frac{i}{2\hbar} G_F(y - x_3) \right] \left[-\frac{i}{2\hbar} G_F(z - x_4) \right]$$

$$+\left[-\frac{i}{2\hbar}G_F(z-x_1)\right]\left[-\frac{i}{2\hbar}G_F(y-x_2)\right]\left[-\frac{i}{2\hbar}G_F(z-x_3)\left[-\frac{i}{2\hbar}G_F(y-x_4)\right]\right\}$$

$$+\cos\left(\frac{\partial_1\theta\partial_4}{2}\right)\cos\left(\frac{\partial_2\theta\partial_3}{2}\right)$$

$$\times\left\{\left[-\frac{i}{2\hbar}G_F(z-x_1)\right]\left[-\frac{i}{2\hbar}G_F(y-x_2)\right]\left[-\frac{i}{2\hbar}G_F(y-x_3)\right]\left[-\frac{i}{2\hbar}G_F(z-x_4)\right]\right.$$

$$+\left[-\frac{i}{2\hbar}G_F(y-x_1)\right]\left[-\frac{i}{2\hbar}G_F(z-x_2)\right]\left[-\frac{i}{2\hbar}G_F(z-x_3)\left[-\frac{i}{2\hbar}G_F(y-x_4)\right]\right\}\right\}+\cdots. \quad (5.11)$$

This total amplitude corresponds exactly to the sum of the partial amplitudes associated to the channels s, t and u. The sum of (5.11) and (5.9) constitutes the 4-point amplitude (at the tree-level). This amplitude agrees with the limit $p\to 1$ of the sum over the permutations of the momenta $\boldsymbol{k}_i$ of the 4-point p-adic amplitudes computed in section 5.5. The details of this calculation are given in section 5.7. Moreover, higher-order amplitudes in the limit $p\to 1$ can be computed similarly, but it will not be done here.

5.2 Twisted Multivariate Local Zeta Functions

We have talked about local zeta functions in 2.3. Here we add to that discussion by considering twisted multivariate local zeta functions. The twist comes from having a multiplicative character. Let $f_1(x),\ldots,f_m(x)\in\mathbb{Q}_p[x_1,\ldots,x_n]$ be non-constant polynomials, and let $\mathbb{D}:=\cup_{i=1}^m f_i^{-1}(0)$ be the divisor attached to them. Let $\chi_1,\ldots,\chi_m$ be multiplicative characters. We set $\boldsymbol{f}:=(f_1,\ldots,f_m)$, $\boldsymbol{\chi}:=(\chi_1,\ldots,\chi_m)$, and $\boldsymbol{s}:=(s_1,\ldots,s_m)\in\mathbb{C}^m$. The multivariate local zeta function attached to $(\boldsymbol{f},\boldsymbol{\chi},\Theta)$, with Θ a test function (i.e. a locally constant function with compact support), is defined as

$$Z_\Theta(\boldsymbol{s},\boldsymbol{\chi},\boldsymbol{f})=\int_{\mathbb{Q}_p^n\smallsetminus\mathbb{D}}\Theta(x)\prod_{i=1}^m\chi_i\left(ac\left(f_i(x)\right)\right)|f_i(x)|_p^{s_i}\prod_{i=1}^n dx_i, \quad (5.12)$$

with $\mathrm{Re}(s_i)>0$ for all i. The angular component $ac(x)$ is defined as $ac(x)=x|x|_p$. In other words, the angular component is a p-adic integer with the same digits as x. Integrals of type (5.12) are holomorphic functions in $\boldsymbol{s}$, which admit meromorphic continuations to the whole $\mathbb{C}^m$, as rational functions in the variables $p^{-s_1},\ldots,p^{-s_m}$ [75, Théorème 1.1.4.], see also [76].

We need to extend the previous result to when each $\chi_i\circ ac$ is the trivial character $\chi_{\mathrm{triv}}(x)$ or $\mathrm{sgn}_\tau(x)$. The difference is that $\mathrm{sgn}_\tau(x)$ depends on both the angular component, and order of x. By Hironaka's resolution of singularities theorem, $Z_\Theta(\boldsymbol{s},\boldsymbol{\chi},\boldsymbol{f})$ is a linear combination of integrals of type

$$\int_{c+p^e\mathbb{Z}_p^n}\prod_{j=1}^r\left\{|y_j|_p^{\sum_{i=1}^m N_{i,j}s_i+v_j-1}\chi_i^{N_{i,j}}(y_j)\right\}dy_j,$$

where $c=(c_1,\ldots,c_n)\in\mathbb{Q}_p^n$, $1\leq r\leq n$, $N_{i,j}$ and v_j are integers such that $N_{i,j}\geq 0$, $v_j>0$, for $i\in\{1,\ldots,m\}$, $j\in\mathcal{T}$(a finite set), see proof of [76, Theorem 8.2.1] and [75, Théorème 1.1.4.].

We only need to study the meromorphic continuation of an integral of the form

$$I(s):=\int\limits_{c_j+p^e\mathbb{Z}_p}|y_j|_p^{\sum_{i=1}^m N_{i,j}s_i+v_j-1}\operatorname{sgn}_\tau^{N_{i,j}}(y_j)\,dy_j,$$

since the case for $\chi_{\text{triv}}(x)$ is already known, see e.g. [76, Lemma 8.2.1]. Several cases occur. If $c_j\notin p^e\mathbb{Z}_p$, then because $|\cdot|_p$ and $\operatorname{sgn}_\tau(\cdot)$ are locally constant functions we get

$$I(s)=p^{-e}\ |c_j|_p^{\sum_{i=1}^m N_{i,j}s_i+v_j-1}\operatorname{sgn}_\tau^{N_{i,j}}(c_j).$$

If $c_j\in p^e\mathbb{Z}_p$, we have

$$\begin{aligned}I(s)&=\sum_{l=e}^{\infty}\int\limits_{p^l\mathbb{Z}_p^\times}|y_j|_p^{\sum_{i=1}^m N_{i,j}s_i+v_j-1}\operatorname{sgn}_\tau^{N_{i,j}}(y_j)\,dy_j\\&=\left\{\sum_{l=e}^{\infty}p^{-l\left(\sum_{i=1}^m N_{i,j}s_i+v_j\right)}\operatorname{sgn}_\tau^{N_{i,j}}\left(p^l\right)\right\}\int\limits_{\mathbb{Z}_p^\times}\operatorname{sgn}_\tau^{N_{i,j}}(u)\,du\\&=:J(s)\int\limits_{\mathbb{Z}_p^\times}\operatorname{sgn}_\tau^{N_{i,j}}(u)\,du,\end{aligned}$$

where $y_j=p^lu$.

Now if $\tau=\varepsilon$, $\operatorname{sgn}_\tau(u)=(-1)^{ord(u)}\equiv 1$ for any $u\in\mathbb{Z}_p^\times$, then $\int_{\mathbb{Z}_p^\times}\operatorname{sgn}_\varepsilon^{N_{i,j}}(u)\,du=1-p^{-1}$. In the case $\tau\neq\varepsilon$,

$$\int\limits_{\mathbb{Z}_p^\times}\operatorname{sgn}_\tau^{N_{i,j}}(u)\,du=\begin{cases}1-p^{-1} & \text{if}\quad N_{i,j}\text{ is even}\\ \\ 0 & \text{if}\quad N_{i,j}\text{ is odd.}\end{cases}$$

By using that

$$\operatorname{sgn}_\tau^{N_{i,j}}\left(p^l\right)=\operatorname{sgn}_\tau^{lN_{i,j}}(p)=\begin{cases}1 & \text{if}\quad l\text{ is even}\\ \\ \operatorname{sgn}_\tau^{N_{i,j}}(p) & \text{if}\quad l\text{ is odd,}\end{cases}$$

we have

$$\begin{aligned}J(s)&=\sum_{l=e}^{\infty}p^{-l\left(\sum_{i=1}^m N_{i,j}s_i+v_j\right)}\operatorname{sgn}_\tau^{lN_{i,j}}(p)\\&=\sum_{k=0}^{\infty}p^{-(k+e)\left(\sum_{i=1}^m N_{i,j}s_i+v_j\right)}\operatorname{sgn}_\tau^{N_{i,j}}\left(p^{k+e}\right)\\&=p^{-e\left(\sum_{i=1}^m N_{i,j}s_i+v_j\right)}\operatorname{sgn}_\tau^{N_{i,j}}(p^e)\sum_{k=0}^{\infty}p^{-k\left(\sum_{i=1}^m N_{i,j}s_i+v_j\right)}\operatorname{sgn}_\tau^{kN_{i,j}}(p).\end{aligned}$$

If $N_{f_i,j}$ is even

$$J(s)=p^{-e\left(\sum_{i=1}^{m}N_{i,j}s_i+v_j\right)}\sum_{k=0}^{\infty}p^{-k\left(\sum_{i=1}^{m}N_{i,j}s_i+v_j\right)}=\frac{p^{-e\left(\sum_{i=1}^{m}N_{i,j}s_i+v_j\right)}}{1-p^{-\sum_{i=1}^{m}N_{i,j}s_i-v_j}}.$$

If $N_{i,j}$ is odd, then $I(s)=0$. In conclusion, since $Z_\Theta\left(\boldsymbol{s},\boldsymbol{\chi},\boldsymbol{f}\right)$ is a finite linear combination of products of integrals of type $I(s)$, then $Z_\Theta\left(\boldsymbol{s},\boldsymbol{\chi},\boldsymbol{f}\right)$ admits a meromorphic continuation as a rational function in the variables $p^{-s_1},\ldots,p^{-s_m}$. More precisely,

$$Z_\Theta\left(\boldsymbol{s},\boldsymbol{\chi},\boldsymbol{f}\right)=\frac{L_{\Theta,\boldsymbol{\chi}}\left(\boldsymbol{s}\right)}{\prod\limits_{j\in\mathcal{T}}\left(1-p^{-\sum_{i=1}^{m}N_{i,j}s_j-v_j}\right)},\tag{5.13}$$

where $L_{\Theta,\boldsymbol{\chi}}\left(\boldsymbol{s}\right)$ is a polynomial in the variables $p^{-s_1},\ldots,p^{-s_m}$, and the real parts of its poles belong to the finite union of hyperplanes

$$\sum_{i=1}^{m}N_{i,j}\operatorname{Re}\left(s_i\right)+v_j=0,\qquad\text{for } j\in\mathcal{T}.$$

This result is a variation of [75, Théorème 1.1.4.].

5.3 The Ghoshal-Kawano local zeta function

In [51] Ghoshal and Kawano proposed the following amplitude (for the N-point tree-level, p-adic open string amplitude, with Chan-Paton rules in a constant B-field):

$$\begin{aligned}A^{(N)}\left(\boldsymbol{k},\theta,\tau;x_1,x_{N-1}\right):=&\int\limits_{\mathbb{Q}_p^{N-3}\smallsetminus\mathbb{D}}\prod_{i=2}^{N-2}|x_i|_p^{\boldsymbol{k}_1\boldsymbol{k}_i}\,|1-x_i|_p^{\boldsymbol{k}_{N-1}\boldsymbol{k}_i}\,H_\tau\left(x_i\right)H_\tau\left(1-x_i\right)\\&\times\prod_{2\le i<j\le N-2}|x_i-x_j|_p^{\boldsymbol{k}_i\boldsymbol{k}_j}\,H_\tau\left(x_i-x_j\right)\\&\times\ \exp\left\{-\frac{\sqrt{-1}}{2}\left(\sum_{1\le i<j\le N-1}(\boldsymbol{k}_i\theta\boldsymbol{k}_j)\operatorname{sgn}_\tau(x_i-x_j)\right)\right\}\prod_{i=2}^{N-2}dx_i,\end{aligned}\tag{5.14}$$

where $N\ge 4$, $\boldsymbol{k}=(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)$, $\boldsymbol{k}_i=(k_{0,i},\ldots,k_{l,i})$, $i=1,\ldots,N$, is the momentum vector of the i-th tachyon (with $\boldsymbol{k}_i\boldsymbol{k}_j=-k_{0,i}k_{0,j}+k_{1,i}k_{1,j}+\cdots+k_{l,i}k_{l,j}$) that satisfy momentum conservation $\sum_{i=1}^{N}\boldsymbol{k}_i=\boldsymbol{0}$ and $\boldsymbol{k}_i\boldsymbol{k}_i=2$, and θ is a fixed antisymmetric bilinear form.

$$\mathbb{D}:=\left\{(x_2,\ldots,x_{N-2})\in\mathbb{Q}_p^{N-3};\ \prod_{i=2}^{N-2}x_i\left(1-x_i\right)\prod_{2\le i<j\le N-2}\left(x_i-x_j\right)=0\right\}.$$

In the bosonic string theory $l=26$, however, this plays no role in our calculations.

In order to study the amplitude $\boldsymbol{A}^{(N)}\left(\underline{\boldsymbol{k}},\theta,\tau;x_1,x_{N-1}\right)$, we introduce

$$\boldsymbol{s}=(s_{ij})=\cup_{i=2}^{N-2}\left\{s_{1i},s_{(N-1)i}\right\}\cup\cup_{2\le i<j\le N-2}\left\{s_{ij}\right\}\in\mathbb{C}^{\mathbf{d}}$$

a list consisting of $\mathbf{d}$ complex variables, where s_{ij} is symmetric, and

$$\mathbf{d} := \begin{cases} 2(N-3) + \begin{pmatrix} N-3 \\ 2 \end{pmatrix} & \text{if } \ N \geq 5 \\ \\ 2 & \text{if } \ N = 4 \end{cases} = \frac{N\,(N-3)}{2}.$$

Furthermore, we introduce the variables $\widetilde{s}_{ij} \in \mathbb{R}$, for $1 \leq i < j \leq N-1$, and denote $\widetilde{\boldsymbol{s}} = (\widetilde{s}_{ij})$. We set

$$F(\boldsymbol{x}, \boldsymbol{s}, \tau) := \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} H_\tau(x_i) H_\tau(1 - x_i) \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} H_\tau(x_i - x_j),$$

and

$$\begin{aligned} E(\boldsymbol{x}, \widetilde{\boldsymbol{s}}, \tau; ; x_1, x_{N-1}) := \exp &\left\{ \frac{-\sqrt{-1}}{2} \left(\sum_{2 \leq j \leq N-1} \widetilde{s}_{1j} \mathrm{sgn}_\tau(x_1 - x_j) \right) \right\} \times \\ \exp &\left\{ \frac{-\sqrt{-1}}{2} \left(\sum_{2 \leq i \leq N-2} \widetilde{s}_{i(N-1)} \mathrm{sgn}_\tau(x_i - x_{N-1}) \right) \right\} \times \\ \exp &\left\{ \frac{-\sqrt{-1}}{2} \left(\sum_{2 \leq i < j \leq N-2} \widetilde{s}_{ij} \mathrm{sgn}_\tau(x_i - x_j) \right) \right\}. \end{aligned} \tag{5.15}$$

Later on, we will fix the points $x_1 = 0$, $x_{N-1} = 1$ and $x_N = \infty$. Now, we define the Ghoshal-Kawano local zeta function as

$$Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}) = \int_{\mathbb{Q}_p^{N-3} \smallsetminus \mathbb{D}} F(\boldsymbol{x}, \boldsymbol{s}, \tau) E(\boldsymbol{x}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}) \prod_{i=2}^{N-2} dx_i. \tag{5.16}$$

For the sake of simplicity, from now on, we will use $\mathbb{Q}_p^{N-3}$ as domain of integration in (5.16).

By using that $|E(\boldsymbol{x}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1})| = 1$, $|H_\tau(x_i)| \leq 1$, $|H_\tau(1 - x_i)| \leq 1$, for any i, and that $|H_\tau(x_i - x_j)| \leq 1$, for any i, j, we have

$$\begin{aligned} \left| Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}) \right| &\leq \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{\mathrm{Re}(s_{1i})} |1 - x_i|_p^{\mathrm{Re}\left(s_{(N-1)i}\right)} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{\mathrm{Re}(s_{ij})} \prod_{i=2}^{N-2} dx_i \\ &= Z^{(N)}\left(\mathrm{Re}(\boldsymbol{s})\right), \end{aligned}$$

where $Z^{(N)}(\boldsymbol{s})$ is the Koba-Nielsen string amplitude studied in [25], see also [26]. Since this last integral is holomorphic in an open set $\mathcal{K} \subset \mathbb{C}^{\mathbf{d}}$, we conclude that

$$Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau) \text{ is holomorphic in } \ \boldsymbol{s} \in \mathcal{K} \text{ for any } \widetilde{\boldsymbol{s}},\ \tau,\ x_1,\ x_{N-1}.$$

The Ghoshal-Kawano local zeta functions seem to miss something, the test functions that limits the integration region to a compact region. However, we will now show that there is no need for them. The functions H_τ do the job for us. To see it we set $T := \{2, \dots, N-2\}$, and define for $I \subseteq T$, *the sector attached to* I as

$$Sect(I) = \left\{ (x_2, \dots, x_{N-2}) \in \mathbb{Q}_p^{N-3}; |x_i|_p \leq 1 \ \Leftrightarrow i \in I \right\}.$$

Then $\mathbb{Q}_p^{N-3} = \bigsqcup_{I \subseteq T} Sect(I)$ and

$$Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}) = \sum_{I \subseteq T} Z_I^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}), \tag{5.17}$$

where $Z_I^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1})$ is like the right hand side of (5.16), but integrating over $Sect(I)$. We now notice that $Z_I^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}) \equiv 0$ if $I \neq T$. Indeed, in the case $I^c = T \smallsetminus I \neq \emptyset$, $F(\boldsymbol{x}, \boldsymbol{s}, \tau) \equiv 0$ due to the fact $H_\tau(x)H_\tau(-x)$ appears as a factor in $F(\boldsymbol{x}, \boldsymbol{s}, \tau)$, and that $H_\tau(x)H_\tau(-x) = 0$. For this reason, we redefine the Ghoshal-Kawano local zeta function as

$$Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}) = \int_{\mathbb{Z}_p^{N-3}} F(\boldsymbol{x}, \boldsymbol{s}, \tau) E(\boldsymbol{x}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}) \prod_{i=2}^{N-2} dx_i. \tag{5.18}$$

5.4 Meromorphic continuation of Ghoshal-Kawano local zeta function

Some formulae

For $\widetilde{s} \in \mathbb{R}$ and $x \in \mathbb{Q}_p \smallsetminus \{0\}$ we have,

$$\exp\left\{-\frac{i}{2}\widetilde{s} \cdot \operatorname{sgn}_\tau(x)\right\} = \cos\left(\frac{\widetilde{s}}{2}\right) - i \operatorname{sgn}_\tau(x) \sin\left(\frac{\widetilde{s}}{2}\right). \tag{5.19}$$

Using this relation, and the convention $x_1 = 0$, $x_{N-1} = 1$, we can see that

$$\exp\left\{-\frac{i}{2}\left(\sum_{2 \leq j \leq N-1} \widetilde{s}_{1j} \cdot \operatorname{sgn}_\tau(-x_j)\right)\right\} = \sum_{I \subseteq \{2,\dots,N-1\}} C_I(\widetilde{\boldsymbol{s}}) \prod_{j \in I} \operatorname{sgn}_\tau(x_j);$$

$$\exp\left\{-\frac{i}{2}\left(\sum_{2 \leq j \leq N-1} \widetilde{s}_{i(N-1)} \cdot \operatorname{sgn}_\tau(x_j - 1)\right)\right\} = \sum_{J \subseteq \{2,\dots,N-1\}} D_J(\widetilde{\boldsymbol{s}}) \prod_{j \in J} \operatorname{sgn}_\tau(1 - x_j);$$

$$\exp\left\{-\frac{i}{2}\left(\sum_{2 \leq i < j \leq N-2} \widetilde{s}_{ij} \cdot \operatorname{sgn}_\tau(x_i - x_j)\right)\right\} = \sum_{K \subseteq \{2 \leq i < j \leq N-2\}} D_K(\widetilde{\boldsymbol{s}}) \prod_{i,j \in K} \operatorname{sgn}_\tau(x_i - x_j),$$

with the convention that $\prod_{j \in \emptyset} \equiv 1$. Using (5.19), one can check the coefficients that C_I, D_J and D_K are complex functions depending on $\cos\left(\frac{\widetilde{s}_{ij}}{2}\right)$ and $\sin\left(\frac{\widetilde{s}_{ij}}{2}\right)$. Therefore,

$$E(\boldsymbol{x}, \widetilde{\boldsymbol{s}}, \tau) := \sum_{I,J,K} E_{I,J,K}(\widetilde{\boldsymbol{s}}) \prod_{j \in I} \operatorname{sgn}_\tau(x_j) \prod_{j \in J} \operatorname{sgn}_\tau(1 - x_j) \prod_{i,j \in K} \operatorname{sgn}_\tau(x_i - x_j). \tag{5.20}$$

In a similar way, we obtain that

$$\prod_{i=2}^{N-2} H_\tau(x_i) H_\tau(1-x_i) \prod_{2\leq i<j\leq N-2} H_\tau(x_i - x_j)$$
$$= \sum_{I,J,K} e_{I,J,K} \prod_{j\in I} \operatorname{sgn}_\tau(x_j) \prod_{j\in J} \operatorname{sgn}_\tau(1-x_j) \prod_{i,j\in K} \operatorname{sgn}_\tau(x_i - x_j), \tag{5.21}$$

where the $e_{I,J,K}$s are constants.

5.4.1 Meromorphic continuation of $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$

We fix the points $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$, and denote the corresponding Ghoshal-Kawano zeta function as $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$. By using formulae (5.19)-(5.21) and (5.18), $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ is a finite sum of integrals of type

$$C(\widetilde{\boldsymbol{s}}) \int_{\mathbb{Z}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1-x_i|_p^{s_{(N-1)i}} \prod_{2\leq i<j\leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{j\in I} \chi_\tau(x_j) \prod_{j\in J} \chi_\tau(1-x_j)$$
$$\times \prod_{i,j\in K} \chi_\tau(x_i - x_j) \prod_{i=2}^{N-2} dx_i,$$

where $C(\widetilde{\boldsymbol{s}})$ is an $\mathbb{R}$-analytic function, χ_τ denotes the trivial character or sgn_τ. This formula implies that $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ is a linear combination of multivariate Igusa local zeta functions with coefficients in the ring of $\mathbb{R}$-analytic functions in the variables $\widetilde{\boldsymbol{s}}$. Consequently, by (5.13), $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ admits a meromorphic continuation as a rational function in the variables $p^{-s_{1j}}$, $p^{-s_{(N-1)j}}$, $p^{-s_{ij}}$ and the real parts of its poles belong to the finite union of hyperplanes of type

$$\mathcal{H} = \left\{ s_{ij} \in \mathbb{C}^{\mathbf{d}};\ \sum_{ij\in M} N_{ij,k} \operatorname{Re}(s_{ij}) + \gamma_k = 0,\ \text{for } k \in T \right\},$$

where $N_{ij,k} \in \mathbb{N}$, $\gamma_k \in \mathbb{N} \smallsetminus \{0\}$, and M, T are finite sets. Furthermore, $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ is holomorphic in

$$\bigcap_{\mathcal{H}} \left\{ s_{ij} \in \mathbb{C}^{\mathbf{d}};\ \sum_{ij\in M} N_{ij,k} \operatorname{Re}(s_{ij}) + \gamma_k > 0,\ \text{for } k \in T \right\}.$$

Meromorphic continuation of $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ for arbitrary x_1, x_{N-1}

Originaly in [51], the Ghoshal-Kawano amplitude fixed the points $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$. However the meromorphic continuation of $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1})$ is valid for arbitrary points x_1, x_{N-1}. Indeed, notice that

$$\mathbb{Z}_p^{N-3} = \bigsqcup_{i=1}^{W} \boldsymbol{a}_i + p^L \mathbb{Z}_p^{N-3},$$

where W, L are positive integers, and $\boldsymbol{a}_i \in \mathbb{Z}_p^{N-3}$ for any i. With this notation, we have

$$Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1}) = \sum_{i=1}^{W} Z_{\boldsymbol{a}_i}^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1}),$$

where

$$Z_{\boldsymbol{b}}^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1}) := \int\limits_{\boldsymbol{b}+p^L\mathbb{Z}_p^{N-3}} F(\boldsymbol{x},\boldsymbol{s},\tau)E(\boldsymbol{x},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})\prod_{i=2}^{N-2} dx_i,$$

see (5.15). For sufficiently large L, the meromorphic continuation of $Z_{\boldsymbol{b}}^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})$ can be obtained by the methods presented in Sections (5.4)-(5.4.1), by computing a Taylor expansion of the polynomial $\prod_{i=2}^{N-2} x_i(1-x_i)\prod_{2\leq i<j\leq N-2}(x_i-x_j)$ near $\boldsymbol{b}$.

5.5 Explicit computation of $Z^{(4)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$

In this section we compute the Ghoshal-Kawano local zeta function for four points:

$$Z^{(4)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau) = \exp\left\{\frac{i}{2}\widetilde{s}_{13}\right\}\int\limits_{\mathbb{Z}_p} |x_2|_p^{s_{12}}\,|1-x_2|_p^{s_{32}}\,H_\tau(x_2)\,H_\tau(1-x_2)\,E^{(4)}(x_2,\widetilde{\boldsymbol{s}},\tau)\,dx_2,$$

where

$$E^{(4)}(x_2,\widetilde{\boldsymbol{s}},\tau) := E^{(4)}(x_2,\widetilde{s}_{12},\widetilde{s}_{32},\tau) = \exp\left\{\frac{i}{2}\Big(\widetilde{s}_{12}\mathrm{sgn}_\tau(x_2)+\widetilde{s}_{23}\mathrm{sgn}_\tau(1-x_2)\Big)\right\}.$$

Using that $\mathrm{sgn}_\tau(y)\in\{1,-1\}$ and $H_\tau(y)\in\{0,1\}$, one verifies that

$$\exp\left\{\frac{i}{2}\Big(\widetilde{s}_{12}\mathrm{sgn}_\tau(x_2)\Big)\right\}H_\tau(x_2) = \exp\left(\frac{i}{2}\widetilde{s}_{12}\right)H_\tau(x_2),$$

$$\exp\left\{\frac{i}{2}\Big(\widetilde{s}_{23}\mathrm{sgn}_\tau(1-x_2)\Big)\right\}H_\tau(1-x_2) = \exp\left(\frac{i}{2}\widetilde{s}_{23}\right)H_\tau(1-x_2),$$

and consequently,

$$E^{(4)}(\widetilde{s}_{12},\widetilde{s}_{32}) = \exp\left\{\frac{i}{2}\Big(\widetilde{s}_{12}+\widetilde{s}_{23}\Big)\right\},$$

and

$$Z^{(4)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau) = \exp\left\{\frac{i}{2}\Big(\widetilde{s}_{13}+\widetilde{s}_{12}+\widetilde{s}_{23}\Big)\right\}\int\limits_{\mathbb{Z}_p} |x_2|_p^{s_{12}}\,|1-x_2|_p^{s_{32}}\,H_\tau(x_2)\,H_\tau(1-x_2)\,dx_2.$$

We first compute some p-adic integrals needed in this section.

Some p-adic integrals

Formula 1

Assume that $S\subset Z_p\smallsetminus\{0\}$ satisfies $-S=S$. Then, for $\tau\in\{p,\varepsilon p\}$

$$\int\limits_{S} |x_2|_p^{s_{12}}\,\mathrm{sgn}_\tau(x_2)dx_2 = 0.$$

This follows from changing variables as $x_2=-y$ and using that $\mathrm{sgn}_\tau(-y)=-\mathrm{sgn}_\tau(y)$.

Formula 2

If $p \equiv 3 \bmod 4$ and $\tau \in \{p, \varepsilon p\}$, then

$$S(\tau,p) := \frac{1}{p}\sum_{j=2}^{p-1} H_\tau(j)H_\tau(1-j) = \frac{p-3}{4p}.$$

From the table 2.3 in section 2.2.3, for $j = 2, \ldots, p-1$,

$$H_\tau(j)H_\tau(1-j) = \frac{1}{4}\left\{1 + \left(\frac{j}{p}\right)\right\}\left\{1 - \left(\frac{j-1}{p}\right)\right\},$$

and thus

$$S(\tau,p) := \frac{1}{4p}\left\{p - 2 + \sum_{j=2}^{p-1}\left(\frac{j}{p}\right) - \sum_{j=2}^{p-1}\left(\frac{j-1}{p}\right) - \sum_{j=2}^{p-1}\left(\frac{j}{p}\right)\left(\frac{j-1}{p}\right)\right\}.$$

Now by using that $\sum\limits_{k=1}^{p-1}\left(\frac{k}{p}\right) = 0$, we get that

$$\sum_{j=2}^{p-1}\left(\frac{j}{p}\right) = -1 \text{ and } \sum_{j=2}^{p-1}\left(\frac{j-1}{p}\right) = \sum_{k=1}^{p-2}\left(\frac{k}{p}\right) = -\left(\frac{p-1}{p}\right) = 1,$$

and thus

$$S(\tau,p) = \frac{1}{4p}\left\{p - 4 - \sum_{k=1}^{p-2}\left(\frac{k+1}{p}\right)\left(\frac{k}{p}\right)\right\}.$$

To compute

$$L(\tau,p) := \sum_{k=1}^{p-2}\left(\frac{k+1}{p}\right)\left(\frac{k}{p}\right),$$

we define

$$A_{ij} = \left\{a \in \{1,\ldots,p-2\}\,;\left(\frac{a}{p}\right) = (-1)^i \text{ and } \left(\frac{a+1}{p}\right) = (-1)^j\right\},$$

then $\{1,\ldots,p-2\} = A_{00}\bigsqcup A_{01}\bigsqcup A_{10}\bigsqcup A_{11}$ and

$$L(\tau,p) = \#A_{00} - \#A_{01} - \#A_{10} + \#A_{11}.$$

Now, if $p \equiv 3 \bmod 4$, then

$$\#A_{00} = \#A_{10} = \#A_{11} = \frac{p-3}{4}, \text{ and } \#A_{01} = \frac{p+1}{4}, \tag{5.22}$$

see e.g. [111, Chapter 9, Exercise 5 in p. 201], and therefore

$$L(\tau,p) = \#A_{00} - \#A_{01} = -1, \text{ and } S(\tau,p) = \frac{1}{4p}(p-3).$$

Formula 3

Set

$$I(\boldsymbol{s},\tau)=\int\limits_{\mathbb{Z}_p}|x_2|_p^{s_{12}}\,|1-x_2|_p^{s_{32}}\,H_\tau\left(x_2\right)H_\tau\left(1-x_2\right)dx_2.$$

Then

$$I(\boldsymbol{s},\tau)=\frac{p-3}{4p}+\frac{p^{-1-s_{12}}\left(1-p^{-1}\right)}{2\left(1-p^{-1-s_{12}}\right)}+\frac{p^{-1-s_{32}}\left(1-p^{-1}\right)}{2\left(1-p^{-1-s_{32}}\right)}.\tag{5.23}$$

By using the partition

$$\mathbb{Z}_p=\bigsqcup\nolimits_{j=0}^{p-1}j+p\mathbb{Z}_p\tag{5.24}$$

and the fact that

$$H_\tau\left(x_2\right)|_{j+p\mathbb{Z}_p}=H_\tau\left(j\right)\ \text{for } j\neq0\ \text{ and } H_\tau\left(1-x_2\right)|_{j+p\mathbb{Z}_p}=H_\tau\left(1-j\right)\ \text{for } j\neq1,\tag{5.25}$$

we have

$$I(\boldsymbol{s},\tau)=\sum\nolimits_{j=0}^{p-1}I_j(\boldsymbol{s},\tau),\tag{5.26}$$

where

$$I_j(\boldsymbol{s},\tau)=\int\limits_{j+p\mathbb{Z}_p}|x_2|_p^{s_{12}}\,|1-x_2|_p^{s_{32}}\,H_\tau\left(x_2\right)H_\tau\left(1-x_2\right)dx_2.$$

If $j\neq0,1$, then

$$I_j(\boldsymbol{s},\tau)=p^{-1}H_\tau\left(j\right)H_\tau\left(1-j\right).\tag{5.27}$$

If $j=0$, then by using Formula 1,

$$\begin{aligned}I_0(\boldsymbol{s},\tau)&=\int\limits_{p\mathbb{Z}_p}|x_2|_p^{s_{12}}\,H_\tau\left(x_2\right)dx_2=\frac{1}{2}\int\limits_{p\mathbb{Z}_p}|x_2|_p^{s_{12}}\,dx_2+\frac{1}{2}\int\limits_{p\mathbb{Z}_p}|x_2|_p^{s_{12}}\,\mathrm{sgn}_\tau(x_2)dx_2\\&=\frac{1}{2}\int\limits_{p\mathbb{Z}_p}|x_2|_p^{s_{12}}\,dx_2=\frac{1}{2}\frac{p^{-1-s_{12}}\left(1-p^{-1}\right)}{1-p^{-1-s_{12}}}.\end{aligned}\tag{5.28}$$

The case $j=1$ is similar to the case $j=0$,

$$I_1(\boldsymbol{s},\tau)=\int\limits_{1+p\mathbb{Z}_p}|1-x_2|_p^{s_{32}}\,H_\tau\left(1-x_2\right)dx_2=\frac{1}{2}\frac{p^{-1-s_{32}}\left(1-p^{-1}\right)}{1-p^{-1-s_{32}}}.\tag{5.29}$$

Formula (5.23) follows from (5.26) by using (5.27)-(5.29) and Formula 2.

Computation of $Z^{(4)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$

In conclusion,

$$Z^{(4)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)=\exp\left\{\frac{i}{2}\left(\widetilde{s}_{13}+\widetilde{s}_{12}+\widetilde{s}_{23}\right)\right\}\left(\frac{p-3}{4p}+\frac{p^{-1-s_{12}}\left(1-p^{-1}\right)}{2\left(1-p^{-1-s_{12}}\right)}+\frac{p^{-1-s_{32}}\left(1-p^{-1}\right)}{2\left(1-p^{-1-s_{32}}\right)}\right)\tag{5.30}$$

is holomorphic in

$$\mathrm{Re}(s_{12})>-1\ \text{and}\ \mathrm{Re}(s_{32})>-1.\tag{5.31}$$

The above expression for $Z^{(4)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$ was also obtained in [51].

