EIKONAL APPROXIMATION TECHNIQUES IN ELASTIC SCATTERING AND PRODUCTION PROCESSES

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There is a considerable attractiveness in any approximation scheme for physical processes that while maintaining some hold on the nature of the approximations, size of the corrections, and significance to real physical phenomena also yields tractable, often analytic expressions for important measurable quantities. The subject of these lectures, the eikonal approximation, in elastic scattering and production processes, has precisely all these virtues and, despite its occasional faults, offers a very useful framework in which to discuss high energy phenomena. Indeed, the whole idea of an eikonal- or straight-line-approximation is generic to high energy or shortwave length physics. As we shall see quite explicitly herein it is when a particle begins moving with extremely large momentum that it makes a great deal of sense to describe, in a first approximation, its path of motion by a straight line. This reduction of a three dimensional motion to one dimensional motion is at the heart of the tractability of the scheme we shall discuss.

The plan of these lectures is to first go back to potential scattering and discuss in some detail the basic ideas of the eikonal technique. (It is my purpose to be frankly pedagogical in these talks and because of that the experts will find the start quite simpleminded, but perhaps they too will find something useful eventually.) After that I shall turn to the eikonal approximation in quantum field theory and discuss how the familiar two dimensional eikonal form emerges for elastic scattering. Finally, we shall discuss a really most interesting topic: inelastic processes and production of particles in an eikonal framework.

A short bibliographical note before we begin. There are innumerable references to the eikonal method in physics and we will refer, without elaborate apology, to a selected subset of them. The standard source of ideas on the eikonal approximation in non-relativistic quantum theory is the set of lectures by Glauber at the 1958 Boulder Summer School[1]. Further development of those ideas was given by Blankenbecler and Sugar[2] and their followers[3] and in a more modern context by Bjorken and his collaborators[4]. For a review of these older ideas I can recommend without hesitation the lectures I gave at the 1971 Boulder Summer School[5] at which time a rather different set of topics was emphasized. The subject of production processes in eikonal approximation has been most attractively pursued by Sugar[6] and his collaborators and we shall follow his lead in our own discussion.

1. POTENTIAL SCATTERING

As for so many ideas we have about modern physics, non-relativistic quantum mechanics provided the ground in which the eikonal approximation was introduced into modern physics[1]. There are at least two ways to state the basic idea and since both are illuminating we will consider both.

The basic problem that we want to address is the evaluation of the scattering matrix element

$$f(\vec{k}_f, \vec{k}_i) \tag{1}$$

for a particle of mass m to go from initial momentum $\vec{k}_i$ to final momentum $\vec{k}_f$ in the presence of a potential $V(\vec{r})$ which we take, for the present, to be energy independent. This matrix element is related to the differential cross section into a solid angle Ω about $\hat{k}_f$[7] by

$$\frac{d\sigma}{d\Omega_{\hat{k}_f}} = \left| f(\vec{k}_f, \vec{k}_i) \right|^2 \tag{2}$$

and is related to the usual T-matrix by

$$T(\vec{k}_f, \vec{k}_i) = -\frac{2\pi}{m} f(\vec{k}_f, \vec{k}_i) \tag{3}$$

This T-matrix element is given in terms of the full solution $\psi^{(+)}_{k_i}(\vec{r})$ to the Schrödinger equation with incoming plane waves, $\exp(i\,\vec{k}_i\cdot\vec{r})$, and outgoing spherical waves by the familiar

$$T(\vec{k}_f,\vec{k}_i)=\int d^3r\; e^{-i\vec{k}_f\cdot\vec{r}}\,V(\vec{r})\,\psi^{(+)}_{k_i}(\vec{r}). \tag{4}$$

It satisfies the standard integral equation

$$T(\vec{k}_f,\vec{k}_i)=\tilde{V}(\vec{k}_f-\vec{k}_i)+ \tag{5}$$

$$+\int\frac{d^3q}{(2\pi)^3}\,\tilde{V}(\vec{k}_f-\vec{q})\,\frac{1}{\frac{k^2-q^2}{2m}+i\varepsilon}\,T(\vec{q},\vec{k}_i),$$

where $k=|\vec{k}_i|=|\vec{k}_f|$, $q=|\vec{q}|$, and

$$\tilde{V}(\vec{\Delta})=\int d^3r\; e^{-i\vec{\Delta}\cdot\vec{r}}\,V(\vec{r}) \quad . \tag{6}$$

Our notation established, we are ready to consider large incident momenta k_i and think how we are to approximate (4) or (5). Of course, we need some criterion of "large" k_i to begin with. One will certainly agree that if k_i is large compared to the primary fourier components in the potential $\tilde{V}(q)$, then the potential will not severely disturb the motion of the projectile and we have the basis for an approximation which is not dependent on the details of $\tilde{V}$. The size of a typical q in $\tilde{V}(q)$ is the inverse of the range, a, of the potential. So let us agree that $k_i a \gg 1$. Furthermore let us imagine that the potential is "smooth" in momentum space (this essentially says $k_i a \gg 1$ for then many particle wave lengths will fit into any variation of the potential), and that its strength V_0 is small compared to the initial energy $E = k^2/2m$. Then, turning to (4) we may imagine approximating $\psi^{(+)}$ by an incident plane wave modulated by some function which takes into account the small disturbance due to V:

$$\psi^{(+)}_{k_i}(\vec{r})=\left(\exp\, i\vec{k}_i\cdot\vec{r}\right)B(\vec{r}). \tag{7}$$

The Schrödinger equation provides an exact equation for $B(\vec{r})$, of course, but we wish to imagine that because V is "smooth" the major variation in $\psi^{(+)}$ comes from the exponential and that spatial derivatives of B are small. One arrives in this way at an approximate equation for B

$$\frac{-2i\vec{k}_i\cdot\nabla_r B(\vec{r})}{2m}+V(\vec{r})\,B(\vec{r})=0, \tag{8}$$

whose solution is direct

$$B(\underset{\sim}{b} + \lambda \hat{k}_i) = \exp \frac{im}{k} \int_{-\infty}^{\lambda} d\tau \, V(\underset{\sim}{b} + \tau \hat{k}_i) \,, \tag{9}$$

choosing $B(-\infty) = 0$ and decomposing $\vec{r}$ into a piece along $\vec{k}_i$ and a piece, $\underset{\sim}{b}$, orthogonal to it. When (7) and (9) are placed in (4) we have an expression for T.

What is the key to our arriving at (9) ? Clearly once we dropped higher derivatives of $B(\vec{r})$, because of smoothness in V and thus in its spatial variation, we arrived at a <u>one dimensional Schrödinger equation along $\hat{k}_i$</u>. This is saying quite directly that propagation along the initial directions is essentially undisturbed by V, only the amplitude of the wave is modulated. This in its simplest form is the <u>eikonal</u> or <u>straight-line</u> or <u>semi-classical approximation</u>. It is valid when the projectile indeed propagates in essentially a straight line which will be true for large k and small scattering angles, $\cos\theta = \hat{k}_i . \hat{k}_f$, in the smooth potentials we have described. Corrections to the basic approximation of a modulated plane wave are elaborated upon in Ref.5.

