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Puzzle of the cornell potential on the heavy and heavy-light meson spectra

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Abstract

We computed the eigenvalues and eigenfunctions via the Cornell potential, approximately, using Numerov approach. The experimental data for the quantum states $^3S_1$ were utilized to fit the mass spectra. Results were compared with observed data and the results obtained from other potential functions in the existing literature for the heavy ($c\bar{c}$, $b\bar{b}$ and $b\bar{c}$) as well as heavy-light (B, Bs, D and Ds) meson states. The yielded meson spectra based on this potential are matching well with the available experimental data.

1. Introduction

Quarkonium systems provide a hot set for studying fundamental features of Quantum Chromodynamics (QCD) and matter's behavior under extreme conditions. The fruitful results of these studies help us to understand both perturbative and non-perturbative processes in particle physics. Within the exciting field of particle physics, quarkonium is a unique kind of bound states which build up of an antiquark plus heavy quark (which can be either a charm or a bottom quark). The word 'quarkonium' refers to the bound state formed by an electron and an anti-electron (positron), analogous to positronium. Quarkonium states featured with parity, spin and angular momentum, in addition to the other quantum numbers. Higher energy states of quarkonium are characterized by a series excited (φ', ψ'', etc) states. Some certain states such as the unknown X (3872) particle remains unverified and still being studied [1, 2]. Strong force interactions between heavy quarks can be better understood by looking at quarkonium states, which show how fundamental particles interact in a complex dance in the quantum realm [3, 4].

In hadron physics, heavy quarkonium have been extensively studied because they involve the non-perturbative part of QCD and have a large amount of experimental data [5–7]. Theoretically, tremendous models have been used to study heavy quarkonium [8–18]. Among these, the non-relativistic potential models stand out for their simplicity, as they simulate quark interaction using potential energy in the standard Schrödinger equation. Within the framework of QCD, the concept of the Cornell potential [19, 20] arises in particle physics. In 1970, researchers at Cornell University recommended the Cornell potential to describe the masses of quarkonium states (such as mesons and baryons) and their correlation with the angular momentum. Cornell potential is a good inception for one or more quark mass expansion in the framework of the non-relativistic QCD (NRQCD) which can be used to justify this image for heavy quarkonium [21–24]. The potential energy that exists between two quarks as a function of their separation distance is represented by Cornell potential. The advantage of using Cornell potential is that the two naturally yielded Hamiltonian possibilities are comparable [25]. Since the spin–orbit and tensor components of the potential are absent for the $l > 0$ states, the mass spectra degenerate, and the hyperfine splitting is only observed for the S-wave states.

The Cornell potential predicted linear confinement behavior, which was validated by lattice QCD simulations (numerical computations utilizing a grid). Researchers have employed Numerical techniques to solve Schrödinger equation based on Cornell potential [26, 27]. The one-dimensional solution to Schrödinger equation will be discussed. This problem is quite similar to calculating the radial wave function for spherically symmetric potentials in two or three dimensions. This equation can be solved by using a fifth-order numerical methodology called Numerov's method [28–34]. The Numerov approach, commonly referred to as Cowell's method, is a numerical method for solving second-order ordinary differential equations (ODEs), which is useful in case of the first-order term is missing.

Boris Vasil'evich Numerov, a famous Russian astronomer, produced and recommended this method. We can convert this equation into one that is compatible with Numerov's technique by introducing a suitable replacement. The outcomes of this approach are very important for predicting the behavior of the quantum systems. For a certain potential, the energy eigenvalues and eigenfunctions (stationary states) can be estimated. We limited our investigation to S-wave states, however, the method presented in this paper can be generalized to the radial and orbitally excited states as well. The main goal of this wirk is to find out an approximate solution based on Cornell potential for the meson bound states, and hence, using the bound state solution to extract the quarkonium system mass spectra.

This paper is organized as follows: section 2, is devoted to draw out the main elements of the potential model used and the approximations of energies and wave functions for heavy quarkonium using Numerov technique. Section 3, is devoted to discuss the resulted meson spectra based on Cornell potential. And finally, a concluding remark are given in section 4.