5.6 Explicit computation of $Z^{(5)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$

In this section we compute explicitly the amplitude for five points:

$$Z^{(5)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau) = \int_{\mathbb{Z}_p^2} E^{(5)}(x_2, x_3, \widetilde{\boldsymbol{s}}, \tau) F^{(5)}(x_2, x_3, \boldsymbol{s}, \tau) dx_2 dx_3,$$

with

$$\begin{aligned} E^{(5)}(x_2, x_3, \widetilde{\boldsymbol{s}}, \tau) = \exp\left\{\frac{-\sqrt{-1}}{2}\Big(\widetilde{s}_{14}\mathrm{sgn}_\tau(-1) + \widetilde{s}_{12}\mathrm{sgn}_\tau(-x_2) + \widetilde{s}_{13}\mathrm{sgn}_\tau(-x_3)\Big)\right\} \\ \times \exp\left\{\frac{-\sqrt{-1}}{2}\Big(\widetilde{s}_{42}\mathrm{sgn}_\tau(1-x_2) + \tilde{s}_{43}\mathrm{sgn}_\tau(1-x_3) + \tilde{s}_{23}\mathrm{sgn}_\tau(x_2 - x_3)\Big)\right\} \end{aligned}$$

and

$$\begin{aligned} F^{(5)}(x_2, x_3, \boldsymbol{s}, \tau) = |x_2|_p^{s_{12}} |x_3|_p^{s_{13}} |1-x_2|_p^{s_{42}} |1-x_3|_p^{s_{43}} |x_2 - x_3|_p^{s_{23}} \\ \times H_\tau(x_2) H_\tau(x_3) H_\tau(1-x_2) H_\tau(1-x_3) H_\tau(x_2 - x_3). \end{aligned}$$

Using the reasoning given at the beginning of the previous section we have

$$E^{(5)}(\widetilde{\boldsymbol{s}}) = \exp\left\{\frac{\sqrt{-1}}{2}\Big(\widetilde{s}_{14} + \widetilde{s}_{12} + \widetilde{s}_{13} + \widetilde{s}_{24} + \tilde{s}_{34} + \tilde{s}_{32}\Big)\right\},$$

and then

$$Z^{(5)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau) = E^{(5)}(\widetilde{\boldsymbol{s}}) L(\boldsymbol{s}, \tau), \tag{5.32}$$

where

$$L(\boldsymbol{s}, \tau) = \int_{\mathbb{Z}_p^2} F^{(5)}(x_2, x_3, \boldsymbol{s}, \tau) dx_2 dx_3. \tag{5.33}$$

More p-adic Sums and Integrals

First we give some formulae needed in the following calculations. Remember, in general we are using that $p \equiv 3 \bmod 4$ and $\tau \in \{p, \varepsilon p\}$, and consequently $\mathrm{sgn}_\tau(-x) = -\mathrm{sgn}_\tau(x)$, where the sign function sgn_τ is given in Table (2.3).

Formula 4

For $A \subset \{1, 2, \ldots, p-1\}$, and $H_\tau(x) = \frac{1}{2}(1 + \mathrm{sgn}_\tau(x))$, we have

$$V(A, p, \tau) := \sum_{\substack{i,j \in A \\ i \neq j}}^{p-1} H_\tau(i - j) = \frac{(\#A)(\#A - 1)}{2} = \binom{\#A}{2}.$$

Indeed

$$\begin{aligned} V(A, p, \tau) &= \sum_{\substack{i,j \in A \\ j<i}} H_\tau(i-j) + \sum_{\substack{i,j \in A \\ i<j}} H_\tau(-(j-i)) = \sum_{\substack{i,j \in A \\ j<i}} [H_\tau(i-j) + H_\tau(-(i-j))] \\ &= \frac{1}{2} \sum_{\substack{i,j \in A \\ j<i}} \left[1 + \left(\frac{i-j}{p}\right) + 1 - \left(\frac{i-j}{p}\right)\right] = \sum_{\substack{i,j \in A \\ j<i}} 1 = \binom{\#A}{2}. \end{aligned}$$

Formula 5

$$T(p,\tau) := \frac{1}{p^2}\sum_{\substack{i,j=2\\ i\neq j}}^{p-1} H_\tau(i)H_\tau(1-i)H_\tau(j)H_\tau(1-j)H_\tau(i-j) = \frac{(p-3)(p-7)}{32p^2}.$$

We define $B := \{k \in \{2,3,\ldots,p-1\}\,;\, H_\tau(k)H_\tau(1-k) = 1\}$. Then, using the results and notation in the proof of Formula 2, we have $\#B = \#A_{10} = \frac{p-3}{4}$, and

$$T(p,\tau) = \frac{1}{p^2}\sum_{\substack{i,j\in B\\ i\neq j}} H_\tau(i-j) = \frac{1}{p^2}\binom{\#B}{2} = \frac{1}{2p^2}\left(\frac{p-3}{4}\right)\left(\frac{p-7}{4}\right) = \frac{(p-3)(p-7)}{32p^2}.$$

Formula 6

For $a,\ b,\ c \in \mathbb{C}$, we set

$$L_{00}(a,b,c) := \frac{1}{8}\int_{(p\mathbb{Z}_p)^2} |x_2|_p^a |x_3|_p^b |x_2-x_3|_p^c dx_2dx_3, \text{ for } \operatorname{Re}(a),\ \operatorname{Re}(b),\ \operatorname{Re}(c) > 0.$$

Then $L_{00}(a,b,c)$ has a meromorphic continuation to the whole complex plane given by

$$L_{00}(a,b,c) = \frac{1}{8}\frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}}\left\{p^{-1}\left(p-2\right) + \frac{p^{-1-a}\left(1-p^{-1}\right)}{1-p^{-1-a}} + \frac{p^{-1-b}\left(1-p^{-1}\right)}{1-p^{-1-b}} + \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}}\right\}.$$

In order to compute $L_{00}(a,b,c)$, we introduce the following subsets:

$$A := \left\{(x_2,x_3)\in (p\mathbb{Z}_p)^2\,;\, \left|\frac{x_2}{x_3}\right|_p \le 1\right\},$$

$$B := \left\{(x_2,x_3)\in (p\mathbb{Z}_p)^2\,;\, \left|\frac{x_3}{x_2}\right|_p < 1\right\}.$$

Then

$$(p\mathbb{Z}_p)^2 \smallsetminus \left\{(x_2,x_3)\in (p\mathbb{Z}_p)^2\,;\, x_2x_3 = 0\right\} = A\bigsqcup B, \tag{5.34}$$

and $L_{00}(a,b,c) = L_{00}^{(A)}(a,b,c) + L_{00}^{(B)}(a,b,c)$, where

$$L_{00}^{(A)}(a,b,c) := \frac{1}{8}\int_A |x_2|_p^a |x_3|_p^b |x_2-x_3|_p^c dx_2dx_3,$$

and

$$L_{00}^{(B)}(a,b,c) := \frac{1}{8}\int_B |x_2|_p^a |x_3|_p^b |x_2-x_3|_p^c dx_2dx_3.$$

We compute first $L_{00}^{(A)}(a,b,c)$, by using the following change of variables:

$$x_2 = uv,\ x_3 = u. \tag{5.35}$$

Then $dx_2dx_3 = |u|_p \, dudv$ and

$$\begin{aligned} L_{00}^{(A)}(a,b,c) &= \frac{1}{8}\int_{p\mathbb{Z}_p\times\mathbb{Z}_p} |u|_p^{a+b+c+1}|v|_p^a|v-1|_p^c dudv \\ &= \frac{1}{8}\left\{\int_{p\mathbb{Z}_p} |u|_p^{a+b+c+1}du\right\}\left\{\int_{\mathbb{Z}_p} |v|_p^a|v-1|_p^c dv\right\} \\ &=: \frac{1}{8}\frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}} J\left(a,c\right). \end{aligned} \tag{5.36}$$

By using partition (5.24),

$$J\left(a,c\right) = \sum_{i=0}^{p-1} J_i\left(a,c\right).$$

For $i \neq 0, 1$,

$$J_i\left(a,c\right) = \int_{i+p\mathbb{Z}_p} |v|_p^a|v-1|_p^c dv = p^{-1},$$

and the contribution of all these integrals is

$$\sum_{i=2}^{p-1} J_i\left(a,c\right) = p^{-1}\left(p-2\right). \tag{5.37}$$

For $i = 0$,

$$J_0\left(a,c\right) = \int_{p\mathbb{Z}_p} |v|_p^a dv = \frac{p^{-1-a}\left(1-p^{-1}\right)}{1-p^{-1-a}}. \tag{5.38}$$

For $i = 1$,

$$J_1\left(a,c\right) = \int_{1+p\mathbb{Z}_p} |v-1|_p^c dv = \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}}. \tag{5.39}$$

Therefore, from (5.36)-(5.39),

$$L_{00}^{(A)}(a,b,c) = \frac{1}{8}\frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}}\left\{p^{-1}\left(p-2\right) + \frac{p^{-1-a}\left(1-p^{-1}\right)}{1-p^{-1-a}} + \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}}\right\}.$$

Now we compute $L_{00}^{(B)}(a,b,c)$, by using the following change of variables:

$$x_2 = t, \quad x_3 = zt. \tag{5.40}$$

Then $dx_2dx_3 = |t|_p \, dzdt$ and

$$\begin{aligned} L_{00}^{(B)}(a,b,c) &= \frac{1}{8}\int_{(p\mathbb{Z}_p)^2} |t|_p^{a+b+c+1}|z|_p^b|1-z|_p^c dzdt = \frac{1}{8}\int_{(p\mathbb{Z}_p)^2} |t|_p^{a+b+c+1}|z|_p^b|dzdt \\ &= \frac{1}{8}\frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}}\frac{p^{-1-b}\left(1-p^{-1}\right)}{1-p^{-1-b}}. \end{aligned}$$

Formula 7

For $a,\ b,\ c \in \mathbb{C}$,

$$L_{00}^{(1)}(a,b,c,\tau) := \frac{1}{8}\int_{(p\mathbb{Z}_p)^2} |x_2|_p^a |x_3|_p^b |x_2 - x_3|_p^c \mathrm{sgn}_\tau(x_2)\mathrm{sgn}_\tau(x_3) dx_2 dx_3$$
$$= \frac{1}{8}\frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}}\left\{-p^{-1} + \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}}\right\},$$

for $\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c) > 0$. By using partition (5.34), we get that $L_{00}^{(1)}(a,b,c,\tau) = L_{00}^{(1,A)}(a,b,c,\tau)$ $+L_{00}^{(1,B)}(a,b,c,\tau)$. We compute integral $L_{00}^{(1,A)}(a,b,c,\tau)$, respectively $L_{00}^{(1,B)}(a,b,c,\tau)$, by using change of variables (5.35), respectively (5.40), as follows:

$$L_{00}^{(1,A)}(a,b,c,\tau) = \frac{1}{8}\left\{\int_{p\mathbb{Z}_p} |u|_p^{a+b+c+1} du\right\}\left\{\int_{\mathbb{Z}_p} |v|_p^a |v-1|_p^c \mathrm{sgn}_\tau(v) dv\right\}$$
$$= \frac{1}{8}\frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}} K(a,c,\tau).$$

By using partition (5.24),

$$K(a,c,\tau) = \sum_{j=0}^{p-1} K_j(a,c,\tau).$$

For $j \neq 0, 1,\ K_j(a,c,\tau) = p^{-1}\mathrm{sgn}_\tau(j)$, thus, the contribution of all these integrals is

$$\sum_{j=2}^{p-1} K_j(a,c,\tau) = p^{-1}\sum_{j=2}^{p-1}\mathrm{sgn}_\tau(j) = p^{-1}\sum_{j=2}^{p-1}\left(\frac{j}{p}\right) = -p^{-1}.$$

For $j = 0$, by using the Formula 1, $K_0(a,c,\tau) = 0$. For $j = 1$,

$$K_1(a,c,\tau) = \int_{1+p\mathbb{Z}_p} |v-1|_p^c \mathrm{sgn}_\tau(v) dv = \int_{1+p\mathbb{Z}_p} |v-1|_p^c dv$$
$$= \int_{p\mathbb{Z}_p} |v|_p^c dv = \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}}.$$

In conclusion,

$$L_{00}^{(1,A)}(a,b,c,\tau) = \frac{1}{8}\frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}}\left\{-p^{-1} + \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}}\right\}.$$

Now, by Formula 1,

$$L_{00}^{(1,B)}(a,b,c,\tau) = \left\{\frac{1}{8}\int_{p\mathbb{Z}_p} |t|_p^{a+b+c+1} dt\right\}\left\{\int_{p\mathbb{Z}_p} |z|_p^b\ \mathrm{sgn}_\tau(z) dz\right\} = 0.$$

Formula 8

For $a,\ b,\ c \in \mathbb{C}$, we set

$$L_{00}^{(2)}(a,b,c,\tau) := \frac{1}{8}\int_{(p\mathbb{Z}_p)^2} |x_2|_p^a |x_3|_p^b |x_2 - x_3|_p^c\ \mathrm{sgn}_\tau(x_2)\mathrm{sgn}_\tau(x_2 - x_3) dx_2 dx_3.$$

Then

$$L_{00}^{(2)}(a,b,c,\tau) = L_{00}^{(1)}(a,c,b,\tau).$$

This identity is obtained by changing variables as $u = x_2$, $v = x_2 - x_3$, and using Formula 7.

Formula 9

For a, b, $c \in \mathbb{C}$, we set

$$L_{00}^{(3)}(a,b,c,\tau) := \frac{1}{8}\int_{(p\mathbb{Z}_p)^2} |x_2|_p^a |x_3|_p^b |x_2 - x_3|_p^c \ \mathrm{sgn}_\tau(x_3)\mathrm{sgn}_\tau(x_2 - x_3) dx_2 dx_3.$$

Then

$$L_{00}^{(3)}(a,b,c,\tau) = -L_{00}^{(2)}(b,a,c,\tau).$$

This formula follows from Formula 8 by changing variables as $(x_2, x_3) \to (x_3, x_2)$.

Computation of $Z^{(5)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$

To get $Z^{(5)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ we must compute the integral $L(\boldsymbol{s}, \tau)$ in (5.33). By using the partition

$$\mathbb{Z}_p^2 = \bigsqcup_{i,j=0}^{p-1} (i + p\mathbb{Z}_p) \times (j + p\mathbb{Z}_p)\,,$$

we have

$$L(\boldsymbol{s}, \tau) = \sum_{i,j=0}^{p-1} L_{ij}(\boldsymbol{s}, \tau),$$

where

$$L_{ij}(\boldsymbol{s}, \tau) = \int_{i+p\mathbb{Z}_p \times j+p\mathbb{Z}_p} F^{(5)}(x_2, x_3, \boldsymbol{s}, \tau) dx_2 dx_3.$$

To compute these integrals we must check different cases for the values of i, j separately.

Case $i, j \in \{2, 3, \ldots, p-1\}$ **and** $i \neq j$.

By using that $H_\tau \mid_{i+p\mathbb{Z}_p} = H_\tau(i)$ for $i \in \{1, \ldots, p-1\}$, we have

$$L_{ij}(\boldsymbol{s}, \tau) = p^{-2} H_\tau(i) H_\tau(1-i) H_\tau(j) H_\tau(1-i) H_\tau(i-j).$$

Now by using Formula 5, the contribution of all these integrals is

$$\sum_{\substack{i,j=2 \\ i \neq j}}^{p-1} L_{ij}(\boldsymbol{s}, \tau) = \frac{(p-3)(p-7)}{32p^2}. \tag{5.41}$$

Case $i, j \in \{2, 3, \ldots, p-1\}$ **and** $i = j$.

In this case, by using (5.28),

$$\begin{aligned}L_{ii}(\boldsymbol{s},\tau) &= H_\tau^2(i)H_\tau^2(1-i)\int_{i+p\mathbb{Z}_p\times i+p\mathbb{Z}_p}|x_2-x_3|_p^{s_{23}}H_\tau(x_2-x_3)dx_2dx_3\\ &= H_\tau(i)H_\tau(1-i)\int_{(p\mathbb{Z}_p)^2}|x_2-x_3|_p^{s_{23}}H_\tau(x_2-x_3)dx_2dx_3\\ &= p^{-1}H_\tau(i)H_\tau(1-i)\int_{p\mathbb{Z}_p}|x_2|_p^{s_{23}}H_\tau(x_2)dx_2 = p^{-1}H_\tau(i)H_\tau(1-i)I_0(s_{23},\tau)\\ &= p^{-1}H_\tau(i)H_\tau(1-i)\frac{p^{-1-s_{23}}(1-p^{-1})}{2(1-p^{-1-s_{23}})}.\end{aligned}$$

Now, by using that $p\equiv 3 \bmod 4$, $\tau\neq\varepsilon$, and Formula 2, the contribution of all these integrals is

$$\frac{p^{-1-s_{23}}(1-p^{-1})}{2(1-p^{-1-s_{23}})}\frac{1}{p}\sum_{j=2}^{p-1}H_\tau(j)H_\tau(1-j) = \left(\frac{p-3}{8p}\right)\frac{p^{-1-s_{23}}(1-p^{-1})}{(1-p^{-1-s_{23}})}. \tag{5.42}$$

Case $i=1$ **and** $j=0$.

In this case by using Formula 1,

$$\begin{aligned}L_{10}(\boldsymbol{s},\tau) &= \int_{1+p\mathbb{Z}_p\times p\mathbb{Z}_p}|1-x_2|_p^{s_{42}}|x_3|_p^{s_{13}}H_\tau(1-x_2)H_\tau(x_3)dx_2dx_3\\ &= \left\{\int_{1+p\mathbb{Z}_p}|1-x_2|_p^{s_{42}}H_\tau(1-x_2)dx_2\right\}\left\{\int_{p\mathbb{Z}_p}|x_3|_p^{s_{13}}H_\tau(x_3)dx_3\right\}\\ &= \left\{\int_{p\mathbb{Z}_p}|x_2|_p^{s_{42}}H_\tau(-x_2)dx_2\right\}\left\{\int_{p\mathbb{Z}_p}|x_3|_p^{s_{13}}H_\tau(x_3)dx_3\right\}\\ &= \left\{\frac{1}{2}\int_{p\mathbb{Z}_p}|x_2|_p^{s_{42}}dx_2\right\}\left\{\frac{1}{2}\int_{p\mathbb{Z}_p}|x_3|_p^{s_{13}}dx_3\right\} = \frac{(1-p^{-1})^2}{4}\frac{p^{-2-s_{42}-s_{13}}}{(1-p^{-1-s_{42}})(1-p^{-1-s_{13}})}.\end{aligned} \tag{5.43}$$

Case $i=0$ **and** $j=1$.

Since

$$H_\tau(x_2-x_3)\Big|_{p\mathbb{Z}_p\times 1+p\mathbb{Z}_p} = H_\tau(-1)=0,$$

we have $L_{01}(\boldsymbol{s},\tau)=0$.

Case $i=j=0$.

In this case,

$$L_{00}(\boldsymbol{s},\tau) = \int_{(p\mathbb{Z}_p)^2}|x_2|_p^{s_{12}}|x_3|_p^{s_{13}}|x_2-x_3|_p^{s_{23}}H_\tau(x_2)H_\tau(x_3)H_\tau(x_2-x_3)dx_2dx_3.$$

By using that

$$\begin{aligned}H_\tau(x_2)H_\tau(x_3)H_\tau(x_2-x_3) = \frac{1}{8}\{&1+\mathrm{sgn}_\tau(x_2)+\mathrm{sgn}_\tau(x_3)+\mathrm{sgn}_\tau(x_2-x_3)\\ &+\mathrm{sgn}_\tau(x_2)\mathrm{sgn}_\tau(x_3)+\mathrm{sgn}_\tau(x_2)\mathrm{sgn}_\tau(x_2-x_3)+\mathrm{sgn}_\tau(x_3)\mathrm{sgn}_\tau(x_2-x_3)\\ &+\mathrm{sgn}_\tau(x_2)\mathrm{sgn}_\tau(x_3)\mathrm{sgn}_\tau(x_2-x_3)\},\end{aligned}$$

and the notation introduced in Formulae 6 to 9, we have

$$\begin{aligned} L_{00}(\boldsymbol{s},\tau) &= L_{00}(s_{12},s_{13},s_{23}) + L_{00}^{(1)}(s_{12},s_{13},s_{23},\tau) + L_{00}^{(2)}(s_{12},s_{13},s_{23},\tau) + L_{00}^{(3)}(s_{12},s_{13},s_{23},\tau) \\ &= L_{00}(s_{12},s_{13},s_{23}) + L_{00}^{(1)}(s_{12},s_{13},s_{23},\tau) + L_{00}^{(1)}(s_{12},s_{23},s_{13},\tau) - L_{00}^{(1)}(s_{13},s_{23},s_{12},\tau), \end{aligned} \tag{5.44}$$

the integrals involving an odd number of sign functions vanish. This fact can be established by a suitable change of variables as in Formula 1.

Case $i = j = 1$.

In this case,

$$L_{11}(\boldsymbol{s},\tau) = \int_{(1+p\mathbb{Z}_p)^2} |1-x_2|_p^{s_{42}} |1-x_3|_p^{s_{43}} |x_2-x_3|_p^{s_{23}} H_\tau(1-x_2)H_\tau(1-x_3)H_\tau(x_2-x_3)dx_2dx_3.$$

Now by changing variables as $u = 1-x_2$, $v = 1-x_3$, we get

$$\begin{aligned} L_{11}(\boldsymbol{s},\tau) &= \int_{(p\mathbb{Z}_p)^2} |u|_p^{s_{42}} |v|_p^{s_{43}} |u-v|_p^{s_{23}} H_\tau(u)H_\tau(v)H_\tau(v-u)dudv \\ &= L_{00}(s_{42},s_{43},s_{23}) + L_{00}^{(1)}(s_{42},s_{43},s_{23},\tau) - L_{00}^{(2)}(s_{42},s_{43},s_{23},\tau) - L_{00}^{(3)}(s_{42},s_{43},s_{23},\tau) \\ &= L_{00}(s_{42},s_{43},s_{23}) + L_{00}^{(1)}(s_{42},s_{43},s_{23},\tau) - L_{00}^{(1)}(s_{42},s_{23},s_{43},\tau) + L_{00}^{(1)}(s_{43},s_{23},s_{42},\tau) \end{aligned} \tag{5.45}$$

Cases $i = 0$ **and** $j \in \{2, 3, \ldots, p-1\}$ **or** $i \in \{2, 3, \ldots, p-1\}$ **and** $j = 1$.

In these cases,

$$L_{0j}(\boldsymbol{s},\tau) = L_{i1}(\boldsymbol{s},\tau) = 0. \tag{5.46}$$

The vanishing of the integral $L_{0j}(\boldsymbol{s},\tau)$ follows from

$$H_\tau(x_3)H_\tau(x_2-x_3)\Big|_{p\mathbb{Z}_p \times j+p\mathbb{Z}_p} = H_\tau(j)H_\tau(-j) = 0.$$

The other case is treated in a similar way.

Case $i \in \{2, 3, \ldots, p-1\}$ **and** $j = 0$.

By using (5.28),

$$\begin{aligned} L_{i0}(\boldsymbol{s},\tau) &= H_\tau^2(i)H_\tau(1-i)H_\tau(1) \int_{i+p\mathbb{Z}_p \times p\mathbb{Z}_p} |x_3|_p^{s_{13}} H_\tau(x_3)dx_2dx_3 \\ &= p^{-1}H_\tau(i)H_\tau(1-i)\int_{p\mathbb{Z}_p} |x_3|_p^{s_{13}} H_\tau(x_3)dx_3 = p^{-1}H_\tau(i)H_\tau(1-i)I_0(s_{13},\tau) \\ &= p^{-1}H_\tau(i)H_\tau(1-i)\frac{p^{-1-s_{13}}(1-p^{-1})}{2(1-p^{-1-s_{13}})}. \end{aligned} \tag{5.47}$$

Now, using Formula 2, the contribution of all these integrals is

$$\sum_{i=2}^{p-1} L_{i0}(\boldsymbol{s},\tau) = \frac{p^{-1-s_{13}}(1-p^{-1})}{2(1-p^{-1-s_{13}})}\frac{1}{p}\sum_{i=2}^{p-1} H_\tau(i)H_\tau(1-i) = \left(\frac{p-3}{8p}\right)\frac{p^{-1-s_{13}}(1-p^{-1})}{(1-p^{-1-s_{13}})}. \tag{5.48}$$

Case $i=1$ and $j\in\{2,3,\ldots,p-1\}$.

This case is similar to the previous one,

$$\sum_{j=2}^{p-1} L_{1j}(\boldsymbol{s},\tau)=p^{-1}I_0(s_{42},\tau)\sum_{j=2}^{p-1}H_\tau(j)H_\tau(1-j)=\left(\frac{p-3}{8p}\right)\frac{p^{-1-s_{42}}(1-p^{-1})}{(1-p^{-1-s_{42}})}. \tag{5.49}$$

In conclusion, from (5.32), (5.33), and (5.41)- (5.49), we have

$$\begin{aligned}Z^{(5)}(\boldsymbol{s},\widetilde{\boldsymbol{s}})=E^{(5)}(\widetilde{\boldsymbol{s}})\Bigg\{&\frac{(p-3)(p-7)}{32p^2}+\left(\frac{p-3}{8p}\right)\Bigg[\frac{p^{-1-s_{23}}(1-p^{-1})}{(1-p^{-1-s_{23}})}\\&+\frac{p^{-1-s_{13}}(1-p^{-1})}{(1-p^{-1-s_{13}})}+\frac{p^{-1-s_{42}}(1-p^{-1})}{(1-p^{-1-s_{42}})}\Bigg]+\frac{(1-p^{-1})^2}{4}\frac{p^{-2-s_{13}-s_{42}}}{(1-p^{-1-s_{13}})(1-p^{-1-s_{42}})}\\&+\frac{1}{4}\frac{p^{-s_{12}-s_{13}-s_{23}-2}\left(1-p^{-1}\right)}{1-p^{-s_{12}-s_{13}-s_{23}-2}}\left[\frac{1}{2}-\frac{3}{2p}+\frac{p^{-1-s_{23}}\left(1-p^{-1}\right)}{1-p^{-1-s_{23}}}+\frac{p^{-1-s_{13}}\left(1-p^{-1}\right)}{1-p^{-1-s_{13}}}\right]\\&+\frac{1}{4}\frac{p^{-s_{42}-s_{43}-s_{23}-2}\left(1-p^{-1}\right)}{1-p^{-s_{42}-s_{43}-s_{23}-2}}\left[\frac{1}{2}-\frac{3}{2p}+\frac{p^{-1-s_{23}}\left(1-p^{-1}\right)}{1-p^{-1-s_{23}}}+\frac{p^{-1-s_{42}}\left(1-p^{-1}\right)}{1-p^{-1-s_{42}}}\right]\Bigg\}.\end{aligned} \tag{5.50}$$

$Z^{(5)}(\boldsymbol{s},\widetilde{\boldsymbol{s}})$ is a holomorphic function in

$$\begin{aligned}&\mathrm{Re}(s_{13})>-1;\quad \mathrm{Re}(s_{23})>-1;\quad \mathrm{Re}(s_{42})>-1;\\&\mathrm{Re}(s_{12}+s_{13}+s_{23})>-2;\quad \mathrm{Re}(s_{42}+s_{43}+s_{23})>-2.\end{aligned}$$

5.7 The limit $p\to 1$ of the Ghoshal-Kawano amplitudes

Using the theory of topological zeta functions of [36], in [35] we define rigorously the tree-level p-adic open string amplitudes in the limit $p\to 1$. Notice that when using the effective action, one calculates in $\mathbb{R}^D$, meanwhile in the case of p-adic string amplitudes one calculates in $\mathbb{Q}_p^D$. In the p-adic topology the limit $p\to 1$ does not make sense. However, surprisingly, the computation of the limit $p\to 1$ (considering p as a real parameter) of the p-adic open string amplitudes gives the right answer! In this subsection we compute the limit $p\to 1$ (considering p as a real parameter) in the cases $N=4$, 5. This limit requires explicit formulas, see [35] for further details. The limit $p\to 1$ of $Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$, $N=4$, 5, with $p\equiv 3 \bmod 4$, are given by

$$\lim_{p\to 1} Z^{(4)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)=\exp\left\{\frac{\sqrt{-1}}{2}\left(\widetilde{s}_{13}+\widetilde{s}_{12}+\widetilde{s}_{23}\right)\right\}\left\{-\frac{1}{2}+\frac{1}{2(s_{12}+1)}+\frac{1}{2(s_{32}+1)}\right\}, \tag{5.51}$$

for $\tau\in\{p,\varepsilon p\}$, and

$$\begin{aligned}\lim_{p\to 1} Z^{(5)}(\boldsymbol{s},\widetilde{\boldsymbol{s}})=E^{(5)}(\widetilde{\boldsymbol{s}})\Bigg\{&\frac{3}{16}-\frac{1}{4(s_{23}+1)}-\frac{1}{4(s_{13}+1)}-\frac{1}{4(s_{42}+1)}\\&+\frac{1}{4\left(s_{42}+1\right)\left(s_{13}+1\right)}+\frac{1}{4\left(s_{12}+s_{13}+s_{23}+2\right)}\left[-1+\frac{1}{(s_{23}+1)}+\frac{1}{(s_{13}+1)}\right]\\&+\frac{1}{4\left(s_{42}+s_{43}+s_{23}+2\right)}\left[-1+\frac{1}{(s_{23}+1)}+\frac{1}{(s_{42}+1)}\right]\Bigg\}.\end{aligned} \tag{5.52}$$

In the case $N=4$, after the appropriate sum over the permutations of the momenta $\boldsymbol{k}_i$, the amplitude agrees with the Feynman amplitude obtained from the deformed Gerasimov-Shatashvili action with a logarithmic potential (5.4).

Chapter 6

Towards Non-Archimedean Superstring Amplitudes

In this chapter, we review the results obtained in the work [49]. We propose a worldsheet action containing bosons and fermions on the p-adic worldsheet projected on the boundary. We will show that this action is supersymmetric and thus might be considered as a p-adic analogue of the worldsheet superstring action in the superconformal gauge [55]. Moreover we will show that this action can be rewritten as an action in a p-adic version of the ordinary superspace. Furthermore we compute the tree-level N-point open string amplitudes of this superstring action and we obtain the corresponding Koba-Nielsen formula of the well known amplitudes in the NSR formalism [55, 56]. This is carried out explicitly by performing the path integration of this superstring action with N tachyonic vertex operators in the spirit of Refs. [20, 51]. We obtain the amplitudes previously found in Refs. [57, 58].

6.1 A p-adic fermionic action

In this section, we consider a p-adic analogue of a fermionic action corresponding to the fermionic sector of the Archimedean worldsheet superstring action in the superconformal gauge. The action already appeared in [58, 63], however it was not studied in detail. The proposed action is the following

$$S_F[\psi] = \frac{\text{sgn}_\tau(-1)p}{2}\frac{1}{\alpha'}\int_{\mathbb{Q}_p^2} \psi^\mu(x)\eta_{\mu\nu}\frac{\text{sgn}_\tau(x-y)}{|x-y|_p^{s+1}}\psi^\nu(y)dydx, \tag{6.1}$$

where $\eta_{\mu\nu}$ =diag$(-1,1,\ldots,1)$ is the Minkowski metric, $\psi : \mathbb{Q}_p \to \Lambda$, is a 'worldsheet' p-adic field valued in the Grassmann number field Λ, and sgn_τ is the p-adic sign function[1]. Some comments regarding this proposal are in order. The action (6.1) requires $\text{sgn}_\tau(-1) = -1$, otherwise it

[1]To get more details on the p-adic sign function see section 2.2.3. Basically there are 3 distinct non-trivial sign functions determined by $\tau \in \{\epsilon, p, \epsilon p\}$ (ϵ is a $(p-1)$-rooth of unity). One doesn't always have $\text{sgn}_\tau(-1) = -1$, such requirement implies the restrictions $\tau \neq \epsilon$ and $p \equiv 3 \bmod 4$.

will vanish identically due to the anti-commutativity of ψ. The authors of [65, 64] considered a complex ψ, that would imply having ψ^* instead of one of the fields ψ in (6.1). This eliminates the need for including sgn_τ at all, but whether or not $\mathrm{sgn}_\tau(-1) = -1$ determines if (6.1) is symmetric or antisymmetric under the exchange $\psi \leftrightarrow \psi^*$. Here we keep the fields real ($\psi* = \psi$) as it is closer to the Archimedean case.

The action (6.1) is closely related to a *twisted* Vladimirov derivative recently studied in [10, 93, 91]. It is straightforward to show that

$$\int_{\mathbb{Q}_p} \psi^\mu(x) D_s^\tau \psi^\nu(x) dx + \int_{\mathbb{Q}_p} \psi^\nu(x) D_s^\tau \psi^\mu(x) dx$$
$$= (1 - \mathrm{sgn}_\tau(-1)) \int_{\mathbb{Q}_p^2} \psi^\mu(x) \frac{\mathrm{sgn}_\tau(x-y)}{|x-y|_p^{s+1}} \psi^\nu(y) dy dx, \tag{6.2}$$

where D_s^τ is the generalized or twisted Vladimirov derivative defined by

$$D_s^\tau \psi^\mu(x) := \int_{\mathbb{Q}_p} \frac{\psi^\mu(y) - \psi^\mu(x)}{\mathrm{sgn}_\tau(x-y)|x-y|_p^{s+1}} dy. \tag{6.3}$$

Notice the two terms on the first line of (6.2) differ only by the exchange of indices $\mu \leftrightarrow \nu$. Thus when we contract with the metric $\eta_{\mu\nu}$ they become equal. Therefore we have

$$\int_{\mathbb{Q}_p} \eta_{\mu\nu} \psi^\mu(x) D_s^\tau \psi^\nu(x) dx = \frac{1 - \mathrm{sgn}_\tau(-1)}{2} \int_{\mathbb{Q}_p^2} \psi^\mu(x) \eta_{\mu\nu} \frac{\mathrm{sgn}_\tau(x-y)}{|x-y|_p^{s+1}} \psi^\nu(y) dy dx. \tag{6.4}$$

Here we can explicitly see that if $\mathrm{sgn}_\tau(-1) = 1$, the right-hand-side of (6.4) would be zero. We can see then that the fermionic action is almost the same as the bosonic action, except for the inclusion of the sign function and the parameter s. To connect with previous work and the Archimedean case, we will eventually make $s = 0$, but for now we leave it general.

Unfortunately, with the inclusion of sgn_τ, there is no value of s for which (6.1) is conformally invariant (invariant under $\mathrm{PGL}(2, \mathbb{Q}_p)$ transformations), we only have translation invariance. Later in section 6.3.1 we will implement conformal symmetry directly in the definition of the amplitudes.

