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Masses of the *u*, *d*, and *s* Quarks

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Ratios m_s/m_d and m_u/m_d of the light quark masses have been determined from expressions for squared masses of pseudoscalar mesons m_π^2 and m_K^2 obtained with an accuracy of the second order in chiral symmetry breaking. The fit of the theoretical expressions for m_π^2 and m_K^2 to their phenomenological values leads to a functional relation between the ratios m_s/m_d and m_u/m_d, which is described by a third-order curve. The application of the lattice calculation result $m_s/m_{ud} = 27.23(10)$, where $m_{ud} = (m_u + m_d)/2$, reported by the flavor lattice averaging group (FLAG) for the case of four quark flavors provides an additional constraint, which significantly reduces the error (~2%) for the ratio $m_u/m_d = 0.455(8)$. The absolute values of the quark masses have been then obtained.

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The determination of the current masses of the *u*, *d*, and *s* quarks is of primary importance for the modern particle physics because they are the fundamental parameters of the Standard Model. In quantum chromodynamics (QCD), they are responsible for the explicit $SU(3)_L \times SU(3)_R$ chiral symmetry breaking, which is described by the Hamiltonian

$$H_m = \sum_{q=u,d,s} m_q \bar{q}q. \quad (1)$$

The product $m_q\bar{q}q$ is invariant under renormalization group transformations (in the leading logarithm approximation). It is known that the light quark masses at the energy scale $\mu \approx 1$ GeV (in the $\overline{MS}$- subtraction scheme) are $m_u \approx 4.2$ MeV, $m_d \approx 7.5$ MeV, and $m_s \approx 150$ MeV [1] (see also [2–5]). Since $m_q/\mu \ll 1$, H_m can be considered as a small perturbation near the chiral limit $m_q = 0$.

The ratio of the quark masses $m_q(\mu)/m_{q'}(\mu)$ in the $\overline{MS}$ scheme is independent of the choice of the subtraction point μ and can be obtained from the mass formulas for the pseudoscalar mesons by the standard current algebra methods [6]:

$$\begin{aligned} m_{\pi^+}^2 &= B_0(m_u + m_d) + \gamma_{\pi^+}, \\ m_{\pi^0}^2 &= B_0(m_u + m_d) + \gamma_{\pi^0}, \\ m_{K^+}^2 &= B_0(m_u + m_s) + \gamma_{K^+}, \\ m_{K^0}^2 &= B_0(m_d + m_s) + \gamma_{K^0}. \end{aligned} \quad (2)$$

Here, $B_0 = -\langle\bar{q}q\rangle/F^2$ is the parameter proportional to the $SU(3)$ quark condensate, F is the pion decay constant in the chiral limit, m_π and m_K are the phenomenological values of π and K meson masses, respectively; and γ_π and γ_K are the electromagnetic contributions to the self-energies of these pseudoscalar mesons. According to Dashen's theorem [7], in the leading order in m_q, $\gamma_{K^0} = \gamma_{\pi^0} = 0$ and $\gamma_{\pi^+} = \gamma_{K^+} = m_{\pi^+}^2 - m_{\pi^0}^2$. Due to these relations, the known Weinberg formulas are valid:

$$\begin{aligned} \frac{m_u}{m_d} &\overset{\text{LO}}{=} \frac{m_{K^+}^2 - m_{K^0}^2 + 2m_{\pi^0}^2 - m_{\pi^+}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} = 0.56, \\ \frac{m_s}{m_d} &\overset{\text{LO}}{=} \frac{m_{K^+}^2 + m_{K^0}^2 - m_{\pi^+}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} = 20.18. \end{aligned} \quad (3)$$

Systematic calculations of the next-to-leading (NLO) correction to the Weinberg result [8, 9] showed that a functional relation between the ratios $x = m_u/m_d$ and $y = m_s/m_d$ appears beyond the current algebra. According to the low-energy Gasser–Leutwyler theorem [8], the curve $f(x, y) = 0$ is an ellipse. On the contrary, the fit of the mass formulas obtained with an accuracy up to the first correction to the current algebra result (2) shows that x and y belong (taking into account approximations accepted in [9]) to a second-order curve whose canonical form is a parabola. Since the function $f(x, y)$ is important for the extraction of theoretical information on the ratios x and y of quark masses, it is reasonable to examine both methods in more detail in order to understand their advantages and disadvantages. The aim of this work is to study this problem.

