Aspects of Brane Dynamics in String Theory

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To my family - for all their support and encouragement

Declaration

I declare that the material in this thesis is a representation of my own work, unless otherwise stated, and resulted from collaborations with Steven Thomas, Burin Gumjudpai, Tapan Naskar, Sudhakar Panda, Mohammad Sami and Shinji Tsujikawa. Much of the material in the thesis is based upon the papers [1] - [9].

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Abstract

This thesis investigates the dynamics and potential application of D-brane systems in Type II String Theory. In the first section we investigate the dynamics of a single brane in the $NS5$-brane ring geometry, and further develop the tachyon-radion correspondance which maps dynamical brane solutions to open string tachyon condensation on unstable branes. We show that the resulting Geometrical Tachyon exhibits many of the properties of the open string tachyon.

In the second section we investigate the dynamics of N coincident Dp-branes in various brane backgrounds. This involves the use of matrix degrees of freedom and non-commutative geometries known as fuzzy spheres. We study the collapse of these fuzzy spheres in Dp-brane and $NS5$-brane backgrounds, before generalising the results. We also examine the $D1-D3$ intersection in a general background, from the macroscopic and microscopic viewpoints. This section closes with the microscopic description of (p,q) strings in the Warped Deformed Conifold. We calculate the tension spectrum and find agreement with the macroscopic solution in the limit of large flux. Using the finite q prescription for the Myers action, we conjecture a form for the string tension when there are a finite number of D-strings.

The final section emphasises the cosmological aspects of such dynamics. We being by using the Geometrical Tachyon as a toy model of tachyonic inflation. We then construct a hybrid inflation scenario using the Geometrical Tachyon coupled to the open string tachyon. Both models compare well to the experimental data. Finally we develop a new model of DBI inflation using multiple branes at the IR tip of a warped throat. We study the theory in the large N limit and show how it is similar to the single brane models. We then extend the analysis to the case of finite N, which we expect to be more phenomenologically viable. In the large N limit we find that inflation is possible, but the amplitude of gravitational waves is large. This is not the case in the finite N limit, which can allow for small perturbations unusual levels of non-gaussianities making it a testable prediction of string theory.

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CHAPTER 1

INTRODUCTION.

Despite the many advances made in theoretical physics over the last half-century, there still remains a fundamental problem - namely the reconciliation of Quantum Mechanics (QM) with General Relativity (GR). These two great paradigms of theoretical physics underpin virtually all our current understanding of the universe around us, but seem almost totally incompatible. This is problematic if our goal is to develop a fully unified theory of physics which describes everything in the universe. Indeed unification has proven to be important in theoretical particle physics ever since it was first realised that electricity and magnetism were two descriptions of the same force, namely electro-magnetism (EM). Later it was demonstrated how to combine the Weak nuclear force with the force of EM into the so-called Electro-Weak (EW) force, and most theorists are confident that there is also a Grand Unified Theory (GUT) which unifies the EW and Strong nuclear forces. However the remaining known force of nature, gravity, stubbornly refuses to be unified with the other forces - lacking a consistent quantum mechanical description. This problem has potentially been resolved through the formulation of string theory [10].

The basic idea of string theory is to give up on the notion of point particles, instead the fundamental objects of the theory become strings. These strings are vanishingly small with a size somewhere around the Planck length, which is $l_p \sim 1.6 \times 10^{-35} m$. The length of the string is denoted by l_s, and an important parameter in the theory is given by $\alpha' = l_s^2$. Just like strings on a guitar, these strings can vibrate at different frequencies with each frequency corresponding to a different fundamental particle to an observer located at some distance from the string. The strings come in essentially two varieties, open strings and closed strings and it was shown that upon quantisation the closed string gave rise to a massless spin-two particle which we identify as the graviton. Thus a quantum theory of gravity emerges naturally within the string theory picture. However there is a price to pay for this description. The original version of the theory, the bosonic string, contained no fermions and was only perturbatively consistent if the dimension of space-time was $d = 25 + 1$. Once fermionic degrees of freedom were included, the string became the superstring to reflect the fact that it was supersymmetric. However the superstring was only consistently defined in $d = 9 + 1$ space-time dimensions, still far removed from the $D = 3 + 1$ dimensions we observe in our universe. Moreover it was shown that the theory was not unique, in fact there were five different superstring theories which were all related to one another via a set of non-perturbative dualities [13]. This fact led to the discovery of an even higher dimensional theory known only as M-theory, which exists in $D = 10 + 1$. Fortunately this theory is expected to be unique and has become the focal point for constructing a Theory of Everything (TOE).

One of the many interesting aspects of these non-perturbative dualities relating the different superstring theories was the discovery of higher-dimensional objects known as branes. These objects had in fact been discovered within the context of supergravity some years before [10, 11], but it was only within the confines of superstring theory that a real understanding of these objects was possible. We will be interested in the so called type II superstring theories for the majority of this thesis, which are closed string theories, and we will sketch out the origins of these branes in flat ten-dimensional space-time.

String theory contains many antisymmetric gauge fields. Now, it is a general rule that in d space-time dimensions a gauge field can have at most $d/2 - 1$ antisymmetric indices. The reason is that the corresponding field strength is written (in the notation of differential geometry) as

$$F = dA \tag{1.1}$$

for an $[n]$ form gauge field, where the operator d maps $[n]$ forms to $[n+1]$ forms. Now field strengths with more than $d/2$ indices are in fact related to field strengths with less indices through contraction with the epsilon tensor. This is known as Hodge duality and implies that

$$F_{(n)} = *F_{(d-n)}. \tag{1.2}$$

For example let us consider a three-form field strength in $d = 4$. By contracting the three form indices with the four-dimensional epsilon tensor we see that it is dual to a one-form field strength. i.e $F_{(3)} = *_4 F_{(1)}$. This implies that the two-form gauge field is dual to a scalar. The benefit of using differential form notation is that the antisymmetric field strength F is independent of any metric [1]. What does this all mean? It seems natural to integrate the antisymmetric gauge fields over some $[n]$ cycle. In the case of a one-form we find

$$\int A = \int A_\mu dx^\mu = \int A_\mu \left(\frac{dx^\mu}{d\tau}\right) d\tau \tag{1.3}$$

which implies that the integral of A_μ is equivalent to the integral of the gauge field contracted with the tangent vector of a curve parameterised by $x^\mu(\tau)$. This effectively means that we are contracting the gauge field with the world-line of a particle, where the world-line is parameterised by τ. This is exactly the prescription for describing a charged particle in standard Quantum Mechanics. If the gauge field has $[n]$ indices, then the extension is obvious. We contract the gauge field with tangent vectors representing some n-dimensional surface, and then integrate over the coordinates of that surface, This can be interpreted as some world-volume rather than a world-line and we say that there must exist some charged

[1] This is not true for $*F$ which does depend on the space-time metric.

n-dimensional surface. The problem now amounts to determining which form fields are present within superstring theory.

The superstring allows for the possibility of differing boundary conditions for the world-sheet fermions. If the fields are periodic then they have so called Ramond (R) boundary conditions, whilst if they are anti-periodic they have so called Neveu-Schwarz (NS) boundary conditions. Now the modes on any closed string can be decomposed into left and right moving parts, and since it is supersymmetric we expect there to be a spinor groundstate for each of these movers which we denote by $|\mathbf{s}>_L$ and $|\bar{\mathbf{s}}>_R$. Each spinor is actually a **32** dimensional Dirac spinor in the ten-dimensional theory, so the ground state is actually in the $\mathbf{32} \otimes \mathbf{32}$ representation. However these spinors are reducible and it can be shown that each of the **32** decomposes into two Weyl spinors **16** and $\mathbf{16}'$ which have opposite chiralities [10]. If we further demand that unphysical states decouple from the spectrum then we have the physical state condition that $\mathbf{16} \to \mathbf{8}$ and $\mathbf{16}' \to \mathbf{8}'$. Basically this means that the original groundstate can be written in terms of products of eight-dimensional spinors. There are actually two inequivalent choices that we can make on the groundstate, namely if the spinors have the same chirality $\mathbf{8} \otimes \mathbf{8}$ then we have the type IIB superstring theory, whereas if the spinors have different chirality $\mathbf{8} \otimes \mathbf{8}'$ then we have type IIA superstring theory. Because the ground state is a tensor product of spinors, this can be decomposed into a sum of antisymmetric tensor representations of the group $SO(8)$ It turns out that both IIA and IIB share the same NS-NS sector[2] where the decomposition gives the following massless fields

$$\mathbf{8} \otimes \mathbf{8} = \phi \oplus \mathbf{B}_{\mu\nu} \oplus \mathbf{G}_{\mu\nu} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \tag{1.4}$$

with ϕ being the dilaton, $B_{\mu\nu}$ being the antisymmetric two form field and $G_{\mu\nu}$ being the symmetric spin-two graviton. For the R-R sector[3] we obtain the following massless fields for type IIA and type IIB respectively;

$$\begin{aligned} \mathbf{8} \otimes \mathbf{8}' &= [1] \oplus [3] \\ \mathbf{8} \otimes \mathbf{8} &= [0] \oplus [2] \oplus [4]_+ \end{aligned} \tag{1.5}$$

where we have introduced the notation $[n]$ to denote an antisymmetric tensor with n indices, and the $+$ subscript denotes that the tensor is self-dual in the sense of Hodge duality. The same decomposition also occurs in the remaining NS-R and R-NS sectors, which give rise to a total of two gravitinos.

Looking at the RR type IIA solutions (1.5) we see that there exists a one-form and a three-form gauge potential. These will clearly couple to a zero (space) dimensional object (or point particle), and a two-dimensional object. By Hodge duality we see that there will

[2]We will refer to this simply as the NS sector.

[3]We will write this explicitly as the RR sector.

also exist a five-form and a seven-form gauge potential, which will couple to a four and six dimensional object. Conversely in IIB we see that there are couplings to an object of dimension minus one (an instanton), a one-dimensional object (a string) and a three-dimensional object. Again under Hodge duality we see that there will also exist a five-dimensional object. Thus IIA has even dimensional objects, whilst IIB has odd dimensional ones. In fact these objects were already known to exist as solutions of classical IIA/B supergravity [11], so it seemed remarkable that they arose naturally in a string theory context. The NS sector also contains an antisymmetric two-form, which couples to a one-dimensional object, which is actually just the fundamental string, and also a five-dimensional one via Hodge duality.

Understanding the physical nature of these objects was the subject of the second string revolution. Polchinski showed [12] that these *membranes* (or branes for short) were actually solitonic hypersurfaces on which the fundamental open string could end. These objects existed even in the purely closed type II string theories. The endpoints of an open string attached to such a p-brane satisfied $p+1$ Neumann boundary conditions, and $9-p$ Dirichlet conditions in the transverse directions. Thus they became known as D(irichlet) p-branes (or Dp-brane for short), carried one unit of charge associated with the $p+1$ form gauge potentials coming from the RR sector and are half-BPS objects. The five-dimensional object in the NS sector was given the name $NS5$-brane, however it is different to the D-branes because the open string cannot end on its worldvolume. However a non-perturbative duality known as S-duality [13]allows us to interchange the $D5$-brane with the $NS5$-brane, aswell as the $D1$-brane with the fundamental string. In fact another duality, known as T-duality, relates all of the D-branes to one another by compactifying various world-volume directions onto a torus.

1.0.1 Introducing the Abelian DBI action

The low energy theory for the open strings on the Dp-brane is given by the Dirac-Born-Infeld (DBI) action, which is a modification of the Born-Infeld action used in the study of non-linear electrodynamics [17]. As we have seen, the Dp-brane is a $(p+1)$-dimensional object that carries RR-charge and allows fundamental open strings to end on its surface. The massless modes of these strings form a $U(1)$ gauge theory with a gauge field A_μ and $(9-p)$ real scalar fields ϕ^i [4]. The action for this $U(1)$ gauge theory was derived by Leigh [15] using CFT techniques, and by the other authors in [15] via the path integral formalism. The CFT approach relies on calculating the α' corrections to the β function in both the open and closed string sector, which can be done by assuming a leading order expansion in the background fields. The vanishing of the β function then exactly matches the equations of motion coming from the DBI action. The path integral approach requires an explicit calculation

[4] There are also world-sheet fermions which will be neglected in this thesis.

of the partition function for open strings with mixed boundary conditions propagating in a background of massless string modes, but yields the same result.

The leading order terms in the action give rise to a dimensionally reduced Super Yang-Mills theory, however there are also higher order α' corrections which are inherently stringy in origin. Provided that the field strength of the gauge field is constant these α' corrections can be resummed and we obtain the following action

$$S_{DBI} = -T_p \int d^{p+1}\xi e^{-\phi} \sqrt{-\det\left(\mathbf{P}[\mathrm{G}+\mathrm{B}]_{\mathrm{ab}} + \lambda \mathrm{F}_{\mathrm{ab}}.\right)} \tag{1.6}$$

The overall coefficient T_p is the tension of the brane [10], which is given by the following expression

$$T_p = \frac{M_s^{p+1}}{(2\pi)^p g_s} \tag{1.7}$$

where $M_s = l_s^{-1}$ is the mass associated with the string scale and g_s is the asymptotic value of the string coupling constant. Note also that $\lambda = 2\pi\alpha'$ is the inverse of the fundamental string tension. The above action describes the coupling of the brane to the massless, closed string modes of the NS sector, which is simply the bulk string frame metric $G_{\mu\nu}$, the NS two-form $B_{\mu\nu}$ and the dilaton ϕ. The symbol $\mathbf{P}$ indicates that we must pull-back the respective tensor fields to the world-volume of the brane. In the canonical basis we see that in the static gauge this operation is given by

$$\mathbf{P}[E]_{ab} = E_{ab} + \lambda E_{ai}\partial_b\phi^i + \lambda E_{ib}\partial_a\phi^i + \lambda^2 E_{ij}\partial_a\phi^i\partial_b\phi^j \tag{1.8}$$

and therefore we expect to recover the usual canonical kinetic terms upon expansion of the action. The massless modes of the RR sector can also be included by introducing the following Chern-Simons term [5]

$$S_{CS} = \mu_p \int \mathbf{P}\left(\sum C_{(n)} e^B\right) e^{\lambda F} \tag{1.9}$$

where $C_{(n)}$ denotes the n form RR potential. There is a summation over all the allowed values of n so that the Dp-brane is charged under the $C_{(p+1)}$ form with charge μ_p. However supersymmetry imposes the important relation that $\mu_p = \pm T_P$.

What is noticeable about this action is that there are couplings to RR potentials of *lower* dimensions. Since these potentials will also couple to branes of lower dimension, this implies that a Dp-brane may be thought of as a bound state of a Dp-brane with $\sum_k D(p-2k)$ branes, where $k \in \mathbb{Z}$ [68], or as an intersection of such branes [16]. Schematically we can see that if there is non-trivial first Chern-class on the brane (i.e non-zero magnetic flux) then

[5]Strictly speaking this is known as the Wess-Zumino action, however we will use the Chern-Simons notation as it is more general.

the Chern-Simons term can be expanded to give couplings of the form

$$S_{CS} \sim \mu_p \int C_{(p+1)} + \lambda C_{(p-1)} \wedge F + \dots \tag{1.10}$$

where the dots denote possible couplings to forms of a lower degree (with suitable numerical factors), and the wedge product is the totally antisymmetric tensor product between form fields.

The fact that the Dp-brane carries a $U(1)$ gauge field allowed for a world-volume description of gauge theories. There was a lot of work concerned with brane intersections, in order to engineer configurations which had the same gauge group as the standard model. The study of time dependence was neglected, primarily for technical reasons related to the CFT description. However time-dependent solutions are important if string theory is to describe dynamical processes such as cosmology [76, 105]. Therefore it is a worthwhile cause to learn as much as possible about the dynamics of branes within a string context. This was the genesis for the work in this thesis.

In chapter two we will discuss the relationship between the dynamics of a probe brane in a certain $NS5$-background, and the condensation of an open string tachyonic mode on an unstable D-brane. This is an example of the Tachyon-Radion correspondance. In chapter three we will introduce the action for multiple coincident branes, and study their dynamics in a variety of non-trivial backgrounds. We will also use this technology to construct the macroscopic and microscopic theories of the BIon spike on a brane in an arbitrary background, and then consider a related physical application in cosmology and gauge theory by modelling strings in the Warped Deformed Conifold. The final chapter is devoted to a study of inflationary cosmology using brane dynamics. We present three different models of inflation and investigate how they can be reconciled with experimental evidence. This thesis summarises the collected works of [1] - [9].

CHAPTER 2

THE TACHYON-RADION CORRESPONDANCE.

2.1 Introduction.

We have already seen that Dp-branes exist in type II string theory due to the fact that there are non-zero RR fields. However we also note that there is the NS two-form field $B_{(2)}$ which acts as the source for fundamental strings. Using ten-dimensional Hodge duality we see that this implies the existence of a field strength $H_7 = *H_3$ (where $H_3 = dB_2$), which is associated with a six-form field $B_{(6)}$. This means that there must be some $(5+1)$ dimensional brane that carries this charge which is called the $NS5$-brane. The $NS5$-brane is a hypersurface in ten dimensions in much the same way as the $D5$-brane of type IIB string theory. However the fact that the NS two-form exists in both type IIA and type IIB means that the $NS5$-brane also exists in both sectors of the theory. One could also have deduced the existence of the $NS5$-brane using the properties of S-duality, beginning with the $D5$-brane of type IIB string theory and then performing duality transformations on all the relevant fields. We find that this implies that the tension of the $NS5$-brane goes like $1/g_s^2$, making it far heavier than the D-branes in the perturbative regime [26]. Thus the $NS5$-brane (or 'fivebrane') is indeed a solitonic object at weak coupling. Another difference between the fivebrane and the D-branes is that the fundamental string cannot end on the world-volume of the fivebrane due to conservation of charge. In contrast we have defined the D-brane as the hypersurface where the fundamental string can end, which implies that the D-brane is a source for closed strings. The fivebrane is clearly not a source for closed strings, however it is possible for D-branes themselves to end on the fivebrane, making it a useful object to study. If we place the fivebrane in flat ten-dimensional spacetime then it preserves exactly half of the supersymmetries, much like the D-branes. Therefore it is again a half-BPS object. However we state without proof that it preserves a different half of the supersymmetries to that preserved by the D-brane, and so if we place a fivebrane and a D-brane so that they are parallel in the spacetime they will completely break the bulk supersymmetry and a non-zero force should exist between them [21]. This means that the D-brane will be attracted towards the fivebranes, and we can study the brane dynamics using the Abelian DBI action.

In the main introduction we briefly explained the existence of the stable D-branes in type II string theory, however there also exist unstable (or non-BPS) branes. In type IIB we saw that only branes of odd dimension were charged under the RR fields, however that is not to say that even dimensional branes do not exist in the theory. In fact they do exist, but are uncharged objects and hence unstable. This instability can be represented by the presence of a tachyonic mode on the worldvolume of the brane and therefore described by a theory of open strings. This was carried out in detail by Sen, where he examined the condensation of this tachyonic mode using boundary conformal field theory. He conjectured that the decay of D-branes can be completely described by tachyon condensation, and that the energy difference between the tachyonic false vacuum and the true open string vacuum should completely account for the tension of the decaying brane [28]. A simple way to study this condensation process relies on the use of an effective action for non-BPS branes [31], the bosonic sector of which can be written as follows

$$S = -T_p \int d^{p+1}\xi V(T)\sqrt{-det(\eta_{\mu\nu} + \partial_\mu T \partial_\nu T + \lambda F_{\mu\nu})} \tag{2.1}$$

where $V(T)$ is the tachyonic potential for the open string mode T - which in this expression has dimensions of length. This is an effective action describing the massless string modes arising from the NS-sector of the theory, and can easily be generalised to non-trivial backgrounds. This action forms the basis of most work on open string tachyon condensation, and there is now a vast amount of literature dealing with this subject [28].

It was first suggested in [26] that this system is remarkably similar to the dynamics of a Dp-brane in the fivebrane background, provided that the symmetry of the transverse space is broken to become $SO(4) \to SO(3) \times U(1)$ and the probe D-brane is confined to motion along the S^1. One can show that there exists a map between the Abelian DBI in this background and the non-BPS DBI in a flat background, and is referred to as the Tachyon-Radion correspondance. At its simplest level this correspondance allows us to identify the condensation of the open-string tachyon with the dynamics of D-branes, at least at the classical level. This leads to the possibility that we can formulate an alternative formalism to describe the open string tachyon vacuum - which is currently only really understood using String Field Theory.

In this chapter we will consider brane dynamics in a different background to the ones presented in [21], namely that generated by a ring of $NS5$-branes [1]. In fact the two backgrounds are related to one another via a non-trivial space-time duality transformation. They are also clearly related in the large distance limit, since the ring distribution will appear to be pointlike to a sufficiently distant observer. We will first investigate the relativistic dynamics of a brane in such a background, before later developing the tachyon-radion correspondance.

[1] See [23] for other related work.

2.2 The $NS5$-brane ring solution.

We wish to study the dynamics of a Dp-brane in the background gravitational potential generated by a ring of k, static $NS5$ branes in type II string theory. In order to do so we must first introduce the solutions for the background fields which were first obtained by CHS [19]. The resulting expressions for the metric, dilaton and NS B-field are as follows

$$\begin{aligned} ds^2 &= dx_\mu dx^\mu + H(x^n) dx^m dx^m \\ \frac{g_s^2(\phi)}{g_s^2} = e^{2(\phi-\phi_0)} &= H(x^n) \\ H_{mnp} &= -\epsilon^q_{mnp}\partial_q \phi \end{aligned} \tag{2.2}$$

where $\mu = 0..5$ label coordinates parallel to the $NS5$-brane, roman indices m, n run over the four transverse dimensions, whilst g_s is the asymptotic value of the string coupling. As usual H_{mnp} is the 3-form NS field strength for the B field, and $H(x^n)$ is a harmonic function describing the location of the $NS5$ branes and which satisfies the Poisson equation in the transverse space. The general solution for a total of k fivebranes located at arbitrary positions with respect to a given origin, can be written as follows

$$H = 1 + l_s^2 \sum_{i=1}^{k} \frac{1}{|\mathbf{x} - \mathbf{x_i}|^2} \tag{2.3}$$

where $l_s = \sqrt{\alpha'}$ is the fundamental string length. This expression clearly simplifies when the branes are all coincident, however we wish to investigate the dynamics of branes in a slightly more complicated geometry such as that described by a ring of branes. The harmonic function for the ring solution can be obtained from the extremal limit of the rotating $NS5$ brane solutions, however this is generally a very complicated expression and does not yield itself to simple analytic solutions.

We can consider a limit of this particular solution where we have a continuous and uniform distribution of branes along a circle of radius R, which we can interpret as a smearing of charge around a ring [20]. Essentially this means that k must be large and thus the individual branes are not resolvable. The harmonic function in this 'continuum' limit is therefore given by

$$H = 1 + \frac{k l_s^2}{\sqrt{(l_1^2 - l_2^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2)^2 - 4(l_1^2 - l_2^2)(x_6^2 + x_7^2)}}. \tag{2.4}$$

where we have written the ring radius as $R = \sqrt{|l_1^2 - l_2^2|}$, which is oriented in the $x_6 - x_7$ plane in the transverse Euclidean space. The full geometry under question originally possesses an $SO(1,5) \times SO(4)$ symmetry, however since we are considering a ring distribution it is evident that this breaks the $SO(4)$ symmetry down to $SO(2) \times SO(2)$. This suggests that we should use polar coordinates rather than Cartesian coordinates to make full use of the symmetry of the problem, as we can identify two distinct planes. Let us introduce the following parameterisations for these planes

$$\begin{aligned} x_6 &= \rho\cos(\theta), \qquad x_7 = \rho\sin(\theta) \\ x_8 &= \sigma\cos(\phi), \qquad x_9 = \sigma\sin(\phi), \end{aligned} \tag{2.5}$$

and so the harmonic function in this instance reduces to the following expression

$$H(\rho,\sigma) = 1 + \frac{kl_s^2}{\sqrt{(R^2+\rho^2+\sigma^2)^2 - 4R^2\rho^2}}. \tag{2.6}$$

We now wish to probe this background using a Dp-brane, described by the DBI action. This is possible provided that the probe brane does not back-react upon the ring geometry. A way to ensure that this doesn't happen is to take the limit $k >> 1$. Since $1/k$ is a measure of the α' corrections to the background, this limit ensures that the supergravity approximation is reliable. Fortunately the $NS5$-brane solution exists for both type IIA and IIB, so we are not forced to specify the dimensionality of the probe brane [2] We will orient the probe parallel to the five-branes, and use the residual reparameterization invariance to go to static gauge. As a result the transverse directions to the $NS5$ branes induce scalar fields on the Dp-brane world volume, whose behaviour is described by the DBI action

$$S = -T_p \int d^{p+1}\zeta e^{-\phi}\sqrt{-det(G_{\mu\nu} + B_{\mu\nu} + 2\pi l_s^2 F_{\mu\nu})}. \tag{2.7}$$

where T_p is the Dp-brane tension, $F_{\mu\nu}$ is the field strength for the $U(1)$ gauge field, whilst $G_{\mu\nu}$ and $B_{\mu\nu}$ are the pullbacks of the metric and the B field to the brane:

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu X^A \partial_\nu X^B G_{AB} \\ B_{\mu\nu} &= \partial_\mu X^A \partial_\nu X^B G_{AB} \end{aligned} \tag{2.8}$$

Here $A, B = 0, 1, 2, \ldots, 9$ run over the full ten dimensional bulk spacetime and G_{AB}, B_{AB} are the bulk closed string fields.

2.2.1 Dynamics of the probe brane.

For simplicity we will be interested in homogeneous solutions to the equations of motion, where the transverse scalars dependent only on time. This will also ensure that coupling to the B field vanishes and so we will ignore it from now on. We will also consider a non zero (but constant) electric field on the brane world volume, i.e $F_{0m} = E_m$. Note that nonzero electric fields on the brane world-volume can be interpreted as dissolved (fundamental)

[2] However in the S-dual picture we have a ring of $D5$-branes and are thus in type IIB string theory, forcing us to restrict our analysis only to odd dimensional branes.

$F1$-strings. So we can write the induced metric on the brane as follows

$$G_{\mu\nu} = \eta_{\mu\nu} + H(x^n)\delta^0_\mu \delta^0_\nu (\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{\sigma}^2 + \sigma\dot{\phi}^2) \tag{2.9}$$

and upon substitution of this into the action (2.7) we obtain

$$S = -T_p \int d^{p+1}\zeta \sqrt{H^{-1} - \dot{\rho}^2 - \dot{\sigma}^2 - \rho^2\dot{\theta}^2 - \sigma^2\dot{\phi}^2 - H^{-1}F^2} \tag{2.10}$$

where we have defined $F^2 = E^m E_m$ as the constant electric field strength in dimensionless units by absorbing the factors of $\lambda = 2\pi l_s^2$. Also we have assumed that F^2 is small in order to obtain the relatively simple form of the action (2.10). The fact that we are using polar coordinates implies that we can consider the probe brane motion in a two-dimensional plane, which significantly simplifies the analysis and also allows us to include angular momentum about the origin. From (2.10) we can deduce the following expressions for the canonical momenta:

$$\begin{aligned}
\Pi_\sigma &= \frac{m\dot{\sigma}}{\sqrt{H^{-1}(1-F^2) - (\dot{\rho}^2 + \dot{\sigma}^2 + \rho^2\dot{\theta}^2 + \sigma^2\dot{\phi}^2)}} \\
\Pi_\rho &= \frac{m\dot{\rho}}{\sqrt{H^{-1}(1-F^2) - (\dot{\rho}^2 + \dot{\sigma}^2 + \rho^2\dot{\theta}^2 + \sigma^2\dot{\phi}^2)}} \\
L_\theta &= \frac{m\rho^2\dot{\theta}}{\sqrt{H^{-1}(1-F^2) - (\dot{\rho}^2 + \dot{\sigma}^2 + \rho^2\dot{\theta}^2 + \sigma^2\dot{\phi}^2)}} \\
L_\phi &= \frac{m\sigma^2\dot{\phi}}{\sqrt{H^{-1}(1-F^2) - (\dot{\rho}^2 + \dot{\sigma}^2 + \rho^2\dot{\theta}^2 + \sigma^2\dot{\phi}^2)}}
\end{aligned} \tag{2.11}$$

where $m = T_p \int d^p\zeta$ represents the effective 'mass' of the probe brane. In what follows it will be useful to rescale these momenta to remove this mass dependence, and so we are left with the following physical expressions

$$\tilde{\Pi}_\rho = \Pi_\rho/m, \;\; \tilde{\Pi}_\sigma = \Pi_\sigma/m, \tag{2.12}$$

$$\tilde{L}_\theta = L_\theta/m, \;\; \tilde{L}_\phi = L_\phi/m.$$

It now becomes a fairly straightforward procedure to calculate the canonical energy density of the brane using the Legendre transform

$$\tilde{E} \equiv \frac{E}{m} = \frac{1}{H\sqrt{H^{-1}(1-F^2) - (\dot{\rho}^2 + \dot{\sigma}^2 + \rho^2\dot{\theta}^2 + \sigma^2\dot{\phi}^2)}}, \tag{2.13}$$

implying that in these conventions the energy is dimensionless. The above expression yields the following equation for the motion of ρ and σ, using the fact that the energy must be conserved:

$$(\dot{\rho}^2 + \dot{\sigma}^2) = \frac{(1-F^2)}{H(\rho,\sigma)} - \frac{1}{H^2(\rho,\sigma)\tilde{E}^2}\left(1 + \frac{\tilde{L}_\theta^2}{\rho^2} + \frac{\tilde{L}_\phi^2}{\sigma^2}\right) \tag{2.14}$$

Since the RHS of this equation must be non-negative, it imposes constraints on the maximal strength of the electric field. In order to see this we first set the angular momentum terms to zero, implying that the following constraint must be satisfied due to the non-negativity of the LHS:

$$H(1-F^2) \geq 1/\tilde{E}^2. \tag{2.15}$$

In order to determine the full constraint it is necessary to specify the trajectory of the probe brane. Since our coordinate choice has broken the original $SO(4)$ symmetry down to $SO(2) \times SO(2)$, it is simpler to treat the dynamics as being confined to one or both of these planes. We will consider the simplest scenario where the motion occurs in only one plane i.e in the plane parallel to the ring (and inside the ring, where $\sigma = 0$), or in the plane transverse to the ring (i.e $\rho = 0$)[3]. In the first instance setting σ to zero reduces the harmonic function (2.6) to the following

$$H(\rho) = 1 + \frac{kl_s^2}{|R^2 - \rho^2|}. \tag{2.16}$$

Which can easily be seen to be singular at $\rho = R$ when the probe brane hits the ring. Upon insertion of the harmonic function into the constraint equation we find:

$$\frac{kl_s^2}{|R^2-\rho^2|}(1-F^2) - F^2 \geq \frac{1}{\tilde{E}^2} - 1. \tag{2.17}$$

which tells us that F^2 must be less than unity, in agreement with what we expect from the definition of the DBI action, and also that the energy density should be large. We can also consider the case of motion transverse to the disk plane by fixing $\rho = 0$ and considering motion in the σ plane. This gives us a new harmonic function, namely

$$H(\sigma) = 1 + \frac{kl_s^2}{|R^2 + \sigma^2|} \tag{2.18}$$

which can be seen to be nowhere singular as expected because the probe brane never will never hit the ring. The constraint condition, however, is essentially the same as that for the case of motion in the plane. Following on from the work of Kutasov [21] we will define the general effective potential for the system to be the equal to the negative of the kinetic term

$$\mathrm{V_{eff}} = -\frac{(1-\mathrm{F}^2)}{\mathrm{H}(\rho,\sigma)} + \frac{1}{\mathrm{H}(\rho,\sigma)^2\tilde{\mathrm{E}}^2}\left(1 + \frac{\tilde{\mathrm{L}}_\theta^2}{\rho^2} + \frac{\tilde{\mathrm{L}}_\phi^2}{\sigma^2}\right). \tag{2.19}$$

[3]The most general solution would obviously correspond to motion in both planes simultaneously.

In general this potential will give rise to interesting dynamical solutions, however we expect to capture most of the important behaviour by restricting our analysis to planar motion. The two most important cases are discussed below.

2.2.2 Probe motion in the ring plane.

As discussed previously the ring plane is identified with the coordinates ρ and θ, and the harmonic function is given by (2.16). It is known that multiple coincident $NS5$-branes produce an infinitely warped throat which is asymptotically connected to Minkowski space. The region of interest for this geometry is essentially that of the throat and so we will define the 'throat geometry' as the solution where we neglect the factor of unity in the harmonic function - which decouples the Minkowski part of the solution. In the case of the ring the warped geometry is far more intricate, however for convenience we will refer to the throat as the region near to the five-branes. In this instance we must ensure that $kl_s^2 >> R^2$ for our analysis to hold near the origin.

Because of the circular distribution of the $NS5$-branes there are two disconnected regions in the plane corresponding to the spacetime inside or outside of the ring. Of course, the full equations of motion are complicated and need to be solved numerically (see later), but we can make some progress by considering various approximating limits. Let us first consider the case of $\rho << R$, which puts the probe brane very close to the centre of the ring, and so we can essentially neglect factors of ρ in the expression for the harmonic function. The equation of motion now reads

$$\dot{\rho}^2 = \frac{(1-F^2)R^2}{kl_s^2} - \frac{R^4}{\tilde{E}^2 k^2 l_s^4}. \tag{2.20}$$

Recall that the electric field is a constant, and also the energy density $\tilde{E}$ is a conserved charge. This allows us to find the following solution (where we have dropped any constants of integration in order to illustrate the dynamical behaviour)

$$\rho = \frac{Rt}{\sqrt{k}l_s}\sqrt{(1-F^2) - \frac{R^2}{\tilde{E}^2 kl_s^2}}, \tag{2.21}$$

i.e is linear in the bulk time t. Thus at t=0 we expect the probe to be at the centre of the ring, which is the furthest distance from the $NS5$ branes in the region under consideration, and as time evolves it moves outwards due to the gravitational force from the fivebranes. Obviously (2.21) will only be valid in the small ρ regime and so this solution can only be trusted for early times. Furthermore, we can see that the solution becomes time independent if the following constraint is satisfied

$$(1-F^2) = \frac{R^2}{\tilde{E}^2 kl_s^2}, \tag{2.22}$$

and the probe will always remain at the origin. This condition acts as a constraint on the possible energy of the probe brane. For the moment let us set the gauge field to zero, therefore for the brane to remain fixed at the origin requires $\tilde{E}^2 = R^2/(kl_s^2)$, and so $\tilde{E}$ must be small in order to be consistent with our assumptions about the harmonic function. Effectively this means that the self energy of the brane is exactly cancelling out the attraction due to gravity. Including the non zero gauge field relaxes the constraint upon the total energy of the brane, such that for a near critical field this condition can be satisfied for larger values of the energy density. We shall not consider such a solution in this thesis, as we are interested in purely dynamical situations.

We can also consider the regime where the probe is located far from the ring, i.e $\rho >> R$, but with kl_s^2 still larger than ρ. The equations of motion are now modified slightly to become

$$\dot{\rho}^2 = \frac{(1-F^2)\rho^2}{kl_s^2} - \frac{\rho^4}{\tilde{E}^2 k^2 l_s^4}, \tag{2.23}$$

which gives us the solution

$$\frac{1}{\rho} = \frac{1}{\tilde{E} l_s \sqrt{k(1-F^2)}} \cosh\left(\frac{t\sqrt{1-F^2}}{\sqrt{k} l_s}\right) \tag{2.24}$$

This is the same expression that Kutasov found for a probe moving in the background of a stack of coincident $NS5$-branes [21], and reinforces our claim that the ring distribution appears pointlike at large distances. The above solution informs us that at $t = 0$ the probe is at its maximum distance from the sources, and as time evolves it is gravitationally attracted towards the ring. Of course we must be aware that this solution is no longer valid in the regime where the probe is near the ring, since the pointlike approximation will clearly no longer be valid.

So far we have made decent progress by simply considering the asymptotic limits of the solution, but in order to understand the ring background we must try and find explicit solutions for the equation of motion in the region close to the ring. In order to do this we have resorted to a numerical approach [4].

Consider first the case with $L_\theta = L_\phi = 0$. Figs 2.1 and 2.2 show numerical solutions for the distance function $\rho(t)$. In Fig 2.1 we have taken the dimensionless energy density to be $\tilde{E} = 0.6$ and the electric flux $F = 0$ or $F = 0.8$. We have assumed a positive initial velocity for the probe brane and a starting value of ρ outside the ring. (In this and all subsequent plots we have taken $kl_s^2/R^2 = 1$ for simplicity). It is clear that in this case trajectories of the probe brane are bound to the ring and cannot escape to infinity. The effect of turning on the electric flux on the probe is to increase its 'effective mass' which results in the maximum distance away from the ring being reduced. Fig 2.2 is the same situation but with $\tilde{E} = 1.5$.

[4] In fact we can find interpolating analytic solutions, which will be discussed in a later section.

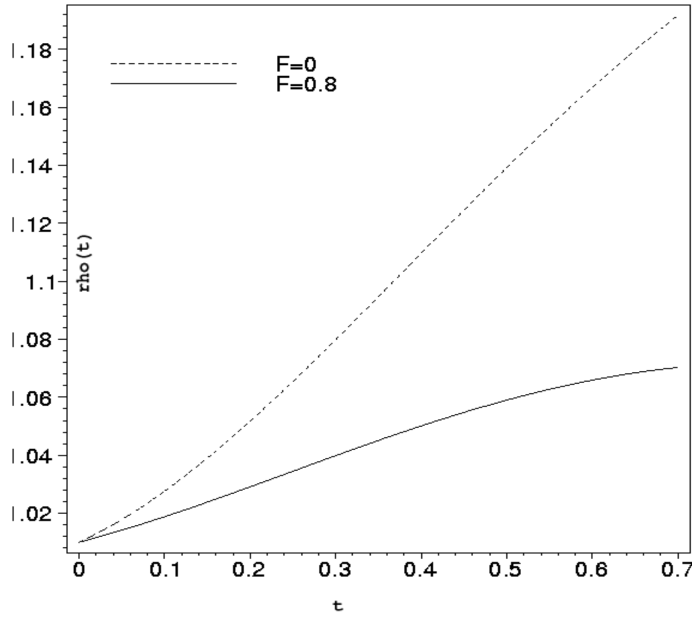

Figure 2.1: Plot of the radial coordinate ρ vs t, for $\tilde{E} = 0.6, L_\theta = L_\phi = 0$ and taking electric flux $F = 0$, 0.8.

In this case both solutions describe a probe that can escape the ring and move to infinity, the case with no electric flux having a greater escape velocity. This is to be expected since we know that the presence of an electric field acts to reduce the velocity of the probe brane. Similar plots for starting positions inside the ring (or trajectories where the initial velocity is towards the ring starting from $\rho > R$) show trajectories that eventually hit the ring at $\rho = R$ although strictly speaking, one cannot follow these trajectories right to the ring location as in this region there are large stringy effects which need to be included in the analysis.

These solutions can be understood in terms of the effective potential $V_{eff}(\rho/R)$ plotted for various values of $\tilde{E}$ and F. Fig 2.3 shows four such plots, taking e.g. $\tilde{E} = 0.6$ or 1.5 and $F = 0$ or 0.8. These plots cover the region from $\rho = 0$ at the centre of the ring, to values outside.

2.2.3 Probe motion transverse to the ring plane.

As in the previous section we initially consider the situation when $\sigma << R$. The equation of motion in this plane is similar to the one for motion in the ring plane, and we obtain the solution

$$\sigma = \frac{Rt}{\sqrt{k}l_s}\sqrt{(1-F^2) - \frac{R^2}{\tilde{E}^2 k l_s^2}}. \tag{2.25}$$

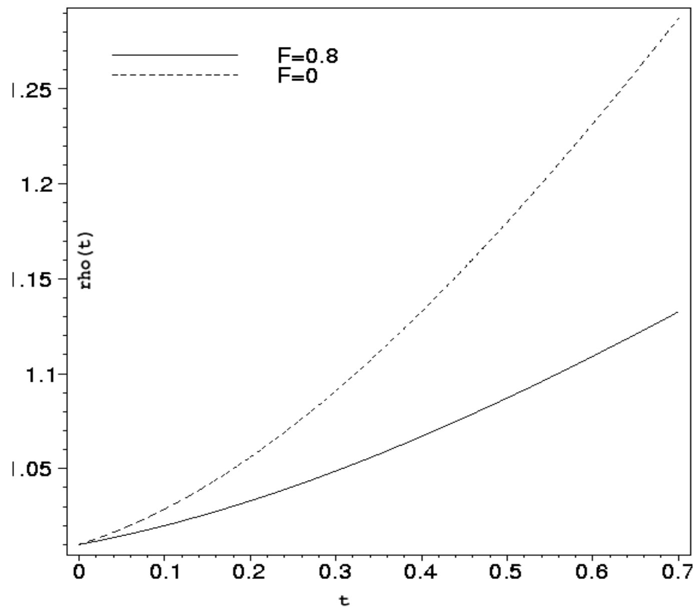

Figure 2.2: Plot of the radial coordinate $\frac{\rho}{R}$ vs t, for $\tilde{E} = 1.5, L_\theta = L_\phi = 0$ and taking electric flux $F = 0,\ 0.8$.

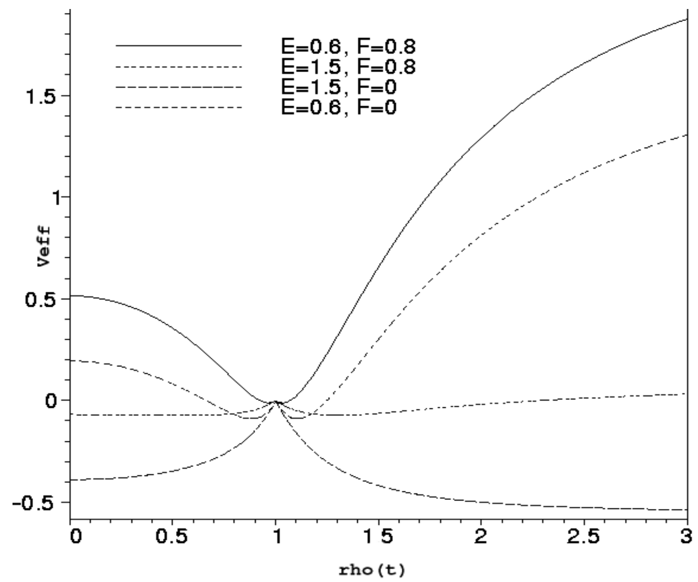

Figure 2.3: Plot of the effective potential V_{eff} vs $\frac{\rho}{R}$, for $\tilde{E} = (0, 1.5), L_\theta = L_\phi = 0$ and taking electric flux $F = 0.6,\ 0.8$.

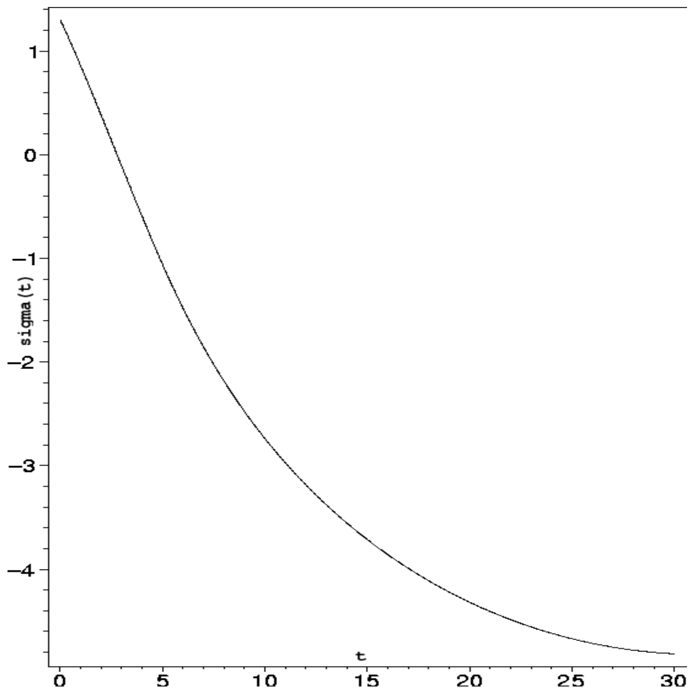

Figure 2.4: Plot of the distance $\frac{\sigma}{R}$ of the probe from the ring plane vs t, for $\tilde{E} = 0.98, L_\theta = L_\phi = 0$ and taking electric flux $F = 0$. This describes motion of the probe through the centre of the ring at $\rho = 0$

The same comments apply to this solution, except that in this instance the probe brane is no longer moving towards the $NS5$ branes as time evolves, since it is moving in the transverse plane to the ring. Again we must be aware that this linear solution is only valid for small σ. If we now consider the case where $\sigma >> R$, then we can again imagine that at large enough distances the ring distribution will appear pointlike and we expect to recover a similar solution to that obtained in the previous section. This is indeed the case, and the solution is

$$\frac{1}{\sigma} = \frac{1}{\tilde{E} l_s \sqrt{k(1-F^2)}} \cosh\left(\frac{t\sqrt{1-F^2}}{\sqrt{k} l_s}\right) \tag{2.26}$$

Where the same comments must apply when considering the critical value of the electric field.