6.1.1 Fermionic Green's function

We can rewrite the action (6.1) in a simpler quadratic form. Defining a suitable operator Δ_s^τ, the action is proportional to $\psi \cdot \Delta_s^\tau \psi$. The purpose of this section is to obtain the inverse operator of Δ_s^τ. We use Fourier analysis, this is close in spirit to the computation done in [51] for bosonic strings in an external B-field. In [92] there is a more rigorous study of Green's functions for simple Vladimirov derivatives.
First, we define the function

$$\mathcal{F}_{\mu\nu}^F(s; x-y) = \eta_{\mu\nu} \mathcal{F}_s^F(x-y) = \eta_{\mu\nu} \frac{\mathrm{sgn}_\tau(x-y)}{|x-y|_p^{s+1}}. \tag{6.5}$$

The superscript F means we are working with the fermionic sector. $\mathcal{F}_s^F$ can be regarded as the integration kernel for the operator Δ_s^τ. Equivalently, we define the operator Δ_s^τ acting as the convolution with the function $\mathcal{F}_s^F(\cdot)$,

$$\Delta_s^\tau \psi^\mu(x) = (\mathcal{F}_s^F * \psi^\mu)(x). \tag{6.6}$$

We also define $\mathcal{G}_F^{\mu\nu}(s; x-y) = \eta^{\mu\nu}\mathcal{G}_F^s(x-y)$ as the inverse of $\mathcal{F}_s^F$, such that

$$\int_{\mathbb{Q}_p} \mathcal{F}_s^F(x-z)\mathcal{G}_F^s(z-y)dz = \delta(x-y). \tag{6.7}$$

In Fourier space (see 2.2.6), this equation reads

$$\widetilde{\mathcal{G}}_F^s(\omega) = \frac{1}{\widetilde{\mathcal{F}}_s^F(\omega)}. \tag{6.8}$$

To obtain $\mathcal{G}_F^s$ we first calculate the Fourier transform of $\mathcal{F}_s^F$,

$$\widetilde{F}_s^F(\omega) = \int_{\mathbb{Q}_p} \chi(\omega x)\frac{\text{sgn}_\tau(x)}{|x|_p^{s+1}}dx$$

$$= \begin{cases} |\omega|_p^s \text{sgn}_\tau(\omega) L(\tau,p) p^{-s-1}, & \tau \neq \epsilon \\ |\omega|_p^s \text{sgn}_\tau(\omega) p^{-s}\frac{1+p^{-s-1}}{p^{-s}+1}, & \tau = \epsilon \end{cases}; \quad \text{Re}(s) < 0. \tag{6.9}$$

Here $L(\tau,p) = \text{sgn}_\tau(p)\sum_{a=1}^{p-1}\text{sgn}_\tau(a)\chi(a/p)$, but the exact value is not relevant. Then we have

$$\mathcal{G}_s^F(x-y) = \int_{\mathbb{Q}_p} \chi(-\omega(x-y))\frac{\text{sgn}_\tau(\omega)}{|\omega|_p^s \mathcal{C}(s,\tau)}d\omega$$

$$= \begin{cases} \text{sgn}_\tau(-1)p|x-y|_p^{s-1}\text{sgn}_\tau(x-y), & \tau \neq \epsilon \\ \text{sgn}_\tau(-1)p|x-y|_p^{s-1}\text{sgn}_\tau(x-y)\frac{(1+p^{-s})^2}{(1+p^{-s-1})(1+p^{-s+1})}, & \tau = \epsilon \end{cases}; \quad \text{Re}(s) < 0, \tag{6.10}$$

where $\mathcal{C}(s,\tau)$ is the coefficient of $|\omega|_p^s\text{sgn}_\tau(\omega)$ in Eq. (6.9). We are interested in the case $\tau \neq \epsilon$. We must cancel the extra factor $\text{sgn}_\tau(-1)p$, which is why it appears as a coefficient in front of the action (6.1). In appendix B we show explicitly that, as usual, the Green's function is the same as the fermion field two-point function.

6.2 A 'worldsheet' action for the p-adic superstring

In this section we propose a prospect of a non-Archimedean superstring action. This action can be compared with the usual Archimedean superstring action in the superconformal gauge. It is also shown that this action satisfies a supersymmetric transformation in the p-adic context. Moreover a superspace formulation is also provided.

We propose the action $I_S[X,\psi] = I_B[X] + I_F[\psi]$, that describes the non-Archimedean superstring. It is given by the sum of a bosonic action $I_B[X]$, defined as

$$I_B[X] = \frac{T_0}{2}\int_{\mathbb{Q}_p^2} \eta_{\mu\nu}\frac{(X^\mu(x)-X^\mu(y))(X^\nu(y)-X^\nu(x))}{|x-y|_p^2}dydx, \tag{6.11}$$

and a fermionic action $I_F[\psi]$ given by Eq. (6.1). Thus $I_S[X,\psi]$ is

$$\begin{aligned} I_S[X,\psi] = &\frac{T_0}{2}\int_{\mathbb{Q}_p^2} \eta_{\mu\nu}\frac{(X^\mu(x)-X^\mu(y))(X^\nu(y)-X^\nu(x))}{|x-y|_p^2}dydx \\ &+ \frac{\text{sgn}_\tau(-1)p}{2\alpha'}\int_{\mathbb{Q}_p^2}\psi^\mu(x)\eta_{\mu\nu}\frac{\text{sgn}_\tau(x-y)}{|x-y|_p^{s+1}}\psi^\nu(y)dydx. \end{aligned} \tag{6.12}$$

This action can be rewritten as

$$I_S = -T_0\int_{\mathbb{Q}_p}\eta_{\mu\nu}X^\mu(x)[D_1^1X^\nu](x)dx + \frac{\text{sgn}_\tau(-1)p}{2\alpha'}\int_{\mathbb{Q}_p}\eta_{\mu\nu}\psi^\mu(x)[\Delta_s^\tau\psi^\nu](x)dx, \tag{6.13}$$

where D_1^1 is the Vladimirov derivative (6.3) with $s=1$ and $\tau=1$ (this just means taking the trivial sign, that is always 1), and $T_0 = \frac{p(p-1)}{4(p+1)\ln p}\frac{1}{\alpha'}$, (see [20, 21]).

6.2.1 Equations of motion

Now we compute the variation of the action for the bosonic and fermionic fields to obtain the equations of motion. The fields are real valued, therefore the variation is the same as in the usual Archimedean case. It is given by

$$\begin{aligned} I_B[X+\delta X] &= -T_0\int_{\mathbb{Q}_p}\eta_{\mu\nu}(X^\mu+\delta X^\mu)(D_1^1X^\nu + D_1^1\delta X^\nu) \\ &= I_B[X] - 2T_0\int_{\mathbb{Q}_p}\eta_{\mu\nu}\delta X^\mu[D_1^1X^\nu] + \mathcal{O}(\delta X^2). \end{aligned} \tag{6.14}$$

This implies that

$$\delta I_B[X] = -2T_0\int_{\mathbb{Q}_p}\eta_{\mu\nu}\delta X^\mu(x)[D_1^1X^\nu](x)dx. \tag{6.15}$$

In the previous computation we used the following fact

$$\int_{\mathbb{Q}_p} f(x)[D_1^1g](x)dx = \int_{\mathbb{Q}_p}[D_1^1f](x)g(x)dx, \tag{6.16}$$

which uses Fubini's theorem[2], $|-1|_p = 1$ and a symmetric $\eta_{\mu\nu}$. Similarly, for the fermionic action we have

$$I_F[\psi+\delta\psi] = \frac{\text{sgn}_\tau(-1)p}{2\alpha'}\int_{\mathbb{Q}_p^2}\eta_{\mu\nu}\Big(\psi^\mu(x)+\delta\psi^\mu(x)\Big)\frac{\text{sgn}_\tau(x-y)}{|x-y|_p^{s+1}}\Big(\psi^\nu(y)+\delta\psi^\nu(y)\Big)dxdy$$

[2]See footnote 3 at the end of page 20.

$$= S_F[\psi] + \frac{\text{sgn}_\tau(-1)p}{2\alpha'} \int_{\mathbb{Q}_p^2} \eta_{\mu\nu} \frac{\text{sgn}_\tau(x-y)}{|x-y|_p^{s+1}} \Big(\delta\psi^\mu(x)\psi^\nu(y) + \psi^\mu(x)\delta\psi^\nu(y)\Big) dxdy + \mathcal{O}(\delta\psi^2). \quad (6.17)$$

This implies that[3]

$$\delta I_F[\psi] = \frac{\text{sgn}_\tau(-1)-1}{2\alpha'} p \int_{\mathbb{Q}_p^2} \eta_{\mu\nu} \frac{\text{sgn}_\tau(x-y)}{|x-y|_p^{s+1}} \delta\psi^\mu(x)\psi^\nu(y) dxdy$$

$$= \frac{\text{sgn}_\tau(-1)-1}{2\alpha'} p \int_{\mathbb{Q}_p} \delta\psi^\mu(x)[\Delta^\tau_{s,\mu\nu}\psi^\nu](x)dx. \quad (6.18)$$

These two variations imply the following equations of motion for the bosonic and fermionic fields, X and ψ respectively

$$-2T_0\eta_{\mu\nu}[D_1^1 X^\nu](x) = 0, \qquad \frac{\text{sgn}_\tau(-1)-1}{2\alpha'} p[\Delta^\tau_{s,\mu\nu}\psi^\nu](x) = 0. \quad (6.19)$$

6.2.2 Supersymmetry transformation

We show here that the proposed action admits an infinitesimal supersymmetric transformation. From (6.18) and (6.15), the variation of the 'worldsheet' action $I_S[X,\psi]$ is written as

$$\delta I_S[X,\psi] = -2T_0 \int_{\mathbb{Q}_p} \eta_{\mu\nu}\delta X^\mu(x)[D_1^1 X^\nu](x)dx + \frac{\text{sgn}_\tau(-1)-1}{2\alpha'} p \int_{\mathbb{Q}_p} \delta\psi^\mu(x)[\Delta^\tau_{s,\mu\nu}\psi^\nu](x)dx. \quad (6.20)$$

Let us insert the following variations:

$$\delta X^\mu = A\lambda\Delta_s^\tau\psi^\mu, \qquad \delta\psi^\mu = B\lambda D_1^1 X^\mu, \quad (6.21)$$

where A and B are to be determined and λ is a Grassmann parameter. Then we have

$$\delta I_S[X,\psi] = \int_{\mathbb{Q}_p} \eta_{\mu\nu}\bigg(-2T_0 A\lambda[\Delta_s^\tau\psi^\mu](x)[D_1^1X^\nu](x) + \frac{\text{sgn}_\tau(-1)-1}{2\alpha'} pB\lambda[D_1^1X^\mu](x)[\Delta_s^\tau\psi^\nu](x)\bigg)dx$$

$$= \int_{\mathbb{Q}_p} \big(\lambda[\Delta_s^\tau\psi]\cdot[D_1^1X]\big)\left(-2T_0A + \frac{\text{sgn}_\tau(-1)-1}{2\alpha'}pB\right)dx, \quad (6.22)$$

where we can see that choosing constants A and B such that $-2T_0A + \frac{\text{sgn}_\tau(-1)-1}{2\alpha'}pB = 0$, we will get $\delta I_S[X,\psi] = 0$. Then for example, when $\text{sgn}_\tau(-1) = -1$, the transformation

$$\delta X^\mu = \lambda\Delta_s^\tau\psi^\mu; \qquad \delta\psi^\mu = -\frac{2\alpha'}{p}T_0\lambda D_1^1X^\mu, \quad (6.23)$$

is an infinitesimal supersymmetric transformation of $I_S[X,\psi]$.

[3] To get (6.17), we could have started with the equivalent definition of $S_F \sim \psi\cdot D_s^\tau\psi$. If one does this, one would eventually get in the integrand $\delta\psi\cdot D_s^\tau\psi + D_s^\tau\psi\cdot\delta\psi + (1-\text{sgn}_\tau(-1))\delta\psi\cdot\Delta_s^\tau\psi$. The first two terms cancel and we end up with the same result as in (6.18). Notice that this is analogous to a boundary term after integration by parts.

6.2.3 Superspace description of the p-adic superstring

It is desirable to have a more efficient and concise approach to describe the p-adic superstring such as the superspace approach. In [62, 61] a notion of superspace over $\mathbb{Q}_p$ is introduced. Motivated by this, we follow [56] (see also [112]) and define a superfield and a superoperator for our model. After some guess work we find that the superoperator and the superfield are of the following form

$$\mathcal{X}^\mu(x,\theta) = AX^\mu(x) + B\theta\psi^\mu(x),$$
$$\mathcal{D}_s^\tau = a\theta D_1^1 + b\Delta_s^\tau\partial_\theta. \tag{6.24}$$

Then

$$\begin{aligned} \mathcal{D}_s^\tau\mathcal{X}^\mu &= aA\theta D_1^1X^\mu + bB\Delta_s^\tau\psi^\mu, \\ \mathcal{X}\cdot\mathcal{D}_s^\tau\mathcal{X} &= aA^2\theta X\cdot D_1^1X + bABX\cdot\Delta_s^\tau\psi + bB^2\psi\cdot\Delta_s^\tau\psi. \end{aligned} \tag{6.25}$$

In order to have the action in the following form

$$I_S[X,\psi] = \int\limits_{\mathbb{Q}_p}\int d\theta\ \eta_{\mu\nu}\mathcal{X}^\mu[\mathcal{D}_s^\tau\mathcal{X}^\mu]dx, \tag{6.26}$$

we need to choose the constants a, b, A and B, such that $aA^2 = -T_0$ and $bB^2 = \mathrm{sgn}_\tau(-1)p/2\alpha'$. This is underdetermined, however, there is a further constraint that can be considered. In order to obtain the amplitudes, we compute correlation functions of vertex operators of the form

$$\mathcal{V}(y_\ell) = k_\ell\cdot\psi(y_\ell)e^{ik_\ell\cdot X(y_\ell)} = \int d\theta_\ell e^{ik_\ell\cdot X(y_\ell)+\theta_\ell k_\ell\cdot\psi(y_\ell)}, \tag{6.27}$$

where y_ℓ is the insertion point, and θ_l are auxiliary Grassmann variables. In the superspace approach this vertex operator is usually given by

$$\mathcal{V}(k_\ell;y_\ell) = \int d\theta_\ell e^{ik_\ell\cdot\mathcal{X}_\ell(y_\ell)}. \tag{6.28}$$

A comparison with Eq. (6.27) shows that we need $A = 1$ and $B = -i$. This determines $a = -T_0$ and $b = -\mathrm{sgn}_\tau(-1)p/2\alpha'$. Thus the appropriate choice of constants in Eq. (6.24) to obtain (6.26) and (6.28) is

$$\mathcal{X}^\mu(x,\theta) = X^\mu(x) - i\theta\psi^\mu(x), \tag{6.29}$$
$$\mathcal{D}_s^\tau = -T_0\theta D_1^1 - \frac{\mathrm{sgn}_\tau(-1)p}{2\alpha'}\Delta_s^\tau\partial_\theta. \tag{6.30}$$

The Green's function of the differential operator $\mathcal{D}$ is given by

$$\begin{aligned} \mathcal{G}_s(x-y;\theta,\theta') &= \frac{\alpha'}{1-s}\ln\left(|x-y|_p^{1-s} + \mathrm{sgn}_\tau(x-y)(1-s)\theta\theta'\right) \\ &= \alpha'\ln|x-y|_p + \alpha'\theta\theta'\frac{\mathrm{sgn}_\tau(x-y)}{|x-y|_p^{1-s}}, \end{aligned} \tag{6.31}$$

and satisfies

$$\int_{\mathbb{Q}_p}\mathcal{D}_s^\tau(x-z;\theta)\mathcal{G}_s(z-y;\theta,\theta')dz = \delta(x-y)(\theta-\theta'). \tag{6.32}$$

6.3 Tree-level amplitudes of the p-adic superstring

In this section we obtain the tree-level amplitudes of our p-adic superstring model. They are obtained through the computation of correlation functions of vertex operators. The N-point function for this system is given by the insertion of N vertex operators of the form shown in Eq. (6.27). Inserting vertex operators inside the path integral is equivalent to having a generating function with appropriately chosen sources for both fermions and bosons. The integration of the bosonic part can be obtained in a standard way using the corresponding Green's function $\mathcal{G}^{B}_{\mu\nu}(x-y) = -\alpha' \log|x-y|_p$, see [51]. Here we will perform the analogous computation carried out in [51], but now for the fermionic sector.

We start by recalling the operator $\Delta^{\tau}_{\mu\nu}$ (from now on we will omit the explicit dependence on the parameter s), and define its inverse $(\Delta^{-1}_{\tau})_{\mu\nu}$ by

$$[\Delta^{\tau}_{\mu\nu}K^{\nu}](x) = (\mathcal{F}^{F}_{\mu\nu} * K^{\nu})(x) = \int_{\mathbb{Q}_p} \mathcal{F}^{F}_{\mu\nu}(x-y)K^{\nu}(y)dy, \tag{6.33}$$

and

$$[(\Delta^{-1}_{\tau})_{\mu\nu}K^{\nu}](x) = (\mathcal{G}^{F}_{\mu\nu} * K^{\nu})(x) = \int_{\mathbb{Q}_p} \mathcal{G}^{F}_{\mu\nu}(x-y)K^{\nu}(y)dy. \tag{6.34}$$

Using Fubini's theorem it is straightforward to check the following equations

$$((\Delta^{-1}_{\tau})^{\mu\alpha}[\Delta^{\tau}_{\alpha\nu}K^{\nu}])(x) = K^{\mu}(x);$$

$$(\Delta^{\tau}_{\mu\alpha}[(\Delta^{-1}_{\tau})^{\alpha\nu}K_{\nu}])(x) = K_{\mu}(x).$$

Now notice the following identity for general functions f and g

$$\int_{\mathbb{Q}_p} f(x)[\Delta^{\tau}_{\mu\nu}g](x)dx = \int_{\mathbb{Q}_p^2} f(x)\mathcal{F}^{F}_{\mu\nu}(x-y)g(y)dydx$$

$$= \int_{\mathbb{Q}_p} \text{sgn}_{\tau}(-1)\left[\int_{\mathbb{Q}_p} f(x)\mathcal{F}^{F}_{\mu\nu}(y-x)dx\right]g(y)dy$$

$$= \text{sgn}_{\tau}(-1)\int_{\mathbb{Q}_p} [\Delta^{\tau}_{\mu\nu}f](x)g(x)dx,$$

where we used $\mathcal{F}^{F}_{\mu\nu}(x-y) = \text{sgn}_{\tau}(-1)\mathcal{F}^{F}_{\mu\nu}(y-x)$. In words, the operator $\Delta^{\tau}_{\mu\nu}$ inside the integral may switch its action to the rest of the integrand at the cost of a sign. From this we can easily get that

$$\int_{\mathbb{Q}_p} [(\Delta^{-1}_{\tau})^{\nu\alpha}K_{\alpha}](x)[\Delta^{\tau}_{\nu\beta}\psi^{\beta}](x)dx = \text{sgn}_{\tau}(-1)\int_{\mathbb{Q}_p} K^{\alpha}(x)\psi_{\alpha}(x)dx. \tag{6.35}$$

With these results one can verify the following identity

$$-\frac{1}{2}\int_{\mathbb{Q}_p} \psi^{\mu}(x)[\Delta^{\tau}_{\mu\nu}\psi^{\nu}](x)dx + \int_{\mathbb{Q}_p} K_{\mu}(x)\psi^{\mu}(x)dx$$

$$= -\frac{1}{2}\int_{\mathbb{Q}_p}\left(\psi^\mu(x) - \text{sgn}_\tau(-1)[(\Delta^{-1})^{\mu\alpha}_\tau K_\alpha](x)\right)\left[\Delta^\tau_{\mu\nu}\left(\psi^\nu - \text{sgn}_\tau(-1)[(\Delta^{-1})^{\nu\beta}_\tau K_\beta]\right)\right](x)dx$$

$$+\frac{1}{2}\int_{\mathbb{Q}_p} K^\mu(x)[(\Delta^{-1})^{\mu\nu}_\tau K_\nu](x)dx, \tag{6.36}$$

that is the fermion analogue of the well known relation for the bosonic finite dimensional case: $-\frac{1}{2}x^T\cdot A\cdot x + K^T\cdot x = -\frac{1}{2}(x^T - K^T\cdot A^{-1})\cdot A\cdot(x - A^{-1}\cdot K) + \frac{1}{2}K^T\cdot A^{-1}\cdot K$, where A is a matrix, x and K are column vectors and T denotes the transpose operation. Thus, we obtain the generating function of the path integral with bosonic sources J^μ and fermionic sources K^μ in this p-adic setting

$$\mathcal{Z}[J,K] = \frac{1}{\mathcal{Z}}\int D\psi\int DX\exp\left\{-I_B[X] - I_F[\psi] + \int_{\mathbb{Q}_p} J_\mu(x)X^\mu(x)dx + \int_{\mathbb{Q}_p} K_\mu(x)\psi^\mu(x)dx\right\}$$

$$= \exp\left\{\frac{1}{2}\int_{\mathbb{Q}_p^2} J^\mu(x)\mathcal{G}^B_{\mu\nu}(x-y)J^\nu(y)dxdy + \frac{\alpha'}{2\text{sgn}_\tau(-1)p}\int_{\mathbb{Q}_p^2} K^\mu(x)\mathcal{G}^F_{\mu\nu}(x-y)K^\nu(y)dxdy\right\}. \tag{6.37}$$

This generating function is equivalent to the N-point function if we use the sources

$$\mathcal{J}^\mu(x) = i\sum_{l=1}^N k_l^\mu\delta(x-y_l), \quad \mathcal{K}^\mu(x) = \sum_{m=1}^N \theta_m k_m^\mu\delta(x-y_m), \tag{6.38}$$

and integrate out the θ_m variables such that

$$\langle V(y_1)\cdots V(y_N)\rangle = \int d\theta_1\ldots d\theta_N\mathcal{Z}[\mathcal{J},\mathcal{K}]. \tag{6.39}$$

The dependence on the insertion points is left implicit in the sources on the right hand side. This is the analogue for the usual basic prescription to obtain N-point amplitudes. It is our starting point in order to get explicit expressions for the tree-level amplitudes.

6.3.1 Integral expression

The next step is to obtain an explicit expression for the correlation functions (6.39). As mentioned in section 6.1, we kept $\text{sgn}_\tau(-1)$ general to exhibit the subtleties that come with it (see Eq. (6.35)). It also allows to use the previous computations when considering complex valued Grassmann fields and any sign function will give non-trivial results. However in what follows and for the rest of the chapter we will take $\text{sgn}_\tau(-1) = -1$, otherwise all of the following computations would be identically 0.
First we obtain

$$\frac{1}{2}\int_{\mathbb{Q}_p^2}\mathcal{K}^\mu(x)\mathcal{G}^F_{\mu\nu}(x-y)\mathcal{K}^\nu(y)dxdy = \frac{1}{2}\sum_{m,n=1}^N\theta_m\theta_n k_m^\mu k_n^\nu\int_{\mathbb{Q}_p^2}\delta(x-y_m)\mathcal{G}^F_{\mu\nu}(x-z)\delta(z-y_n)dxdz$$

$$= \frac{1}{2}\sum_{\substack{m,n=1\\ m\neq n}}^N\theta_m\theta_n k_m^\mu k_n^\nu\mathcal{G}^F_{\mu\nu}(y_m-y_n)$$

$$= \frac{1}{2}\sum_{m<n} k_m \cdot k_n \left[\theta_m\theta_n \mathcal{G}^F(y_m - y_n) + \theta_n\theta_m \mathcal{G}^F(y_n - y_m)\right]$$

$$= \sum_{m<n} k_m \cdot k_n \theta_m\theta_n \mathcal{G}^F(y_m - y_n) = \operatorname{sgn}_\tau(-1)p \sum_{m<n} \theta_m\theta_n \; k_m \cdot k_n \frac{\operatorname{sgn}_\tau(y_m - y_n)}{|y_m - y_n|_p^{1-s}}. \tag{6.40}$$

Similarly[4]

$$\frac{1}{2}\int_{\mathbb{Q}_p^2} \mathcal{J}^\mu(x)\mathcal{G}^B_{\mu\nu}(x-y)\mathcal{J}^\nu(y)dxdy = \alpha' \sum_{m<n} k_m \cdot k_n \log|y_m - y_n|_p. \tag{6.41}$$

Using this and the properties of Grassmann variables (see Eq. (2.59) in Appendix B) we can write

$$\mathcal{Z}[\mathcal{J},\mathcal{K}] = \exp\left\{\sum_{m<n} \alpha' k_m \cdot k_n \left[\log|y_m - y_n|_p + \theta_m\theta_n \frac{\operatorname{sgn}_\tau(y_m - y_n)}{|y_m - y_n|_p^{1-s}}\right]\right\}$$

$$= \exp\left\{\sum_{m<n} \frac{\alpha'}{1-s} k_m \cdot k_n \left[\log\left(|y_m - y_n|_p^{1-s} + \operatorname{sgn}_\tau(y_m - y_n)(1-s)\theta_m\theta_n\right)\right]\right\}$$

$$= \prod_{m<n} \left(|y_m - y_n|_p^{1-s} + \operatorname{sgn}_\tau(y_m - y_n)(1-s)\theta_m\theta_n\right)^{k_m\cdot k_n \frac{\alpha'}{1-s}}. \tag{6.42}$$

We will also make now $s=0$ for the rest of the chapter[5]. This is not necessary, but it relates better to the Archimedean case and simplifies the expressions. Then finally we define the N-point amplitudes as follows

$$\mathcal{A}_p^{(N)}(\mathbf{k}) := \mathcal{N}\int_{\mathbb{Q}_p^N} \langle V(k_1;y_1)\cdots V(k_N;y_N)\rangle \prod_{m<n} \operatorname{sgn}_\tau(y_m - y_n) \prod_{i=1}^N dy_i$$

$$= \mathcal{N}\int_{\mathbb{Q}_p^N}\int \prod_{j=1}^N d\theta_j \prod_{m<n} \left(|y_m - y_n|_p + \operatorname{sgn}_\tau(y_m - y_n)\theta_m\theta_n\right)^{\alpha' k_m\cdot k_n} \operatorname{sgn}_\tau(y_m - y_n)\prod_{i=1}^N dy_i, \tag{6.43}$$

where $\mathcal{N}$ is a normalization constant. This is the analogue of the usual Archimedean result. The factor $\prod_{m<n} \operatorname{sgn}_\tau(y_m - y_n)$ is added to implement conformal invariance. These amplitudes were also proposed in [57, 58]. As we can see it is quite similar to the usual result [55, 112], the only difference is the appearance of the sign functions.

[4]The astute reader may have noticed that we seemingly forgot the case $m = n$ in the sum. A careful calculation shows that those terms vanish. The subtlety lies in the fact that in order to have a well defined integration of the Green's functions, we must integrate over the p-adic plane such that $x \neq y$. This is because $\mathcal{G}^B$ is singular when its argument is 0, and $\mathcal{G}^F$ is ill-defined at 0 ($\operatorname{sgn}_\tau(0)$ is ill-defined). Then when we integrate one of the deltas in the first line of (6.3.1), we are left with something proportional to $\delta(x - y_n)\Big|_{x\neq y_n}$, that is always 0.

[5]Again, a careful reader might doubt the validity setting $s = 0$, since in (6.10) we determined that $\text{Re}(s) < 0$. This issue is common in theoretical physics, by $s = 0$, we mean to take the analytic continuation of the result (6.10).

6.3.2 Integrating the Grassmann variables

The purpose of this subsection is to get the tree-level open string amplitudes even more explicitly by integrating the Grassmann variables in (6.43). If we go back to the first line of (6.42), we can factorize the exponential and expand the fermionic part in the following way

$$\exp\left\{\sum_{m<n}\alpha' k_m\cdot k_n\frac{\mathrm{sgn}_\tau(y_m-y_n)}{|y_m-y_n|_p}\theta_m\theta_n\right\}=\prod_{m<n}\left(1+\alpha' k_m\cdot k_n\frac{\mathrm{sgn}_\tau(y_m-y_n)}{|y_m-y_n|_p}\theta_m\theta_n\right). \quad (6.44)$$

We can show that the only nonvanishing amplitudes are for even insertions. This follows from noticing that there are N Grassmann integrals while the terms on the right hand side of (6.44) will always have an even number of θ_n variables. Using the rules for integrating these variables presented in section 2.4 we see that if N is odd the amplitude vanishes. From now on we consider to have an even N. Remember that $\theta_m^2=0$ and terms with less than N θs will be annihilated by the integrals. Therefore only terms with N distinct θ_n variables survive.

Expanding the product in (6.44) and considering what we just mentioned, amounts to the amplitudes being composed of $(N-1)!!$ terms, this is Wick's theorem. The terms differ in specific permutations of the N vertex operators. We will define these permutations a few lines below, but for now consider them of the form

$$\theta_{m_1}\theta_{n_1}\cdots\theta_{m_{N/2}}\theta_{n_{N/2}},\qquad m_i<n_i,\ m_i\neq m_j,\ n_i\neq n_j. \quad (6.45)$$

Integrating the θ_m variables in (6.44), we are left with

$$\alpha'\sum_P(-)^P\prod_{i=1}^{N/2}k_{P(2i-1)}\cdot k_{P(2i)}\frac{\mathrm{sgn}_\tau(y_{P(2i-1)}-y_{P(2i)})}{|y_{P(2i-1)}-y_{P(2i)}|_p}, \quad (6.46)$$

where P are permutations of the form (6.45) and $(-)^P$ is its sign. With this the amplitudes (6.43) are now in the Koba-Nielsen form

$$\mathcal{A}_p^{(N)}(\mathbf{k})=\mathcal{N}\int_{\mathbb{Q}_p^N}\prod_{m<n}|y_m-y_n|_p^{\alpha' k_m\cdot k_n}\mathrm{sgn}_\tau(y_m-y_n)$$
$$\times\sum_P(-)^P\prod_{i=1}^{N/2}k_{P(2i-1)}\cdot k_{P(2i)}\frac{\mathrm{sgn}_\tau(y_{P(2i-1)}-y_{P(2i)})}{|y_{P(2i-1)}-y_{P(2i)}|_p}\prod_{i=1}^N dy_i. \quad (6.47)$$

This is made cleaner by defining the following amplitude

$$A_p^{(N)}(\mathbf{k},P_N):=\mathcal{N}\prod_{m<n}(k_m\cdot k_n)^{q_{mn}(P_N)}$$
$$\times\int_{\mathbb{Q}_p^N}\prod_{m<n}|y_m-y_n|_p^{\alpha' k_m\cdot k_n-q_{mn}(P_N)}[\mathrm{sgn}_\tau(y_m-y_n)]^{1+q_{mn}(P_N)}\prod_{i=1}^N dy_i, \quad (6.48)$$

where P_N is a general permutation of N elements and $q_{mn}(P_N)$ is defined as follows

$$q_{mn}(P) := \begin{cases} 1, \ \exists\, i \in \{1,\ldots,N/2\}; \quad (m,n) = (P(2i-1), P(2i)) \\ 0, \text{ otherwise} \end{cases}$$

$$= \sum_{i=1}^{N/2} \delta_m^{P(2i-1)} \delta_n^{P(2i)}, \tag{6.49}$$

notice that $q_{mn} \neq q_{nm}$. Now we define more rigorously the permutations in the sum (6.47). Let $\widetilde{P}$ be the set of permutations of N elements such that for every $i \in \{1,\ldots,N/2\}$ we have $\widetilde{P}(2i-1) < \widetilde{P}(2i)$ and $\widetilde{P} \equiv \widetilde{P}'$ if $\widetilde{P}(2i-1) = \widetilde{P}'(2j-1), \widetilde{P}(2i) = \widetilde{P}'(2j)$ for $1 \leq j \leq N/2$. (All permutations that differ by the exchange of any two pairs of consecutive elements are equivalent) Notice that we have in total $\frac{N!}{(N/2)!2^{N/2}} = (N-1)!!$ elements in $\widetilde{P}$, as we should. Now the amplitudes have a more elegant and deceivingly concise form

$$\mathcal{A}_p^{(N)}(\mathbf{k}) = \sum_{\widetilde{P}} (-)^{\widetilde{P}} A_p^{(N)}(\mathbf{k}, \widetilde{P}). \tag{6.50}$$

Conformal Symmetry

In this section, we check expicitly the conformal invariance of the amplitudes (6.50). In the usual case, one can carry out the gauge fixing of the symmetries of worldsheet diffeomorphisms and Weyl transformations. However, it is not completely fixed and there is a remnant symmetry on the two-sphere, this is the $\mathrm{PSL}(2,\mathbb{C})$ symmetry [55]. In the present case, even though the action does not have conformal symmetry, we will find the conditions under which it can be implemented at the level of the amplitudes. These conditions involved the sign functions. The procedure works for the p-adic bosonic string where $k^2 = 2/\alpha'$ [18]. In our case we have $k^2 = 1/\alpha'$, however, the factors $|y_m - y_n|_p^{-1}$ coming from the fermionic sector described above save the day. Something similar will happen to the sgn_τ functions. If you accept this procedure can be done for our case, go ahead to the next section.
Consider the transformation $y_m = \frac{a\bar{y}_m + b}{c\bar{y}_m + d}$; with $ad - cb = 1$ for the integrand in (6.48) with a given permutation $\widetilde{P}$. Then we have

$$|y_m - y_n|_p^{s_{mn}} = |\bar{y}_m - \bar{y}_n|_p^{s_{mn}} |c\bar{y}_m + d|_p^{-s_{mn}} |c\bar{y}_n + d|_p^{-s_{mn}}, \quad dy_m = |c\bar{y}_m + d|_p^{-2} d\bar{y}_m. \tag{6.51}$$

Applying the change of variables, in the integrand we will encounter the following product

$$\prod_{m<n} |c\bar{y}_m + d|_p^{-s_{mn}} |c\bar{y}_n + d|_p^{-s_{mn}} = \prod_{m=1}^{N-1} |c\bar{y}_m + d|_p^{-\sum_{n>m} s_{mn}} \prod_{n=2}^{N} |c\bar{y}_n + d|_p^{-\sum_{m<n} s_{mn}}$$

$$= \prod_{m=1}^{N} |c\bar{y}_m + d|_p^{-\sum_{n\neq m} s_{mn}}. \tag{6.52}$$

Of course we must set $s_{mn} = \alpha' k_m \cdot k_n$. We use the momenta conservation $\sum_m k_m^\mu = 0$, and that for open superstrings we have $k_m^2 = 1/\alpha'$, to obtain

$$- \sum_{n \neq m} s_{mn} = -\alpha' k_m \cdot \sum_{n \neq m} k_n = -\alpha' k_m \cdot (-k_m) = \alpha' k_m^2 = 1. \tag{6.53}$$

Then for a fractional linear transformation the integrand of (6.47) in the new variables will have the extra factor

$$\prod_{m=1}^{N} |c\bar{y}_m + d|_p \prod_{m=1}^{N} |c\bar{y}_m + d|_p^{-2} \prod_{m=1}^{N} |c\bar{y}_m + d|_p = 1. \tag{6.54}$$

Here the first product is a result of (6.52) and (6.53), the second product comes from the second equality in (6.51) and third product is the contribution of the fermionic sector (6.46)[6].
We now deal with the sign functions, which is easier. We have

$$\begin{aligned}
\prod_{m<n} [\mathrm{sgn}_\tau(y_m - y_n)]^{1+q_{mn}} &= \prod_{m<n} [\mathrm{sgn}_\tau(\bar{y}_m - \bar{y}_n)]^{1+q_{mn}} \mathrm{sgn}_\tau(c\bar{y}_m + d) \mathrm{sgn}_\tau(c\bar{y}_n + d) \\
&\quad \times \prod_{m<n} [\mathrm{sgn}_\tau(c\bar{y}_m + d)]^{q_{mn}} [\mathrm{sgn}_\tau(c\bar{y}_n + d)]^{q_{mn}} \\
&= \prod_{m<n} [\mathrm{sgn}_\tau(\bar{y}_m - \bar{y}_n)]^{1+q_{mn}} \prod_{m=1}^{N} [\mathrm{sgn}_\tau(c\bar{y}_m + d)]^{N-1} \prod_{m=1}^{N} \mathrm{sgn}_\tau(c\bar{y}_m + d) \\
&= \prod_{m<n} [\mathrm{sgn}_\tau(\bar{y}_m - \bar{y}_n)]^{1+q_{mn}},
\end{aligned} \tag{6.55}$$

the second product of the third line comes from realizing that for any point y_a, the product $\prod_{m<n} \mathrm{sgn}_\tau(c\bar{y}_m + d)$ will give us $[\mathrm{sgn}_\tau(c\bar{y}_a + d)]^{N-a}$. Similarly for $\prod_{m<n} \mathrm{sgn}_\tau(c\bar{y}_n + d)$ we have $[\mathrm{sgn}_\tau(c\bar{y}_a + d)]^{a-1}$. Since q_{mn} is non-zero for each unique pair mn, only one such factor appears per point, this explains the last product in the third line. In the last line we used that N is even and $[\mathrm{sgn}_\tau(\cdot)]^2 = 1$.
With this we have proven the symmetry of the amplitudes. Conformal invariance allows us to fix three insertion points. It is customary to take such points as 0, 1, and ∞. Here is the explicit transformation that does the job

$$x_i = \frac{(y_{N-1} - y_N)(y_1 - y_i)}{(y_1 - y_{N-1})(y_i - y_N)} \quad \Leftrightarrow \quad y_i = \frac{x_i\, y_N(y_1 - y_{N-1}) + y_1(y_{N-1} - y_N)}{y_{N-1} - y_N + x_i(y_1 - y_{N-1})}.$$

This sends $x_1 = 0,\ x_{N-1} = 1,\ x_N = \infty$. We only transform y_i for $i \in \{2, \ldots, N-2\}$. We have

$$\begin{aligned}
y_1 - y_i &= \frac{(y_1 - y_N)(y_1 - y_{N-1})}{y_{N-1} - y_N + x_i(y_1 - y_{N-1})} x_i, \quad y_i - y_{N-1} = \frac{(y_1 - y_{N-1})(y_{N-1} - y_N)}{y_{N-1} - y_N + x_i(y_1 - y_{N-1})}(1 - x_i) \\
y_i - y_N &= \frac{(y_1 - y_N)(y_{N-1} - y_N)}{y_{N-1} - y_N + x_i(y_1 - y_{N-1})}, \quad dy_i = \frac{|y_1 - y_{N-1}|_p |y_1 - y_N|_p |y_{n-1} - y_N|_p}{|y_{N-1} - y_N + x_i(y_1 - y_{N-1})|_p^2} dx_i \\
y_i - y_j &= \frac{(y_1 - y_{N-1})(y_1 - y_N)(y_{n-1} - y_N)}{(y_{N-1} - y_N + x_i(y_1 - y_{N-1}))(y_{N-1} - y_N + x_j(y_1 - y_{N-1}))}(x_i - x_j).
\end{aligned} \tag{6.56}$$

[6] Had we kept s arbitrary, we would have gotten $\prod_{m=1}^{N} |c\bar{y}_m + d|_p^{-s}$ on the right side of (6.54).