2. Methodology

The effective potential models are utilized in an antiquated efficient method to determine the masses of quarkonium states. According to this method, the quarks are stated in a static potential because their motion is non-relativistic, which is similar to the non-relativistic version of hydrogen atom. In 1970, Godfrey and Isgur [35] provided one of the most often used potential models, Cornell potential, and expressed as [36–40]

$$V(r) = \frac{-4\ \alpha_s}{3\ r} + br + c, \tag{1}$$

where r is the quarkonium state effective radius, the $-4/3$ is a colour factor, α_s, b, and c are parameters that will be found later by fitting the states' meson mass spectra with values that have been measured by the Particle Data Group (PDG) [41, 42]. This potential is divided into three parts: The first part ($-4\ \alpha_s\ /3$r), represents the one-gluon exchange potential between the quark and antiquark, which is also called the Coulombic potential because of the $1/\ r$ form is the same as the well-known Coulombic potential caused by the electromagnetic force. The second part is the confinement potential, br, this part is used to parameterize the non-perturbative effects of QCD, which are not well understood.

The third part, c, represents the relativistic correction and other effects, which can be incorporated into this approach by adding additional terms to the potential, similar to the hydrogen atom behavior in the non-relativistic quantum mechanics.

When employing this method, a convenient form for the quarks' wave function is picked, and α_s and b are obtained by fitting the calculated results of the masses for the well-measured quarkonium states. Although there is no strong theoretical basis for this approach, it is widely used because it makes accurate quarkonium parameter predictions without requiring a complicated lattice computations and distinguishes between the long- and short-range confinement effects, which are helpful in understanding the quark/anti-quark force produced by QCD. The two-body system radial Schrödinger equation (RSE) in a spherical symmetric potential is represented based on wave function $R_{nl}(r)$, energy eigenvalue (E_{nl}), and centrifugal barrier term as

$$-\frac{\hbar^2}{2\mu}\frac{d^2R_{nl}(r)}{dr^2} + \left(V(r) + \frac{l(l+1)}{2\mu r^2}\right)R_{nl}(r) = E_{nl}R_{nl}(r), \tag{2}$$

where μ is meson's reduced mass and l is the orbital quantum number. Naturally, the non-relativistic Hamiltonian provided by [43–47] is the left term.

$$H = M + \frac{P^2}{2\mu} + V(r), \tag{3}$$

$$M = M_q + M_{\bar{q}}, \tag{4}$$

$$\mu = \frac{M_q\ M_{\bar{q}}}{M_q + M_{\bar{q}}}, \tag{5}$$

$M_q/M_{\bar{q}}$ denotes the quark/antiquark mass parameters. The symbol ' P ' stands for the relative momentum of each quark.

By adopting natural units and substituting equation (1) into equation (2), we get

$$-\frac{1}{2\mu}\frac{d^2R_{nl}(r)}{dr^2}+\left(\frac{-4}{3}\frac{\alpha_s}{r}+br+\frac{l(l+1)}{2\mu r^2}\right)R_{nl}(r)=E_{nl}R_{nl}(r), \quad (6)$$

In addition, we take c = 0 in the potential (1) into consideration for the verification of quantum mechanical expectation. Our approach is based on the numerical solution of equation (6) as a matrix eigenvalue problem. The radial second derivative finite difference approximation can be simplified by converting it into tridiagonal matrix form. Because of the presence of the Gaussian function, the analytical solution of equation (6) is not possible using the potential from (1). So, we numerically solved this equation using the Numerov technique [33] to derive the eigenvalue and eigenfunction equations for the ground state ($l = 0$ and $S = 1$) with the Cornell potential, as well as the heavy quarkonium spectrum and wave functions.

The time-independent one-dimensional Schrödinger equation can be written as follows

$$f(r)=\frac{2m}{\hbar^2}(E_{nl}-V(r)), \quad ;\hbar=1 \quad (7)$$

With a distance d between each point on the lattice, x_i, equally spaced, we can write the integration formula as

$$\psi_{i+1}=\frac{\psi_{i-1}(12-d^2f_{i-1})-2\psi_i(5d^2f_i+12)}{d^2f_{i+1}-12}, \quad (8)$$

From the above equation

$$\psi_{i+1}=\frac{12\psi_{i-1}-d^2f_{i-1}\psi_{i-1}-10d^2f_i\psi_i-24\psi_i}{d^2f_{i+1}-12},$$

Hence

$$d^2f_{i+1}\psi_{i+1}-12\psi_{i+1}=12\psi_{i-1}-d^2f_{i-1}\psi_{i-1}-10d^2f_i\psi_i-24\psi_i, \quad (9)$$

Applying equation (7), we obtain

$$-2md^2/\hbar^2[(E\psi_{i-1}-V_{i-1}\psi_{i-1})+(10E\psi_i-10V_i\psi_i)+(E\psi_{i+1}-V_{i+1}\psi_{i+1})]=12\psi_{i-1}-2\psi_i+\psi_{i+1}, \quad (10)$$

where $\psi_i = \psi(x_i)$. After rearranging the equation above, we get:

$$\frac{-1}{2m}\frac{(\psi_{i-1}-2\psi_i+\psi_{i+1})}{d^2}+\frac{(V_{i-1}\psi_{i-1}+10V_i\psi_i+V_{i+1}\psi_{i+1})}{12}=E\frac{(\psi_{i+1}+10\psi_i+\psi_{i-1})}{12}, \quad (11)$$