Once again we can understand the solutions inbetween small or large values of σ/R by using numerical methods. Given that a probe is attracted to the NS5 ring if it is positioned above it, we might guess that a brane with small enough energy, falling towards the centre of the ring from above the plane of the ring, would pass through its centre and then extend below it to some maximum distance and then be attracted back through the centre of the ring and so on. That is we might expect a special solution describing the oscillatory motion of the probe through the ring centre. Such a solution should match on to the linear solution described above when the probe is at a small distance either above or below the ring plane, i.e when $\sigma/R << 1$. Fig 2.4 shows a plot of the numerical solution in this instance.

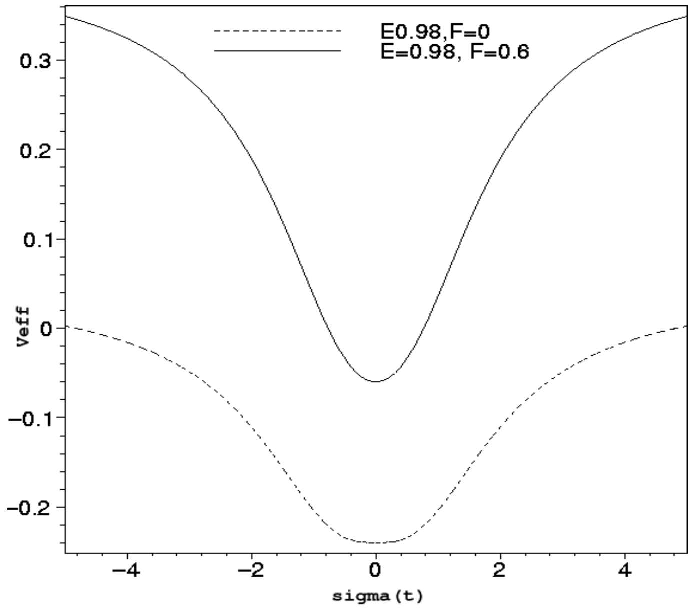

Figure 2.5: Plot of $V_{eff}(\rho = 0, \sigma)$ relevant to the study of probe motion through the centre of the ring. $\tilde{E} = 0.98, L_\theta = L_\phi = 0$, whilst the electric flux is taken to be $F = 0, 0.6$.

The linear behaviour of $\sigma(t)$ as a function of t, for small values of σ/R is clearly evident from Fig 2.4, and so matches our analytic solution. In this plot the probe starts from $\frac{\sigma}{R} = 1.3$ at $t = 0$ and reaches a maximum distance below the ring of about $4.5R$. After this time the probe is attracted back up through the ring and the process repeats. The motion thus describes oscillation between the two zeros of the effective potential $V_{eff}(\sigma)$ which is plotted in Fig 2.5. What is also noteworthy about this particular solution is that it is stable to stringy corrections since we can control the minimum distance the probe comes to the ring by making the ring radius sufficiently large. In reality however, we know that the D-brane carries charge which we would expect to be radiated away during this oscillatory phase as closed string modes [21, 30].

In the plot of Fig 2.5 we have also shown the effect turning on some electric flux has on V_{eff}. It is clear that it results in making this potential more positive everywhere and hence reduces the range and period of the oscillation through the center of the ring. What is particularly interesting is that there exists a critical value of the flux F (in fact this is around 0.6) beyond which oscillation is not possible and the probe is stuck at the ring center. The existence of this critical value of the flux F (for a given fixed energy density $\tilde{E}$) can easily be understood.

The energy density of a probe brane carrying flux F, which is at rest above the ring plane at distance σ (with $\rho = 0$) must satisfy the following equation

$$\frac{1}{\tilde{E}^2} - (1 - F^2)H(R, \sigma) = 0 \tag{2.27}$$

where the harmonic function $H(R, \sigma)$ is given in (2.18). Now we see that for given $\tilde{E}$, turning on the flux, F, on a probe brane which was initially at some point σ above the ring plane means that in order to satisfy this equation the probe has to move closer to the plane, thus increasing the value of $H(R, \sigma)$. But as we keep increasing F this cannot carry on indefinitely as there is a maximum value that $H(R, \sigma)$ can take (for fixed value of the background charge), which is its value at the centre of the ring. Thus there is a critical value of flux for a given $\tilde{E}$. Of course one has to bear in mind that we cannot make the factor of $1 - F^2$ too small as we are assuming that our derivations are only perturbative in F.

So far we have only considered radial trajectories with vanishing angular momentum, at this point we must also consider the probe dynamics when the momenta are non zero.

2.2.4 Motion in the ring plane with $\tilde{L}_\theta \neq 0$

If we retain the angular momentum term in (2.14) we must try to solve the expression

$$\dot{\rho}^2 = \frac{(1 - F^2)}{H(\rho)} - \frac{1}{\tilde{E}^2 H(\rho)^2}\left(1 + \frac{\tilde{L}_\theta^2}{\rho^2}\right) \tag{2.28}$$

As in the previous sections we begin by considering the limit $\rho << R$, which puts the probe brane inside the ring. We find that the solution in this instance is given by the following expression;

$$\rho^2 = R^2 \frac{t^2(\tilde{E}^2 k l_s^2(1 - F^2))^2 - 2R^2\tilde{E}^2 k l_s^2 t^2(1 - F^2) + R^4 t^2 + \tilde{E}^2 k^2 l_s^4 \tilde{L}_\theta^2}{\tilde{E}^2 k^2 l_s^4(\tilde{E}^2 k l_s^2(1 - F^2) - R^2)} \tag{2.29}$$

This somewhat complicated expression reduces to (2.21) in the limit $\tilde{L}_\theta = 0$. At the opposite end of the spectrum in the $\rho >> R$ regime, we find that the solution is given by

$$\frac{1}{\rho} = \frac{1}{\tilde{E}\sqrt{k} l_s \sqrt{(1 - F^2) - \tilde{L}_\theta^2/k l_s^2 \tilde{E}^2}} \cosh\left(\frac{t}{\sqrt{k} l_s}\sqrt{(1 - F^2) - \frac{\tilde{L}_\theta^2}{k l_s^2 \tilde{E}^2}}\right) \tag{2.30}$$

which again reproduces the earlier result in the limit of no angular momentum, and shows us that the momentum term has the effect of slowing the decrease of ρ in the $t \to \infty$ limit. Furthermore this equation provides us with bounds on the angular momentum, since it must satisfy the constraint

$$\tilde{L}_\theta^2 < (1 - F^2) k l_s^2 \tilde{E}^2. \tag{2.31}$$

If $\tilde{L}_\theta$ saturates this bound then the only solution is $1/\rho = 0$. Thus we see that increasing the flux automatically leads to a reduction in the angular momentum. This is in agreement with our intuitive picture of the flux providing extra mass on the brane.

In order to study trajectories of the full theory without resorting to the special limits in ρ discussed above we again look to numerical solutions. We expect that solutions to the full theory will describe the probe brane in an unstable orbit about the ring. This is confirmed in Fig 2.6 which is a parametric plot in the (ρ, θ) plane of a solution which starts at $(\frac{\rho}{R} = 1.1, \theta = 0)$ at $t = 0$. In this plot we took $\tilde{E} = 1.02, L_\theta = 0.98R$ and $F = 0.2$ and we see the trajectory spiralling outwards from the ring. Starting with different initial conditions would produce e.g. trajectories that spiral towards the ring (either starting from inside or outside) and eventually ending on there.

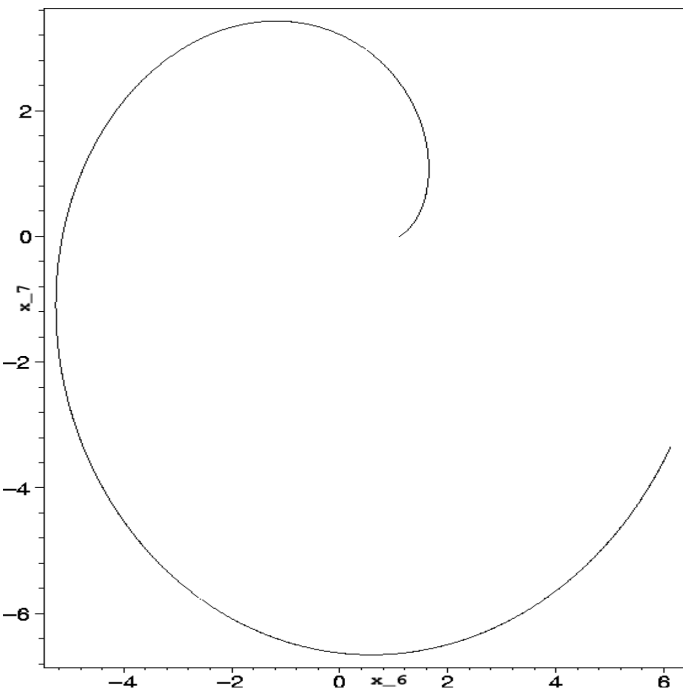

Figure 2.6: Plot of brane trajectory in the $x_6 - x_7$ (i.e ρ, θ) plane, for $\tilde{E} = 1.02$, $\frac{L_\theta}{R} = 0.98$ and taking electric flux $F = 0.2$

2.2.5 Motion transverse to the ring plane with $\tilde{L}_\phi \neq 0$.

Since we expect the analytic solutions to be similar to those discussed in the ring plane, we present the numerical solutions associated with this case.

Fig 2.7 shows the plot of V_{eff} vs $\frac{\rho}{R}$ with all other values as above. The same function is also shown for $\tilde{E} = 0.75$, in which case the spiral trajectories cannot escape to infinity.

Figure 2.8 shows a 3-d plot of the effective potential for non-zero values for L_θ and L_ϕ and $\tilde{E} = 3.29$ respectively. For this value of the energy we expect trajectories corresponding

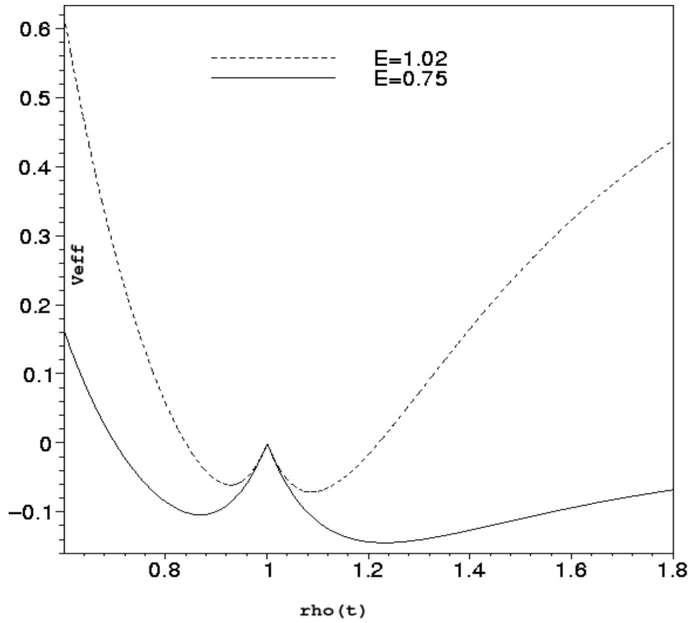

Figure 2.7: Plot of $V_{eff} vs \frac{\rho}{R}$ with $\tilde{E} = (0.75, 1.02)$, $\frac{L_\theta}{R} = 0.98$ and taking electric flux $F = 0.2$

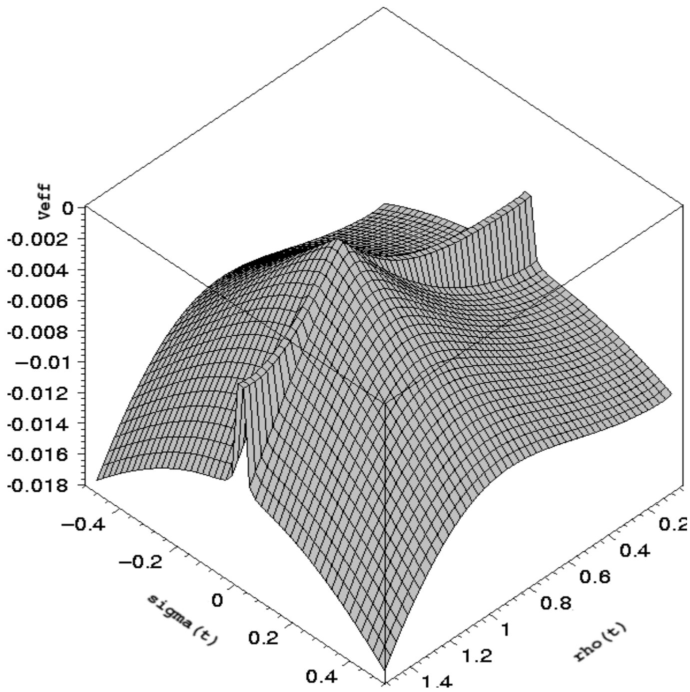

Figure 2.8: A 3-d plot of V_{eff} vs $\frac{\rho}{R}$ and $\frac{\sigma}{R}$ with $\tilde{E} = 3.29$, $\frac{L_\theta}{R} = 1.2$, $\frac{L_\phi}{R} = 1.4$ and taking electric flux $F = 0$

to the probe brane moving away from the ring to escape to infinity, which corresponds to V_{eff} becoming negative at large distances in σ, ρ as can be seen in the plot. On the other hand probe branes moving towards the ring feel a generic repulsion due to the presence of a centrifugal barrier coming from the angular momentum terms in V_{eff}. This would lead to scattering of the probe brane off the ring which happens also in the case of point like source of NS5-branes, in the presence of angular momentum. The plot also shows a 'gap' in the centrifugal barrier located at the ring location $\sigma = 0, \rho = 1$ so that its possible for some trajectories to still end on the ring itself (ignoring possible stringy corrections). A numerical study is needed to distinguish these various possibilities. Unfortunately this requires solving the full set of non-linear equations of motion for ρ, σ, θ and ϕ which requires methods that go beyond those we used earlier. Nevertheless it would be interesting to explore the nature of trajectories in this case.

2.3 Tachyon map.

The dynamics of a probe brane in the ring background were discussed in the previous section using a combination of analytic and numerical methods. In this section we introduce the concept of the 'tachyon map', which we hope will shed new light on these solutions, and also give us more understanding of the behaviour of the open string tachyon in string theory. The reasoning for this was suggested by Kutasov [26] and extended by Sen [27], namely that the rolling tachyon [28] and the late time dynamics of D-branes in non-trivial backgrounds lead to vanishing of the spatial components of the energy-momentum tensor.

Upon substitution of the background metric into our DBI action (2.10) for time dependent scalar fields, and setting the $U(1)$ gauge field to zero (we will discuss non-vanishing electric fields later) we obtained the following action

$$S = -T_p \int d^{p+1}\zeta H^{-1/2}\sqrt{1 - H(\rho,\sigma)(\dot{\rho}^2 + \dot{\sigma}^2)}, \tag{2.32}$$

where we have also set the angular terms to zero to consider purely radial motion. It is important to remember that the action is only well defined if the higher order derivatives are vanishingly small, i.e we are demanding that $\partial_t \partial_t \rho \sim 0$. The crucial point is that this form of the action is reminiscent of that describing the open string tachyon, which is governed by a Born Infeld action of the form

$$S = -\int d^{p+1}\zeta V(T)\sqrt{1 + \partial_\mu T \partial^\mu T}, \tag{2.33}$$

where $V(T)$ is the tachyon potential (there are also other actions describing the behaviour of the open string tachyon [5] which are more appropriate in other regions of field space). We

[5] See [28] and references therein.

are also absorbing factors of λ into the definition of the tachyon field to give it dimensions of length. Using the techniques of boundary conformal field theory Sen argued that the tachyon potential is an even, runaway function of T, with the maximum value occurring at $T = 0$, and tending to zero as $T \to \pm\infty$. One particular representation of this potential is given by

$$V(T) \sim \frac{1}{\cosh(\mathrm{T}/\mathrm{T}_0)}, \tag{2.34}$$

where T_0 is a theory dependent constant.

In fact it is possible for us to define a map from one action to the other, whereby we rescale our 'radion' fields (ρ, σ) to become 'tachyonic' fields with a potential given by

$$V(T) = \frac{T_p}{\sqrt{H(\rho, \sigma)}}. \tag{2.35}$$

This mapping can be thought of as a worldsheet duality map. On one hand we begin with a theory where there is a Dp-brane moving in the background of a ring of fivebranes, whose solutions we have already analysed. Performing the tachyon map is equivalent to considering a non-BPS brane moving in 10D Minkowski space, but now with a non-trivial, tachyonic open string mode on its world-volume. This new mode is a Geometrical (or Geometric) Tachyon, as it clearly has a geometrical origin. The string mode will be tachyonic in the sense that it's dynamics are governed by an unstable potential which forces the field to roll toward the true vacuum - exactly as in the case for the open string tachyon. In fact we will treat this Geometrical Tachyon as being as fundamental as the open string tachyon in the rest of this chapter. One may ask about the justification for such a proposal. However it is clear that once we start thinking in terms of the Geometrical Tachyon as a field in its own right, we can use much of the technology associated with tachyon condensation to probe the dynamics of the theory. This will also allow us to 'lift' our theory back to the bulk supergravity picture and shed new light on the fivebrane background. In addition it was shown by Kutasov how to construct a theory which mimicked Sen's tachyon potential using coincident fivebranes - where the transverse symmetry was broken from $SO(4) \to SO(3) \times U(1)$. The suggestion was that this provided a geometrical origin for the open string tachyon [26]. Another reason is that massive open string states have also been found using string field theory, which have a potential of the form $V(\phi) \sim \exp(\frac{m^2}{2}\phi^2)$ and so it may well be that there are a whole class of other tachyonic theories, with different potentials, that remain to be discovered. In any event, we expect that our analysis will shed new light on the conjecture by Kutasov, and also be an alternative way of discussing brane dynamics in non-trivial backgrounds.

Referring back to our ring solution it is clear that we can consider 3 different types of motion for the probe brane, namely motion in the ring plane with $\rho < R$, motion in the ring plane with $\rho > R$ and motion completely perpendicular to the ring plane. We will study each of these cases separately for simplicity. In the following chapter we will often switch between the tachyon picture (in Minkowski space) and the bulk supergravity picture (ring background) in order to better understand the physics of a given solution. We hope that this will not confuse the reader.

2.3.1 Inside the ring.

Setting $\sigma = 0$ and assuming that $\rho < R$ we find that the harmonic function reduces to (2.16), where we are assuming the throat approximation

$$H(\rho) = \frac{kl_s^2}{R^2 - \rho^2}. \tag{2.36}$$

Although the use of polar coordinates was employed in the first section it will actually be more convenient to revert back to the full Cartesian form, where $\rho^2 = \sum_i x_i^2, \;\; i = 6, 7$. It is more usual to consider the tachyon mapping as being one dimensional, as this is essentially the case for the open string tachyon. In what follows we will consider the brane to start at $x_i = -R$, and follow its motion through the origin until it reaches $x_i = +R$ i.e the entire interior diameter of the ring. The tachyon map in this instance is given by the following expression

$$T(x_i) = \int \sqrt{H(x_i)} dx_i, \tag{2.37}$$

which can be trivially integrated to give

$$T(x_i) = \sqrt{kl_s^2} \arcsin(x_i/R), \tag{2.38}$$

and therefore the harmonic function written in terms of the Geometrical Tachyon field becomes

$$H(T) = \frac{kl_s^2}{R^2 \cos^2(T/\sqrt{kl_s^2})}. \tag{2.39}$$

We know that $\rho = 0$ is an unstable point, since a probe brane initially located at the origin will move toward the ring if perturbed due to its gravitational attraction, and from the tachyon map we find that $T(x_i) = 0$ at this point. The maximum values of the field are therefore $T_{\rm max} = \pm\pi\sqrt{kl_s^2}/2$, which occur when the probe brane strikes the ring, of course we are neglecting stringy corrections which renders this limit invalid. The corresponding potential for this Geometrical Tachyon field is given by

$$V(T) = T_p^{\rm unstable} \cos(T/\sqrt{kl_s^2}), \tag{2.40}$$

where we are defining the unstable brane tension through the following expression

$$T_p^{\rm unstable} = \frac{T_p R}{\sqrt{kl_s^2}}. \tag{2.41}$$

It is interesting to note that the tension of the unstable brane at this point is proportional to the radius of the ring. The tachyon potential clearly has its maximum at $T = 0$, and tends to the value $\pm\pi\sqrt{kl_s^2}/2$ as $\rho \to \pm R$, corresponding to the point where the probe is attached to the ring. This agrees with the general descriptions of the potential proposed in [28]if we consider $kl_s^2 >> 1$ (strictly speaking this should be the $kl_s^2 \to \infty$ limit). The potential contains the mass of this tachyonic field, which can be seen by expanding about $T = 0$, corresponding to the perturbative vacuum. The resulting action is given by

$$S \approx -\int d^{p+1}\zeta T_p^{\mathrm{unstable}} \left(1 - \frac{T^2}{2kl_s^2} + \dots\right)\sqrt{1-\dot{T}^2}, \tag{2.42}$$

which implies that

$$M_T^2 = -\frac{1}{kl_s^2}.$$

Note that this is extremely small when compared to the usual $(\text{mass})^2$ term for the open string tachyon, $M_T^2 = -1/2$ (in units where $\alpha' = 1$) due to the additional suppression by the factor kl_s^2, which in the supergravity picture is simply the charge of the background branes.

We can also calculate the components of the energy momentum tensor associated with this tachyonic field which will become relevant later. The expression is exactly the same as that for the open string tachyon, derived from the effective action

$$T_{00} = \frac{V(T)}{\sqrt{1-\dot{T}^2}}$$

$$T_{ij} = -\delta_{ij}V(T)\sqrt{1-\dot{T}^2}, \tag{2.43}$$

where the pressure goes to zero at late times as expected for tachyonic matter. This can easily be seen since $V(T)$ tends to zero as the tachyon rolls towards either of its maximal values.

2.3.2 Outside the ring.

In this case we have $\rho > R$ and so the harmonic function simplifies to the following

$$H(\rho) = \frac{kl_s^2}{\rho^2 - R^2}. \tag{2.44}$$

Using the same method of analysing the tachyon map as in the previous section, with $x_i \geq R$ and $x_i \leq -R$ being the allowed range of the probe, we obtain

$$T(x_i) = \ln\left(\frac{|x_i|}{|R|} + \sqrt{\frac{x_i^2}{R^2} - 1}\right). \tag{2.45}$$

This tachyon is clearly zero at the ring $x_i = \pm R$, and tends to ∞ as $|x_i| >> |R|$ in accordance with our boundary conditions. For simplicity we split the solution into two different branches, namely domains where $|x_i| >> |R|$, and $|x_i| \approx |R|$.

$$\begin{aligned} T(x_i) &\rightarrow \ln\left(\frac{|2x_i|}{|R|}\right) \qquad |x_i| >> |R| \\ T(x_i) &\rightarrow \ln\left(\frac{|x_i|}{|R|}\right) \qquad |x_i| \approx |R| \end{aligned} \tag{2.46}$$

and consequently the tachyon potential can be approximated by

$$\begin{aligned} V(|x_i| >> |R|) &= \frac{R}{2\sqrt{kl_s^2}}\sqrt{e^{2T}-4} \\ V(|x_i| \approx |R|) &= \frac{R}{\sqrt{kl_s^2}}\sqrt{e^{2T}-1}. \end{aligned} \tag{2.47}$$

This potential vanishes at $T = 0$ where the probe brane hits the ring, and as expected it gives us a pressure-less fluid at late times. The form of the potential indicates that the probe brane will be gravitationally attracted to the ring, which is what we would expect from the analysis of the previous section. However if it is expanded for small T we find that the field is actually massive, but we will continue to refer to it as a Geometrical Tachyon field so as to avoid any possible confusion.

2.3.3 Transverse to the ring.

If we now consider the case of motion transverse to the ring, i.e with $\rho = 0$ and $\sigma = \sqrt{x_8^2 + x_9^2}$, the harmonic function becomes (2.18) where we have dropped the absolute value notation as this is a strictly positive definite function

$$H(\sigma) = \frac{kl_s^2}{(R^2+\sigma^2)}. \tag{2.48}$$

Since, in the bulk picture, we are considering brane motion passing directly through the origin of the ring, we again resort to Cartesian coordinates to parameterise the space. Performing the tachyon map yields the following solutions as a function of $x_j, j = 8, 9$.

$$\begin{aligned} T(x_j) &= \sqrt{kl_s^2}\text{arcsinh}(x_j/R), \\ V(T) &= \frac{R}{\sqrt{kl_s^2}}\text{csgn}\left[\cosh(T/\sqrt{kl_s^2})\right]\cosh(T/\sqrt{kl_s^2}), \end{aligned} \tag{2.49}$$

where the csgn function is defined as follows

$$\text{csgn}[y] = \begin{cases} 1 & \text{Re(y)} > 1 \\ -1 & \text{Re(y)} < 1 \end{cases}. \tag{2.50}$$

Thus we find that at $T = 0$ the tachyon potential is at a minimum, whilst for $T \to \pm\infty$ we have $V(T) \to \pm\infty$. Immediately this suggests that the scalar mode must be massive, i.e its mass^2 is positive definite, mimicking the solution [82]. Furthermore the fact that the potential has a clearly identifiable minimum allows for the prospect of field oscillation around this point - which is obviously not realised in the case of the open string tachyon. In our idealised scenario this oscillation can happen indefinitely since we are neglecting stringy effects, however as we have already pointed out - a realistic analysis must account for the emission of closed strings as the brane passes back and forth through the origin which will modify our tachyon solution by a decay factor. Eventually the field will come to reside at the minimum of the potential (the origin in the ring picture), which we know to be the unstable point of the theory and thus susceptible to quantum fluctuations which will destabilise the field.

In this section we have seen that the tachyon map can be defined in each of the three solution branches, but that only in the case where we have motion inside the ring do we actually recover a 'real'(i.e. negative mass^2) field. We will examine this field in more detail in the next section.

2.4 Geometrical Tachyon kinks.

Using our tachyon map we are able to map the open string 'radion' to a Geometrical Tachyon field. We want to argue that this is due to the fact that we are considering brane motion in a compact, bounded space where the ring of $NS5$-branes acts as the boundary to the solution located at the radial distance R from the origin. For future convenience we will represent this field by $\tilde{T}$ rather than T, which we will reserve exclusively to denote the open string tachyon. We will also write the unstable tension as being T_p^u in order to simplify the expressions. Let us recapitulate by considering the following action for our new tachyon field, given by

$$S = -\int d^{p+1}\zeta V(\tilde{T})\sqrt{-det(\eta_{\mu\nu} + \partial_\mu \tilde{T}\partial_\nu \tilde{T} + \lambda F_{\mu\nu})}, \tag{2.51}$$

where

$$V(\tilde{T}) = T_p^u \cos\left(\frac{\tilde{T}}{\sqrt{kl_s^2}}\right) \tag{2.52}$$

Note that this action is slightly different to that of the usual open string tachyon because this is obtained from a duality map from a Dp-brane to a Dp-brane. In reality we know that the tachyonic mode on a non-BPS brane occurs because we are embedding the brane into the wrong theory - for example if we embedded a $D2$-brane into type IIB string theory. The fact that the NS-fivebranes exist in both IIA and IIB, and are related to one another only through a T-duality in the transverse directions, implies that our Geometrical action can have the same dimensionality as the original probe brane action. Also note that we have added the gauge field in by hand in order to make the action look the same as that for a non-BPS brane.

In general the scalar field will be a function of all the world-volume coordinates, however for simplicity we will consider the case directly related to the rolling radion mode, namely a time dependent field, and will also set the gauge field to zero. The energy momentum tensor associated with such a tachyon field is given by (2.43) with T replaced by $\tilde{T}$. We could solve the full equations of motion to determine the dynamical behaviour of the tachyon field, however it is far simpler to use the conservation of the energy momentum tensor

$$\partial_\mu T_{\mu\nu} = 0, \tag{2.53}$$

which shows us that the T_{00} component must be conserved in time. Strictly speaking this should be the covariant derivative acting on the energy-momentum tensor, but as we are in Minkowksi space with no gauge field this reduces to an ordinary partial derivative. This allows us to write the first equation above as

$$\frac{V(\tilde{T})}{\sqrt{1-\dot{\tilde{T}}^2}} = \gamma = \text{constant}. \tag{2.54}$$

Upon substitution of the potential we can integrate this equation to determine the full time dependence of the tachyon [29]. As an intermediate step we write

$$(\partial_t \tilde{T})^2 + \frac{V(\tilde{T})^2}{\gamma^2} = 1. \tag{2.55}$$

This expression tells us immediately that there are no kink solutions[6] possible when the following condition is satisfied

$$\left(\frac{T_p^u}{\gamma}\right)^2 > 1 \tag{2.56}$$

using the classification in [29]. Additionally we see that if the above condition becomes an equality, then the only solution we expect to obtain will be the trivial $\tilde{T}(t) = 0$ - which is the constraint that the probe brane is fixed at the origin when we revert to the bulk picture. Performing the integral gives us, up to any arbitrary constants which we neglect,

$$\frac{\sin(\tilde{T}/\sqrt{kl_s^2})}{\sqrt{1-u^2\cos^2(\tilde{T}/\sqrt{kl_s^2})}} = Sn\left[\frac{t}{\sqrt{kl_s^2}}, u\right] \tag{2.57}$$

where we have written $u = T_p^u/\gamma$, and Sn is the Jacobi Elliptic function. Fortunately this equation is invertible and we obtain the following solution for the evolution of the

[6]Timelike kinks usually correspond to S-branes [85], with a Euclidean DBI action. They are intimately related to the open string tachyon, however we will not investigate the analogue of the Geometrical S-brane in this thesis.

Geometrical Tachyon.

$$\tilde{T}(t) = \sqrt{kl_s^2}\arcsin\left(\frac{Sn\left[\frac{t}{\sqrt{kl_s^2}}, u\right]\sqrt{1-u^2}}{\sqrt{1-u^2 Sn^2\left[\frac{t}{\sqrt{kl_s^2}}, u\right]}}\right). \tag{2.58}$$

Now using the conservation equation we see that at $t = 0, \tilde{T} = 0$, which gives us a constraint on the allowed values of the parameter u. In fact we find the velocity condition

$$(\partial_t \tilde{T})^2 = (1 - u^2), \tag{2.59}$$

implying u lies in the range $0 \le u \le 1$ in agreement with (2.56). We also remind the reader that $u = 1$ corresponds to $\tilde{T} = 0$, and that if $u = 0$ then the tachyon is moving at the speed of light. Thus although our tachyon has negative mass-squared, it is still causal.

The solution (2.58) is effectively the solution to the full equation of motion for the Dp-brane discussed in the first section. In that analysis we resorted to numerical simulation to determine the probe brane dynamics, however using the tachyon map has yielded an explicit solution for the whole region of parameter space consistent with our approximations (and neglecting stringy effects). We plot solutions for various values of u in Figure 2.9. It is interesting to see that in the $u \to 0$ limit, the motion of the probe brane is ultra relativistic. Whilst for $u \to 1$ the probe accelerates toward the ring with much smaller velocities. This solution is intuitively understood since, if there is tachyon rolling, the decreasing potential must be compensated by an increase in the derivative in order for γ to remain constant (unless, of course, the tachyon field is moving at the speed of light). Not plotted in this figure is the $T(t) = 0$ solution, which corresponds to the probe brane being trapped at the origin.

Recall that our solution in this region was valid between $x_i = -R$ and $x_i = R$, indicating that at some point the field must pass through the origin. Typically we expect this to give rise to some kind of topological defect, which is precisely what occurs in the case of the open string tachyon. In that instance the tachyon field yields a kink solution on the world-volume, which can be interpreted in the effective theory as a co-dimension one object - namely a $D(p-1)$-brane. However the original action corresponded to a non-BPS brane, so once the open string tachyon condenses the brane can become stable and will therefore become BPS. Therefore it is interesting to find the equivalent relation in our picture, knowing that our tachyon field has a potentially much smaller mass. We may expect to find a kink solution if we consider the tachyon to be dependent upon a solitary spatial direction, namely $\tilde{T} = \tilde{T}(x)$.

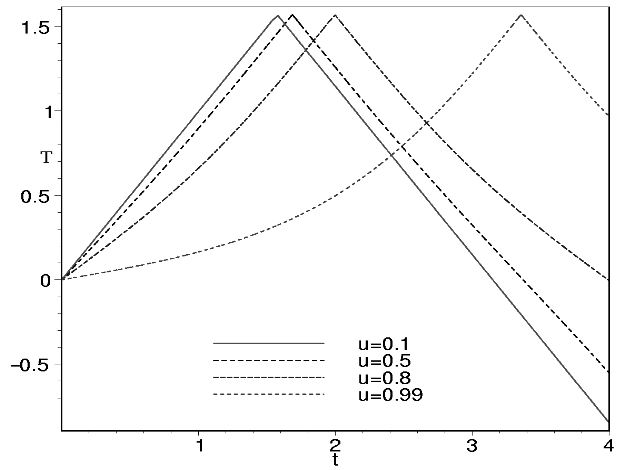

Figure 2.9: Solution curves for the evolution of the time dependent tachyon $\tilde{T}$ with varying values of u. The maximum value of the tachyon field is $\pi/2$, which corresponds to the probe brane being stuck to the ring, thus the continuation of the curves beyond the time this happens is unphysical. For simplicity we have set $\sqrt{kl_s^2} = 1$.

In this case conservation of the energy momentum tensor implies

$$\frac{V(\tilde{T})}{\sqrt{1+(\partial_x \tilde{T})^2}} = \gamma = \text{constant}. \tag{2.60}$$

An initial inspection of this equation reveals that there is no kink solution if $u < 1$, and that if this constraint becomes an equality then again, the only solution will be $\tilde{T}(x) = 0$. From the conservation equation we expect that γ will be zero. This is because a kink solution requires $\tilde{T}(x) = \pm\pi\sqrt{kl_s^2}/2$ for $x \neq 0$, and consequently the potential must be zero. This implies that γ is zero since the derivative of $\tilde{T}$ will be finite. At $x = 0$ we find that the derivative term blows up in the denominator of the energy momentum tensor, and so once again we find that $\gamma \to 0$. It will transpire that γ corresponds to the width of the kink, and since we expect it to be zero this implies that a BPS $D(p-1)$ brane has zero thickness. This discussion has actually been general, and is also valid for the open string tachyon.

If we now proceed with our integration, we find that the solution is given by

$$\tilde{T}(x) = \sqrt{kl_s^2}\arcsin\left(\frac{\sqrt{u^2-1}}{u}Sn\left[\frac{xu}{\sqrt{kl_s^2}}, \frac{\sqrt{u^2-1}}{u}\right]\right), \tag{2.61}$$

where once again we have defined $u = T_p^u/\gamma$ and Sn is the Jacobi elliptic function. Now we make an important observation. For small γ we find $u >> 1$ and so the second term in the Jacobi function can be approximated by unity. Using the properties of Sn, namely that

$$Sn(z,1) = \tanh(z), \tag{2.62}$$

our expression for the tachyon reduces to

$$T(x) = \sqrt{kl_s^2} \arcsin\left(\tanh\left[\frac{xu}{\sqrt{kl_s^2}}\right]\right). \tag{2.63}$$

This is clearly a kink solution which interpolates between $\pm\sqrt{kl_s^2}\pi/2$ due to the arcsin function, for non-zero x, whilst at $x = 0$ we find $T(x) = 0$. Furthermore by differentiating the full solution (2.61) we find that at $x = 0$ we have

$$(\partial_x \tilde{T})|_{x=0} = \sqrt{u^2 - 1} \tag{2.64}$$

This can be made infinite by sending u to infinity, and so we recover the usual solution for tachyonic kinks as in the case of the open string tachyon. In terms of the bulk picture, this corresponds to a brane attached to the ring at $-R$ for $x < 0$, and at $+R$ for $x > 0$. At $x = 0$ we obtain the usual soliton solution which stretches across the entire diameter of the ring. This kink solution is interesting since the Geometrical Tachyon only oscillates between the two zeros of the potential, and not the two minima. The open string tachyon also has this behaviour, but it has a runaway potential which is effectively of zero width, whereas the Geometrical Tachyon potential is clearly of finite width. This is not the case for topological defect solutions in field theory, which tend to stretch from one minima to another in order for them to be stable. Furthermore we can compute the energy density $\mathcal{E}$ of the kink using

$$\mathcal{E} = \int_{-\infty}^{\infty} dx V(\tilde{T})\sqrt{1 + (\partial_x \tilde{T})^2}. \tag{2.65}$$

In the large u limit we obtain, after some algebra

$$\begin{aligned} \mathcal{E} &= T_p^u \sqrt{kl_s^2} \int_{-\infty}^{\infty} dy \ \ \mathrm{sech}^2(y) \\ &= 2T_p^u \sqrt{kl_s^2} \\ &= 2RT_p. \end{aligned} \tag{2.66}$$

Clearly the energy corresponds to a kink solution which is stretched across the diameter of the ring. If we compare this to the energy bound we find

$$T_{\alpha\beta}^{kink} = -\eta_{\alpha\beta} \int_{-\pi\sqrt{kl_S^2}/2}^{\pi\sqrt{kl_s^2}/2} d\tilde{T} \ V(\tilde{T}), \tag{2.67}$$

which reduces to

$$T_{\alpha\beta} = -2\eta_{\alpha\beta} T_p^u \sqrt{kl_s^2}. \tag{2.68}$$

Thus we can see that both integrals yield the exact same result, implying that this is indeed the lowest energy configuration for the solution.

We now investigate what happens when we couple an electric field to the kink to create a charged soliton. For simplicity we choose a constant electric field which is perpendicular to $\partial_x \tilde{T}$, so e.g. $E_i = E\delta_{ik}$ where $x^k \neq x$ Expanding the action (2.51) for small λ, and incorporating the factors of l_s^2 into the field definition allows us to write the components of the energy-momentum tensor as

$$\begin{aligned} T_{\alpha\beta} &= -\eta_{\alpha\beta} V(\tilde{T})\sqrt{1 - E^2 + (\partial_x \tilde{T})^2} \\ T_{xx} &= \frac{V(\tilde{T})}{\sqrt{1 - E^2 + (\partial_x \tilde{T})^2}}. \end{aligned} \tag{2.69}$$

There will also be a constant conserved displacement field, which can be derived by varying the action with respect to $\dot{A}_k$.

$$D = \frac{EV(\tilde{T})}{\sqrt{1 - E^2 + (\partial_x \tilde{T})^2}}. \tag{2.70}$$

We note that using a perturbative expansion in λ allows us to consider $E \to 1$, where $E = 1$ is the critical value for the field. Using the conservation of T_{xx} we can write

$$E = \frac{D}{\gamma} = \text{constant}. \tag{2.71}$$

The presence of the electric field modifies the kink solution only slightly, and in fact we find the following expression for the scalar profile

$$\tilde{T}(x) = \sqrt{kl_s^2}\text{arcsin}\left(\frac{\sqrt{u^2 + E^2 - 1}}{u} Sn\left[\frac{xu}{\sqrt{kl_s^2}}, \frac{\sqrt{u^2 + E^2 - 1}}{u}\right]\right). \tag{2.72}$$

There is an interesting case where the electric field takes its critical value, as we no longer have to consider the large u limit in our solution since it reproduces (2.63) exactly [7]. This generally implies that the kink solution can be non-singular. If we allow $u = 1$ in the full solution (2.72), then the tachyon is entirely dependent upon the electric field, E. In fact it reduces to

$$T(x) = \sqrt{kl_s^2}\,\text{arcsin}\left(ESn\left[\frac{x}{\sqrt{kl_s^2}}, E\right]\right). \tag{2.73}$$

Using the expansion properties of the Jacobi function we find that for E close to unity, we have a solitary kink solution of finite width, whilst for small E we find an tiny array of small kink-antikink solutions which have period $x = 2n\pi\sqrt{kl_s^2}$.

We want to know which of these solutions are stable, so we must integrate the energy

[7] Although we have to be careful about the validity of our action in such a limit.

momentum tensor over the x direction on the world sheet. The final result valid for all u is

$$T_{\alpha\beta} = -2\eta_{\alpha\beta}T_p^u\sqrt{kl_s^2}\text{EllipticE}\left(\frac{\sqrt{u^2+E^2-1}}{u}\right), \tag{2.74}$$

where we have used the periodic properties of the Jacobi functions. This clearly shows us that the minimum energy configuration occurs when

$$\frac{(u^2+E^2-1)}{u^2} = 1 \tag{2.75}$$

which implies that $u \to \infty$ or $E = 1$, with u being arbitrary. The first case corresponds to the uncharged kink solution of infinitesimal width, implying that the electric flux is diluted to the point where it is effectively zero. The second solution requires that the electric field takes its critical value, and the resulting kink solution can be deformed to one of finite width. Furthermore all the possible kink configurations will have the exact same energy. Thus the introduction of electric flux introduces a fixed point into the theory, since a small electric field will find it energetically more favourable to increase to its maximum size. This tells us that the stable kink solutions will either be uncharged, or fully charged under the $U(1)$ gauge field. By this we mean that the stability requires either a zero field, or a critical field.

Interestingly in the time dependent kink solution with critical field strength, we find that $\tilde{T}(t) = \sqrt{kl_s^2}\pi/2$ for all time, corresponding to the probe brane being permanently attached to the ring in the bulk picture. Physically this is to be expected since flux on the brane effectively increases the 'mass', forcing the probe further into the throat generated by the fivebranes.

2.5 Dynamics of Non-BPS branes.

The existence of the unstable Dp-brane at the point $\rho = 0, \sigma = 0$ is reminiscent of a Non-BPS brane. Thus it is also useful to consider the coupled dynamics on a Non-BPS brane in this ring background along the lines of [22]. Recall that a Non-BPS Dp-brane is related to a BPS $D(p-1)$-brane, since the latter is a soliton solution on the world-volume of the former. The Non-BPS brane action is similar to the usual DBI governing the behaviour of the BPS Dp-brane, except that it has a tachyon on its world-volume, and an additional tachyon potential controlling the overall brane tension. The action[8] is of the form (2.33)

$$S = -\int d^{p+1}\zeta V(T)\sqrt{-det(G_{\mu\nu} + \partial_\mu T\partial_\nu T)}. \tag{2.76}$$

[8]Note that we choose this form of the action rather than the alternative proposal in Kutasov [31], since the form of the harmonic function makes it difficult to find space-time symmetries. It would be interesting to look for symmetries, if any, using the alternative form of the action, and compare the results to those obtained in the following section.

The form of the action suggests that the tachyon is effectively playing the role of an extra direction in field space. We shall insert this brane into the $NS5$-ring background and examine the dynamics. Now there will be two scalar fields present, the open string tachyon and also the radion field parameterising fluctuations of the brane in the transverse directions. The action for this system becomes, upon using (2.2)

$$S = -\int d^{p+1}\zeta \frac{V(T)}{\sqrt{H}} \sqrt{-det(\eta_{\mu\nu} + H\partial_\mu X^m \partial_\nu X_m + \partial_\mu T \partial_\nu T)}. \tag{2.77}$$

Where we employed the static gauge, and X^m are the transverse scalars on the world-volume. We now use the tachyon map to redefine the radion field, recalling that $\tilde{T}$ is the geometrical tachyon the action becomes:

$$S = -\int d^{p+1}\zeta V(T,\tilde{T}) \sqrt{-det(\eta_{\mu\nu} + \partial_\mu T \partial_\nu T + \partial_\mu \tilde{T} \partial_\nu \tilde{T})}. \tag{2.78}$$

Note that we are able to couple the two tachyon potentials together into product form, with $V(T,\tilde{T}) = V(T)\tilde{V}(\tilde{T})$, $\tilde{V}(\tilde{T})$ being the potential of the Geometric Tachyon. This is already suggestive of something interesting. The Geometrical Tachyon potential is that already derived in (2.40). We have implicitly assumed that the fields do not mix at the leading order classical level in order to justify the tachyon mapping. This is actually a highly non-trivial step since we are demanding that only half the scalar fields transform under this mapping operation. We will try and remain general about the form of the tachyon potential $V(T)$ by insisting that it meets the criteria described in the previous section. Note that this behaviour is easily satisfied by the $1/\cosh(T/\sqrt{2})$ potential which is valid for the superstring tachyon in units of the string length. In addition we note that the form of the action allows the two tachyons to decouple from each others equation of motion. Thus we may look at the dynamics by explicitly solving these equations, or by conservation of the energy momentum tensor.