Suppose, in fact, that the scattering angle is small so we may write $\vec{r} \cdot (\vec{k}_f - \vec{k}_i)$ as

$$\vec{r} \cdot (\vec{k}_f - \vec{k}_i) \approx \left[\underset{\sim}{b} + \frac{\lambda (\vec{k}_f + \vec{k}_i)}{|\vec{k}_f + \vec{k}_i|} \right] \cdot (\vec{k}_f - \vec{k}_i) \tag{10}$$

$$\approx \underset{\sim}{b} \cdot (\vec{k}_f - \vec{k}_i) \equiv \underset{\sim}{b} \cdot \underset{\sim}{\Delta} \,, \tag{11}$$

where $\vec{\Delta} = \vec{k}_f - \vec{k}_i$ is the momentum transfer to V in the scattering. One may now cast the expression for T into

$$T(k,\Delta) = \int d^2b \int_{-\infty}^{+\infty} d\lambda \left[V(\underset{\sim}{b} + \hat{k}\lambda) \exp \frac{im}{k} \int_{-\infty}^{\lambda} d\tau \, V(\underset{\sim}{b} + \tau \hat{k}) \right] e^{-i\underset{\sim}{\Delta} \cdot \underset{\sim}{b}} \tag{12}$$

$$= -\frac{ik}{m} \int d^2b \, e^{-i \underset{\sim}{\Delta} \cdot \underset{\sim}{b}} \left[\exp \frac{im}{k} \int_{-\infty}^{+\infty} d\tau \, V(\underset{\sim}{b} + \tau \hat{k}) - 1 \right] , \tag{13}$$

recognizing the total derivative in the brackets of (12). This last form is what is generally referred to as the eikonal approximation, and the phase $\chi(k,\underset{\sim}{b}) = \frac{m}{k} \int_{-\infty}^{\infty} d\tau \, V(\underset{\sim}{b} + \tau \hat{k})$ is called the eikonal phase. It is the phase up by the projectile in traversing the straight line path along $\hat{k}$ during the scattering process[8].

This seems as good a time as later to remark on the result of our modulated wave approximation. First, suppose we hold Δ fixed and let $k \to \infty$, then the only term in (13) which survives is

$$\int d^2b \; d\lambda \; e^{-i\Delta\cdot \underset{\sim}{b}} \, V(\underset{\sim}{b} + \lambda \hat{k}) \tag{14}$$

which is the Born approximation. This is the correct result for potentials like $V(\vec{r})$ which do not depend on k. However, there is a value in (13) which is not possessed by the Born approximation, namely the eikonal approximation satisfies unitarity in the direct (k) channel. To see this, write

$$T(k,\Delta) = -\frac{ik}{m}\int d^2b \; e^{-i\underset{\sim}{\Delta}\cdot \underset{\sim}{b}} \left[S(\underset{\sim}{b}, k) - 1 \right] \tag{15}$$

and note that since $S = \exp(i\chi)$, unitarity will follow in the high energy limit. To be more precise, let the potential depend only on $|\vec{r}|$, then χ and S depend only on $|\underset{\sim}{b}| = b$. We may do the angular integration in (15) to find

$$T(k,\Delta) = -\frac{2\pi i k}{m}\int_0^\infty b \, db \, J_0(\Delta b) \left[S(b,k) - 1 \right] \tag{16}$$

The unitarity relation in the high energy limit becomes diagonal, to order 1/k, when transformed with $J_0(b\Delta)$, and reads

$$\mathrm{Im}\, T(\underset{\sim}{b}, k) \propto |T(\underset{\sim}{b}, k)|^2 + O(1/k) , \tag{17}$$

which means

$$S(b,k)\, S^{+}(b,k) = 1 + O(1/k), \tag{18}$$

which is true for our solution.

This would seem to provide a very general framework for approximations to elastic scattering which satisfy direct channel unitarity. Indeed, it does. Any choice of V(even energy dependent) leads to a unitary S in the elastic scattering hilbert space. And there is the rub. For a large variety of scattering processes: $\pi^{\pm}p$, $K^{\pm}p$, pp, and $\bar{p}p$, the ratio of $\sigma_{elastic}/\ \sigma_{total}$ at even moderate laboratory momenta is of the order of 1/7 or so[9]. This means there is a lot of inelasticity and we do not want to satisfy unitarity in the elastic sector only. We shall suggest a solution to this later on.

Next suppose the potential V is proportional to the momentum k. Then χ is independent of k and

$$\lim_{\substack{k\to\infty \\ \Delta \text{ fixed}}} T_{k^1}(k,\Delta) = ik\, F(\Delta) \tag{19}$$

from (13). That is amusing since $\sigma_{total} \propto \frac{1}{k}\mathrm{Im}\ T(k,0)$ is then constant. And what if $V \propto k^{1+\varepsilon}$, $\varepsilon > 0$? Then the exponential in the eikonal form (13) oscillates like crazy unless $b \approx \log k$, and one finds

$$\lim_{\substack{k\to\infty \\ \Delta \text{ fixed}}} T_{k^{1+\varepsilon}}(k,\Delta) \sim k\,(\log k)^2 , \tag{20}$$

which is not accidentally reminiscent of the Froissart bound[10]. This shows us how unitarity plays its important role.

Next let us turn to a more formal deduction of the eikonal approximation (13) - one which stresses the straight line approximation. We begin with the Schrödinger integral equation (5) in operator form

$$T = V + V\, G_0(E)\, T , \tag{21}$$

where $G_0(E)^{-1} = E - \frac{p^2}{2m} + i\varepsilon$, $\vec{p} = -i\nabla_r$ and $E = \frac{k^2}{2m}$.

The solution to this is

$$T = V + VGV = VG\,G_0^{-1} = G_0^{-1}\,GV , \tag{22}$$

where

$$G(E)^{-1} = E - \frac{p^2}{2m} - V + i\varepsilon . \tag{23}$$

We want to approximate G, the full Green function, and evaluate T as best we can using(22).

We imagine, therefore, that the projectile is incident upon the potential with large momentum k_i. Under the smoothness assumptions we made earlier it behoves us to guess that during the scattering process the values of momentum encountered do not appreciably deviate from k_i. Thus we write $G(E)^{-1}$ in a form which emphasizes the closeness of p (albeit an operator) to k_i (a c-number)

$$G(E)^{-1} = E - \frac{p^2}{2m} - V + \frac{(\vec{p}-\vec{k}_i)^2}{2m} - \frac{(\vec{p}-\vec{k}_i)^2}{2m} \tag{24}$$

$$= \frac{k^2 + k_i^2 - 2\vec{p}\,\vec{k}_i}{2m} - V - \frac{(\vec{p}-\vec{k}_i)^2}{2m} \tag{25}$$

$$= G_i(E)^{-1} - H_i \tag{26}$$

taking
$$H_i = \frac{(\vec{p}-\vec{k}_i)^2}{2m} \tag{27}$$

and for the moment staying off the energy shell $k_i^2 = k^2$. The idea is to treat deviations from k_i, as embodied in H_i, as perturbations, although H_i is a fairly singular operator, and to perform a perturbation expansion of G beginning with

$$G = G_i + G_i H_i G = G_i + G H_i G_i \tag{28}$$

The first term is clearly G_i which upon examination of its structure as given in (25) represents propagation of the particle in one dimension along k_i. This should be no surprise, since we constructed it to do this, but the connection with the equation (8) for the amplitude modulating functions should be noted.