Using these expressions one can check that the integrand factorizes into

$$dy_1 dy_{N-1} dy_N \prod_{m=2}^{N-2} |x_m|_p^{s_{1m}+q_{1m}} |1-x_m|_p^{s_{m(N-1)}+q_{m(N-1)}} [\mathrm{sgn}_\tau(x_m)]^{1+q_{1m}} [\mathrm{sgn}_\tau(1-x_m)]^{1+q_{m(N-1)}}$$

$$\times \prod_{2\leq m<n\leq N-2} |x_m - x_n|_p^{s_{mn}+q_{mn}} [\mathrm{sgn}_\tau(x_m - x_n)]^{1+q_{mn}} \prod_{m=2}^{N-2} dx_m.$$

We leave the details to the reader. Having $dy_1 dy_{N-1} dy_N$ means that we'll have the factor $[\mathrm{Vol}(\mathbb{Q}_p)]^3$ but this gets canceled after normalizing. Therefore we redefine the amplitudes (6.48) as

$$A_p^{(N)}(\mathbf{k}, P) = \prod_{m<n} (k_m \cdot k_n)^{q_{mn}}$$

$$\times \int_{\mathbb{Q}_p^{N-2}} \prod_{m=2}^{N-2} |x_m|_p^{s_{1m}-q_{1m}} |1-x_m|_p^{s_{m(N-1)}-q_{m(N-1)}} [\mathrm{sgn}_\tau(x_m)]^{1+q_{1m}} [\mathrm{sgn}_\tau(1-x_m)]^{1+q_{m(N-1)}}$$

$$\times \prod_{2\leq m<n\leq N-2} |x_m - x_n|_p^{s_{mn}-q_{mn}} [\mathrm{sgn}_\tau(x_m - x_n)]^{1+q_{mn}} \prod_{m=2}^{N-2} dx_m. \tag{6.57}$$

6.4 Four-point amplitudes

As an illustrative example, we show the case $N = 4$ in (6.50) with the definition (6.57). The first ingredients are the permutations $\widetilde{P}$, these are $\widetilde{P} = \{(1234),\ (1324)\ (1423)\}$ with signs $\{1, -1, 1\}$. Next we determine the non-zero components of q_{mn}. They are different depending on the permutation, for example for (1234) only q_{12} and q_{34} are 1 while the other components equal 0.

Then the amplitude (6.50), after gauge fixing three points as shown in (6.57), is

$$\begin{aligned}\mathcal{A}_p^{(4)}(\mathbf{k}) = (k_1\cdot k_2)(k_3\cdot k_4) \int_{\mathbb{Q}_p} |x_2|_p^{\alpha' k_1\cdot k_2 - 1} |1-x_2|_p^{\alpha' k_2\cdot k_3} \mathrm{sgn}_\tau(1-x_2) dx_2 \\ - (k_1\cdot k_3)(k_2\cdot k_4) \int_{\mathbb{Q}_p} |x_2|_p^{\alpha' k_1\cdot k_2} |1-x_2|_p^{\alpha' k_2\cdot k_3} \mathrm{sgn}_\tau(x_2)\mathrm{sgn}_\tau(1-x_2) dx_2 \\ + (k_1\cdot k_4)(k_2\cdot k_3) \int_{\mathbb{Q}_p} |x_2|_p^{\alpha' k_1\cdot k_2} |1-x_2|_p^{\alpha' k_2\cdot k_3 - 1} \mathrm{sgn}_\tau(x_2) dx_2.\end{aligned} \tag{6.58}$$

Going for efficiency, we do the following integration

$$\int_{\mathbb{Q}_p} |x|_p^u |1-x|_p^v [\mathrm{sgn}_\tau(x)]^{t_1} [\mathrm{sgn}_\tau(1-x)]^{t_2} dx = \int_{p\mathbb{Z}_p} |x|_p^u [\mathrm{sgn}_\tau(x)]^{t_1} dx$$

$$+ \int_{\mathbb{Z}_p^\times} |1-x|_p^v [\mathrm{sgn}_\tau(x)]^{t_1} [\mathrm{sgn}_\tau(1-x)]^{t_2} dx + \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p^\times} |x|_p^{u+v} [\mathrm{sgn}_\tau(x)]^{t_1+t_2} dx$$

$$= \int_{p\mathbb{Z}_p} |x|_p^u [\mathrm{sgn}_\tau(x)]^{t_1} dx + [\mathrm{sgn}_\tau(-1)]^v p^{-1} \sum_{a=2}^{p-1} [\mathrm{sgn}_\tau(a)]^u [\mathrm{sgn}_\tau(a-1)]^v$$

$$+[\mathrm{sgn}_\tau(-1)]^v\int_{p\mathbb{Z}_p}|x|_p^v[\mathrm{sgn}_\tau(x)]^{t_2}dx+[\mathrm{sgn}_\tau(-1)]^v\int_{p\mathbb{Z}_p}|x|_p^{-u-v-2}[\mathrm{sgn}_\tau(x)]^{t_1+t_2}dx$$

$$=\begin{cases}\frac{1-p^{-1}}{p^{u+1}-1}+\frac{1-p^{-1}}{p^{v+1}-1}+\frac{1-p^{-1}}{p^{-u-v-1}-1}+1-2p^{-1}, & (t_1,t_2)=(0,0)\\ \frac{1-p^{-1}}{p^{u+1}-1}-p^{-1}=\frac{1-p^{u}}{p^{u+1}-1}, & (t_1,t_2)=(0,1)\\ \frac{1-p^{-1}}{p^{v+1}-1}-p^{-1}=\frac{1-p^{v}}{p^{v+1}-1}, & (t_1,t_2)=(1,0)\\ p^{-1}-\frac{1-p^{-1}}{p^{-u-v-1}-1}=\frac{p^{-u-v-2}-1}{p^{-u-v-1}-1}, & (t_1,t_2)=(1,1)\end{cases}. \tag{6.59}$$

With this result, we can easily see that

$$\mathcal{A}_p^{(4)}(\mathbf{k})=(k_1\cdot k_2)(k_3\cdot k_4)\left[\frac{1-p^{\alpha' k_1\cdot k_2-1}}{p^{\alpha' k_1\cdot k_2}-1}\right]-(k_1\cdot k_3)(k_2\cdot k_4)\left[\frac{p^{\alpha' k_1\cdot k_3-1}-1}{p^{\alpha' k_1\cdot k_3}-1}\right]$$
$$+(k_1\cdot k_4)(k_2\cdot k_3)\left[\frac{1-p^{\alpha' k_2\cdot k_3-1}}{p^{\alpha' k_2\cdot k_3}-1}\right]. \tag{6.60}$$

This is the same result reported in [58, 57], where it was computed as a direct analogue of the Archimedean expressions. Comparing to the Archimedean result from [56] our amplitudes are very similar in the integral form, and will likely be so for arbitrary points. The main difference is the presence of sign functions, these functions annihilate several terms in the amplitudes when compared to the p-adic bosonic string case.

A Vertex operators

We briefly review the process for the basic tree-level amplitudes in the Archimedean superstrings done in Ref. [55], that is the analogue of the computations done above. Consider the vertex operator

$$V(k;X,\psi)=k\cdot\psi:e^{ik\cdot X}=\int d\theta e^{ik\cdot X+\theta k\cdot\psi}, \tag{6.61}$$

where θ is an auxiliary Grassmann variable. We highlight that the far right hand side is a consequence of the Grassmann variables properties, see section 2.4.
Now we use the following two-point functions [55]

$$\langle X^\mu(y_i)X^\nu(y_j)\rangle = -\eta^{\mu\nu}\log(y_i-y_j), \tag{6.62}$$
$$\left\langle\frac{\psi^\mu(y_i)}{\sqrt{y_i}}\frac{\psi^\nu(y_j)}{\sqrt{y_j}}\right\rangle = \frac{\eta^{\mu\nu}}{y_i-y_j}, \tag{6.63}$$

to get that

$$\left\langle\frac{V(k_i;y_i)}{\sqrt{y_i}}\frac{V(k_j;y_j)}{\sqrt{y_j}}\right\rangle = \int d\theta_i d\theta_j e^{k_i\cdot k_j\left(\log(y_i-y_j)-\frac{\theta_i\theta_j}{y_i-y_j}\right)}$$
$$=\int d\theta_i d\theta_j(y_i-y_j-\theta_i\theta_j)^{k_i\cdot k_j}. \tag{6.64}$$

For multiple vertex operators we have

$$\prod_{l=1}^{N} V(k_l; y_l) = \int d\theta_1 \cdots d\theta_N \exp\left\{\sum_{l=1}^{N} \boldsymbol{k}_l \cdot (i\boldsymbol{X}(y_l) + \theta_l \boldsymbol{\psi}(y_l))\right\}. \tag{6.65}$$

Then

$$\begin{aligned}\left\langle \prod_{l=1}^{N} \frac{V(k_l; y_l)}{\sqrt{y_l}} \right\rangle &= \int d\theta_1 \cdots d\theta_N \\ &\times \exp\left\{\sum_{l,m=1}^{N} k_{l,\mu} k_{m,\nu} \left(-\langle X^\mu(y_l) X^\nu(y_m)\rangle - \theta_l\theta_m \left\langle \frac{\psi^\mu(y_l)}{\sqrt{y_l}} \frac{\psi^\nu(y_m)}{\sqrt{y_m}} \right\rangle\right)\right\} \\ &= \int d\theta_1 \cdots d\theta_N \exp\left\{\sum_{l,m=1}^{N} \boldsymbol{k}_l \cdot \boldsymbol{k}_m \left(\log(y_l - y_m) - \frac{\theta_l\theta_m}{y_l - y_m}\right)\right\} \\ &= \int d\theta_1 \cdots d\theta_N \prod_{l<m} (y_l - y_m - \theta_l\theta_m)^{\boldsymbol{k}_l \cdot \boldsymbol{k}_m}.\end{aligned} \tag{6.66}$$

This derivation demanded only Fubini's theorem and changes of variables, both are well defined over the p-adics. Thus in the non-Archimedean setting, we can follow this same path, the only difference are the two-point functions, that are described in the sections above.
The last equality of (6.66) used the following identity for Grassmann variables θ_i

$$\begin{aligned}\log(y_i - y_j) + \frac{\theta_i\theta_j}{y_i - y_j} &= \log(y_i - y_j) + \log\left(1 + \frac{\theta_i\theta_j}{y_i - y_j}\right) \\ &= \log(y_i - y_j + \theta_i\theta_j).\end{aligned} \tag{6.67}$$

This is quite general, in fact, one can check that for constants A, B and s, the following holds

$$\begin{aligned}A \log|y_i - y_j| + B\frac{\theta_i\theta_j}{|y_i - y_j|^s} &= \frac{A}{s}\left(\log|y_i - y_j|^s + \frac{Bs}{A}\frac{\theta_i\theta_j}{|y_i - y_j|^s}\right) \\ &= \frac{A}{s}\log\left(|y_i - y_j|^s + \frac{Bs}{A}\theta_i\theta_j\right).\end{aligned} \tag{6.68}$$

This more general identity is used above to obtain (6.42).

B Functional derivatives

In this appendix we define in more detail a functional derivative for p-adic fermion fields. It is done in a very similar way to the usual bosonic variables. We also use it to obtain the fermion propagator as a two point function. Even though we are using Grassmann variables, commutativity issues do not arise because we use only pairs of Grassmann variables.
We define the functional derivative for a Grassmann valued field $K: \mathbb{Q}_p \to \Lambda$, and Grassmann variable θ

$$\frac{\delta Z[K]}{\delta K^\mu(y)} = \int d\theta \lim_{\varepsilon \to 0} \frac{Z[K + \varepsilon\theta\delta^{\cdot}_{\mu}\delta(\cdot - y)] - Z[K]}{\varepsilon}, \tag{6.69}$$

where ϵ is a real parameter. The dots indicate a missing argument in the deltas. Consider the following partition function with a propagator $G_{\mu\nu}(x)$ that is antisymmetric (it satisfies $G_{\mu\nu}(-x) = -G_{\mu\nu}(x)$)

$$Z[K] = \exp\left\{\frac{1}{2}\int_{\mathbb{Q}_p^2} K^\mu(x) G_{\mu\nu}(x-y) K^\nu(y) dx dy\right\}. \tag{6.70}$$

Now let's first see that

$$\begin{aligned}
Z[K + \varepsilon\theta\delta_\alpha^{\cdot}\delta(\cdot - z)] &= \exp\left\{\frac{1}{2}\int_{\mathbb{Q}_p^2}(K^\mu(x) + \varepsilon\theta\delta_\alpha^\mu\delta(x-z))G_{\mu\nu}(x-y)(K^\nu(y) + \varepsilon\theta\delta_\alpha^\nu\delta(y-z))dxdy\right\} \\
&= \exp\left\{\frac{1}{2}\int_{\mathbb{Q}_p^2} K^\mu(x) G_{\mu\nu}(x-y) K^\nu(y) dx dy\right\} \\
&\times \exp\left\{\frac{1}{2}\varepsilon\int_{\mathbb{Q}_p^2}\left[K^\mu(x)G_{\mu\nu}(x-y)\theta\delta_\alpha^\nu\delta(y-z) + \theta\delta_\alpha^\mu\delta(x-z)G_{\mu\nu}(x-y)K^\nu(y)\right]dxdy\right\} \\
&= Z[K]\exp\left\{\frac{1}{2}\varepsilon\int_{\mathbb{Q}_p} G_{\mu\alpha}(x-z)\left[K^\mu(x)\theta - \theta K^\mu(x)\right]dx\right\} \\
&= Z[K]\exp\left\{\varepsilon\theta\int_{\mathbb{Q}_p} G_{\mu\alpha}(z-x)K^\mu(x)dx\right\} \\
&= Z[K]\left[1 + \varepsilon\theta\int_{\mathbb{Q}_p} G_{\mu\alpha}(z-x)K^\mu(x)dx + \mathcal{O}(\varepsilon^2)\right].
\end{aligned} \tag{6.71}$$

Thus, we can now obtain the functional derivative

$$\begin{aligned}
\frac{\delta Z[K]}{\delta K^\mu(y)} &= \int d\theta \lim_{\varepsilon\to 0}\left[\theta\int_{\mathbb{Q}_p} G_{\mu\alpha}(z-x)K^\mu(x)dx Z[K] + \mathcal{O}(\varepsilon)\right] \\
&= \int_{\mathbb{Q}_p} G_{\mu\alpha}(y-x)K^\mu(x)dx Z[K].
\end{aligned} \tag{6.72}$$

Our functional derivative (6.69) follows the Leibniz rule, one can easily check this. Now we obtain the two point function

$$\begin{aligned}
\left.\frac{\delta^2 Z[K]}{\delta K^\mu(y_1)\delta K^\nu(y_2)}\right|_{K=0} &= \left[G_{\mu\nu}(y_1 - y_2) + \int_{\mathbb{Q}_p} G_{\mu\alpha}(y_1 - x)K^\alpha(x)dx\right. \\
&\times \left.\int_{\mathbb{Q}_p} G_{\nu\beta}(y_2 - x)K^\beta(x)dx\right] Z[K]\Big|_{K=0} = G_{\mu\nu}(y_1 - y_2).
\end{aligned} \tag{6.73}$$

One also can check that

$$\left.\frac{\delta}{\delta K^\mu(y)}\exp\left\{\int_{\mathbb{Q}_p} K_\nu(x)\psi^\nu(x)dx\right\}\right|_{K=0} = \psi_\mu(x). \tag{6.74}$$

Looking at the fermionic part in (6.37) coming from the action $I_F[\psi]$, we see that indeed

$$\langle\psi^\mu(x)\psi^\nu(y)\rangle = \frac{\alpha'}{\mathrm{sgn}_\tau(-1)p}\mathcal{G}_F^{\mu\nu}(x-y). \tag{6.75}$$

Chapter 7

Local Zeta Functions and Koba-Nielsen Amplitudes

This chapter is a brief review of section 8 in [27] that is itself a review of [26]. In these works the Koba-Nielsen type amplitudes defined over local fields of characteristic zero ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$ and its extensions) are treated on an equal footing. They show that these type of integrals have a connected region of convergence and thus admit meromorphic continuations in their arguments. We include it because these results involve the usual Archimedean string amplitudes. It is a product of studying p-adic theories that found its way back to Archimedean physics.

7.1 Meromorphic Continuation of Koba-Nielsen Amplitudes Defined on Local Fields of Characteristic Zero

As mentioned earlier in section 2.3, the problem with the string amplitudes as mathematical objects, is that they are not directly local zeta functions. They lack a compact support, however in this section we take a brief view on how nevertheless one can show that the theorems on analytic continuation are extended to include these type of integrals. We start with the following theorem that merely states the convergence of the amplitudes in the case in which the region of integration is compact.

Theorem 7.1.1. *[26, Lemma 6.4, Remark 2] Let $f_1(x), \dots, f_m(x) \in \mathbb{K}[x_1, \dots, x_n]$ be non-constant polynomials over the local field $\mathbb{K}$, and $\Phi : \mathbb{K}^n \to \mathbb{C}$ a smooth function with compact support, to which we associate the multivariate local zeta function $Z_\Phi(\boldsymbol{f}, \boldsymbol{s})$. Fix an embedded resolution $\sigma : X \to \mathbb{K}$ of $D_{\mathbb{K}} = \cup_{i=1}^m f_i^{-1}(0)$ as in Theorem 2.3.1 . Then*
(i) $Z_\Phi(\boldsymbol{f}, \boldsymbol{s})$ is convergent and defines a holomorphic function in the region

$$\sum_{j=1}^{m} N_{f_j,i} \operatorname{Re}(s_j) + v_i > 0, \qquad \text{for } i \in T;$$

(ii) $Z_{\Phi}(\boldsymbol{f},\boldsymbol{s})$ *admits a meromorphic continuation to the whole* $\mathbb{C}^m$*, with poles belonging to*

$$\bigcup_{i\in T}\bigcup_{t\in\mathbb{N}}\left\{\sum_{j=1}^{m}N_{f_j,i}s_j+v_i+t=0\right\},$$

with $t\in\mathbb{N}$ *if* $\mathbb{K}=\mathbb{R}$ *and* $t=\frac{1}{2}\mathbb{N}$ *if* $\mathbb{K}=\mathbb{C}$*, and with poles belonging to*

$$\bigcup_{1\leq i\leq r}\left\{\sum_{j=1}^{m}a_{j,i}s_j+b_i=0\right\},$$

in the non-Archimedean case. In addition, in the p-adic case the multivariate local zeta function has a meromorphic continuation as a rational function

$$Z_{\Phi}(\boldsymbol{f},\boldsymbol{s})=\frac{P_{\Phi}(\boldsymbol{s})}{\prod_{i\in T}\left(1-q^{-\left(\sum_{j=1}^{m}N_{f_j,i}s_j+v_i\right)}\right)}$$

in $q^{-s_1},\ldots,q^{-s_m}$*, where* $P_{\Phi}(\boldsymbol{s})$ *is a polynomial in the variables* q^{-s_i}*.*

Now we will see that for the specific case of Koba-Nielsen type integrals one may obtain similar results. The Koba-Nielsen open string amplitudes for N-points over a local field $\mathbb{K}$ of characteristic zero are defined as

$$A_{\mathbb{K}}^{(N)}(\boldsymbol{k}):=\int_{\mathbb{K}^{N-3}}\prod_{i=2}^{N-2}|x_j|_{\mathbb{K}}^{\boldsymbol{k}_1\boldsymbol{k}_j}|1-x_j|_{\mathbb{K}}^{\boldsymbol{k}_{N-1}\boldsymbol{k}_j}\prod_{2\leq i<j\leq N-2}|x_i-x_j|_{\mathbb{K}}^{\boldsymbol{k}_i\boldsymbol{k}_j}\prod_{i=2}^{N-2}dx_i, \tag{7.1}$$

where $\boldsymbol{k}=(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)$, $\boldsymbol{k}_i=(k_{0,i},\ldots,k_{l,i})\in\mathbb{C}^{l+1}$, for $i=1,\ldots,N$, is the momentum vector of the i-th tachyon (with the product defined as $\boldsymbol{k}_i\boldsymbol{k}_j=-k_{0,i}k_{0,j}+k_{1,i}k_{1,j}+\cdots+k_{l,i}k_{l,j}$), obeying

$$\sum_{i=1}^{N}\boldsymbol{k}_i=\boldsymbol{0},\qquad \boldsymbol{k}_i\boldsymbol{k}_i=2\ \text{ for } i=1,\ldots,N. \tag{7.2}$$

Four our purposes, the dimension l can be any positive number, and the product $\boldsymbol{k}_i\boldsymbol{k}_i$ can be any positive integer depending on the theory, for instance 8 for closed strings. In [113, Section 2], it is shown that the N-point tree-level closed string amplitude is the product of $A_{\mathbb{C}}^{(N)}(\boldsymbol{k})$ times a polynomial in the momenta $\boldsymbol{k}$. Hence, the results of [26] are still valid for closed string amplitudes.

We take the product of the momenta as $\boldsymbol{k}_i\boldsymbol{k}_j=s_{ij}\in\mathbb{C}$. Then the string amplitude (7.1) becomes a type of multivariate local zeta function as follows

$$Z_{\mathbb{K}}^{(N)}(\boldsymbol{s}):=\int_{\mathbb{K}^{N-3}\smallsetminus D_N}\prod_{i=2}^{N-2}|x_j|_{\mathbb{K}}^{s_{1j}}|1-x_j|_{\mathbb{K}}^{s_{(N-1)j}}\prod_{2\leq i<j\leq N-2}|x_i-x_j|_{\mathbb{K}}^{s_{ij}}\prod_{i=2}^{N-2}dx_i, \tag{7.3}$$

where $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure on $\mathbb{K}^{N-3}$, $\boldsymbol{s} := (s_{ij}) \in \mathbb{C}^D$, with $D = \frac{N(N-3)}{2}$ denotes the total number of indices ij, and

$$D_N := \left\{ x \in \mathbb{K}^{N-3}; \prod_{i=2}^{N-2} x_i \prod_{i=2}^{N-2} (1-x_i) \prod_{2 \leq i < j \leq N-2} (x_i - x_j) = 0 \right\}.$$

We name these type of integrals as *Koba-Nielsen local zeta functions*. For simplicity of notation, we put $\mathbb{K}^{N-3}$ instead of $\mathbb{K}^{N-3} \smallsetminus D_N$ in (7.3).

To prove the meromorphic continuation of (7.3), we express it as linear combinations of local zeta functions. Here we show the real case, the p-adic case is seen in [25] and see [26] for the complex case. We consider $\mathbb{R}^{N-3}$ as an $\mathbb{R}$-analytic manifold (see the paragraph above Theorem 2.3.1), with $N \geq 4$, and use $\{x_2, \ldots, x_{N-2}\}$ as a coordinate system. We make a partition of $\mathbb{R}^{N-3}$ constructed using a smooth function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\chi(x) = \begin{cases} 1 & \text{if} \quad x \in [-2, 2] \\ 0 & \text{if} \quad x \in (-\infty, -2-\epsilon] \cup [2+\epsilon, +\infty), \end{cases}$$

for some fixed positive ϵ sufficiently small. This function is well-known, see e.g. [76, Section 5.2]. The number 2 is arbitrary, however we must ensure the interval $[0, 1]$ is included in the locus where $\chi \equiv 1$. The idea is to use the function χ to split the integral into many regions and see that in each region one can reduce it to an actual local zeta function so that we simply use the theorems at hand to regularize the integral (7.3). Now, we write

$$Z^{(N)}(\boldsymbol{s}) = \sum_I Z_I^{(N)}(\boldsymbol{s}), \tag{7.4}$$

with

$$Z_I^{(N)}(\boldsymbol{s}) := \int_{\mathbb{R}^{N-3}} \varphi_I(x) \prod_{j=2}^{N-2} |x_j|^{s_{1j}} \prod_{j=2}^{N-2} |1-x_j|^{s_{(N-1)j}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|^{s_{ij}} \prod_{i=2}^{N-2} dx_i, \tag{7.5}$$

where the functions $\{\varphi_I\}$ are defined as

$$\varphi_I : \mathbb{R}^{N-3} \to \mathbb{R};\ x \mapsto \prod_{i \in I} \chi(x_i) \prod_{i \notin I} (1 - \chi(x_i)), \tag{7.6}$$

for $I \subseteq \{2, \ldots, N-2\}$, including the empty set, with the convention that $\prod_{i \in \emptyset} \cdot \equiv 1$. Notice that $\varphi_I \in C^\infty(\mathbb{R}^{N-3})$ and $\sum_I \varphi_I(x) \equiv 1$, for $x \in \mathbb{R}^{N-3}$, that is, the functions $\{\varphi_I\}$ form a partition of the unity.

In the case $I = \{2, \ldots, N-2\}$, $Z_I^{(N)}(\boldsymbol{s})$ is a classical multivariate Igusa local zeta function (since $\varphi_I(x)$ has compact support). It is well known that these integrals are holomorphic functions in $\mathrm{Re}(s_{ij}) > 0$ for all ij, and they admit meromorphic continuations to the whole $\mathbb{C}^D$, see [26, Theorem 3.2].

In the case $I \neq \{2,\ldots,N-2\}$, by changing variables in (7.5) as $x_i \to \frac{1}{x_i}$ for $i \notin I$, and $x_i \to x_i$ for $i \in I$, we have $\prod_{i=2}^{N-2} dx_i \to \prod_{i\notin I} \frac{1}{|x_i|^2} \prod_{i=2}^{N-2} dx_i$, and by setting $\widetilde{\chi}(x_i) := 1 - \chi\left(\frac{1}{x_i}\right)$ for $i \notin I$, i.e.,

$$\widetilde{\chi}(x_i) = \begin{cases} 1 & \text{if} \quad |x_i| \leq \frac{1}{2+\epsilon} \\ \\ 0 & \text{if} \quad |x_i| \geq \frac{1}{2}, \end{cases}$$

we have that supp $\widetilde{\chi} \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\widetilde{\chi} \in C^\infty(\mathbb{R})$. Now setting $\widetilde{\varphi}_I(x) := \prod_{i\notin I} \widetilde{\chi}(x_i) \prod_{i\in I} \chi(x_i)$, and

$$F_I(x, \boldsymbol{s}) := \prod_{j\in I} |x_j|^{s_{1j}} \prod_{j=2}^{N-2} |1 - x_j|^{s_{(N-1)j}} \prod_{\substack{2\leq i<j\leq N-2 \\ i,\, j\in I}} |x_i - x_j|^{s_{ij}}$$
$$\times \prod_{\substack{2\leq i<j\leq N-2 \\ i,\, j\notin I}} |x_i - x_j|^{s_{ij}} \prod_{\substack{2\leq i<j\leq N-2 \\ i\notin I,\, j\in I}} |1 - x_i x_j|^{s_{ij}} \prod_{\substack{2\leq i<j\leq N-2 \\ i\in I,\, j\notin I}} |1 - x_i x_j|^{s_{ij}},$$

we have

$$Z_I^{(N)}(\boldsymbol{s}) = \int_{\mathbb{R}^{N-3}\smallsetminus D_I} \frac{\widetilde{\varphi}_I(x)\, F_I(x, \boldsymbol{s})}{\prod_{i\notin I} |x_i|^{s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2,\, j\neq i} s_{ij}+2}} \prod_{i=2}^{N-2} dx_i, \tag{7.7}$$

where D_I is the divisor defined by the polynomial

$$\prod_{i=2}^{N-2} x_i \prod_{i=2}^{N-2} (1 - x_i) \prod_{\substack{2\leq i<j\leq N-2 \\ i,\, j\in I}} (x_i - x_j) \prod_{\substack{2\leq i<j\leq N-2 \\ i,\, j\notin I}} (x_i - x_j)$$
$$\times \prod_{\substack{2\leq i<j\leq N-2 \\ i\notin I,\, j\in I}} (1 - x_i x_j) \prod_{\substack{2\leq i<j\leq N-2 \\ i\in I,\, j\notin I}} (1 - x_i x_j).$$

The integrals $Z_I^{(N)}(\boldsymbol{s})$, with $I \neq \{2,\ldots,N-2\}$, are not classical multivariate local zeta functions. Thus, in [26] it is shown that they define holomorphic functions on some nonempty open in $\mathbb{C}^D$, and admit meromorphic continuations to the whole $\mathbb{C}^D$. As in the p-adic case, we use the meromorphic continuation of (7.3) to the whole $\mathbb{C}^D$, which is denoted by $Z_{\mathbb{K}}^{(N)}(\boldsymbol{s})$, as regularizations of the amplitudes $A_{\mathbb{K}}^{(N)}(\boldsymbol{k})$ by redefining

$$A_{\mathbb{K}}^{(N)}(\boldsymbol{k}) = Z_{\mathbb{K}}^{(N)}(\boldsymbol{s})\,|_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j},$$

see [26, Theorem 6.1]. It is important to mention that we do not use the kinematic restrictions (7.2) to get our results.

To illustrate the process, next we show explicitly the cases $N = 4, 5$.

7.1.1 4-point Koba-Nielsen string amplitude

The 4-point Koba-Nielsen open string amplitude is defined as

$$Z^{(4)}(s) = \int_{\mathbb{R}} |x_2|^{s_{12}} |1 - x_2|^{s_{32}} dx_2.$$

By using the function

$$\chi\left(x_{2}\right)=\begin{cases}1 & \text{if} \quad x_{2}\in[-2,2]\\ \\ 0 & \text{if} \quad x_{2}\in(-\infty,-2-\epsilon]\cup[2+\epsilon,+\infty),\end{cases} \tag{7.8}$$

where $\epsilon>0$ is sufficiently small, we construct the following partition of unity:

$$\begin{array}{llll}\varphi_{\{2\}}: & \mathbb{R} & \rightarrow & \mathbb{R}; \quad x_{2}\mapsto\chi\left(x_{2}\right)\\ \varphi_{\varnothing}: & \mathbb{R} & \rightarrow & \mathbb{R}; \quad x_{2}\mapsto 1-\chi\left(x_{2}\right).\end{array}$$

Notice that $\varphi_{\varnothing},\varphi_{\{2\}}\in C^{\infty}(\mathbb{R})$ and $\varphi_{\{2\}}(x)+\varphi_{\varnothing}(x)\equiv 1$, for $x\in\mathbb{R}$. Hence,

$$Z^{(4)}(\boldsymbol{s})=Z_{\{2\}}^{(4)}(\boldsymbol{s})+Z_{\varnothing}^{(4)}(\boldsymbol{s}),$$

with

$$Z_{\{2\}}^{(4)}(\boldsymbol{s})=\int_{\mathbb{R}}\chi(x_{2})|x_{2}|^{s_{12}}|1-x_{2}|^{s_{32}}dx_{2} \tag{7.9}$$

and

$$Z_{\varnothing}^{(4)}(\boldsymbol{s})=\int_{\mathbb{R}}(1-\chi(x_{2}))|x_{2}|^{s_{12}}|1-x_{2}|^{s_{32}}dx_{2}. \tag{7.10}$$

The integral (7.9) is a multivariate local zeta function since $\chi(x_2)$ has compact support. By a classical result of local zeta functions, $Z_{\{2\}}^{(4)}(\boldsymbol{s})$ converges when $\operatorname{Re}(s_{12})>-1$ and $\operatorname{Re}(s_{32})>-1$. Furthermore, it admits a meromorphic continuation to the whole $\mathbb{C}^2$, see [26, Theorem 3.2].

The second integral (7.10) is not a multivariate local zeta function, but it can be transformed into one by changing $x_2\rightarrow\frac{1}{x_2}$. Then $dx_2\rightarrow\frac{1}{|x_2|^2}\,dx_2$, and by setting $\widetilde{\chi}\left(x_{2}\right):=1-\chi\left(\frac{1}{x_{2}}\right)$, we have that

$$\widetilde{\chi}\left(x_{2}\right)=\begin{cases}1 & \text{if} \quad |x_{2}|\leq\frac{1}{2+\epsilon}\\ \\ 0 & \text{if} \quad |x_{2}|\geq\frac{1}{2}.\end{cases}$$

Notice that the support of $\widetilde{\chi}$ is contained in $\left[-\frac{1}{2},\frac{1}{2}\right]$ and $\widetilde{\chi}\in C^{\infty}(\mathbb{R})$. Thus integral (7.10) becomes

$$Z_{\emptyset}^{(4)}(\boldsymbol{s})=\int_{\mathbb{R}}\widetilde{\chi}(x_{2})|x_{2}|^{-s_{12}-s_{32}-2}|1-x_{2}|^{s_{32}}dx_{2},$$

which is analytic when $-\operatorname{Re}(s_{12})-\operatorname{Re}(s_{32})-1>0$ and $\operatorname{Re}(s_{32})+1>0$. We concluded that $Z^{(4)}(s)$ is analytic in the region

$$\operatorname{Re}(s_{12})>-1,\quad \operatorname{Re}(s_{32})>-1,\quad \operatorname{Re}(s_{12})+\operatorname{Re}(s_{32})<-1.$$

Which contains the open set $-1<\operatorname{Re}(s_{12})<-\frac{1}{2}$ and $-1<\operatorname{Re}(s_{32})<-\frac{1}{2}$.