The form of the action in (2.78) suggests that we could obtain this action by demanding that the open string tachyon is a complex field, where we have factorised the tachyon into its real parts. As such this suggests that the best way to carry out our analysis should be to (re) introduce a complexified field given via $U = T + i\tilde{T}$ This is usually done when we have a Dp and a $\bar{D}p$-brane and are looking for vortex solutions [28], since the brane/anti-brane pair will annihilate with each other to form co-dimension two objects such as fundamental strings and $D1$-branes (or D-strings). One can see this from K-theory which tells us that only objects of *even* co-dimension can form in such a process.

This is interesting in its own right, however for simplicity we will consider the evolution of the two fields separately to emphasise their differing origins. We can now proceed with our analysis of solutions to the equations of motion for the non-BPS action.

2.5.1 One spatial direction.

To begin with we will consider the simple case where $T = T(x)$ and $\tilde{T} = \tilde{T}(x)$, where x is an arbitrary direction on the world-volume. Here and in the remainder of the section we shall set $l_s^2 = 1$. Denoting the derivative with respect to x by a prime, we can write the action as follows.

$$S = -\int d^{p+1}\zeta V(T,\tilde{T})\sqrt{1+T'^2+\tilde{T}'^2} \tag{2.79}$$

which allows us to calculate the associated energy momentum tensor, with components

$$\begin{aligned} T_{\alpha\beta} &= -\eta_{\alpha\beta}V(T,\tilde{T})\sqrt{1+T'^2+\tilde{T}'^2} \\ T_{xx} &= \frac{-V(T,\tilde{T})}{\sqrt{1+T'^2+\tilde{T}'^2}} \end{aligned} \tag{2.80}$$

Where α, β run over the $0, 2 \ldots p-1$ directions perpendicular to x. We will assume that the open string tachyon has the usual kink solution, namely that as $x \to \pm\infty$, $T(x) \to \pm\infty$ and $V(T) \to 0$. Using the conservation of the energy momentum tensor, $\partial_x T_{xx} = 0 \quad \forall x$, we see that this is automatically satisfied by the kink solution since the open string tachyon component of the potential rolls to zero as the tachyon condenses. Furthermore, this is true irrespective of the behaviour of the Geometrical Tachyon since they are decoupled fields. In fact it turns out that the xx component of the tensor must vanish for all x, not just the derivative. In any case, this physically corresponds to the appearance of a co-dimension one brane located at the origin of the ring, which is just the BPS $D(p-1)$ probe brane used to probe the background. Alternatively we may find that it is the Geometrical Tachyon which condenses first, in which case the brane will be stretched across the diameter of the ring leaving an unstable soliton at the origin. It would be interesting to see what happens when both fields condense at the same time as this is the more general expectation - however the resulting analysis is complicated.

2.5.2 Two (or more) spatial directions.

We can now extend the analysis to consider dependence upon two (or more) spatial directions, namely $T = T(x)$, $\tilde{T} = \tilde{T}(y_j)$, where $j = 2 \ldots p+1$. The associated components of the energy momentum tensor in this instance are

$$\begin{aligned} T_{\alpha\beta} &= -\eta_{\alpha\beta}V(T,\tilde{T})\sqrt{(1+(\partial_x T)^2)(1+(\partial_{y_j}\tilde{T})^2)} \\ T_{xx} &= -V(T,\tilde{T})\frac{\sqrt{1+(\partial_{y_j}\tilde{T})^2}}{\sqrt{1+(\partial_x T)^2}} \\ T_{y_j y_j} &= -V(T,\tilde{T})\frac{\sqrt{1+(\partial_x T)^2}}{\sqrt{1+(\partial_{y_j}\tilde{T})^2}} \end{aligned} \tag{2.81}$$

Now we find there are two conservation conditions, $\partial_x T_{xx} = 0$ and $\partial_{y_j} T_{y_j y_j} = 0$. These simply state that T_{xx} is independent of x and $T_{y_j y_j}$ is independent of the y_j's. We look for the usual tachyon kink solution in the x direction, which satisfies both conditions in the limit that $x \to \pm\infty$. At the point $x = 0$ we expect that the derivative of the tachyon field becomes infinite, which means that T_{xx} vanishes. But there is still the conservation of $T_{y_j y_j}$ to consider. In order for this to hold for all y_j we must ensure that $\tilde{V}(\tilde{T}) \to 0$, which means that the Geometrical Tachyon must also condense. But this tachyon also has a kink solution associated with it, and so the conservation conditions are automatically satisfied.

We know that the condensation of the open string tachyon yields a BPS $D(p-1)$ brane, but we may well enquire about what the subsequent condensation of the geometrical tachyon correspond to? Naively we may assume that it gives rise to a Non-BPS $D(p-2)$-brane, but this brane would be unstable and have no tachyonic modes left which could condense and stabilise it. Therefore we must look for an alternative explanation

More generally, using the factorization properties of the Non-BPS brane action, we may write the brane descent relations for both kink solutions and find the total energy,

$$\mathcal{E} = \int_{-\infty}^{\infty} V(T)dT \int_{-\pi\sqrt{k}/2}^{\pi\sqrt{k}/2} \tilde{V}(\tilde{T})d\tilde{T}. \tag{2.82}$$

In order to do this integration we must first specify the form of the open string tachyon potential, which we will take to be

$$V(T) = \frac{T_p^{non}}{\cosh(\frac{T}{\sqrt{2}})}. \tag{2.83}$$

with T_p^{non} the non-BPS brane tension. The first integration yields

$$\mathcal{E} = \pi\sqrt{2}T_p^{non} \int_{-\pi\sqrt{k}/2}^{\pi\sqrt{k}/2} \tilde{V}(\tilde{T})d\tilde{T} \tag{2.84}$$

However we have already seen that the configuration of the BPS D(p-1) brane is itself unstable due to the geometrical tachyonic mode also forming a kink. If we integrate over this potential we find

$$\mathcal{E} = 2\pi R T_p^{non}\sqrt{2} \tag{2.85}$$

This can be written as

$$\mathcal{E} = (2RT_p) \times 2\pi \tag{2.86}$$

where we have used the relation $T_p^{non} = \sqrt{2}T_p$, T_p being the usual BPS D-brane tension. One possible interpretation of the form of the energy (2.86) is as follows. The first factor in (2.86) is the energy of D-brane stretched along the diameter of the ring which we calculated earlier (2.66). The additional factor of 2π can be thought of as coming from 'smearing' the stretched brane around the inside of the ring. Thus the process of condensation acts to deform the brane when we view it in the bulk picture.

2.5.3 Space and time dependence.

Let us now consider mixed dependence for the two fields. We will assume that the Geometrical Tachyon is time dependent, whilst the open string tachyon is spatially dependent. This allows us to write the the components of the energy momentum tensor as follows

$$\begin{aligned} T_{00} &= V(T,\tilde{T})\frac{\sqrt{1+(\partial_x T)^2}}{\sqrt{1-(\partial_0\tilde{T})^2}} \\ T_{ij} &= -\delta_{ij}V(T,\tilde{T})\sqrt{(1+(\partial_x T)^2)(1-(\partial_0\tilde{T})^2)} \\ T_{xx} &= -V(T,\tilde{T})\frac{\sqrt{1-(\partial_0\tilde{T})^2}}{\sqrt{1+(\partial_x T)^2}}. \end{aligned} \tag{2.87}$$

We again appeal to the conservation equations to determine the behaviour of the kink solution. If we assume there is a kink in the x direction, then we find that there are only two possibilities for the Geometrical Tachyon. We either have a kink in the time direction, or the field must condense. Since we have already established that there is no stable kink solution, we again find that the tachyon condenses, and this implies that the Dp-brane moves toward the ring in the bulk picture. Looking explicitly at the conservation equation for the T_{xx} component we find the expected decoupling behaviour

$$\partial_x\left(\frac{V(T)}{\sqrt{1+(\partial_x T)^2}}\right)=0. \tag{2.88}$$

This allows us to integrate the equation to determine the x dependence of the open string tachyon, provided we specify the explicit form of the potential. If we choose the usual form (2.83) then upon integration we obtain the solution

$$\sinh\left(\frac{T}{\sqrt{2}}\right)=\frac{\sqrt{1-p^2}}{p}\sin\left(\frac{x}{\sqrt{2}}\right) \tag{2.89}$$

where p is an arbitrary constant of integration. If we now substitute for $\partial_x T$ in the T_{00} component of the stress tensor we find

$$T_{00}=\frac{\tilde{V}(\tilde{T})}{\sqrt{1-(\partial_0\tilde{T})^2)}}\frac{p}{p^2+(1-p)^2\sin^2\left(\frac{x}{\sqrt{2}}\right)}. \tag{2.90}$$

This is the same result that Kluson derived for branes moving on a transverse $\mathbf{R}^3\times\mathbf{S}^1$ [22], and can be interpreted as an array of $D(p-1)$-branes and $D(p-1)$-antibranes. Since there is a map between the rolling of the time dependent Geometrical Tachyon and the motion of a probe Dp-brane we can see that these branes simply move toward the ring i.e they are gravitationally unstable as one would expect.

2.6 Compactification in a transverse direction.

In [26] Kutasov established the relationship between BPS and Non-BPS branes by compactifying one of the transverse directions in the coincident $NS5$-brane background. Since we have already seen that Geometric Tachyons can exist when the brane is probing a compact space, we may well enquire if there will be tachyonic modes if we compactify one of the transverse directions in the ring background.

Due to the symmetry of the transverse space, it is easiest to consider a compactification in the σ plane. We remind the reader that the harmonic function in this case is given by (2.18). We will choose to compactify the x_8 direction into a circle of radius L. The resultant expression for the harmonic function becomes

$$H = \sum_{m=-\infty}^{\infty} \frac{kl_s^2}{(R^2 + x_9^2) + (x_8 - 2\pi Lm)^2}. \tag{2.91}$$

This sum is easy to do, since it is very similar in form to the one in [26, 27], and we obtain the final form of the function

$$H = \frac{kl_s^2}{2L|z|} \frac{\sinh(|z|/L)}{(\cosh(|z|/L) - \cos(y/L)),} \tag{2.92}$$

where we have defined $z = \sqrt{R^2 + x_9^2}$ and $y = x_8$. In this form the harmonic function is exactly the same as that for a coincident fivebrane background where we compactify a direction within the S^3 onto a S^1 , except that if we set the x_9 fields to their minimum value we find $z = R$.

As usual we insert a Dp-brane into this background and then perform the tachyon map which leads to the following solution for the Geometrical Tachyon field

$$\tilde{T}(y) = \sqrt{\frac{2Lkl_s^2 \sinh(|z|/L)}{|z|(1+a)}} \mathrm{EllipticF}(\delta, \mathrm{r}) \tag{2.93}$$

where EllipticF is the incomplete elliptic integral of the second kind and we have made the following definitions:

$$\begin{aligned} a &= \cosh(|z|/L) \\ \delta &= \arcsin\left(\sqrt{\frac{(1+a)(1-\cos(y/L))}{2(a-\cos(y/L))}}\right) \\ r &= \sqrt{\frac{2}{1+a}}. \end{aligned} \tag{2.94}$$

By setting the x_9 fields to zero we see that the behaviour of the tachyon is dependent upon the ratios R/L and y/L. We can see that as $y \to 0$, the numerator of $\delta \to 0$ and using the properties of elliptic integrals we find that $\tilde{T} \to 0$. This is to be expected since Kutasov [26] essentially argues the same thing, namely that the tachyon starts at some initial value $\tilde{T} = \tilde{T}_{max}$ at the point $y = \pi L$, and then rolls toward zero as $y \to 0$. However we see that there is also the ratio R/L in the tachyon solution, which we would expect to explicitly determine the value of $\tilde{T}$ since this term dominates the cosine term associated with the motion around the compact dimension. We can calculate the maximum value for the tachyon

$$\tilde{T}_{max} = \sqrt{\frac{2Lkl_s^2 \sinh(R/L)}{R(1+\cosh(R/L))}} \text{EllipticK}\left(\sqrt{\frac{2}{1+\cosh(R/L)}}\right), \tag{2.95}$$

where we have introduced the elliptic integral of the first kind. We can make some approximations to determine the behaviour of the field. Firstly we take the limit $R/L << 1$

$$\tilde{T}_{max} \approx \sqrt{\frac{2Lkl_s^2 \sinh(R/L)}{R(1+\cosh(R/L))}} \ln\left(4\sqrt{\frac{1+\cosh(R/L)}{\cosh(R/L)-1}}\right) \tag{2.96}$$

which can be seen to tend to infinity. In the converse limit we can approximate the tachyon field by

$$\tilde{T} \approx \sqrt{\frac{2Lkl_s^2}{R}} \arcsin\left(\sqrt{\frac{a(1-\cos(y/L))}{2(a-\cos(y/L))}}\right). \tag{2.97}$$

which yields a maximum value of

$$\tilde{T}_{max} \approx \sqrt{\frac{2Lkl_s^2}{R}} \frac{\pi}{2}, \tag{2.98}$$

and will roll to zero as $y \to 0$. Clearly the maximum value of the tachyon field will be determined by the exact ratio of R/L and also the number of source branes.

We can also determine the tachyon potential in this instance by inverting the solution (2.93). After some manipulation using elliptic functions we obtain

$$\tilde{V}(\tilde{T}) = \sqrt{\frac{2LR}{kl_s^2 \sinh(R/L)}} \sqrt{\frac{(1-a)(1+a)}{1-a-2Sn^2[\Delta(\tilde{T}), r]}}. \tag{2.99}$$

where a and r are the same as before, whilst $\Delta(\tilde{T})$ is defined to be

$$\Delta(\tilde{T}) = \tilde{T}\sqrt{\frac{R(1+a)}{kl_s^2 L \sinh(R/L)}}.$$

This is a complicated potential, and does not yield simple analytic solutions. Furthermore we see that it is not defined for $R/L << 1$ or even $R \approx L$ due to the presence of the Jacobi function in the denominator. Thus, it is only valid for the $R/L >> 1$ case and we must also assume that the tachyon never becomes too large! The potential can only be zero if $L = 0$ which means that the compact dimension is of zero size, and a probe brane moving along it will be essentially stuck at the origin. For all other values of R and L the minimum of the potential is at some fixed non-zero value. The unstable maximum of the potential should occur when $\tilde{T} = \tilde{T}_{max}$. For the case when $R/L >> 1$ we find that

$$V_{max}(\tilde{T}) \approx \sqrt{\frac{2LR\cosh(R/L)}{kl_s^2\cosh(R/L) + 2\tilde{T}^2 R}}. \tag{2.100}$$

As the tachyon field decreases, the potential decreases, passing through its minimum at $\tilde{T} = 0$ which in the brane picture corresponds to the probe passing through the origin. Thus we anticipate that the tachyon field in this instance will be massive, as it was for the case in an earlier section. This again suggests that we will obtain massive fields when we compactify, unless the compact space is bounded by fivebranes.

Compactifying one of the directions in the plane of the ring is also possible, however we will not consider it here. The main difficulty lies in the fact that there is a crossover between harmonic functions in different regions of the covering space, but there is also the additional problem of the complicated form of the functions. It appears likely that there will be a geometrical tachyon in this instance when the probe is confined to the region $y = R \ldots 2\pi L - R$, since it is bounded by $NS5$ branes. But there is also the possibility of new Geometrical Tachyons in the region $2\pi L - R \ldots R$ which should map onto the tachyon field discussed earlier. Once again we would expect the ratio R/L to fully specify the scalar dynamics.

2.7 Discussion

In this chapter we have investigated the dynamics of probe branes in a non-trivial background, and shown how this leads to an alternate description in terms of a tachyonic field. In particular we have focused on the fivebrane ring solution, which extends the work of [21]. Most of the dynamical trajectories are attracted to the ring, however we found an interesting oscillatory solution which is stable to stringy corrections. The dynamics of the brane are more easily described in terms of the Geometrical Tachyon solution, obtained by mapping the BPS D-brane action to that of a non-BPS brane in flat space.

If we restrict ourselves to dynamics in the plane (and inside) of the ring, then we find a Geometrical Tachyon, which has a mass^2 given by $m^2 = -m_s^2/k$, which is significantly smaller than that associated with the usual open string tachyon in the large k limit. The dynamics in the transverse direction, or outside of the ring and in the ring plane, could be mapped to massive scalar fields on the worldvolume. Therefore it appears that the tachyonic nature of the scalar field is entirely due to the trajectory of the brane being bounded so that there exists a $\mathbb{Z}_2$ symmetry. We then restricted ourselves to a more detailed analysis of the Geometrical Tachyon using the methods proposed in [29]. We found that the solution admits similar co-dimension one defects to those obtained in the case of the open string tachyon, and therefore suggests a novel description for the process of tachyon condensation. The corresponding kink solution also appears to have a geometrical description in terms of the fivebrane background, although the width of the kink was found to be significantly larger than those associated with the open string tachyon, and also more sensitive to the strength of any coupled gauge field. We also discussed a hybrid theory where the Geometrical Tachyon and open string tachyon were coupled together in a modified non-BPS action.

The main objective of this chapter was to emphasise the tachyon-radion correspondance using a non-trivial but highly geometric background. The correspondance relates the dynamics of Dp-branes with the condensation of an open string tachyonic mode on a non-BPS Dp-brane. In fact these two descriptions are thought to be equivalent through a new conjecture due to Sen [27], and based on the original proposal by Kutasov [26]. Recall that the string background is taken to be the coincident fivebrane background, where one of the transverse directions is compactified on a circle of radius r. The Geometrical Tachyon corresponds to a Dp-brane located on this circle at the point πr, and moving towards the fivebranes. It has a negative mass^2 given by $m^2 = -m_s^2/k^2$ exactly as in our solution. In the language of Sen the Geometrical field is known as G-type, whilst the open string tachyon on a non-BPS brane in flat space is known as U-type. If we take the limit of $k \to 2$ of the G-type solution, then we recover the U-type solution which suggests that they are different descriptions of the same underlying theory. However we know that the α' corrections to the DBI action vary like $1/k$, which implies that one cannot naively take the $k \to 2$ limit. Fortunately Sen identified a third kind of solution known as S-type, which corresponds to a non-BPS brane in the coincident fivebrane background, which wraps the transverse circle. His conjecture is that this S-type solution is *exactly* dual to the G-type solution. The interesting feature is that one can safely take the $k \to 2$ limit of the S-type theory, and therefore there is indeed a direct relationship between the G-type solutions and the open string tachyon. This is important because the tachyon vacuum in the G-type solution corresponds to a D-brane localised in the core of the $NS5$-branes, and therefore using the conjecture, we may learn something new about the U-type vacuum.

This conjecture has been applied to the original configuration due to Kutasov, however it has not been developed in the ring case. We would expect the conjecture to hold however, since the coincident fivebrane background is clearly related to the ring background, and in fact can be mapped to it in the near horizon region [9]. The S-type solutions in this case would correspond to non-BPS branes wrapping the radius of the ring - between the fivebranes. However a more detailed analysis of this scenario remains an outstanding problem.

[9] However there are some technical issues involved with this mapping.

CHAPTER 3
COINCIDENT D-BRANES AND THE NON-ABELIAN DBI ACTION

3.1 Introduction

One of the most interesting and possibly most phenomenologically important aspects of D-brane physics is that each brane carries a $U(1)$ gauge group. As demonstrated in the previous chapter, one can learn a great deal about the classical dynamics of solitary branes in various supergravity backgrounds - which also implies that we are learning about the dynamics of $U(1)$ gauge theory. However the Standard Model of particle physics is not a simple Abelian gauge group, instead it is the non-Abelian group $SU(3) \times SU(2) \times U(1)$ [1]. Therefore it is important that non-Abelian gauge groups can be realised in a string theory context, which is possible using multiple coincident branes. However the full description of multiple branes is still unresolved

The work in this chapter will study the dynamics of multiple branes in various brane backgrounds, where we again use the probe approximation to simplify the analysis. We will also study multiple non-BPS branes in these backgrounds, which have the additional complication of open string tachyon fields on their worldvolume. After this we will write down the dynamical solutions valid for more general backgrounds, with and without world-volume gauge fields. We will also construct the microscopic and macroscopic descriptions of the $D1$-$D3$ intersection in the general case. Finally we will discuss a relevant physical application of the non-Abelian DBI action when we develop a model of cosmic (p, q)-strings in the Warped Deformed Conifold.

3.1.1 The Non-Abelian DBI action.

To begin this section we will introduce the effective action for N coincident Dp-branes. As branes approach one another we know that the massless string states form representations of a $U(N)$ gauge group. We can see this in a simple example where two branes approach one another, which we denote by [1] and [2]. At large distances the $1-2$ and $2-1$ strings are massive and dominate the dynamics giving rise to a $U(1)^2$ gauge theory. There are also massless $1-1$ and $2-2$ string states which start and end on the same brane and have

[1] In fact there is also an additional discrete Z_6 subgroup required for invariance of all the SM fields.

a negligible contribution. Once the branes reach the order of the string scale, the $1-2$ and $2-1$ strings also become massless and therefore combine with the $1-1$ and $2-2$ strings to form four (i.e 2^2) massless modes, which is the dimension of the group $U(2)$. Thus we have an enhancement of the symmetry from $U(1)^2 \rightarrow U(2)$ as the branes coalesce. Generalising this, we anticipate that for N branes we should find the symmetry enhanced from $U(1)^N \rightarrow U(N)$. This is a non-Abelian gauge group and so we have the following transformation for the world-volume vector field

$$A_a = A_a^i T_i \qquad F_{ab} = \partial_{[a} A_{b]} + i[A_a, A_b] \tag{3.1}$$

where T_i are N^2 generators satisfying $\mathrm{Tr}(T_i T_j) = N\delta_{ij}$. The scalar fields must also now transform under the gauge group, where we choose them to transform in the adjoint representation. Gauging the symmetry implies that the covariant derivative is now

$$D_a \phi^i = \partial_a \phi^i + i[A_a, \phi^i]. \tag{3.2}$$

If the scalars commute with each other then we may simultaneously diagonalise them. The resulting eigenvalues can be interpreted as being the individual positions of each of the N branes [33]. Therefore even though the scalars in the non-Abelian DBI are matrix valued, they still allow us to interpret them as fluctuations of the branes in analogy with the Abelian description. However the fact that the transverse scalars are matrix valued suggests that we should consider space-times with non-commutative geometries, which is a generalisation of the more familiar commutative (classical) geometry. In fact calculations involving Matrix theory [38] further solidify this notion, and thus it appears that non-commutative geometry is an essential ingredient when considering multiple branes.

Calculating the precise form of the non-Abelian action is extremely difficult due to the matrix nature of the fields, however it can be determined from the vector scattering amplitude on the disc [35]. Then using T-duality it is possible to obtain the non-Abelian DBI action for branes of all dimensionalities, noting the fact that under T-duality we see that $\phi^{p+1} \rightarrow A_{(p+1)}$ and $A_p \rightarrow \phi^p$ with the other components remaining unchanged. It was proposed that the resultant non-Abelian action takes the following form which we will refer to as the Myers action [33] [2]

$$S_{BI} = -T_p \int d^{p+1}\zeta \mathrm{STr}\left(\mathrm{e}^{-\phi}\sqrt{\det(\mathrm{Q^i_j})}\sqrt{-\det([\hat{\mathrm{E}}_{\mathrm{ab}} + \hat{\mathrm{E}}_{\mathrm{ai}}(\mathrm{Q}^{-1}-\delta)^{\mathrm{ij}}\hat{\mathrm{E}}_{\mathrm{jb}}] + \lambda \mathrm{F}_{\mathrm{ab}})}\right)$$

where we have made the following definitions

$$\lambda = 2\pi l_s^2, \qquad \hat{E}_{\mu\nu} = \mathcal{P}[G_{\mu\nu} + B_{\mu\nu}], \quad \text{and} \quad \mathrm{Q^i}_{\ \mathrm{j}} \equiv \delta^{\mathrm{i}}_{\ \mathrm{j}} + \mathrm{i}\lambda[\phi^{\mathrm{i}}, \phi^{\mathrm{k}}]\mathrm{E_{kj}}. \tag{3.3}$$

[2] See [34] for an alternative proposal.

The coupling of the closed string RR fields to the action is given by an analogous modification of the Abelian Chern-Simons action

$$S_{CS} = \mu_p \int STr(\mathcal{P}[e^{i\lambda i_\phi i_\phi} \sum C^{(n)} e^B] e^{\lambda F}). \tag{3.4}$$

As usual $\mathcal{P}[\ldots]$ represents the pullback of the spacetime tensors to the brane worldsheet, which is now non-Abelian of course

$$\mathcal{P}[E]_{ab} = E_{ab} + \lambda E_{ai} D_b \phi^i + \lambda E_{ib} D_a \phi^i + \lambda^2 E_{ij} D_a \phi^i D_b \phi^j \tag{3.5}$$

with a, b corresponding to worldvolume indices, and i, j being transverse directions. Note that we are employing the relation $x^i = \lambda \phi^i$ in order to work with scalar fields having canonical mass dimension. The validity of the Myers action, together with its origin, have been discussed in a number of papers [36], with an interesting result coming from the work of Howe at al, who showed that by including boundary fermions and quantising using a suitable gauge choice, it is possible to exactly reproduce the Myers action. This suggests that the Myers action is a good starting point for discussing the theory of coincident branes.

In the Chern-Simons term (3.4) there is also a new interior product which is given by the following expression acting on an arbitrary p form

$$i_\phi i_\phi C^{(p)} = \frac{1}{2(p-2)!} [\phi^i, \phi^j] C^{(p)}_{ji\alpha_3\alpha_4\ldots\alpha_p} dx^{\alpha^3} \ldots dx^{\alpha^p}. \tag{3.6}$$

Note that this is explicitly non-zero because the scalar fields are matrix valued. The action is further complicated by the symmetrized gauge trace, which is the maximally symmetric trace over the fields [33]. This prescription involves taking the symmetric average over all the possible orderings of the $F_{ab}, D_a\phi^i, [\phi^i, \phi^j]$ and any individual scalars arising from performing a non-Abelian Taylor expansion of the background fields. The low energy expansion of the resulting theory agrees with the results obtained by string scattering amplitudes up to order F^4, however at sixth-order and above we need to incorporate additional commutators of the field strength in order to obtain the correct physics [33, 37]. Fortunately the STr prescription can be simplified by consideration of the large N limit. In this case we can replace the symmetric trace by a trace, since the corrections will be subleading in powers of $1/N$. This limit will be the main assumption employed in this chapter.

It is well known, and we have already stated, that a Dp-brane is electrically charged under the $(p+1)$ form RR potential, with a charge μ_p. As usual supersymmetry constraints impose the additional condition that $\mu_p = \pm T_p$. Whilst the Abelian Chern-Simons action included couplings to RR charges of lower dimension, the non-Abelian Chern-Simons action shows that a Dp-brane can couple to RR charges of higher dimensionality, and thus permits the possibility of a brane dielectric effect whereby a lower dimensional brane expands into a higher dimensional one. For example if we expand the Chern-Simons action to leading order with no B field and no worldvolume gauge field, we find

$$S_{CS} = \mu_p \int STr(\mathcal{P}[C^{(p+1)} + i\lambda i_\phi i_\phi C^{(p+3)} - \frac{\lambda^2}{2}(i_\phi i_\phi)^2 C^{(p+5)}]). \tag{3.7}$$

where the first term acts as a source for Dp-branes, whilst the second term is clearly sourcing $D(p+2)$-branes. Performing the pullback operation on these fields leads to expressions involving covariant derivatives of the scalars, and is thus generally complicated. Turning on non-vanishing Chern classes and including non-zero B fields further increases the complexity of the theory, and leads to more interesting brane couplings.

3.1.2 Fuzzy spheres as non-commutative geometry.

At this stage we should briefly introduce the concept of the fuzzy sphere, as this is integral to much of the work in this chapter [18]. Consider first the classical unit two-sphere S^2, which has an algebra of complex valued functions that can be expanded as follows

$$f(x^i) = f_0 + f_i x^i + \frac{1}{2} f_{ij} x^i x^j + \dots \tag{3.8}$$

where roman indices run only over $1, 2, 3$ and we must satisfy the additional constraint that $x^i x^i = 1$. Each of the coefficients in the expansion are traceless and symmetric, since we know that the Euclidean coordinates are commutative. We can now consider truncating this expansion to a finite number of terms using a series of non-commutative approximations. Consider first the zeroth order term f_0. This reduces the algebra of the sphere to the algebra of complex numbers $\mathcal{A}_0(S^2) = \mathbb{C}$ which means we can only identify a single point on the sphere. Let us now also include the linear term in the expansion, which implies that we now have a four-dimensional vector space. If we endow this vector space with an appropriate product then we can form another algebra $\mathcal{A}_1(S^2)$. The simplest option is to choose this vector product to be isomorphic to the algebra of 2×2 complex matrices. This can be done by identifying the coordinates x^i with the $SU(2)$ Pauli matrices σ^i, as this continues to respect the original $SO(3)$ symmetry of the sphere. This means that we are now able to distinguish *two* points on the sphere, one for each of the eigenvalues of the matrix σ^3. The next truncation is to include the quadratic terms. If we now identify the coordinates with the three dimensional representation of the $SU(2)$ algebra, then we can identify three points on the sphere. Thus we describe the sphere with only a finite number of identifiable points as being 'fuzzy'. This truncation can clearly be continued up to the nth term which gives us the algebra $\mathcal{A}_n(S^2) = N^2$, implying that the associated vector space is N^2 dimensional. As before we identify the coordinates with matrices in some N dimensional representation of the gauge group In this thesis our convention for the $SU(2)$ algebra will be to use $[T^i, T^j] = 2i\epsilon_{ijk}T^k$, where the T^i are the N dimensional generators.

Identifying the coordinates with these generators yields the following constraint

$$T^iT^i = C\mathbf{1}_N \rightarrow \frac{1}{N}Tr(T^iT^i) = N^2 - 1 \tag{3.9}$$

where we have introduced the $N \times N$ identity matrix $\mathbf{1}_N$, and the quadratic Casimir $C = N^2 - 1$. In order to normalise this with the spherical constraint we must identify the coordinates in the following manner. Firstly we promote the x^i to matrices X^i, and then we see that the appropriate normalisation requires $X^i = T^i/N$ in the large N limit. Therefore we find that the spherical constraint in this language becomes

$$\frac{1}{N}Tr(X^iX^i) = 1 - \mathcal{O}\left(\frac{1}{N^2}\right) \tag{3.10}$$

which agrees with the classical solution up to $1/N^2$ terms as expected. If we look at the commutator of these coordinates we find that

$$\begin{aligned} [X^i, X^j] &= \frac{1}{N^2}[T^i, T^j] \\ &= \frac{1}{N}2i\epsilon_{ijk}X^k \end{aligned} \tag{3.11}$$

and therefore in the limit that $N \rightarrow \infty$ we again recover the classical notion of commutative space. Of course as N gets larger the sphere becomes 'less fuzzy' since we can identify an increasing number of points.

The example of the fuzzy S^2 serves to introduce the basic concept of a non-commutative geometry, however one must be careful because this is a particularly simple example where we can safely take the $N \rightarrow \infty$ limit. In general this may not be the case, as can be seen when one considers higher dimensional even fuzzy spheres i.e S^{2k} where $k \in \mathbb{Z}$. We will comment on these solutions in a later section. The fuzzy sphere description arises in the Myers action because we will (initially) choose three transverse coordinates to be non-commutative - whilst ignoring those parallel to both sets of brane worldvolumes. This allows us to consider the problem either through the dynamics of multiple brane configurations, or the expansion/collapse of a fuzzy sphere.

Pre-empting much of the work in this chapter, we will employ the following definition for the physical radius of a general fuzzy sphere which is valid in the large N limit

$$r^2 = \frac{1}{N}Tr(X^iX^i) = \frac{\lambda^2}{N}Tr(\phi^i\phi^i), \tag{3.12}$$

where we have implicitly identified the transverse scalars with the $SU(2)$ generators as in (3.10)

Now that we have set the stage, we can use our Dp-brane solutions to determine the dynamics of a collapsing fuzzy sphere in various backgrounds. Some early work on this topic included [40]. We will assume that the N branes will not backreact on the background geometries, which implies that the background charges must be substantially larger than N if we are to neglect any $1/N$ corrections to the Myers action in what follows.

3.2 Dynamics in Dp and $NS5$-brane Backgrounds

3.2.1 D-Brane Backgrounds.

We consider the standard type II supergravity background solutions for M coincident Dp-branes, where p is even for type IIA and odd for type IIB string theory. These source branes are all assumed to be parallel in the sense that their world volumes are oriented in the same directions, and they are all static. This will ensure that our solutions are as simple as possible. Furthermore we will only consider Dp-brane backgrounds where $p < 7$. The reason for this is to allows for the possibility of decoupling the open/closed string interactions in the study of gauge/gravity duality [14]. Decoupling the massive string modes requires us to send $\alpha' \to 0$, however if we also want to decouple interactions we additionally need to send $l_p \to 0$ (typically we require that g_{YM}^2 remains finite where $g_{YM}^2 = g_s(2\pi)^{p-2}(\alpha')^{\frac{p-3}{2}}$ in the study of gauge/gravity duality). However since $l_p^4 = g_s\alpha'^2$, we see that requiring g_{YM} to remain finite in the decoupling limit implies that $l_p \sim \alpha'^{\frac{7-p}{4}}$. Although we will not decouple the string modes in this thesis, we will still use this as a guiding principle for our background solutions. It will then be straight-forward to modify our analysis in this limit to study the dynamics of gauge theories. The 10-dimensional bulk spacetime is assumed to be infinite in extent, and we assume that there are no gravitational moduli in the problem. The solutions for the metric, dilaton and R-R field are given by the following expressions [10, 11, 23]

$$\begin{aligned} ds^2 &= H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}dx^m dx^n \\ e^\phi &= H^{(3-p)/4} \\ C_{0\ldots p} &= 1 - H^{-1}, \end{aligned} \tag{3.13}$$

where μ, ν represent directions parallel to the background branes, whilst m, n are transverse directions transforming under an $SO(9-p)$ symmetry The harmonic function H again satisfies a Laplace equation in the transverse Euclidean space. In general it can be written as a multi-centred function of the transverse coordinates:

$$H = 1 + \sum_{i=1}^{M} \frac{\tilde{k}_p}{|\mathbf{x} - \mathbf{x}_i|^{7-p}} \tag{3.14}$$

which for coincident D-branes clearly reduces to

$$H = 1 + \frac{k_p}{r^{7-p}}. \tag{3.15}$$

where, $r = \sqrt{x_m x^m}$ and $k_p = (2\sqrt{\pi})^{5-p} M\Gamma(\frac{7-p}{2}) g_s l_s^{7-p}$. As usual l_s is the string length and g_s is the asymptotic value of the string coupling.

Into this background we wish to insert N probe Dp'-branes where we must ensure that $N < M$ and also that $p \geq p'$ in order to satisfy the supergravity constraints (note that we will neglect the case of $p' = -1$ in IIB, which corresponds to the D-instanton). Because there is more than a single probe brane we can no longer use the Abelian DBI action, as the extra massless string modes enhance the gauge symmetry on the world-volume. This can clearly be regarded as an extension of [23] to the non-Abelian case.

Most of the interesting physics will take place in the 'near horizon geometry', which is an approximation that we will frequently make reference to. This simply means that we will neglect the factor of unity in the harmonic function and just concentrate on the radial dependence. Not only will this simplify the analysis, but it effectively decouples the asymptotic Minkowski space from the problem.

In this analysis we are assuming that all the probe branes are parallel to the source branes, therefore we find that the leading order contribution to the Chern-Simons coupling reduces to:

$$S_{CS} = \mu_p \int STr(\mathcal{P}[C^{p+1}]) \tag{3.16}$$

which, upon insertion of the background solutions, becomes

$$S_{CS} = +q \int dtNH^{-1} \tag{3.17}$$

up to an arbitrary constant, where $q = +1$ corresponds to a D-brane probe and $q = -1$ corresponds to an antibrane. Clearly we could consider more general couplings in this action even without turning on any gauge fields, but this is left for future endeavour. Now, in the Abelian case we know that there is only a coupling if $p = p'$ or if $p = 6, p' = 0$ [23]. Since we are neglecting higher order corrections to the Chern-Simons action, we effectively have the same situation and so we must remember to include these couplings in our effective theory.

To simplify the analysis as much as possible we will only consider time dependent solutions for the transverse scalars. This will ensure that there are no caustics in the action. We will also set F_{ab} to zero, and allow the only fields to be excited on the branes to be those which are not in the angular directions. This will also ensure that the B field will drop out of the action.

3.2.2 Radial Collapse.

In this section, we will consider the purely radial motion of N Dp'-branes in the background of M Dp-branes, where we must ensure that $M >> N$ for us to be able to trust our effective action. Provided we can perform this tuning of the background charge, we will treat the

N-branes as being probes of the geometry. Physically this corresponds to the leading order dynamics of the theory, where there are expected to be back-reactive corrections which may in principle be calculated. We anticipate that these corrections would be most important when the probe branes are close to the background branes, as it is here that the warped geometry would be most sensitive to other branes. The detailed calculations relevant here are beyond the scope of the current work, where we will concentrate on what are effectively the 'zeroth' order solutions.

For simplicity we begin with the $p = p'$ case, where the dimensions of the branes are all equal. In general there are difficulties associated with the $p \neq p'$ cases which also arise when using the Abelian DBI action. It becomes necessary to search for various conserved charges in order to solve the equations of motion. However we should also note that in the Abelian case there exist supersymmetric solutions when $p - p' = 4$, and also when $p = 6, p' = 0$ where the probe brane doesn't feel the gravitational force from the background branes [23].

3.2.3 Dynamics in the $p = p'$ case.

Inserting the background solutions (3.13) into the non-Abelian DBI and Chern-Simons actions we find that at leading order

$$S = -T_{p'} \int d^{p'+1}\zeta STr \left(H^{-1} \sqrt{(1 - H\lambda^2 \dot{\phi}^i \dot{\phi}^j \delta_{ij})(1 - \frac{1}{2}\lambda^2 H[\phi^i, \phi^j][\phi^j, \phi^i])} \right)$$

$$S_{CS} = +T_{p'} \int d^{p'+1}\zeta \frac{qN}{H}, \tag{3.18}$$

where we have made the approximation $Q^{ij} \sim \delta^{ij}$, and only expanded the second square root term to leading order. Our approximation that the inverse matrix Q^{ij} is treated as unity to leading order in lambda is consistent as long as our solution only probes distances greater than the string length. As the fuzzy sphere radius starts approaching l_s we anticipate that higher order terms in Q^{ij} (and in the square root of det(Q)) would need to be kept for consistency. This approximation has been used by other authors who have investigated fuzzy spheres in the non-Abelian DBI theory, see [33] for example.

In order to simplify the expression to something more useful we need to expand the commutator terms. The simplest ansatz possible is to make the transverse scalars all commuting, however it has been shown that the system will be unstable since it will not be at its minimal energy. This can be easily be verified by expanding out the last term in the action. Instead we opt for the more familiar $SU(2)$ ansatz which parameterises a non-commutative object known as a fuzzy 2-sphere as described in the introduction. The reason for choosing $SU(2)$ is that it is the simplest non-Abelian group one could consider. Moreover as we are limiting our analysis to values of $p \leq 6$ we see that there will always exist at least an $SO(3)$ isometry group in the transverse directions as can be read off from the metric in (3.13). We could also consider higher dimensional gauge groups, however they will lead to restrictions on the dimension of the supergravity background as we will see later. It is also possible to allow the scalars to transform in reducible representations of $SU(2)$, which could lead to very interesting dynamical effects and even fuzzy sphere nucleation however we will not study this possibility in this thesis.

After this motivation we choose to identify our scalar fields with the gauge group generators as follows

$$\phi^i = R(t)T^i, \quad i = 1, 2, 3 \tag{3.19}$$

where the T^i are in the $N \times N$ irreducible matrix representation of $SU(2)$ algebra

$$[T^i, T^j] = 2i\varepsilon_{ijk}T^k. \tag{3.20}$$

Note that since the scalar has canonical mass dimensions in (3.19), so too must the radius $R(t)$ - however we will see shortly that this is not a physical radius and therefore is an acceptable parameterisation. The remaining fields $\phi^i, i = 4, 5...$ are set to zero or more generally to constant matrices that commute with the $SU(2)$ generators. Let us make some comments concerning the generality and validity of this 'round' fuzzy sphere ansatz in (3.19). Our ansatz sets the non-Abelian transverse fields ϕ^i either to be $SU(2)$ valued fields (the fuzzy sphere ansatz) or to constant commuting matrices. The latter are taken to commute with both the $SU(2)$ generators and themselves. These latter fields have no potential because they commute with everything, so the assumption that they are constant is consistent with their equations of motion; they simply parameterise flat directions of the theory.

There remains the related issue of what is the most general time dependent configuration for these fields. For example one would anticipate that there will be non-spherical fluctuations due to tidal effects in the direction of motion in the curved backgrounds which should alter the geometry of the fuzzy sphere...maybe leading to a fuzzy 'egg'. But these are deformations of the spherical solution, so we would argue that in the first instance one should study these first and then investigate fluctuations about the solutions. This is what we will concern ourselves with in this thesis

To check that our spherical ansatz is at least a consistent one, we consider the equations of motion for the non-Abelian fields ϕ^i in a general curved background as will be derived in a later section in this chapter. Let us consider a background metric of the form

$$ds^2 = -g_{00}dt^2 + g_{xx}dx^a dx^b \delta_{ab} + g_{zz}dz^i dz^j \delta_{ij} \tag{3.21}$$

where a, b run over the p worldvolume directions and i, j are transverse directions to the source. This background could obviously be generated by a stack of coincident branes, or something more exotic. The NS-sector of the non-Abelian action then takes the form

$$S = -T'_p \int d^{p'+1}\zeta STr\left(e^{-\phi}\sqrt{g^p_{xx}g_{00}(1-\lambda^2 g_{zz}g_{00}^{-1}\dot{\phi}^i\dot{\phi}^j\delta_{ij})}\sqrt{1-\frac{1}{2}\lambda^2 g^2_{zz}[\phi^i,\phi^j][\phi^j,\phi^i]}\right) \tag{3.22}$$

Note that restricting the metric components $g_{00} = g_{xx} = g_{zz}^{-1} = H^{1/2}$ the above action reproduces that in (3.2.3) above. Working to leading order in λ the equations of motion for ϕ^i from this action can be seen to yield

$$\frac{d}{dt}(e^{-2\phi}g^{p/2}_{xx}g_{00}^{-1/2}g_{zz}\dot{\phi}^i) = g^2_{zz}[\phi^i,[\phi^j,\phi^i]], \tag{3.23}$$

where we are implicitly using the large N limit to ensure that the STr reduces to a trace - which must be imposed on both sides of the equation. Now let us consider the more general ansatz for the ϕ^i

$$\phi^i = R(t)T^i + \beta(t)Y^i, \quad i = 1,2,3 \tag{3.24}$$

where the matrices Y^i represent some non-spherical orthogonal directions to the $SU(2)$ generators T^i, and without loss of generality we assume that $Tr(T^iY^i) = 0$. Using this property one can easily obtain the equations of motion for $R(t)$ and $\beta(t)$ by substituting the above ansatz into (3.23). In the limit when we send $\beta(t) \to 0$ (i.e our original fuzzy sphere ansatz) the equation of motion for $\beta(t)$ becomes

$$\frac{d}{dt}(e^{-2\phi}g^{p/2}_{xx}g_{00}^{-1/2}g_{zz}\dot{\beta}) = \frac{1}{Tr(Y^iY^j)}g^2_{zz}Tr([T^i,[T^j,T^i]]Y^j) \tag{3.25}$$

Due to the orthogonality of both T^i and Y^j, the second trace factor in (3.25) vanishes so $e^{-2\phi}g^{p/2}_{xx}g_{00}^{-1/2}g_{zz}\dot{\beta}$ is a constant. We can choose this constant to be zero and hence $\dot{\beta}$ also vanishes. It *is* therefore consistent to set $\beta(t) = 0$ at the outset as in our spherical fuzzy sphere ansatz (3.19).

Returning then to our spherical fuzzy sphere ansatz for ϕ^i as argued before, we choose the generators to be in the irreducible representation of the algebra since this will correspond to the minimum energy configuration [33]. We will repeat the following expressions again for convenience. The physical radius of the fuzzy sphere is given by the following relationship at large N

$$r^2 = \frac{\lambda^2}{N}Tr(\phi^i\phi^j\delta_{ij}) = \lambda^2 R(t)^2 C, \tag{3.26}$$

where C is the quadratic Casimir of the representation defined by

$$\sum_i^3 (T^i)^2 = C1_N, \tag{3.27}$$

and 1_N is the $N \times N$ identity matrix. We also note that for the irreducible representation, $C = N^2-1$, which can be approximated by N^2 in the large N limit. For most of our analysis this will be the limit of interest, since there are many complications with the symmetrized trace prescription at finite N [3].