Now, using only the grid number d and matrix size N, we will convert the well-known Numerov approach into a representation of matrix form on a discrete lattice. In order to accomplish that, ψ will be defined as a matrix and represented by a column vector $(\ldots.\psi_{i-1}, \psi_i, \psi_{i+1}\ldots.)$

$$A_{N,N}=\frac{(I_{-1}-2I_0+I_1)}{d^2}, B_{N,N}=\frac{(I_{-1}+10I_0+I_1)}{12}, V_N=diag(\ldots.,V_{i-1}, V_i, V_{i+1}),$$

where I_p is a matrix of 1S along the pth diagonal. The matrix version of equation (11) could be created, and zeros elsewhere, as follows

$$\frac{-1}{2m}A_{N,N}\psi_i+B_{N,N}V_N\psi_i=E_iB_{N,N}\psi_i, \quad (12)$$

Multiplying by $B_{N,N}^{-1}$ yields

$$\frac{-1}{2m}A_{N,N}B_{N,N}^{-1}\psi_i+V_N\psi_i=E_i\psi_i, \quad (13)$$

For the 3D radial Schrödinger equation, equation (13) could be written as

$$\frac{-\hbar^2}{2\mu}A_{N,N}B_{N,N}^{-1}\psi_i+\left[V_N(r)+\frac{l(l+1)}{r^2}\right]\psi_i=E_i\psi_i, \quad (14)$$

This numerical technique allows us to solve the eigenvalue issue for any possible hadron-hadron bound states.

3. Results and discussion

The Numerov approach has been employed extensively to obtain the heavy mesons' bound states. We applied the Cornell potential to solve Schrödinger equation. The interpolated function representing the reduced wave function and energy eigenvalue are the outputs that are automatically generated. In addition to the heavy-light

Table 1. Optimal values of parameters used to get the best values of heavy and heavy -light mesons.

Parameters	$m_{c/\bar{c}}$ (GeV)	$m_{b/\bar{b}}$ (GeV)	$m_{u/\bar{u}}$ (GeV)	$m_{s/\bar{s}}$ (GeV)	α_s (GeV^{-1})	b (GeV^2)
Charmonium	1.44	—	—	—	0.48	0.149
Bottomonium	—	4.58	—	—	0.119	0.24
Bottom-charm	1.44	4.58	—	—	0.126	0.168
Charmed/Charmed-strange	1.44	—	0.295	0.376	0.586	0.109
bottomonia	—	4.58	0.295	0.376	0.87	0.18

Table 2. Charmonium mesons mass spectra. the units for all the masses are MeV.

State	Charmonium				
$J^{pc}=1$	Our	[49]	[50]	[48]	Exp.[41,42]
1S	3.076	3.081	3.1404	3.096	3.097
2S	3.656	3.717	3.7017	3.685	3.686
3S	4.064	4.003	4.0502	4.039	4.039
4S	4.408	4.156	4.4185	4.427	4.421
5S	4.714	4.247	4.6591	4.837	4.63
6S	4.996	4.305	4.8801	5.167	—

Table 3. Bottomonium mesons mass spectra. the units for all the masses are MeV.

State	Bottomonium				
$J^{pc}=1^-$	our	[51]	[50]	[49]	Exp.[41,42]
1S	9.556	9.460	9.49081	9.465	9.460
2S	10.012	10.064	10.01257	10.003	10.023
3S	10.365	10.355	10.32775	10.354	10.355
4S	10.671	10.517	10.5461	10.635	10.579
5S	10.947	10.617	10.82628	10.878	10.885
6S	11.204	10.682	10.97061	11.102	11.000

mesons, the mass spectra for the charmonium, bottomonium and bottom-charm mesons were obtained for the triplet ($S=1$) states.