We could just as well have made our expansion about k_f since the smoothness of the potential assures us that p will not significantly deviate from it either. So we would write

$$G(E)^{-1} = G_f^{-1} - H_f \tag{29}$$

where

$$G_f^{-1} = \frac{(k^2 + k_f^2 - 2\vec{p}\cdot\vec{k}_f)}{2m} - V(\vec{r}) \tag{30}$$

and

$$H_f = \frac{(\vec{p} - \vec{k}_f)^2}{2m} . \tag{31}$$

Of course,

$$G = G_f + G_f H_f G = G_f + G H_f G_f , \tag{32}$$

which together with (28) yields

$$G = \tfrac{1}{2}(G_i + G_f) + \tfrac{1}{2} G_f (H_i + H_f) G_i + G_f H_f G H_i G_i , \tag{33}$$

suggesting as a first approximation to G the eikonal Green function

$$G_{Eikonal} = \tfrac{1}{2}(G_i + G_f) , \tag{34}$$

and corresponding T-matrix

$$T_{Eikonal} = \tfrac{1}{2}(T_{Ei} + T_{Ef})$$

and

$$T_{E(i\ \mathrm{or}\ f)} = V + V\, G_{(i\ \mathrm{or}\ f)} V . \tag{35}$$

We will evaluate the incoming eikonal T-matrix, leaving T_{Ef} for the diligent student. First write

$$T_{Ei} = V \left(1 + \frac{1}{\frac{k^2 + k_i^2 - 2\vec{p}\cdot\vec{k}_i}{2m} - V + i\varepsilon} V\right) \tag{36}$$

$$= V\, G_i \frac{k_i^2 + k^2 - 2\vec{p}\cdot\vec{k}_i}{2m} . \tag{37}$$

We desire the matrix element

$$T_{Ei}(\vec{k}_f, \vec{k}_i) = \langle \vec{k}_f | T_{Ei} | \vec{k}_i \rangle \tag{38}$$

$$= \lim_{k_i^2 \to k^2} \langle \vec{k}_f \,|\, V G_i \,|\, \vec{k}_i \rangle \left(\frac{k^2 - k_i^2}{2m} \right) \tag{39}$$

$$= \lim_{k_i^2 \to k^2} \int d^3r \, e^{-i\vec{k}_f \cdot \vec{r}} \, V(\vec{r}) \langle \vec{r} \,|\, G_i \,|\, \vec{k}_i \rangle \left(\frac{k^2 - k_i^2}{2m} \right) , \tag{40}$$

which should be very familiar in its last form. Indeed we expect that $\lim_{k_i^2 \to k^2} \frac{(k^2 - k_i^2)}{2m} \langle \vec{r} \,|\, G_i \,|\, \vec{k}_i \rangle = (\exp i \vec{k}_i \cdot \vec{r}) \, B(\vec{r})$, and we won't be disappointed. There are many ways to see this. Perhaps the most straightforward is to note that the differential equation for $\langle \vec{r} \,|\, G_i \,|\, \vec{k}_i \rangle$ is

$$\left[\frac{k_i^2 + k^2 - 2\vec{p} \cdot \vec{k}_i}{2m} - V(\vec{r}) \right] \langle \vec{r} \,|\, G_i \,|\, \vec{k}_i \rangle = e^{i\vec{k}_i \cdot \vec{r}} , \tag{41}$$

which has the solution

$$\langle \vec{r} \,|\, G_i \,|\, \vec{k}_i \rangle = \exp i \vec{k}_i \cdot \vec{r} \;\; i \int_0^\infty dt \, \times$$

$$\times \exp\left\{ \frac{i(k^2 - k_i^2)t}{2m} - i \int_0^t d\tau \, V\left(\vec{r} - \vec{k}_i \frac{\tau}{m}\right) \right\} \tag{42}$$

in convenient parametric form. The on energy shell T-matrix is

$$T_{Ei}(\vec{k}_f, \vec{k}_i) = \int d^3r \, e^{-i\vec{\Delta} \cdot \vec{r}} \, V(\vec{r}) \exp\left[-i \int_0^\infty d\tau \, V(\vec{r} - \vec{k}_i \, \tau/m) \right] , \tag{43}$$

which is just what we expect.

The eikonal T-matrix referring to k_f is similarly found to be

$$T_{E_f}(\vec{k}_f, \vec{k}_i) = \int d^3r \, e^{-i\vec{\Delta} \cdot \vec{r}} \, V(\vec{r}) \exp\left[-i \int_0^\infty d\tau \, V(\vec{r} + \vec{k}_f \, \tau/m) \right] . \tag{44}$$

As one can discover in a straight forward calculation, each of T_{Ei} and T_{Ef} reduce to the expression (13) we have called the eikonal form, when the scattering angle between $\hat{k}_i$ and $\hat{k}_f$ is small.

Having commented on some of the implications of the eikonal approximations before, let me close this section with a few remarks.

(1) One can clearly perform the expansion of G in power of a perturbation H about any linear combination $ak_i + bk_f$ of directions which are close to the initial or final path. The literature contains several discussions of the virtues of various choices[8] especially the average momentum[3]

$$\vec{k} = \frac{1}{2}(\vec{k}_i + \vec{k}_f) \tag{45}$$

whose main claim a priori seem to be connected with being able to satisfy time reversal invariance, but which appears (for reasons mysterious to me) to possess some superiority on numerical and other grounds.

(2) Despite the fact that (13) satisfies unitarity in the elastic sector it may prove useful in nuclear physics problems where the elastic channel is important or the form may be appropriate for a few channel problem by replacing the real potential by some sort of optical potential

$$V(\vec{r}) = V_1 + iV_2$$

using the V_2 term both to represent absorption out of the elastic channel and to define $_{\text{inelastic}}$.

(3) The correction term indicated in (33) to the basic eikonal approximation may be evaluated term by term. The first term

$$\frac{1}{2} G_f \,(H_i + H_f)\, G_i \tag{46}$$

leads to the so-called Saxon-Schiff correction[2,3,5] to the eikonal approximation. If the fourier transform of $V(r)$ falls off as a power in q momentum space, then simple estimates[2] indicate that the eikonal plus first correction is an excellent representation of $T(k_f, k_i)$ over all angles. This says physically that for such potentials the projectile chooses to make its momentum transfer $\vec{\Delta}$ in one or two steps (no H or one H) rather than multiple scattering. For a gaussian potential; this is not the case. Then a separate analysis[11] reveals that the basic eikonal is most important.