7.1.2 5-point Koba-Nielsen string amplitude

We define the following partition of the unity:

$$\begin{array}{rrcll} \varphi_{\{2,3\}}: & \mathbb{R}^2 & \to & \mathbb{R}; & (x_2,x_3)\mapsto \chi(x_2)\,\chi(x_3)\,, \\ \varphi_{\{2\}}: & \mathbb{R}^2 & \to & \mathbb{R}; & (x_2,x_3)\mapsto \chi(x_2)\,(1-\chi(x_3)), \\ \varphi_{\{3\}}: & \mathbb{R}^2 & \to & \mathbb{R}; & (x_2,x_3)\mapsto \chi(x_3)\,(1-\chi(x_2)), \\ \varphi_{\varnothing}: & \mathbb{R}^2 & \to & \mathbb{R}; & (x_2,x3)\mapsto (1-\chi(x_2))(1-\chi(x_3)), \end{array}$$

where $\chi(x)$ is defined in (7.8). Then $Z^{(5)}(\boldsymbol{s}) = Z^{(5)}_{\{2,3\}}(\boldsymbol{s}) + Z^{(5)}_{\{3\}}(\boldsymbol{s}) + Z^{(5)}_{\{2\}}(\boldsymbol{s}) + Z^{(5)}_{\emptyset}(\boldsymbol{s})$. First we see that

$$Z^{(5)}_{\{2,3\}}(\boldsymbol{s}) = \int\limits_{\mathbb{R}^2} \chi(x_2)\chi(x_3)|x_2|^{s_{12}}|x_3|^{s_{13}}|1-x_2|^{s_{42}}|1-x_3|^{s_{43}}|x_2-x_3|^{s_{23}}dx_2dx_3, \tag{7.11}$$

is a local zeta function since $\varphi_{\{2,3\}}$ has compact support. Thus, we use resolution of singularities of the divisor D_5 defined by $x_2x_3(1-x_2)(1-x_3)(x_2-x_3)=0$. The integral (7.11) is not locally monomial only at the points $(0,0)$ and $(1,1)$, and therefore we cannot use the theorems in a region containing them. Hence, we pick a partition of the unity, $\sum_{i=0}^{2}\Omega_i(x_2,x_3)=1$, and we write

$$Z^{(5)}_{\{2,3\}}(s) = \sum_{j=0}^{2} Z^{(5)}_{\Omega_j}(s),$$

where

$$Z^{(5)}_{\Omega_j}(s) = \int_{\mathbb{R}^2} \Omega_j(x_2,x_3)|x_2|^{s_{12}}|x_3|^{s_{13}}|1-x_2|^{s_{42}}|1-x_3|^{s_{43}}|x_2-x_3|^{s_{23}}dx_2dx_3.$$

Here, Ω_0 and Ω_1 are smooth functions supported in a small neighborhood of $(0,0)$ and $(1,1)$, respectively. We only need to analyse the integrals $Z^{(5)}_{\Omega_0}(s)$ and $Z^{(5)}_{\Omega_1}(s)$.

In terms of convergence and holomorphy, around $(0,0)$, the factor $|1-x_2|^{s_{42}}|1-x_3|^{s_{43}}$ can be neglected. Then we only need an embedded resolution of $x_2x_3(x_2-x_3)=0$, obtained by a blow-up at the origin. To do it we make the changes of variables

$$\begin{aligned} \sigma_0:\mathbb{R}^2\to\mathbb{R}^2;\ u_2 &\mapsto x_2 = u_2 \\ u_3 &\mapsto x_3 = u_2u_3. \end{aligned}$$

The integral $Z^{(5)}_{\Omega_0}(s)$ becomes

$$\int\limits_{\mathbb{R}^2} (\Omega_0\circ\sigma_0)(u_2,u_3)\,|u_2|^{s_{12}+s_{13}+s_{23}+1}\,|u_3|^{s_{13}}|1-u_3|^{s_{23}}g(u,\boldsymbol{s})du_2du_3,$$

where $g(u,s)$ is invertible on the support of $\Omega_0\circ\sigma_0$ and is irrelevant for convergence and holomorphy. By [26, Lemma 3.1], we obtain the convergence conditions:

$$\operatorname{Re}(s_{12})+\operatorname{Re}(s_{13})+\operatorname{Re}(s_{23})+2>0, \quad \operatorname{Re}(s_{23})+1>0, \quad \operatorname{Re}(s_{13})+1>0. \tag{7.12}$$

If we consider the other chart of the blow-up, i.e. the change of variables

$$\sigma_0' : \mathbb{R}^2 \to \mathbb{R}^2;\ u_2 \mapsto x_2 = u_2 u_3$$
$$u_3 \mapsto x_3 = u_3,$$

we get the first and second condition in (7.12) and also

$$\operatorname{Re}(s_{12}) + 1 > 0. \tag{7.13}$$

Similarly, for the convergence of $Z^{(5)}_{\Omega_1}(s)$, we need also the new conditions

$$\operatorname{Re}(s_{42}) + \operatorname{Re}(s_{43}) + \operatorname{Re}(s_{23}) + 2 > 0, \quad \operatorname{Re}(s_{42}) + 1 > 0, \quad \operatorname{Re}(s_{43}) + 1 > 0. \tag{7.14}$$

The conditions coming from the locally monomial integral $Z^{(5)}_{\Omega_2}(s)$ are already included.

The integral $Z^{(5)}_{\{3\}}(\boldsymbol{s})$ is not a multivariate local zeta function, so we take the change of variables $x_2 \to \frac{1}{x_2}$. Then we have $dx_2 \to \frac{1}{|x_2|^2}\, dx_2$, and by setting $\widetilde{\chi}\,(x_2) := 1 - \chi\left(\frac{1}{x_2}\right)$, we have that

$$\widetilde{\chi}\,(x_2) = \begin{cases} 1 & \text{if} \quad |x_2| \leq \frac{1}{2+\epsilon} \\ \\ 0 & \text{if} \quad |x_2| \geq \frac{1}{2}, \end{cases}$$

Then

$$Z^{(5)}_{\{3\}}(\boldsymbol{s}) = \int_{\mathbb{R}^2} \widetilde{\chi}(x_2)\chi(x_3)|x_2|^{-s_{12}-s_{42}-s_{23}-2}|x_3|^{s_{13}}|1-x_2|^{s_{42}}|1-x_3|^{s_{43}} \times |1-x_2x_3|^{s_{23}} dx_2 dx_3.$$

Since $x_2x_3(1-x_2)(1-x_3)(1-x_2x_3)$ is locally monomial in the support of $\widetilde{\chi}(x_2)\chi(x_3)$, the only new condition is

$$-\operatorname{Re}(s_{12}) - \operatorname{Re}(s_{42}) - \operatorname{Re}(s_{23}) - 1 > 0. \tag{7.15}$$

Doing the same procedure, $Z^{(5)}_{\{2\}}(\boldsymbol{s})$ induces the extra condition

$$-\operatorname{Re}(s_{13}) - \operatorname{Re}(s_{43}) - \operatorname{Re}(s_{23}) - 1 > 0. \tag{7.16}$$

For the last integral, we set $\widetilde{\chi}\,(x_2) := 1 - \chi\left(\frac{1}{x_2}\right)$ and $\widetilde{\chi}\,(x_3) := 1 - \chi\left(\frac{1}{x_3}\right)$. Thus,

$$Z^{(5)}_{\emptyset}(\boldsymbol{s}) = \int_{\mathbb{R}^2} \widetilde{\chi}(x_2)\widetilde{\chi}(x_3)|x_2|^{-s_{12}-s_{42}-s_{23}-2}|x_3|^{-s_{13}-s_{43}-s_{23}-2}|1-x_2|^{s_{42}}|1-x_3|^{s_{43}} \times |x_2-x_3|^{s_{23}} dx_2 dx_3.$$

In this case, we have the same divisor as for $Z^{(5)}_{\{2,3\}}(\boldsymbol{s})$; the differences are the powers of $|x_2|$ and $|x_3|$, and the function $\widetilde{\chi}(x_2)\widetilde{\chi}(x_3)$, that does not contain $(1,1)$ in its support. So, the only new condition will arise from the blow-up at the origin, namely

$$-\operatorname{Re}(s_{12}) - \operatorname{Re}(s_{42}) - \operatorname{Re}(s_{23}) - \operatorname{Re}(s_{13}) - \operatorname{Re}(s_{43}) - 2 > 0. \tag{7.17}$$

Consequently $Z^{(5)}(\boldsymbol{s})$ is analytic in the region defined by conditions (7.12)-(7.17), that is,

$$\begin{aligned}
&\mathrm{Re}(s_{ij}) > -1, \text{ for all ij} \\
&\mathrm{Re}(s_{12}) + \mathrm{Re}(s_{13}) + \mathrm{Re}(s_{23}) > -2, \quad \mathrm{Re}(s_{42}) + \mathrm{Re}(s_{43}) + \mathrm{Re}(s_{23}) > -2, \\
&\mathrm{Re}(s_{12}) + \mathrm{Re}(s_{42}) + \mathrm{Re}(s_{23}) < -1, \quad \mathrm{Re}(s_{13}) + \mathrm{Re}(s_{43}) + \mathrm{Re}(s_{23}) < -1, \\
&\sum_{ij} \mathrm{Re}(s_{ij}) < -2.
\end{aligned}$$

This region of convergence contains the open subset defined by

$$-\frac{2}{3} < \mathrm{Re}(s_{ij}) < -\frac{2}{5} \qquad \text{for all } ij.$$

Then, in particular, $Z^{(5)}(s)$ is analytic in the interval $-\frac{2}{3} < \mathrm{Re}(s) < -\frac{2}{5}$. This result is important as it shows that the Archimedean bosonic string amplitudes are well defined integrals in a mathematically rigorous way. The result is also very relevant in showing that using p-adic numbers and non-Archimedean physics models are useful as toy models of Archimedean physics. They can inspire new ideas, to say the least. They can also give new tools to prove or clarify old problems, such as the one presented here.

Part III

AdS/CFT

Chapter 8

The Archimedean and non-Archimedean AdS/CFT correspondence

In this part we will discuss some work about the recently developed AdS/CFT correspondence using p-adic numbers. First we will go over the basics of the correspondence in the real scenario, emphasizing the key ideas on the subject, the correlation functions and the basic procedure of holographic renormalization. Then we will review the p-adic analogues of the correspondence, as usual it is very close in spirit to the real case but with simplified calculations and relations. Finally we will present some new results that connect two articles about the p-adic AdS/CFT correspondence, we will show that although they seem barely related to each other, in fact they are very closely related, and can be seen as complementary to each other. Finally we explore the topic of holographic renormalization in the p-adic context.

8.1 Essentials of the AdS/CFT correspondence

The AdS/CFT correspondence is a duality between two theories. Dualities have been around for quite some time, there are plenty of known relations between different theories. For example the T-duality that relates two string theories, and there are other dualities that relate string theories. There are also dualities for Super Yang-Mills theories like the Montonen-Olive duality, or the procedure known as bosonization, that relates fermionic degrees of freedom to bosonic ones in another theory. For all these cases we can say that the fundamental nature of the theories that are being related doesn't change. However the story is very different for AdS/CFT, in this case the relation is between a quantum field theory with conformal symmetry in d-dimensional flat spacetime, also known as a Conformal Field Theory (CFT), and a quantum gravity theory over Anti de Sitter space (AdS) in $d+1$-dimensions, a spacetime with constant negative curvature. Not only are we changing the number of dimensions, we are going from one quantum theory without gravity, to a theory with gravity. Furthermore it is also a weak/strong duality, meaning that the weakly coupled (perturbative) regime of one theory is related to the strongly coupled (non-perturbative) regime of the other.

The correspondence has acquired many names due to its characteristics. It is called sometimes a Gauge/Gravity duality because it relates a gauge theory with a gravity theory. Another name is a holographic correspondence or principle, this is because it relates a theory in $d+1$ dimensions with one in d dimensions at its boundary. It is like a hologram in the sense that all the information of a 3d object is captured by its boundary 2d surface. There are many works where one can learn about the correspondence, here we follow closely [114], but one may also see [115, 116, 117, 118, 119, 38] to name a few. The correspondence was first discovered by Maldacena in 1998 [120]. Studying two equivalent descriptions of the type IIB string theory different regimes, he realized that the theories obtained by varying the regime led to very different theories. In particular we consider the configuration of having $N \gg 1$ coincident D3-branes on 10-dimensional type IIB superstring theory. On the one hand, when the coupling constant g_s for open and closed strings is small, and we consider only massless excitations, or low energies $E \ll \alpha'^{-1/2}$, we obtain an effective description that contains $\mathcal{N}=4$ Super Yang Mills coupled to supergravity in $\mathbb{R}^{9,1}$; this perspective is valid for $g_sN \ll 1$. On the other hand, the other side of the regime is when $g_sN \gg 1$, this corresponds to the low energy limit of the the string theory, that is supergravity, and also having weak curvature, meaning that the characteristic length measured in string lengths $L^4/\alpha'^2 \sim g_sN$ is large.

Furthermore, in each of these descriptions, we can take the point-particle limit $\alpha' \to 0$. For the case $g_sN \ll 1$ we are left with two decoupled theories, Supergravity in $\mathbb{R}^{9,1}$ and $\mathcal{N}=4$ Super Yang Mills. For the case $g_sN \gg 1$, the description corresponds to closed string excitations in a background with two distinguished regions. The decoupling limit then decouples these two regions, leaving us with Supergravity in $\mathbb{R}^{9,1}$ on one side and fluctuations about the $AdS_5 \times S^5$ solution of IIB supergravity on the other side. On the pure supergravity side ($g_sN \gg 1$), we make the distinction between two regions, one asymptotically flat, and the other is referred to as the *throat*. The gravity theory is said to live in *the bulk* and the gauge theory lives on *the boundary*.

If we take the decoupling limit first, we should be able to interpolate between the two sides of pairs of decoupled theories just described. Both have supergravity in 10 flat dimensions, however the other decoupled theories do not match, but one is looking at two different regimes of the same theory. This led Maldacena to the conjecture that there is a deep connection between $\mathcal{N}=4$ Super Yang-Mills in $\mathbb{R}^{9,1}$ and classical Supergravity in $AdS_5 \times S^5$, one is related to the other taking an appropriate limit. This is the weak form of the conjecture. A strong form of it is to assume the relation for any g_sN, which means there is a duality between the theories, now having a string theory instead of supergravity. And there is an even stronger version in which we have a duality between superYang-Mills and a quantum string theory.

To be a little more explicit, there is a dictionary between objects of the theories. The basic relations is through the fundamental parameters of the theories. For instance SuperYang-Mills has the gauge group $SU(N)$ and coupling constant g_{YM}, while the superstring theory has a string length $l_s = \sqrt{\alpha'}$, coupling constant g_s and AdS radius L (remember the string theory lives in $AdS_5 \times S^5$). These are related by

$$g_{YM}^2 = 2\pi g_s, \quad 2g_{YM}^2 N = 2\lambda = L^4/\alpha'^2, \tag{8.1}$$

Where $\lambda = g_{YM}^2 N$ is known as the t'Hooft coupling. As one goes further into this correspondence one finds eventually relations between other objects in the theory. A first example is that a bulk scalar field ϕ, on the boundary acts as a source field for a boundary scalar operator $\mathcal{O}$. One is able to find relation between the bulk field mass and the dimension of the boundary operator. Other examples include relations between a boundary stress-energy tensor $T^{\mu\nu}$ and current J^μ with the bulk gravitational field h_{MN} and Maxwell field A_M, respectively. In this thesis we are only interested in the first example involving the scalar bulk field and the boundary operator, as it is the simplest to transform into a p-adic version.

8.1.1 Correlation Functions

Ok, that last section was very hand wavy, but it is how the correspondence was discovered. Here we will show a much more explicit form in which we can relate the two theories, this is due to Gubser, Klebanov, Polyakov [118], and Witten [119]. They relate both sides of the correspondence by equating their partition functions,

$$Z_{CFT}[\phi_{(0)}] = \exp\left\{ -S_{SUGRA}[\phi]\Big|_{\lim\limits_{z\to 0} z^{d-\Delta}\phi(z,x)=\phi_{(0)}(x)} \right\}. \tag{8.2}$$

This expression is known as the GKPW relation, in honor of their authors. The left hand side is the generating function of the CFT with source $\phi_{(0)}$, on the right hand side is the exponential of the gravity action evaluated at the classical solution subject to the boundary behavior shown. With this explicit realization of the correspondence, it is now very clear how to compute something in both sides and compare them. To obtain the CFT correlation functions from the gravity theory, one must first obtain a classical solution to the supergravity equations of motion subject to the boundary condition $\lim\limits_{z\to 0} z^{d-\Delta}\phi(z,x) = \phi_{(0)}(x)$. Then one must evaluate the supergravity action with this solution. Finally, connected correlations functions are obtained by functionally differentiating the action with respect to $\phi_{(0)}$.

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{CFT} = -\frac{\delta S_{SUGRA}[\phi]}{\delta\phi^1_{(0)}(x_1)\cdots\delta\phi^n_{(0)}(x_n)}\Big|_{\phi^i_{(0)}=0}, \tag{8.3}$$

where the index on each operator indicates that we can have multiple operators other than scalars, the generic notation $\phi_{(0)}$ can be suppressing some further gauge indices. As you may have noticed, on the gravity side the correlation functions are at tree level. In the same way as usual, there are Feynman diagrams associated to these correlation functions, but in this context they are known as Witten diagrams, because they involve points at the boundary that carry a different propagator. Witten diagrams are like Feynman diagrams but they have a circle surrounding the diagram where the points can end, this circle represents the boundary. One can deduce some Feynman rules for each diagram, broadly speaking they indicate that lines connecting boundary and bulk points mean a *bulk-to-boundary propagator*, lines connecting points inside the bulk are known as *bulk-to-bulk propagators*. Interior vertices carry the interaction information of the supergravity theory.

For our purposes it is enough to consider a toy model of a scalar theory in Euclidean AdS_{d+1} space dual to a CFT_d. The CFT is not specified because all we care about are the lowest correlation functions and by symmetry they all posses the same behavior. We will be thinking about a scalar field ϕ of mass m that is dual to an operator $\mathcal{O}$ of conformal dimension Δ with the action

$$S[\phi] = \frac{C}{2} \int d^d x dz \sqrt{g} \left(g^{ab} \partial_a \phi \partial_b \phi + m^2 \phi^2 \right) \tag{8.4}$$

where C is a constant and the metric is

$$ds^2 = \frac{L^2}{z^2} \left(dz^2 + \delta_{\mu\nu} dx^\mu dx^\nu \right), \tag{8.5}$$

and $x \in \mathbb{R}^d$. The field ϕ has mass $L^2 m^2 = \Delta(d-\Delta)$ that is dual to an operator $\mathcal{O}$ of dimension Δ. The equation of motion for this action is

$$(\Box_g - m^2)\phi(x) = 0; \quad \Box_g = \frac{1}{\sqrt{g}} \partial_a \left(\sqrt{g} g^{ab} \partial_b \right). \tag{8.6}$$

For the AdS metric (8.5) the laplacian is

$$\Box_g \big|_{AdS} = \frac{1}{L^2} \left(z^2 \partial_z^2 - (d-1) z \partial_z + z^2 \eta^{\mu\nu} \partial_\mu \partial_\nu \right). \tag{8.7}$$

In the solution to the equation of motion ϕ_Δ we will indicate the dependence on Δ from the mass by a subscript. If we wanted a solution to this equation with the boundary condition $\phi_\Delta(z,x) \to z^{d-\Delta} \phi_{\Delta,(0)}(x)$ as $z \to 0$, we need to use an integral kernel $K_\Delta(z,x;y)$ that we refer to as the bulk-to-boundary propagator. It satisfies

$$\phi(z,x) = \int_{\partial AdS} d^d y K_\Delta(z,x;y) \phi_{(0),\Delta}(y), \tag{8.8}$$

where y is a boundary point. For consistency, this propagator has to satisfy

$$\lim_{z \to 0} z^{d-\Delta} K_\Delta(z,x;y) = \delta(x-y). \tag{8.9}$$

Similarly, we can use the bulk kernel $G(z,x;w,y)$, to solve the bulk equations of motion with a source $J(z,x)$, that is $(\Box_g - m^2)\phi(z,x) = J(z,x)$. This leads to have

$$\phi(z,x) = \int_{AdS} dw d^d y \sqrt{g} G_\Delta(z,x;w,y) J(w,y), \tag{8.10}$$

where (z,x) and (w,y) are bulk points. Of course, G should satisfy

$$(\Box_g - m^2)G_\Delta(z,x;w,y) = \frac{\delta(z,w)\delta^d(x,y)}{\sqrt{g}}. \tag{8.11}$$

The solution to this equation is a hypergeometric function, and it is best expressed by what is known as a *chordal distance* ξ defined by

$$\xi = \frac{2zw}{x^2+y^2-(z-w)^2}. \tag{8.12}$$

Then the solution to (8.11) is

$$G_\Delta(z,x;w,y) = \frac{\Gamma(\Delta)}{2^\Delta(2\Delta-d)\pi^{d/2}\Gamma(\Delta-d/2)}\xi^\Delta \cdot {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}, \Delta - \frac{d}{2}+1, ; \xi^2\right). \tag{8.13}$$

Both propagators are related by

$$K_\Delta(z,x;y) = \lim_{w\to 0}(2\Delta-d)w^{-\Delta}G_\Delta(z,x;w,y). \tag{8.14}$$

Using (8.13) and (8.14) we can get explicitly the bulk-to-boundary propagator

$$K_\Delta(z,x;y) = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-d/2)}\left(\frac{z}{z^2+(x-y)^2}\right)^\Delta. \tag{8.15}$$

Now we can give a concrete expression for the solution to the equation (8.6)

$$\phi(z,x) = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-d/2)}\int_{\partial AdS} d^d y \left(\frac{z}{z^2+(x-y)^2}\right)^\Delta \phi_{(0),\Delta}(y). \tag{8.16}$$

As a check of the correspondence we can compute the two point function of the boundary CFT from information on the bulk. If you recall from the previous section, we need the action evaluated at a classical solution with the appropriate boundary condition. The on-shell action (8.4) is integrated by parts, where the following boundary term appears

$$-\frac{C}{2}\int d^d x\sqrt{g}g^{zz}\phi(z,x)\partial_z\phi(z,x)\big|_{z=\epsilon}. \tag{8.17}$$

The boundary term at $z\to\infty$ vanishes because we demand regularity at the interior of the bulk. We do not go to exactly $z=0$ because $\sqrt{g}g^{zz}$ diverges and we need to regularize. It is best to do this procedure in Fourier space, but only in the variable x. The solution to the equation of motion (8.6) in Fourier space can be arranged as a modified Bessel function. Demanding regularity in the interior of AdS, the solution takes the form

$$\phi(z,p) = A_p z^{d/2}K_\nu(z|p|), \tag{8.18}$$

where $\nu = \Delta - d/2 = \sqrt{d^2/4+m^2L^2}$. The factor A_p can be determined by the boundary condition, resulting in

$$\phi(z,p) = \frac{z^{d/2}K_\nu(z|p|)}{\epsilon^{d/2}K_\nu(\epsilon|p|)}\phi_{(0)}(p)\epsilon^{d-\Delta}. \tag{8.19}$$

Now we want to insert this into the action (8.4). Taking advantage that it will be an on-shell action, we can integrate by parts to reduce the on-shell action to just the following boundary term

$$S[\phi_{cl}] = \frac{C}{2}\int d^dx\sqrt{g}g^{zz}\phi_{cl}(z,x)\partial_z\phi_{cl}(z,x)|_{z=\epsilon}, \tag{8.20}$$

where a term proportional to $\phi(\Box + m^2)\phi$ was omitted as it is zero for an on-shell field ϕ_{cl}. Going to momentum space and inserting (8.19) into the action we get

$$S[\phi_{(0)}] = -\frac{CL^{d-1}}{2\epsilon^{d-1}}\int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}(2\pi)^d\delta(p+q)\phi(z,p)\partial_z\phi(z,p)|_{z=\epsilon}. \tag{8.21}$$

With this we can compute the two-point function of the dual operator $\mathcal{O}$ of ϕ as

$$\begin{aligned}\langle\mathcal{O}(p)\mathcal{O}(q)\rangle_\epsilon &= -(2\pi)^d\frac{\delta^2 S[\phi_{(0)}]}{\delta\phi_{(0)}(-p)\delta\phi_{(0)}(-q)}\\ &= -\frac{(2\pi)^d\delta(p+q)CL^{d-1}}{\epsilon^{2\Delta-d}}\left(\frac{d}{2}\frac{\epsilon|p|K'_\nu(\epsilon|p|)}{K_\nu(\epsilon|p|)}\right).\end{aligned} \tag{8.22}$$

The last step is to take the limit $\epsilon \to 0$. In order to do it one must expand K in powers of z. The behavior of $K_\nu(u)$ for small u and positive integer ν is

$$K_\nu(u) \sim u^{-\nu}(a_0(\nu) + a_1(\nu)u^2 + \mathcal{O}(u^4)) + u^\nu \ln u(b_0(\nu) + b_1(\nu)u^2 + \mathcal{O}(u^4)). \tag{8.23}$$

With this behavior we can now determine the limit $\epsilon \to 0$ in the 2 point function. After ignoring some scheme dependent contact terms we end up with something $\sim |p|^{2\nu}\ln|p|$, that after transforming back to position space we get

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = CL^{d-1}\frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)}\frac{2\Delta - d}{\pi^{d/2}|x-y|2\Delta}. \tag{8.24}$$

This is the expected position dependence for the two-point function of a CFT, determined by symmetry. This is an explicit check of the correspondence and in particular of the ansatz (8.3).

8.1.2 Holographic Renormalization

Although the previous computation was satisfactory, there are still some issues. One of them is that some divergencies are present, and we need a way to deal with them. There is a general process known as *holographic renormalisation*, that is used to regularize the divergencies in the physical quantities in holographic theories, see for instance [67]. In particular here we will overview the process to renormalise the action (8.4) as we approach the boundary. The boundary is at $z \to 0$, so we focus on the z dependence of the on-shell field. We will first solve the z dependence of the equations of motion. As a first step, it is convenient to make the simple change of coordinates to $\rho = z^2$. With this the metric becomes

$$ds^2 = L^2\left(\frac{d\rho^2}{\rho^2} + \frac{1}{\rho}\delta_{\mu\nu}dx^\mu dx^\nu\right). \tag{8.25}$$

Next we propose the following ansatz for the scalar field

$$\phi(\rho, x) = \rho^{(d-\Delta)/2}\bar{\phi}(z, x) = \rho^{(d-\Delta)/2}\left(\phi_{(0)}(x) + \rho\phi_{(2)}(x) + \cdots\right). \tag{8.26}$$

After inserting this into the equation of motion (8.6) we have

$$\left[(m^2L^2 - \Delta(\Delta - d))\bar{\phi}(\rho, x) - \rho\left(\Box_0\bar{\phi} + 2(d - 2\Delta + 2)\partial_\rho\bar{\phi}(\rho, x) + 4\rho\partial_\rho^2\bar{\phi}(\rho, x)\right)\right] = 0, \tag{8.27}$$

where $\Box_0 = \delta^{\mu\nu}\partial_\mu\partial_\nu$ is the Laplacian on the transversal direction. This equation is solved order by order. First we can set $\rho = 0$ and get the mass relation $m^2L^2 = \Delta(d - \Delta)$. That relation makes the first term vanish and leaves us with the following equation for the field $\bar{\phi}$ defined in (8.26)

$$\Box_0\bar{\phi} + 2(d - 2\Delta + 2)\partial_\rho\bar{\phi}(\rho, x) + 4\rho\partial_\rho^2\bar{\phi}(\rho, x) = 0. \tag{8.28}$$

This can be solved for each $\phi_{(2n)}$ in terms of $\phi_{(0)}$ by iteratively taking derivatives of (8.28) with respect to ρ and then setting $\rho = 0$. This gives us the following recursive relation

$$\phi_{(2n)}(x) = \frac{1}{2n(2\Delta - d - 2n)}\phi_{(2n-2)}(x). \tag{8.29}$$

This procedure will stop if the denominator becomes zero for some integer $n = k$. This would mean having $\Delta = k + d/2$. If this happens it is necessary to add a logarithmic term to the expansion of $\bar{\phi}$ at order ρ^k in the form

$$\bar{\phi}(x) = \phi_{(0)}(x) + \cdots + \rho^k\left(\phi_{(2k)}(x) + \chi_{(2k)}(x)\log\rho\right) + \cdots \tag{8.30}$$

where $\chi_{(2k)}$ will be determined by

$$\chi_{(2k)}(x) = -\frac{1}{2^{2k}\Gamma(k)\Gamma(k+1)}(\Box_0)^k\phi_0(x), \tag{8.31}$$

and $\phi_{(2k)}$ remains undetermined. After finding the form of the on-shell field, we want to evaluate the regulated boundary action (8.20) with this solution. Once one has dealt with the algebra, we end up with

$$S_{reg}[\phi] = CL^{d-1}\int d^dx\left(\epsilon^{-\Delta+\frac{d}{2}}a_{(0)} + \epsilon^{-\Delta+\frac{d}{2}+1}a_{(2)} + \cdots - \ln\epsilon\, a_{(2\Delta-d)}\right), \tag{8.32}$$

where $a_{(2n)}$ are local functions depending only on $\phi_{(0)}$. The subscript reg in S_{reg} references having the cutoff at $\rho = \epsilon$. These coefficients $a_{(2i)}$ are given by

$$\begin{aligned} a_{(0)} = -\frac{1}{2}(d - \Delta)\phi_{(0)}^2; \quad a_{(2)} &= -\frac{d - \Delta + 1}{2(2\Delta - d - 2)}\phi_{(0)}\Box_0\phi_{(0)}; \\ a_{(2\Delta-d)} &= -\frac{d}{2^{2k+1}\Gamma(k)\Gamma(k+1)}\phi_{(0)}(\Box_0)^k\phi_{(0)}. \end{aligned} \tag{8.33}$$

Now of course in general we are considering $2\Delta - d \geq 0$, and then the terms shown in (8.32) are all divergent. In fact, because of the negative sign of Δ in the exponent of ϵ, depending on its value we will have more or less divergent terms. Concretely assuming that $\Delta = d/2 + k$ with $k \in \mathbb{N}$ we will have k divergent terms. These are problematic and need to be removed, this is what we call renormalisation. The common strategy is to simply use counterterms, that is defining a renormalised action S_{ren} that subtracts the divergent terms as we go to the boundary. The counter terms are collected in a counterterms action S_{ct}. It is desirable to have the form of the action in a diffeomorphism invariant way, this mainly means writing the actions in terms of the original field $\phi(\epsilon, x)$, and use the induced Laplacian $\Box_\gamma = \gamma^{\mu\nu}\partial_\mu\partial_\nu$ with $\gamma_{\mu\nu} = L^2\delta_{\mu\nu}/\epsilon$ being the induced metric on the hyperplane $\rho = \epsilon$. After inverting the series (8.26) we end up with

$$S_{ct}[\phi] = \frac{C}{L}\int d^dx\sqrt{\gamma}\left(\frac{d-\Delta}{2}\phi^2(\epsilon,x) + \frac{1}{2(2\Delta-d-2)}\phi(\epsilon,x)\Box_\gamma\phi(\epsilon,x) + \cdots\right). \tag{8.34}$$

We now can define the renormalised action as simply

$$S_{ren}[\phi] = S_{reg}[\phi] + S_{ct}[\phi]. \tag{8.35}$$

This action can be thought of as the physical action for the theory as it contains no divergencies when going to the boundary. It is remarkable that the counterterms can be written in a diffeomorphism invariant manner, one may say this makes the whole action S_{ren} look nice and not forced to cancel the infinities. This action can now be used without trouble to obtain boundary quantities in terms of bulk quantities such as n-points functions, conformal anomalies, Ward identites, renormalization group equations and more. This section showed the procedure for a scalar field in a fixed AdS background, more complex examples and further details may be found in [114, 68, 67].

8.2 The p-adic AdS/CFT correspondence

In the previous decade, the works [39, 40] pioneered a description of a holographic principle using p-adic numbers. They realized that the Bruhat-Tits tree actually had holographic properties. In particular a formula was given back in 1989 in [20] that reconstructed the field inside the tree from the field in the boundary. Back then it was called a Poisson formula, but now we would call it bulk-to-boundary propagator in the context of AdS/CFT. The construction from [40] is from a more mathematical perspective and they make connections with tensor networks. The one in [39] is closer to the usual formulation of AdS/CFT in a more physical spirit (this shouldn't be surprising as Steven Gubser, was one of the developers of the usual AdS/CFT correspondence). Both of these articles have sprouted a plethora of works both in connection with tensor networks [40, 83, 87, 88, 89, 90] and extending the work done by Gubser [84, 41, 64, 85, 65, 42, 86, 63]. Here we will follow more closely [39] as it is closer to what we need.