By employing the Mathematica package reduction strategy, we were able to extract the Cornell potential free parameters α_s and b numerically from the mass spectra equation provided by the relation.

$$M_{nl} = m_q + m_{\bar{q}} + E_{nl}, \tag{15}$$

Utilizing this approach, we formulate the equation $(M_i\ (\alpha_s,\ b),—m_i)$ in a way that matches the global minimum equation.

$$\sum_{i=0}^{3}(M_i(m_q,\ \alpha_s\ ,\ b) - m_i)^2 = 0. \tag{16}$$

where the meson triplet state constant fitting masses are represented by the m_i. In order to conduct an acceptable fit. we set additional constraints on α_s, b, m_q and $m_{\bar{q}}$. A global minimum, also known as an absolute minimum, is the smallest overall value of a set, function, etc, over its entire range. It is impossible to construct an algorithm that will find a global minimum for an arbitrary function. Table 1 outlined the fitted potential parameters. For the quantum states listed in tables 2–6, we utilized the fitted masses of the PDG. Several model predications of mass spectra for heavy flavored mesons that are accessible in the literature are included in tables 2–6.

The charmonium mesons mass spectra are found in table 2 and compared with the findings of some previous studies published in the literature [48–50] and the experimental data that are accessible on the PDG [41, 42]. The computed mesons masses for S-wave states with quantum number $J^{PC}=1^{-}$ are consistent with the measured data and some other published results.

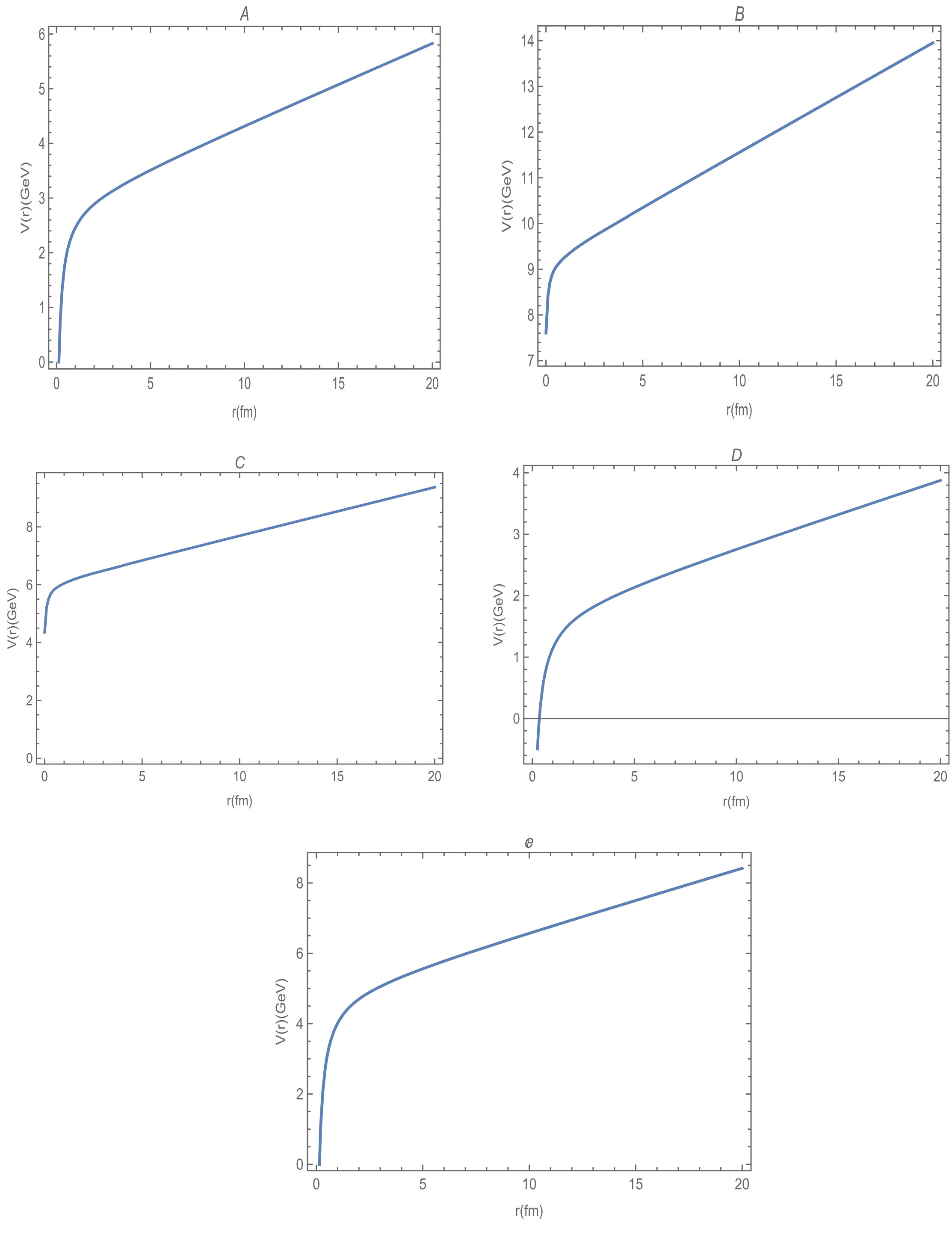

Figure 1. Potential energy curves for *S*-wave state with spin one of heavy and heavy-light mesons.