2. FIELD THEORY

As we well know field theory is much richer than potential scattering because of the ability to produce particles. In this section we will concentrate on the extraction of the high energy elastic scattering amplitude in a relativistic framework and in the next section we will turn to consideration of how particle production may be treated.

The basic thing to realize is that the field theory may be reduced to a discussion of potential scattering and that our experience with the latter, now enhanced by our eikonal knowledge, will serve as a guide to physical approximations.

So, in what way is field theory just potential scattering? The answer[12] is that if we know the amplitude for a particle of momentum p_1 [four momentum = (p_t, p_x, p_y, p_z)] to go to momentum p_1' in the presence of a c-number external potential $A(x)$, then we may answer any question involving the interaction of those same particles and the quanta associated with the potential. Essentially the amplitude $T_A(p_1', p_1)$ acts as a generating function[13] for emission and absorption of such quanta.

Let us illuminate these remarks by considering the quantum electrodynamics of a dirac particle moving in an external c-number electromagnetic potential $A_\alpha(x)$. Suppose we have calculated $T_A(p_1', p_1)$ to second order in A. This is given by

$$\bar{u}(p_1')\, e\gamma\, \tilde{A}(q_1')\, \frac{1}{m - \gamma\cdot(p_1' + q_1')}\, e\gamma\cdot\tilde{A}(q_1)\, u(p_1) \tag{47}$$

or in configuration space

$$T_A(p_1', p_1) = \int d^4z\, d^4y\, \bar{u}(p_1')\, e^{-ip_1'\cdot z}\, e\gamma\cdot A(z) S_+(z-y)\, \times$$
$$\times\, e\gamma\cdot A(y)\, e^{+ip_1\cdot y}\, u(p_1)\,, \tag{48}$$

where u and $\bar{u}$ are the standard dirac spinors, γ_α the usual 4x4 matrices, $\tilde{A}(q)$ the fourier transform of $A(x)$, and $S_+(z)$ the standard causal propagator for a dirac particle.

Now knowing this suppose we ask: what is the probability amplitude for the particle to scatter in the potential $A_\alpha(x)$, to first order, and in the same act radiate a photon of momentum k with spin wave function $\epsilon_\alpha(k)$? The answer, as we well know,

$$T_A(p_1 \to p_1' + k) = e^2 \epsilon_\alpha^*(k) \int d^4z\, d^4y \{ \bar{u}(p_1') e^{-i(p_1'+k)\cdot z} \gamma_\alpha \times$$

$$\times S_+(z-y)\, \gamma\cdot A(y)\, e^{ip_1\cdot y} u(p_1) + \bar{u}(p_1') e^{-ip_1'\cdot z} \times \tag{49}$$

$$\times \gamma\cdot A(z)\, S_+(z-y)\, \gamma_\alpha\, e^{i(p_1-k)y} u(p_1) \},$$

representing emission after and before the scattering on A. It is the relation between these two answers that we seek. If we replace in (48) $A_\alpha(z)$ by $\epsilon_\alpha^*(k)\, e^{-ik\cdot z}$, we obviously arrive at the first term of (49), and similarly the second term comes from replacing A(y) by $\epsilon_x^\alpha(k) e^{-ik\cdot y}$.

Both of these things can be done at a stroke by taking a derivative of (48) with respect to $A_\alpha(x)$ and in each of the two terms inserting the appropriate photon wave function $a_\alpha^*(k,x) = \epsilon_\alpha^*(k) e^{-ik\cdot x}$, then integrating over all possible space-time points x where the insertion might have occurred. That is

$$T_A(p_1 \to p_1' + k) = \int d^4x \frac{\delta T_A(p_1 \to p_1')}{\delta A_\alpha(x)} a_\alpha^*(k,x), \tag{50}$$

which it is easy enough to see is generally true. The funny δ means literally: replace $A_\alpha(z)$ under an integral by $\delta^4(x-z)$; that is

$$\delta A_\alpha(z) / \delta A_\beta(x) = g_{\alpha\beta}\, \delta^4(z-x). \tag{51}$$

Another question one may ask of (48) is what is the amplitude, to second order in e, for the particle to emit a photon and then reabsorb it? We must open up the potential at $A_\alpha(z)$, say, propagate a photon from z to y via the causal propagator $D_+(z-y)$ and then allow the particle to reabsorb the photon at y. This yields

$$T_{\substack{\text{No} \\ \text{Emission}}}(p_1 \to p_1') = \frac{1}{2} \int d^4w\, d^4z \frac{\delta}{\delta A_\alpha(w)} D_+(w-z) \frac{\delta}{\delta A_\alpha(z)} T_A(p_1 \to p_1'), \tag{52}$$

the 1/2 comes from the possibility of the photon's beginning at either y or z in $T_A(p_1 \to p_1')$. The general answer for $T(p_1 \to p_1')$ no emission is gotten by some simple counting

$$T_{\substack{No \\ \text{Emission}}}(p_1 \to p_1') =$$

$$\left[\exp -\frac{1}{2}\int d^4z\, d^4w \frac{\delta}{\delta A_\alpha(z)} D_+(z-w) \frac{\delta}{\delta A_\alpha(w)}\right] T_A(p_1 \to p_1'), \tag{53}$$

and after taking all derivatives, set $A_\alpha = 0$.

Interaction between particles which is mediated by the quanta of A_α, namely photons, comes from two particles moving in two potentials A_1 and A_2 and from evaluating the probability amplitude to open up an A_1 spot and propagate a photon, via D_+, over to an opened A_2 spot. Thus the amplitude for electron scattering $T(p_1 + p_2 \to p_1' + p_2')$ in the absence of corrections to the photon propagator D_+ coming from internal electrons is

$$T(p_1 + p_2 \to p_1' + p_2') =$$

$$\left[\exp -\int d^4w\, d^4z \frac{\delta}{\delta A_{1\alpha}(w)} D_+(w-z) \frac{\delta}{\delta A_{2\alpha}(z)}\right] \times \tag{54}$$

$$\times \left[T_{A_1}(p_1 \to p_1')\, T_{A_2}(p_2 \to p_2') - T_{A_1}(p_1 \to p_2')\, T_{A_2}(p_2 \to p_1')\right]_{A_1 = A_2 = 0}.$$

As promised we now see that a large variety of field theoretic phenomena follow from potential scattering probability amplitudes. Indeed, if one takes (54) and performs exactly the same kind of eikonal approximation on T_{A_1} and T_{A_2} as we have done in the previous section, then it follows[3,5] after some straightforward computation that exactly the form (13) emerges. The "potential" is just that of the photon exchange which yields the Born approximation. Rather than present this derivation here, we will follow a somewhat different route which will lead more smoothly into the discussion of production. All the principles of the direct calculation from (54)[5] are illuminated.