Combining all this information allows us to write the final form of the action as follows

$$S = -T_{p'} \int d^{p'+1}\zeta N H^{-1} \sqrt{(1 - H\lambda^2 \dot{R}^2 C)(1 + 4\lambda^2 HCR^4)} + T_{p'} \int d^{p'+1}\zeta \frac{qN}{H}. \tag{3.28}$$

which is obtained by expanding the square root and dropping all the terms that are order $1/N^2$ or higher. This ensures that the remaining terms are all proportional to the identity matrix and therefore the symmetrization is not necessary. Taking the trace and then re-summing the terms should re-produce (3.28). Note that we have not restricted ourselves to a specific orientation for the brane, therefore we have written the action in terms of the integer $q = \pm 1$. From the definition of the harmonic function we see that the large r limit corresponds to Minkowski space, and the non-Abelian action reduces to its usual flat space form [33, 45, 50]. We can now calculate the associated canonical momentum and energy density from the action, which are defined as follows

$$\tilde{\Pi} = \frac{\Pi}{T_p'} = N\lambda^2 \dot{R} C \sqrt{\frac{(1 + 4\lambda^2 HCR^4)}{(1 - H\lambda^2 \dot{R}^2 C)}} \tag{3.29}$$

$$\tilde{E} = \frac{E}{T_p'} = \frac{N}{H} \sqrt{\frac{(1 + 4\lambda^2 CHR^4)}{(1 - H\lambda^2 \dot{R}^2 C)}} - \frac{qN}{H}, \tag{3.30}$$

where the momentum is the derivative of the Lagrangian with respect to $\dot{R}$, and the energy is constructed via Legendre transform. In addition we have divided out by a factor of $\int d^{p'}\zeta$ which loosely corresponds to the 'volume' of each Dp'-brane. To construct the potential energy we will find it useful to switch to the Hamiltonian formalism, where we write the energy in terms of the conjugate variables.

$$\tilde{E} = \sqrt{N^2 H^{-2}(1 + 4\lambda^2 CHR^4) + \frac{\tilde{\Pi}^2}{H\lambda^2 C}} - \frac{qN}{H}, \tag{3.31}$$

which allows us to define the non-Abelian static potential via $V_{\text{eff}} = \tilde{E}(\tilde{\Pi} = 0)$.

$$V_{\text{eff}} = NH^{-1} \left(\sqrt{1 + 4\lambda^2 CHR^4} - q \right), \tag{3.32}$$

In order to consider the collapse of the fuzzy sphere, it will be more convenient to work in term of the physical radius r rather than R. In which case the potential can be written

$$V_{\text{eff}} = NH^{-1} \left(\sqrt{1 + \frac{4Hr^4}{\lambda^2 C}} - q \right), \tag{3.33}$$

[3] The work in this section was performed well before [53], which conjectured an exact prescription for the symmetrized trace of $SU(2)$ generators.

which is the gravitational potential generated by the background branes located at $r = 0$.

It is useful to compare this result with that coming from the Abelian case [23]. The most direct means of comparing the two cases is to consider the scenario where there are N independent probe branes separated by a distance larger than the string length. The corresponding potential would then simply be N times the potential due to a single probe brane, at leading order.

$$V^{abelian} = \frac{N(1-q)}{H} \tag{3.34}$$

This potential vanishes for $q = 1$ reflecting the BPS nature of the branes, but a potential is generated for anti-branes as we would anticipate. Clearly we see that there is an additional term present in (3.33) compared to (3.34) arising from the non-Abelian nature of the effective action, which we interpret as the energy of the fuzzy sphere. This extra term generates a non-zero potential regardless of the value of q, indicating that the solution is no longer BPS. It is instructive to consider the behaviour of the potential in the different regions of spacetime, but first we must ensure that there are no limiting constraints to be imposed on the solution.

Since the energy is conserved in time we can solve this equation for the physical velocity to obtain the equation of motion, which in turn will yield a constraint on the dynamics.

$$\dot{r}^2 = \frac{1}{H}\left(1 - \frac{N^2}{(\tilde{E}H + qN)^2}\left\{1 + \frac{4Hr^4}{\lambda^2 C}\right\}\right). \tag{3.35}$$

Since this equation is non-negative we see that the following constraint must be satisfied, when we set the Chern-Simons part to zero,

$$1 \geq \frac{N^2}{\tilde{E}^2 H^2}\left\{1 + \frac{4Hr^4}{\lambda^2 C}\right\}.$$

We consider what happens when we are in the near horizon geometry, as the constraint reduces to the following expression

$$1 \geq \frac{N^2}{\tilde{E}^2}\left(\frac{r^{7-p}}{k_p}\right)^2\left\{1 + \frac{4k_p r^{p-3}}{\lambda^2 C}\right\}, \tag{3.36}$$

For $p \geq 3$ the leading term in the expression is expected to be dominant and so we are effectively left with the following constraint

$$1 \geq \frac{N^2}{\tilde{E}^2}\left(\frac{r^{7-p}}{k_p}\right)^2. \tag{3.37}$$

Now the near horizon approximation implies that the term in parenthesis is already vanishingly small, which in turn implies that the ratio $N/\tilde{E}$ can take a wide range of values and still satisfy this constraint. We must emphasise at this point that the classical analysis may break down as the fuzzy sphere collapses toward zero size, since the back reaction upon the source branes will no longer be negligible and there will doubtless be correction terms to the energy which will invalidate this constraint. Furthermore there will also be the problem of open string tachyon modes, which will arise as the branes approach distances comparable to the string length. If we now consider the limiting case where $p < 3$, the constraint equation becomes

$$1 \geq \frac{4}{\tilde{E}^2}\left(\frac{r^{7-p}}{k_p}\right)\frac{r^4}{\lambda^2}, \tag{3.38}$$

when we take the large N limit. This solution has explicit dependence upon the ratio of the radius to the string length, which we would expect to be larger than unity in order for us to have any faith in the effective field theory description. This implies that the energy density can again be reasonably arbitrary as the supergravity constraint implies that the other term is already vanishingly small. To be safe we will assume that $\tilde{E} >> N$ in what follows, as there is no ambiguity in the constraints if this is fulfilled. Interestingly if we reinstate the Chern-Simons contribution we find, to leading order, that the same constraints apply.

For completeness we now turn out attention to the large r region i.e asymptotically flat space. This means that we are approximating the harmonic function by unity. In the Abelian case there are no constraints to be imposed, and so the probe branes can move to an infinitely large distance from the sources [23]. In the non-Abelian case however, we can obtain an equation for the maximum radius of the fuzzy sphere which can be written (dropping a factor of unity)

$$r^4_{max} \sim \frac{\lambda^2 C\tilde{E}^2}{4N^2}\left(1+\frac{2qN}{\tilde{E}}\right), \tag{3.39}$$

from which we deduce that the orientation of the Dp'-branes plays the role of a small correction term provided we take our $\tilde{E} > N$ approximation. This maximal distance represents the limit of our effective action, and it is likely that higher order correction terms will allow us to consider limits such as $r_{max} \to \infty$. We note, however, that this maximal distance is dependent upon the energy of the probe branes, and that by tuning the energy we can effectively consider an unbounded solution in Minkowski space. If we take the large N limit and use the binomial theorem for the Chern-Simons contribution, this equation simplifies to

$$r_{max} \sim \sqrt{\frac{\tilde{E}\lambda}{2}}\left(1+\frac{qN}{2\tilde{E}}+\dots\right) \qquad = \sqrt{\tilde{E}\pi l_s^2}\left(1+\frac{qN}{2\tilde{E}}+\dots\right) \tag{3.40}$$

which shows that the size of the fuzzy sphere is only dependent upon the energy of the solution. This is what we expect from our knowledge of dielectric branes and Giant Gravitons [57], which are expanding brane solutions sourced by non-trivial background fields. Even though we are only looking at a relatively simple example, we would expect to find some similarities between these problems.

Armed with this knowledge from the constraints we may proceed to investigate the behaviour of the effective potential. A quick calculation shows that the potential has no turning point, therefore we shouldn't expect any stable bound states between the fuzzy sphere and the background branes. It will be easier to consider the throat region and the Minkowski region as two decoupled regions of space-time, as this will simplify our analysis. Any solutions to the equations of motion can then be patched together using appropriate boundary conditions. In the throat region, for vanishing r we find the potential becomes

$$V_{\text{eff}} \sim \frac{Nr^{7-p}}{k_p}\left(\sqrt{1+\frac{4k_p r^{p-3}}{\lambda^2 C}}-q\right). \tag{3.41}$$

Now for $p \geq 3$ we may again ignore the radial contribution in the square root, provided that

$$r << \left(\frac{\lambda^2 C}{4k_p}\right)^{3-p},$$

and correspondingly we find that the potential is well approximated by

$$V_{\text{eff}} \sim \frac{Nr^{7-p}}{k_p}(1-q), \tag{3.42}$$

which we can see is identically zero if $q = 1$, and is attractive if $q = -1$. This is the same behaviour as seen for arbitrary p in the Abelian case, and implies that the configuration can become BPS at sufficiently small distances. However the size of this stabilisation radius is likely to be smaller than the string length, where our effective action is not valid. Now if we consider $p < 3$ we find the potential is given by

$$V_{\text{eff}} \sim \frac{Nr^{7-p}}{k_p}\left(\sqrt{\frac{4k_p}{\lambda^2 C r^{3-p}}}-q\right). \tag{3.43}$$

which is attractive for all valid p in this region. Therefore we see that to leading order, the probe branes are always gravitationally attracted toward the source branes.

In the large r limit, remembering that there is a maximum radius for the fuzzy sphere solution to hold, the potential becomes.

$$V_{\text{eff}} \sim N\left(\sqrt{\frac{4r^4}{\lambda^2 C}}-q\right), \tag{3.44}$$

which we see will tend to a positive constant depending upon the exact size of the maximum radius. If we substitute our solution (3.39) into the potential, we find

$$V_{\text{eff}} \sim \tilde{E}\sqrt{1+\frac{2Nq}{\tilde{E}}}-qN \sim \tilde{E}, \tag{3.45}$$

where we have explicitly expanded out the square root term using our energy constraints. Thus the potential energy is effectively the energy density at large r. Before proceeding to solve (3.35), it is worth mentioning that the 'velocity' of the collapse is a decreasing function of time. This is in stark contrast to the fuzzy sphere in a flat Minkowski background, where we find that at small r, the velocity is a substantial fraction of the speed of light [50]. The curved geometry of spacetime in the near horizon limit acts in such a way as to slow the rate of collapse, in fact for an observer on the background branes it would take an infinite amount of time for the sphere to reach zero size. Only if we switch to conformal time will we see a finite time solution. This is an example of the usual red shift phenomenon in General Relativity.

In the large r region, we see that the harmonic function is well approximated by unity and we would expect to find the usual equations of motion for collapsing fuzzy spheres in flat space [51]. Using the fact that the energy is conserved in time, we can integrate the equation of motion to obtain the general form of the radial collapse in terms of Jacobi elliptic functions. By carefully selecting our initial value of r_0 to be

$$r_0^4 = \frac{\lambda^2 C\tilde{E}}{4N}\left(\frac{\tilde{E}}{N}+2q\right), \tag{3.46}$$

we find that the equation of motion is given by

$$r(t) = \pm r_0 \mathrm{JacobiCN}\left[2\sqrt{\frac{2}{C}}\frac{r_0 t}{\sqrt{1+\frac{4r_0}{\lambda^2 C}}}, \frac{1}{\sqrt{2}}\right] \tag{3.47}$$

The form of this solution has been extensively discussed in [51], and so we will not say much about it here. In this instance we know that the regime of validity for the solution is $r^{7-p} >> k_p$ and so we find a simple monotonically expanding/contracting solution without collapse toward zero size. Thus the effective action should remain a valid description of the dynamics, and we do not have to worry about the physical nature of the coordinate system being employed. Interestingly this solution appears to be valid for arbitrary values of p since all the p dependence arises in the form of the harmonic function, and gives rise to another example of the so called p-brane democracy. Unfortunately the precise form of the equation of motion in this limit will make it difficult to obtain smooth analytic solutions interpolating between flat space and the near horizon geometry.

Turning our attention to the throat solutions, we see that the complicated form of the equation of motion also makes analytic solutions difficult to obtain. One case where we can make some progress is the $p = 3$ background, as the 'fuzzy' term loses all radial dependence in this instance. The time evolution is given in terms of a hypergeometric function, and it thus difficult to invert to obtain a closed form expression for r

$$t - t_0 \sim \pm\frac{\sqrt{k_3}}{r}\,{}_2F_1\left(\frac{1}{2}, \frac{-1}{8}, \frac{7}{8}, \frac{N^2 r^8}{\tilde{E}^2 k_3^2}\left\{1+\frac{4k_3}{\lambda^2 C}\right\}\right). \tag{3.48}$$

In the limit that the sphere collapses toward zero size, we can expand the hypergeometric function using the well known series expansion

$$t - t_0 \sim \pm\frac{\sqrt{k_3}}{r}\left(1 - \frac{N^2r^8}{14\tilde{E}^2k_3^2}\left\{1 + \frac{4k_3}{\lambda^2 C}\right\}\right), \tag{3.49}$$

which implies that at very late times the solution behaves as

$$r \sim \pm\frac{\sqrt{k_3}}{t - t_0}. \tag{3.50}$$

The collapse of the sphere is described by the positive branch of the above solution, and is in fact an example of a simple power law solution. This power law behaviour can be explicitly seen at late times in the general case by assuming that the dominant contribution to the denominator of (3.35) is unity. The resulting integral is trivial to perform and we obtain the general late time solution (dropping constants of integration)

$$r \sim \pm\left(\frac{(p-5)(t-t_0)}{2\sqrt{k_p}}\right)^{2/(p-5)}, \tag{3.51}$$

the solution for $p = 5$ must be calculated separately, but is simply proportional to an exponential

$$r \sim \exp\left(\pm\frac{t}{\sqrt{k_5}}\right). \tag{3.52}$$

Thus we have shown that the solutions obey simple power law equations of motion as $r \to 0$. Of course, we must be careful in our interpretation of these results as we expect corrections to affect the validity of our effective action as the fuzzy sphere collapses.

Whilst we cannot analytically solve the equations of motion in the throat limit unless we assume the $r \to 0$ limit, we can attempt to solve them numerically, which gives us some indication of the late time dynamics as measured by observers on the source branes. Figure 3.1, for example, shows the numerical solution for $D0$ and $\bar{D}0$ branes. In order to generate this solution we took $l_s = 1$, $g_s = 0.1$, $N = 100$, $\tilde{E} = 200$ and $M = 1000$, whilst retaining the full form of the harmonic function but taking the large N limit. Although the parameter space of solutions is large, we expect the numerical solutions to be representative of more general behaviour. In fact we investigated the dynamics for various ranges of energy, and found approximately the same solutions with all the solution curves collapsing onto one another at very small distances. The analytic solution clearly shows that the radius collapses rapidly when the metric is approximately flat, but decelerates as the sphere enters the near horizon geometry. We expect that our solutions will break down as the probes near the source branes, although it is useful to recall that $D0$-branes can probe distances smaller than the string length and so the solution may be valid for some time [44]. The plot shows that the brane and anti-brane follow similar trajectories as they cross into the near horizon region and are thus indistinguishable. Our analysis of the potential suggests that it should

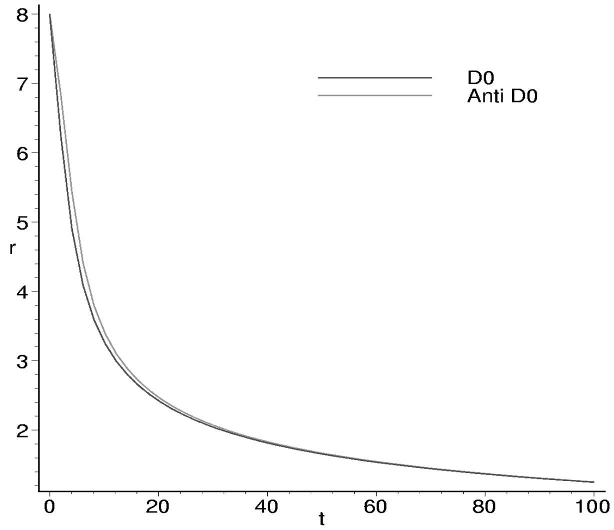

Figure 3.1: Numerical solution to the equations of motion for the $D0$-brane background.

vanish for the $D0$-brane solution as $r \to 0$. Clearly our plot shows that this must happen at a distance smaller than the string scale.

Figure 3.2 shows the solutions for the $D4$ and $D5$-brane backgrounds using the same parameters, but ignoring the Chern-Simons term. The five brane solution indeed tends toward an exponential at late times as expected from our simplified analytic solution.

Figure 3.3 shows the solution for the $D3$ and $\bar{D}3$-branes. In this instance we can clearly see that the fuzzy sphere associated with the $D3$ solution collapses faster than the $\bar{D}3$ solution when they are in flat space. This is because the $D3$-branes are more strongly attracted to the sources than the $\bar{D}3$-branes. However as they cross into the near horizon geometry, both spheres tend to the same radius as the Chern-Simons term becomes negligible which accounts for the similarity in their dynamics.

The difficulty in analytically solving the integral equation of motion is related to the fact that it describes curves on hyper-elliptic Riemann surfaces, with the infinitesimal time playing the role of a holomorphic differential [51]. The velocity and the radius can each be regarded as two complex variables related by a single constraint. We can use the simplified Riemann Hurwitz formula to calculate the genus, g, of the underlying surface

$$g = \frac{1}{2}(B - 2), \tag{3.53}$$

where B refers to the number of branch points - which is the same as the degree of the corresponding polynomial in the equation of motion. This is a special case of the general

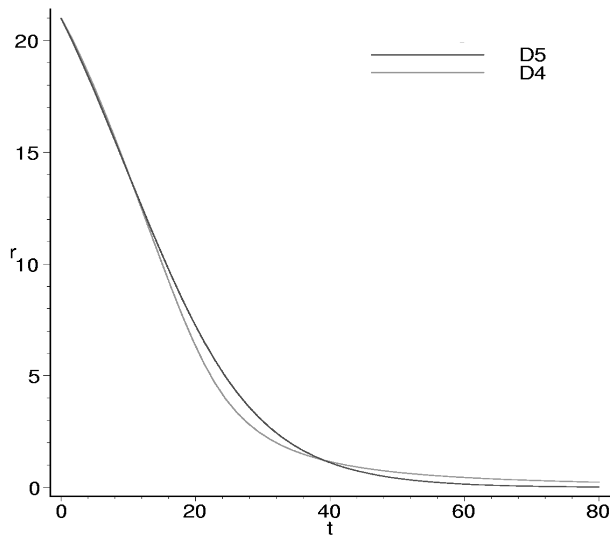

Figure 3.2: Solutions for $D4$ and $D5$ brane backgrounds, ignoring the Chern-Simons coupling.

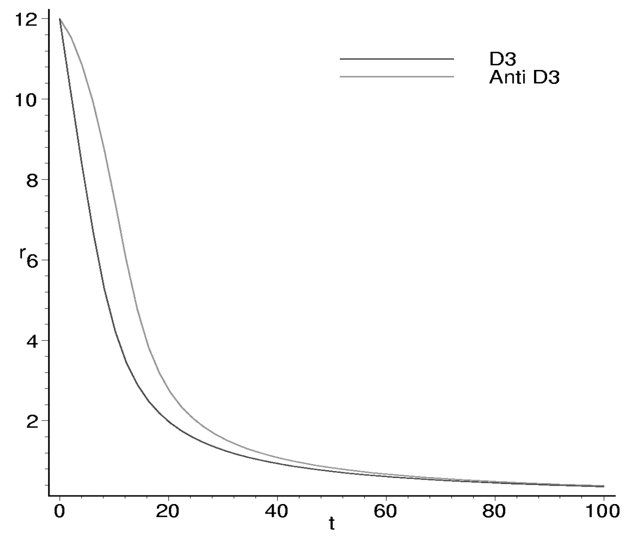

Figure 3.3: Solutions for the fuzzy sphere sourced by $D3$ and anti $D3$-branes.

expression

$$g = n(G-1) + 1 + \frac{B}{2}, \tag{3.54}$$

where G is the target space genus - which is zero because the target space object is a sphere, and n is the degree of the mapping between the covering space and the target space - which is two in this case. It is fairly straight forward to see that the $p = 6$ and $p = 5$ cases correspond to genus 2 surfaces, $p = 3, 4$ give rise to genus 3 surfaces, $p = 2, 1$ are genus 5 surfaces and $p = 0$ defines a genus 7 surface. Thus as we decrease the dimensionality of the background branes, we find surfaces of higher and higher genus. Obviously this leads to the difficulty in obtaining an analytic solution to the equation of motion. Even if we include the Chen-Simons term in the equation of motion, this doesn't modify the number of branch points. As in [51]it may be possible to reduce the integral for the genus 3 and 5 surfaces into integrals over products of genus 1 surfaces using the special symmetries present.

It is important to note that the fuzzy sphere solutions in flat space correspond to a genus 1 surface (a torus), which provides a hint as to why it is possible to find an an explicit solution to the equation of motion. By contrast the Riemann surfaces corresponding to collapse in curved backgrounds are actually of a higher genus, with the branch points on the complex plane being totally unresolvable when the cycles are large. This is suggestive that the higher genus surfaces naturally have more structure, which makes analytic solutions difficult. However this may be resolved by keeping higher order terms in the effective action, or by including finite N effects.

3.2.4 Dynamics in the $p \neq p'$ case.

We now turn our attention to the more general case where $p \neq p'$. However, as we are only looking at the leading order terms in the action we find that there is no Chern-Simons term except for the $p = 6, p' = 0$ case. But we will neglect this contribution in this thesis, and focus solely on the NS sector. The action in this instance is a simple extension of (3.28), and can be written in the following way

$$S = -T_{p'} \int d^{p'+1}\zeta N H^{(p-p'-4)/4} \sqrt{(1+4H\lambda^2 C R^4)(1-H\lambda^2 C\dot{R}^2)}, \tag{3.55}$$

which clearly reduces to the expression in the previous section when taking the $p = p'$ limit. We will again divide out by the 'mass' of the brane to find a closed form expression for the canonical momentum, which turns out to be

$$\tilde{\Pi} = N H^{(p-p'-4)/4} \lambda^2 C \dot{R} \sqrt{\frac{1+4H\lambda^2 C R^4}{1-H\lambda^2 C\dot{R}^2}}, \tag{3.56}$$

and the corresponding energy is obtained via Legendre transformation in the usual manner.

$$\begin{aligned} \tilde{E} &= NH^{(p-p'-4)/4}\sqrt{\frac{1+4H\lambda^2CR^4}{1-H\lambda^2C\dot{R}^2}} \\ &= \sqrt{N^2H^{(p-p'-4)/2}(1+4H\lambda^2CR^4)+\frac{\tilde{\Pi}^2}{H\lambda^2C}}. \end{aligned} \tag{3.57}$$

Extending the results from the previous section we will define the effective potential as follows

$$V_{eff} = NH^{(p-p'-4)/4}\sqrt{1+\frac{4Hr^4}{\lambda^2C}}, \tag{3.58}$$

which is clearly the general extension of (3.32) when there is no Chern-Simons coupling term. Once again we see that the additional energy density due to the fuzzy sphere is responsible for generating a non-trivial potential.

Using the conservation of energy we also have a modified constraint condition

$$1 \geq \frac{N^2H^{(p-p'-4)/2}}{\tilde{E}^2}\left(1+\frac{4Hr^4}{\lambda^2C}\right). \tag{3.59}$$

In the near horizon geometry we see that the RHS blows up as as the radius tends to zero when $p-p' > 4$ which, because of the dimensionality of the branes, implies that for the $p=6, p'=0$ case the energy must go to infinity as the fuzzy sphere collapses in order to satisfy the constraint. All of the other solutions are satisfied for arbitrary energy in this limit. This tells us that the $D6$-$D0$ solution will not collapse to zero size, instead it will be energetically favourable for the fuzzy sphere to expand in the near horizon geometry. In the large r limit we again expect there to be a maximum size for the fuzzy sphere solution, which is given by (3.40) when we take the large N limit.

By analysing the behaviour of the effective potential we should get a general understanding of the dynamics of the fuzzy sphere as the probe branes are attracted towards the source branes. In general we see that the potential is always attractive, implying that the fuzzy sphere will eventually collapse down toward zero size. The case where this isn't true is when $p=6$ and $p'=0$, which has a repulsive potential at small radius exactly as expected from the energy constraints. We will have more to say about the $D6$-$D0$ configuration in a later section as this is an example of the gravitational Myers effect [55], and is relevant for constructing a microscopic description of the Quantum Hall soliton. The other case where the potential does not vanish is when $p-p'=4$, corresponding to the cases $p=6, p'=2$; $p=5, p'=1$ and $p=4, p'=0$. In these cases we see that the potential tends to N with vanishing radius. Again this should be expected as the branes are all parallel and this is precisely the supersymmetry preserving condition in the Abelian theory, however this may well occur at distances beyond the regime of validity of our effective theory.

Solving the equations of motion in the general case is far from trivial, as the integral equation gives rise to a description of Riemann surfaces of varying genus. For completeness we have written the genus associated with all the possible values of p, p' in the following table. Note that as the factor $p - p'$ increases, the genus of the surface associated with the solution decreases. For example in the $p - p' = 4$ case (not including $p = 6$), we see that the Riemann surface becomes a simple two-sphere. This is interesting as we know that this is exactly the supersymmetry preserving condition in the Abelian theory, and a quick calculation verifies that the Abelian equation also yields a genus 0 surface even in the $p = 6, p' = 2$ case. This poses the question of whether there is some deeper connection between the preservation of supersymmetry and the underlying Riemannian geometry. An example solution can be found in the $p = 4, p' = 0$ case which will be valid when r satisfies the following constraint, $\lambda^2 \tilde{E}^2 >> 4k_4 r$. Upon integration we find

$$r \sim r_0 \pm \frac{4\tilde{E}k_4}{(\tilde{E}^2 - N^2)t^2}, \tag{3.60}$$

where we must take the negative branch of the solution to approximate the collapsing fuzzy sphere.

p	6				5			4			3		2		1	0
p'	6	4	2	0	5	3	1	4	2	0	3	1	2	0	1	0
genus	2	2	1	1	2	1	0	3	2	0	3	1	5	2	5	7

3.2.5 Corrections from the symmetrized trace.

In our work so far we have only considered the leading order Lagrangian, and neglected any $1/N$ corrections. This was essential in order to avoid the complications introduced by the symmetrization over the fields. However some of these terms can be calculated allowing corrections to the effective potential to be found [50]. We remind the reader that to lowest order, we have calculated the energy density to be

$$\tilde{E} = \frac{\delta \mathcal{L}}{\delta \dot{R}} \dot{R} - \mathcal{L}.$$

Based upon arguments in [50] we know that the corrections to next order are given by

$$\tilde{E}_1 = \left(1 - \frac{2}{3} C \frac{\partial^2}{\partial C^2}\right) \tilde{E}, \tag{3.61}$$

We differentiate our expression for $\tilde{E}$ in order to find the next order corrections to the effective potential. Note that for static BPS configurations such as the $D1$-$D3$ intersection in flat space, all the symmetrized trace correction terms are zero [51]. This will be important for later discussions. We don't anticipate the same situation occurring here because the Chern-Simons coupling is independent of C at leading order and will drop out when we differentiate the Lagrangian. Since it is this coupling which (in the Abelian case at least) preserves the bulk supersymmetries, we expect that higher order corrections will, in general, not be BPS configurations and so we should find non-zero correction terms to all orders.

Our calculation gives us the following first order correction to the potential (where we write $V_1 = V_{\rm eff} + \Delta V_{\rm eff}$)

$$\Delta V_{\rm eff} = \frac{8NH^{(p-p'+4)/4}r^8}{3\lambda^4 C^3\left(1+\frac{4Hr^4}{\lambda^2 C}\right)^{3/2}}, \tag{3.62}$$

where we have made explicit use of the near horizon approximation to simplify the result. Once more we find that the solution depends heavily upon the dimensionality of the branes involved. Firstly we will consider the case when $p \geq 3$. In this instance the correction term becomes;

$$\Delta V \sim \frac{8Nr^8 b}{3\lambda^4 C^3}\left(\frac{1}{r^{7-p}}\right)^{(p-p'+4)/4}, \tag{3.63}$$

where we have introduced the quantity $b = k^{(p-p'+4)/4}$ to simplify the notation. In general the factor of $p-p'$ can only take the integral values of $6, 4, 2$ or 0, and so it is easily noted that the potential tends to zero as $r \to 0$ for all values of p and p' in this particular range. If on the other hand we consider the case where $p < 3$ then $p - p'$ is limited to be either 2 or 0. The correction term in this instance reduces to

$$\Delta V \sim \frac{8Nr^8 b}{3\lambda^4 C^3}\left(\frac{1}{r^{7-p}}\right)^{(p-p'+4)/4}\left(\frac{\lambda^2 C}{4k_p r^{p-3}}\right)^{3/2} \tag{3.64}$$

This potential again tends to zero with r for all values of p and p', which is in agreement with our general expectations from the behaviour of the leading order term. Thus the correction doesn't alter the overall dynamics of the fuzzy sphere, and we don't find any bounce solutions. However it should be noted that if we relax our throat approximation and look at large r, we would expect to find differing behaviour. For example [50]showed that there are bounce solutions for the $D0$-solution in flat Minkowski space when the $1/N$ sub leading order terms are taken into account. It is well known that $D0$-branes may probe distances much smaller than the string length, however the curved backgrounds we have been studying in this section appear to impose constraints upon this behaviour [4].

3.2.6 Remarks on the $D6$-$D0$ solution.

In this section we will briefly comment on the $p = 6, p' = 0$ solution as there is a similarity with the Quantum Hall Soliton [56], which we will briefly introduce below.

The stringy QHS was introduced as a way of establishing the link between condensed

[4] In fact it was shown that this 'bounce' was dependent on the order of the correction, and in fact vanishes when computed using the finite N proposal for the symmetrized trace.

matter physics and string theory. To construct the QHS we imagine a type IIA background of k coincident $D6$-branes with k strings emerging from them. The transverse space to the $D6$-branes can be parameterised by $\mathbb{R}\times\mathbf{S}^2$, and we wrap a $D2$-brane over the $\mathbf{S}^2$. However it is known that this configuration is unstable, and so we are forced to introduce $ND0$-branes, which are dissolved into the $D2$-brane world volume. Since it is well known that $D6$ and $D0$-branes repel each other (due to the energy becoming infinite at small distances), this extra repulsion stabilises the QHS. The $D6$ is charged under the seven-form RR gauge field $*C_{(1)}$, which is the dual of the field charging the $D0$-branes. This is analogous to the fact that F-strings cannot end on $NS5$-branes since they are either electrically or magnetically charged under the NS two form field. It was further shown that the world-volume of the spherical $D2$-brane is the surface where the quantum hall fluid lives.

This is a purely macroscopic Abelian construction in terms of the $D2$ picture, however our Non-Abelian construction can provide information on the dual microscopic picture. This is because we can consider $ND0$-branes in the supergravity background of M coincident $D6$-branes. We expect that the fuzzy sphere ansatz will play the role of the $D2$-brane with flux on the Abelian side, furthermore we anticipate that the $D0$-branes can be regarded as being the endpoints of fundamental strings which start on the background $D6$-branes. The only difference is that we are neglecting the open string contributions from the background branes to the probe branes. This is similar to the Myers Dielectric effect [33], except that it is the gravitational field that 'puffs up' the $D0$-branes rather than non-trivial Chern-Simons terms. Consequently this phenomenon has been labelled the 'Gravitational Myers Effect' in the literature [55]. From the previous section we know that the effective potential for this (bosonic) configuration can be written as follows

$$V = N\sqrt{H}\sqrt{1+\frac{4Hr^4}{\lambda^2 c}}, \tag{3.65}$$

where the harmonic function, H, can be approximated in the near horizon limit by

$$H \sim \frac{Mg_s l_s}{2r}.$$

We now determine that the potential has a minimum at the distance

$$r_{min} = \left(\frac{\pi^2 l_s^3 N^2}{Mg_s}\right)^{1/3} \tag{3.66}$$

where we have explicitly employed the use of the large N limit. This is exactly the same result that was obtained for the stability of the spherical $D2$-brane with flux in terms of the Gravitational Myers effect effect [55]. We wish to compare this result to the one calculated in [56], which relied on the $D2$-brane theory. In that paper they used a coordinate rescaling to simplify the initial background metric. The scaling is given by

$$r = \rho \left(\frac{M g_s}{2} \right)^{-1/3},$$

and consequently the equation for the stabilisation radius (denoted by ρ_*) is given by

$$\rho_* = \frac{(N\pi)^{2/3} l_s}{2}. \tag{3.67}$$

Performing the same rescaling in our Non-Abelian dual picture gives the result

$$\rho_* = \frac{(N\pi)^{2/3} l_s}{2^{1/3}} \tag{3.68}$$

which is almost identical to the Abelian theory. In fact the discrepancy between the two radii is due to the contribution from the k strings on the Abelian side, which has been neglected in our analysis. In fact the string contribution appears to alter the stabilization radius only by a factor of $2^{-2/3}$. If we reconstruct the Abelian action for the QHS, but neglect the stringy contribution and allow for time dependent radial solutions we obtain the following

$$S = -T_2 \int d^3\zeta H r^2 \sin(\theta) \sqrt{(1 - H\dot{r}^2)\left(1 + \frac{\lambda^2 N^2}{4Hr^4}\right)}, \tag{3.69}$$

where we are using spherical coordinates to parameterise the $D2$-brane worldvolume and the flux on the brane is given by

$$F_{\theta\phi} = \frac{N \sin(\theta)}{2}, \tag{3.70}$$

which satisfies the usual quantization conditions. For a more rigorous explanation of the derivation we refer the reader to [56] for more details. We can integrate out the angular dependence to find an exact expression for the Lagrangian

$$\mathcal{L} = -T_2 4\pi r^2 H \sqrt{(1 - H\dot{r}^2)\left(1 + \frac{\lambda^2 N^2}{4Hr^4}\right)}. \tag{3.71}$$

Using this we can easily construct the static potential for the Abelian theory in the near horizon geometry, which we find to be

$$V = \frac{kr}{\lambda} \sqrt{1 + \frac{\lambda^2 N^2}{2k g_s l_s r^3}}. \tag{3.72}$$

Although this appears to be different from the non-Abelian potential, they are in fact identical as can be verified with a simple expansion. Thus the theories are in fact dual to one another, which we can further exhibit by analysing the equations of motion for the radion fields. Using subscripts A (for Abelian) and N (for non-Abelian) to represent the two theories, we find the result

$$\begin{aligned}\dot{r}_A^2 &= \frac{1}{H}\left(1 - \frac{16\pi^2 T_2^2 H^2 r^4}{E^2}\left\{1 + \frac{\lambda^2 N^2}{4Hr^4}\right\}\right), \\ \dot{r}_N^2 &= \frac{1}{H}\left(1 - \frac{T_0^2 N^2 H}{E^2}\left\{1 + \frac{4Hr^4}{\lambda^2 C}\right\}\right).\end{aligned} \tag{3.73}$$

If we take the large N limit and carefully expand these equations, using the definition of the brane tension in each case, we see that they are identical. This was noted for the case of a fuzzy sphere in flat space, and as expected this duality continues to hold in a curved geometry. On the Abelian side we find an explicit example of the gravitational dielectric effect, whilst on the non-Abelian side we have the gravitational Myers effect. It would be useful to include the terms coming from the strings in our work, as this would be the dual of the QHS, however this is expected to be complicated as the strings are charged under $U(M)$ on one end and $U(N)$ on the other. The corresponding trace over the Chan-Paton factors will be expected to yield an extra term in the DBI forcing the fuzzy sphere to stabilise at a smaller radius due to the tension of the strings.

As a further remark we should note that this macroscopic/microscopic duality only holds for the $p = 6, p' = 0$ case. We could consider a different background source such as $D4$, $D2$ or $D0$-branes, with a single probe $D2$-brane wrapped over a transverse $\mathbf{S}^2$ whilst the remaining transverse coordinated are set to zero. Unfortunately the corresponding solutions do not map across to the non-Abelian construction where we would have $D0$-branes probing each of these background solutions. This is because we are losing information about the theory by setting some of the Abelian degrees of freedom to zero.

It is interesting to examine the stability of our solution with regards to $D0$-brane emission. It was argued for the QHS that there is an energy barrier proportional to N, preventing the tunnelling of $D0$-branes out of the $D2$ brane. In fact it requires energy to be put into the system to remove the $D0$-brane. Therefore the QHS appears to be stable with respect to particle emission [5]. The potential at the stable radius in our dual picture can be written explicitly as

$$V = N\sqrt{\frac{(Mg_s)^{4/3}}{2(N\pi)^{2/3}}}\sqrt{1 + \frac{N^2}{2C}}, \tag{3.74}$$

where we are using the dimensionless potential obtained from $\tilde{E}$. We now revert to proper time as measured by an observer on the fuzzy sphere, which allows us to re-write the

[5] [56] also noted that there could be possible nucleation of the $D2$-brane causing another $D2$ brane to appear inside the original one. Although we can consider multiple fuzzy spheres by selecting an ansatz which is a reducible representation, this does not correspond to the QHS picture on the Abelian side. It would be certainly interesting to consider a non-Abelian description of this.

minimised potential with respect to proper time

$$V_T(N) = \sqrt{\frac{N}{\pi}} \frac{Mg_s}{2^{3/4}} \sqrt{1 + \frac{N^2}{2C}}. \tag{3.75}$$

Now imagine that the soliton emits a single $D0$-brane into the bulk, the change in the potential - to leading order in $1/N$, and taking the large N limit - can be approximated by

$$V_T(N) - V_T(N-1) \sim \sqrt{\frac{3}{N\pi}} M g_s. \tag{3.76}$$

We now need to compare this with the potential energy of a single $D0$-brane attached to a fuzzy sphere located at the stabilisation radius. Although our effective action is valid as a large N expansion, we can use it to determine the potential for a single brane provided that we neglect the back reaction terms between brane and fuzzy sphere. By adding this contribution to the one calculated in the previous line we see that

$$V_{tot} \sim \frac{Mg_s}{\sqrt{\pi}} \left(\sqrt{\frac{3}{N}} + \frac{1}{\sqrt{2}} \right), \tag{3.77}$$

which is larger than the potential of the stable fuzzy sphere. Thus we conclude that the solution appears to be stable with regard to emission. This gives us an estimate of the binding energy of the $D0$-branes in the near horizon region, which we interpret as the energy barrier needed for quantum tunnelling

$$E_{\text{binding}} \sim \nu g_s \sqrt{N}, \tag{3.78}$$

where we have made use of the ratio $\nu = M/N$ to simplify the result. In the QHS picture this corresponds to the definition of the filling ratio. Clearly the barrier is an increasing function of N, thus in the large N limit we would expect the fuzzy sphere to be stable.

The supergravity description is then the following. If the fuzzy sphere is initially large, then the metric is approximately Minkowski and we have our usual collapsing solution with velocity approaching that of light. As the $D0$-branes enter the near horizon geometry they decelerate (from the $D6$ viewpoint) until they oscillate around the minimum of the potential, eventually forming a bound state at r_{min}. If on the other hand, the fuzzy sphere is initially small, then the gravitational dielectric effect forces the configuration to expand until it reaches the stabilisation radius - at which point it settles into its bound state after oscillation.

3.2.7 Inclusion of Angular Momentum.

Our analysis thus far has only considered radial dynamics, which is not the most general solution possible. Therefore it would be interesting to include something akin to angular momentum in our theory - which will be the focus of the next section. In the Abelian case the inclusion of angular momentum terms in the action is essentially trivial since all the coordinates commute [23]. This will clearly not be the case in the Non-Abelian version and so we must choose a specific ansatz for the scalar fields. A fuzzy cylinder ansatz was introduced in, which was able to rotate about three independent axes. However this ansatz proves to be restrictive on the dimensionality of the background brane solutions limiting them to $p \leq 3$, although it may be useful in describing dual versions of supertubes and we will have a closer look at it in the next section. Instead we choose a different ansatz corresponding to rotation in the $\phi^6 - \phi^7$ plane [42],

$$\begin{aligned} \phi^6 &= R(t)\cos(\theta)T_3, \\ \phi^7 &= R(t)\sin(\theta)T_3, \\ \phi^8 &= R(t)T_1, \\ \phi^9 &= R(t)T_2. \end{aligned} \tag{3.79}$$

This means that the resulting theory will only be valid for $p < 6$ - as we require four transverse directions for this ansatz, and so we will not be able to consider rotation in the Gravitational Myers effect. The action for this particular ansatz can be calculated and we find (again assuming large N)

$$S = -T_{p'} \int d^{p'+1}\zeta \sum_{j=0}^{N} NH^{(p-p'-4)/4} \sqrt{(1+4H\lambda^2 CR^4)(1-H\lambda^2 C\dot{R}^2 - H\lambda^2 R^2\dot{\theta}^2\lambda_j^2)}. \tag{3.80}$$

where λ_j is the jth eigenvalue of the matrix $(T_3)^2$ (using a matrix representation for the diagonal generator). If we expand the action out to leading order this enables us to isolate the λ_j dependence and we can perform the sum to obtain

$$\sum_{j=0}^{N} \lambda_j^2 = \frac{N}{12}(N^2-1) = \frac{CN}{12}. \tag{3.81}$$

In general, the inclusion of angular momentum for the fuzzy sphere is non-trivial. If we employ a convention where the subscript on the λ implies summation over that variable then we find the exact solution for the static potential in physical radius is given by

$$V_{eff} = \frac{NH^{(p-p'-4)/4}}{\sqrt{1-H\lambda^2R^2\dot{\theta}^2\lambda_j^2}} \sqrt{1+\frac{4Hr^4}{\lambda^2 C}} \left(\frac{HNr^2\dot{\theta}^2}{12} + \sqrt{1-H\lambda^2R^2\dot{\theta}^2\lambda_k^2}\sqrt{1-H\lambda^2R^2\dot{\theta}^2\lambda_j^2} \right), \tag{3.82}$$

where $\dot{\theta}$ corresponds to the angular velocity of the fuzzy sphere. By setting this term to zero we recover the result for the purely radial collapse, as we would anticipate. Even though we cannot find a closed form solution for the potential we can still make some comments about the dynamics of the fuzzy sphere. Interestingly we expect that the potential will vanish in the $r \to 0$ limit, as the only case where there is the possibility of a bound state is when $p - p' > 4$ corresponding to the $p = 6, p' = 0$ case we investigated in the previous section. Unfortunately our choice of ansatz doesn't allow for this to be investigated here. This tells us that the angular momentum term cannot counteract the gravitational force exerted by the source branes, and the fuzzy sphere will always collapse.

3.2.8 Alternative ansatz.

Thus far our analysis has been exact but does not allow us to obtain closed form solutions describing the full dynamics, so it is useful to consider an alternative ansatz which allows us to incorporate angular momentum in a clear manner. Since we require two transverse scalars to define a plane in the transverse space, and at most each plane is parameterised by one of the generators of the representation, we are led to the conclusion that we should use six transverse scalars to introduce angular momentum. This will obviously place severe restriction upon the dimensionality of the branes that we can consider in our solution, since not all background solutions admit spacetimes with six-transverse directions. In fact we find that at most we can consider a $D3$-brane background. We choose to parameterise the six transverse scalars as follows - temporarily dropping the group generators from our notation:

$$\begin{aligned} \phi^1 &= R(t)\cos(\theta) \qquad \phi^2 = \mathrm{R(t)sin}(\theta) \\ \phi^3 &= R(t)\cos(\theta) \qquad \phi^4 = \mathrm{R(t)sin}(\theta) \\ \phi^5 &= R(t)\cos(\theta) \qquad \phi^6 = \mathrm{R(t)sin}(\theta) \end{aligned} \tag{3.83}$$

Thus we are breaking the $SO(6)$ symmetry of the transverse space down to $SO(2)\times SO(2)\times SO(2)$, and choosing the same angle θ to parameterise the three planes. This may seem a rather restrictive ansatz but it will actually allow us to make some progress. The action in this case becomes

$$S = -T_{p'}\int d^{p'+1}\zeta STr\left(H^{(p-p'-4)/4}\sqrt{1 - H\lambda^2 C(\dot{R}^2 + R^2\dot{\theta}^2))(1 + 4\lambda^2 HR^4 C)}\right), \tag{3.84}$$

with a possible Chern-Simons term, defined up to a constant factor

$$S_{CS} = +T_{p'}\delta^p_{p'}\int dt\frac{q}{H}. \tag{3.85}$$

Since both terms in the Born-Infeld part of the action are proportional to the identity matrix, we find that the STr reduces to Tr to leading order in large N. Finally we obtain

$$S = -T_{p'}\int d^{p'+1}\zeta NH^{(p-p'-4)/4}\sqrt{(1 - H\lambda^2 C(\dot{R}^2 + R^2\dot{\theta}^2))(1 + 4\lambda^2 HR^4 C)}. \tag{3.86}$$

We can now proceed as usual by switching to the Hamiltonian formalism and writing the canonical energy density as

$$\tilde{E} = \sqrt{N^2 H^{(p-p'-4)/2}(1 + 4\lambda^2 CHR^4) + \frac{1}{H\lambda^2 C}\left(\tilde{\Pi}^2 + \frac{\tilde{L}^2}{R^2}\right)}. \tag{3.87}$$

Reverting to the formulation in terms of the physical radius r, we find that the effective potential becomes

$$V_{\text{eff}} = \sqrt{N^2 H^{(p-p'-4)/2}\left(1 + \frac{4Hr^4}{\lambda^2 C}\right) + \frac{\tilde{L}^2}{Hr^2}} \tag{3.88}$$

Where we must remember that this equation is only valid for $p \leq 3$, and that both the energy density and the angular momentum are conserved charges.