The bottomonium and bottom-charm meson masses are shown in tables 3 and 4, respectively. The bottomonium meson results matched well with experimental data [41, 42] and previously published works [49–51]. The masses of bottom-charm mesons matched also well with the previous theoretically-obtained masses [49, 50, 52].

The heavy-light mesons masses are shown in tables 5 and 6, respectively. The heavy-light meson results matched well with the measured data [41, 42] and previously research works [53, 54]. This research work produces a comparable result for the heavy and heavy-light meson masses spectra in comparison to the available experimental data and some other published values as mentioned in the literatures [48–54].

Using the potential parameters obtained in table 1, the potential energy curves were plotted for heavy mesons, as well as heavy-light mesons (See figure 1). In figure 1, (A) corresponds to $c\bar{c}$, (B) corresponds to $b\bar{b}$, (C) corresponds to $b\bar{c}$, (D) corresponds to charmed meson, and (e) corresponds to bottomonia meson.

Table 4. Bottom-charm mesons mass spectra. the units for all the masses are MeV.

State $J^{pc}=1^-$	Bottom-charm				
	Our	[49]	[50]	[52]	Exp.[41,42]
1S	6.462	6.2808	6.6225	6.321	6.375
2S	6.906	6.8523	6.9169	6.900	—
3S	7.256	7.1179	7.1858	7.338	—
4S	7.561	7.2626	7.431	7.714	—
5S	7.838	7.3500	7.654	8.054	—
6S	8.095	7.4068	7.858	8.368	—

Table 5. Charmed and charmed-strange mesons mass spectra. the units for all the masses are MeV.

State $J^{pc}=1^-$	D				Ds			
	our	[53]	[54]	Exp.	Our	[53]	[54]	Exp.[41,42]
1S	2.135	2.225	1.973	2.006	2.149	2.253	2.075	2.112
2S	2.738	2.724	2.732	—	2.730	2.726	2.781	2.708
3S	3.190	3.104	3.325	—	3.159	3.072	3.326	—
4S	3.589	—	—	—	3.528	—	—	—
5S	4.015	—	—	—	3.889	—	—	—
6S	4.532	—	—	—	4.308	—	—	—

Table 6. Bottomonia mesons mass spectra. the units for all the masses are MeV.

State $J^{pc}=1^-$	B				Bs			
	our	[53]	[54]	Exp.	Our	[53]	[54]	Exp.[41,42]
1S	5.251	5.330	5.313	5.324	5.194	5.429	5.403	5.415
2S	6.126	5.911	6.064	—	6.04 1	5.957	6.088	—
3S	6.752	—	6.640	—	6.623	—	6.612	—
4S	7.283	—	—	—	7.106	—	—	—
5S	7.762	—	—	—	7.539	—	—	—
6S	8.23	—	—	—	7.937	—	—	—

Figure 1 depicts the effect of Cornell potential on the different bound states. It is clear that the bound states exhibit mathematically similar behavior with slight variations.

4. Conclusion

This work focuses on the ground state ($l=0$ and $S=1$) quantum number $J^{PC}=1^{-}$for heavy ($c\bar{c}$, $b\bar{b}$, and $b\bar{c}$), as well as heavy-light (D, D_s, B, B_s) meson spectra. However, $c\bar{c}$ and $b\bar{b}$ have several excited states for different values of l and S. The Cornell potential works well for highly heavy quarks moving in a non-relativistic framework (static quarks). Our calculated meson spectra were used successfully to analyze the recent observed values on the PDG by embracing the global minimization approach in Mathematica software to deduce the free the potential parameters.

The potential shape and nominated method are affecting the accuracy of the mass spectra. Resulted behavior of the meson mass spectra based on Cornell potential is matched well with the measured energy levels. The validity of our method is verified with the calculated results by implying the outlined approximation technique. Due to the approximated method in solving Schrödinger equation, the considered potential is effective in case of lower states whereas, it lacks some additional mathematical treatments to exhibit reasonable values in case of higher states. Although, the success of our technique in determining the mass spectra with high accuracy for the heavy mesons, newly measured data for meson spectra are recommended on the near future to verify the reliability of that technique in determining the heavy-light meson mass spectra.

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Data availability statement

All data that support the findings of this study are included within the article

Declarations

Conflict of interest

The authors declare that no conflicts of interest or personal relationships have influenced this work.

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