In the p-adic AdS/CFT correspondence, the role of AdS spacetime is played by the Bruhat-Tits tree, and its boundary, that is identified with the projective line $\mathbb{P}(\mathbb{Q}_p)$, is assumed to be where a p-adic analogue of a CFT lives[1]. Although strictly speaking the tree is a 2d space, there is a way to interpret it as a d-dimensional space using algebraic extensions of $\mathbb{Q}_p$, this was reviewed in 2.2.4. In particular the Bruhat-Tits tree $\mathcal{T}_{p^d}$ associated to the unique unramified extension of degree d of the p-adics, $\mathbb{Q}_{p^d}$, is interpreted as the analogue of AdS$_d$. It is common to use $q = p^d$ and leave the fixed degree of the extension implicit. This interpretation is illustrated in Figure 8.1. Using this idea one can obtain remarkably close results to the Archimedean case. The

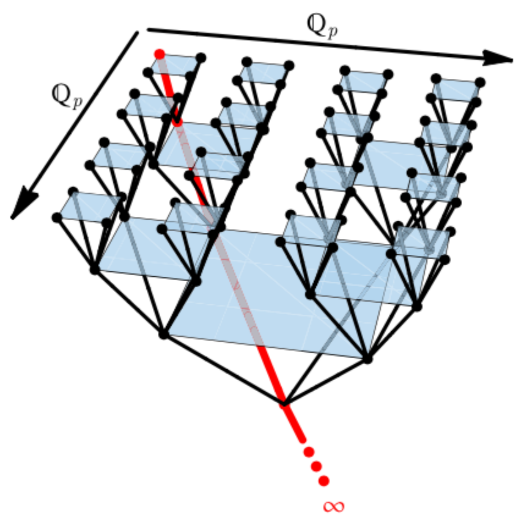

Figure 8.1: The Bruhat-Tits tree shown here is of $\mathbb{Q}_{2^2}$, that is the unramified extension of degree 2 with $p = 2$. An element of the residue group $x \in \mathbb{F}_{2^2} \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ has the form $x = a + \sqrt{\tau}b$ with $a, b \in \mathbb{Q}_2$. Then naturally we can interpret the point x as living in a 2d p-adic space with coordinates a and b. (Image taken from [39])

holographic model over $\mathbb{Q}_p$ is the simplest action for which holographic properties are known, i.e. a scalar on a fixed background. In this case, we are going to define an action for a scalar field defined on the Bruhat-Tits tree denoted by $\mathcal{T}_q$. The action is

$$S[\phi] = \sum_{a \in \mathcal{T}_q} \left(\frac{1}{4} \sum_{b \sim a} (\phi_a - \phi_b)^2 + \frac{1}{2} m^2 \phi_a^2 - J_a \phi_a \right), \tag{8.36}$$

where ϕ_a is a scalar field defined on the vertex a, J_a is a source field and $b \sim a$ means that b runs through the q nearest neighbors of a. Taking a usual variation $\phi_a + \delta\phi_a$ and keeping first order only, one can find the equation of motion

$$(\Box + m^2)\phi_a = J_a, \tag{8.37}$$

[1]There aren't a lot of works that construct a p-adic CFT, one important is [44] that constructs it from the operators algebra, and in [40] the simplest Lagrangian with conformal invariance is obtained. Other works have also discussed it [121]. There has been some difficulty in keeping conformal invariance while extending some known results for string in the p-adic case [51, 49].

with the Laplacian operator

$$\Box\phi_a = \sum_{b\sim a}(\phi_a - \phi_b) = (q+1)\phi_a - \sum_{b\sim a}\phi_b. \tag{8.38}$$

A general solution to the equation (8.37) involves getting a Green's function $G(a,b)$ that satisfies

$$(\Box_a + m^2)G(a,b) = \delta(a,b); \quad \phi_a = \sum_{b\in\mathcal{T}_q} G(a,b)J_b, \tag{8.39}$$

with $\delta(a,b)$ a Kronecker delta and the second equality is the general solution to the equation of motion. $G(a,b)$ is a power of p depending on the distance between the points a and b. Concretely it is

$$G(a,b) = p^{-\Delta}\zeta_p(2\Delta)p^{-\Delta D(a,b)}, \tag{8.40}$$

where the function $D(a,b)$ is the distance between the vertices a and b, that is the number of edges one needs to walk on to go from one vertex to the other, and the function ζ_p is defined as

$$\zeta_p(x) = \frac{1}{1-p^{-x}}. \tag{8.41}$$

Here every edge is taken to have a unit length, one could consider assigning a constant length L to every edge, this would only amount to a rescaling of the mass[2]. The parameter Δ is again associated to the conformal dimension of a operator in the CFT and obeys the mass relation

$$m^2 = -\frac{1}{\zeta_p(\Delta - d)\zeta_p(-\Delta)}. \tag{8.42}$$

This equation is quadratic in p^Δ and therefore has two solutions that satisfy

$$\Delta_\pm = \log_p \frac{1}{2}\left(q+1+m^2 \pm \sqrt{(q+1+m^2)^2 - 4q}\right). \tag{8.43}$$

One can check that $\Delta_+ + \Delta_- = d$, and $m^2 \geq m_{BF}^2 = 1/\zeta_p(-d/2)^2$. If $m_{BF}^2 \leq m^2 \leq 0$ then both $\Delta_\pm$ are positive ($d/2 \leq \Delta_+ \leq d$, and $0 \leq \Delta_- \leq d/2$), if $m^2 > 0$ then $\Delta_+ > 0$ and $\Delta_- < 0$. All of this is very similar to the Archimedean case.

At this point we introduce the coordinate system used in [39]. Looking to be close to the Archimedean case, we choose two coordinates labeled $(z,x) \in p^{\mathbb{Z}} \times \mathbb{Q}_p$ to represent points on the tree. This coordinates work as follows, consider the path on the tree from the point at infinity to the boundary point x, each vertex in this path is associated to a coefficient in the series expansion of x in powers of p. The coordinate z indicates precisely which vertex to choose from the path, the one associated to the coefficient of the power z in the series for x (See Figure 8.2). There is an obvious problem with this coordinate system, given a vertex on the tree, there

[2]There is of course the more exotic possibility of considering a variable length for different edges. This would be analogous to having gravity in our model. Little work has been done with this idea, in [41] this idea is explored in the context presented here, in [88] it is explored in a perturbative fashion in connection to tensor networks.

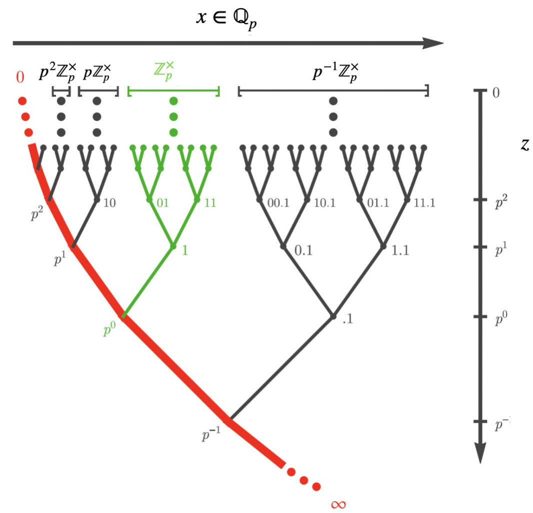

Figure 8.2: Illustration of the coordinates used for the tree. In red (thick line) is the path from 0 to ∞, sometimes called the trunk. In green ($\mathbb{Z}_p^\times$) is the branch rooted in the trunk associated to the unit circle at the boundary. The figure shows the case $p = 2$. [Image adapted from [39]

are infinitely many boundary points x that can ve used to represent the same vertex. These coordinates allow us to write things in a very similar way as in the Archimedean case.
Now we will obtain the bulk-to-boundary propagator. Insisting on translation invariance, and the fact that the bulk-to-boundary propagator must be a limit of the bulk-to-bulk propagator, one can deduce the dependence on the bulk and boundary points (z, x) and y respectively. The bulk-to-boundary propagator has the form

$$K(z, x; y) \sim \frac{|z|_p^\Delta}{|(z, x-y)|_s^{2\Delta}}; \quad |(z, x-y)|_s = \sup\{|z|_p, |x-y|_p\}. \tag{8.44}$$

The strange dependence using the supremum norm $|\cdot|_s$ is necessary because of the ambiguity of the coordinate system mentioned earlier. To get the proportionality constant we use the normalization

$$\int_{\mathbb{Q}_q} dy K(z, x; y) = |z|_p^{d-\Delta}. \tag{8.45}$$

Then we get that

$$K(z, x; y) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - d)} \frac{|z|_p^\Delta}{|(z, x-y)|_s^{2\Delta}}.^{3} \tag{8.46}$$

This propagator can be obtained also from a suitable limit of the bulk-to-bulk propagator $G(a, b)$, one needs to put both propagator using the coordinate system, however we will not show it explicitly here. It is interesting to note that the expressions for the bulk-to-boundary

[3] As an interesting comment, this exact expression with $\Delta = d = 1$ was found a long time ago in [20] as the integral kernel for the Poisson formula to write harmonic functions ϕ on the tree, subject to the boundary condition $\phi_{(0)}$, as $\phi(z, x) = \int_{\partial \mathcal{T}_p} dy P(a, x; , y)\phi_{(0)}(y)$, with $P(z, x, y) = K(z, x; y)|_{\Delta = d = 1}$.

propagators in the Archimedean case in (8.15) and non-Archimedean case in (8.46) are quite similar to each other. One needs only to define the zeta function in the Archimedean case as

$$\zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2).$$

Then the expressioin (8.15) becomes

$$K(z,x;y) = \frac{\zeta_\infty(2\Delta)}{\zeta_\infty(2\Delta-d)}\frac{z^\Delta}{(z^2+(x-y)^2)^\Delta}.$$

8.2.1 The 2-point function

To obtain the correlation functions we need the Fourier transform of $K(z,x;y)$ in the variable y. This turns out to be

$$K(z,k) \equiv \int_{\mathbb{Q}_q} dy\chi(kx)K(z,0;y) = \left(|z|_p^{d-\Delta} + |k|_p^{2\Delta-d}|z|_p^\Delta \frac{\zeta_p(n-2\Delta)}{\zeta_p(2\Delta-n)}\right)\gamma_q(kz), \tag{8.47}$$

where γ_q is the characteristic function of the unit ball. The strategy is the same as in the Archimedean case, we need the action evaluated on shell, only that for this case we only insert the relevant part of the on-shell field as

$$\phi_{cl}(z,x) = \lambda_1\chi(k_1x)K_\epsilon(z,k_1) + \lambda_2\chi(k_1x)K_\epsilon(z,k_2), \tag{8.48}$$

with

$$K_\epsilon(z,k) = \frac{|z|_p^{\Delta-d} + \zeta_R|k|_p^{2\Delta-d}|z|_p^\Delta}{|\epsilon|_p^{\Delta-d} + \zeta_R|k|_p^{2\Delta-d}|\epsilon|_p^\Delta}\gamma_q(kz), \quad \zeta_R = \frac{\zeta_p(n-2\Delta)}{\zeta_p(2\Delta-n)} = -p^{n-2\Delta}.$$

Now we will use an adaptation of the GKPW relation above for the non-Acrhimedean case

$$-\log\left\langle \exp\left\{\int_{\mathbb{Q}_q} dx\phi_\epsilon(x)\mathcal{O}_\epsilon(x)\right\}\right\rangle_p = \underset{\phi(\epsilon,x)=\phi_\epsilon(x)}{\text{extremum}} S_\epsilon[\phi], \tag{8.49}$$

with

$$S_\epsilon[\phi] = \sum_{\substack{a\in\mathcal{T}_q \\ |z(a)|_p\geq|\epsilon|_p}} \left(\frac{1}{4}\sum_{b\sim a}(\phi_a-\phi_b)^2 + \frac{1}{2}m^2\phi_a^2\right), \tag{8.50}$$

where $z(a)$ is the z coordinate of the vertex a. In other words S_ϵ is the action on a regulated tree that has a cutoff at $z=\epsilon$. We can also see that the on-shell action is reduced to a boundary term. Using the (easily proven) following identity

$$\frac{1}{4}\sum_{b\sim a}(\phi_a-\phi_b)^2 + \frac{1}{2}m^2\phi_a^2 - \frac{1}{2}\phi_a(\Box+m^2)\phi_a = -\frac{1}{4}\Box\phi_a^2, \tag{8.51}$$

the on-shell action becomes

$$S_\epsilon[\phi] = -\frac{1}{4}\sum_{|z(a)|_p\geq|\epsilon|_p}\Box\phi_a^2. \tag{8.52}$$

Now it is straightforward to check that summing the Laplacian of a function on the tree reduces to summing over the boundary, however one must do it carefully as the result is not entirely obvious or expected. When evaluating the Laplacian on a given inner vertex, say a, one subtracts the field at the vertex a from its neighbors, say b, leaving us with something like $\phi_a - \phi_b$. But when the Laplacian is evaluated at the neighbors b, the term $\phi_b - \phi_a$ appears, as a is a neighbors of b. This then cancels the evaluation of the field on the edge connecting a and b (See Fig. 8.3 (a)). This process will keep happening until we reach the end of the tree. Having a hard cutoff at $z = \epsilon$ means that the last place where we evaluate the Laplacian is at $z = \epsilon/p$[4] and end up having a boundary term of the *edges* between $z = \epsilon$ and $z = \epsilon/p$. Something similar will happen if we had a cutoff for some $z = z_l$ with $|z_l|_p \gg 1$, the boundary term will be over the edges at the bottom of the tree. See Figure 8.3 (b) for a graphical depiction. Finally we have

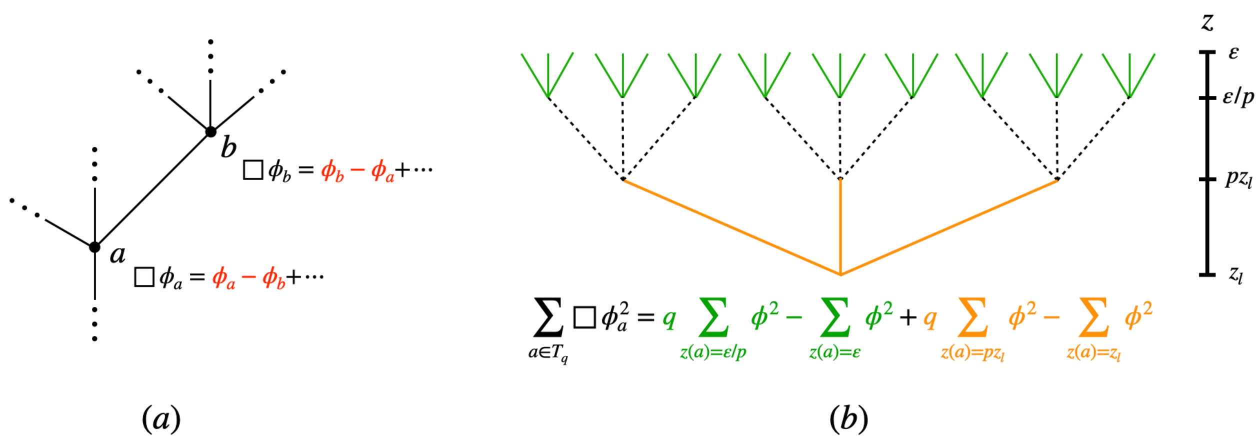

Figure 8.3: (a) When evaluating the laplacian in neighboring vertices, the terms cancel each other, as shown with the terms in red. (b) The cancellation when evaluating the laplacian occurs in the interior of the tree, and we are only left with evaluating the difference between vertices at $z = \epsilon/p$ and $z = pz_l$ minus their respective boundaries.

the on-shell action

$$S_\epsilon[\phi] = -\frac{1}{4}\left[q \sum_{z(a)=\epsilon} \phi_a^2 - \sum_{z(a)=\epsilon/p} \phi_a^2 - \left(q \sum_{z(a)=z_l} \phi_a^2 - \sum_{z(a)=pz_l} \phi_a^2\right)\right]. \tag{8.53}$$

We consider a tree without an IR boundary and hence we can dismiss the last two terms. Another way of thinking about it is that the cutoff is large enough so that $|k_i z|_p > 1$ and K_ϵ vanishes there. Now we can obtain the 2-point function by inserting the expression (8.48) into (8.53) and calculating

$$\langle \mathcal{O}_\epsilon(k_1)\mathcal{O}_\epsilon(k_2)\rangle = -\frac{\partial^2 S_\epsilon[\phi_{cl}]}{\partial\lambda_1\partial\lambda_2} = \frac{\delta(k_1+k_2)}{|\epsilon|_p^d}\left[-\frac{1}{2\zeta_p(2\Delta-2d)} + \frac{p^d\zeta_R}{\zeta_p(2\Delta-d)}|k_1\epsilon|_p^{2\Delta-d} + \cdots\right]. \tag{8.54}$$

[4]We cannot evaluate at $z = \epsilon$ in the cutoff tree because then the whole expression becomes zero.

Now defining the following operators

$$\mathcal{O}(x) = \lim_{|\epsilon|_p \to 0} |\epsilon|_p^{d-\Delta} \mathcal{O}_\epsilon(x), \tag{8.55}$$

transforming back to coordinate space and ignoring a divergent term proportional to $\delta(x)$ we get the result

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{p^{2\Delta-d}\zeta_p(2\Delta)}{\zeta_p(2\Delta-d)^2}\frac{1}{|x_1-x_2|_p^{2\Delta}}. \tag{8.56}$$

We have skipped a lot of details, the interested reader may see [39]. The main point is to see that we get the expected result for the coordinate dependence. So we can see that in the p-adic case, using the Bruhat-Tits tree as an analogue for the AdS space works quite well. Using the coordinate system shown here we can obtain some results that greatly resemble the Archimedean case. This is a little surprising considering two main differences, first that the Bruhat-Tits tree is flat, meaning that the distance between neighboring vertices is the same throughout the tree. The second difference is that the nature of the spaces changes from the bulk to the boundary, the tree is discrete whereas the boundary is continuous, so we are linking one discrete theory with one continuous using holographic techniques. I suspect this is intimately connected to the fact that $\mathbb{Q}_p$ has a totally disconnected topology.

8.3 A coarse-grained realization of p-adic AdS/CFT

In this section we are going to present an alternative way to find a solution to the equation of motion for the action on the tree (8.36) in the above section. At first it would seem completely different from what we have done, however a first result is to show that in fact the procedure done in this section generalizes the above results. We will start by generalizing the results from [66] where a solution to the equation of motion in the tree, $(\Box + m^2)\phi = 0$, with UV and IR boundary conditions, is built for the massless case ($m = 0$) and working with $\mathbb{Q}_p$, or equivalently setting $d = 1$. Here we will do the same procedure but for general m (and therefore general Δ) and in any dimension d, again using the unramified extension of the tree $\mathcal{T}_q$. Also the procedure for the most part will keep the notation used so far, as opposed to the one used in [66].
First notice that the equation of motion for the scalar action

$$S[\phi] = \sum_{a\in\mathcal{T}_q}\left(\frac{1}{4}\sum_{b\sim a}(\phi_a-\phi_b)^2 + \frac{1}{2}m^2\phi_a^2\right),$$

can be written as

$$(\Box + m^2)\phi_a = 0; \quad \Rightarrow \quad (q+1+m^2)\phi_a = \sum_{b\sim a}\phi_b. \tag{8.57}$$

We are going to consider a finite tree that has at the bottom an edge whose endpoints have the radial coordinates $z = L$ and $z = pL$, with $L = p^{-l}$ for some large integer $l \in \mathbb{N}$. From the top vertex sprouts the tree up to the coordinate $z = \epsilon$, see Figure 8.4. We introduce a little bit of

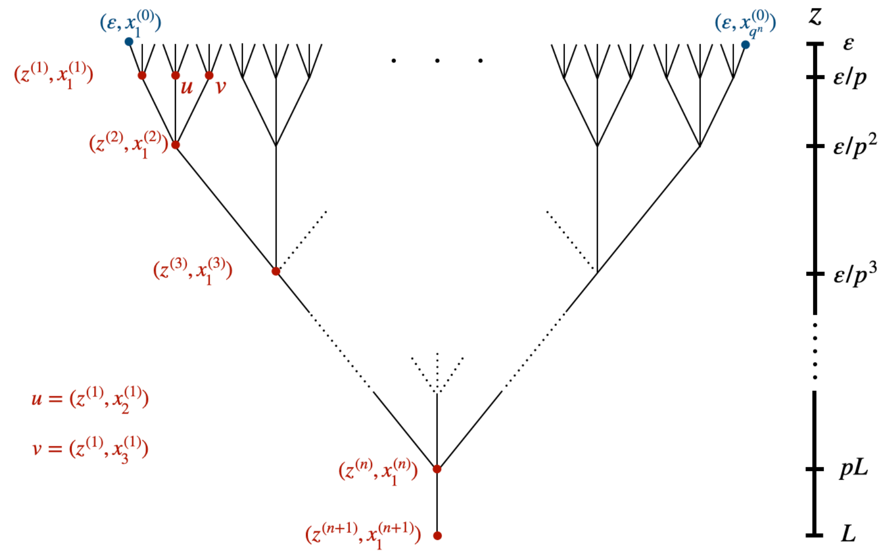

Figure 8.4: The finite tree considered for $q = 3$. The range of the radial coordinate is from $z = \epsilon$ to $z = L$ and we denote by $n + 1$ the total height of the tree, that is $z/L = p^{n+1}$. The on-shell field at the point $(z^{(1)} x_1^{(1)})$ can be written in terms of the 3 vertex above it and the one below it $(z^{(2)} x_1^{(2)})$ using the equation of motion. The same holds for every vertex all the way down to the vertex $(z^{(n)} x_1^{(n)})$ in terms of the field at the vertices $(z^{(n-1)} x_1^{(n-1)})$ and $(z^{(n+1)} x_1^{(n+1)})$.

notation, $(z^{(i)}, x_k^{(i)})$ is the point on the extreme left at the radial coordinate $z^{(i)} = \epsilon/p^i$, that is i edges below the top vertices. The integer k indicates the horizontal positioning of the vertex from left to right. Then (8.57) looks like

$$(q + 1 + m^2)\phi(z^{(i)}, x_1^{(i)}) = \sum_{k=1}^{q} \phi(z^{(i-1)}, x_k^{(i-1)}) + \phi(z^{(i+1)}, x_1^{(i+1)}). \tag{8.58}$$

Remember we have the mass relation $1 + q + m^2 = p^{\Delta} + p^{d-\Delta}$, and let us define

$$a_k := p^{k\Delta} - p^{k(d-\Delta)}, \tag{8.59}$$

Then

$$q + 1 + m^2 = p^{\Delta} + p^{d-\Delta} = \frac{a_2}{a_1}.$$

To avoid clutter, we make a further use of notation, we define $\phi^{(i,k)} := \phi(z^{(i)}, x_k^{(i)})$, the index i runs vertically along the left side of the tree in Fig. 8.4 and k runs horizontally from the extreme left to q vertices to the right. Now consider (8.58) for $i = 1$, if we multiply it times a_1 we get

$$a_2 \phi^{(1,1)} = a_1 \phi^{(2,1)} + a_1 \sum_{k=1}^{q} \phi^{(0,k)}. \tag{8.60}$$

This relation is generalized to any depth as

$$a_{i+1}\phi^{(i,1)} = a_i\phi^{(i+1,1)} + a_1 \sum_{k=1}^{q^i} \phi^{(0,k)}, \tag{8.61}$$

notice that this relates a given vertex with the one below it and the boundary vertices at the top. We prove it by induction, (8.61) is our induction hypothesis and we show that it is also true for $i \to i+1$. Consider the equation of motion (8.58) with $i \to i+1$, then use (8.61) for the fields $\phi^{(i,k)}$ in the sum over k. This results in

$$(p^\Delta + p^{d-\Delta})\phi^{(i+1,1)} = q\frac{a_i}{a_{i+1}}\phi^{(i+1,1)} + \frac{a_1}{a_{i+1}}\sum_{k=1}^{q^{i+1}} \phi^{(0,k)} + \phi^{(i+2,1)}.$$

Multiplying times a_{n+1} and rearranging we get

$$[(p^\Delta + p^{d-\Delta})a_{i+1} - qa_i]\phi_{i+1} = a_{i+1}\phi^{(i+2,1)} + a_1 \sum_{k=1}^{q^{i+1}} \phi^{(0,k)}.$$

It is not hard to show that $(p^\Delta + p^{d-\Delta})a_{i+1} - qa_i = a_{i+2} = p^{(i+2)\Delta} - p^{(i+2)(d-\Delta)}$. This completes the proof of (8.61) by induction. The objective is to be able to write the field at any vertex inside the tree in terms of the field at the top and at the bottom, to do it we can recursively use (8.61) starting from a given depth h. Start with (8.61) with $i = h$, then substitute $\phi^{(h+1,1)}$ with

$$\phi^{(h+1,1)} = \frac{a_{h+1}}{a_{h+2}}\phi^{(h+2,1)} + \frac{a_1}{a_{h+2}}\sum_{k=1}^{q^{h+1}} \phi^{(0,k)},$$

that is again (8.61) but rearranged and with $i = h+1$. We can keep recursively doing so, this results in

$$\begin{aligned}\phi^{(h,1)} &= \frac{a_h}{a_{h+1}}\phi^{(h+1,1)} + \frac{a_1}{a_{h+1}}\sum_{k=1}^{q^h}\phi^{(0,k)} = \frac{a_h}{a_{h+1}}\left(\frac{a_{h+1}}{a_{h+2}}\phi^{(h+2,1)} + \frac{a_1}{a_{h+2}}\sum_{k=1}^{q^{h+1}}\phi^{(0,k)}\right) + \frac{a_1}{a_{h+1}}\sum_{k=1}^{q^h}\phi^{(0,k)} \\ &= \frac{a_h}{a_{h+1}}\left(\frac{a_{h+1}}{a_{h+2}}\dots\left(\frac{a_n}{a_{n+1}}\phi^{(n+1,1)} + \frac{a_1}{a_{n+1}}\sum_{k=1}^{q^n}\phi^{(0,k)}\right)\dots\right) + \frac{a_1}{a_{h+1}}\sum_{k=1}^{q^h}\phi^{(0,k)} \\ &= \frac{a_h}{a_{n+1}}\phi^{(n+1,1)} + \sum_{i=h}^{n}\frac{a_1 a_h}{a_i a_{i+1}}\sum_{k=1}^{q^i}\phi^{(0,k)}.\end{aligned}$$

The last sum can be rearranged, first, we introduce the last bit of notation

$$\sum_{k=1}^{q^i}\phi^{(0,k)} = \sum_{k=1}^{i}\sum_{j=q^{k-1}+1}^{q^k}\phi^{(0,j)} \equiv \sum_{k=1}^{i}\sum_{j\in k\setminus k-1}\phi^{(0,j)},$$

then we use the following identities for double finite sums

$$\sum_{i=1}^{n}\sum_{k=1}^{i} a_i b_k = \sum_{i=1}^{n}\sum_{k=i}^{n} a_k b_i; \qquad \sum_{i=h}^{n}\sum_{k=1}^{i} a_i b_k = \sum_{i=1}^{n}\sum_{k=1}^{i} a_i b_k - \sum_{i=1}^{h-1}\sum_{k=1}^{i} a_i b_k.$$

Then we have

$$\sum_{i=h}^{n} \frac{a_1 a_h}{a_i a_{i+1}} \sum_{k=1}^{q^i} \phi^{(0,k)} = \sum_{i=1}^{n}\sum_{k=i}^{n} \left(\frac{a_1 a_h}{a_k a_{k+1}}\right) \sum_{j\in i\setminus(i-1)} \phi^{(0,j)} - \sum_{i=1}^{h-1}\sum_{k=i}^{h-1} \left(\frac{a_1 a_h}{a_k a_{k+1}}\right) \sum_{j\in i\setminus(i-1)} \phi^{(0,j)}$$

$$= a_1 a_h \left(\sum_{i=1}^{n} A_i \sum_{j\in i\setminus(i-1)} \phi^{(0,j)} - \sum_{i=1}^{h-1} (A_i - A_h) \sum_{j\in i\setminus(i-1)} \phi^{(0,j)} \right),$$

where we defined

$$A_i \equiv \sum_{k=i}^{n} \frac{1}{a_i a_{i+1}}. \tag{8.62}$$

We gather all of this and write a general expression for the field at any depth on the left side of the tree

$$\phi^{(h,1)} = a_1 a_h \sum_k A_{\frac{h+d(h,k)}{2}} \phi^{(0,k)} + \frac{a_h}{a_{n+1}} \phi^{(n+1,1)}, \tag{8.63}$$

where $d(h,k)$ is the number of edges separating $\phi^{(h,1)}$ from $\phi^{(0,k)}$ in the sum. We worked assuming that the field was at the extreme left of the tree, however one can easily see that this applies to all the vertices in the tree, the horizontal dependence is through the distance $d(h,k)$ in the first term. This first term is a weighted sum of the boundary field with the weight depending on the distance from the vertex in the tree to the boundary point. Said distance will be different depending on where is the vertex inside the tree. That way the result (8.63) is valid for all the vertices in the tree. Now this is a solution to the equation of motion in terms of the field at $z = \epsilon$ and at $z = L$, that we can interpret as a solution to a boundary value problem. The notation used is unusual and we are going to change it to something that is clearer.

Like in the previous sections, we use (z, x_Λ) as the coordinates for the bulk point, where $x_\Lambda \in \mathbb{Q}_q/p^\Lambda\mathbb{Z}_q$ is a point at the top of the tree. The quantity $h + d(h,k)$ is actually the distance $d(x_\Lambda, y_\Lambda)$ between two boundary points. We assign $\epsilon = p^\Lambda$ and $L = p^l$, then $n+1 = \Lambda - l$, and $z = p^{v(z)}$ where $v(z)$ is the valuation of the radial coordinate z. With this notation we can see that $h + v(z) = \Lambda$, therefore

$$a_h = p^{h\Delta} - p^{h(d-\Delta)} = p^{(\Lambda - v(z))\Delta} - p^{(\Lambda-v(z))(d-\Delta)} = \frac{|z|_p^{\Delta}}{|\epsilon|_p^{\Delta}} - \frac{|z|_p^{d-\Delta}}{|\epsilon|_p^{d-\Delta}}. \tag{8.64}$$

Similarly

$$a_{n+1} = \frac{|L|_p^{\Delta}}{|\epsilon|_p^{\Delta}} - \frac{|L|_p^{d-\Delta}}{|\epsilon|_p^{d-\Delta}}. \tag{8.65}$$

Using all of this we can write the solution (8.63) as

$$\phi(z,x_\Lambda)=\left(\frac{|z|_p^\Delta}{|\epsilon|_p^\Delta}-\frac{|z|_p^{d-\Delta}}{|\epsilon|_p^{d-\Delta}}\right)\left(a_1\sum_{y_\Lambda}A_{\frac{d(x_\Lambda,y_\Lambda)}{2}}\phi(\epsilon,y_\Lambda)+\frac{|\epsilon|_p^d}{|L|_p^\Delta|\epsilon|_p^{d-\Delta}-|L|_p^{d-\Delta}|\epsilon|_p^d}\phi(L,x_\Lambda)\right). \tag{8.66}$$

A clarification is in order, in principle there is only one tree that represents all of $\mathbb{Q}_q$ at its boundary. We have just removed the top and bottom of it. Because of the nature of the tree, having an IR cutoff, even if we remove the UV one, means that we do not recover the entire field $\mathbb{Q}_q$ at the boundary with the single tree, just a ball of radius $z=L$, where the IR cutoff is. This conflicts with desired properties like translation invariance. To recover the entire $\mathbb{Q}_q$ at the boundary, even with a cutoff, we need to consider infinitely many trees as shown in Figure 8.5.

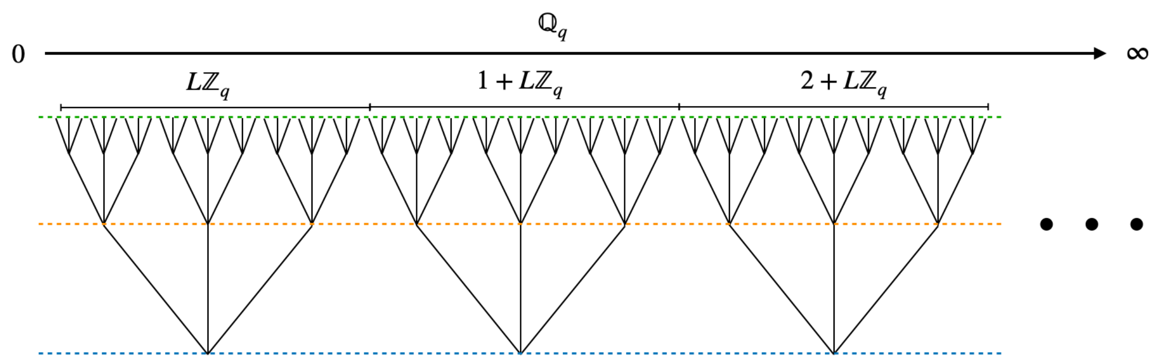

Figure 8.5: A cutoff tree can only be identified with a ball of $\mathbb{Q}_q$ at the boundary. To recover all $\mathbb{Q}_q$ one needs to have infinitely many trees, one per ball missing, that is, one per element of $\mathbb{Q}_q/L\mathbb{Z}_q$. The horizontal lines show different cutoffs and how they split a single tree into many.

8.3.1 Removing the cutoffs

In this section we are going to see that the result (8.66) is actually a generalization of some previous results in the literature. In particular we will remove the cutoffs and see that we recover exactly the bulk-to-boundary propagator in [39]. We want to see the solution (8.66) in the limit where we recover the entire Bruhat-Tits tree by removing the cutoffs. We will keep using the conventions for the cutoff tree from the previous section, i.e. that $n+1=\Lambda-l$. And we will use $h+d(h,0)=d(x_\Lambda,y_\Lambda)\equiv d$. To avoid confusion, we will denote the dimension of the tree (the degree of the algebraic extension on $\mathbb{Q}_p$) by D.

As we remove the UV cutoff by taking $\Lambda\to\infty$, we have that $a_n\to p^{n\Delta}$ as $n\to\infty$. This is valid when $\Delta>D/2$ because $a_n=p^{n\Delta}(1-p^{n(D-2\Delta)})$. The distance between boundary points

d will go to infinity as well. Then we have

$$A_{\frac{h+d(h,0)}{2}} \to \sum_{i=d/2}^{\Lambda-l-1} \frac{1}{p^{i\Delta}p^{(i+1)\Delta}} = p^{-\Delta} \sum_{i=d/2}^{\Lambda-l-1} (p^{-2\Delta})^i$$

$$= p^{-\Delta} \frac{p^{-2\Delta(\Lambda-l)} - p^{-\Delta d}}{p^{-2\Delta} - 1} = p^{-\Delta}\zeta_p(2\Delta)(p^{-\Delta d} - p^{-2\Delta(\Lambda-l)}).$$

It is not hard to see that $p^{d/2} = p^{\Lambda}|(z, x_\Lambda - y_\Lambda)|_s$, where the supremum norm $|\cdot|_s$ was introduced in (8.44). Also we have $a_1 p^{-\Delta} = (1 - p^{D-2\Delta}) = 1/\zeta_p(2\Delta - D)$, and $p^h = p^{(\Lambda - v(z))} = p^{\Lambda}|z|_p$, (see Figure 8.6). Putting all of this together we have

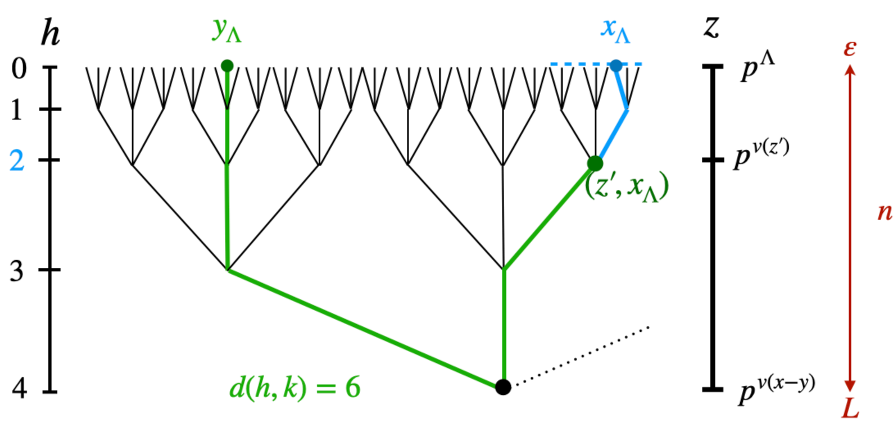

Figure 8.6: The relative depth of a vertex h can be mapped to a coordinate (z, x_Λ), as well as the edge distance between x_Λ and y_Λ. The tree shown is for $q = 3$ and its height is $4 = \Lambda - l$. The dotted blue line means that any point in that range works as a longitudinal coordinate for the point marked (z', x_Λ).

$$\phi(z, x_\Lambda) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - D)} \sum_{y_\Lambda} (p^{\Lambda\Delta}|z|_p^{\Delta} - p^{\Lambda(D-\Delta)}|z|_p^{D-\Delta}) p^{-2\Delta\Lambda} \left(\frac{1}{|(z, x_\Lambda - y_\Lambda)|_s^{2\Delta}} - p^{2\Delta l} \right) \phi(\epsilon, y_\Lambda)$$
$$+ \frac{(p^{\Lambda\Delta}|z|_p^{\Delta} - p^{\Lambda(D-\Delta)}|z|_p^{D-\Delta})}{p^{(\Lambda-l)\Delta}} \phi(p^l, x_\Lambda)$$
$$= \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - D)} \sum_{y_\Lambda} |\epsilon|_p^D (|z|_p^{\Delta} - |\epsilon|_p^{D-4\Delta}) \left(\frac{1}{|(z, x_\Lambda - y_\Lambda)|_s^{2\Delta}} - p^{2\Delta l} \right) |\epsilon|_p^{-(D-\Delta)} \phi(\epsilon, y_\Lambda)$$
$$+ (|z|_p^{\Delta} - |\epsilon|_p^{2\Delta - D}|z|_p^{D-\Delta}) p^{l\Delta} \phi(p^l, x_\Lambda).$$

In this last expression we separated the factor $|\epsilon|_p^D$ next to the sum, as we remove the cutoffs the sum over y_Λ tends to a sum over $\mathbb{Q}_q$. In order to properly identify it as an integral, we need the factor $|\epsilon|_p^D$ to complete the measure

$$\sum_{y_\Lambda} |\epsilon|_p^D \underset{|\epsilon|_p \to 0}{\longrightarrow} \int_{\mathbb{Q}_q} dy. \tag{8.67}$$

This was shown in [20] and later used in [65, 66]. Now we can take the limits $|\epsilon|_p \to 0$ and $l \to -\infty$, that remove the UV and IR cutoffs, respectively. Many terms vanish and we are left with

$$\phi(z,x) = \int_{\mathbb{Q}_q} K(z,x;y)\phi_{(0)}(y)dy, \tag{8.68}$$

where

$$\phi_{(0)}(x) = \lim_{|\epsilon|\to 0} |\epsilon|_p^{-(d-\Delta)}\phi(\epsilon,x), \tag{8.69}$$

$$K(z,x;y) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta-d)} \frac{|z|_p^{\Delta}}{|(z,x-y)|_s^{2\Delta}}. \tag{8.70}$$

We assumed that $p^{l\Delta}\phi(p^l,x) \to 0$ as $l \to -\infty$. The integral kernel (8.70) is exactly the bulk-to-boundary propagator from [39] reviewed in section 8.2, and (8.68) is the p-adic analogue of (8.8). With this we see that the solution (8.66) is a generalization of the p-adic bulk-to-boundary propagator. It is a coarsed-grained version that works when there is a cutoff and the normalizable mode of the solution is present. The solution also includes a boundary condition at the bottom of the tree that we identify as the IR region.