Our analysis is based on the formulas

$$\begin{aligned} m_{\pi^+}^2 &= B_0(m_u + m_d)[1 + \Delta(m_u + m_d)] + \gamma_{\pi^+}, \\ m_{K^+}^2 &= B_0(m_u + m_s)[1 + \Delta(m_u + m_s)] + \gamma_{K^+}, \\ m_{K^0}^2 &= B_0(m_d + m_s)[1 + \Delta(m_d + m_s)] + \gamma_{K^0}, \end{aligned} \quad (4)$$

where the NLO contribution to the eigenenergy of the pseudoscalar mesons is taken into account. These relations can be obtained in the $1/N_c$ chiral perturbation theory [10, 11]; in this case, the parameter $\Delta = 8B_0(2L_8 - L_5)/F^2$ is expressed in terms of the low-energy constants of the effective chiral Lagrangian. Formulas (4) can also be derived in the Nambu–Jona-Lasinio model using the count rules $1/N_c$, p^2, $m_q = \mathcal{O}(\delta)$ accepted in the $1/N_c$ chiral perturbation theory. In this case, $\Delta = \delta_M/(2M_0)$ [12], where M_0 is the gap in the fermion spectrum and δ_M is the dimensionless constant by the entire set of the parameters of the model. Further, the value of the parameter Δ is fixed in terms of the modern lattice QCD estimates. It is also noteworthy that the values of the Δ and B_0 do not affect the form of the function $f(x, y)$.

It is worth emphasizing that the derivation of Eqs. (4) involves not only expansions in powers of quark masses and momenta but also the classification of the vertices of the effective meson Lagrangian according to their behavior in the limit of a large number N_c of color degrees of freedom. This allowed one to extend the symmetry group of the theory to $U(3)_L \times U(3)_R$, by including the η' meson in the consideration, thus taking into account the known solution of the $U(1)$ problem [13–18] and suppressing the processes violating the Zweig rule. Due to the $1/N_c$ expansion, Eqs. (4) do not contain the contribution from chiral logarithms, which has the next order $\mathcal{O}(\delta^3)$. This important circumstance makes it possible to avoid ambiguity in calculations of x and y, which appears due to the unphysical symmetry of the Lagrangian of chiral perturbation theory under the Kaplan–Manohar transformations [9].

Dashen's theorem is violated beyond the current algebra. This primarily concerns the contributions γ_{π^+} and γ_{K^+}. For the neutral modes, $\gamma_{K^0} = \gamma_{\pi^0} = 0$ can still be accepted in Eqs. (4). The contribution from virtual photons to the self-energy of the charged pion is given by the formula

$$\gamma_{\pi^+} = (m_{\pi^+}^2 - m_{\pi^0}^2) - (\bar{m}_{\pi^+}^2 - \bar{m}_{\pi^0}^2). \quad (5)$$

The pure QCD contribution is small $(\bar{m}_{\pi^+}^2 - \bar{m}_{\pi^0}^2) \sim (m_d - m_u)^2$ and can be calculated in the chiral perturbation theory, which gives the estimate $\bar{m}_{\pi^+} - \bar{m}_{\pi^0} = 0.17(3)$ MeV [8]. Consequently,

$$\gamma_{\pi^+} = 1.21(1) \times 10^{-3}\ \text{GeV}^2\ [19, 20]. \quad (6)$$

For the charged kaon, deviation from Dashen's theorem ($\gamma_{K^+} = \gamma_{\pi^+}$) can be characterized by the parameter ϵ as follows [21]:

$$\gamma_{K^+} = \gamma_{\pi^+} + \epsilon(m_{\pi^+}^2 - m_{\pi^0}^2). \quad (7)$$

The flavor lattice averaging group (FLAG) in recent review [22] of the lattice QCD results obtained by various collaborations presents the averaged values

$$\epsilon = 0.79(6)\ (N_f = 2 + 1 + 1), \quad (8)$$

$$\epsilon = 0.73(17)\ (N_f = 2 + 1). \quad (9)$$

Below, the former estimate is used because it has a smaller error and corresponds to simulations with four quark flavors $m_u = m_d \neq m_s \neq m_c$. This gives

$$\gamma_{K^+} = 2.21(8) \times 10^{-3}\ \text{GeV}^2, \quad (10)$$

which indicates a significant violation of Dashen's theorem compared to Eq. (6).