Since we are interested in the scattering of a particle in an external potential let us consider some particle currents $J_\alpha(x)$ interacting with the c-number potential $A_\alpha(x)$. The interaction Lagrangian

$$\mathcal{L}_I(x) = J_\alpha(x) A_\alpha(x) \tag{55}$$

leads via the standard rules of field theory to the S-matrix element to go from some state i to another f

$$S_{fi} = \langle f \,|\, T(\exp - i\int d^4x\, J_\alpha(x) A_\alpha(x)) \,|\, i \rangle \tag{56}$$

We want to evaluate S_{fi} when the states $|i\rangle$ and $|f\rangle$ are moving very fast in, say, the three direction [14]. So we take a standard state, say a rest state for a single particle, and boost it along the 3-axis very fast, so

$$|i\rangle = \exp - i K_3 \theta \;\; |i_0\rangle,$$

where θ is a boost angle and K_3, the generator of 3-boosts. A particle at rest is taken by this operation from $p_0 = (m, 0, 0, 0)$ to

$$e^{-iK_3\theta}\, p_0 = (m\cosh\theta, 0, 0, m\sinh\theta). \tag{57}$$

Under such a transformation it is convenient not to consider separately p_t and p_z, but the combinations

$$p_\pm = p_t \pm p_z \tag{58}$$

for

$$e^{-iK_3\theta}\, p_\pm = e^{\pm\theta}\, p_\pm \tag{59}$$

since p_+ gets large and p_- small when θ is large. How does $J_\alpha(x)$ transform? Of course, its components in the α or 1 direction are untouched but

$$e^{iK_3\theta}\, J_\pm(x_+, x_-, \underset{\sim}{x})\, e^{-iK_3\theta} = e^{\pm\theta}\, J_\pm(e^{-\theta}x_+, e^{\theta}x_-, \underset{\sim}{x}), \tag{60}$$

where $J_\pm = J_t \pm J_3$ and $x_\pm = x^0 \pm x^3$ while $\underset{\sim}{x}$ is the two vector (x^1, x^2).

The matrix element S_{fi} becomes

$$S_{fi} = \langle f_0 | \, e^{i\theta K_3} \, T\left(\exp -i \int \frac{dx_+ dx_- d^2x}{2} \left[\frac{J_+ A_- + J_- A_+}{2} - \underset{\sim}{J} \cdot \underset{\sim}{A} \right] \right) \times$$
$$\times \, e^{-iK_3\theta} \, | i_0 \rangle \tag{61}$$

$$= \langle f_0 | \, T\left(\exp\left[-\frac{i}{4} \int dx_+ dx_- d^2x \, e^{\theta} J_+ (e^{-\theta} x_+, e^{\theta} x_-, \underset{\sim}{x}) A_-(x_+, x_-, \underset{\sim}{x}) \right]\right) | i_0 \rangle$$
$$+ \, O(e^{-\theta}) , \tag{62}$$

or on changing integration variables to $\tilde{x}_- = e^{+\theta} x_-$,

$$S_{fi} = \langle f_0 | \, T\left(\exp -\frac{i}{4} \int dx_+ d\tilde{x}_- d^2x \; J_+(0, \tilde{x}_-, \underset{\sim}{x}) A_-(x_+, 0, \underset{\sim}{x})\right) | i_0 \rangle$$
$$+ \, O(e^{-\theta}) \tag{63}$$

which if we define

$$\mathcal{J}(\underset{\sim}{x}) = \frac{1}{2} \int dx_- \; J_+(0, x_-, \underset{\sim}{x}) \tag{64}$$

and

$$\mathcal{A}(\underset{\sim}{x}) = \frac{1}{2} \int dx_+ \; A_-(x_+, 0, \underset{\sim}{x}) \tag{65}$$

takes on the two dimensional form

$$S_{fi} = \langle f_0 | \, T\left(\exp -i \int d^2x \, \mathcal{J}(\underset{\sim}{x}) \, \mathcal{A}(\underset{\sim}{x})\right) | i_0 \rangle + O(e^{-\theta}) , \tag{66}$$

however, the time ordering which remains must be discussed before we are so excited about (66).

To get rid of the time ordering, which we must do in order for S_{fi} to have the true exponential form exhibited by the eikonal approximation, it is necessary to make some <u>ansatz</u> about the component J_+ of the particle source operator. The only "time" left to be ordered is the dependence of J_+ on x_-, so if we make J_+ a c-number with respect to its dependence on x_- we may remove the T operation and have essentionally an eikonal or more precisely an exponential form for S_{fi}. This requires that $J_+(x_+, x_-, \underset{\sim}{x})$ and $J_+(y_+, y_-, \underset{\sim}{y})$ commute at $x_- = y_-$

$$\left[J_+(x_+, x_-, \underset{\sim}{x}),\ J_+(y_+, y_-, \underset{\sim}{y}) \right] \Big|_{x_- = y_-} = 0. \tag{67}$$

Now $J_\alpha(x)$ commutes with $J_\beta(y)$ for $(x-y)^2 < 0$, already if it is a local operator. So since

$$(x-y)^2 = (x-y)_+ (x-y)_- - (\underset{\sim}{x} - \underset{\sim}{y})^2 , \tag{68}$$

we see that automatically (67) is satisfied if $(\underset{\sim}{x}-\underset{\sim}{y})^2$ is not zero. The additional assumption[15] is that <u>if x-y is lightlike $[J_+(x), J_+(y)] = 0.$</u>

If this is true, then

$$S_{fi} = \langle f \,|\, \exp -i\int d^2x\, \mathcal{J}(\underset{\sim}{x})\, \mathcal{A}(\underset{\sim}{x}) \,|\, i \rangle , \tag{69}$$

which is essentially the eikonal answer.

It is not obvious that all currents $J_\alpha(x)$ are such that their plus components commute on the light cone. Indeed, as Weinberg[15] has shown, this places strong dynamical constraints on the matrix elements of J which are tantamount to generalized sum rules of the Drell-Hearn-Gerasimov variety. One implication of this is that if we return to our problem of a dirac particle scattering in an external potential and do not add radiative corrections of a self-energy or vertex correction variety, then the particle must have no anomalous magnetic moment and the eikonal technique "works" only if g = 2. This is consistent with more direct approaches to this question.

In the early work on eikonal approximations the c-number nature of J_+ was arranged in a very simple fashion: it was taken to be a c-number. The physical meaning of this is straightforward. If the particle which is being scattered neither produces other particles nor any quanta of A_α (that is, there are no radiative corrections on the particle line) nor does it change its co-ordinates such as a spin, helicity or, if it is taken composite, its internal wave function, then the transition operator causing the scattering is acting in a one dimensional space of particle variables and it is essentially a c-number. The assumption that all these conditions be met is generic to older treatments of the eikonal approximation. Our exercise above shows us that a more general class of scattering processes may take the exponential form of the eikonal. Clearly if J_{α} (x) is an operator still, the particle scattering in the potential can make many intermediate states of varying character before it emerges in state $|f\rangle$.