8.4 p-adic Holographic Renormalization

In this section we make an analogous result to the one in section 8.1.2, where we constructed the renormalized boundary action. The p-adic case is very close in spirit to the Archimedean case, however there are some key differences which we will see on the way. The most important difference is that in this case there is a fixed number of counterterms needed for the counterterm action. In other words the number of counterterms does not depend on the value of Δ, that is in contrast with the Archimedean case.

We start by evaluating the on-shell action like we did in the section 8.2.1, only this time we are going to consider (8.66) as the classical solution and insert it into the action. We use again the identity (8.51) that we repeat here, for any $a \in \mathcal{T}_q$ we have

$$\frac{1}{4}\sum_{b\sim a}(\phi_a - \phi_b)^2 + \frac{1}{2}m^2\phi_a^2 = \frac{1}{2}\phi_a(\Box + m^2)\phi_a - \frac{1}{4}\Box\phi_a^2.$$

By construction (8.66) satisfies the equation of motion $(\Box + m^2)\phi_h = 0$ (see the appendix A for an explicit proof). In this section we will use the shorter notation $\phi_h = \phi^{(h,1)}(z,x)$ whenever there is no confusion, the subindex h refers to the depth on the finite tree as shown in Fig. 8.6. The on-shell action then reduces to a boundary term. We set cutoffs at $|\epsilon|_p = p^{\Lambda}$ and $|L|_p = p^l$. The action is

$$S[\phi_{cl}] = -\frac{1}{4}\sum_{\mathcal{T}_q}\Box\phi_h^2 = -\frac{1}{4}\left[q\sum_{|z|_p=|\epsilon|_p p}\phi_h^2 - \sum_{|z|_p=|\epsilon|}\phi_h^2 - \left(q\sum_{|z|_p=|L|}\phi_h^2 - \sum_{|z|_p=|L|_p/p}\phi_h^2\right)\right], \tag{8.71}$$

see Fig. 8.3. It is important to say that in the sum the Laplacian is evaluated up to the radial coordinates $z = p^{\Lambda-1}$ and $z(a) = p^{l+1}$. Now we need the square of the on-shell field

$$\phi_h^2 = a_1^2 a_h^2 \left(\sum_{y_\Lambda} A_{\frac{d(x_\Lambda, y_\Lambda)}{2}} \phi(\epsilon, y_\Lambda) \right)^2 + 2\frac{a_1 a_h^2}{a_{n+1}} \sum_{y_\Lambda} A_{\frac{d(x_\Lambda, y_\Lambda)}{2}} \phi(\epsilon, y_\Lambda) \phi_{n+1} + \frac{a_h^2}{a_{n+1}^2} \phi_{n+1}^2. \tag{8.72}$$

Inserting this into (8.71) gives

$$\begin{aligned} S[\phi_{cl}] = -\frac{1}{4} \Bigg[q \sum_{x_{\Lambda-1}} \left(a_1^2 \sum_{y_\Lambda} A_{\frac{1+d(x_{\Lambda-1}, y_\Lambda)}{2}} \phi(\epsilon, y_\Lambda) + \frac{a_1}{a_{n+1}} \phi(L, x_\Lambda) \right)^2 - \sum_{x_\Lambda} \phi(\epsilon, x_\Lambda)^2 \\ -q \sum_{z=L} \phi(L, x_\Lambda)^2 + \sum_{z=pL} \left(\frac{a_1}{a_{n+1}} \sum_{x_{\Lambda-1}} \phi(\epsilon, x_\Lambda) + \frac{a_n}{a_{n+1}} \phi(L, x_\Lambda) \right)^2 \Bigg], \end{aligned} \tag{8.73}$$

where the sum over $x_{\Lambda-1}$ means over the vertices at depth $h = 1$ or with radial coordinate $z = \epsilon/p = p^{\Lambda-1}$, and the sum over $z = L$ and $z = pL$ means summing the vertices with that radial coordinate. For a single connected tree there is only one vertex with coordinate $z = L$ and one with $z = pL$, we indicate a sum to remind us that we should be considering infinite trees to recover all $\mathbb{Q}_q$, see Figure 8.5. However since trees that are disconnected from each other have no effect between them, we omit this sum in the following and will only consider one finite tree.

We collect the factors for each power of $\phi(L, x_\Lambda)^2$ in (8.73) (omitting the $-1/4$ for now). Remember that $\Lambda - l = n + 1$, these factors are

$$\sim \phi(L, x_\Lambda)^2 : \quad q^{n+1} \frac{a_1^2}{a_{n+1}^2} + q \frac{a_n^2}{a_{n+1}^2} - q = q \frac{a_n(a_n - a_{n+2})}{a_{n+1}^2}; \tag{8.74}$$

$$\begin{aligned} \sim 2\phi(L, x_\Lambda) : \quad & \frac{a_1 a_n}{a_{n+1}^2} \sum_{x_\Lambda} \phi(\epsilon, x_\Lambda) + q \frac{a_1^3}{a_{n+1}} \sum_{x_{\Lambda-1}} \sum_{y_\Lambda} A_{\frac{1+d(x_{\Lambda-1}, y_\Lambda)}{2}} \phi(\epsilon, y_\Lambda) \\ & = \frac{(q+1) a_1 a_n}{a_{n+1}^2} \sum_{x_\Lambda} \phi(\epsilon, x_\Lambda); \end{aligned} \tag{8.75}$$

$$\begin{aligned} \sim \phi(L, x_\Lambda)^0 : \quad & q a_1^4 \sum_{x_{\Lambda-1}} \left(\sum_{y_\Lambda} A_{\frac{1+d(x_{\Lambda-1}, y_\Lambda)}{2}} \phi(\epsilon, y_\Lambda) \right)^2 + \frac{a_1^2}{a_{n+1}^2} \left(\sum_{x_\Lambda} \phi(\epsilon, x_\Lambda) \right)^2 - \sum_{x_\Lambda} \phi(\epsilon, x_\Lambda)^2 \\ & = \sum_{y_\Lambda, \tilde{y}_\Lambda} \phi(\epsilon, y_\Lambda) \phi(\epsilon, \tilde{y}_\Lambda) \left(q a_1^4 \sum_{x_{\Lambda-1}} A_{\frac{1+d(x_{\Lambda-1}, y_\Lambda)}{2}} A_{\frac{1+d(x_{\Lambda-1}, \tilde{y}_\Lambda)}{2}} + \frac{a_1^2}{a_{n+1}^2} - \delta_{y_\Lambda, \tilde{y}_\Lambda} \right). \end{aligned} \tag{8.76}$$

We want to see what divergencies arise in these terms when we go to the boundary at $\Lambda \to \infty$. It is in principle very straightforward considering that $a_n \to p^{n\Delta}$, as was done in section 8.3.1, and it works well for the factors of $\phi(L, x_\Lambda)^2$ and $2\phi(L, x_\Lambda)$. The first term (8.74) simplifies to

$$q \frac{a_n(a_n - a_{n+2})}{a_{n+1}^2} \to q \frac{p^{n\Delta}(p^{n\Delta} - p^{(n+2)\Delta})}{p^{2(n+1)\Delta}} = q \frac{1 - p^{2\Delta}}{p^{2\Delta}} = -\frac{q}{\zeta_p(2\Delta)}.$$

And the second (8.75) to

$$\frac{(q+1)a_1a_n}{a_{n+1}^2}\sum_{x_\Lambda}\phi(\epsilon,x_\Lambda)=\frac{(q+1)a_1p^{(\Lambda-l-1)\Delta}}{p^{2(\Lambda-l)\Delta}}\sum_{x_\Lambda}|\epsilon|_p^{D-\Delta}[|\epsilon|_p^{\Delta-D}\phi(\epsilon,x_\Lambda)]$$
$$\to\frac{q+1}{\zeta_p(2\Delta-D)}|L|_p^{-\Delta}\int_{L\mathbb{Z}_p}\phi_{(0)}(x)dx,$$

where we used interchangeably that $|\epsilon|_p=p^{-\Lambda}$. Both cases are finite, so there is no need to renormalize them. If we do this for the last case (8.76), we will see that indeed we have divergent terms. And we will notice that we only have a few divergent terms, not depending on the value of Δ, contrasting with the Archimedean case. However to be sure, we need to do the sum $\sum_{x_{\Lambda-1}}A_{\frac{1+d(x_{\Lambda-1},y_\Lambda)}{2}}A_{\frac{1+d(x_{\Lambda-1},\tilde{y}_\Lambda)}{2}}$ like we did $\sum_{x_{\Lambda-1}}A_{\frac{1+d(x_{\Lambda-1},y_\Lambda)}{2}}$ in (8.75) (we shamelessly skipped this but don't worry, we will see how this is done in the next section). Taking the asymptotic $a_n\to p^{n\Delta}$ means considering only the leading terms going to the boundary. Then is possible we missed subleading terms that would give rise to further divergencies, and possibly recover what happens in the Archimedean case. To rule this out, we take a different approach for the computation in the next section.

8.4.1 Extracting the counterterms

We will take a different approach in the computation of divergent terms in the on-shell action. The idea is to keep things exact without taking any asymptotic behavior. We will focus on the last term in (8.76). To this end, we write the exact solution (8.66) purely in terms of powers of $|z|_p$ and $|\epsilon|_p$, without fractions. The idea is rooted in the fact that

$$\frac{1}{a_i}=\frac{1}{p^{i\Delta}-p^{i(D-\Delta)}}=\frac{p^{-i\Delta}}{1-p^{i(D-2\Delta)}}=p^{-i\Delta}\sum_{m=0}^{\infty}p^{mi(D-2\Delta)}=p^{-i\Delta}[1+p^{i(D-2\Delta)}+p^{2i(D-2\Delta)}+\cdots].$$

This assumes that $\Delta>D/2$. For brevity we will make $s\equiv D-2\Delta$. Then we have

$$\begin{aligned}\frac{1}{a_ia_{i+1}}&=p^{-(2i+1)\Delta}[1+p^{is}+p^{2is}+\cdots][1+p^{(i+1)s}+p^{2is}+\cdots]\\&=p^{-(2i+1)\Delta}[1+p^{is}+p^{2is}(1+p^s)+p^{3is}(1+p^s+p^{2s})+\cdots]\\&=p^{-(2i+1)\Delta}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}p^{ks}\right)p^{ims}=p^{-(2i+1)\Delta}\sum_{m=0}^{\infty}\frac{p^{(m+1)s}-1}{p^s-1}p^{ims}.\end{aligned}\tag{8.77}$$

You may see where this is going. Let us take the third line of (8.77) and see how A becomes,

$$\begin{aligned}A_k=\sum_{i=k}^{n}\frac{1}{a_ia_{i+1}}&=\frac{p^{-\Delta}}{p^s-1}\sum_{t=0}^{\infty}(p^{(t+1)s}-1)\sum_{i=k}^{n}p^{(ts-2\Delta)i}\\&=\frac{p^{-\Delta}}{p^s-1}\sum_{t=0}^{\infty}(p^{(t+1)s}-1)\frac{p^{(ts-2\Delta)(n+1)}-p^{(ts-2\Delta)k}}{p^{ts-2\Delta}-1}.\end{aligned}\tag{8.78}$$

Although this does not look very friendly (and indeed it isn't), we have what we wanted, the leading term ($t=0$) is exactly what one gets when making $a_n \to p^{n\Delta}$. This means that the subsequent terms are corrections to the limiting simplifications done above. The good thing is that it is exact. Now we want to apply this to the solution $\phi(z, x_\Lambda)$ in (8.66). First remember $p^h = |\epsilon|_p^{-1}|z|_p$ and $p^{n+1} = |\epsilon|_p^{-1}|L|_p$; then

$$a_h = \left|\frac{z}{\epsilon}\right|_p^{\Delta} - \left|\frac{z}{\epsilon}\right|_p^{D-\Delta}, \qquad \frac{1}{a_{n+1}} = p^{-(n+1)\Delta}\sum_{m=0}^{\infty} p^{m(n+1)s} = \left|\frac{\epsilon}{L}\right|_p^{\Delta}\sum_{m=0}^{\infty}\left|\frac{\epsilon}{L}\right|_p^{-sm}$$

$$\begin{aligned}\frac{a_h}{a_{n+1}} &= \left|\frac{z}{L}\right|_p^{\Delta}\sum_{m=0}^{\infty}\left|\frac{\epsilon}{L}\right|_p^{-sm} - |z|_p^{D-\Delta}|\epsilon|_p^{-s}|L|_p^{-\Delta}\sum_{m=0}^{\infty}\left|\frac{\epsilon}{L}\right|_p^{-sm} \\ &= \left|\frac{z}{L}\right|_p^{\Delta} + |L|_p^{-\Delta}\sum_{t=1}^{\infty}\left(|z|_p^{\Delta}|L|_p^{st} - |z|_p^{D-\Delta}|L|_p^{s(t-1)}\right)|\epsilon|_p^{-st}.\end{aligned}$$

Recall from section 8.3.1 that $p^{(h+d(h,0))/2} = p^{d(x,y)/2} = |\epsilon|_p^{-1}|x_\Lambda - y_\Lambda|_p$. Then using (8.78) we get

$$A_{\frac{d(x_\Lambda,y_\Lambda)}{2}} = p^{-\Delta}\zeta_p(2\Delta - D)\sum_{t=0}^{\infty}\frac{\zeta_p(2\Delta - ts)}{\zeta_p(-s(t+1))}\left[\left(\frac{|\epsilon|_p}{|(z, x_\Lambda - y_\Lambda)|_s}\right)^{2\Delta - st} - \left|\frac{\epsilon}{L}\right|_p^{2\Delta - st}\right]. \tag{8.79}$$

Then

$$\begin{aligned} a_1 a_h \sum_{y_\Lambda} A_{\frac{d(x_\Lambda,y_\Lambda)}{2}}\phi(\epsilon, y_\Lambda) &= \left(\left|\frac{z}{\epsilon}\right|_p^{\Delta} - \left|\frac{z}{\epsilon}\right|_p^{D-\Delta}\right)\sum_{t=0}^{\infty}\frac{\zeta_p(2\Delta - ts)}{\zeta_p(-s(t+1))} \\ &\quad\times\sum_{y_\Lambda}\left[\left(\frac{|\epsilon|_p}{|(z, x_\Lambda - y_\Lambda)|_s}\right)^{2\Delta - st} - \left|\frac{\epsilon}{L}\right|_p^{2\Delta - st}\right]\phi(\epsilon, y_\Lambda) \\ &= |z|_p^{\Delta}\sum_{t=0}^{\infty}\frac{\zeta_p(2\Delta - ts)}{\zeta_p(-s(t+1))}\sum_{y_\Lambda}\left[\frac{1}{|(z, x_\Lambda - y_\Lambda)|_s^{2\Delta - st}} - \frac{1}{|L|_p^{2\Delta - st}}\right]\phi(\epsilon, y_\Lambda)|\epsilon|_p^{\Delta - st} \\ &\quad -|z|_p^{D-\Delta}\sum_{t=0}^{\infty}\frac{\zeta_p(2\Delta - ts)}{\zeta_p(-s(t+1))}\sum_{y_\Lambda}\left[\frac{1}{|(z, x_\Lambda - y_\Lambda)|_s^{2\Delta - st}} - \frac{1}{|L|_p^{2\Delta - st}}\right]\phi(\epsilon, y_\Lambda)|\epsilon|_p^{\Delta - s(t+1)} \\ &= |z|_p^{\Delta}\frac{\zeta_p(2\Delta)}{\zeta_p(-s)}\sum_{y_\Lambda}\left[\frac{1}{|(z, x_\Lambda - y_\Lambda)|_s^{2\Delta}} - \frac{1}{|L|_p^{2\Delta}}\right]\phi(\epsilon, y_\Lambda)|\epsilon|_p^{\Delta} \\ &\quad + \sum_{t=1}^{\infty}\left\{|z|_p^{\Delta}\frac{\zeta_p(2\Delta - ts)}{\zeta_p(-s(t+1))}\sum_{y_\Lambda}\left[\frac{1}{|(z, x_\Lambda - y_\Lambda)|_s^{2\Delta - st}} - \frac{1}{|L|_p^{2\Delta - st}}\right]\phi(\epsilon, y_\Lambda)\right. \\ &\quad \left. -|z|_p^{D-\Delta}\frac{\zeta_p(2\Delta - (t-1)s)}{\zeta_p(-st)}\sum_{y_\Lambda}\left[\frac{1}{|(z, x_\Lambda - y_\Lambda)|_s^{2\Delta - s(t-1)}} - \frac{1}{|L|_p^{2\Delta - s(t-1)}}\right]\phi(\epsilon, y_\Lambda)\right\}|\epsilon|_p^{\Delta - st}\end{aligned}$$

Let's be the closest to [68, 67], where there is no IR cutoff. This means taking $|L|_p \to \infty$ for our notation, it also implies having a vanishing IR boundary condition, i.e. $\phi(L, x) = 0$. Now we can expand the solution squared from the last three lines of the previous expression

$$\left(a_1 a_h\sum_{y_\Lambda}A_{\frac{d(x_\Lambda,y_\Lambda)}{2}}\phi(\epsilon, y_\Lambda)\right)^2 = |z|_p^{2\Delta}\left(\frac{\zeta_p(2\Delta)}{\zeta_p(-s)}\right)^2\left(\sum_{y_\Lambda}\frac{\phi(\epsilon, y_\Lambda)}{|(z, x_\Lambda - y_\Lambda)|_s^{2\Delta}}\right)^2|\epsilon|_p^{2\Delta}$$

$$+2|z|_p^{\Delta}\frac{\zeta_p(2\Delta)}{\zeta_p(-s)}\sum_{y_\Lambda}\frac{\phi(\epsilon,y_\Lambda)}{|(z,x_\Lambda-y_\Lambda)|_s^{2\Delta}}|\epsilon|_p^{\Delta}\times\sum_{t=1}^{\infty}\left\{|z|_p^{\Delta}\frac{\zeta_p(2\Delta-ts)}{\zeta_p(-s(t+1))}\sum_{\tilde{y}_\Lambda}\frac{\phi(\epsilon,\tilde{y}_\Lambda)}{|(z,x_\Lambda-\tilde{y}_\Lambda)|_s^{2\Delta-st}}\right.$$
$$\left.-|z|_p^{D-\Delta}\frac{\zeta_p(2\Delta-(t-1)s)}{\zeta_p(-st)}\sum_{\tilde{y}_\Lambda}\frac{\phi(\epsilon,\tilde{y}_\Lambda)}{|(z,x_\Lambda-\tilde{y}_\Lambda)|_s^{2\Delta-s(t-1)}}\right\}|\epsilon|_p^{\Delta-st}$$
$$+\left[\sum_{t=1}^{\infty}\left\{|z|_p^{\Delta}\frac{\zeta_p(2\Delta-ts)}{\zeta_p(-s(t+1))}\sum_{y_\Lambda}\frac{\phi(\epsilon,y_\Lambda)}{|(z,x_\Lambda-y_\Lambda)|_s^{2\Delta-st}}\right.\right.$$
$$\left.\left.-|z|_p^{D-\Delta}\frac{\zeta_p(2\Delta-(t-1)s)}{\zeta_p(-st)}\sum_{y_\Lambda}\frac{\phi(\epsilon,y_\Lambda)}{|(z,x_\Lambda-y_\Lambda)|_s^{2\Delta-s(t-1)}}\right\}|\epsilon|_p^{\Delta-st}\right]^2.$$

Now we evaluate this at $|z|_p=p|\epsilon|_p$ and sum over all the vertices with that radial coordinate denoted by $x_{\Lambda-1}$. This becomes

$$\begin{aligned}
&p^{2\Delta}\left(\frac{\zeta_p(2\Delta)}{\zeta_p(-s)}\right)^2\sum_{x_{\Lambda-1}}\left(\sum_{y_\Lambda}\frac{\phi(\epsilon,y_\Lambda)}{|(\epsilon/p,x_{\Lambda-1}-y_\Lambda)|_s^{2\Delta}}\right)^2|\epsilon|_p^{4\Delta}+\sum_{x_{\Lambda-1}}2p^{\Delta}\frac{\zeta_p(2\Delta)}{\zeta_p(-s)}\sum_{y_\Lambda}\frac{\phi(\epsilon,y_\Lambda)}{|(\epsilon/p,x_{\Lambda-1}-y_\Lambda)|_s^{2\Delta}}|\epsilon|_p^{2\Delta}\\
&\times\sum_{t=1}^{\infty}\left\{\frac{\zeta_p(2\Delta-ts)}{\zeta_p(-s(t+1))}\sum_{\tilde{y}_\Lambda}\frac{\phi(\epsilon,\tilde{y}_\Lambda)p^{\Delta}|\epsilon|_p^{\Delta}}{|(\epsilon/p,x_{\Lambda-1}-\tilde{y}_\Lambda)|_s^{2\Delta-st}}\right.\\
&\left.-\frac{\zeta_p(2\Delta-(t-1)s)}{\zeta_p(-st)}\sum_{\tilde{y}_\Lambda}\frac{\phi(\epsilon,\tilde{y}_\Lambda)p^{D-\Delta}|\epsilon|_p^{D-\Delta}}{|(\epsilon/p,x_{\Lambda-1}-\tilde{y}_\Lambda)|_s^{2\Delta-s(t-1)}}\right\}|\epsilon|_p^{\Delta-st}\\
&+\sum_{x_{\Lambda-1}}\left[\sum_{t=1}^{\infty}\left\{\frac{\zeta_p(2\Delta-ts)}{\zeta_p(-s(t+1))}\sum_{y_\Lambda}\frac{\phi(\epsilon,y_\Lambda)p^{\Delta}|\epsilon|_p^{\Delta}}{|(\epsilon/p,x_{\Lambda-1}-y_\Lambda)|_s^{2\Delta-st}}\right.\right.\\
&\left.\left.-\frac{\zeta_p(2\Delta-(t-1)s)}{\zeta_p(-st)}\sum_{x_\Lambda}\frac{\phi(\epsilon,x_\Lambda)p^{D-\Delta}|\epsilon|_p^{D-\Delta}}{|(\epsilon/p,x_{\Lambda-1}-y_\Lambda)|_s^{2\Delta-s(t-1)}}\right\}|\epsilon|_p^{\Delta-st}\right]^2.
\end{aligned}\tag{8.80}$$

There is a particular sum that appears 6 times and is somewhat tricky to evaluate. We introduce a bit of notation and treat it separately as follows

$$\begin{aligned}
E(\phi,\epsilon,a,b)&:=\sum_{x_{\Lambda-1},y_\Lambda,\tilde{y}_\Lambda}\frac{\phi(\epsilon,y_\Lambda)}{|(\epsilon/p,x_{\Lambda-1}-y_\Lambda)|_s^{a}}\frac{\phi(\epsilon,\tilde{y}_\Lambda)}{|(\epsilon/p,x_{\Lambda-1}-\tilde{y}_\Lambda)|_s^{b}}\\
&=:\sum_{y_\Lambda,\tilde{y}_\Lambda}\phi(\epsilon,y_\Lambda)\phi(\epsilon,\tilde{y}_\Lambda)H(\epsilon,y_\Lambda,\tilde{y}_\Lambda,a,b).
\end{aligned}\tag{8.81}$$

We focus on the functions H for which we label only the entries $H(a,b)$. To simplify H, we sum over the variable $x_{\Lambda-1}$ and use Fig. 8.7 to help us do that. It is a matter of close inspection and counting the number of edges there are from a given vertex in the dotted line in Fig. 8.7 to the boundary vertices y and $\tilde{y}$. One quickly realizes that there is a pattern and the function H is

$$\begin{aligned}
H(a,b)=&\,p^{-a}|\epsilon|_p^{-a}|x_\Lambda-y_\Lambda|_p^{-b}+(q-1)|\epsilon|_p^{-a}|y_\Lambda-\tilde{y}_\Lambda|_p^{-b}\sum_{j=2}^{d(y,\tilde{y})/2-1}q^{j-2}p^{-aj}+(a\leftrightarrow b)\\
&+(q-2)/q^2|y_\Lambda-\tilde{y}_\Lambda|_p^{D-a-b}|\epsilon|_p^{-D}+(q-1)|\epsilon|_p^{-a-b}\sum_{j=d(y,\tilde{y})/2+1}^{\infty}q^{j-2}p^{-(a+b)j}.
\end{aligned}\tag{8.82}$$

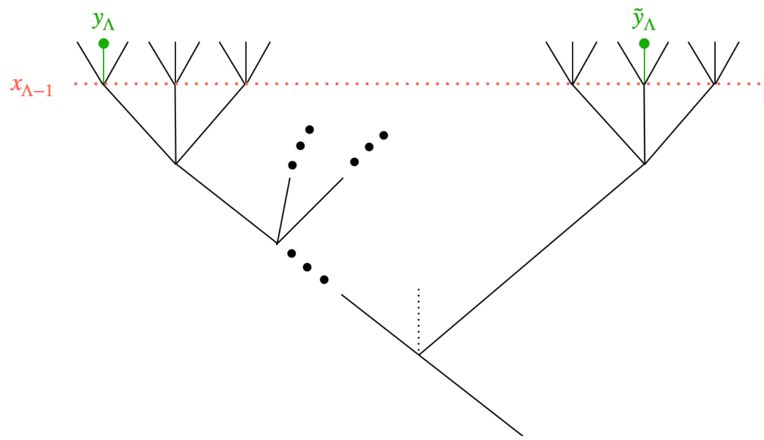

Figure 8.7: The sum (8.81) is over the vertices along the dotted red line, one needs to carefully consider each case and the values of the functions $|(\epsilon/p, x_{\Lambda-1} - y_\Lambda)|_s$ and $|(\epsilon/p, x_{\Lambda-1} - \tilde{y}_\Lambda)|_s$. The sum is made from left to right. This image is for $q = 3$, but for general q one needs to be careful about including the parts indicated by black dots.

Let's work out the sums that just appeared above:

$$\sum_{j=2}^{d(y,\tilde{y})/2-1} q^{j-2}p^{-aj} = p^{-2D} \sum_{j=2}^{d(y,\tilde{y})/2-1} p^{(D-a)j} = p^{-2D}\frac{p^{(D-a)d(y,\tilde{y})/2} - p^{2(D-a)}}{p^{(D-a)} - 1}$$

$$= \zeta_p(a-D)(p^{-2a} - q^{-2}|y_\Lambda - \tilde{y}_\Lambda|_p^{D-a}|\epsilon|_p^{a-D}).$$

$$\sum_{j=d(y,\tilde{y})/2+1}^{\infty} q^{j-2}p^{-(a+b)j} = p^{-2D} \sum_{j=d(y,\tilde{y})/2+1}^{\infty} p^{(D-a-b)j}$$

$$= p^{-2D}\frac{p^{(D-a-b)(d(y,\tilde{y})/2+1)}}{1-p^{D-a-b}} = p^{-(D+a+b)}\zeta_p(a+b-D)|x_\Lambda - y_\Lambda|_p^{D-a-b}|\epsilon|_p^{a+b-D}.$$

Substituting in (8.82) we get

$$\left[p^{-a}|x_\Lambda - y_\Lambda|_p^{-b} + (q-1)p^{-2a}\zeta_p(a-D)|y_\Lambda - \tilde{y}_\Lambda|_p^{-b}\right]|\epsilon|_p^{-a} - (q-1)q^{-2}\zeta_p(a-D)|y_\Lambda - \tilde{y}_\Lambda|_p^{D-a-b}|\epsilon|_p^{-D}$$

$$+(a \leftrightarrow b) + (q-2)q^{-2}|x_\Lambda - y_\Lambda|_p^{D-a-b}|\epsilon|_p^{-D} + (q-1)p^{-(D+a+b)}\zeta_p(a+b-D)|x_\Lambda - y_\Lambda|_p^{D-a-b}|\epsilon|_p^{-D}.$$

Finally we can rearrange the result to show the powers of ϵ that appear in H as follows

$$\begin{aligned} H(\epsilon, y, \tilde{y}, a, b) = p^{-a}\frac{\zeta_p(a-D)}{\zeta_p(a)}|y_\Lambda - \tilde{y}_\Lambda|_p^{-b}|\epsilon|_p^{-a} + p^{-b}\frac{\zeta_p(b-D)}{\zeta_p(b)}|y_\Lambda - \tilde{y}_\Lambda|_p^{-a}|\epsilon|_p^{-b} \\ +\Big[(q-2)q^{-2} + (q-1)\left(p^{-(D+a+b)}\zeta_p(a+b-D) - \right. \\ \left. q^{-2}(\zeta_p(a-D) + \zeta_p(b-D))\right)\Big]|y_\Lambda - \tilde{y}_\Lambda|_p^{D-a-b}|\epsilon|_p^{-D}. \end{aligned} \tag{8.83}$$

Now, there is a subtlety with the sum for H we just did. It assumed that the boundary points y, $\tilde{y}$ were *sufficiently separated* as they look in Fig. 8.7. However when the boundary points are close to each other, the sum for H is different. We denote by $y \approx \tilde{y}$ the case in which $d(y_\Lambda, \tilde{y}_\Lambda) \leq 2$, in other words, when the points share their bottom neighbor vertex. In this case, from the definition of E in (8.81), we notice that $|(\epsilon/p, x_{\Lambda-1} - y_\Lambda)|_s = |(\epsilon/p, x_{\Lambda-1} - \tilde{y}_\Lambda)|_s$, and H simply becomes

$$\sum_{x_{\Lambda-1}}^{q^{n-1}} |(\epsilon/p, x_{\Lambda-1} - y_\Lambda)|_s^{-a-b}.$$

This is the same as $H(\epsilon, y, \tilde{y}, a+b, 0)$ and is equal to

$$H(\epsilon, y, \tilde{y}, a, b)\big|_{y\approx\tilde{y}} = H(\epsilon, y, \tilde{y}, a+b, 0) = p^{-a-b}\frac{\zeta_p(a+b-D)}{\zeta_p(a+b)}|\epsilon|_p^{-a-b}.$$

Then the function E actually is

$$E(\phi, \epsilon, a, b) = \sum_{y_\Lambda \not\approx \tilde{y}_\Lambda} \phi(\epsilon, y_\Lambda)\phi(\epsilon, \tilde{y}_\Lambda) H(\epsilon, y_\Lambda, \tilde{y}_\Lambda, a, b) + p^{-a-b}\frac{\zeta_p(a+b-D)}{\zeta_p(a+b)}|\epsilon|_p^{-a-b} \sum_{y_\Lambda \approx \tilde{y}_\Lambda} \phi(\epsilon, y_\Lambda)\phi(\epsilon, \tilde{y}_\Lambda), \tag{8.84}$$

where $y_\Lambda \not\approx \tilde{y}_\Lambda$ means $d(y_\Lambda, \tilde{y}_\Lambda) > 2$. When going to the boundary we will have $y \approx \tilde{y} \to y = \tilde{y}$ (with an additional factor of q as there are q $\tilde{y}$ per boundary point y), and $y_\Lambda \not\approx \tilde{y}_\Lambda \to y \neq \tilde{y}$. At the boundary, the second term in (8.84) will contribute with something proportional to $\phi(y)^2$, but more on that later.

We proceed to write the boundary term using the function E, resulting in

$$\begin{aligned}
q\sum_{z_{\Lambda-1}} \phi_{cl}^2 - \sum_{x_\Lambda} \phi_{cl}^2 &= p^{D+2\Delta}\left(\frac{\zeta_p(2\Delta)}{\zeta_p(-s)}\right)^2 E(2\Delta, 2\Delta)|\epsilon|_p^{4\Delta} \\
&+ \sum_{t=1}^{\infty}\Bigg\{2p^{D+2\Delta}\frac{\zeta_p(2\Delta)}{\zeta_p(-s)}\frac{\zeta_p(2\Delta-ts)}{\zeta_p(-s(t+1))}E(2\Delta, 2\Delta-st)|\epsilon|_p^{4\Delta-st} \\
&-2p^{2D}\frac{\zeta_p(2\Delta)}{\zeta_p(-s)}\frac{\zeta_p(2\Delta-(t-1)s)}{\zeta_p(-st)}E(2\Delta, 2\Delta-s(t-1))|\epsilon|_p^{D+2\Delta-st}\Bigg\} \\
&+ \sum_{t,w=1}^{\infty}\Bigg\{p^{D+2\Delta}\frac{\zeta_p(2\Delta-st)}{\zeta_p(-s(t+1))}\frac{\zeta_p(2\Delta-sw)}{\zeta_p(-s(w+1))}E(2\Delta-st, 2\Delta-sw)|\epsilon|_p^{4\Delta-s(t+w)} \\
&-p^{2D}\frac{\zeta_p(2\Delta-ts)}{\zeta_p(-s(t+1))}\frac{\zeta_p(2\Delta-(w-1)s)}{\zeta_p(-sw)}E(2\Delta-st, 2\Delta-s(w-1))|\epsilon|_p^{D+2\Delta-s(t+w)} + (t\leftrightarrow w) \\
&p^{3D-2\Delta}\frac{\zeta_p(2\Delta-(t-1)s)}{\zeta_p(-st)}\frac{\zeta_p(2\Delta-(w-1)s)}{\zeta_p(-sw)}E(2\Delta-s(t-1), 2\Delta-s(w-1))|\epsilon|_p^{2D-s(t+w)}\Bigg\} \\
&- \sum_{x_\Lambda}\phi(\epsilon, x_\Lambda)^2.
\end{aligned} \tag{8.85}$$

To look for boundary divergent terms we only need to check the powers of $|\epsilon|_p$. Using (8.83) and inspecting closely (8.85) we can construct the table 8.1 that organizes the powers of $|\epsilon|_p$ that appear in each term in (8.85).