For ease of calculation we choose to rescale the potential by a factor of N. This is possible because there is an N^2 term in the angular momentum density. The resulting non-Abelian and Abelian potentials are written below for comparative purposes

$$\begin{aligned} \bar{V}_{\text{eff}} &= \sqrt{H^{(p-p'-4)/2}\left(1 + \frac{4Hr^4}{\lambda^2 C}\right) + \frac{\tilde{L}^2}{Hr^2}}, \\ V^{\text{abelian}} &= \sqrt{H^{(p-p'-4)/2} + \frac{\tilde{L}^2}{Hr^2}}. \end{aligned} \tag{3.89}$$

Simple analysis of the potential in the non-Abelian case shows that it is a monotonically decreasing function for all valid p and p' in this regime. Therefore there is no possibility of the formation of bound states, in the same way that there are no bound orbits in the Abelian theory. Once again it is useful to look at the equations of motion to determine if there are any constraints to be imposed on the solution. We wish to consider a case where the energy density and the angular momentum density are constant. Thus we find the following expression

$$\dot{r}^2 = \frac{1}{H}\left(1 - \frac{1}{\tilde{E}^2}\left[N^2 H^{(p-p'-4)/2}\left\{1 + \frac{4Hr^4}{\lambda^2 C}\right\} + \frac{\tilde{L}^2}{Hr^2}\right]\right). \tag{3.90}$$

If we assume that the angular momentum takes some fixed, non-zero value - then we can consider how the constraint equation is modified in the asymptotic limit of $r \to 0$

$$1 \geq \frac{1}{\tilde{E}^2}\left(\frac{4N^2 k_p^{(p-p'-4)/4} k_p}{\lambda^2 C r^{((7-p)(p-p'-4)+6-2p)/2}} + \frac{\tilde{L}^2 r^{5-p}}{k_p}\right). \tag{3.91}$$

This appears to have a complicated dependence upon r, however because of the restrictions from the ansatz we know that there are only two possible cases we can consider, $i)$ $p - p' = 2$

and *ii*) $p - p' = 0$. The first case reduces the constraint to the following

$$1 \geq \frac{1}{\tilde{E}^2}\left(r^4 + \tilde{L}^2 r^{5-p}\right). \tag{3.92}$$

It is clear that as r vanishes the contribution from the angular momentum term also vanishes and the energy density can be relatively arbitrary, as already discussed. The second condition implies a similar result, however the dimensionalities of the branes involved plays a role in determining how quickly the leading term vanishes.

3.2.9 Non-BPS branes.

It is well known that BPS branes can be interpreted as solitonic solutions of Non-BPS branes [28], so it is natural to enquire about the dynamics of these branes in various backgrounds. In this section we will look at the action for N Non-BPS branes in the Dp-brane background and try and study the dynamical evolution of the fuzzy sphere in this instance. This will not be as straightforward to analyse as the BPS case, as there is the additional complication of open string tachyon modes condensing on the world volume. We begin this section by introducing the non-Abelian extension of the non-BPS Dp-brane action [39].

$$S = -\int d^{p+1}\zeta STrV(T)e^{-(\phi-\phi_0)}\sqrt{-\det(\mathcal{P}[E_{ab}+E_{ai}(Q^{-1}-\delta)^{ij}E_{jb}]+\lambda F_{ab}+T_{ab})}$$
$$\times\sqrt{\det Q^i_j}. \tag{3.93}$$

As usual there is an overall factor of the tachyon potential multiplying the entire action, and the rest of the fields are defined in the same way as before. In addition there is a new tensor, T_{ab}, containing all the open string tachyon terms. The expansion of this tensor can be written as follows

$$T_{ab} = \lambda D_a T D_b T - D_a T[x^i, T](Q^{-1})_{ij}[x^j, T]D_b T + \ldots. \tag{3.94}$$

In what follows we will assume that the tachyon potential takes the same form as the single brane case, namely

$$V(T) = \frac{1}{\cosh(T/\sqrt{2})}, \tag{3.95}$$

which tends to an exponential for large T in agreement with calculations from BSFT [28].

Inserting the D-brane background solutions, we find that at leading order the action reduces to

$$S = -T_{p'}\int d^{p'+1}xSTrV(T)H^{(p-p'-4)/4}\sqrt{(1-\lambda^2 H\dot{\phi}^2 - H^{1/2}\lambda\dot{T}^2)(1-\frac{\lambda^2 H}{2}[\phi^i,\phi^j][\phi^j,\phi^i])}. \tag{3.96}$$

We are absorbing α' factors into the definition of the tachyon (T) to make it dimensionless and we are assuming, like the transverse scalars, that it is purely time dependent. We are also assuming that the tachyon is real in this model. Generally we would expect the tachyon to be a complex field, however for our purposes this only served to make things even more analytically complicated. The explicit time dependence of the tachyon also ensures that the Chern Simons term vanishes to lowest order when we employ the static gauge [6].

The overall multiplication factor, $V(T)$, is the potential for the tachyon field which describes the changing tension of each of the branes. We will again rescale it so that it is dimensionless. The form of the tachyon potential is not known in general, however using arguments from boundary string field theory, Sen argued that it is exponentially vanishing at large values. As a result it has become somewhat standard in the literature to use an inverse cosh potential to describe the rolling tachyon. This is perfectly valid for flat space calculations, however the form of the potential may well be modified in non-trivial backgrounds. For simplicity we will assume that the inverse cosh potential is a reasonable description of the open string decay even in curved space.

As usual, the simplest ansatz for the transverse scalars is to use the standard $SU(2)$ ansatz, $\phi^i = R(t)T^i$. In which case we find that the action becomes

$$S = -T_{p'} \int d^{p'+1}x N V(T) H^{(p-p'-4)/4} \sqrt{(1-\lambda^2 C H \dot{R}^2 - \lambda H^{1/2}\dot{T}^2)(1+4H\lambda^2 C R^4)}, \quad (3.97)$$

where we have traced over the $SU(2)$ indices. As it stands, this form of the action is perfectly acceptable for us to analyse the dynamics of the brane. However the presence of the world-volume tachyon makes things difficult since it will not decouple from the equation of motion for the radion. It turns out that we can re-write the action in an equivalent form which is more useful for analysing the dynamics. In order to do this we choose to rescale the tachyon field such that

$$\frac{\tilde{T}}{\sqrt{2}} = \sinh\left(\frac{T}{\sqrt{2}}\right), \quad (3.98)$$

which transforms the action into the following form

$$S = -T_{p'} \int d^{p'+1}x \frac{N H^{(p-p'-4)/4}}{\sqrt{F}} \sqrt{1+\frac{4Hr^4}{\lambda^2 C}} \sqrt{1 - H\dot{r}^2 - \frac{H^{1/2}\lambda \dot{T}^2}{F}}. \quad (3.99)$$

Where we have introduced a new parameter, F, which controls the changing tension of the probe branes

$$F(T) = 1 + \frac{T^2}{2}, \quad (3.100)$$

and we have also chosen to write the *rescaled* tachyon field in terms of T for ease of notation. This form of the action allows us to investigate the dynamics of the Non-BPS brane when

[6] This is because the CS term goes like $\int dT \wedge C$ where our parameterisation of C ensures that it has one leg in the time direction.

the tachyon field is large. At this juncture we must point out that there may also be objections to using this form of the tachyon potential, as we are assuming that it will hold true in a gravitating background. It is well known that there are many effective descriptions for the tachyon field, with each one defined on a specific section of tachyon moduli space [31]. However as there has been little progress in constructing non-Abelian versions of these effective actions, we must use the DBI and hope that it provides an adequate description of the physics at late time.

It turns out that making the field redefinition will still not be enough to simplify the problem, and so we are also forced to consider the throat geometry around the source branes. In terms of field space definitions we are probing the large T, small r region of the theory. We can now use the Noether method to find the charge associated with a scaling symmetry on the brane world volume. We postulate that the fields and the time scale as follows:.

$$t' = \Gamma^{\alpha} t, \quad r' = \Gamma^{\beta} r, \quad T' = \Gamma^{\gamma} T. \tag{3.101}$$

Inserting these transformations into the action yields the following constraints,

$$\beta(p-3) = 0, \quad \alpha = -\beta, \quad \gamma = -\alpha p'. \tag{3.102}$$

The first of these is the most important, since we have two possible solution branches. Firstly we can have $\beta = 0$, which automatically implies that $\alpha = \gamma \to 0$ and so there are no field symmetries. However the second solution gives $p = 3$, which implies that the scaling variables are arbitrary. What we have found is that the symmetry on the world-volumes of the probe branes imposes a constraint on the allowed dimensionality of the background. If we were to allow extended transformations, for example a rescaling of the string coupling, we find that the background constraint becomes $p = 5$. Only in the case where we rescale all the fields, the string coupling *and* the string length can we eliminate this background constraint and have a theory valid in all backgrounds.

For brevity we will only consider the basic case in this thesis. Extending this to the more general case remains an outstanding problem. As the scaling variables are arbitrary, we find it convenient to choose $\alpha = -1$, thus the scalings become

$$t' = \Gamma^{-1} t, \quad r' = \Gamma r, \quad T' = \Gamma^{p'} T, \tag{3.103}$$

and we find a representation for the conserved charge generating these transformations, which is

$$D = t\tilde{E} + r\tilde{\Pi} + p' T P_T, \tag{3.104}$$

where $\tilde{E}, \tilde{\Pi}$ and P_T are the canonical energy density, radial momentum and tachyon momentum respectively. Now it is useful to write the energy density in canonical form

$$\tilde{E} = \sqrt{\frac{2N^2}{T^2}\left(\frac{k_3}{r^4}\right)^{-(1+p')/2}\left\{1+\frac{4k_3}{\lambda^2 C}\right\} + \frac{\tilde{\Pi}^2 r^4}{k_3} + \frac{T^2 P_T^2 r^2}{2\lambda\sqrt{k_3}}}, \tag{3.105}$$

where we have written k_3 to denote the constant charge of the $D3$-brane background. Using this expression we find the equations of motion for the radion and tachyon fields reduce to

$$\dot{r} = \frac{\tilde{\Pi} r^4}{\tilde{E} k_3}, \qquad \dot{T} = \frac{T^2 P_T r^2}{2\tilde{E}\lambda\sqrt{k_3}}. \tag{3.106}$$

Note that in this instance, neither $\tilde{\Pi}$ or P_T is a conserved charge which makes it difficult to solve the equations of motion. However due to our world-sheet transformations we have discovered a charge, D, that is conserved and so we can use this to simplify the equations of motion. In order to do this we will have to consider specific decompositions of the symmetry charge, as the general expression does not lead to simple analytic solutions. This will be the focus of the next subsection.

3.2.10 Decomposition of charge.

Even with the existence of the conserved charge (3.104), this does not allow an easy split between the variables r and T which would allow us to solve (3.106).[7]. In order to try and find analytic solutions (even approximate ones) we need to impose further conditions on the canonical variables in a manner consistent with the equations of motion. Let us write the conserved scaling charge D in (3.104) as the condition

$$\Phi = (t\tilde{E} + r\tilde{\Pi} + p' T P_T - D) = 0. \tag{3.107}$$

This constraint is preserved under Hamiltonian flow since it can be verified that $\dot{\Phi} = d\Phi/dt + \{H, \Phi\} = 0$ where $\{,\}$ defines the usual Poisson bracket and H is the Hamiltonian defined in (3.105) . Now let us decompose $\Phi = \Phi_1 + \Phi_2$ where

$$\begin{aligned} \Phi_1 &= \tilde{E}_1 t + r\tilde{\Pi} - D_1 \\ \Phi_2 &= \tilde{E}_2 t + p' T P_T - D_2 \end{aligned} \tag{3.108}$$

with $\tilde{E}_1 + \tilde{E}_2 = \tilde{E}$ and $D_1 + D_2 = D$. If we now impose, for example, the additional constraint that $\Phi_1 = 0$ (and hence $\Phi_2 = 0$ as a consequence) then this would allow us to solve for $r(t)$ and $T(t)$. However we must check that this additional constraint is preserved

[7]The only exception is the case $p' = 0$ which we shall discuss later.

under Hamiltonian flow, i.e that

$$\dot{\Phi}_1 = \frac{d\Phi_1}{dt} + \{H, \Phi_1\} = 0 \tag{3.109}$$

This leads to the following algebraic constraint between r and T :-

$$\tilde{E}_1 - \tilde{E} - \frac{2N^2 p'}{\tilde{E}T^2}\left(\frac{k_3}{r^4}\right)^{-(1+p')/2}\left\{1 + \frac{4k_3}{\lambda^2 C}\right\} = 0 \tag{3.110}$$

The case $p' = 0$ is special in that the original constraint, $\Phi = 0$, can be used to solve the r, T system completely (see later). For now we will assume that $p' \neq 0$. Since we are considering $p' < p = 3$ we only need consider the case when $p' = 2$. It's clear from (3.110) that $\tilde{E}_1 > \tilde{E}$ if this constraint is to be solved exactly. But one can then show an inconsistency appears when this algebraic constraint is applied to (3.106). Thus at best we can only solve (3.110) approximately. One such solution is to take $\tilde{E}_1 \approx \tilde{E}$ and assume T is large. We remind the reader that we already assumed that T is large in order to obtain the scaling symmetries earlier. We can now proceed to solve the r, T system of equations.

Solving for the radial equation of motion we find

$$\frac{1}{r^2} = \frac{1}{r_0^2} - \frac{t}{\tilde{E}k_3}(2D_2 - \tilde{E}t) \tag{3.111}$$

Now for small values of D_2 the dynamics of the probe obeys a $1/t$ relationship. The exact description of the dynamics will depend on the relative sizes of D_2 and $\tilde{E}$. If $\tilde{E} >> D_2$, then the quadratic term will be dominant. This ensures that the solution starts at some maximal distance and tends to zero. Conversely if D_2 is much larger than $\tilde{E}$, then the linear term is dominant and this describes an expanding solution which will break down when the supergravity constraint is no longer satisfied. However when the two charges are of the same order of magnitude we find a turning solution. The sphere initially expands from $t = 0$ until it reaches a stationary point at $t = D_2/\tilde{E}$, before collapsing toward zero size.

Using the second constraint to solve for the tachyon momentum yields the solution to the tachyon equation of motion

$$T \sim T_0 \exp\left(\frac{\sqrt{k_3}r_0^2}{4\lambda} f(t)\right) \tag{3.112}$$

where the function $f(t)$ is proportional to $\mathrm{arctanh(t\tilde{E} - D_2)}$. Thus the general behaviour of the tachyon solution is that it is an exponential function of time.

The results obtained so far have all been for the case $p' = 2$. In order to determine the dynamics of the $p' = 0$ case corresponding to N coincident D-particles we see that the tachyon dependence drops out of of the conserved charge. This is an unusual case, since when the tachyon condenses we will be left with D-instantons. Initially solving for the radial equation of motion, we find the expression

$$\frac{1}{r^2} = \frac{1}{r_0^2} - \frac{t}{\tilde{E}k_3}(2D - t\tilde{E}) \tag{3.113}$$

which is a similar solution as the one obtained in the charge decomposition above for $p' = 2$. Therefore we also expect to find a similar turning solution for the fuzzy sphere parameterised by the time $t = D/\tilde{E}$. We have no other constraint to impose on the equation of motion for the tachyon field, but we can write the tachyon momentum in terms of the other canonical forms

$$P_T^2 = \frac{2\sqrt{k_3}\lambda}{T^2 r^2}\left(\tilde{E}^2 - \frac{2r^2}{T^2\sqrt{k_3}}\left\{1 + \frac{4k_3}{\lambda^2 C}\right\} - \frac{\tilde{\Pi}^2 r^4}{k_3}\right). \tag{3.114}$$

In general we can use this solution to exactly solve for the tachyon field, however this is extremely difficult and we will find it much more useful to find an approximate solution. From the above equation we see that the supergravity solution implies $r^4/k_3 << 1$, and so we can effectively neglect the contribution from the final two terms. Inserting this into the equation of motion yields the solution

$$T \sim T_0 \exp\left(\left(\frac{\sqrt{k_3}}{2\lambda}\right)^{1/2} \ln\left[\sqrt{\mathrm{\tilde{E}k_3 - r_0^2 t(2D - t\tilde{E})}} + \frac{\mathrm{r_0(tE - D)}}{\sqrt{\mathrm{\tilde{E}}}}\right]\right), \tag{3.115}$$

which we expect to provide a reasonable approximation as $r \to 0$, and once again shows the increasing exponential dependence of the tachyon field. Again the contribution from the two charges can change the dynamics of the field, as described earlier.

The general solution for the tachyon field is expected to be background dependent, however we see that in the $D3$-case it is roughly exponential in all cases. The fuzzy sphere appears to always collapse, but there is an intricate relationship between the tachyon condensation and the radial modes which depends upon the conserved charges. When both terms appear in the radial equation of motion we see that there can be turning solutions describing an initial expansion which eventually contracts within finite time. This is a result of the tachyon condensation which decreases the tension of the branes so that they feel a weaker gravitational attraction. However the combination of the charges in the tachyon solution also implies a turning point for the tachyon field and so the tension eventually increases and the fuzzy spheres collapses - provided that the tachyon solution still remains valid.

3.2.11 NS5-brane background.

The analysis in the previous section was concerned with coincident Dp-brane backgrounds, but we wish to extend this to the $NS5$-brane background. This particular background is important for several reasons as in many cases there is an exact conformal field theory description allowing BCFT calculations [21]. Secondly there is an interesting duality which relates six dimensional string theory on the $NS5$-brane world-volume, known as Little String Theory (LST) [43], to supergravity in the bulk permitting an understanding of the dynamics in terms of defects of the LST. Aside from this - as we saw in the first chapter of this thesis, there is an interesting relationship between brane dynamics and tachyon condensation in these backgrounds. Much of the construction of the non-Abelian theory follows a similar line to that of the D-brane backgrounds

For simplicity we will restrict ourselves to the analysis of coincident fivebranes. The background supergravity solution was presented in the first chapter, see (2.2). Note that there are no bulk RR fields in this solution because the $NS5$-branes don't carry these charges. The probe branes themselves do carry RR charge, which would be emitted as closed string radiation as the branes move through the background [30]. This has important consequences as we know that the Abelian theory is only valid for probe brane dimensions $3 \leq p < 5$, due to the emission of these string modes. This result was calculated from the CFT perspective, so we expect it to be correct. Performing the equivalent calculation for multiple branes is significantly harder, and remains an open problem.

We now insert the background solution (2.2) into our non-Abelian action. Once again we expand everything to leading order and assume that the transverse scalars are time dependent, which will ensure that our solutions are homogeneous and thus there will be no formation of caustics. Finally we arrive at the following form of the action

$$S = -T_p \int d^{p+1}\zeta STr \left(H^{-1/2}\sqrt{1 - H\lambda^2\dot{\phi}^i\dot{\phi}^j\delta_{ij}}\sqrt{1 - 1/2\lambda^2 H^2[\phi^i,\phi^j][\phi^j,\phi^i]} \right). \quad (3.116)$$

Recall that the $NS5$-branes have a tension that goes as $1/g_s^2$, whilst the Dp-branes each have tensions proportional to $1/g_s$, thus the five-branes are heavier in the large k limit, however as we are considering the large N limit we may find there is considerable back reaction upon the throat in the target space which may deform it substantially. As far as this thesis is concerned we will always assume $k >> N$ in order to minimise this effect, however we should bear in mind that this may not always lead to a physical solution. Following on from the previous sections we will assume the usual $SO(3)$ ansatz for the transverse scalars and the resulting action reduces to[8]

$$S = -T_p \int d^{p+1}\zeta \frac{N}{\sqrt{H}} \sqrt{(1 - H\lambda^2\dot{R}^2 C)(1 + 4\lambda^2 C H^2 R^4)}, \quad (3.117)$$

with C being the usual Casimir of the N-dimensional representation. Switching now to

[8]For simplicity we do not include angular momentum though this can be done as in the case of the D-brane backgrounds.

physical coordinates we arrive at the final form of the action useful for our analysis

$$S = -T_p \int d^{p+1}\zeta \frac{N}{\sqrt{H}} \sqrt{(1 - H\dot{r}^2)\left(1 + \frac{4H^2r^4}{\lambda^2 C}\right)}, \tag{3.118}$$

from which we can derive the usual canonical momenta and energy densities, where we have explicitly divided out the 'mass' of each brane.

$$\begin{aligned} \tilde{\Pi} &= \frac{NH\dot{r}}{\sqrt{H}} \sqrt{1 + \frac{4H^2r^4}{\lambda^2 C}} \frac{1}{\sqrt{1 - H\dot{r}^2}} \\ \tilde{E} &= \frac{N}{\sqrt{H}} \sqrt{1 + \frac{4H^2r^4}{\lambda^2 C}} \frac{1}{\sqrt{1 - H\dot{r}^2}}. \end{aligned} \tag{3.119}$$

We solve the equation for the energy, which is conserved, to obtain the following constraint on the dynamics of the probe branes assuming a fixed energy density

$$1 \geq \frac{N^2}{\tilde{E}^2 H}\left(1 + \frac{4H^2r^4}{\lambda^2 C}\right). \tag{3.120}$$

As usual the interesting physics occurs in the near horizon geometry of the five-brane background, and so we will drop the factor of unity from the harmonic function when referring to this limit. We will also anticipate that there is a maximum size for the fuzzy sphere in the large r region, which is again identified with Minkowski space. In the throat we find that the constraint becomes

$$1 \geq \frac{N^2 r^2}{\tilde{E}^2 k l_s^2}\left(1 + \frac{4k^2 l_s^4}{\lambda^2 C}\right) \tag{3.121}$$

which is automatically satisfied for the radial part since we know that $H >> 1$ in this region. This actually allows us to find the following constraint on the energy density

$$\frac{\tilde{E}^2}{N^2} \geq \frac{r^2}{k l_s^2}\left(1 + \frac{k^2}{\pi^2 C}\right). \tag{3.122}$$

The supergravity solution tells us that r^2/kl_s^2 must be extremely small, and we can select k/N to be small even when k and N are individually large, thus the last term can be tuned to be $\mathcal{O}(1)$ and implies that $\tilde{E}$ is larger than N. Thus like the majority of the Dp-brane solutions we find that the fuzzy sphere can collapse down toward zero size.

One caveat to this assumption is that the dynamical D-branes will shed their energy as they fall toward the five-branes, and could eventually form a (k, N) bound state in analogy to the $(k, 1)$ state in the Abelian case. As we have already seen, one of the main differences between the usual fuzzy sphere solutions in flat space and those in curved background is that the velocity term decreases with the radius. In flat space we find that the collapsing configuration approaches the speed of light at late times and thus the $1/N$ corrections due to the symmetrized trace become important. Clearly we don't see the same behaviour in this case, in fact a six dimensional observer on the $NS5$-brane world volume will record that it takes an infinite amount of time for the fuzzy sphere to collapse to zero size. This is interesting as it appears that the energy of a collapsing fuzzy sphere in flat space is the same as an essentially static sphere in a space-time throat [9], and is clearly related to the formation of a bound state of (p, q) fivebranes.

In the large r region corresponding to Minkowski space, we find that the constraint becomes

$$1 \geq \frac{4N^2 r^4}{\tilde{E}\lambda^2 C} \tag{3.123}$$

which translates into the condition that the fuzzy sphere has a maximum radius given by

$$r_{max} = \sqrt{\frac{\tilde{E}\lambda C^{1/2}}{2N}}. \tag{3.124}$$

which, as expected, is the same as the value obtained in the Dp-brane backgrounds in the absence of Chern-Simons terms.

We now look at the static potential associated with the fivebrane background. Following the parameter conventions employed in the Abelian cases [21], we easily find that the potential can be written in the following form

$$V_{eff} = \frac{N^2}{\tilde{E}^2 H^2}\left(1 + \frac{4H^2 r^4}{\lambda^2 C}\right) - \frac{1}{H} \tag{3.125}$$

The interesting question is what happens in the throat, since we know that in the large r region the potential will be a simple monotonically increasing/decreasing function, which goes like

$$V_{eff} \sim \frac{4r^4}{\lambda^2 \tilde{E}^2}. \tag{3.126}$$

Dropping the factor of unity as before we find that in the $r \to 0$ limit the potential becomes

$$V_{eff} \sim \frac{N^2 r^4}{\tilde{E}^2 k^2 l_s^4}\left(1 + \frac{4k^2 l_s^4}{\lambda^2 C}\right) - \frac{r^2}{k l_s^2}. \tag{3.127}$$

which indeed tends to zero with r, for fixed $\tilde{E}$ as expected

In any case, we wish to solve the equation of motion for the probe branes in the throat. Because the energy is conserved, the solution - up to constants of integration - is simply

$$\frac{1}{r} = \sqrt{\frac{N^2}{\tilde{E}^2 k l_s^2}\left(1 + \frac{4k^2 l_s^4}{\lambda^2 C}\right)} \cosh\left(\frac{t}{\sqrt{k l_s^2}}\right), \tag{3.128}$$

[9] Provided one is in the correct reference frame.

which is actually a simple extension of the solution for a single probe Dp-brane in the Abelian theory [21], which was shown to be

$$\frac{1}{r} = \frac{1}{\sqrt{\tilde{E}^2 k l_s^2 - \tilde{L}^2}} \cosh\left(\frac{t}{\sqrt{kl_s^2}}\sqrt{1-\frac{\tilde{L}^2}{kl_s^2\tilde{E}^2}}\right). \tag{3.129}$$

where we have also included the angular momentum contribution for completeness. This extra term acts as a deceleration parameter, slowing the velocity of the probe brane.

Let us make some brief remarks on the solution (3.128). If we perform a Wick rotation on the time coordinate for the collapsing solution, we find a periodic solution in terms of a cosine function. This can be interpreted as the collapse and subsequent bounce of the fuzzy sphere in imaginary time - although the physical interpretation of this solution is not clear we expect it to approximate the time dependent solution for Euclidean branes. This sinusoidal behaviour can also be seen if we switch to conformal time where an observer sees that the collapse occurs in finite time. In this case we would expect $1/r$ to be proportional to $\sin(t)$ which again is suggestive of a periodic collapse and expansion. However this solution would indeed probe the non-perturbative region of the theory, and it is not clear if the corrections (e.g quantum, $1/k$ and back-reaction) would admit such a solution. One further thing to note is that using S-duality we may map this solution to that of the coincident $D5$-brane background being probed by coincident $D3$-branes. This agrees with our expectation that the $D5$-brane background yields exponential solutions at late times.

We may enquire about the validity of this classical solution in the throat region. Using our time dependent ansatz we see that the dilaton is also a time dependent function, in fact for a radially collapsing solution we find that the dilaton behaves as

$$e^{\phi} = \frac{Ng_s}{\tilde{E}}\sqrt{1+\frac{4k^2 l_s^4}{\lambda^2 C}} \cosh\left(\frac{t}{\sqrt{kl_s^2}}\right). \tag{3.130}$$

Note that quantum effects can be neglected provided that $g_s = e^{\phi} << 1$, however as we know that $\tilde{E} >> N$ from our constraint equation we expect that the classical analysis will provide an accurate description of the solution, at least for early times. This can be 'fine tuned' for specific values of k and N so that the classical solution continues to hold at late times. If we do not wish to fine tune the solution, then we must uplift to M-theory in order to study the dynamics. However there is no M-theory analogue of the non-Abelian action[10], which makes this step impossible. Simple analysis shows that a theory of coincident $M5$-branes must also contain a theory of $M2$-branes which intersect along one dimension of the $M5$-branes. This means that the massles degrees of freedom will not vary as N^2 like in ten-dimensions, in fact they vary like $N^{3/2}$.

[10]However see [47] for a recent development.

3.2.12 Correction from symmetrized trace.

Thus far we have investigated the dynamics of the action at leading order, and seen that the fuzzy sphere will generally collapse down to small size. It is expected that the effective action will break down at distances comparable with the string length, and thus $1/N$ corrections will become important. In order to deal with this situation we will look at the next order terms arising from corrections to the symmetrized trace. As we have already seen we can write the first order correction to the energy as [50]

$$\tilde{E}_1 = \left(1 - \frac{2}{3}C\frac{\partial^2}{\partial C^2}\right)\tilde{E}_0,$$

which yields the corrected energy

$$\tilde{E}_1 = \frac{N}{\sqrt{H}\sqrt{1-H\dot{r}^2}}\left(W(k,C) + \frac{2Ck^4}{3W(k,C)^{3/2}\pi^4C^4} - \frac{2k^2C}{3W(k,C)\pi^2C^3}\right). \tag{3.131}$$

Where we have introduced the following simplifying notation

$$W(k,C) = \sqrt{1 + \frac{k^2}{\pi^2 C}}. \tag{3.132}$$

This term can be thought of as a mass term, by seeing how it arises in the context of the energy. In the Dp-brane case (and in flat space) this term is an explicit function of the radius, and we have the notion of a position dependent mass. However in the near horizon of the $NS5$-brane background this term reduces to a constant. Because we are using the supergravity approximation in our analysis, we are taking k and N to both be large, and so this 'mass' term is positive but small since we are also demanding k/N to be small. If we now employ the canonical formulation of the energy, we can set the $\tilde{\Pi}$ terms to zero to find the corrected potential for the probe branes up to leading order in $1/C$

$$V_1 = \frac{N}{\sqrt{H}}\left(W(k,C) - \frac{2k^2}{3W(k,C)\pi^2C^2}\right). \tag{3.133}$$

The potential clearly does not vanish with this correction because the new term is suppressed by k/N. In fact, even taking into account higher order corrections, the potential is nowhere vanishing since the corrections are not functions of r. Thus the symmetrized trace correction does not appear to yield exotic behaviour, such as bounce solutions.

3.2.13 Non-Abelian tachyon map.

It has been shown in the case of a single probe brane, that the unstable dynamics in the $NS5$-brane background are more easily understood in terms of the rolling tachyon, since the energy momentum tensors have similar behaviour at late times. We may ask what the implications are when we have multiple coincident branes with a $U(N)$ symmetry on their worldvolumes. This relationship can be explicitly demonstrated by mapping the probe brane action into that of the tachyon action in flat Minkowski space. This is particularly simple in the Abelian case, but we wish to show that it is also possible in our non-Abelian construction. The corresponding non-Abelian action for tachyons in a flat background, to leading order, can be written

$$S = -T_p V_p \int dt N V(T)\sqrt{1-\dot{T}^2}. \tag{3.134}$$

Because the tachyon field does not take values in the $SU(2)$ algebra we find that the action is simply N times that of a single non-BPS brane. In fact this corresponds to a configuration of branes each separated by distances larger than the string length, as found in constructions of Assisted Inflation [97]. In this scenario each of the tachyons is assumed to follow a similar trajectory toward the late time attractor point, namely $T_1 \sim T_2 \ldots \sim T_N \equiv T$. Here V_p is the effective 'volume' of each brane, whilst $V(T)$ is the tachyon potential which we will again assume to be of the form;

$$V(T) = \frac{1}{\cosh(T/T_0)}, \tag{3.135}$$

We remind the reader of the action for the probe brane in the $NS5$-background, which we have already shown to be of the form

$$S = -T_p V_p \int dt \frac{NW(r)}{\sqrt{H}}\sqrt{1-H\dot{r}^2}, \tag{3.136}$$

where, for simplicity, we have absorbed the potential term into our definition of $W(r)$. Clearly we can map this action to that of the non-Abelian tachyon by making the identification

$$d\tilde{T} = \sqrt{H}dr, \qquad V(\tilde{T}) = \frac{W(r)}{\sqrt{H}}. \tag{3.137}$$

Using the near horizon approximation we can solve for the Geometrical Tachyon in terms of the physical radius of the fuzzy sphere. The result, up to arbitrary constants of integration, is simply an exponential as expected from the Abelian case which allows us to write the tachyon field as

$$\tilde{T} \sim \sqrt{kl_s^2}\ln(r). \tag{3.138}$$

The solution tells us that as $r \to 0, \tilde{T} \to -\infty$ as expected, whilst as $r \to r_{max}$ we find $\tilde{T} \to \tilde{T}_{max}$. Clearly this is not the general behaviour associated with the open string tachyon solution, which we should have expected from the Abelian theory, but we may anticipate that the decay of the fuzzy sphere will also be describable in terms of this rolling tachyon solution. Using our field redefinition, we write the expression for the tachyon potential as

$$V(\tilde{T}) = \sqrt{\frac{1}{kl_s^2}\left(1+\frac{k^2}{\pi^2 C}\right)} \exp\left(\frac{\tilde{T}}{\sqrt{kl_s^2}}\right). \tag{3.139}$$

The form of the potential shows that it had its maximum at $\tilde{T} = 0$, and tends to zero for $\tilde{T} \to -\infty$. The exact maximum will be defined by the number of source branes, as expected from the Abelian case. However note that there is a correction term present here due to the fuzzy sphere, which does not occur in the leading order tachyon action as we know that the tachyonic scalar field is a commuting variable. Therefore although we can capture the general behaviour of the tachyon action, we must go beyond leading order to find closer agreement. If we construct the Energy-Momentum tensor associated with this rolling tachyon solution, omitting the delta functions which localise the tensor on the brane world-volumes, we obtain

$$\begin{aligned} T_{00} &= \frac{NV(\tilde{T})}{\sqrt{1-(\partial_t \tilde{T})^2}} \\ T_{ij} &= -NV(\tilde{T})\sqrt{1-(\partial_t \tilde{T})^2}, \end{aligned} \tag{3.140}$$

which shows that the pressure tends to zero as the potential tends to zero, i.e, when the probe branes approach the fivebranes at late times. This is because the probe brane will emit energy in closed string modes as the fuzzy sphere collapses, and the resulting matter will be non-Abelian pressureless fluid. One must also imagine that because the fuzzy sphere collapses in the near throat region of the fivebranes, becoming pointlike at distances approaching the string length, the harmonic function approximation may fail, and there will certainly be quantum corrections to take into account. This is due in part to the back reaction of the probes on the source branes and the throat, therefore in order to determine the physics of this non-Abelian fluid it will be necessary to calculate this back reaction term and incorporate it into the action. In any case, it would be useful to compute the dynamics of this configuration using the exact CFT on the world volume which would help shed further light on the validity of the classical solution.

3.2.14 Non-BPS branes in fivebrane backgrounds.

We have already started developing the technology to deal with this solution in the case of Dp-brane backgrounds, where the action of our coincident probe branes is once again given by (3.93). As usual we expand everything to leading order, and we will drop any gauge field term so that all the covariant derivatives reduce to normal derivatives. The resulting action can be written as follows

$$S = -\int d^{p+1}\zeta STr \frac{V(T)}{\sqrt{H}} \sqrt{(1 + H\lambda^2 \partial_0 \phi^i \partial^0 \phi^j \delta_{ij} + \lambda \partial_0 T \partial^0 T)\left(1 - \frac{1}{2}H^2[\phi^i, \phi^j][\phi^j, \phi^i]\right)}. \tag{3.141}$$

We again use the fuzzy sphere ansatz for the radially dependent transverse scalars which reduces the action to a more tractable form

$$S = -\int d^{p+1}\zeta N \frac{V(T)}{\sqrt{H}} \sqrt{(1 - H\lambda^2 C\dot{R}^2 - \lambda \dot{T}^2)(1 + 4H^2\lambda^2 CR^4)}. \tag{3.142}$$

The presence of the open string tachyon will again generally prohibit exact solutions to the equations of motion for the radion field unless we take various asymptotic limits. This is obvious, as the form of the action shows that the conjugate momenta associated with the radion and tachyon fields will not be conserved. Once again it will be useful to perform symmetry transformations on the various fields, as in the Dp-brane case, which will allow us to solve the equations in specific regions of field space.

We will assume the 'canonical' form for the tachyon potential as in (3.95). We insert this into the action, and once again switch to using physical coordinates. Again note that the current form of the potential will make it difficult to find symmetries of the action as it stands, thus it will be more useful to perform the same field redefinition as we did for the coincident Dp-brane background

$$\frac{\tilde{T}}{\sqrt{2}} = \sinh\left(\frac{T}{\sqrt{2}}\right),$$

and for convenience we re-write $\tilde{T} = T$ for ease of calculation, although we will always imply that this is the re-definition of the original tachyon field. As mentioned previously there may be objections to performing this kind of field redefinition using the non-Abelian action in this gravitational background. Assuming that this won't be too problematic, we can now proceed to analyse the resulting action,

$$S = -T_p \int d^{p+1}\zeta \frac{N}{\sqrt{HF}} \sqrt{\left(1 - H\dot{r}^2 - \frac{\lambda \dot{T}^2}{F}\right)\left(1 + \frac{4H^2 r^4}{\lambda^2 C}\right)}, \tag{3.143}$$

where we have introduced the following definitions

$$F = \left(1 + \frac{T^2}{2}\right), \qquad H = 1 + \frac{kl_s^2}{r^2}. \tag{3.144}$$

We can now try to find the conserved charge associated with transformations of this action, and use that in conjunction with the energy density to solve the equations of motion. Unfortunately we see that this is still non trivial unless we make further approximations, thus we will look at the theory in the large T and small r limit. Since the large tachyon field gives rise to a gas of closed strings arising due to tachyon condensation, we expect to find that the radial field on the probe branes will describe the late time dynamics of this gas. The action in this instance, reduces to

$$S = -T_p \int d^{p+1}\zeta \frac{\sqrt{2}Nr}{\sqrt{kl_s^2}T}\sqrt{1 - \frac{kl_s^2\dot{r}^2}{r^2} - \frac{2\lambda\dot{T}^2}{T^2}}\sqrt{1 + \frac{k^2}{\pi^2 C}}, \tag{3.145}$$

At this juncture we will revert back to the $W(k, C)$ notation to simplify things, and furthermore, we postulate that the action be invariant under the following transformations [22] $T = \lambda T, r = \lambda r$, where $\lambda = 1 + \epsilon$ and infinitesimally we find

$$\delta T = \epsilon T, \qquad \delta r = \epsilon r, \tag{3.146}$$

for some parameter ϵ. Note that this is a scaling transformation that acts both on the world-volume fields and the transverse scalars. Presumably there is some relation here to the space-time uncertainty principle [48]

$$\Delta t \Delta X \geq \alpha' \tag{3.147}$$

where distances on the world-sheet are inversely related to distances in the bulk. Since the $NS5$-brane world-volume theory is related to a Little String Theory (LST), it would be interesting to find out the implications of the transformations for fields in the LST.

Varying the action, we determine that the charge associated with this symmetry is given by

$$D = \frac{Nr\sqrt{2}}{T\sqrt{kl_s^2}}\left(\frac{kl_s^2\dot{r}}{r} + \frac{2\lambda\dot{T}}{T}\right)\frac{W(k,C)}{\sqrt{1 - \frac{kl_s^2\dot{r}^2}{r^2} - \frac{2\lambda\dot{T}^2}{T^2}}}, \tag{3.148}$$

which can be seen to have dimensions of length. We also determine the canonical energy density associated with the action, using the canonical momenta of the radion and the tachyon fields. For brevity we will simply state the resultant dimensionless energy density and not the individual momenta

$$\tilde{E} = \frac{Nr\sqrt{2}W(k,C)}{T\sqrt{kl_s^2}}\frac{1}{\sqrt{1 - \frac{kl_s^2\dot{r}^2}{r^2} - \frac{2\lambda\dot{T}^2}{T^2}}}. \tag{3.149}$$

It can be seen that both $\tilde{E}$ and D are conserved, as expected, and it will be useful to combine both of these charges to form a solitary conserved charge

$$Q = \frac{D}{\tilde{E}} = \frac{kl_s^2\dot{r}}{r} + \frac{2\lambda\dot{T}}{T}, \tag{3.150}$$

which after some manipulation can be used to define the tachyon field via

$$T = C_o \exp\left(\frac{Qt}{2\lambda}\right) r^{-k/4\pi}, \tag{3.151}$$

where C_0 is a constant of integration. Furthermore from (3.150) we can also find the time dependence of the tachyon field in this condensing limit.

$$\dot{T} = T\left(\frac{Q}{4\pi l_s^2} - \frac{k\dot{r}}{4\pi r}\right). \tag{3.152}$$

As we are probing the large T region of field space, we expect that the dominant contribution to the charge will come from the radial modes. Now that we have written the tachyon field in terms of this conserved charge we can attempt to solve the radial equations of motion. Note that this would be extremely challenging if we had tried to proceed from the original form of the action without finding another conserved charge. We will initially consider the case where $Q = 0$. This obviously implies that we are setting $D \to 0$ (or taking the energy density to be extremely large), which may seem strange, however we have used the charge to construct an expression for the tachyon field and so it is valid. By setting $Q = 0$ we are identifying the condensation of the tachyon field with the inverse of the radion field on the probe branes (up to some power), and so small r will automatically imply large T. The simplicity of this approach is now clear, since we began with two distinct fields and have effectively coupled them via the conserved charge thus only requiring us now to solve for one of the fields. We now substitute the expressions for the tachyon into the energy equation, which will now solely be a function of r.

$$\tilde{E} = \frac{NW(k,C)r^y\sqrt{2}}{C_0\sqrt{kl_s^2}} \frac{1}{\sqrt{1 - \frac{kl_s^2\dot{r}^2}{r^2}\left(1+\frac{k}{4\pi}\right)}} \tag{3.153}$$

and for future reference, we will introduce the simplifying notation

$$B = \frac{NW(k,C)\sqrt{2}}{\tilde{E}C_0\sqrt{kl_s^2}}, \qquad y = 1 + \frac{k}{4\pi}, \qquad x = kl_s^2\left(1+\frac{k}{4\pi}\right) \tag{3.154}$$

Rearranging the energy equation allows us to solve for $r(t)$, which we find to be, up to constants of integration

$$\frac{1}{r} \sim \left(B\cosh\left[\frac{\pm y(t-t_0)}{\sqrt{x}}\right]\right)^{1/y}, \tag{3.155}$$

where t_0 parameterises some initial time for the dynamics. This solution describes an expanding fuzzy sphere which reaches its maximum size at $t = t_0$ and thereafter collapses down to zero size. We easily find that the maximum radius will be given by

$$r_{\max} = \left(\frac{\tilde{E}C_0\sqrt{kl_s^2}}{NW(k,C)\sqrt{2}}\right)^{1/y}. \tag{3.156}$$

The physics behind this solution can be understood. As the fuzzy sphere expands the tension of the non-BPS branes is increased as the tachyon moves closer to the top of its potential (assumed to be located at $T = 0$). Thus the expanding solution has a natural 'braking force' that restricts it to expand to a certain size. Conversely in the collapsing phase, the non-BPS branes feel a decreasing tension which goes to zero as the solution collapses toward the origin.

We can also determine the constant of integration by demanding that $T = T_0$ at $t = t_0$, and since we are in the large T region of field space we will assume that $|T_0| >> 1$. After some manipulation we find

$$C_0 = T_0^y \left(\frac{NW(k,C)\sqrt{2}}{\tilde{E} l_s^2 \sqrt{k}} \right)^{k/4\pi}, \tag{3.157}$$

and therefore we can completely determine the behaviour of the tachyon near condensation, in the approximation where $Q = 0$. It is natural to now consider the case where $Q \neq 0$, however we should note that this case is not solvable exactly, and we must be forced into approximations. If we insert the full expression for the tachyon field into the energy equation we find

$$1 - \frac{k l_s^2 \dot{r}^2}{r^2} - \frac{l_s^2}{4\pi} \left(\frac{Q^2}{l_s^4} - \frac{2Qk\dot{r}}{l_s^2 r} + \frac{k^2 \dot{r}^2}{r^2} \right) = B^2 e^{-Qt/\lambda} r^{2y}. \tag{3.158}$$

Now at late times we see that the RHS of this equation will become vanishingly small, and so we neglect it in our analysis. This allows us to rewrite the LHS as a quadratic equation, which we solve to find

$$\frac{\dot{r}}{r} = \frac{Qk \pm 2\sqrt{k\pi(4\pi l_s^2 + k l_s^2 - Q^2)}}{(4\pi k + k^2) l_s^2} = \beta, \tag{3.159}$$

and upon integration we can determine the late time behaviour of the fuzzy sphere

$$r \simeq r_0 \exp(\beta t), \tag{3.160}$$

with the corresponding late time solution for the tachyon field given by

$$T \simeq \exp\left(\frac{Qt}{4\lambda}\right) \exp(-k\beta t/8\pi). \tag{3.161}$$

Now if we look for a collapsing solution we must take β to be negative in (3.160), where we must bear in mind that the solution is only valid for late times. In this case the tachyon field will be large even if the charge Q is small, and so our analysis is consistent. Furthermore having non-zero Q appears to imply that there will not be a bounce solution, rather the probe branes will eventually approach the source branes and the fuzzy sphere will collapse toward zero size. This can be seen from (3.159) which suggests that for a real solution, we must ensure that $(4\pi + k) l_s^2 \geq Q^2$. In the large k limit this is approximated by the constraint $kl_s^2 \geq Q^2$. Clearly if this is saturated then we find

$$\beta \rightarrow \frac{Q}{(4\pi + k)l_s^2}, \tag{3.162}$$

which is dependent upon the sign of Q. If we accept the constraint, then for β to be negative we require

$$4\pi(4\pi l_s^2 + kl_s^2 - Q^2) > Q^2 k, \tag{3.163}$$

which becomes

$$4\pi l_s^2 > Q^2 \tag{3.164}$$

when we ensure $k >> 1$. Clearly the only way to satisfy this constraint is to assume that Q is vanishingly small. This is inconsistent with (3.150) for both expanding and contracting solutions. It would be interesting to see if this holds when we keep higher order tachyon terms in the action.