The appearance of the commutator of the particle source, J, on the light-cone might at first seem strange. However, just this kind of quantity is to be expected in relativistic theories on the following heuristic grounds. When we deal with very fast particles, say $p_3 \to \infty$, on the mass shell, $p^2 = m^2$, then we are specifying two out of four components of p; namely p_3 and $p_t = p_3 + (\underset{\sim}{p}^2 + m^2) / 2p_3$ (to leading order), where $\underset{\sim}{p} = (p_x, p_y)$. In co-ordinate space we are thus providing information on the variables conjugate to p_t and p_3; namely t and x_3.

Indeed, we are constraining $x_3 \approx \pm t$ for forward or backward going particles respectively. This places us very near the light cone in a space time diagram and tells us that for $p_3 \to \infty$, the dynamics of scattering is gathering information from the whole light cone and not just the canonical space-like surface of equal t. To put it another way, the "natural" variables for $p_3 \to \infty$ processes would appear to be $x_t \pm x_3$ and $\underset{\sim}{x}$, not separately x_t and x_3. Of course, any description must be equivalent; it is just that one's good sense suggests it will be simpler in these light cone variables.

One more word before going on to production. This same heuristic argument tells us why we keep finding two dimensional integrals. It's simple; we have given p_3 and p_t, thus the dynamics lies in the two dimensional subspace we have said nothing about.

3. PRODUCTION PROCESSES

In view of some of the fairly drastic approximations which are going to be made in this section, I feel it may be worthwhile to recount a few of the salient features of data on production processes which, thanks to the wide attention directed in that direction for the past several years, is now available in useful form. First, as we have noted before, the study of production or inelastic reactions is bound to be important for high energy physics because experimentally even at AGS or CERN - PS or Serpukhov energies, $p_{lab} \simeq$ 30-70 GeV/c, the total cross section is on the order of 7 to 10 times

$\sigma_{elastic}$. It is unlikely that we will understand the latter or quasi-two body processes taken alone. Second, σ_{total} appears to be extraordinarily constant over enormous ranges of incident energy. Strictly speaking, this is known for pp scattering from $p_{lab} \approx 25$ GeV/c to $p_{lab} \approx 1500$ GeV/c as measured at the CERN-ISR. For Kp and πp scatterings the detailed situation is more fluid. Third, the average transverse momentum, which we have called $\underset{\sim}{p}$, seems to be quite small even at the highest available energies. The distribution in $|\underset{\sim}{p}| = p_T$ of detected particles tends to be

$$\frac{dN}{dp_T^2} \sim exp(-ap_T^2) \quad or \tag{70}$$

$$\sim exp(-bp_T) \ , \tag{71}$$

with $a \approx 3$ or 4 $(GeV/c)^{-2}$ or $b \approx 6$ $(GeV/c)^{-1}$. So most hadron physics is within $0 \leq p_T \leq 0.5$ GeV/c. Fourth, and this is most important for what follows, the distribution in momentum of particles detected at high energies is quite different for particles which are definitely produced (e.g., π or K or $\bar{p}$ in pp collisions) from the distribution of particles which can "come through", e.g. protons in pp collisions. The effect seen is that for "through going" particles there is a pronounced maximum for very large or very small longitudinal momenta when measured in the center of mass. That is, the beam or the target particles go through. The distribution of produced particles, on the other hand, is largest at small center of mass p_{long}. This phenomenon, which is very distinct in the ISR data at $p_{lab} \approx 1500$ GeV/c is popularly known as the leading particle effect.

The rest of this section will be devoted to a model which is eikonalistic and attempts to incorporate many of the features just described[16]. The particular type of process which I have in mind will be Nucleon + Nucleon $\rightarrow$ Nucleon + Nucleon + some produced things (π or K or what you like). Taking a strong hint from the data we treat the nucleons as leading particles and treat their co-ordinates as c-numbers. That is, the nucleons are treated as through going objects whose variables, if they are altered at all, are not altered appreciably. This c-number nature of the nucleons is, of course, precisely the trick that allows us to eikonalize this process.

Since there are always 2 nucleons in both the initial and final states let us label the amplitude for n pions + 2N $\rightarrow$ m pions + 2N as T_{mn}, and let us agree to call all produced particles pions[17]. We know from earlier work the T_{oo} (elastic scattering) takes the form

$$T_{oo}(s,t) = is\int d^2B\, e^{i\underset{\sim}{Q}\cdot\underset{\sim}{B}}\; T_{oo}(s,\underset{\sim}{B}), \tag{72}$$

where

$$T_{oo}(s,B) = e^{i\chi(s,B)} - 1 , \tag{73}$$

and s is the usual square of the incoming c.m. energy, and $t = -|\underset{\sim}{Q}|^2$ is the four momentum transferred. This suggests that we operate in a space where the nucleon co-ordinates s and $\underset{\sim}{B}$ are specified, for there we will have a good chance to construct a set of T_{mn} which are unitary. (Recall that unitarity was a nice feature of eikonal approximations.)

The nucleons, then, will be treated as a source, a <u>c-number source</u>, for pions which will be parameterized by the co-ordinates s and $\underset{\sim}{B}$. The leading particles carry off a large fraction of the initial energy, and we will imagine that the pions can move freely in phase space with no special constraints on them due to energy momentum conservation. That is to say, since the pions come out with small p_{long}, in the c.m., and if their number is not large, as we will shortly impose, then energy momentum constraints are essentially negligible on them. Our problem then is to find the S-matrix which comes from a c-number source $\rho(s,\underset{\sim}{B};x)$ which can emit and absorb pions.

A digression on convenient notation before we solve this problem. It is useful to use instead of s and p_{long}, the dimensionless variable <u>rapidity</u>, commonly called y. The rapidity of a particle of mass m, momentum ($\underset{\sim}{p}_T$, p_{long}) is defined to be

$$y = \frac{1}{2}\log\left[\frac{E+p_{long}}{E-p_{long}}\right], \tag{74}$$

and is essentially

$$\tanh y = v_{long}/c , \tag{75}$$

or the angle of "rotation" in the time-z plane to produce a particle of momentum p_{long}.

The incident nucleons have center of mass rapidity $\pm\frac{y_o}{2}$ for the beam and target respectively, where $y_o = \log s$ to an excellent approximation. The values of y_o in the real world are not overwhelming: $y \approx 4$ at the AGS, $y_o \approx 6$ at NAL, $y_o \approx 8$ at the ISR, and at the "planned" ISABELLE 200 GeV/c colliding beam facility $y_o \approx 11.5$. The real advantage of rapidity is that invariant phase space is simple

$$\frac{d^3p}{E} = dy\, d^2p_T \quad . \tag{76}$$

Usually, energy momentum conservation is complicated in terms of y, but we have just agreed not to worry about this, so we are spared that misery.

Now we are ready to calculate T_{mn}. We want to determine the pion field $\phi(x)$ in the presence of the c-number source $\rho(y_o, \underset{\sim}{B}; x)$ [18]. To do this we must solve the equation

$$[m^2 + \partial^2]\,\phi(x) = \rho(y_o, \underset{\sim}{B}; x). \tag{77}$$

The solution, of course, is

$$\phi(x) = \int d^4y\, D(x-y)\, \rho(y), \tag{78}$$

where

$$(m^2 + \partial_x^2)\, D(x) = \delta^4(x), \tag{79}$$

and we have temporarily dropped the y_o, $\underset{\sim}{B}$ parameters.