To see the divergent terms we remember that in order to convert to boundary terms, we must

$E(2\Delta, 2\Delta)\lvert\epsilon\rvert_p^{4\Delta}$	$\lvert\epsilon\rvert_p^{-2\Delta}$ $\lvert\epsilon\rvert_p^{-2\Delta}$ $\lvert\epsilon\rvert_p^{-D}$	$\lvert\epsilon\rvert_p^{2\Delta}$ $\lvert\epsilon\rvert_p^{2\Delta}$ $\lvert\epsilon\rvert_p^{4\Delta-D}$
$E(2\Delta, 2\Delta - st)\lvert\epsilon\rvert_p^{4\Delta-st}$	$\lvert\epsilon\rvert_p^{-2\Delta}$ $\lvert\epsilon\rvert_p^{-(2\Delta-st)}$ $\lvert\epsilon\rvert_p^{-D}$	$\lvert\epsilon\rvert_p^{2\Delta+(2\Delta-D)t} \to \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \lvert\epsilon\rvert_p^{8\Delta-3D}, \ldots$ $[t]\lvert\epsilon\rvert_p^{2\Delta}$ $\lvert\epsilon\rvert_p^{4\Delta-D+(2\Delta-D)t} \to \lvert\epsilon\rvert_p^{6\Delta-2D}, \lvert\epsilon\rvert_p^{8\Delta-3D}, \ldots$
$E(2\Delta, 2\Delta - s(t-1))\lvert\epsilon\rvert_p^{D+2\Delta-st}$	$\lvert\epsilon\rvert_p^{-2\Delta}$ $\lvert\epsilon\rvert_p^{-(2\Delta+s-st)}$ $\lvert\epsilon\rvert_p^{-D}$	$\lvert\epsilon\rvert_p^{D+(2\Delta-D)t} \to \lvert\epsilon\rvert_p^{2\Delta}, \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \ldots$ $[t]\lvert\epsilon\rvert_p^{2\Delta}$ $\lvert\epsilon\rvert_p^{2\Delta+(2\Delta-D)t} \to \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \ldots$
$E(2\Delta - st, 2\Delta - sw)\lvert\epsilon\rvert_p^{4\Delta-s(t+w)}$	$\lvert\epsilon\rvert_p^{-(2\Delta-st)}$ $\lvert\epsilon\rvert_p^{-(2\Delta-sw)}$ $\lvert\epsilon\rvert_p^{-D}$	$[t]\lvert\epsilon\rvert_p^{(2\Delta-D)w} \to \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \lvert\epsilon\rvert_p^{8\Delta-3D}, \ldots$ $[w]\lvert\epsilon\rvert_p^{(2\Delta-D)t} \to \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \lvert\epsilon\rvert_p^{8\Delta-3D}, \ldots$ $\lvert\epsilon\rvert_p^{(2\Delta-D)(t+w+1)} \to \lvert\epsilon\rvert_p^{6\Delta-3D}, 2\lvert\epsilon\rvert_p^{8\Delta-4D}, \ldots$
$E(2\Delta-st, 2\Delta-s(w-1))\lvert\epsilon\rvert_p^{D+2\Delta-s(t+w)}$	$\lvert\epsilon\rvert_p^{-(2\Delta-st)}$ $\lvert\epsilon\rvert_p^{-(2\Delta+s-sw)}$ $\lvert\epsilon\rvert_p^{-D}$	$[t]\lvert\epsilon\rvert_p^{D+(2\Delta-D)w} \to \lvert\epsilon\rvert_p^{2\Delta}, \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \ldots$ $[w]\lvert\epsilon\rvert_p^{2\Delta+(2\Delta-D)t} \to \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \lvert\epsilon\rvert_p^{8\Delta-3D}, \ldots$ $\lvert\epsilon\rvert_p^{2\Delta+(2\Delta-D)(t+w)} \to \lvert\epsilon\rvert_p^{6\Delta-2D}, 2\lvert\epsilon\rvert_p^{8\Delta-3D}, \ldots$
$E(2\Delta-s(t-1), 2\Delta-sw)\lvert\epsilon\rvert_p^{D+2\Delta-s(t+w)}$	$\lvert\epsilon\rvert_p^{-(2\Delta+s-st)}$ $\lvert\epsilon\rvert_p^{-(2\Delta-sw)}$ $\lvert\epsilon\rvert_p^{-D}$	$[t]\lvert\epsilon\rvert_p^{2\Delta+(2\Delta-D)w} \to \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \lvert\epsilon\rvert_p^{8\Delta-3D}, \ldots$ $[w]\lvert\epsilon\rvert_p^{D+(2\Delta-D)t} \to \lvert\epsilon\rvert_p^{2\Delta}, \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \ldots$ $\lvert\epsilon\rvert_p^{2\Delta+(2\Delta-D)(t+w)} \to \lvert\epsilon\rvert_p^{6\Delta-2D}, 2\lvert\epsilon\rvert_p^{8\Delta-3D}, \ldots$
$E(2\Delta-s(t-1), 2\Delta-s(w-1))\lvert\epsilon\rvert_p^{2D-s(t+w)}$	$\lvert\epsilon\rvert_p^{-(2\Delta+s-st)}$ $\lvert\epsilon\rvert_p^{-(2\Delta+s-sw)}$ $\lvert\epsilon\rvert_p^{-D}$	$[t]\lvert\epsilon\rvert_p^{D+(2\Delta-D)w} \to \lvert\epsilon\rvert_p^{2\Delta}, \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \ldots$ $[w]\lvert\epsilon\rvert_p^{D+(2\Delta-D)t} \to \lvert\epsilon\rvert_p^{2\Delta}, \lvert\epsilon\rvert_p^{4\Delta-D}, \lvert\epsilon\rvert_p^{6\Delta-2D}, \ldots$ $\lvert\epsilon\rvert_p^{D+(2\Delta-D)(t+w)} \to \lvert\epsilon\rvert_p^{4\Delta-D}, 2\lvert\epsilon\rvert_p^{6\Delta-2D}, \ldots$

Table 8.1: Organization of the powers of $|\epsilon|_p$. The first column shows the function E with the arguments that appear in (8.85). The second column represents the term with the shown power of $|\epsilon|_p$ associated to the function E on the left. The third column shows the powers of $|\epsilon|_p$ of each term by expanding the series in t or w. The symbols $[t]$ and $[w]$ mean that there is an entire sum over t or w respectively as a factor of the power shown.

accommodate the necessary powers of $|\epsilon|_p$ to account for the measure and the boundary behavior of ϕ (8.69). This demands to have $|\epsilon|_p^{2\Delta}$, and to make the substitutions $\sum \to \int$ and $\phi \to \phi_{(0)}$ we pay the price of multiplying the whole expression times $|\epsilon|_p^{-2\Delta}$. Also remember that $\Delta > D/2$. After a careful inspection, we realize that in fact we only have one divergent type of term, the one that goes as ϕ^2. These terms are from the last line in (8.85) and the terms coming from the right hand side of (8.84). These latter is quite difficult to obtain explicitly, as seen from the table 8.1 (or from (8.85)), we always have $E(a,b)|\epsilon|_p^{a+b}$, and from the second term in the right hand side of (8.85) we know that $E(a,b)|\epsilon|_p^{a+b} \sim \phi^2$ without a power of $|\epsilon|_p$. Then the correct coefficient that multiplies the soon-to-be counterterm is the right-hand-side of (8.85) with the replacement

$$E(a,b) \to p^{-a-b}\frac{\zeta_p(a+b-D)}{\zeta_p(a+b)}.$$

This is very lengthy, but after some simplifications we have that it is equal to

$$\mathcal{C} := p^{D-2\Delta}\left(1-p^{D-2\Delta}\right)^2\left(\frac{\zeta_p(2\Delta)}{\zeta_p(-s)}\right)^2\frac{\zeta_p(4\Delta-D)}{\zeta_p(4\Delta)}$$

$$\sum_{t=1}^{\infty}2p^{D-2\Delta}\left(1-p^{D-2\Delta}\right)^2 p^{st}\frac{\zeta_p(2\Delta)}{\zeta_p(-s)}\frac{\zeta_p(2\Delta-ts)}{\zeta_p(-s(t+1))}\frac{\zeta_p(4\Delta-D-ts)}{\zeta_p(4\Delta-ts)} \tag{8.86}$$

$$+\sum_{t,w=1}^{\infty}p^{D-2\Delta}\left(1-p^{D-2\Delta}\right)^2 p^{s(t+w)}\frac{\zeta_p(2\Delta-st)}{\zeta_p(-s(t+1))}\frac{\zeta_p(2\Delta-sw)}{\zeta_p(-s(w+1))}\frac{\zeta_p(4\Delta-s(t+w)-D)}{\zeta_p(4\Delta-s(t+w))}-1.$$

Unfortunately, simplifying this any further seems rather difficult and we are forced to leave it like this at this time. $\mathcal{C}$ is the coefficient of the counterterm that diverges as $|\epsilon|_p^{-2\Delta}$.
The boundary convergent terms are the ones marked in red in table 8.1, all of the rest will vanish at the boundary. These convergent terms are

$$|\epsilon|_p^{2\Delta}\sum_{y_\Lambda,\tilde{y}_\Lambda}\frac{\phi(\epsilon,y_\Lambda)\phi(\epsilon,\tilde{y}_\Lambda)}{|y_\Lambda-\tilde{y}_\Lambda|_p^{2\Delta}}\left\{2q\frac{\zeta_P(2\Delta)}{\zeta_p(2\Delta-D)}+2q\frac{\zeta_P(2\Delta)}{\zeta_p(2\Delta-D)}\sum_{t=1}^{\infty}p^{(D-2\Delta)t}-2p^{2D-2\Delta}\frac{\zeta_P(2\Delta)}{\zeta_p(2\Delta-D)}\right.$$

$$\left.-2q\frac{\zeta_P(2\Delta)}{\zeta_p(2\Delta-D)}\sum_{t=1}^{\infty}p^{(D-2\Delta)t}-2p^{2D-2\Delta}\frac{\zeta_P(2\Delta)}{\zeta_p(2\Delta-D)}\sum_{t=1}^{\infty}p^{(D-2\Delta)t}+2p^{2D-2\Delta}\frac{\zeta_P(2\Delta)}{\zeta_p(2\Delta-D)}\sum_{t=1}^{\infty}p^{(D-2\Delta)t}\right\}$$

$$=|\epsilon|_p^{2\Delta}\sum_{y_\Lambda,\tilde{y}_\Lambda}\frac{\phi(\epsilon,y_\Lambda)\phi(\epsilon,\tilde{y}_\Lambda)}{|y_\Lambda-\tilde{y}_\Lambda|_p^{2\Delta}}\left\{2q\frac{\zeta_P(2\Delta)}{(\zeta_p(2\Delta-D))^2}\right\}=2p^{D-2\Delta}a_1^2\zeta_P(2\Delta)|\epsilon|_p^{2\Delta}\sum_{y_\Lambda\neq\tilde{y}_\Lambda}\frac{\phi(\epsilon,y_\Lambda)\phi(\epsilon,\tilde{y}_\Lambda)}{|y_\Lambda-\tilde{y}_\Lambda|_p^{2\Delta}}. \tag{8.87}$$

We used that

$$\sum_{t=0}^{\infty}p^{st}=\frac{p^s}{1-p^s}=p^{D-2\Delta}\zeta_p(2\Delta-D).$$

The form of the convergent term should be familiar. At the boundary it becomes proportional to the unregulated version of a Vladimirov derivative (see, for instance [41, Appendix B])

$$|\epsilon|_p^{2\Delta}\sum_{x_\Lambda\neq y_\Lambda}\frac{\phi(\epsilon,y_\Lambda)\phi(\epsilon,\tilde{y}_\Lambda)}{|y_\Lambda-\tilde{y}_\Lambda|_p^{2\Delta}}\to\int_{\mathbb{Q}_p}\phi_{(0)}(y)\int_{\substack{\mathbb{Q}_p\\ \tilde{y}\neq y}}\frac{\phi_{(0)}(\tilde{y})}{|y-\tilde{y}|_p^{2\Delta}}d\tilde{y}dy.$$

In order to have a well defined operator we must regularize it with an additional term that cancels the divergencies. So in fact we can take it to be another necessary counterterm! This will be

$$\frac{1}{2}p^{D-2\Delta}a_1^2\zeta_P(2\Delta)|\epsilon|_p^{2\Delta}\sum_{y_\Lambda\neq\tilde{y}_\Lambda}\frac{\phi(\epsilon,y_\Lambda)^2}{|y_\Lambda-\tilde{y}_\Lambda|_p^{2\Delta}}. \tag{8.88}$$

Renormalized Action

Finally, as we can see, there are two divergent terms, and they are both proportional to ϕ^2. The divergent term is composed of the one appearing with a negative sign in (8.85) and the other from 8.84. This term diverges as $|\epsilon|_p^{-2\Delta}$ as we approach the boundary. Then the counterterm action to construct the renormalized action is

$$S_{ct}[\phi] = -\frac{1}{4}\mathcal{C}\sum_{\substack{x_\Lambda \\ z=\epsilon}}\phi(\epsilon, x_\Lambda)^2 + \frac{1}{2}p^{D-2\Delta}a_1^2\zeta_P(2\Delta)|\epsilon|_p^{2\Delta}\sum_{\substack{y_\Lambda \\ z=\epsilon}}\sum_{\tilde{y}_\Lambda\neq y_\Lambda}\frac{\phi(\epsilon, y_\Lambda)^2}{|y_\Lambda - \tilde{y}_\Lambda|_p^{2\Delta}}, \tag{8.89}$$

where $\mathcal{C}$ is given in (8.86). As usual, it turns out that the p-adic case is computationally simpler than the Archimedean case. What is interesting is that in the real correspondence, there is also a divergent term proportional to ϕ^2, it is the leading divergent term, but some others appear depending on the value of Δ, see section 8.1.2. This difference is because in the usual correspondence, the general solution to the bulk equations (8.19) has the the two leading behaviors $z^{d-\Delta}$ and z^Δ for small z (see (8.23)). However in the p-adic case, the functions $|z|_p^{D-\Delta}$ and $|z|_p^\Delta$ are **exact** solutions to the bulk equation of motion. Therefore it is no surprise that only seemingly leading terms appear in the renormalized action (8.89). The fact is that there are no subleading terms to add.
This result agrees with the one in [65], where they showed that the boundary effective theory of the scalar massive theory has a divergent term $\sim \phi^2$. There is no mention of the process of holographic renormalization, and like in our case, they add the necessary term to regularize the Vladimirov derivative. Here we interpret these terms as counterterms to have a well behaved boundary renormalized action. Although still not developed fully, our work also generalizes to have an IR cutoff and boundary condition, that leaves the road ready to start the process of Wilsonian renormalization to obtain the effective action that captures the UV behavior of our theory.
We end with the renormalized action

$$S_{ren}[\phi] = \lim_{|\epsilon|_p\to 0}\left[\sum_{\substack{a\in\mathcal{T}_q \\ |z(a)|_p\geq|\epsilon|_p}}\left(\frac{1}{4}\sum_{b\sim a}(\phi_a-\phi_b)^2 + \frac{1}{2}m^2\phi_a^2\right) + S_{ct}[\phi]\right], \tag{8.90}$$

where the counterterm action $S_{ct}[\phi]$ is action over the boundary given in (8.89).

A Explicit proof that both terms in the solution satisfy the equation of motion

By construction (8.63) (or (8.66)) should satisfy the equation of motion $(\Box + m^2)\phi^{(h,1)} = 0$. However just to be sure we are going to check it explicitly, and there is one interesting thing that comes up from this. The interesting part is that both terms separately satisfy the equation, so in fact it is the sum of two linearly independent solutions. First let's check the easy one, the second term $\frac{a_h}{a_{n+1}}\phi^{(n+1,1)}$, remember, h is the height of the vertex of interest measured from the top of the tree. And the Laplacian $\Box_v$ is just the sum of the difference $\phi_v - \phi_u$ for the neighboring vertices u of v. We can write the equation in a more convenient way

$$(\Box + m^2)\phi_v = \sum_{u\in\sim v}(\phi_v - \phi_u) + m^2\phi_v = (q+1+m^2)\phi_v - \sum_{u\sim v}\phi_u = 0 \Rightarrow \sum_{u\sim v}\phi_u = (q+1+m^2)\phi_v.$$

Therefore it is enough to check that the sum over neighboring vertices multiplies the field times $q+1+m^2$ (or if you want to sound fancier you can say that ϕ is an eigenfunction of the operation of summing over neighboring vertices with eigenvalue $q+1+m^2$). For the second term the only thing that changes is the factor that depends on $h(v)$, (we indicate explicitly that the height from above $h(v)$ depends on the vertex v in the tree, however, to avoid clutter we omit this dependence, keep it in mind though),that is a_h, and we'll have q vertices with a_{h-1} and 1 with a_{h+1}, then

$$\sum_{u\sim v} a_{h(u)} = qa_{h-1} + a_{h+1} = q(p^{(h-1)\Delta} - p^{(h-1)D-\Delta}) + p^{(h+1)\Delta} - p^{(h+1)(D-\Delta)}$$

$$= p^{h\Delta}(qp^{-\Delta} + p^{\Delta}) - p^{h(D-\Delta)}(qp^{-(D-\Delta)} + p^{D-\Delta}) = p^{h\Delta}(p^{\Delta} + p^{D-\Delta}) - p^{h(D-\Delta)}(p^{\Delta} + p^{D-\Delta})$$

$$= (p^{\Delta} + p^{D-\Delta})a_h = a_h\frac{a_2}{a_1} = (q+1+m^1)a_h.$$

We used that $q = p^{\Delta + D-\Delta}$ and that Δ is related to the mass by $p^{\Delta} + p^{D-\Delta} = q+1+m^2$. This proves the result.

We are going to now do it for the first term of (8.63). Evidently it is harder because of the sum, the dependence on h is not trivial. Consider the vertex v at the height h and its neighboring vertices u. Now consider the set of boundary vertices (height h=0) that are above each of the vertices u of height $h-1$, these are shown in green in Fig. 8.8. We will keep referring to this image. When summing the on shell field over the neighboring vertices, we will have a total of $q+1$ weights A_i adding to the same boundary region. The subindex i for the weight depends on the point where it's being evaluated, specifically on the length from the vertex to the boundary. For the vertices at height $h\pm 1$ we have the rule $A_{\frac{h\pm 1 + d(u,y_\Lambda)}{2}}$, where y_Λ is a point on the cutoff boundary. Then isolating a set of boundary vertices above one of the upper neighbors of v (shown in green), such boundary set of vertices will have the weight A_{h-1} from the vertex u directly below it, $q-1$ weights of A_h coming from the other same height neighbors, and finally one weight A_{h+1} from the vertex at height $h+1$. The solutions has the factor a_h as well, then the coefficient for the given boundary set is

$$a_{h-1}A_{h-1} + (q-1)a_{h-1}A_h + a_{h+1}A_{h+1}. \tag{8.91}$$

This is true for every set above the original vertex v. Now for the vertices marked in pink in Fig 8.8, they have q weights of A_{h+1} coming from the $h-1$ vertices because $h-1+d(u,y_\Lambda) = h-1+h-1+4 = 2h+2$, and one A_{h+1} from the bottom neighbor because $h+1+d(u,y_\Lambda) = h+1+h+1 = 2h+2$. Giving us the coefficient

$$(qa_{h-1} + a_{h+1})A_{h+1}.$$

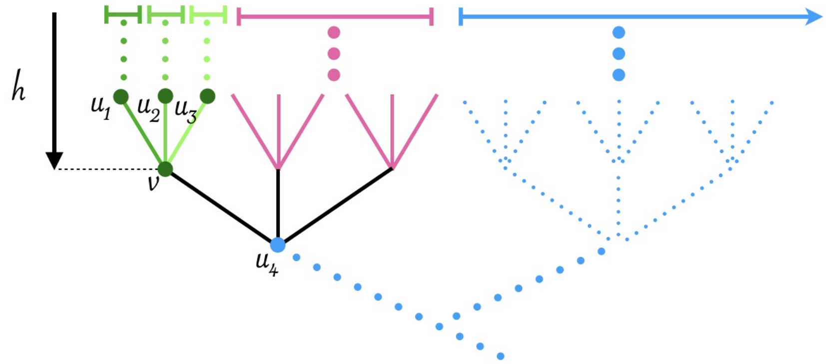

Figure 8.8: The vertices u_k are the neighboring vertices of v. The different colors indicate the regions with different weights A_i in the on-shell field evaluated at v. When evaluating the Laplacian, we evaluate the field on the vertices u_k, $k \leq 3$, the different shades of green (left) represent sets above these vertices, that will have the same weight A_i. For the evaluation at u_4 the pink (middle) and green regions all have the same weight. The boundary sets in blue (right) all have the same weight for all vertices u_k and v in the Laplacian.

Similar reasoning will show that the blue boundary set in Fig. 8.8 has the weight A_{h+2}. The rest of the boundary set will keep increasing the subindex of the weight, however it can be put in a nice form by measuring the distances relative to the original centered vertex v. This is done by noticing that $d(v, y_\Lambda) = d(u, y_\Lambda) + 1 = d(w, y_\Lambda) - 1$. Then the coefficient is

$$(qa_{h-1} + a_{h+1})A_{\frac{h+d(v,y_\Lambda)}{2}}.$$

Let's simplify the common factor $qa_{h-1} + a_{h+1}$,

$$qa_{h-1} + a_{h+1} = p^{(h+1)\Delta} - p^{(h+1)(D-\Delta)} + q(p^{(h-1)\Delta} - p^{(h-1)(D-\Delta)})$$

$$= p^{h\Delta}(qp^{-\Delta} + p^{\Delta}) - p^{hD-\Delta}(qp^{-(D-\Delta)} + p^{D-\Delta}) = (p^{\Delta} + p^{D-\Delta})a_h = (q + 1 + m^2)a_h.$$

This is good news, but we still need to manage the first coefficient (8.91). First we make all the capital A into A_{h+1} by making explicit the extra terms. With this we have

$$a_{h-1}\left(\frac{1}{a_{h-1}a_h} + \frac{1}{a_h a_{h+1}} + (q-1)\frac{1}{a_h a_{h+1}}\right) + (qa_{h-1} + a_{h+1})A_{h+1}$$

$$= \frac{a_{h-1}}{a_h}(\frac{1}{a_{h-1}} + \frac{q}{a_{h+1}}) + (q + 1 + m^2)a_h A_{h+1} = \frac{a_{h+1} + qa_{h-1}}{a_h a_{h+1}} + (q + 1 + m^2)a_h A_{h+1}$$

$$= (q + 1 + m^2)a_h\left(\frac{1}{a_h a_{h+1}} + A_{h+1}\right) = (q + 1 + m^2)a_h A_h.$$

Interpreting these results a little bit we see that all the green regions will have the coefficient $(q + 1 + m^2)a_h A_h$. The pink region will have $(q + 1 + m^2)a_h A_{h+1}$ as a coefficient, and the coefficient of the blue region is $(q+1+m^2)a_h A_{\frac{h+d(v,y_\Lambda)}{2}}$. But this is exactly the on shell solution $\phi^{(h,1)}$ with the additional factor $(q+1+m^2)$. Therefore

$$\sum_{u\sim v} a_{h(u)} \sum_{y_\Lambda} A_{\frac{h(u)+d(u,y_\Lambda)}{2}} \phi(\epsilon, y_\Lambda) = (q+1+m^2) a_{h(u)} \sum_{y_\Lambda} A_{\frac{h(u)+d(u,y_\Lambda)}{2}} \phi(\epsilon, y_\Lambda). \tag{8.92}$$

This proves that both terms of $\phi^{(h,1)}$ in (8.63) satisfy the equation of motion independently.

B The case $\Delta = D/2 = D - \Delta$

This is an interesting case because a plain substitution gives all as equal to zero and we would have $\frac{0}{0}$, something to be determined. We will do the usual and use L'Hôpital rule to get the result. First

$$\lim_{\Delta\to D/2} \frac{a_h}{a_{n+1}} = \lim_{\Delta\to D/2} \frac{h(\ln p)p^{h\Delta} - (-h\ln p)p^{h(D-\Delta)}}{(n+1)\ln p(p^{(n+1)\Delta} + p^{(n+1)(D-\Delta)})} = \frac{h}{n+1}\frac{2p^{hD/2}}{2p^{(n+1)D/2}} = \frac{h}{n+1}p^{(h-n-1)D/2}.$$

Now we remember that $v(z)+h = \Lambda$ and $n+1 = \Lambda - l$. Then we have

$$\lim_{\Delta\to D/2} \frac{a_h}{a_{n+1}} = \frac{\ln p}{\ln p}\frac{\ln|z/\epsilon|_p}{\ln|L/\epsilon|_p}\left|\frac{z}{L}\right|_p^{D/2} = \frac{\ln|z|_p - \ln|\epsilon|_p}{\ln|L|_p - \ln|\epsilon|_p}\left|\frac{z}{L}\right|_p^{D/2}.$$

In a more general way we have

$$\lim_{\Delta\to D/2} \frac{a_x}{a_w} = \frac{x}{w}p^{(x-w)D/2}. \tag{8.93}$$

Now specifically we have

$$\lim_{\Delta\to D/2} \frac{a_1 a_h}{a_i a_{i+1}} = \frac{h}{i(i+1)}p^{(h-2i)D/2}. \tag{8.94}$$

With this we see that

$$\begin{aligned}\lim_{\Delta\to D/2} a_1 a_h A_k &= \sum_{i=k}^{n} \frac{h}{i(i+1)}p^{(h-2i)D/2} = hp^{hD/2}\sum_{i=k}^{n} q^{-i}\left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= hp^{hD/2}\left(\sum_{i=k}^{n}\frac{q^{-i}}{i} - q\sum_{i=k}^{n}\frac{q^{-(i+1)}}{i+1}\right) = hp^{hD/2}\left(\frac{q^{-k}}{k} - \frac{q^{-n}}{n+1} + (1-q)\sum_{i=k+1}^{n}\frac{q^{-i}}{i}\right).\end{aligned} \tag{8.95}$$

An interesting thing is

$$\sum_{i=1}^{\infty}\frac{q^{-i}}{i} = -\ln(1-q^{-1}).$$

So we do have a logarithmic behavior all over the place. We simply mention that $hp^{hD/2} = \frac{1}{\ln p}\left(\ln\left|\frac{z}{\epsilon}\right|_p\right)\left|\frac{z}{\epsilon}\right|_p^{D/2}$. Considering all of this we have

$$\begin{aligned}\phi(z,x_\Lambda)\Big|_{\Delta=D/2} &= \frac{1}{\ln p}\left(\ln\left|\frac{z}{\epsilon}\right|_p\right)\left|\frac{z}{\epsilon}\right|_p^{D/2}\sum_{y_\Lambda}\left(\ln p\frac{|\epsilon|_p^D|(z,x_\Lambda-y_\Lambda)|_s^{-D}}{\ln(|\epsilon|_p^{-1}|(z,x_\Lambda-y_\Lambda)|_s)} - q^{-1}\ln p\frac{|\epsilon|_p^D|L|_p^{-D}}{\ln(|\epsilon|_p^{-1}|L|_p)}\right. \\ &\left.+(1-q)\sum_{i=k+1}^{n}\frac{q^{-i}}{i}\right)\phi(\epsilon,y_\Lambda) + \frac{\ln|z|_p - \ln|\epsilon|_p}{\ln|L|_p - \ln|\epsilon|_p}\left|\frac{z}{L}\right|_p^{D/2}\phi(L,x_\Lambda).\end{aligned} \tag{8.96}$$

Chapter 9

Conclusion

In this thesis we reviewed the older and more recent developments in the use of p-adic numbers in physics. It is a fruit of the works [48, 49, 50, 27] in which the author was involved. In [48], starting from the noncommutative effective action (5.1) discussed in [33, 51], we obtain the corresponding tree-level four-point amplitudes (5.30) in the limit $p \to 1$. This result was achieved by adapting the heuristic approach given in [29] for the noncommutative case. By an explicit computation using the noncommutative field theory [109, 110], we determine the four-point amplitude at the tree level coming from the noncommutative Gerasimov-Shatashvili Lagrangian. This tree-level amplitude is the sum of the expressions (5.9) and (5.11) and is completely described by planar Feynman diagrams, consequently the noncommutativity effect arises as a global phase factor in front of the amplitude. The study of the p-adic Ghoshal-Kawano amplitudes requires the use of multivariate local zeta functions involving multiplicative characters and a phase factor including the noncommutative parameter θ. These are new mathematical objects. We prove that these integrals admit meromorphic continuation as complex functions in the external momenta of the N external particles using Hironaka's resolution of singularities theorem. It would be very interesting to study the possibility of finding a non-trivial noncommutative effect, as the IR/UV mixing, as a result of the contribution of one-loop non-planar diagrams. Probably the multi-loop analysis of the p-adic string theory studied in [104], will play an important role for the analysis of the IR/UV mixing and other interesting effects of the B-field in p-adic string theory amplitudes.

On the other hand, we think that the study of the amplitudes (5.14) without the ad hoc normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$ may provide new insights on the effects of the B-field in p-adic string theory amplitudes.

In [27] we made a survey of the relations between local zeta functions and string amplitudes. The Archimedean case is easier to compute explicitly, and the techniques developed in that case can be applied to the Archimedean case. This makes some previously intractable problems more accessible, such as the rigorous regularization of the Archimedean Koba-Nielsen type amplitudes done in [26]. Here we emphasize on the examples for the simple cases, we can see that the procedure is straightforward, however it quickly becomes very lengthy to do explicitly.

In [49] we propose a theory of free p-adic worldsheet superstrings. An action analogous to the Archimedean case in the superconformal gauge was considered. As usual, the action consists of two terms, a bosonic and a fermionic part. We based our proposal in different works that proposed a fermionic propagator or action. This implies the use of Grassmann valued p-adic fields. To prevent the fermionic term from vanishing identically, it is necessary to insert an antisymmetric sign function, *i.e.* $\text{sgn}_\tau(-1) = -1$. This restricts the possible values of p to roughly half the primes, and τ to 2 of its non-trivial values. We noticed that the fermionic term is in fact very similar to the bosonic one, the only two differences being the use of a generalized Vladimirov derivative (that includes sgn_τ) and the order of the derivative is decreased by 1. From this action we were able to find a supersymmetry transformation, and write the action in a superspace formalism, defining a p-adic superfield and a derivative superoperator.

Using standard field theory techniques, we obtained the tachyon N-point tree amplitudes. We checked that the fermion propagator is equivalent to the corresponding two-point function. This required a functional derivative for p-adic fermion fields. Like in the Archimedean case, these amplitudes are non-vanishing only for even N. A neat and simple integral form for these amplitudes that is analogous to the Archimedean case can be given, albeit not very useful for computations. Explicit results can be obtained by manipulating the expressions. The procedure is similar to the one of the Archimedean case. Previous works have shown that the type of amplitudes obtained here are integrable and convergent in a certain region of momenta space [25]. The work done was for $\mathbb{Q}_p$, but in principle one can apply it to unramified extensions of the p-adic field $\mathbb{Q}_{p^n}$. In the spirit of p-adic AdS/CFT this would mean having multiple worldsheet coordinates. We have restricted the value of p, yet the case $p = 2$ remains to be explored. $\mathbb{Q}_2$ admits antisymmetric sign functions, but they behave very differently from their odd primed partners. Finding more vector amplitudes like the ones proposed in [58] can also be useful to understand better the theory.

Another future direction is including a B-field as in [51] in the context of superstrings. Recently it has been explored the idea of the p-adic bosonic string as a $p2$-brane defined on the Bruhat-Tits tree [122]. Further generalizations of this idea require the extension to the supersymmetric case.

In [50] we show that the bulk-to-boundary of [39] and the bulk reconstruction in the Bruhat-Tits tree of [66] is in fact the same thing. The first establishes the propagator in the infinite tree for a massive scalar in many dimensions, the second uses the propagator in a cutoffed tree in the massless case in a 2d bulk. We showed this explicitly generalizing both works. Additionally, we implemented the process of holographic renormalization for the bulk action as it approaches the boundary. This meant finding all divergencies and renormalizing them using a counterterm actions. In the p-adic case the number of counter terms needed is fixed. This differs from the Archimedean case, where the number of counterterms depends on the conformal dimension Δ. It would be interesting to continue this analysis obtaining several quantities of interest from this renormalized action such as correlation functions and anomalies. One can use the cutoff tree to implement the Wilsonian renormalization on the tree, obtaining the effective action that captures the UV region along the lines shown in [123, 124, 125]. Another project is to reproduce the argument in [126] to prove that in the p-adic case the correlators are related also by simpler limits with a scaling factor in the holographic correspondence, first shown in [127].

The use of p-adic numbers in physics has been around for several decades now. At the end of the last century there was great interest and many works on different subjects around p-adic string theory. Most of them were about making analogue computations of Archimedean results. There was a particular focus on the amplitudes of different string theories. Some of them explored the interesting relations one finds between Archimedean and non-Archimedean models through adelic products and the limit $p \to 1$. All of these works are very interesting and represents the extend of the broad search that occurs when there are novel developments in theoretical science. One needs to explore all possibilities and take them seriously, including changing the very numbers that one uses in its computations.

After this burst of interest, came a cold winter in the interest of these ideas. It was until the mid 2010s that a resurgence in the use of p-adic numbers came to physics, now in a different context, the newest revolution in theoretical physics, the AdS/CFT correspondence or holographic principle. This is happening right now, once again, there is a lot of interest and many works on different subjects around holography. I dare say that once again it is in the search of all possibilities to shed some light into the deep mysteries of the correspondence. There is no way to tell if this interest will fade yet again, it is the author's hope that it doesn't.

There is a common theme in the use of p-adics, the models that are built with them are simpler and the computations are easier. Yet the physics that one obtains from them is very similar to the usual case. This is part of the reason why there is interest in developing these ideas as toy models. Another healthy benefit is that since even the word p-adic is mostly unknown to physicists and not so unfamiliar to mathematicians, it forces a communication between the two disciplines. This is benefitial as it provides different approaches and techniques.

Surely much work remains to be done regarding p-adic physics. The extend to which it can be useful in physics is yet to be seen. For the moment it has proved to be a more manageable analog or toy model of ordinary string theory. This is due to the fact that $\mathbb{Q}_p$ and $\mathbb{R}$ behave similarly in mathematical analysis. Hopefully p-adic physics can bring new ideas to the table that can bring us a step closer to solving nature's most profound of mysteries.

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