The system of three equations (4) has four free parameters (e.g., m_dB_0, $m_d\Delta$, x, and y). Therefore, its solution is the relation $f(x, y) = 0$. To determine the function $f(x, y)$, the following two ratios are considered

$$\begin{aligned} r_x &= \frac{m_{K^+}^2 - m_{K^0}^2 + m_{\pi^+}^2 - \gamma_{K^+} - \gamma_{\pi^+}}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2 + \gamma_{K^+} - \gamma_{\pi^+}} \\ &= \frac{x + m_d\Delta[y(x-1) + x(x+1)]}{1 + m_d\Delta[y(1-x) + 1 + x]}, \end{aligned} \quad (11)$$

$$\begin{aligned} r_y &= \frac{m_{K^+}^2 + m_{K^0}^2 - m_{\pi^+}^2 - \gamma_{K^+} + \gamma_{\pi^+}}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2 + \gamma_{K^+} - \gamma_{\pi^+}} \\ &= \frac{y + m_d\Delta[y(y+x) + y - x]}{1 + m_d\Delta[y(1-x) + 1 + x]}, \end{aligned} \quad (12)$$

where Eqs. (4) were used in the second step. On the one hand, the substitution of the experimental masses of the π^+, K^+, and K^0 mesons and estimates (6) and (10) into Eqs. (11) and (12) gives $r_x = 0.498(5)$ and $r_y = 19.32(6)$, respectively. On the other hand, the exclusion of the parameter $m_d\Delta$ from Eqs. (11) and (12) provides the relation

$$(y^2 - 1)(1 - xr_x) = (1 - x^2)(yr_y - 1). \quad (13)$$

This relation specified a third-order (cubic) curve of genus $g = 1$. It has three independent branches including two hyperbolic and one straight-line (hyperbolic-type) branches. The point $(x, y) = (r_x, r_y)$ belongs to the last branch.

Result (13) is independent of the choice of the ratios r_x and r_y. After the exclusion of the parameter Δ, any arbitrarily chosen pair

$$r_\alpha = \frac{(\alpha_P, \bar{m}_P^2)}{(\bar{\alpha}_P, \bar{m}_P^2)}, \quad r_\beta = \frac{(\beta_P, \bar{m}_P^2)}{(\bar{\beta}_P, \bar{m}_P^2)}, \tag{14}$$

where $\bar{m}_P^2 = m_P^2 - \gamma_P$ and $(\alpha_P, \bar{m}_P^2) = \alpha_{K^+}\bar{m}_{K^+}^2 + \alpha_{K^0}\bar{m}_{K^0}^2 + \alpha_{\pi^+}\bar{m}_{\pi^+}^2$, leads to the equation

$$\begin{aligned} &(\alpha_P, \mu_P^2)(\beta_P, \mu_P) - (\alpha_P, \mu_P)(\beta_P, \mu_P^2) \\ &+ r_\alpha[(\bar{\alpha}_P, \mu_P)(\beta_P, \mu_P^2) - (\bar{\alpha}_P, \mu_P^2)(\beta_P, \mu_P)] \\ &+ r_\beta[(\bar{\beta}_P, \mu_P^2)(\alpha_P, \mu_P) - (\bar{\beta}_P, \mu_P)(\alpha_P, \mu_P^2)] \\ &= r_\alpha r_\beta[(\bar{\beta}_P, \mu_P^2)(\bar{\alpha}_P, \mu_P) - (\bar{\beta}_P, \mu_P)(\bar{\alpha}_P, \mu_P^2)]. \end{aligned} \tag{15}$$

Here, $\mu_P = m_i + m_j$, where the subscripts (i, j) specify the quark composition of a particular meson state P: $(u, d) \to \pi^\pm$, $(u, s) \to K^\pm$, and $(d, s) \to K^0, \bar{K}^0$.

The dependence on the arbitrary parameters α_P, β_P, $\bar{\alpha}_P$, and $\bar{\beta}_P$ is factorized. To show this factorization, it is necessary to express r_α and r_β in Eq. (15) in terms of r_x and r_y. As a result, the equation takes the form

$$\begin{aligned} &\mathcal{F}(\alpha_P, \beta_P, \bar{\alpha}_P, \bar{\beta}_P, r_x, r_y) \\ &\times[(y^2 - 1)(1 - x r_x) - (1 - x^2)(y r_y - 1)] = 0. \end{aligned}$$

If $\mathcal{F} \neq 0$, Eq. (13) is valid. Otherwise, r_α and r_β are not independent variables. Thus, the shape of the curve specified by Eqs. (14) is independent of the choice of the pair r_α and r_β. Furthermore, when $\bar{\alpha}_P = \bar{\beta}_P$, as, e.g., in (11) and (12), the ratios lead to Eq. (13) regardless of the δ expansion of the ratios given by Eqs. (14).