Which function D(x) we want is dictated as usual by the boundary conditions of the problem.

We wish to express the out field $\phi_{out}(x)$ for departing pions in terms of $\phi_{in}(x)$

$$\phi_{out}(x) = \phi_{in}(x) + \int d^4z\, \hat{D}(x-z)\, \rho(z), \tag{80}$$

$$(\partial^2 + m^2)\,\phi_{in}(x) = (\partial^2 + m^2)\,\phi_{out}(x) = 0, \tag{81}$$

so

$$(\partial^2 + m^2)\,\hat{D} = 0. \tag{82}$$

Since

$$\phi(x) = \phi_{in}(x) + \int d^4z\; D_R(x-z)\,\rho(z), \tag{83}$$

and

$$\phi(x) = \phi_{out}(x) + \int d^4z\; D_A(x-z)\,\rho(z), \tag{84}$$

where D_R and D_A are the retarded and advanced Green functions satisfying (79), we have

$$\hat{D}(z) = D_R(z) - D_A(z). \tag{85}$$

The integral representation of this difference Green function is

$$\hat{D}(z) = \int \frac{d^4q}{2(2\pi)^3}\, e^{iq\cdot z}\, \delta(q^2 - m^2). \tag{86}$$

It is useful, then, to decompose ϕ_{out} and ϕ_{in} into creation and annihilation operators in momentum space

$$\phi_{in}(x) = \int \frac{d^4q}{2(2\pi)^3}\, \delta(m^2 - q^2)\left[a_{in}(y, q_T)\, e^{iq\cdot x} + a^+_{in}(y, q_T)\, e^{-iq\cdot x}\right], \tag{87}$$

and

$$\phi_{out}(x) = \int \frac{d^4q}{2(2\pi)^3}\, \delta(m^2 - q^2)\left[a_{out}(y, q_T)\, e^{iq\cdot x} + a^+_{out}(y, q_T)\, e^{-iq\cdot x}\right], \tag{88}$$

where we remember that

$$\int \frac{d^4q}{2(2\pi)^3}\, \delta(m^2 - q^2) = \int \frac{d^3q}{2q_0\, 2(2\pi)^3} \tag{89}$$

$$= \int \frac{d^2 q_T \, dy}{4(2\pi)^3} \quad . \tag{90}$$

To guarantee the proper equal time commutation relations for $\phi_{in}(x)$ and $\phi_{out}(x)$ we take

$$[a_{in}(y, q_T), a_{in}^+(z, \underset{\sim}{k}_T)] = 4(2\pi)^3 \delta(y-z) \delta^2(q_T - \underset{\sim}{k}_T) , \tag{91}$$

and

$$[a_{out}(y, q_T), a_{out}^+(z, \underset{\sim}{k}_T)] = 4(2\pi)^3 \delta(y-z) \delta^2(q_T - \underset{\sim}{k}_T) . \tag{92}$$

From the solution to our problem, Eq.(85), we now learn

$$a_{out}(y, q_T) = a_{in}(y, q_T) + \frac{1}{2} \rho(y, q_T) \tag{93}$$

where

$$\rho(y, q_T) = \int d^4x \, \rho(x) \, e^{-iq\cdot x} \tag{94}$$

with

$$q = (q_0, q_T, q_3) , \qquad q_0 = +\left[|q_T|^2 + q_3^2 + m^2 \right]^{1/2} , \tag{95}$$

and

$$y = \frac{1}{2} \log \left[\frac{q_0 + q_3}{q_0 - q_3} \right] \tag{96}$$

as before.

The S matrix is defined to be the operator which takes a_{in} to a_{out} so we have

$$S \, a_{in}(y, q_T) \, S^+ = a_{out}(y, q_T) \tag{97}$$

or

$$[S, a_{in}(y, q_T)] = \frac{1}{2} \rho(y, q_T) \, S . \tag{98}$$

This form suggests we seek a solution of (98) as

$$S = C \exp\int\frac{dy\,d^2q_T}{4(2\pi)^3}\lambda(y,q_T)a_{in}^{+}(y,q_T)\times \times\exp\int\frac{dy\,d^2q_T}{4(2\pi)^3}\lambda^{*}(y,q_T)\,a_{in}(y,q_T), \tag{99}$$

where C is a c-number normalization constant. Then using the operator identity

$$[e^A, B] = [A, B]\,e^A \tag{100}$$

which is true when $[A,B]$ commutes with A and B, we find

$$[S, a_{in}(y,q_T)] = \lambda(y,q_T) = \frac{1}{2}\rho(y,q_T), \tag{101}$$

which solves our problem. The normalization C is found by requiring

$$SS^{+} = 1, \tag{102}$$

that is, unitarity. This makes C take the value

$$C = \exp -\frac{1}{8}\int\frac{dy\,d^2q_T}{4(2\pi)^3}\,|\rho(y,q_T)|^2, \tag{103}$$

using (101). So we have constructed an explicity unitary S-matrix operator which acts in pion space (the nucleons are always present so the "vacuum" is the two nucleon state). Of course if we want T_{mn} we must evaluate

$$T_{mn} = \langle in, m\,|\,S-1\,|\,in, n\rangle \tag{104}$$

$$= \langle in, m\,|\,S\,|\,in, n\rangle - \delta_{mn}, \tag{105}$$

where the n pion state $|n,in\rangle$ is

$$\frac{1}{\sqrt{n!}}\,a_{in}^{+}(y_1, q_{T_1})\cdots a_{in}^{+}(y_n, q_{T_n})\,|0\rangle \tag{106}$$

in a standard manner.

It is amusing to ask for the elastic scattering matrix which is the vacuum to vacuum transition; i.e., no pions in, no pions out:

$$T_{00}(y_0, \underset{\sim}{B}) = \langle 0 | S | 0 \rangle - 1 \tag{107}$$

$$= C - 1 , \tag{108}$$

and reference to (103) shows that indeed we no longer have a unitary S matrix in the elastic sub-space. In fact, since C = exp-(real number), the eikonal phase is pure imaginary and thus the "potential" has become completely absorbing!

How are we to regard this solution of the eikonal production problem? Since the source function $\rho(y_0, \underset{\sim}{B}; y, \underset{\sim}{q}_T)$ is unspecified there is an enormous freedom in possible unitary answers. One must think of this as a class of solutions which for any ρ yields a unitary S-matrix and thereby provides an attractive framework into which one may put his best guess or theory for ρ . There are certain constraints on ρ which come from experimental knowledge on cross sections, multiplicity, etc., and we shall now look at some of these quantities.

The total cross section is given by the optical theorem as

$$\sigma_T(s) = \frac{1}{s} \operatorname{Im} T_{00}(y_0, 0) \tag{109}$$

for s large, and in the present solution is

$$\sigma_T(s) = \int d^2B \left[C(y_0, \underset{\sim}{B}) - 1 \right] , \tag{110}$$

and this only suggests that for large s or $y_0 = \frac{1}{2}\log s$, $C(y_0,\underset{\sim}{B})$ is independent of y_0 in order to reproduce the constant total cross sections that are observed.