One way of interpreting the physical aspect of the conserved charge is that it parameterises the deviation from the single field duality we found when we identified the tachyon field with the inverse of the radial mode.

3.2.15 Higher (even) dimensional fuzzy spheres.

So far our analysis has dealt with collapsing fuzzy two-spheres in curved backgrounds, thus it would be useful to extend this to higher (even) dimensional fuzzy spheres [52]. The procedure for dealing with higher dimensional fuzzy spheres is essentially the same as for the $\mathbf{S}^2$, although technically it is far more complicated. We will briefly sketch out how one can do this, but our interest lies in the dynamics of such solutions and not in their mathematical construction. So many results will simply be stated without proof. We will briefly look at the fuzzy 4-sphere before commenting on how our analysis can generalise to the fuzzy $2k$ sphere where k is an integer. In the following discussion we will concern ourself with D-brane backgrounds for simplicity. We cannot consider the $NS5$-brane backgrounds in this instance because we require at least $2k+1$ transverse scalars to describe the fuzzy $\mathbf{S}^{2k}$.

Let us begin by constructing the fuzzy $\mathbf{S}^4$ solution, where we need five transverse scalar fields satisfying the following ansatz

$$\phi^i = \pm R G^i, \qquad i = 1 \ldots 5. \tag{3.165}$$

This will obviously imply that we can only look at $p \leq 4$ backgrounds. The G^i matrices above arise through the totally symmetric n-fold tensor product of the gamma matrices of

$SO(5)$, which have dimension

$$N = \frac{(n+1)(n+2)(n+3)}{6}. \tag{3.166}$$

For a detailed description of these constructions we refer the interested reader to [51, 52] and the references therein. In terms of the physical radius we find a similar relationship to the case of the $SU(2)$ algebra, where we write

$$r = \lambda\sqrt{C}R, \tag{3.167}$$

note that in this instance R must be positive definite and the Casimir is given by products of the G^i, as usual, where we have $G^iG^i = C\mathbf{1}_{N\times N} = n(n+4)\mathbf{1}_{N\times N}$. We can now use this ansatz in our non-Abelian DBI effective action, which we again treat as a lowest order expansion. The resultant action may be written [51]

$$S = -T_{p'}\int d^{p'+1}\zeta N H^{(p-p'-4)/4}\sqrt{1-H\lambda^2C\dot{R}^2}\left(1+4H\lambda^2CR^4\right) + T_{p'}\delta_{pp'}\int d^{p'+1}\zeta\frac{qN}{H}, \tag{3.168}$$

where the Chern-Simons term only couples to the action for $p = p'$ as usual. From this action we can derive the usual canonical momenta and energy, which yields the following static potential in terms of physical distances

$$V_{eff} = T_{p'}NH^{(p-p'-4)/4}\left(1+\frac{4Hr^4}{\lambda^2C}\right), \tag{3.169}$$

note that this appears to gave exactly the same basic structure as the fuzzy $\mathbf{S}^2$ potential except that now $p \leq 4$ because of our ansatz. Before we comment on this solution, we should discuss the extension to the fuzzy $\mathbf{S}^6$. We again use the G^i matrices which are now representations of $SO(7)$ as i runs over seven transverse dimensions. Again the G's arise from the action of gamma matrices on the traceless, symmetric n-fold tensor product of the spinor, and we have the following relationship between the dimension of the matrices and the number of branes

$$N = \frac{(n+1)(n+2)(n+3)^2(n+4)(n+5)}{360}. \tag{3.170}$$

The relationship between the physical radius and the transverse scalar ansatz is the same as before except that the Casimir has a different definition $G^iG^i = C\mathbf{1}_{N\times N} = n(n+6)\mathbf{1}_{N\times N}$. This suggests that we can make the following generalisation. For the fuzzy $\mathbf{S}^{2k}$ sphere in ten dimensions, where $k \leq 4$ we require $2k+1$ transverse scalar fields which can be parameterised by the action of $SO(2k+1)$ gamma matrices on tensor products of the spinor. If we assume that this is correct then we propose to write the general form of the action for fuzzy $\mathbf{S}^{2k}$ in a curved D-brane background

$$S = -T_{p'} \int d^{p'+1}\zeta N H^{(p-p'-4)/4} \sqrt{(1 - H\lambda^2 C_k \dot{R}^2)(1 + 4H\lambda^2 C_k R^4)^k} + T_{p'}\delta_{pp'} \int d^{p'+1}\zeta \frac{qN}{H}. \tag{3.171}$$

Where we have written C_k to indicate that the Casimir refers to the gauge group $SO(2k+1)$. The factor of k imposes restrictions upon the dimensionality of the background branes, in fact the maximum value of p is $p_{max} = 8 - 2k$. Thus we see that for the fuzzy $\mathbf{S}^8$ we can only consider $D0$-branes probing the $D0$-brane background. Using the general form of the action we define the effective potential, in physical coordinates, to be

$$V_{eff} = NT_{p'} \left\{ H^{(p-p'-4)4} \left(1 + \frac{4Hr^4}{\lambda^2 C_k}\right)^{k/2} - q\delta_{pp'} \right\}. \tag{3.172}$$

In general we see that the bosonic part of the potential will always tend to zero in the near horizon region, implying that the fuzzy spheres will collapse toward zero size. Thus the only case of interest relates to $p = p'$ when there is the additional term coming from the bulk RR charge of the background branes. In the small radius limit we find that the potential reduces to

$$V_{eff} = \frac{NT_{p'}}{H} \left\{ \left(1 + \frac{4k_p r^{p-3}}{\lambda^2 C_k}\right)^{k/2} - q \right\}. \tag{3.173}$$

We can differentiate this potential to see if there are any solutions corresponding to stable minima at which point the fuzzy sphere may stabilise, however we see that there are no real solutions (as leading order) again implying that all fuzzy spheres are unstable in D-brane backgrounds with the exception of $p = 6, p' = 0$ which we discussed in a previous section.

The generalised form of the equation of motion can be written as

$$\dot{r}^2 = \frac{1}{H} \left\{ 1 - \frac{N^2 T_{p'}^2 H^{(p-p'-4)/2}}{E^2} \left(1 + \frac{4Hr^4}{\lambda^2 C_k}\right)^{k/2} \right\}, \tag{3.174}$$

where we are using a generalised expression for the energy. If we again assume that the velocity and the radius can be treated as complex variables with the equation of motion as a constraint, we can calculate the genus of the underlying Riemannian surface. Interestingly the results are similar to those obtained in an earlier part of this chapter, with the number of branch points being the same, though the the genus is dependent upon the dimensionality of the branes and on the non-Abelian group structure.

3.3 Dynamics in more general backgrounds.

In section 3.2 we considered specific background solutions, however we now wish to develop a more general formalism to deal with arbitrary backgrounds of a specific type. Namely backgrounds whose metrics are block diagonal. We also want to investigate solutions where there is a non-zero, homogeneous $U(1)$ gauge field on each of the branes. In general because of our matrix ansatz, we know that the gauge fields are generically non-Abelian, so this is somewhat of an idealised solution. However it is the first step in constructing a more complete description of non-Abelian brane dynamics.

We will begin by restricting ourselves to type II string theory in ten dimensions, and assume that there is a curved background generated by some source with M units of flux. The only constraint we will impose on the form of the background metric is that it is diagonal, with a symmetry group given by $SO(1,q) \times SO(9-q)$

$$ds^2 = -g_{00}dt^2 + g_{xx}dx^a dx^b \delta_{ab} + g_{zz}dz^i dz^j \delta_{ij} \tag{3.175}$$

where a, b run over the q worldvolume directions and i, j are transverse directions to the source. This background could obviously be generated by a stack of coincident branes, or something more exotic. Although there will generally be RR or NS charge generated by this solution we will only focus on the NS sector of the DBI action for simplicity.

As usual we will use the Myers action to describe the coincident Dp-branes, where the open string couplings on the world-volume are controlled by the inverse of the F-string tension as $\lambda = 2\pi\alpha'$. Recall that $\alpha' = l_s^2$ is the slope of the Regge trajectory and equal to the square of the string length.

We are interested in the dynamics of this configuration, and so we will demand that our transverse scalar fields are time dependent only, namely $\phi = \phi(t)$. Additionally we will begin by using diffeomorphism invariance to position the branes parallel to the gravitational source, but displaced along one of the transverse directions. On each of the world volumes we will also turn on an electric field using $F_{0a} = \varepsilon_a$ where $a, b = 1 \ldots p$ are world-volume directions, and we implicitly assume that we take the $A_0 = 0$ gauge and that the gauge field commutes with itself. It will often be convenient to write $\varepsilon^2 = \sum_a \varepsilon_a \varepsilon^a$ for simplicity. After calculating the determinant using (3.175), the kinetic part of the action can be seen to reduce to the following form

$$S_{kin} = -T_p \int d^{p+1}\zeta STr \left(e^{-\phi} \sqrt{g_{xx}^p g_{00}(1 - \lambda^2 g_{zz} g_{00}^{-1} \dot{\phi}^i \dot{\phi}^j \delta_{ij} - \lambda^2 \varepsilon^2 g_{xx}^{-1} g_{00}^{-1})} \right), \tag{3.176}$$

where we must still perform the symmetrized trace over the adjoint indices. As usual let us take the scalar fields to be valued in $SO(3)$, the fuzzy sphere ansatz. Strictly speaking this ansatz should be imposed upon the complete equations of motion and not upon the action, however it transpires that the ansatz is indeed consistent. Upon substitution into the kinetic part of the action written above we find it reduces to

$$S_{kin} = -T_p \int d^{p+1}\zeta STr \left(e^{-\phi} \sqrt{g_{xx}^p g_{00}(1 - \lambda^2 g_{zz} g_{00}^{-1} \dot{R}^2 \alpha^i \alpha^i - \lambda^2 \varepsilon^2 g_{xx}^{-1} g_{00}^{-1})} \right). \tag{3.177}$$

We also note that the metric components are generally functions of the transverse coordinates, which implies that they will be proportional to a trace over the group generators. However the radial coordinate implicit in the ansatz is not of the correct dimensionality and thus we are forced to revert to the physical distance in the metric. The implications for this are potentially far reaching, as we are assuming that the metric (and dilaton) terms are singlets with respect to the symmetrized trace. With these remarks in mind, and using the definition of the quadratic Casimir $C\mathbf{1}_N = (N^2-1)\mathbf{1}_N$, we can pull various terms through the trace operation and write the full action as follows

$$S = -T_p \int d^{p+1}\zeta N g_{xx}^{p/2} g_{00}^{1/2} e^{-\phi} \sqrt{(1 - g_{zz}g_{00}^{-1}\lambda^2 C\dot{R}^2 - g_{xx}^{-1}g_{00}^{-1}\lambda^2\varepsilon^2)(1+4g_{zz}^2\lambda^2 CR^4)}, \tag{3.178}$$

where we are making the reasonable assumption that the dilaton is a scalar function, and of course - we are taking the large N limit. Varying the Lagrangian density with respect to $\dot{R}$ and ε^a yields the canonical momenta for the radial mode and the displacement field respectively, the latter term being

$$D^a = \frac{T_p V_p e^{-\phi} g_{xx}^{p/2} g_{00}^{1/2} \sqrt{1+4\lambda^2 CR^4 g_{zz}^2}}{\sqrt{1 - g_{zz}g_{00}^{-1}\lambda^2 C\dot{R}^2 - g_{xx}^{-1}g_{00}^{-1}\lambda^2\varepsilon^2}} \left(\frac{\lambda^2\varepsilon^a}{g_{xx}g_{00}}\right), \tag{3.179}$$

where we note that D^a is the electric flux along the x^a direction on each of the world-volumes and is related to the charge of the fundamental string. As usual the canonical momenta allows us to construct the Hamiltonian via Legendre transform

$$\mathcal{H} = \frac{T_p V_p N e^{-\phi} g_{xx}^{p/2} g_{00}^{1/2} \sqrt{1+4g_{zz}^2\lambda^2 CR^4}}{\sqrt{1 - g_{zz}g_{00}^{-1}\lambda^2 C\dot{R}^2 - g_{xx}^{-1}g_{00}^{-1}\lambda^2\varepsilon^2}}. \tag{3.180}$$

At this juncture we note that R is not the physical distance of the probe branes from the source, however the two distances are related via the usual expression

$$r^2 = \frac{\lambda^2}{N} Tr(\phi^i\phi^j\delta_{ij}) = \lambda^2 CR^2$$

and so we may write the physical Hamiltonian as follows

$$\mathcal{H}_{phys} = \frac{T_p V_p N e^{-\phi} g_{xx}^{p/2} g_{00}^{1/2}}{\sqrt{1 - g_{zz}g_{00}^{-1}\dot{r}^2 - g_{xx}^{-1}g_{00}^{-1}\lambda^2\varepsilon^2}} \sqrt{1 + \frac{4g_{zz}^2 r^4}{\lambda^2 C}}, \tag{3.181}$$

or we can write it in the often more convenient Hamiltonian formalism

$$\mathcal{H}_{phys} = \sqrt{\left(T_p V_p N e^{-\phi} g_{xx}^{p/2} g_{00}^{p/2}\right)^2 \left(1 + \frac{4r^4 g_{zz}^2}{\lambda^2 C}\right) + \frac{g_{00}\Pi^2}{g_{zz}\lambda^2 C} + \frac{D^2 g_{xx}g_{00}}{\lambda^2}}. \tag{3.182}$$

In the above expressions we have defined V_p as the p-dimensional volume element of the branes. Note that when $p = 0$, corresponding to coincident $D0$-branes, the electric field contribution vanishes, as it must since the world-volume cannot support a rank two field strength tensor. In general the Hamiltonian will be conserved, however ε^a will not. This is because it is the flux that is the conserved charge on the D-brane, and not the gauge field. However because of our homogeneous ansatz we find that the electric field is conserved in this instance, and so we may write it in the following suggestive way

$$\varepsilon^a = \frac{D^a}{\tilde{\mathcal{H}}}, \tag{3.183}$$

which shows us that the electric field is conserved and quantised with D units of charge.

3.3.1 Minkowski space dynamics.

We have tried to keep the background space-time as general as possible, however in this section we will consider the dynamics of these branes in the flat space limit. The situation can be described as follows. We have N coincident Dp-branes with three excited transverse scalar fields parameterising a fuzzy two-sphere, the physical radius of which is given by r. The flat space Hamiltonian can be written simply as

$$\tilde{\mathcal{H}} = \frac{1}{\sqrt{1-\dot{r}^2-\lambda^2\varepsilon^2}}\sqrt{1+\frac{4r^4}{\lambda^2 C}}, \tag{3.184}$$

where we introduce the simplifying notation $\tilde{\mathcal{H}} = \mathcal{H}/(T_p V_p N)$, and note that $\dot{r}$ corresponds to the velocity of the collapsing fuzzy sphere. Furthermore with this definition of the Hamiltonian we lose all dependence on the dimensionality of the probe Dp-branes. Thus in the Minkowski limit all the p-branes yield the same equations of motion. As is usual with this type of problem it is far more convenient for us to use dimensionless variables. By making the following definitions

$$z = \sqrt{\frac{2}{\lambda\sqrt{C}}}r, \qquad \tau = \sqrt{\frac{2}{\lambda\sqrt{C}}}t, \qquad e = \lambda\varepsilon, \tag{3.185}$$

the Hamiltonian and effective potential can be written as follows

$$\begin{aligned} \tilde{\mathcal{H}} &= \sqrt{\frac{1+z^4}{1-\dot{z}^2-e^2}} \\ V_{eff} &= \sqrt{\frac{1+z^4}{1-e^2}}. \end{aligned} \tag{3.186}$$

The electric field must satisfy the usual constraint $e^2 \le 1$ in order for the theory to remain valid. The other constraint can be seen to be $1 \ge \dot{z}^2 + e^2$, which implies that the velocity of the collapse is reduced by a factor $\sqrt{1-e^2}$, which is less than the speed of light. For an arbitrary field strength we see that the fuzzy sphere will tend to collapse down to zero size as expected.

Our Hamiltonian has no explicit time dependence and is therefore a conserved charge which will allow us to obtain a solution to the equation of motion. We choose the initial conditions $\dot{z}(0) = 0$ and $z(0) = z_0$ to indicate an initially static configuration at some arbitrary distance z_0. By integrating the equation of motion and using the many properties of Jacobi Elliptic functions, we arrive at the solution [50, 51]

$$z(\tau) = \pm z_0 \text{JacobiCN}\left[\frac{\sqrt{2(1-e^2)}\tau z_0}{\sqrt{1+z_0^4}}, \frac{1}{\sqrt{2}}\right]. \tag{3.187}$$

Note that z_0 corresponds to the initial radius of the fuzzy sphere (in dimensionless variables). Taking the positive sign initially, one sees that as time evolves the fuzzy sphere collapses. The speed of the collapse is dependent upon the strength of the electric field, because an increasing field implies that the branes move more slowly. The physical interpretation of this is that the extra flux on the world-volume acts as extra 'mass', which acts to reduce the velocity. If there is a critical electric field which saturates the bound $e^2 = 1$ then the fuzzy sphere will be static for all time. This is different to the result obtained when considering the dynamics without gauge fields, which always implied collapsing solutions - at least to leading order in $1/N$. Eventually the sphere reaches zero size, however the periodic nature of the solution appears to imply re-expansion into a region of negative z. This is due to the ambiguity in taking the positive sign for the physical radius. A similar remark applies when taking the minus sign in the above solution. Note that in both cases, it is the R^2 term that appears in the DBI action and therefore no potential for discontinuities when we use the different sign choices for the physical radius.

The zeros of the elliptic function occur when the amplitude equals $K(k)$, where K is the complete elliptic integral of the first kind. This allows us to calculate the collapse time t_* for the fuzzy sphere to be

$$\tau_* = \sqrt{\frac{1+z_0^4}{2(1-e^2)}}\frac{1}{z_0}K\left(\frac{1}{\sqrt{2}}\right), \tag{3.188}$$

which agrees with our intuitive notion that by increasing the electric field, the collapse takes longer to occur.

3.3.2 1/N Corrections in Minkowski space.

In this section we will investigate the corrections to the theory arising from the symmetrized trace prescription. These corrections were first derived in [50], and we refer the interested reader to that paper for more details. In flat space it was emphasised that as the fuzzy sphere collapses its velocity approaches the speed of light, and therefore higher order terms in $1/N$ ought to become important in order to fully describe the dynamics. This is due to the fact that the energy will increase as the velocity increases. However the presence of an electric field on the brane world-volumes reduces the velocity of the collapse by the factor $\sqrt{1-e^2}$ and thus the leading order Lagrangian may remain valid - although there are difficulties associated with near critical electric fields and the DBI. In curved space the gravitational red shift appears to reduce the velocity of the fuzzy sphere to sub-luminal speeds, however there was found to be no turning point solution in the static potential and therefore no formation of non-Abelian bound states (with the exception of $D0$-branes in the $D6$-brane background.)

The important result from is that the corrections to the Lagrangian an be written as a series expansion in powers of C, thus our Hamiltonian can be shown to be the $0th$ order in this expansion

$$\tilde{\mathcal{H}} = \left(1 - \frac{2C}{3}\frac{\partial^2}{\partial C^2} + \frac{14}{45}C^2\frac{\partial^4}{\partial C^4} + \dots\right)\tilde{\mathcal{H}}_0. \tag{3.189}$$

It will be convenient in what follows to return the original action for a flat background, and define the following dimensionless parameters

$$\begin{aligned} \tilde{r}^4 &= 4\lambda^2 C R^4 \\ \tilde{s}^2 &= \lambda^2 C \dot{R}^2 \\ e^2 &= \lambda^2 \varepsilon^2 \end{aligned} \tag{3.190}$$

where the last expression has already been introduced in the previous section. The first two equations can be regarded as defining complex parameters, constrained by a single equation - namely the conservation of energy, and can be regarded as a 'radial' variable and a 'velocity' variable respectively. In terms of these complex parameters we can define the Hamiltonian to be

$$\tilde{\mathcal{H}} = \sqrt{\frac{1+\tilde{r}^4}{1-\tilde{s}^2-e^2}} = U\gamma, \tag{3.191}$$

where U can be regarded as a position dependent mass term, whilst γ is the modified relativistic factor. If we now apply the leading order symmetrized trace correction to this form of the Hamiltonian we obtain the following solution

$$\tilde{\mathcal{H}}_1 = U\gamma - \frac{\gamma}{6CU^3}\left[3U^4\gamma^4(1-e^2)^2 - 4U^4\gamma^2(1-e^2) - 2U^2\gamma^2(1-e^2) + 4U^2 - 1\right], \tag{3.192}$$

which represents the $1/N$ correction to the Hamiltonian in flat space. The first thing to note is that when there is a critical (or near critical) electric field, the corrected Hamiltonian reduces to

$$\tilde{\mathcal{H}}_1 \sim \frac{U}{i\tilde{s}}\left(1 - \frac{(4U^2-1)}{6CU^4}\right), \tag{3.193}$$

which is clearly imaginary and therefore does not correspond to a physical solution. We can avoid this problem by rotating the background metric to a Euclidean signature and studying the effects of over-critical electric fields, however we will not do that in this instance [11].

More generally we will have an arbitrary non-critical electric field, however we can still learn about the physical interpretation of the energy corrections. We first consider the static solution, i.e zero velocity, in which case the Hamiltonian becomes

$$\mathcal{H}_1 = \frac{U}{\sqrt{1-e^2}}\left(1 - \frac{(2U^2 - U^4 - 1)}{6CU^4}\right). \tag{3.194}$$

The correction terms will be non-zero except for when we choose $U = 1$, or when $U \to \infty$ corresponding to large radius. In this latter limit we would expect the geometry to resemble the classical geometry of the two-sphere. It should be noted that there is no value of r for which the energy will vanish. If we now consider the case where $R \to 0$, the energy reduces to

$$\mathcal{H}_1 = \gamma\left(1 - \frac{(\gamma^2(1-e^2)^2 - 2\gamma^2(1-e^2) + 1)}{2C}\right). \tag{3.195}$$

The correction term will be minimised by sending $\tilde{s} \to 0$, however it can be seen that the Hamiltonian itself will vanish if the velocity term satisfies

$$\tilde{s}^2 = (1-e^2)\left(1 - \frac{1}{1 \pm \sqrt{2}N}\right) \sim (1-e^2) \tag{3.196}$$

where we have explicitly taken the large N limit. Note that when the electric field is zero this condition reduces to $\tilde{s}^2 = 1$, implying that the branes are moving at the speed of light. Therefore in general we see that increasing the strength of the electric field reduces the velocity of the branes, as expected, and therefore can reduce the energy of the configuration when it is located at the origin.

3.3.3 Curved space dynamics.

The dynamics of the fuzzy sphere in curved backgrounds are generally non-trivial due to the additional contributions from the metric and dilaton. Thus we can only obtain exact solutions by specifying the form of the background. We repeat the physical Hamiltonian here for convenience.

$$\tilde{\mathcal{H}} = \frac{e^{-\phi} g_{xx}^{p/2} g_{00}^{1/2}}{\sqrt{1 - g_{zz} g_{00}^{-1} \dot{r}^2 - g_{xx}^{-1} g_{00}^{-1} \lambda^2 \varepsilon^2}} \sqrt{1 + \frac{4 g_{zz}^2 r^4}{\lambda^2 C}},$$

which allows us to define the static potential as follows

$$V = \frac{e^{-\phi} g_{xx}^{p/2} g_{00}^{1/2}}{\sqrt{1 - g_{xx}^{-1} g_{00}^{-1} \lambda^2 \varepsilon^2}} \sqrt{1 + \frac{4 g_{zz}^2 r^4}{\lambda^2 C}} \tag{3.197}$$

[11] We refer the interested reader to the recent work [60] for more information.

The unknown dependence of the metric components upon the physical radius prevents us from determining the general behaviour of the fuzzy sphere in this background. However we can see that the maximum value for the electric field will be a function of the transverse variables and therefore the radius of the fuzzy sphere. The general solution for the maximal field value can be seen to be

$$\varepsilon_{max} \leq \frac{\sqrt{g_{00}g_{xx}}}{\lambda}. \tag{3.198}$$

In our analysis we will assume that the electric field does not saturate this bound in order to keep the action finite and real. There has been extensive work on overcritical fields on D-branes, but this will not be relevant here. Using the conservation of the Hamiltonian we find the general expression for the velocity of the collapsing fuzzy sphere

$$\dot{r} = \frac{\delta\mathcal{H}}{\delta\Pi} = \left(\frac{\Pi g_{00}}{\mathcal{H}g_{zz}\lambda^2 C}\right). \tag{3.199}$$

Now for general supergravity solutions we expect the metric components corresponding to the $SO(1,q)$ directions to correspond to either flat, or decreasing monotonic functions of the physical radius. Conversely we would anticipate that the g_{zz} functions are either flat, or increasing monotonic functions of r - becoming singular when we reach zero radius. Therefore the general expression for the velocity suggests that it is a decreasing function of the physical radius regardless of the specific values of the ratio of $\Pi/\mathcal{H}$, provided that it is finite. The implication for this is that the sphere would take an infinite amount of time to collapse to zero size, neglecting any open string effects at short distances. This 'braking' behaviour is in contrast to what happens in flat space, where the fuzzy sphere collapses at an ever increasing velocity. However this is in a gravitational background and we expect the velocity term to be red shifted by the factor g_{00}/g_{zz}, thus by switching to proper time variables we would find that the collapse occurs in finite time.

The acceleration of the sphere turns out to be

$$\ddot{r} = \frac{\Pi\dot{r}}{\mathcal{H}g_{zz}\lambda^2 C}\left(g_{00}' - \frac{g_{zz}'}{g_{zz}}\right), \tag{3.200}$$

where primes denote derivatives with respect to the physical radius. The equation can be seen to be zero in three cases, firstly when $\dot{r}$ is zero which is the trivial solution as the sphere is static. Secondly when $g_{zz} \to \infty$ which implies that we must take $r \to 0$ and so the effective action breaks down, and finally when we have the case $g_{00} = \ln(\mathrm{g_{zz}})$. Provided the derivatives of the metric function are continuous, we see that the acceleration will never become singular and so we would expect the DBI to provide a reasonable description of the dynamics of the coincident branes.

At this point it is useful to consider some concrete examples of non-trivial backgrounds in order to fully understand the dynamical collapse of the fuzzy sphere.

Dq-brane background

The supergravity solutions are simply those presented in the previous section, which we will not repeat here. We will however switch notation to use q instead of p to denote the spatial dimensionality of the D-branes. We can read off the the various metric components and insert them into the physical Hamiltonian to obtain

$$\tilde{\mathcal{H}} = \frac{H^{(q-p-4)/4}}{\sqrt{1 - H\dot{r}^2 - H\lambda^2\varepsilon^2}}\sqrt{1 + \frac{4Hr^4}{\lambda^2 C}} \tag{3.201}$$

and the expression for the static potential becomes

$$V_{eff} = \frac{H^{(q-p-4)/4}}{\sqrt{1 - H\lambda^2\varepsilon^2}}\sqrt{1 + \frac{4Hr^4}{\lambda^2 C}} \tag{3.202}$$

The last expression tells us that the electric field can diverge as the radius of the fuzzy sphere collapses.

$NS5$-brane background.

Again this background was investigated in some detail in the previous chapter, so we do not re-write the supergravity solutions. The expression of interest for us is the static potential, which can be seen to reduce to

$$V_{eff} = \frac{1}{\sqrt{H}\sqrt{1 - \lambda^2\varepsilon^2}}\sqrt{1 + \frac{4H^2r^4}{\lambda^2 C}}, \tag{3.203}$$

implying that the maximal electric field bound is $\varepsilon_{max} \le \lambda^{-1}$. It is straight-forward to see that there is no turning point for the potential, except when we take r to be large which corresponds to the global maximum. The implication is that there is no radius at which the fuzzy sphere may stabilise at, and therefore nothing to halt the progress of the probe branes toward the five-branes even with the inclusion of an electric field.

F-string background.

We can also consider the background sourced by M fundamental strings, where for consistency we should limit the dimensionality of the probe branes to $p \le 1$ in order to fully justify our assumption about neglecting backreaction effects, the resulting late time configuration is a bound state of fundamental strings and D-strings more commonly referred to as a (p,q)-string [68, 69]. The supergravity background solution is [10, 11]

$$\begin{aligned} ds^2 &= H^{-1}\eta_{\mu\nu} + dz^a dz^b \delta_{ab} \\ e^{-\phi} &= H^{1/2} \\ H &= 1 + \frac{2^5\pi^2 g_s^2 l_s^6 M}{r^6}, \end{aligned} \tag{3.204}$$

where now μ, ν run over one temporal and one spatial dimension. The static potential for the bound state can be written

$$V_{eff} = \frac{1}{H}\sqrt{H\tilde{\tau}_1^2\left(1+\frac{4r^4}{\lambda^2 C}\right) + \frac{\Pi^2 H}{\lambda^2 C} + \frac{D^2}{\lambda^2}}, \tag{3.205}$$

where we have rescaled the $D1$-brane tension such that $\tilde{\tau}_1 = T_1 V_1 N$. Thus we effectively have a (D,N)-string bound state. We are at liberty to consider various limits of the potential, however the general behaviour is that it is always a monotonically decreasing function of the radius. For the D-string dominated solution we find the Hamiltonian scales like the tension of the string on a fuzzy sphere, namely

$$\mathcal{H} \sim \sqrt{\frac{\tilde{\tau}_1^2}{H}\left(1+\frac{4r^4}{\lambda^2 C}\right)}. \tag{3.206}$$

Conversely, taking the F-string dominated solution we find that the Hamiltonian scales with the displacement field

$$\mathcal{H} \sim \frac{|D|}{H\lambda}, \tag{3.207}$$

which shows that both configurations will be gravitationally attracted toward the F-string background as this is the lowest energy state. The background string coupling tends to zero with the physical radius of the fuzzy sphere, which means that our world-volume description can be trusted to very late times. As the strings move closer together we expect the formation of a new $(D+M,N)$-string bound state. The binding energy of which can be shown to be of the form

$$E_{\text{bind}} \sim \sqrt{\tilde{\tau}_1^2 + (D+M)^2}. \tag{3.208}$$

This result mimics the behaviour in the Abelian theory [24] where it was shown that the condition $g_s \to 0$ with $g_s N >> 1$ prevented the emission of closed string states and as such could be regarded as a semi-classical field theory. We close this section with a remark about the electric field in this instance. In the large radius limit we find that the displacement field can be well approximated by

$$D \sim \frac{2\tilde{\tau}_1 \lambda \varepsilon_\infty r^2}{\sqrt{C}\sqrt{1-\lambda^2\varepsilon_\infty^2}}, \tag{3.209}$$

where ε_∞ reflects the strength of the field at large distances where the harmonic function is approximately unity. As the sphere collapses the electric field is driven to its critical value, resulting in an increase in the displacement field. This behaviour can be seen via the expression

$$D^a = \mathcal{H}\lambda^2 H^2 \varepsilon^a, \tag{3.210}$$

where the right hand side naturally becomes large as the radius shrinks. This tells us that exactly at the threshold point of the bound state, the electric field reaches its critical value and the string becomes tensionless. A more detailed analysis with the inclusion of angular momentum modes would tell us a great deal about the formation of this bound state. We have also assumed here that the closed string modes will be suppressed, however a more detailed investigation would be useful as the supergravity constraints impose the strong condition $M >> N$ if we are to neglect back reaction. This is potentially useful in the investigation of cosmic superstring networks [67].

3.3.4 1/N corrections in Curved space.

As in the flat space case we can consider higher order corrections to the energy in powers of $1/N$ coming from the application of the symmetrized trace. We will find it convenient to define the following variables

$$\alpha = e^{-\phi} g_{xx}^{p/2} g_{00}^{1/2} \qquad \beta = \sqrt{1 + 4g_{zz}^2 R^4 \lambda^2 C} \qquad \gamma = (1 - e^2 - g_{zz} g_{00} \dot{R}^2 \lambda^2 C)^{-1/2}, \tag{3.211}$$

where $e^2 = g_{xx}^{-1} g_{00}^{-1} \lambda^2 \varepsilon^2$ and therefore the Hamiltonian reduces to $\tilde{\mathcal{H}} = \alpha\beta\gamma$. We know that this energy is the zeroth order expansion in powers of $1/N$ and using (3.189) we find that the first order Hamiltonian is remarkably similar to that constructed in the flat space instance

$$\tilde{\mathcal{H}}_1 = \alpha\beta\gamma - \frac{\alpha\gamma}{6C\beta^3}\left(3\beta^4\gamma^2(1-e^2) - 4\beta^4\gamma^2(1-e^2) - 2\beta^2\gamma^2(1-e^2) + 4\beta^2 - 1\right) \tag{3.212}$$

In deriving this expression we are explicitly assuming that the metric components are unaffected by the symmetrized trace prescription. Note that the energy to all orders will depend on the α factor, and therefore when this is zero the energy of the configuration will be zero. As we have argued, in general the metric functions g_{00} and g_{xx} are decreasing functions of r so that they vanish when $r \to 0$, thus we may expect that the energy will always tend to zero. However we have no way of knowing the general behaviour of the dilaton term with respect to the radial distance. What is clear is that minimising α is equivalent to minimising the energy. We begin by considering the case of zero electric field. We choose to set $\varepsilon = 0$ rather than taking the limit of the metric components to zero as this will imply that α, and therefore the energy, is zero. We further wish to consider the static case with the branes at an arbitrary distance away from the source. This reduces the Hamiltonian to the following form

$$\tilde{\mathcal{H}}_1 = \alpha\beta - \frac{\alpha}{6C\beta^3}(2\beta^2 - \beta^4 - 1). \tag{3.213}$$

There will be no correction terms when $\beta = 1$, which corresponds to the two cases $R \to 0$ or $g_{zz}^2 \to 0$. The first of these implies that $r \to 0$ and so the branes will be on top of the sources where we expect the DBI to break down. The second case corresponds to sending $r \to \infty$ because the metric component is generally an increasing function as $r \to 0$. This latter limit is unphysical in our situation, and so we see that the sphere energetically favours collapse from a static position. There will also not be any corrections as $\beta \to \infty$, which implies that either $r \to \infty, 0$ leading to the same remarks as above.

We now insist on keeping the electric field turned on, although the modification to the Hamiltonian in the static limit is very similar to the zero field case. The solution reduces to

$$\tilde{\mathcal{H}}_1 = \frac{\alpha\beta}{\sqrt{1-e^2}}\left(1 - \frac{(2\beta^2 - \beta^4 - 1)}{6C\beta^4}\right). \tag{3.214}$$

In this case the correction terms will only vanish as $\beta \to \infty$, which corresponds to the case of infinite energy for the fuzzy sphere. Thus for finite electric field we see that the solution will still collapse toward zero size, provided that the dilaton term does not blow up in the small r limit. In fact this is what distinguishes the $D6$-$D0$-brane system from the others as this is precisely where the dilaton term becomes large as the same time that the other metric components are going to zero. The resultant energy profile is not monotonic but yields a stable minimum in which a bound state can form. The case of critical, or almost critical field, is similar to the flat space scenario, where the energy becomes imaginary.

3.3.5 Brane Intersections in Curved Space.

Thus far our our analysis has dealt with parallel brane configurations, however this is not the only place non-commutative geometry enters into string theory, as we can also consider intersecting branes. The simplest intersections have been investigated in a series of papers where $ND1$-branes intersect with either $D3$, $D5$ or $D7$-branes in flat space-time. There are two dual world-volume descriptions of the intersection. The first is from the higher dimensional brane viewpoint, where the $D1$-brane is realised as an Abelian BIon spike solution protruding in a transverse direction [49]. The second, dual, description is from the non-Abelian viewpoint of the D-strings, which can be seen to blow up into the higher dimensional branes when we use non-commutative co-ordinates to parameterise a fuzzy funnel [45] The simplest and most investigated example of these has been the $D1-D3$ intersection, where the Abelian side consists of a BIon solution on the $D3$-brane world-volume. For the $D3$ case it is necessary to turn on a homogeneous magnetic field on the brane, since the D-string acts as a magnetic monopole solution on the world-volume. By contrast the $D5$ world-volume description is more complicated because there is a non vanishing second Chern class on the world-volume which alters the description of the BIon solution.

In this section we will investigate the $D1-D3$ intersection in the generic, static curved background labelled by the metric solution (3.175), with the inclusion of a constant electric field along the string world-volume [46]. As such, in contrast to the previous section, we will not consider ε as a dynamical degree of freedom. In fact the addition of a constant electric field turns the D-string into a (p,q)-string as the electric field can be interpreted as the dissolving of the fundamental string degrees of freedom into the world-volume. We will assume that the string is oriented in the $X_0 - X_9$ plane, where we will take $X_0 = t$ and $X_9 = \sigma$ to parameterise the embedding coordinates. We will again take the gauge $A_0 = 0$ and assume that the gauge field commutes with the transverse scalars. The kinetic part of the action reduces to the following expression

$$S = -T_1 \int d^2\sigma STr \left(e^{-\phi} \sqrt{g_{00}g_{zz}(1 - \lambda^2 g_{xx} g_{00}^{-1} \dot{\phi}_a \dot{\phi}_a + \lambda^2 g_{xx} g_{zz}^{-1} \phi'_a \phi'_a \lambda^2 \varepsilon^2 g_{00}^{-1} g_{zz}^{-1})} \right), \tag{3.215}$$

where a dot denotes derivatives with respect to time, and primes are derivatives with respect to σ. In the above we use the standard notation of representing the matrix-valued world volume scalar fields as ϕ_a which are not to be confused with the dilaton field ϕ. As in [45] we simplify our analysis by only considering fluctuations of the D-strings perpendicular to their world sheet that are also parallel to the world volume of the source branes. As such we look to employ the modified $SU(2)$ ansatz

$$\phi^a = R(t,\sigma) T^i, \qquad a = 1,2,3,$$

where the T^i again are the generators of the algebra and $a = 1,2,3$ label coordinates parallel to the source branes. Inserting the ansatz into the full action, and taking the large N limit produces the following

$$S = -T_1 \int d^2\sigma \quad N \quad e^{-\phi} \sqrt{(g_{00}g_{zz})(1 - \lambda^2 C g_{xx} g_{00}^{-1} \dot{R}^2 + \lambda^2 C g_{xx} g_{zz}^{-1} R'^2 - \lambda^2 \varepsilon^2 g_{00}^{-1} g_{zz}^{-1})} \sqrt{(1 + 4\lambda^2 C R^4 g_{xx}^2)}, \tag{3.216}$$

where we have neglected higher order corrections to the DBI, and also ignored any potential Chern-Simons term which may arise from the background source.

The metric components are typically functions of the $9-q$ transverse coordinates to the source branes. By our simplification above, we can consistently set the transverse coordinates to zero with the exception of $x_9 = \sigma$, and thus all the metric components are now explicit functions of σ. We will also assume that any dilaton term is purely a function of σ in order to simplify our analysis. In most of what follows we will only consider the near horizon approximation, however we will occasionally make reference to the Minkowski limit.

In what follows we shall be interested in either the time dependent solution or the spatial solution. It will be the latter that defines the fuzzy funnel. In any case the diagonal components of the energy-momentum tensor for the above action can be written as follows

$$\begin{aligned} T_{00} &= \frac{e^{-\phi}\sqrt{g_{00}g_{zz}(1+4\lambda^2 CR^4 g_{xx}^2)}(1+\lambda^2 CR'^2 g_{xx}g_{zz}^{-1} - g_{00}^{-1}g_{zz}^{-1}\lambda^2\varepsilon^2)}{\sqrt{1-\lambda^2 C\dot{R}^2 g_{xx}g_{00}^{-1} + \lambda^2 CR'^2 g_{xx}g_{zz}^{-1} - g_{00}^{-1}g_{zz}^{-1}\lambda^2\varepsilon^2}} \\ T_{\sigma\sigma} &= \frac{e^{-\phi}\sqrt{g_{00}g_{zz}(1+4\lambda^2 CR^4 g_{xx}^2)}(1-\lambda^2 C\dot{R}^2 g_{xx}g_{00}^{-1} - g_{00}^{-1}g_{zz}^{-1}\lambda^2\varepsilon^2)}{\sqrt{1-\lambda^2 C\dot{R}^2 g_{xx}g_{00}^{-1} + \lambda^2 CR'^2 g_{xx}g_{zz}^{-1} - g_{00}^{-1}g_{zz}^{-1}\lambda^2\varepsilon^2}}, \end{aligned} \tag{3.217}$$

where we have explicitly divided out each term by the factor $T_1 N V_1$ which is independent of any space-time coordinates and will not affect the equations of motion. We must now consider the static and dynamical cases separately if we wish to find simple solutions to the equations of motion.

3.3.6 Funnel solutions.

We can now attempt to find solutions by specifying the background explicitly. We know that in flat Minkowski space the solutions correspond to funnels, where the lower-dimensional branes blow up into a solitary $D3$-brane. We may expect these funnel type solutions to occur in curved space as well, however the form of the solution will be different. Firstly consider a stack of Dq-branes, which have the usual supergravity solution

$$ds^2 = H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}dx^i dx^j \delta_{ij}, \qquad e^{-\phi} = H^{(q-3)/4},$$

where μ, ν are world-volume directions and i, j are transverse directions. The warp-factor H is a harmonic function in the transverse directions, which since we are only considering fluctuations of the D-string parallel to the Dq world volume implies they are only dependent on $\sum_{i=9-q}^{9}(x^i)^2 = \sigma^2$ and we assume $q = 1, 3, 5$ only because we are looking at type IIB string theory. The equation of motion can be satisfied by the following expression

$$R'^2 = \frac{1}{\lambda^2 CH^{-1}}\left(H^{(q-3)/2}\{1+4\lambda^2 CR^4 H^{-1}\}(1-\lambda^2\varepsilon^2)^2 - (1-\lambda^2\varepsilon^2)\right). \tag{3.218}$$

Note that for critical electric fields the RHS of the expression vanishes which implies that R =constant and therefore no funnel solution exists regardless of the background. For near critical fields the solution is approximately constant until we reach the point where R diverges. Thus the general behaviour is that increasing the strength of the gauge field forces the funnel to alter its shape. The stronger the field, the wider the funnel and the larger the fuzzy sphere radius. Temporarily setting the electric field to zero brings us back to the D-string solution, and the equation of motion reduces as follows

$$R'^2 = \frac{1}{\lambda^2 C H^{-1}} \left(H^{(q-3)/2} \{ 1 + 4\lambda^2 C R^4 H^{-1} \} - 1 \right), \tag{3.219}$$

which can be seen to be trivially solved when $q = 3$ since the eom reduces to $R' = 2R^2$ and we recover the funnel solution [12]

$$R(\sigma) = \frac{-1}{2(\sigma - \sigma_0)}. \tag{3.220}$$

The radius of the funnel diverges at $\sigma = \sigma_0$ where the D-strings blow up into a $D3$-brane. Note that the minus sign indicates this is a $D3$-brane and not a $\bar{D}3$-brane, since the latter will be unstable in the background. In fact the harmonic function drops out of the equations implying the funnel solution is insensitive to the curved background. This is due to the vanishing dilaton term. If we insist on the inclusion of the electric field in the $D3$-brane solution then we can shift variables in the integration to obtain a solution, which is a simple deformation of the standard funnel as we would anticipate

$$R(\sigma) \sim \frac{-1}{2\sqrt{1 - \lambda^2 \varepsilon^2}(\sigma - \sigma_0)}. \tag{3.221}$$

The effect of increasing the electric field is to force the funnel to open up more at smaller values of σ. In fact for near critical fields we expect the funnel to diverge before the point σ_0, implying that the $D3$-brane is located at a different position to the case of zero field. The structure of the equation of motion prohibits us from finding an exact solution in the $D5$ and $D1$-backgrounds.