Indeed, the δ expansion gives

$$r_\alpha = k_\alpha[1 + l_\alpha(m_d\Delta) + \mathcal{O}(\delta^2)], \tag{16}$$

where the coefficients k_α and l_α are functions of the ratios of quark masses x and y and the parameters α_P and $\bar{\alpha}_P$:

$$k_\alpha = \frac{(\alpha_P, \mu_P)}{(\bar{\alpha}_P, \mu_P)}, \quad l_\alpha = \frac{1}{m_d}\left[\frac{(\alpha_P, \mu_P^2)}{(\alpha_P, \mu_P)} - \frac{(\bar{\alpha}_P, \mu_P^2)}{(\bar{\alpha}_P, \mu_P)}\right].$$

Considering the pair r_α and r_β, it is obviously possible to exclude the parameter $m_d\Delta$ and thus to arrive at the following relation between y and x:

$$k_\alpha k_\beta(l_\alpha - l_\beta) = k_\alpha l_\alpha r_\beta - k_\beta l_\beta r_\alpha. \tag{17}$$

If $\bar{\alpha}_P = \bar{\beta}_P$, Eq. (17) describes the cubic curve specified by Eq. (13). When $\bar{\alpha}_P \neq \bar{\beta}_P$, different pairs r_α and r_β gives different curves. Some consequences of this behavior were studied in [12, 23]. The case $l_\alpha = l_\beta$ is the most interesting here. Then, according to Eq. (17), $r_\alpha/r_\beta = k_\alpha/k_\beta$; i.e., NLO contributions are absent in these two ratios. This statement is the essence of the well-known low-energy Gasser–Leutwyler theorem [8]. For example, consider the chiral expansion of the ratios

$$\begin{aligned} r_1 &= \frac{m_{K^+}^2 - \gamma_{K^+}}{m_{\pi^+}^2 - \gamma_{\pi^+}} \\ &= \frac{m_s + m_u}{m_d + m_u}\Big[1 + \Delta(m_s - m_d) + \mathcal{O}(\delta^2)\Big], \\ r_2 &= \frac{m_{K^0}^2 - m_{K^+}^2 + \gamma_{K^+}}{m_{K^0}^2 - m_{\pi^+}^2 + \gamma_{\pi^+}} \\ &= \frac{m_d - m_u}{m_s - m_u}\Big[1 + \Delta(m_s - m_d) + \mathcal{O}(\delta^2)\Big]. \end{aligned} \tag{18}$$

Here, $l_1 = l_2$ and, consequently,

$$\frac{r_1}{r_2} = \frac{m_s^2 - m_u^2}{m_d^2 - m_u^2} \equiv Q_1^2. \tag{19}$$

Here, the variables x and y obviously belong to an ellipse. In this case, there are three parameters $m_d\Delta$, x, and y to fit two phenomenological values r_1 and r_2. The dimensionless parameter $m_d\Delta$ specifies the position of a point on the curve (unlike a cubic curve, this requires two parameter $m_d\Delta$ and m_dB_0). The equation of ellipse (19) in the variables r_x and r_y takes the form

$$(y^2 - 1)(1 - r_x^2) = (1 - x^2)(r_y^2 - 1). \tag{20}$$

Both Eqs. (13) and (20) specifying curves plotted in Fig. 1 include the same parameters r_x and r_y. The difference is that these parameters in the case of the ellipse constitute the parameter

$$Q_1 = \sqrt{\frac{r_y^2 - r_x^2}{1 - r_x^2}} = 22.28(15), \tag{21}$$

which determines the major semiaxis of this ellipse. The error indicated in parentheses in Eq. (21) is due to the error in ϵ given by Eq. (8). This result is in excellent agreement with the estimate $Q_{\mathrm{GL}} \equiv (m_s^2 - m_{ud}^2)/(m_d^2 - m_u^2) = 22.1(7)$ [19], where $m_{ud} = (m_u + m_d)/2$, obtained from experimental data on $\eta \to 3\pi$ decays. It is also agreement with the FLAG estimate $Q_{\mathrm{GL}} = 22.5(5)$ [22]. A high accuracy of the numerical estimate (21) is due to modern precise lattice QCD calculations of the parameter ϵ.