The distribution of produced pions, that is, the single particle inclusive spectrum, is defined to be

$$\frac{d\sigma\left[NN \rightarrow NN + \text{one pion} + \text{anything}\right]}{dy\, d^2q_T \big/ \left(4[2\pi]^3\right)} = \tag{111}$$

$$\int d^2B\, e^{i\underset{\sim}{Q}\cdot\underset{\sim}{B}} \sum_n \int \frac{dy_1 d^2q_{T1} \ldots dy_n d^2q_{Tn}}{[4(2\pi)^3]^n} \left| \langle in, n | a_{in}(y, q_T) S | 0 \rangle \right|^2 .$$

Now

$$|\langle in, n \,|\, a_{in}(y, q_T) S \,|\, 0\rangle|^2 = \langle 0|[a^{+}_{in}, S^{+}]|n, in\rangle\langle in, n|[a_{in}, S]|0\rangle \tag{112}$$

and summing on n in (111) we have

$$\frac{d\sigma}{dy\,d^2q_T/(4(2\pi)^3)} = \int d^2B\, e^{i\underset{\sim}{Q}\cdot\underset{\sim}{B}}\, \frac{1}{4}\,|\rho(y_0, \underset{\sim}{B}; y, q_T)|^2 . \tag{113}$$

So knowledge about the inclusive distribution is direct knowledge on the source function ρ .

Another interesting quantity is the n pion cross section $\sigma_n(y_0)$ which is

$$\frac{\sigma_n(y_0)}{\sigma_T(y_0)} = \int \frac{d^2B\, dy_1\, d^2q_{T1} \cdots dy_n\, dq_{Tn}}{[4(2\pi)^3]^n}\, |\langle in, n\,|S\,|\,0\rangle|^2 \tag{114}$$

$$= \int d^2B\;\; C(y_0, \underset{\sim}{B})^2 \left[\log\frac{1}{C(y_0, \underset{\sim}{B})^2}\right]^n \Big/ n! \; , \tag{115}$$

<u>and shows that in $\underset{\sim}{B}$, y_0 space one has a Poisson distribution in n</u> with mean

$$\langle n(y_0, \underset{\sim}{B})\rangle = \log\frac{1}{C(y_0, \underset{\sim}{B})^2} \tag{116}$$

$$= \frac{1}{4}\int \frac{dy\, d^2q_T}{4[2\pi]^3}\, |\rho(y_0, \underset{\sim}{B}; y, q_T)|^2 , \tag{117}$$

which should really come as very little surprise. That is to say, once we ignored the constraint of energy momentum conservation on pion production in y_0, $\underset{\sim}{B}$ space, the pions were emitted independently and this is precisely the condition under which a Poisson distribution follows.

The mean multiplicity of particles is given by

$$\langle n(y_0)\rangle = \sum_n n\,\sigma_n \,/\, \sigma_T \tag{118}$$

$$= \int d^2B \, \langle n(y_0, \underset{\sim}{B}) \rangle \tag{119}$$

$$= \int d^2B \, \frac{dy\, d^2q_T}{4(2\pi)^3} \, \frac{1}{4} \left| \rho(y_0, \underset{\sim}{B}; y, q_T) \right|^2 \tag{120}$$

This quantity is known to be approximately log s = y_0, from both cosmic ray and CERN-ISR experiments. By the way, this fact of "small" average multiplicity (relative to the $\sqrt{s}$ which could occur if all the incoming energy went into particle production) provides justification for the input of the present model that a only small number of particles with low center of mass momentum are produced, and so one may approximately ignore the conservation of energy momentum for produced particles, once the leading particles are accounted for.

This really completes our discussion of eikonalized production processes. One might proceed further in two directions; one formal, one phenomenological. The first would consist of generalizing the c-number source approximation to some kind of light cone commutator statement, as was done for elastic processes. The other would be to find attractive phenomenological forms for $\rho(y_0, \underset{\sim}{B}; y, \underset{\sim}{q}_T)$ and predict the results of correlation phenomena to be seen at NAL in, say, two particle inclusive processes. At this time I shall refrain from answering these interesting questions both to give the student something to work on and to give the lecturers at the next summer school something to talk about.

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   Essentially an identical construction has been made by R. Jengo, C. Calluci, and C. Rebbi in various contributions to be published in Nuovo Cimento during 1972 and 1972. H. Leutwyler also informed me that much work along these lines has been carried out by H. Kastrup during the past five years.

7. Three vectors are denoted $\vec{k}$ and a unit vector in the direction of $\vec{k}$ by $\hat{k}$.

8. Because $\hat{k}_i \approx \hat{k}_f$ when θ is small we have ceased distinguishing between them and set either equal to $\hat{k}$. The actual choice of $\hat{k}$ may make some difference in actual uses of the result (13). This has been considered in the recent literature by S.J. Wallace, Phys. Rev. Letters 27, 622 (1971) and E. Kujawski, Phys. Rev. D4, 2573 (1971).

   I would like to thank A. Gal for a valuable discussion on this matter.

9. Compare the elastic total cross sections $\sigma_{elastic}$ taken from G. Giacomelli, CERN-HERA 69-3 (December 1969) to σ_{total} as reported by G. Fox and C. Quigg in their compilation of Elastic Scattering Data, UCRL-20001 (January 1970).

10. In fact this "derivation" was essentially given by Froissart in his original paper, M. Froissart, Phys. Rev. 123, 1053 (1961).

11. R.L. Sugar, private communication.

12. See the reprints of R.P. Feynman and J.Schwinger in Quantum Electrodynamics (Dover Publications Inc., N. Y., (1958), J. Schwinger, editor.

13. Actually a generation functional since it depends on a function: A(x).

14. We are more or less following the development in Ref.4

15. This observation seems to have been made first by B.W. Lee, Phys. Rev. D1, 2361 (1970) and S.J. Chang, Phys. Rev. D2, 2886 (1970).

It was made more explicit by S. Weinberg, MIT-CTP preprint # 231, September, 1971 and by E. Eichten, MIT-CTP preprint # 237, October, 1971.

16. Although the presentation I will give is rather different from Ref. 6, the ideas are the same. Indeed, it was the work of Sugar and his collaborators which suggested the present chapter.

17. This is also not such a dumb idea. At the CERN-ISR the ratio of produced π's to K's is 8/1 and the π/p ratio is 17/1 and the $\pi/\bar{p}$ ratio is 34/1. This is the result of the British-Scandinavian collaboration at p_{lab} = 1500 GeV/c as reported by H. Bøggild at the 3rd International Collogquium on Multiparticle Reactions at Zakopane, Poland; June, 1972.

18. This is an ancient and honorable problem which is treated with great physical sense in the book by E. Henley and W.Thirring, Elementary Quantum Field Theory, (McGraw Hill Book Co.,1962).