We can also look at the $NS5$-brane background, where the supergravity solutions are given in (2.2), but where we now use M instead of k. The solution with zero electric field can be parameterised by $R' = 2R^2\sqrt{H}$, with $H(\sigma)$ given by [19] with $r^2 = \sigma^2$. In the first instance, if we look in the throat approximation (i.e dropping the factor of unity in H) we find the funnel solution

$$R(\sigma) = \frac{-1}{2\sqrt{M l_s^2} \ln(\sigma/\sigma_0)}. \tag{3.222}$$

Here we have selected the cut-off distance σ_0 to represent the location of the $D3$-brane in the transverse space. Because the dilaton term tends to blow up as we approach the fivebranes, we must worry that our solution (being weakly coupled to neglect backreaction) may not be valid deep in the throat geometry. Therefore this solution can be trusted when the curvature of the bulk geometry is relatively small. Interestingly we see the funnel solution is invariant (up to a sign) under $\sigma \to 1/\sigma$, which is related to the large/small duality problem and standard T-duality solutions in type II string theory [51]. The change in sign reflects the

[12] Note that this is also the BPS condition in flat space, however the $D3$-brane will also be supersymmetric in the $D3$-brane background and so this is also the BPS condition in this instance.

change in orientation of the $D3$-brane, however as both D and $\bar{D}$-branes are unstable in the fivebrane background the minus sign is technically irrelevant. It may be possible to probe further into the throat using the corrections from the symmetrized trace. The idea would be to use the fact that g_sN is constant, but take a slightly larger value for the string coupling. In order to compensate for this we must reduce the number of D-strings and therefore extra $1/N$ terms will become important. Using the prescription developed in [50] we can calculate these corrections and check to see how the funnel solution is modified.

We can also extend our solution above to the case where we keep the full expression for H. This yields an interpolating solution between the throat solution and Minkowski space, given by

$$R(\sigma) = \frac{\mp 1}{2\left(\sqrt{Ml_s^2+\sigma^2} - \sqrt{Ml_s^2+\sigma_0^2} + \sqrt{Ml_s^2}\ln\left\{\frac{\sigma[\sqrt{\mathrm{Ml_s^2}}+\sqrt{\mathrm{Ml_s^2}+\sigma_0^2}]}{\sigma_0[\sqrt{\mathrm{Ml_s^2}}+\sqrt{\mathrm{Ml_s^2}+\sigma^2}]}\right\}\right)} \tag{3.223}$$

which can be seen to yield the two asymptotic solutions when we take the appropriate limit. This solution is particularly interesting because of the cut-off imposed in the integral. On one side of the $D3$-brane we have a semi-infinite string solution (solution with +sign in (3.223) whilst on the other (- sign choice) we have a string of finite length. In the throat approximation we can relate the two solutions through a $\sigma \to 1/\sigma$ duality. The finite length of the string implies that the energy of the solution is finite. This differs dramatically from the Minkowski space solution where the energy will be infinite as the string is of infinite length. The profile of the solution therefore relates a finite energy configuration to an infinite energy configuration. This behaviour may well have an interesting analogue in the Abelian world-volume theory.

The corresponding funnel solution in the background of fundamental strings (3.204) can be obtained from the following expression $R'^2 = 4R^4/H$, which gives, in the throat approximation,

$$R(\sigma) = \frac{-2\sqrt{k}}{\sigma^4 - \sigma_0^4}, \tag{3.224}$$

obviously diverging strongly in the limit that $\sigma \to \sigma_0$. Of course there are many other kinds of backgrounds that we are free to consider. As an example we could look at the static Maki-Shiraishi solutions [59] corresponding to a static black hole geometry. In this case we see that $R(\sigma) \propto \sigma^{5/2} - \sigma_0^{5/2}$ which implies that the funnel only diverges at large values of σ, very far from the event horizon of the black hole. This class of metrics also allows for time dependent solutions, corresponding to gases of $D0$-branes and may play an important role in the study of matrix cosmology.

Finally we note that even though it is difficult to obtain an analytic solution of the funnel profile in Dq-brane (for $q \neq 3$)- backgrounds, progress can be made in the large R

approximation. In this case we find from (3.218)

$$R' \approx 2H^{(q-3)/4}R^2 \tag{3.225}$$

which can be integrated to yield approximate (large R) solutions.

3.3.7 1/N Corrections to the Fuzzy Funnel.

We are interested in the corrections to the funnel solutions we have found, particularly those arising from the symmetrized trace prescription. In flat space the funnel is a BPS configuration and thus insensitive to any corrections to all orders. In curved space we have seen that the funnel solution will not generally correspond to a BPS configuration as the bulk supersymmetries will be broken, or at least non-linearly realised. Using (3.189) we can calculate the leading $1/N$ corrections to the Hamiltonian. As usual it is convenient to introduce the following expressions to simplify the results

$$\alpha = e^{-\phi}\sqrt{g_{00}g_{zz}}, \quad \beta = \sqrt{1+4\lambda^2 CR^4 g_{xx}^2}, \quad \gamma = \sqrt{1+\lambda^2 CR'^2 g_{xx}g_{zz}^{-1} - e^2},$$

where we have also introduced the simplification $e^2 = \lambda^2\varepsilon^2 g_{00}^{-1}g_{zz}^{-1}$. This allows us to write the first correction to the Hamiltonian, assuming of course that the dilaton term is not a function of the Casimir

$$\mathcal{H}_1 = \alpha\beta\gamma - \frac{\alpha}{6C}\left\{\frac{2(\beta^2-1)(\gamma^2-1+e^2)}{\beta\gamma} - \frac{\gamma(\beta^2-1)^2}{\beta^3} - \frac{\beta(\gamma^2-1+e^2)^2}{\gamma^3}\right\}. \tag{3.226}$$

Now, setting the electric field to zero implies that the correction terms will cancel out to zero when $\beta = \gamma$. This can actually be seen just by demanding minimisation of $\mathcal{H}_0$, however we can also see that the correction terms vanish upon implementation of the symmetrized trace. The minimisation yields a constraint on the curvature which is given by the following

$$R'^2 = 4R^4 g_{xx}g_{zz}. \tag{3.227}$$

In flat space this is just the BPS condition which leads to the simple funnel solution. In certain backgrounds where the g_{xx} components equal the inverse of the g_{zz} components - for example Dq-brane backgrounds - we also recover the simple funnel solution. However we know that this is only a solution to the equation of motion in the $D3$-brane background, and so we seem to have found solutions satisfying the minimal energy condition but which do not solve the equations of motion. In the $NS5$ and F-string backgrounds we see that this energy condition coincides with a solution to the equations of motion, and so we expect those particular funnel solutions to be minimal energy solutions. This suggests that the symmetrized trace corrections are zero for configurations which are in their minimal energy states. In flat space the minimal energy state coincides with the BPS condition which is why we do not have corrections. In general the lowest energy configuration may not be BPS but will still receive no corrections from the symmetrized trace. The general solution consistent with energy minimisation can be written as

$$R(\sigma) = \frac{\mp 1}{2\int d\sigma \sqrt{g_{xx}g_{zz}}} \quad = \quad \frac{\mp 1}{2\int d\sigma f(\sigma)}. \tag{3.228}$$

We expect simple power law behaviour for $f(\sigma) \sim \sigma^n$ and so the solution can be written as

$$R(\sigma) \sim \frac{\mp(n+1)}{2(\sigma^{n+1} - \sigma_0^{n+1})}, \tag{3.229}$$

where n can be positive or negative, but not equal to -1. The case where $n = 1$ corresponds to flat space. In the above expression we have neglected the dimensionality constant coming from the function f. When $n = -1$ the solution reduces to the inverse logarithm solution we find in the $NS5$-brane background. Note that when n is negative we do not obtain funnel solutions as the radius of the fuzzy sphere never diverges, instead it monotonically increases with the distance from the sources. This indicates that these solutions do not expand into higher dimensional branes, and will not have an Abelian world-volume description.

Even though the funnel configuration appears to satisfy the energy minimisation condition, the energy itself still has dependence on the location of the funnel in the throat through the α term. For the three cases where we find explicit brane solutions, namely the $D3$, $NS5$ and F-string backgrounds this term reduces to unity. In the $D5$-brane background we see that $\alpha \propto 1/\sigma$ and so the solution minimises its energy when it is far from the sources and thus is well approximated by the simple funnel solution. The $D1$-brane background yields $\alpha \propto \sigma^3$ and so the funnel is only a solution when it is on top of the background branes, which is where our effective action will no longer be valid. This perhaps explains why we were unable to find analytic solutions to the funnel equation of motion. We should note at this point that $g_s \to 0$ with σ in the $D5$ case, implying that the tension of the branes will become infinite and again our action will be invalidated. In the $D1$ case we see that the coupling becomes strong as $\sigma \to 0$, therefore the tension of the branes is small but our assumption that $g_s N < 1$ must be violated. It appears that both these backgrounds cause the effective action to break down and so we cannot trust our solutions except at large σ, where the background is essentially flat and we recover our simple funnel solution. The reason why this is not the case in the $NS5$-brane background is because their tension goes as $1/g_s^2$, and so the coincident brane solution has a much larger mass than the N D-strings.

Setting aside the minimal energy condition for a moment we can make some observations about the energy of the funnel including the leading order correction terms. Firstly we consider the case $R' = 0$ corresponding to no curvature. The energy can be written as

$$\mathcal{H}_1 = \alpha\beta\sqrt{1-e^2}\left(1 + \frac{\alpha(\beta^2-1)^2}{6C\beta^4}\right). \tag{3.230}$$

Clearly when $\beta^2 = 1$ there will be no (leading order) corrections to the energy, a condition that can be satisfied either by taking $R \to 0$ or $g_{xx}^2 \to 0$. The first condition corresponds to no curvature, with the strings located at an infinite distance away from the background source. The second condition is the more interesting as it generally implies that $\sigma \to 0$, or that the strings are located at the source. The resultant energy for the strings is then determined by α - provided we have a sub-critical electric field, and so we see that minimising α is equivalent to minimising the energy. We can also consider the case where we take $R = 0$, to see the effect this has on the energy and its corrections. The resultant expression becomes

$$\mathcal{H}_1 = \alpha\gamma \left(1 + \frac{\lambda^2 R'^2 g_{xx} g_{zz}^{-1}}{6\gamma^4}\right). \tag{3.231}$$

Again we see that the correction term vanishes if we demand the curvature to be zero, or alternatively we can set $g_{xx}g_{zz}^{-1} \to 0$ either as a product or individually, which basically implies that $\sigma \to 0$ as usual. We see once more that α plays the dominant role in determining the energy, and that if this term can vanish then so can the energy. This helps to explain why we cannot obtain analytic solutions for the $D5$ and $D1$-backgrounds, as in these cases the α term is a function of σ which implies that the energy will either diverge, or tend to zero with σ, depending on the dimensionality of the source branes. Therefore the energy is dependent upon the space-time variables. For the $D3$, $NS5$ and F-string backgrounds we find that $\alpha = 1$ and thus it is the shape of the funnel itself which dictates the minimal energy configuration.

3.3.8 Time Dependence and Dualities.

In the time dependent case we again use the conservation of the energy-momentum tensor to obtain the equation of motion

$$\dot{R}^2 = \frac{g_{00}(1 - g_{00}^{-1} g_{zz}^{-1}\lambda^2\varepsilon^2)A}{\lambda^2 C g_{xx}\bar{g}_{00}\bar{g}_{zz}(1 + 4\lambda^2 C R_0^4 \bar{g}_{xx}^2)(1 - \bar{g}_{00}^{-1}\bar{g}_{zz}^{-1}\lambda^2\varepsilon^2)}, \tag{3.232}$$

where the coefficient A is written as follows

$$A = -(e^{2(\phi_0-\phi)} g_{00}g_{zz}(1+4\lambda^2 C R^4 g_{xx}^2)(1-g_{00}^{-1}g_{zz}^{-1}\lambda^2\varepsilon^2) - \bar{g}_{00}\bar{g}_{zz}(1+4\lambda^2 C R_0^4 \bar{g}_{xx}^2)(1-\bar{g}_{00}^{-1}\bar{g}_{zz}^{-1}\lambda^2\varepsilon^2)).$$

In deriving this expression we have imposed the initial conditions that $R(t = 0) = R_0$ when $\dot{R} = 0$, and the metric components at this initial point have been denoted by a bar. Note also the factor of e^{ϕ_0} in the solution which reflects the initial value of the dilaton subject to these boundary conditions. In fact this equation is remarkably similar to the static one, which can be calculated to yield

$$R'^2 = \frac{g_{zz}(1 - g_{00}^{-1} g_{zz}^{-1}\lambda^2\varepsilon^2)B}{\lambda^2 C g_{xx}\bar{g}_{00}\bar{g}_{zz}(1 + 4\lambda^2 C R_0^4 \bar{g}_{xx}^2)(1 - \bar{g}_{00}^{-1}\bar{g}_{zz}^{-1}\lambda^2\varepsilon^2)}, \tag{3.233}$$

where the coefficient B turns out to be simply $-A$. If we consider the case where the D-string is located far from the sources in flat Minkowski space, the metric components and the dilaton can be set to unity. In this limit the two equations of motion reduce to

$$\begin{aligned} R'^2 &= \frac{4(1-\lambda^2\varepsilon^2)(R_0^4 - R^4)}{1+4\lambda^2 C R_0^4} \\ \dot{R}^2 &= \frac{4(1-\lambda^2\varepsilon^2)(R^4 - R_0^4)}{1+4\lambda^2 C R_0^4}, \end{aligned} \tag{3.234}$$

which are clearly invariant under the following invertible world-sheet transformation $t \to i\sigma$, which is nothing more than Wick rotation [51]. If we re-write these equations using dimensionless variables as in (3.185), introducing a similar transformation on the σ coordinate, then we find that the two equations of motion are related via $\dot{z} = iz'$. Therefore knowledge of one of the solutions (3.187) automatically implies knowledge of the other solution as follows

$$\begin{aligned} z(\tau) &= \pm z_0 \mathrm{JacobiCN}\left[\frac{\sqrt{2(1-e^2)}\tau z_0}{\sqrt{1+z_0^4}}, \frac{1}{\sqrt{2}}\right] \\ z(\sigma) &= \pm \frac{z_0}{\mathrm{JacobiCN}\left[\frac{\sqrt{2(1-e^2)}\sigma z_0}{\sqrt{1+z_0^4}}, \frac{1}{\sqrt{2}}\right]}. \end{aligned} \tag{3.235}$$

In the last line we have used one of the various properties of elliptic functions. As discussed in an earlier chapter, the last equation defines a periodic array of $D3/\bar{D}3$-branes connected by the fuzzy $D1$-funnels. There are two important comments to be made at this point. Firstly that the equation of motion for a collapsing fuzzy sphere is the same as that of a time-dependent funnel in Minkowski space. Secondly the world-sheet transformation we employed on the equation of motion has a geometric interpretation. Instead of performing a Wick rotation on the time variable, we can instead identify τ with σ provided we also send $z \to 1/z$. Using the definition of the elliptic function we can easily verify that this is true. Therefore we have a concrete example of the so called large/ small duality that pervades all string theories, as a collapsing fuzzy sphere of radius R is dual to a brane-anti-brane array with interpolating funnel solutions of maximal radius $1/R$ [51].

In the more general case it is clear to see that we recover the static equation from the time dependent one by performing the following transformation

$$\varepsilon \to 0, \qquad t \to i\sigma, \qquad g_{00} \to g_{zz}. \tag{3.236}$$

This corresponds to a Wick rotation on the worldsheet and a space-time transformation in the bulk, and is therefore a highly non-trivial symmetry. However we can see that the transformation is not invertible, unlike in Minkowski space, due to the σ dependence of the metric components. If we start with the static equation and rotate the spatial coordinate such that $\sigma \to -i\tau$, then the metric components (as well as the curvature term) become time dependent - corresponding to some form of time dependent background [13]. If we take this solution and then Wick rotate the time variable again we recover the spatial dependent equation. Thus it appears there is a mapping from the time dependent equation to the static one, but not vice-versa. The static equation is invariant under a double Wick rotation, which appears to be the only automorphism of that particular equation. This implies that the large/small duality is broken in this instance by the presence of curved spacetime, which we ought to expect since the time-like and space-like Killing vectors cannot be rotated into one another due to the additional spatial dependence of the metric components. In flat space the metric, and therefore the Killing vectors, are invariant under Wick rotation and so the field theory solutions ought to respect this symmetry.

There is, however, a particularly interesting transformation in curved space when the metric components g_{00} and g_{zz} are inverses of each other - as in the case for Dq-brane backgrounds in the near horizon limit. Writing the harmonic function in terms of the dimensionless distance variable $\tilde{z}$

$$H \sim \frac{1}{\tilde{z}^{7-q}}, \tag{3.237}$$

then it is straightforward to see that the transformation to the static equation is nothing more than T-duality, taking $\tilde{z} \to 1/\tilde{z}$.

One further comment should be made here with regard to the interpretation of the dynamical solution. In the Minkowski limit we saw that the time dependent funnel solution yielded the same equations of motion as the collapsing fuzzy sphere. This led [51] to postulate the existence of a duality between contracting fuzzy spheres and funnels. In curved space we see that this interpretation is no longer valid, since the equations of motion coming from the collapsing fuzzy sphere are different - as shown in section 2.

3.3.9 The Dual Picture - $D3$ world-volume theory.

Our work on constructing funnel solutions in curved space has yielded some interesting results. At this stage we would like to check our assumption that the funnels do in fact lead to the emergence of $D3$-branes, which can be done in the dual $D3$ world-volume theory [49]. We begin with the effective action for a solitary $D3$-brane in a general background with vanishing Kalb-Ramond two form

$$S = -T_3 \int d^4\zeta e^{-\phi} \sqrt{-det(G_{ab} + \lambda F_{ab})},$$

The D-strings in this theory will appear as magnetic monopoles on the $D3$-brane, thus we must ensure a non-trivial magnetic field is turned on. We choose this to be $F_{ab} = \epsilon_{abc} B_c$, with roman indices running over the world-volume. Finally we must also ensure that one of

[13] Unfortunately these are not the spacelike D-brane supergravity solutions constructed in [85].

the transverse scalars - σ is excited. As usual we neglect higher derivative terms in the DBI action, and employ the use of static gauge. The result for the static solution is as follows

$$S = -T_3 \int d^4\zeta e^{-\phi} \sqrt{g_{00} g_{xx}^3 (1 + \lambda^2 g_{zz} g_{xx}^{-1} (\vec{\nabla}\sigma)^2 + \lambda^2 g_{xx}^{-2} \vec{B}^2 + \lambda^4 g_{zz} g_{xx}^{-3} (\vec{B}.\vec{\nabla}\sigma)^2)}. \quad (3.238)$$

It should be noted that the scalar field has canonical dimension of L^{-1}, which we need to be careful of when interpreting our solutions - particularly when trying to show that this is indeed the dual picture configuration. The equation of motion for the transverse scalar is complicated in curved space, and not readily amenable to analytic solutions. Thus we will attempt to find the spike profiles by searching for configurations which minimise the energy, a tactic which worked for several backgrounds in the non-Abelian case where the energy minimisation condition corresponded to the equations of motion [45]. The energy density in the static case simply equals $-\mathcal{L}$ therefore we may write

$$\begin{aligned} \mathcal{H} &= T_3 \int d^3\zeta e^{-\phi} \sqrt{g_{00} g_{xx}^3 (1 + \lambda^2 g_{zz} g_{xx}^{-1} (\vec{\nabla}\sigma)^2 + \lambda^2 g_{xx}^{-2} \vec{B}^2 + \lambda^4 g_{zz} g_{xx}^{-3} (\vec{B}.\vec{\nabla}\sigma)^2)} \\ &= T_3 \int d^3\zeta e^{-\phi} \sqrt{g_{00} g_{xx}^3} \sqrt{\lambda^2 |\sqrt{g_{zz} g_{xx}^{-1}} \vec{\nabla}\sigma \pm g_{xx}^{-1} \vec{B}|^2 + (1 \mp \lambda^2 g_{zz}^{1/2} g_{xx}^{-3/2} \vec{B}.\vec{\nabla}\sigma)^2}, \end{aligned}$$

where in the last line we have written the determinant as the sum of two squares. We see that there is an energy bound given by

$$\mathcal{H} \geq T_3 \int d^3\zeta e^{-\phi} \sqrt{g_{00} g_{xx}^3} |1 \mp \lambda^2 g_{zz}^{1/2} g_{xx}^{-3/2} \vec{B}.\vec{\nabla}\sigma|, \quad (3.239)$$

which is saturated provided that the σ-field satisfies the following constraint

$$\vec{B} = \mp \vec{\nabla}\sigma \sqrt{g_{xx} g_{zz}}, \quad (3.240)$$

which can be seen to reduce to the usual flat space constraint $\vec{B} = \mp\vec{\nabla}\sigma$ as required. The expression for the energy bound (3.239) seems to be the sum of two terms where the second one is topological in nature. We wish to show that this expression has a simple interpretation in terms of the energy of the $D3$-brane and the energy of a warped spike solution. We will write the first term as follows

$$\mathcal{H}_{D3} = T_3 \int d^3\zeta e^{-\phi} \sqrt{g_{00} g_{xx}^3}. \quad (3.241)$$

Now in flat space the energy of the $D3$-brane is simply $T_3 \int d^3\zeta$, however as we are in a generic curved background we must also include the contribution from a non-trivial dilaton. This means the energy is modified to become $T_3 \int d^3\zeta e^{-\phi}$ which is exactly the equation we wrote down for the energy of a warped $D3$-brane. Thus our intuition about the first term is correct, namely that it corresponds to the energy of the brane in curved space. The second term is a simple extension of the BIon spike solution, generalised to a curved background.

We return now to (3.240) which gives us important information about the profile of the spike solution. It is clear that the second term here is a total derivative if the B field satisfies the modified Gauss law equation

$$\vec{\nabla}.(\sqrt{g_{00}g_{zz}}e^{-\phi}\vec{B}) = 0. \tag{3.242}$$

This modification of the Gauss law appears to be due to red-shifting of the magnetic field for an observer in the UV end of the background geometry. Such red-shifting effects are common in warped metrics. Under these circumstances the second term would then be determined by the boundary values of $\sigma(r)$ and so we would find a contribution to the energy proportional to $\sigma(r=\infty) - \sigma(r=0)$ which could be interpreted as the energy of a string stretching along the σ direction i.e the D-strings of the non-Abelian theory.

In the case of background $NS5$ branes or F-strings, which are both charged under the NS field, it is easy to check that (3.242) reduces to the familiar flat-space Gauss constraint due to the cancellation with the metric components i.e $\vec{\nabla}.\vec{B} = 0$. In the case of Dq-brane backgrounds, which are charged under the RR fields, such a cancellation between the dilaton and metric components does not occur and the Gauss law condition reduces to

$$\vec{\nabla}.(e^{-\phi}\vec{B}) = 0. \tag{3.243}$$

We wish to solve the general spike solution using (3.240). In general we may expect a power series solution for the metric functions which will be given by $f(\tilde{\sigma})$, where $\tilde{\sigma}$ refers to the physical coordinate distance. Note that σ is related to the physical distance via $\tilde{\sigma} = l_s^2\sigma$. As in the non-Abelian section we will take $f(\tilde{\sigma}) \sim \tilde{\sigma}^n$, where n can be positive or negative but not unity. It will be convenient to switch to spherical coordinates in which case the magnetic field will only have a radial dependence, and we will take the traditional ansatz for the field to be

$$B = \pm\frac{Q}{4\pi r^2}, \tag{3.244}$$

where Q corresponds to the magnetic charge of the $U(1)$ field. Equating both sides of (3.240) gives us the physical solution for the spike

$$\tilde{\sigma}^{n+1} - \tilde{\sigma}_0^{n+1} \propto \pm\frac{Ql_s^2(n+1)}{4\pi r}, \tag{3.245}$$

where we have neglected a dimensionality factor which makes $f(\tilde{\sigma})$ dimensionless. With reference to the general solution on the non-Abelian side (3.228) in physical coordinates we find

$$\sigma^{n+1} - \sigma_0^{n+1} \sim \pm\frac{\pi N l_s^2(n+1)}{r}. \tag{3.246}$$

If we demand that both of these solutions are equal - to leading order in N - we need to impose the following quantisation condition on the magnetic charge, namely $Q = 4\pi^2 N$. This condition, with the appropriate choice of sign, ensures that the equations from the non-Abelian and Abelian theories are the same in an arbitrary background. The $n = -1$ case, which arises in the fivebrane backgrounds, will give rise to a logarithmic funnel profile and not the simple power law solution.

In the specific case of the $NS5$-brane background we find that the spike solution from the Abelian action, $\tilde{\sigma}(r)$ satisfies the following equation

$$-\frac{1}{r} - \frac{4\pi\sqrt{M}}{Q}\sqrt{\frac{\tilde{\sigma}^2}{l_s^2 M} + 1} + \frac{2\pi\sqrt{M}}{Q}\ln\left(\frac{\sqrt{\frac{\tilde{\sigma}^2}{l_s^2 M} + 1}}{\sqrt{\frac{\tilde{\sigma}^2}{l_s^2 M} - 1}}\right) = c \tag{3.247}$$

where c is a arbitrary constant of integration. In the throat approximation where $\frac{\tilde{\sigma}^2}{l_s^2 M} \ll 1$ this equation can be solved explicitly for the spike profile

$$\tilde{\sigma} = \tilde{\sigma}_0 \exp\left(\frac{-Ql_s}{4\pi r\sqrt{M}}\right) = \tilde{\sigma}_0 \exp\left(\frac{-\pi l_s N}{\sqrt{M} r}\right), \tag{3.248}$$

More generally the complete solution above in (3.247) can be seen to be exactly equivalent to the solution for the fuzzy funnel discovered on the non-Abelian side in (3.223) with an appropriate definition of the constant c in terms of the D3-brane location parameter σ_0 and using the quantisation of magnetic charge Q found earlier.

Now in flat space the fact that a spike profile saturates the energy bound is normally sufficient to argue that such a profile solves the equations of motion. However in the case where there is a throat present due to the $NS5$ source branes, this is not the case. From equation (3.240) with g_{xx} and g_{zz} appropriate to the throat geometry, we can scale $\tilde{\sigma} \to l\tilde{\sigma}$ and still satisfy this equation. However under the same scaling, the energy of the warped $D3$-brane scales like

$$\mathcal{H}_{D3} \to l\mathcal{H}_{D3} \tag{3.249}$$

and so the energy of the brane can now be reduced by sending $l \to 0$, indicating that the $D3$-brane - or funnel solution on the non-Abelian side - will be unstable. This shows that the static spike profile (3.247) is unstable and wants to decay. Thus by considering a time-dependent profile rather than static, we can find a solution to the equations of motion.

In general looking for analytic t and r-dependent solutions to the equations of motion looks very difficult. However assuming the throat approximation, a simple solution, which describes the motion of the funnel as a whole, can be obtained by using separation of variables. Such a solution can be expressed as $F(t)\tilde{\sigma}(r)$, where we have introduced a dimensionless time-dependent profile, $F(t)$ for the spike. It is easy to see that $F(t)$ drops out of (3.240) so that $\tilde{\sigma}(r)$ still describes a static spike profile as in (3.248). $F(t)$ is determined by demanding $F(t)\tilde{\sigma}(r)$ solves the complete equations of motion. We find that the energy density of the brane reduces to the simple form

$$\mathcal{E} = T_3 V_3 \frac{F\tilde{\sigma}}{\sqrt{M l_s^2}} \left| 1 \pm \frac{\lambda^2}{l_s} \sqrt{M} \vec{B}.\vec{\nabla} \ln\left(\frac{\tilde{\sigma}}{\tilde{\sigma}_0}\right) \right| \left(1 - \frac{\lambda^2 M \dot{F}^2}{l_s^2 F^2}\right)^{-1/2}, \qquad (3.250)$$

where V_3 is the volume element of the $D3$-brane. Demanding the conservation of energy (equivalent to solving the equations of motion) we can solve for for $F(t)$, noting that the absolute value of the second term is independent of time. The solution can be seen to yield

$$\frac{1}{F(t)} = \frac{1}{F_0} \cosh\left(\frac{t l_s}{\lambda\sqrt{M}}\right), \qquad (3.251)$$

where F_0 is the initial condition on the profile. There are two important comments to make here. Firstly that the solution appears to be valid for any point on the world-volume, even at the location of the monopole $r = 0$. Secondly the solution for the profile is exactly the same functional form as that of a $D3$-brane with no magnetic flux in the same background, as shown by Kutasov in [21]. This suggests that the BIon spike will not feel any tidal forces due to the gravitational attraction of the fivebranes. We may now write the full solution to the equation of motion (again in the throat approximation) as follows

$$r(\sigma, t) = \frac{N\pi\lambda l_s}{\lambda\sqrt{M} \ln\left(\frac{\tilde{\sigma}}{\tilde{\sigma}_0}[1 + e^{-tl_s/\sqrt{M}\lambda}]\right) + tl_s + \lambda\sqrt{M}\ln(2)}, \qquad (3.252)$$

which we can simplify by considering the solution at late times - and neglecting the constants arising from the initial conditions

$$r(\sigma, t) \sim \frac{N\pi\lambda l_s}{\lambda\sqrt{M} \ln\left(\frac{\sigma}{\sigma_0}\right) + tl_s} \qquad (3.253)$$

which shows that the radion field is proportional to $1/t$ in this limit. We now want to consider how this appears on the non-Abelian side, however we note that even when we include time dependence in the action the equations of motion are highly non trivial and do not yield a simple analytic solution. We should check that the solution (3.253) is actually a solution of the theory. We again factorise the scalar field into a time dependent piece and a spatial piece and make the ansatz

$$R(\sigma, t) = \frac{1}{2\sqrt{M l_s^2} \ln\left(\frac{\sigma}{\sigma_0}\right) + BF(t)}, \qquad (3.254)$$

where B is some arbitrary constant. It can easily be seen that $R' = 2R^2\sqrt{H}$ and $\dot{R} = B\dot{F}R^2$, where H is the usual harmonic function for the $NS5$-brane solution. If we substitute these two equations into the energy density equation for the fuzzy funnel we obtain

$$\mathcal{H} = \frac{T_1 V_1 N (1 + 4\lambda^2 C R^4)^{3/2}}{\sqrt{1 + 4\lambda^2 C R^4 - \lambda^2 C B^2 R^4 \dot{F}^2}}, \tag{3.255}$$

which must be conserved in time. This requires that the $\dot{F}$ term must vanish from the expression. The simplest solution is to take $\dot{F} = 0$, however this implies that F is constant in time and so we are just introducing a constant shift into the equation of motion. A non-trivial solution can be obtained by setting $\dot{F}^2 B^2 = 4$, which has the solution $F(t) = 2t/B$. This reproduces the same functional form for the equation of motion as we derived from the Abelian theory, however we need to check the interpretation of the resultant expression for the energy density, which can be seen to yield

$$E \to T_1 V_1 N (1 + 4\lambda^2 C R^4)^{3/2}. \tag{3.256}$$

Expanding the solution we can see the first term corresponds to the energy density of N coincident D-strings, as we would expect. The higher order terms correspond to non-linearities arising from the fuzzy funnel solution representing the warping of the D-strings in the transverse space. Thus we argue that this ansatz for the equation of motion is a solution of the theory since we are left with the minimal energy configuration. Therefore both solutions agree at late times. Furthermore it was argued in [21] that we can trust the macroscopic description even deep in the IR end of the geometry provided that the energy of the brane is large enough. Therefore we expect our solution to capture the vast majority of the evolution of the system. Of course, our analysis is based upon the fact that we are ignoring the back reaction upon the geometry. Again this requires fine tuning of the various parameters in the theory to accomplish this. Hopefully using the prescription for the symmetrized trace at finite N will alleviate this problem entirely.

Similar analysis can be carried out for both the F-string and Dq-brane backgrounds. The static spike profile in the F-string background, obtained by solving (3.240) is consistent with the static funnel profile obtained in the same background on the non-Abelian side. The same scaling argument about such static solutions being unstable, as discussed in the $NS5$ case above, is not naively applicable here. What we can verify is that at least in the static case, the equation for the spike on the Abelian $D3$-world volume side and the fuzzy funnel on the non-Abelian side agree.

Finally we discuss the situation for Dq-background geometry with $q \neq 3$. Here things are obviously more complicated due to the red-shift of the magnetic field. However we can use some intuition from our knowledge of the Abelian theory to understand the physics. It is known that for supersymmetry to be preserved we require the $D3$-brane to be embedded in either a $D3$-brane or $D7$-brane background. In this case the funnel solution will be completely solvable. For all other brane backgrounds the supersymmetry is broken, and the $D3$ feels a gravitational potential drawing it toward the background branes. Thus our static funnel solution will not be compatible with the full equations of motion, and so we would require a time dependent ansatz. Interestingly in the $D5$-brane background we know that open string modes stretching between the funnel and the source branes will become tachyonic at late times, potentially distorting the funnel.

3.3.10 Higher Dimensional Fuzzy Funnels.

We can generalise the non-Abelian results we have obtained to the higher dimensional theory using the results from the previous chapter. This means we are considering the fuzzy $\mathcal{S}^{2k}$ spheres, which are labelled by the group structure of $SO(2k+1)$ in ten dimensions [51, 52]. This will obviously imply that we require $2k+1$ transverse scalars in the DBI action, where $k \leq 3$ and the funnels are now blowing up into $nD(2k+1)$-branes in an arbitrary background. Of course the higher number of transverse directions will impose serious constraints upon the dimensionality of the possible background sources, in many cases we will be left with unphysical situations such as type IIA, or potentially non braney solutions. The geometry of these higher dimensional fuzzy spheres is interesting to study in its own right, for example we know that the fuzzy S^6 can be written as a bundle over the classical six-sphere. In the classical limit we find that the fibre over the sphere belongs to the group $SO(6)/U(3)$, which implies that constructing a dual picture is non-trivial [52]. The geometrical analysis is revealing as we can calculate the charge of the branes directly from the base space. The general topology of our higher dimensional funnel configuration will now be $\mathbb{R} \times S^{2k}$, and we must modify our gauge group ansatz to read

$$\phi^i = \pm R G^i, \tag{3.257}$$

where the G^i matrices satisfy $G^i G^i = C_k \mathbf{1}_N$ and lie in the irreducible representation of the particular gauge group. The Casimir in this case will be labelled by a k index so that we know which group structure it conforms to. The relationship between N and n means that the dual picture is far more complicated. For example in the $k=2$ case we know that the D-strings blow up to form several $D5$-branes, which have a non-trivial second Chern Class on the world-volume. This makes the dual picture difficult to analyse and we will not do it here - but see [45] for a more detailed derivation of the $D1-D5$ and $D1-D7$ solutions in flat space. The general relationship between the physical distance and the scalar field ansatz can be written as follows

$$r = k\sqrt{C_k}\lambda R, \tag{3.258}$$

which is similar to the $SU(2)$ case, except there is no ambiguity over the choice of sign, and we emphasise that the Casimir will be dependent upon the number of higher dimensional branes in the funnel solution.

The generalisation of the non-Abelian action to leading order is expected to be given by

$$S = -T_1 \int d^2\sigma N e^{-\phi} \sqrt{g_{00} g_{zz}(1+\lambda^2 C_k g_{xx} g_{zz}^{-1} R'^2)} (1+4\lambda^2 C_k R^4 g_{xx}^2)^{k/2}, \tag{3.259}$$

and therefore with our usual rescaling of the tension we can find the spatial component of the energy momentum tensor

$$T_{\sigma\sigma} = \frac{e^{-\phi}\sqrt{g_{00} g_{zz}}(1+4\lambda^2 C_k R^4 g_{xx}^2)^{k/2}}{\sqrt{1+\lambda^2 C_k R'^2 g_{xx} g_{zz}^{-1}}}. \tag{3.260}$$

Our work in the lower dimensional case has shown that we can obtain solutions to the equations of motion, consistent with the energy minimisation principle, when the α term is constant. If we assume that this is true for our background metric then we can write the general equation of motion for the funnel as follows

$$R_k'^2 = \frac{g_{zz}}{\lambda^2 C_k g_{xx}} \left((1+4\lambda^2 C_k R^4 g_{xx}^2)^k - 1 \right). \tag{3.261}$$

A quick check shows that with $k=1$ the solution reduces to $R_1'^2 = 4R^4 g_{xx} g_{zz}$ as expected from our efforts in the preceding sections. Of course setting α to be constant also imposes additional constraints on the possible supergravity backgrounds that exist. Interestingly the higher dimensional solutions will all have a variant of this solution as their lowest order expansion in λ. The $k=2$ and $k=3$ solutions can be written as follows

$$\begin{aligned} R_2'^2 &= 8\left(R^4 g_{xx} g_{zz} + 2\lambda^2 C_2 R^8 g_{xx}^3 g_{zz}\right) \\ R_3'^2 &= 12\left(R^4 g_{xx} g_{zz} + 4\lambda^2 C_3 R^8 g_{xx}^3 g_{zz} + \frac{16}{3}\lambda^4 C_3^2 R^{12} g_{xx}^5 g_{zz}\right) \end{aligned} \tag{3.262}$$

which shows that there are apparent recursive properties for these equations. Note that these expression agree exactly with the ones derived in [45] when taking the flat space limit, where these results were obtained via minimisation of the energy and found to be perturbatively stable. Clearly we do not expect this to be the case in a general background due to the additional σ dependence of the metric components.

In general these equations are difficult to solve, but can in principle be written in terms of elliptic functions. We will try and make some progress by assuming trivial solutions for the g_{xx} components which can be absorbed into a redefinition of R, and power law behaviour for the g_{zz} components. In the $k=2$ case we can find approximate solutions to the equation of motion. In the large R region, the second term is dominant and a quick integration yields the following solution

$$R_2(\sigma) = \left(\frac{\mp 1}{4\lambda\sqrt{C_2}(\sigma^{m+1} - \sigma_0^{m+1})}\right)^{1/3} \qquad m \neq -1 \tag{3.263}$$

$$R_2(\sigma) = \left(\frac{\mp 1}{4\lambda\sqrt{C_2}\ln(\sigma/\sigma_0)}\right)^{1/3} \qquad m = -1.$$

Note that $m = 0$ corresponds to the flat space limit and agrees with the solution in [45]. When R is small the solution is dominated by the leading term and we recover the usual funnel solution derived in previous sections. Clearly this implies the existence of an interpolating region where the solutions cross over from one another. Upon equating the two terms we find that the cross over occurs at

$$R_{cr} \sim \left(\frac{1}{2\lambda^2 C_2}\right)^{1/4}, \tag{3.264}$$

which implies, in physical coordinates, that $r >> l_s$. Moving on to the $k = 3$ case we find it complicated by the appearance of an extra term. Of course in the large R limit this will be the dominant contribution to the integral and we find a similar solution to the one sketched out above with the power now being $1/5$ rather than $1/3$, and the dependence on λ and C will also be slightly altered. The crossover in this case will happen at the point

$$R_{cr} \sim \left(\frac{3}{8\lambda^2 C_3}\left\{1 + \sqrt{\frac{7}{3}}\right\}\right)^{1/4}, \tag{3.265}$$

which will again imply that the physical distance is much larger than the string scale. The general conclusion here is that higher dimensional fuzzy spheres lead to funnel solutions which are modified version of the lower dimensional ones, although we ought to bear in mind that these solutions are potentially only valid in flat space as physical brane sources satisfying the background constraints may not exist. The general behaviour for the funnel in the large R limit can be seen to be

$$R \sim \sigma^{-(m+1)/(2k-1)}, \tag{3.266}$$

and so the higher dimensional effects play a more important role as $\sigma \to \sigma_0$.

We now switch our attention to the leading order $1/N$ corrections for the general fuzzy funnel. As usual we choose to work in terms of the variables α, β, γ, where now β is the general function for arbitrary k. The leading order correction can be calculated to give

$$\tilde{\mathcal{H}_1} = \alpha\beta\gamma\left\{1 - \frac{1}{3\gamma C_k}\left(\frac{k(\gamma^2-1)(\beta^{2/k}-1)}{\gamma\beta^{2/k}} - \frac{(\gamma^2-1)^2}{2\gamma^3} + \frac{k\gamma(k-2)(\beta^{2/k}-1)^2}{2\beta^{4/k}}\right)\right\}$$

which clearly reduces to the standard expression when $k = 1$. This is actually valid for

$k = 4$ provided we take the flat space limit. Now we see that in general the correction terms will be non-zero, even if we assume the funnel configuration where $\beta = \gamma$. This is actually reminiscent of the flat space solutions where the higher dimensional fuzzy funnels are corrected under the symmetrized trace. Taking $k = 2$ for example we find that the corrected energy becomes

$$\tilde{\mathcal{H}}_1 = \alpha\beta\gamma \left\{ 1 - \frac{1}{3\gamma C_2} \left(\frac{2(\gamma^2-1)(\beta-1)}{\gamma\beta} - \frac{(\gamma^2-1)^2}{2\gamma^3} \right) \right\}, \tag{3.267}$$

which implies that the correction terms only vanish for $\gamma^2 = 1$. The non-trivial solution to this implies that $R' = 0$, or that the radius of the sphere is a constant function of σ. Furthermore we see that the correction term will always be positive, therefore the higher order corrections reduce the energy and so we expect the solution to be unstable. It is only the $D1 - D3$ funnel which is the lowest energy configuration in an arbitrary background.

Many of the arguments presented in this section have been quite technical, and one may wonder whether there is any physical application for the Myers action beyond the intrinsic mathematical aspects. In the next section we will suggest an application of this formalism by taking a closer look at the (p, q)-string system that was briefly mentioned before. Because of the gauge/gravity duality, this string configuration is of potential interest from both a cosmological and a gauge theory perspective.

3.4 Application: (p,q) strings in the Warped Deformed Conifold.

An interesting inflationary model within string theory is the hybrid inflation scenario of the $D3$-$\bar{D}3$ annihilation [87]. The different branes experience a Coulombic attraction forcing them towards one another, and so the inter-brane distance can play the role of the inflaton. Once the branes reach distances comparable to the string scale, an open string tachyon forms which causes inflation to end. Despite many technical difficulties with this model, it has been placed on reasonably sound footing by embedding the mechanism into semi-realistic models arising from flux compactifications [108]. The branes annihilate in a warped throat, which is separated from the Standard Model throat by the internal compactification manifold. One of the predictions of this model, however, is that strings will be created during the annihilation process - after inflation has ended. To illustrate this let us consider the annihilation of one $D3$-brane with a $\bar{D}3$-brane. K theory tells us that the remnant brane must have co-dimension two [10], which means that there will be a remnant D-string left after annihilation which is charged under a linear combination of the original $U(1) \times U(1)$ symmetry. The other linear combination of $U(1)$ charge must vanish (as there is only a single residual brane). However the annihilation also leads to the creation of confining flux lines, which can be interpreted as (fundamental) F-strings. The result is that generically we will have remnant cosmic (super) strings [66]. This is clearly incompatible with the experimental data, which can be seen to fit a ΛCDM inflationary model with great accuracy [88–91]. In fact the current observational bounds impose a constraint on any cosmic string tension through the relationship $G\mu \leq 10^{-6}$, where μ is the string tension [67, 76]. Thus we are left with one of three likely possibilities:

- $D3$-$\bar{D}3$ inflation is not a viable mechanism.
- Only a few strings are formed during this process, which mean they contribute negligibly to the CMB.
- The warping of the throat is so severe that the tension of these objects is essentially zero.

Although it may well turn out that the first of these is correct, we consider the other two possibilities in this thesis. In fact one may argue that it is indeed a combination of these two statements which leads to the correct answer. Namely that a few strings are formed after inflation, and the warping does indeed reduce their tension. In the next subsection we will introduce the background supergravity solutions which we need to analyse this problem

3.4.1 The Warped Deformed Conifold.

The conifold is a non-compact, singular manifold which is topologically a cone over the base $T^{1,1}$ [61]. There are classes of more general conical geometries, called $L^{a,b,c}$ manifolds - however we will not discuss them in this thesis [62]. Conifold singularities are the most common type of singularities arising from Calabi-Yau compactifications of string theory, which makes them interesting objects to study. Mathematically it is convenient to define the conifold as a complex algebraic curve satisfying the following equation

$$f(z_1, z_2, z_3, z_4) = \sum_{i=1}^{4} z_i^2 = 0. \tag{3.268}$$

Note that this defines a cone because if z_i lies on the cone then so does λz_i where $\lambda \in \mathbb{C}$. If we intersect the polynomial with a seven-sphere of radius r, then we see that near the intersection we can write the z_i in terms of $x_i + iy_i$ to find $x_i x^i = y_i y^i = r^2/2$ and $x_i y^i = 0$. These equations give us an S^2 which is fibred over an S^3. However such fibrations are trivial and so the resulting bundle must be a product bundle $S^2 \times S^3$. The singularity can be resolved by allowing either the S^3 or the S^2 to shrink to remain finite. When the S^2 shrinks the resulting manifold is the deformed conifold, whilst if it is the S^3 that degenerates then we obtain the resolved conifold. A cartoon of the deformed conifold is sketched in Fig 3.4 which is taken from Majumdar [66].