In the case of the cubic curve, the parameters r_x and r_y are independent. As a result, the ratio of Q_1^2 quark masses varies along the cubic curve and its determination requires an additional assumption, which leads, as shown below, to the result $Q_1 = 22.24(16)$ in the case of physical values of quark masses.

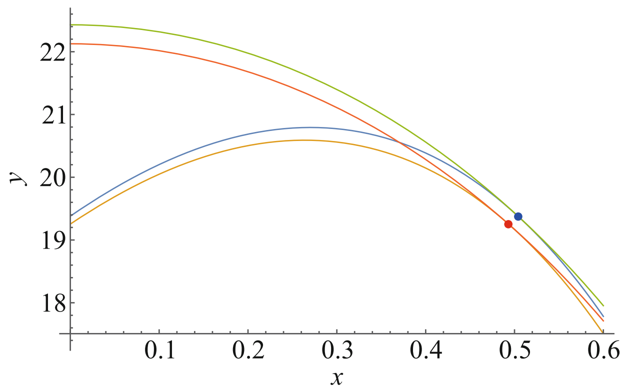

Fig. 1. (Color online) Ratio $y = m_s/m_d$ versus $x = m_u/m_d$ in the next-to-leading order under the conditions of (upper elliptic lines) the low-energy Gasser–Leutwyler theorem and (lower cubic lines) mass formulas. Since Dashen's theorem is violated, two lines covering the interval ϵ = 0.79(6) are plotted for each case. The lines have a common tangent at the point (r_x, r_y). The imaginary horizontal straight line $y = r_y$ intersects the cubic line at the points (r_x, r_y) and $(0, r_y)$.

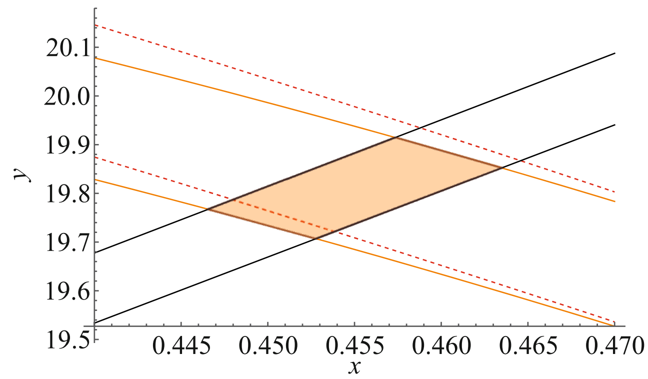

Fig. 2. (Color online) Allowed values for quark mass ratios $x = m_u/m_d$ and $y = m_s/m_d$: the dashed elliptic lines corresponding to the region Q_1 = 22.24(16) in comparison with the solid cubic lines corresponding to the region ϵ = 0.79(6). The thin straight lines separate the region S = 27.23(10) obtained in the FLAG processing of the lattice data for $N_f = 2 + 1 + 1$ [22]. The intersection of the band corresponding to the considered S values and the cubic lines separates the region (filled) of the physical values for the ratios $x = m_u/m_d$ and $y = m_s/m_d$.

The mentioned assumption concerns the ratio $S = m_s/m_{ud}$ = 27.23(10) [22], which is obtained in lattice QCD calculations and is weakly sensitive to corrections caused by the violation of Dashen's theorem. The electromagnetic contribution to this ratio is ≈0.18% [24]; i.e., it is indeed very small. The knowledge of S allows one to significantly limit the region of allowed values for x and y and thus to increase the accuracy of theoretical calculations of the ratio m_u/m_d. This idea was already used in [19], where the ratio m_u/m_d = 0.44(3) was determined from the parameters $Q_{\rm GL}$ = 22.1(7) and S = 27.23(10). This estimate can now be improved due to a high accuracy of Eq. (8).

The result of these calculations is presented in Fig. 2, which demonstrates excellent agreement of data obtained independently using both the low-energy Gasser–Leutwyler theorem and the cubic curve

$$\begin{aligned} m_u/m_d &= 0.455(8)\ \text{(cubic curve)}, \\ m_u/m_d &= 0.456(8)\ \text{(ellipse)}. \end{aligned} \tag{22}$$

As mentioned above, the quark condensate $\langle\bar{q}q\rangle(\mu)$ is related to the low-energy constant $B_0(\mu)$ and depends on the renormalization group scale μ in the $\overline{MS}$ subtraction scheme. However, the product $m_q(\mu)B_0(\mu)$ is invariant under transformations of the renormalization group. Since this product is one of the internal parameters of the cubic curve, the masses of individual quarks can be estimated if B_0 is known, and vice versa. The accuracy of lattice estimates of $B_0(\mu)$ is currently low. The calculations below were performed with the result B_0(2 GeV) = 2.682(36)(39) GeV [26] whose total error of ~2% is comparatively small.

To determine the absolute values of the quark masses, it is necessary to solve the system of three equations (4). To this end, at a given B_0(2 GeV) value, the parameter Δ is varied so that the solution lies in the separated filled region in Fig. 2. The corresponding results are summarized in Table 1.

The first row in Table 1 presents the solutions of the system of Eqs. (4) with the value B_0 = 2.682(53) GeV. To estimate the error introduced by the parameter B_0, the second row of Table 1 presents the results obtained with the fixed value B_0 = 2.66 GeV, which is the arithmetic mean of the central values B_0 = 2.682(53) GeV from [26] and B_0 = 2.64(20) GeV from [27]. The third row presents the values proposed by PDG [25]. The fourth and fifth rows present FLAG estimates [22] made for the cases of four and three flavors of quarks, respectively, by averaging various lattice calculations according to the FLAG selection criteria.

The comparison of the results presented in the first two rows in Table 1 shows that the error ≈2% in B_0 together with the existing error ≈1.5% in the determination of ϵ and S values gives the total error ≈3% for quark masses. This accuracy is sufficient to state that the results of this work completely agrees with the PDG values and with average values presented by FLAG. Moreover, it can be concluded that the δ expansion, which underlies the initial formulas (4) for our analysis, is an efficient tool to calculate the light quark masses.

Since numerous estimates of the quark masses reported in the literature are referred to the scale μ = 1 GeV, the corresponding our results are presented here. Weinberg's results for the quark masses pre-

Table 1. Masses of the light quarks u, d, and s, isospin-averaged value $m_{ud} = (m_u + m_d)/2$, m_u/m_d, Q_1 given by Eq. (21), and $R = (m_s - m_{ud})/(m_d - m_u)$. All parameters correspond to the scale $\mu = 2$ GeV in the $\overline{MS}$ subtraction scheme

Reference	Conditions	m_u, MeV	m_d, MeV	m_{ud}, MeV	m_s, MeV	m_u/m_d	Q_1	R
This work	$B_0 = 2.682(53)$ GeV	2.14 ± 0.07	4.70 ± 0.12	3.42(7)	93.13 ± 2.25	0.455(8)	22.23(16)	35.02(61)
	$B_0 = 2.66$ GeV	2.16 ± 0.03	4.74 ± 0.03	3.447(2)	93.85 ± 0.41	0.455(8)	22.23(16)	35.02(61)
PDG [25]	–	$2.16^{+0.49}_{-0.26}$	$4.67^{+0.48}_{-0.17}$	$3.45^{+0.35}_{-0.15}$	$93.4^{+8.6}_{-3.4}$	$0.474^{+0.056}_{-0.074}$	–	–
FLAG [22]	$N_f = 2 + 1 + 1$	2.14 ± 0.08	4.70 ± 0.05	3.410(43)	93.44 ± 0.68	0.465(24)	22.5(5)	35.9(1.7)
	$N_f = 2 + 1$	2.27 ± 0.09	4.67 ± 0.09	3.364(41)	92.03 ± 0.88	0.485(19)	23.3(5)	38.1(1.5)

sented in the beginning of this article allow the estimate $B_0(1\text{ GeV}) \approx 1.58$ GeV. Indeed, the substitution of this B_0 into Eqs. (2) gives $m_u = 4.1$ MeV, $m_d = 7.4$ MeV, and $m_s = 149$ MeV. The inclusion of the NLO correction leas to the estimates (for the allowed region in Fig. 2) m_u =3.63(5) MeV, m_d = 7.98(4) MeV, and m_s = 158.0(7) MeV. Correspondingly, it can be concluded that the NLO contribution to the light quark masses varies here from 11 to 6%.

To summarize, the cubic curve is an additional useful source to extract theoretical information on the light quark masses. On the one hand, this curve in the region of physical quark masses is excellently consistent with the low-energy Gasser–Leutwyler theorem; on the other hand, it allows one to estimate not only ratios of quark masses but also their absolute values. The estimates obtained in this work seem reasonable and their accuracy will increase with decreasing error in the determination of the low-energy constant B_0.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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