As it stands this is a good string background, however things become even more interesting when we consider the backreaction of the fluxes, which are turned on in the compact

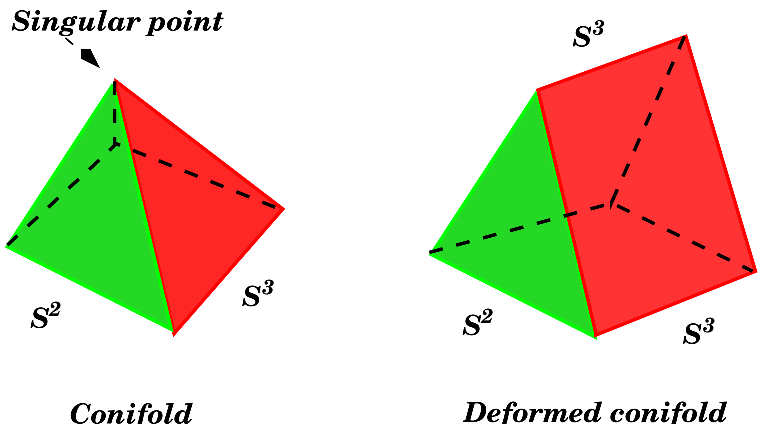

Figure 3.4: Sketch of the conifold and the deformed conifold.

directions. The backreaction can cause some of the three-cycles to collapse, forming finitely warped throats [63]. This was first discussed by Klebanov and Strassler, and is more commonly referred to as the KS throat. More importantly the throats naturally reduce the amount of supersymmetry that the solution preserves. In fact for the warped deformed conifold we find that only $\mathcal{N} = 1$ susy is preserved. The general metric for the conifold can be written as follows

$$ds_6{}^2 = dr^2 + \frac{r^2}{9}\left(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i\right)^2 + \frac{r^2}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right), \tag{3.269}$$

which clearly shows that there is an S^2 fibred over an S^3. The warped deformed conifold emerges when we turn on the RR flux F_3 through the S^3, and the NS-NS flux H_3 through the dual three-cycle which creates the warped throat[14]. The fluxes must satisfy the Dirac quantisation condition

$$\frac{1}{4\pi^2\alpha'}\int_A F_3 = M, \qquad \frac{1}{4\pi^2\alpha'}\int_B H_3 = -K, \tag{3.270}$$

where $M, K \in \mathbf{Z}$ and A, B are the dual three-cycles. We can now deform the equation for the algebraic curve by a small (and real) parameter ϵ using

$$\sum_{i=1}^{4} z_i^2 = \epsilon^2. \tag{3.271}$$

[14]In fact this cycle is basically $r \times S^2$.

Now it is convenient to re-write the metric using a basis of one-forms

$$\begin{aligned} g_1 &= \frac{e^1 - e^2}{\sqrt{2}}, \qquad g_2 = \frac{e^2 - e^4}{\sqrt{2}} \\ g_3 &= \frac{e^1 + e^3}{\sqrt{2}}, \qquad g_4 = \frac{e^2 + e^4}{\sqrt{2}} \\ g_5 &= e^5 \end{aligned} \tag{3.272}$$

where we have defined

$$\begin{aligned} e^1 &= -\sin\theta_1 d\phi_1 \\ e^2 &= d\theta_1 \\ e^3 &= \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2 \\ e^4 &= \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2 \\ e^5 &= d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2. \end{aligned} \tag{3.273}$$

In this basis the metric (3.269) can be seen to take the following form

$$ds_6{}^2 = \frac{\epsilon^{4/3}K(\tau)}{2}\left[\frac{d\tau^2 + g_5^2}{3K^3(\tau)} + \cosh^2(\frac{\tau}{2})[g_3^2 + g_4^2] + \sinh^2\left(\frac{\tau}{2}\right)[g_1^2 + g_2^2]\right], \tag{3.274}$$

where the overall warp factor is given by

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh(\tau)}. \tag{3.275}$$

Note that τ parameterises the size of the S^2 fibration, and as $\tau \to 0$ we are left with the metric for the three-sphere - i.e the warped deformed conifold. We will be mostly interested in the physics of this solution near the tip of the warped throat, once the S^2 has shrunk to zero size. The full ten-dimensional metric in this instance reduces to something like

$$ds_{10}{}^2 \sim a_0^2 \eta_{\mu\nu} dx^\mu dx^\nu + g_s M b_0 \alpha' \left(\frac{1}{2}dr^2 + r^2 d\tilde{\Omega}_2^2 + d\Omega_3^2\right) \tag{3.276}$$

where now r is the radius of the shrinking two-sphere, parameterised by θ, ϕ coordinates using $\theta_1 = -\theta_2$ and $\phi_1 = -\phi_2$. The warp factor a_0^2 arises from the fluxes and can be tuned to be exponentially small. The threading of the flux through the three cycle can be viewed as arising from $N = MK$ $D3$-branes localised at the origin, where the warp factor has been truncated at some finite value - rather than leading to an infinitely warped throat.

The solution is in Einstein frame, and the RR 2-form at the tip of the throat can be written as follows

$$C_2 = M\alpha'\left(\psi - \frac{\sin(2\psi)}{2}\right)\sin(\theta) d\theta d\phi. \tag{3.277}$$

Because the warp factor is smallest at the tip of the throat, we expect that any residual objects will localise themselves here in order to minimise their energy. This implies that we only need consider the physics near the tip, and far away from the gluing region where the KS throat is 'glued' smoothly onto the conifold. One further thing to note is that the dilaton and B_2 are zero at the tip of the throat. The vanishing B field simplifies our analysis however the vanishing dilaton implies that the string coupling $g_s = e^{\phi} = 1$ in this region. Although this means that both Einstein and string frames coincide in this instance, it also means that we are in a strongly coupled regime.

Having stated the supergravity background let us make some comments before proceeding. For cosmological purposes we want to consider the KS throat as being the inflationary throat. Thus after brane annihilation we will end up with cosmic strings at the tip. Although this solution is ten-dimensional, we expect it to be representative of the exact solution obtained upon compactification. Provided we restrict our analysis to the tip of the throat, we are essentially insensitive to UV physics - which is where the model dependent phenomena come into play. This ensures that our analysis will be reasonably robust.

We can also interpret strings in this background in another way [73]. Due to gauge/gravity duality this gravity theory is actually dual to a confining gauge theory. It was shown in [63] that if the fluxes are integer multiples of one another, then the gauge theory undergoes a cascade from $SU(K+M) \to SU(M)$ in the deep IR, i.e near the tip. Because this is a confining gauge theory F-strings in this solution acts as quark-antiquark strings, whilst D-strings will be dual to axionic strings. This means that studying stringy objects in this background has dual application, either for cosmology or for gauge theory. Our primary focus is on a model for cosmology, however we will often make reference to the gauge theory interpretation as there is an abundance of literature on the subject.

3.4.2 Macroscopic (p,q)-strings.

We have already mentioned that we expect F and D-strings to be formed in annihilation processes in the KS throat, but more generally we expect these strings to combine to form a bound state of pF-strings and qD-strings - or a (p,q)-string. This is simply due to $SL(2,Z)$ symmetry which interchanges the $B_{(2)}$ field with the $C_{(2)}$ field, thus transforming F-strings into D-strings [69].

A nice way of obtaining the tension spectrum for such objects in the warped deformed conifold was derived in [70], who considered a $D3$-brane wrapping a two-cycle within the S^3 and extending along two of the four Minkowski directions [15]. As we have mentioned many times in this thesis, a non-zero electric field on the brane is equivalent to dissolved F-string flux so we can simulate a (p,q) string by turning on a non-zero $U(1)$ gauge field. [70] also turned on non-zero magnetic flux on the world-volume to stabilise the brane on the

[15] See [71] for related work.

S^2. For simplicity both these fields are assumed parallel, and we choose a gauge such that only F_{23} and F_{01} are non-zero. The brane is extended in two of the non-compact directions and wrapped on the two-cycle inside the S^3. After collecting all the terms and integrating over the compact directions, one can obtain the Hamiltonian density for the (p,q) bound state via Legendre transform

$$H = \frac{a_0^2}{\lambda}\sqrt{\frac{q^2}{g_s^2} + \frac{b_0^2 M^2}{\pi^2}\sin^4(\psi) + \left[\frac{M}{\pi}\left(\psi - \frac{\sin(2\psi)}{2}\right) - (p - qC_0)\right]^2}. \tag{3.278}$$

At this stage we must minimise the energy with respect to the radius of the S^2. The result is

$$\psi_{min} + \frac{b_0^2 - 1}{2}\sin(2\psi_{min}) = \frac{(p - qC_0)\pi}{M}. \tag{3.279}$$

However the $b_0^2 - 1$ factor is almost zero and so to leading order we can drop this term [16]. Then we see that $\psi_{min} \sim \frac{(p-qC_0)\pi}{M}$ which yields a value r_{min} for the S^2 radius at the minimum

$$r_{\rm min} = \sqrt{b_0 g_s M \alpha'}\left|\sin\left(\frac{\pi(p - qC_0)}{M}\right)\right| \tag{3.280}$$

If we now insert this into our expression for the Hamiltonian we obtain the tension of the (p,q) bound state

$$H_{\rm min} = \frac{a_0^2}{\lambda}\sqrt{\frac{q^2}{g_s^2} + \left(\frac{b_0 M}{\pi}\right)^2 \sin^2\left(\frac{\pi(p - qC_0)}{M}\right)}, \tag{3.281}$$

which is in excellent agreement with the results obtained in [69, 73], when one sets $p = 0$ or $q = 0$ yielding the tension spectrum for the D and F-strings respectively. If we take the large (background) flux limit then we can expand the final term in the square root to the next to leading order in powers of $1/M$. This yields the following expression

$$H_{\rm min} \sim \frac{a_0^2}{\lambda}\sqrt{\frac{q^2}{g_s^2} + b_0^2(p - qC_0)^2\left(1 - \frac{\pi^2(p - qC_0)^2}{3M^2} + \frac{2\pi^4(p - qC_0)^4}{45M^4} - \ldots\right)}, \tag{3.282}$$

where we recall that $b_0 \sim 1$. Thus we see that in the large M limit one recovers the expected result for the (p,q) string tension in a non-trivial background. It is worth mentioning that if we take the limit $q \to 0$, leaving us with only the fundamental string contribution, that the corrections to the tension scale as $1/M^2$. This was first noted by Douglas and Shenker [74] and is different to the $1/M$ correction that arises due to Casimir scaling. Note that in this limit the minimal radius, $r_{\rm min}$, can be approximated by the following

$$r_{\rm min} \sim \sqrt{b_0 g_s \alpha'}\frac{\pi(p - qC_0)}{\sqrt{M}}\left|\left\{1 - \frac{\pi^2(p - qC_0)^2}{6M^2} + \mathcal{O}(\frac{1}{M^4})\right\}\right|, \tag{3.283}$$

[16] Strictly speaking this means that the supergravity solution is now that of [75], which was recently rediscovered by Maldacena-Nunez [64].

where the terms inside the brackets are a power series in even powers of $1/M$.

3.4.3 Microscopic (p, q)-strings.

The macroscopic description proposed in the last section gives a nice result, however from the cosmological perspective we can only use this as a tool. In order for the theory to be 'physical' - i.e arising as a natural consequence of inflation - we need to deal with the effective action for the strings themselves. This is even more important when one wants to study the formation of (p, q) string networks, because typically the dominant contributions come from the lowest lying string modes. However the macroscopic description needs to be modified in this instance, since turning on flux induces non-commutative effects on the brane worldvolume [68]. Seiberg and Witten [58] showed that these effects can be included by the introduction of a suitable star-product, rather than the normal product. For large values of the flux it can be shown that the effects of the star-product are negligible, but for small fluxes they provide an important contribution to the theory.

Using the Myers action [33] to construct the microscopic description allows us to include some of the non-commutative effects, although they are 'washed out' in the limit of a large number of strings (which is where the action is simplest). However using $1/N$ corrections [50], and the conjectured finite N prescription [53], we can attempt to include these effects in order to develop a more realistic model. The starting point is to construct a theory of q coincident D-strings, with non-zero electric flux on their worldvolumes. Since the number of D-strings must be an integer, this automatically accounts for the quantisation of q and therefore only p needs to be quantised.

We can use the results from the previous chapters to determine the action of the strings in this background. It should be remembered that the two-form field C_{ij} is a function of all the background coordinates, and in particular its dependence on the non-commuting coordinates ϕ^i is obtained through the non-Abelian Taylor series expansion. For the $U(1)$ gauge field we will fix the gauge $A_0 = 0$ implying that $F_{01} \neq 0 = \varepsilon$. Effectively this means that the gauge field is proportional to the identity matrix in this picture, breaking the $U(q)$ symmetry group of the coincident branes down to $SU(q) \times U(1)$, where the gauge field now commutes with the $SU(q)$ sector.

We choose to orient the D-strings along two of the Minkowski directions of the non-compact spacetime in order to make contact with the Abelian theory of the wrapped $D3$-brane. Recall that the $D3$-brane wraps an S^2 inside the S^3. Since this S^2 is thus magnetized, it suggests that on the non-Abelian side we should attempt to describe this wrapped S^2 via a fuzzy sphere ansatz for our transverse scalars [65], as we know that in the large q limit we should recover the classical two-sphere geometry with q units of magnetic flux.

Our goal is thus to try and describe, in the non-Abelian theory, a fuzzy two-sphere embedded not in flat space but in a round S^3 geometry, where we capture the essential physics of the solution presented in [70] but do not construct the dual microscopic model. Let us begin by only taking the transverse coordinates ϕ^i to be non-vanishing in the direction of this S^3 whose coordinates we label as $y^a, a = 1,2,3$. The metric on this S^3 can now be obtained after performing a non-Abelian Taylor expansion [17]

$$ds_3^2 = g_{ab}dy^a dy^b, \qquad g_{ab}(\phi) \sim g_{ab}(y) + \dots . \tag{3.284}$$

Since we are not looking for dynamical solutions we can regard the scalar fields as static which simplifies the dynamical portion of the action. If we calculate the determinant in the potential piece then to leading order we find (using the property that $g_{ab}(y)$ is diagonal)

$$\det Q^i_j = 1 - \frac{\lambda^2}{2}[\phi^a, \phi^b][\phi^c, \phi^d]g_{ac}(y)g_{bd}(y) + .. \tag{3.285}$$

and so the DBI contribution to the effective action can be written as

$$S = -T_1 \int d^2\sigma STr\left(a_0^2\sqrt{1 - \frac{\lambda^2\varepsilon^2\mathbf{1}}{a_0^4}}\sqrt{1 - \frac{\lambda^2}{2}[\phi^a, \phi^b][\phi^c, \phi^d]g_{ac}g_{bd}}\right) \tag{3.286}$$

Let us now consider the fuzzy sphere ansatz for the transverse scalars by imposing the following condition[18]

$$\phi^a = \hat{R}e^a_i\alpha^i, \tag{3.287}$$

where the α^i are the generators of the $SU(2)$ algebra, which is isomorphic to $SO(3)$ and satisfies the commutation relation $[\alpha^i, \alpha^j] = 2i\epsilon^{ijk}\alpha^k$, and e^a_i are vielbeins on the round three-sphere. Using this notation the indices i, j label coordinates in the tangent space to this S^3. As before we will take these generators to be in the q dimensional irreducible representation in order for them to yield the lowest energy configuration. If we now impose this ansatz on our fields in the action we find

$$S = -T_1 \int d^2\sigma STr\left(a_0^2\sqrt{1 - \frac{\lambda^2\varepsilon^2}{a_0^4}}\sqrt{1 + 4\lambda^2\hat{R}^4\hat{C}}\right), \tag{3.288}$$

where $\hat{C}$ is the usual quadratic Casimir of the representation given by $\hat{C}I_q = \alpha^i\alpha^j\delta_{ij}$, where I_q is the rank q identity matrix.

It follows from our choice of ansatz in (3.287) that there is no explicit dependence of the metric $g_{ab}(y)$ in the above action. With the inclusion of the S^3 veilbeins in the fuzzy sphere ansatz (3.287), the $SU(2)$ matrices α^i arrange themselves into the Casimir invariant $\alpha^i\alpha^i\delta_{ij}$ in the action (3.288). This feature simplifies the calculation of the symmetrized trace both

[17] In fact we can do better than this by noting that the metric on the S^3 is flat, this means we can avoid having to Taylor expand the metric and the fields [72].

[18] We use α^i to denote the group generators in this section rather than T^i and $\hat{C}$ for the Casimir. We hope this will not confuse the reader.

at large q and also for finite q (see next section). It is plausible that there exists a more general choice of ansatz for the transverse scalars than our proposed solution (3.287), but the resulting symmetrized trace computation may be rather difficult. Another motivation for (3.287) is that it is easy to see that the equations of motion (assuming constant matrices ϕ^a) are satisfied for the S^3 background since the resulting algebraic equations are formally equivalent to those obtained in a flat background, using the ansatz $\phi^i = \hat{R}\alpha^i$.

In the large q limit the symmetrized trace can be approximated by a trace over the gauge group. We can expand the square roots, take the trace and then re-sum the resultant solution to get a closed form expression for the action. Later, when we restrict ourselves to the case of finite q this will change, as we must take the symmetrization over the scalars into account.

From the expansion of the Chern-Simons action we can see that the leading order non-zero contributions are

$$S_{cs} = T_1 g_s \int STr\left(\mathbf{P}[C_0 + e^{i\lambda i_\phi i_\phi} C_{(2)}]\lambda F\right) + \dots \tag{3.289}$$

where $C_{(2)}$ has only non zero components in the spherical directions. Note that we are working in the conventions of [33] in which the normalization of the RR $C_{(n)}$ differs from the canonical one, which accounts for the factor of g_s in (3.289).

After expanding the action to include the interior derivatives, and performing the pull-back operation we find the action reduces to

$$S_{cs} = T_1 g_s \int d^2\sigma STr\left(\lambda\varepsilon C_0 - i\lambda^2\varepsilon \frac{C_{ab}}{2}[\phi^a, \phi^b]\right). \tag{3.290}$$

In order to make any further progress we must Taylor expand the RR two-form which yields a term which will vanish, and also a term $\lambda\partial_c C_{ab}\phi^c$. However under the STr operation this terms is proportional to the field strength F_{abc} which gives rise to quantised flux when integrated over S^3. We write $F_{abc} = f\Omega_{abc}$ where Ω_{abc} is the volume element of S^3, and using the flux normalisation condition $\int_A F_3 = 4\pi^2\alpha' M$ we find

$$f = \frac{2}{(b_0 g_s)^{3/2}\sqrt{M\alpha'}}. \tag{3.291}$$

Combining this with the standard relation $\Omega_{abc}\Omega^{abc} = 6$ implies that the large q limit of the Chern-Simons action reduces to

$$S_{cs} = T_1 \int d^2\sigma \left(g_s\lambda q C_0\varepsilon + \frac{4}{3}\frac{q g_s \hat{C}\lambda^3\varepsilon\hat{R}^3}{(b_0 g_s)^{3/2}\sqrt{M\alpha'}}\right). \tag{3.292}$$

We can now construct the canonical momentum density of the electric field by varying the Lagrangian density. As there is no explicit dependence of the action upon the gauge potential, we expect the resulting quantity to be conserved and also quantised in units of the string tension. The resultant displacement field, p, is found to be

$$p = T_1 q a_0^2 \sqrt{1+4\lambda^2 \hat{C}\hat{R}^4}\frac{\lambda^2\varepsilon}{a_0^4}\left(1-\frac{\lambda^2\varepsilon^2}{a_0^4}\right)^{-1/2} + T_1 q g_s \lambda C_0 + \frac{4 q g_s T_1 \lambda^3 \hat{R}^3 \hat{C}}{3(b_0 g_s)^{3/2}\sqrt{M\alpha'}}. \qquad (3.293)$$

Note that we are using the canonical radius $\hat{R}$ in this expression, which is related to the physical radius, r, of the fuzzy sphere through the expression $r^2 = \hat{R}^2\lambda^2\hat{C}$. If we make this substitution and construct the canonical Hamiltonian density via the Legendre transformation we find (using the relation $T_1\lambda = 1/g_s$)

$$H = \frac{a_0^2}{\lambda}\sqrt{\frac{q^2}{g_s^2}\left(1+\frac{4r^4}{\lambda^2\hat{C}}\right) + \left(p - qC_0 - \frac{4qr^3\sqrt{2\pi}}{3(b_0 g_s)^{3/2}\sqrt{\hat{C}M\lambda^3}}\right)^2}. \qquad (3.294)$$

The overall factor multiplying the square root is simply the warped tension of a fundamental string. At this juncture we should minimise the energy in (3.294) to compare with that predicted in the Abelian theory of the last section. We first concentrate on the large M approximation, which implies that there is a large flux on the three-sphere. Naively one might assume that the energy is minimised when $r = 0$, however we can easily see that this corresponds to a saddle point. In fact a quick calculation shows that in this approximation, the energy is minimised at the following radius

$$r_{\min} = \frac{(p - qC_0)}{2b_0^{3/2}}\sqrt{\frac{2\pi g_s \lambda}{M}}. \qquad (3.295)$$

This should be compared to the Abelian result (3.280) of the last section. We see that approximating $b_0 = 1$, to leading order in $1/M$ both expressions for $r_{\min}$ are in precise agreement. Whilst this result is encouraging, what we are really interested in is comparison of the tension of the (p,q) strings in the two formulations. Substituting (3.295) back into (3.294) we find that keeping terms to $\mathcal{O}(1/M^2)$ the energy density at the minimum becomes

$$H_{\min} \sim \frac{a_0^2}{\lambda}\sqrt{\frac{q^2}{g_s^2}\left(1+\frac{(p-qC_0)^4\pi^2 g_s^2}{b_0^6 M^2 \hat{C}}\right) + (p-qC_0)^2\left\{1 - \frac{4(p-qC_0)^2\pi^2}{3b_0^6 M^2}\right\}}. \qquad (3.296)$$

If we now again work in the approximation where $b_0 = 1$ we can see that the predicted (p,q) string tension agrees exactly with that predicted in the Abelian theory (3.282) up to and including terms of $\mathcal{O}(1/M^2)$. This result is further strong evidence that the non-Abelian DBI description of (p,q) strings through the fuzzy sphere ansatz is capturing the correct physics. This is particularly so of the $\mathcal{O}(1/M^2)$ terms in the tension formula above as these are sensitive to the r^4 and r^3 terms in the non-Abelian DBI and to our choice of fuzzy sphere ansatz.

One may wonder if the predicted tension of the (p,q) strings in the non-Abelian formulation agrees to $\mathcal{O}(1/M^2)$ even if the parameter $b_0 \neq 1$. (In fact the two tensions agree to all orders [72].) To check this one needs the corresponding expression for the tension in the Abelian formulation, expanded as a power series in $1/M$. This is obtained by first solving the minimization equation (3.279) after expanding the $\sin(2\psi_{\rm min})$ term to cubic order. This gives

$$\psi_{\rm min} = \frac{(p-qC_0)\pi}{b_0^2 M} + \frac{2}{3}\frac{(p-qC_0)^3\pi^3(b_0^2-1)}{b^8 M^3} + \mathcal{O}(1/M^5). \tag{3.297}$$

Substituting this value of $\psi_{\rm min}$ into (3.278) and expanding in powers of $1/M$ one find precise agreement with the terms arising from a similar expansion of (3.296).

A further calculation of the $\mathcal{O}(1/M^4)$ in the tension formula shows a discrepancy with the Abelian result. The latter predicts corrections $\frac{2}{45M^4}\pi^4(p-qC_0)^6$ whereas the non-Abelian theory gives a factor $\frac{4}{9M^4}\pi^4(p-qC_0)^6$. An investigation of the algebraic structure of sub-leading corrections in (3.294) shows that they take the form $(\frac{\pi^{2k}(p-qC_0)^{2k+2}}{M^{2k}})$ for $k = 1,2,3..$ (taking $b_0 = 1$). This is exactly the structure one finds on the Abelian side by expanding out the $\sin^2$ term in (3.281).

One may also consider comparing the tension obtained above, in the case where the quantised flux M is not necessarily large. In this case, $r_{\rm min}$ can be obtained by solving the depressed cubic coming from energy minimization. Two of the solutions are imaginary, however the physical solution can be written as follows

$$r_{\rm min} = \left(-\frac{\alpha_0}{2} + \sqrt{\frac{\alpha_1^3}{27} + \frac{\alpha_0^2}{4}}\right)^{1/3} - \left(\frac{\alpha_0}{2} + \sqrt{\frac{\alpha_1^3}{27} + \frac{\alpha_0^2}{4}}\right)^{1/3} \tag{3.298}$$

where for large q, $\alpha_0 = -\frac{3}{4}(p-qC_0)\lambda(g_s M\alpha')^{1/3}$ and $\alpha_1 = \frac{3}{2}M\alpha' g_s$. It should be noted that to avoid large back reaction corrections to the metric of the S^3, M should be taken to be large in order for us to trust the effective action. Then the perturbative analysis of the string tension in powers of $1/M$ is a good approximation.

We have seen that the large q limit agrees with the macroscopic picture. However we should typically regard this with a little suspicion, since a large number of D-strings at the tip of the throat will have a large backreaction effect which we have neglected. Furthermore a large number of strings should, in principle, be visible and contribute to the CMB. From the cosmology perspective it seems far more natural for a small number of strings to be created after inflation. Combined with sufficient warping, this would explain why they are not visible to us. However describing a small number of strings is difficult. In the macroscopic theory this would mean turning on small flux quanta on the worldvolume, which is typically sensitive to non-commutative effect as we have already pointed out. It seems more natural to describe the strings using the Myers action, however this is just as difficult. Our analysis has assumed that q is large, which means that we neglect terms like $1/q^2$. However to describe a small number of strings requires explicit knowledge of the expansion for the symmetrized trace which is unknown.

Recently, however, a prescription for the symmetrized trace in the case of $SO(3)$ has been proposed [53] which we will employ here as a preliminary analysis to the finite p, q problem. Let us write the general form for the Hamiltonian density H, where we continue to use the fuzzy S^2 ansatz but work in terms of the canonical radius $\hat{R}$

$$H = \frac{a_0^2}{\lambda}\sqrt{\frac{1}{g_s^2}\left(STr\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left[-\frac{1}{2}\right]_k(4\lambda^2\hat{C}\hat{R}^4)^k\right)^2 + \left(p - \frac{qC_0}{g_s} - \frac{4q\hat{C}\lambda^2\hat{R}^3}{3(b_0 g_s)^{3/2}\sqrt{M\alpha'}}\right)^2}$$

where we have used the following definition

$$\left[-\frac{1}{2}\right]_k = \frac{\Gamma(k-1/2)}{\Gamma(-1/2)}. \tag{3.299}$$

Now the symmetrized trace acts on the Casimir of the representation in two different ways, depending on whether the spin representation is odd or even. There is a simple relationship between the spin and the number of branes, namely $n = 2j = q - 1$, which will play a role in what follows. The symmetrized trace acts on the 'Casimir'[19] in the following manner

$$\begin{aligned} STr[\hat{C}^m] &= 2(2m+1)\sum_{i=1}^{n/2}(2i)^{2m} & n = \text{even} \qquad (3.300)\\ &= 2(2m+1)\sum_{i=1}^{(n+1)/2}(2i-1)^{2m} & n = \text{odd}. \end{aligned}$$

This prescription implies the following definition for the physical radius of the fuzzy sphere

$$r^2 = \lambda^2\hat{R}^2\mathrm{Lim}_{m\to\infty}\left(\frac{STr\hat{C}^{m+1}}{STr\hat{C}^m}\right) = \lambda^2\hat{R}^2 n^2, \tag{3.301}$$

where the quadratic Casimir is now $\hat{C}I_q = (q^2-1)I_q = n(n+2)I_{n+1}$ in terms of the spin representation.

We can now consistently take the limit of small q using this prescription. To illustrate this we consider the first non-trivial solution where there are two coincident D-strings. Expansion of the symmetrized trace leads to the following expression for H

$$H = \frac{a_0^2}{\lambda}\sqrt{\frac{4}{g_s^2}\left(1+\frac{8r^4}{\lambda^2}\right)^2\left(1+\frac{4r^4}{\lambda^2}\right)^{-1} + \left(p - 2C_0 - \frac{8r^3}{(b_0 g_s\lambda)^{3/2}}\sqrt{\frac{2\pi}{M}}\right)^2} \tag{3.302}$$

where there is a potential sign ambiguity in the r^3 term due to the definition of the physical radius. However we have chosen the minus sign in order for the solution to agree with that

[19]Recall that we really mean T^iT^i.

of the large q limit. Once again we can search for a minimal radius constraint by considering the large flux limit, which is a useful simplification. However as there are now only two branes, the backreaction upon the background is more under control.

Writing the full constraint equation for the minimisation of H for the case $q = 2$, without demanding that $1/M$ terms are negligible, we find

$$\frac{32r}{g_s^2\lambda^2}F(r) = \frac{8(p-2C_0)}{(bg_s\lambda)}\sqrt{\frac{2\pi}{M}}G(r)\left(1 - \frac{8r^3}{(p-2C_0)(bg_s\lambda)^{3/2}}\sqrt{\frac{2\pi}{M}}\right) \tag{3.303}$$

where we have introduced the following simplifications

$$F(r) = 1 + \frac{32}{3}\sum_{k=1}^{2} k\left(\frac{r^4}{\lambda^2}\right)^k, \qquad G(r) = 1 + 8\sum_{k=1}^{2} k\left(\frac{r^4}{\lambda^2}\right)^k. \tag{3.304}$$

Clearly (3.303) is difficult to solve analytically. To simplify the task, we drop all terms of order $1/M$ as in any case they should be insignificant in the large flux solutions we are considering in this note. There are now two limiting cases of interest for us. The first is when the physical radius namely $r^4 << \lambda^2$, which allows us to find the solution

$$r = \frac{r'}{2} \tag{3.305}$$

where r' is a shorthand notation for the corresponding minimal radius in the large q case (3.295) (where we would set q=2). This clearly shows that the minimum energy configuration occurs at a smaller radius. However we should be careful about interpreting this result, as the Myers action may not actually be valid in such a limit. Moreover it would also seem to suggest that the S^2 embedded within the S^3 of the conifold geometry has shrunk to zero - which would imply a further non-trivial topology change. The second limit of interest is when the summation is dominated by the r^8 terms. Again it is easy to see that the minimal radius occurs at

$$r = \frac{3r'}{8} \tag{3.306}$$

which is again smaller than the radius in the large q limit. In fact evaluating the minimum of H for various values of q shows that this radius is always smaller than the corresponding radius in the large q limit, which is what we would naively expect. Fig 3.5 illustrates this in a plot of the tension, H, against the physical radius r for the three values $q = 2, 4, 10$. Here, we chose for convenience $p = 100, g_s = 0.1, C_0 = 0.1, M = 100, a_0 = 1$ and work with units where $\alpha' = 1$. Typically C_0 will be small in this solution, however the results are applicable even when we consider an odd number of branes. This shift in radius arises due to the symmetrization prescription for pairs of fields in the Myers action. Furthermore we see that the tension at the minimum is smaller for finite q, which is phenomenologically interesting from a cosmological perspective as we can imagine a situation where very few F and D strings are formed at the end of brane/anti-brane annihilation. The strings will tend to move together to minimise their energy at the tip of the conifold, and for cases where there are only a couple of D-strings, their respective tensions could easily satisfy the observational bounds.

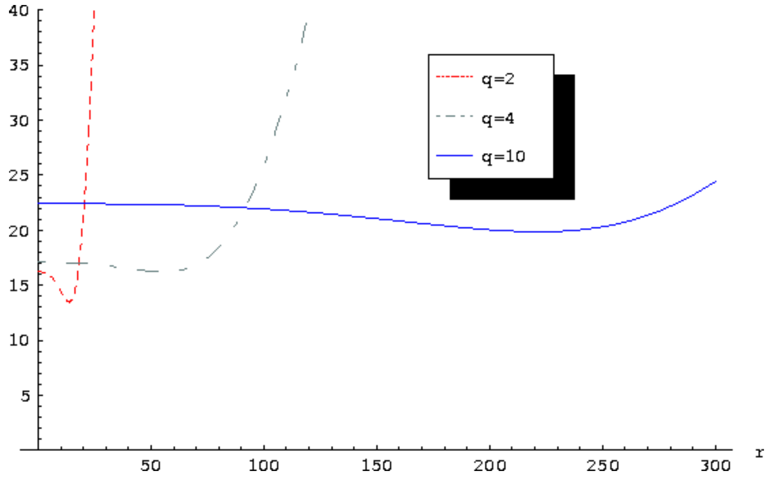

Figure 3.5: string tension H vs r for $q = 2, 4, 10$.

3.5 Discussion.

In this chapter we investigated the dynamics of the Myers action in a variety of different backgrounds. In terms of the fuzzy spherical description we found that there were only collapsing solutions except for the case where coincident $D0$-branes are in the $D6$-brane background. This is the gravitational Myers effect [55], and is the microscopic dual of the gravitational dielectric effect. We saw that there appears to be a complex relationship between the dynamics in non-trivial backgrounds and algebraic geometry, which deserves further analysis as this could also be related to supersymmetry preservation. We also studied how the introduction of the open string tachyon on the worldvolumes alters these dynamics, leading to interesting bounce solutions, and also conjectured solutions for all the fuzzy even spheres S^{2k}. We then generalised these solutions to general curved metrics exhibiting an $SO(1,q) \times SO(9-q)$ symmetry, and also included a commuting electric field. Furthermore we constructed both the macroscopic and microscopic $D1$-$D3$ intersection in this background, which led to unusual fuzzy funnel solutions for certain backgrounds. Using this technology we studied in detail the specific construction of (p,q)-strings in the Warped Deformed Conifold, which has potential applications for a viable model of cosmic superstrings, or as confining strings in an $\mathcal{N} = 1$, $SU(M)$ gauge theory.

The tension spectrum for the microscopic (p,q) string was found to deviate from that predicted by the macroscopic model at higher orders in the $1/M$ expansion. However this is due to the non-Abelian Taylor expansion of all the background fields. In fact we can exactly reproduce the tension spectrum to all orders in $1/M$ by a more selective choice of ansatz for the fuzzy sphere [72], this also suggests that it may be possible to obtain simpler closed form expressions for the tension at finite q and is therefore relevant for gauge theory, and also cosmic string network modelling, as it may provide an experimentally testable prediction of string theory.

CHAPTER 4

BRANE DYNAMICS AND COSMOLOGY.

4.1 Introduction.

Standard Big Bang cosmology is based on the cosmological principle, namely that the universe is homogeneous and isotropic on sufficiently large distance scales, and is well described by the Friedmann Robertson Walker (FRW) metric

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \tag{4.1}$$

where k represents the spatial curvature, and $a(t)$ is the overall scale factor. For flat universes $k = 0$, for positively curved universes $k > 0$ whilst for negatively curved universes $k < 0$. The dynamics of the universe are effectively governed by the energy density it contains through the Einstein equations. In the absence of a cosmological term the Einstein equation is simply $G_{\mu\nu} = T_{\mu\nu}/M_p^2$ [1], and therefore the evolution of the scale factor is given by [76]

$$H^2 = \frac{\rho}{3M_p^2} - \frac{k}{a^2} \tag{4.2}$$

where ρ is the energy density of the universe. If we know the explicit dependence of the energy density on the scale factor, then we can use this expression to obtain a closed form solution for $a(t)$.

The Big Bang model of the universe, developed from these equations, has been experimentally tested and verified during the last century. However despite its many successes, there remain many puzzling questions. It is not the aim of this introduction to go into them all with great detail, but we will briefly mention a couple relevant for our purposes [2].

- The Flatness Problem
  If we define the ratio $\Omega = \rho/\rho_c$ then we can re-write (4.2) as $\Omega - 1 = k/\dot{a}^2$. However because the universe is expanding with decreasing velocity, $\dot{a}^2$ is always decreasing. Recent measurements suggest that Ω is very close to one [88–91], which implies that it must have been even closer to unity - and therefore the universe even flatter - in the

[1] We are using natural units and the reduced Planck scale, which is related to the traditional one via the relation $M_p^2 = m_p^2/8\pi$.

[2] See [76] for a more complete discussion.

distant past. At the time of Nucleosynthesis [76] this implies that $|\Omega - 1| \leq \mathcal{O}(10^{-16})$. Extrapolating this back to Planck times requires this tuning to be $\mathcal{O}(10^{-64})$.

- Defects
  Particle physics suggests that there should be some Grand Unified Theory, which has a large gauge group to encompass all the relevant fields. However when this symmetry is broken we typically obtain magnetic monopoles and other topological defects. If they exist at early times then their energy density varies like matter i.e $\rho \sim a^{-3}$, whereas radiation varies like a^{-4}. Therefore these defects would be dominant form of energy density in the early universe, and could potentially overclose the universe before structure had time to form.

- The Horizon Problem
  The CMB measures photons at the epoch of decoupling. However they are essentially all thermalised at the same temperature of 2.73K across the sky. According to standard lore the physical wavelength of these photons ($a\lambda$) exhibited power law behaviour at early times, $a\lambda \propto t^p$ where $0 < p < 1$. By contrast the Hubble radius satisfied a linear relationship, $H^{-1} \propto t$, therefore the physical wavelength of the photons should be much smaller than the Hubble radius, and causal effects should only cover a small Hubble patch. This clearly disagrees with the CMB observation.

These problems basically arise because of the assumption that $\ddot{a} < 0$ for the entire evolution of the universe. A simple solution is to postulate a period where $\ddot{a} > 0$, implying that $\dot{a}$ *increases* during this period with $a \sim e^{Ht}$. This period is what we call inflation. Because $\dot{a}$ increases during inflation this rapidly drives Ω to unity, and then after inflation it evolves as in the standard Big Bang paradigm. Thus the flatness problem is naturally resolved. The topological defects can be seen to be red-shifted away during this inflationary phase, which prevents them from overclosing the universe. So this solves the defect problem aswell. The horizon problem is solved in a similar fashion, since now the physical wavelength of photons has power law behaviour with $p > 1$. This means that the physical wavelength grows faster than the Hubble radius, and is in fact pushed outside the radius so that the causal horizon is larger than the Hubble horizon. This allows radiation to be correlated on much larger scales.

Inflation appears to be a nice solution to these long standing problems, however the mechanism by which $\ddot{a} > 0$ occurs appears to require the existence of a scalar field known as the inflaton. At the simplest level, we require the potential energy of the inflaton to dominate over the kinetic energy which indicates that it must be reasonably flat in order to ensure that enough inflation occurs. This is more commonly referred to as 'slow roll' inflation, since the inflaton field is non-relativistic. To parameterise the amount of inflation obtained, we introduce the number of e-foldings defined through

$$N = \ln\left(\frac{a_f}{a_i}\right) = \int_{t_i}^{t_f} H dt \tag{4.3}$$

where the subscripts i and f denote values at the start and end of inflation respectively. To solve the flatness and horizon problems it turns out that we require $N \geq 60$ [76].

In recent years we have witnessed an explosion in the number of proposed inflationary models, but essentially these are all special cases of only three kinds of inflationary behaviour which can be classified as (i) small field inflation, (ii) large field inflation and (iii) hybrid inflation. The first class of models have a small initial value for the inflaton field, and include models such as natural [101] and tachyonic inflation (see [80] for a substantial but far from complete set of references.). The second class of models require the inflaton field to have a large initial value, and includes eternal inflation (see Linde [76]). The final class of models is a model of double inflation, where inflation ends when one scalar field rolls to its minimum and forces another field to being rolling [76]. Each model has slightly different physical characteristics, which allows us to distinguish between them.

Of course despite its beautiful simplicity, inflation is still a paradigm in search of a concrete framework. Over the past few years we have seen many attempts to embed inflation naturally into the MSSM and other grand unified theories, but the abundance of free parameters makes few concrete predictions in the absence of experimental evidence. Since superstring theory pertains to be a theory of everything and typically contains many scalar fields, it makes sense to search for inflationary mechanisms within a stringy context. There are many potential benefits to such a program, which are currently under active investigation. But it is fair to say that inflationary model building within a string theory context (typically referred to as String cosmology) is still in its infancy.

There are three main threads in string cosmology. Firstly there is the string gas approach [78], which is purely stringy and relies heavily on the ideas of T-duality and enhanced symmetries. The second approach is through closed string modes, typically moduli fields, and includes models such as modular inflation [79]. The final approach is through the open string sector and relies on brane dynamics [86], or tachyon condensation [80, 82, 87]. It is this final approach which is of relevance to this thesis.

In this chapter we will develop a simple model of inflation/dark energy based on the Geometrical Tachyon. We will also consider a hybrid inflation scenario which can be modelled using similar techniques. Finally we will consider a more robust mechanism relying on the non-linear nature of the DBI action itself, called DBI inflation [92].

4.2 Geometrical Tachyon Inflation.

In Chapter 2 we saw that it was possible to define a map between brane dynamics in curved space and a scalar field on a brane worldvolume in flat space through the radion-tachyon correspondance [26]. We called this the Geometrical Tachyon, as in some sense it was a holographic scalar field, yet encoded all of the classical dynamics of the original theory. This scalar field has a vanishingly small mass, which distinguishes it from the open string tachyon which has a mass around the string scale. In the absence of warping this large mass prevents tachyon inflation from generating the requisite number of e-foldings, and therefore doesn't appear to be a likely candidate for the inflaton field. One may ask what happens to the Geometrical Tachyon field when we treat it as a candidate for the inflaton. This motivates our analysis in this section. However it should be noted that we are no longer considering a pure string cosmology scenario, rather we are in the realms of string *inspired* cosmology - in much the same vein as the Randall-Sundrum models of braneworld cosmology [105].

Recall that our Geometrical Tachyon arose from considering D-brane dynamics in the $NS5$-ring background of type II string theory. The branes are on a circle of radius R, and we will treat the solution (initially) as a smearing of the charge of k branes on an S^1. We will again write the supergravity equations for such a solution for convenience [19]

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + H(x^n)dx^m dx^m \\ e^{2(\phi-\phi_0)} &= H(x^n) \\ H_{mnp} &= -\varepsilon^q_{mnp}\partial_q\phi, \end{aligned} \tag{4.4}$$

where μ, ν parameterize the directions parallel to the fivebranes whilst m, n are the transverse directions and ϕ is the dilaton. As usual the harmonic function $H(x^n)$ describes the orientation of the fivebranes in the transverse space. Part our assumption is that the number of branes must be large in order to neglect α' corrections, so we must take $k >> 1$ in what follows.

$$H = 1 + \frac{kl_s^2}{\sqrt{(R^2+\rho^2+\sigma^2)^2 - 4R^2\rho^2}}. \tag{4.5}$$

In this equation we have switched to polar coordinates to simplify the expression. This parameterisation breaks the transverse symmetry from $SO(4) \to SO(2) \times SO(2)$

$$\begin{aligned} x^6 &= \rho\cos(\theta), \qquad x^7 = \rho\sin(\theta) \\ x^8 &= \sigma\cos(\psi), \qquad x^9 = \sigma\sin(\psi), \end{aligned} \tag{4.6}$$

and now the harmonic function describes a ring oriented in the $x^6 - x^7$ plane. Note that we are also able to switch between IIA and IIB theory because the $NS5$-brane background solution is insensitive to a single T-duality transformation, as the harmonic function couples only to the transverse parts of the metric.

We now insert a probe Dp-brane in this background whose low energy effective action is the DBI action, which we again write as

$$S = -T_p \int d^{p+1}\zeta e^{-\phi} \sqrt{-det(G_{\mu\nu} + B_{\mu\nu} + \lambda F_{\mu\nu})}, \tag{4.7}$$

where both $G_{\mu\nu}$ and $B_{\mu\nu}$ are the pullbacks of the space-time tensors to the brane, $\lambda = 2\pi l_s^2$ is the usual coupling for the open string modes, $F_{\mu\nu}$ is the field strength of the $U(1)$ gauge field on the world volume, and T_p is the tension of the brane. As discussed in Chapter 2, we should only consider classical solutions where $3 \leq p \leq 5$ in order to neglect backreactive effects, and divergent closed string emission. We will assume that the transverse scalars are time dependent only, and set the gauge field and Kalb-Ramond field to zero for simplicity. Upon insertion of the $NS5$-brane background solution, we see the action in static gauge reduces to

$$S = -T_p \int d^{p+1}\zeta \sqrt{H^{-1} - \dot{X}^2}, \tag{4.8}$$

with X^m parameterizing the transverse scalar fields, and there is no Chern-Simons coupling because the fivebrane background does not act as a source for RR-fields.

4.2.1 Cosmology in the ring plane.

We now restrict ourselves to motion within the plane of the ring, and map the DBI action to the tachyon action as before. It is convenient to simply restate the results from chapter 2. The action can be written as follows

$$S = -\int d^{p+1}\zeta V(T)\sqrt{1 - \dot{T}^2}, \tag{4.9}$$

where the potential and unstable tension are given by

$$\begin{aligned} V(T) &= T_p^u \cos\left(\frac{T}{\sqrt{kl_s^2}}\right) \\ T_p^u &= \frac{T_p R}{\sqrt{kl_s^2}}. \end{aligned} \tag{4.10}$$

In obtaining the above, we have used the throat approximation for the harmonic function, which means neglecting the factor of unity in (4.5). Under this assumption we see that taking $\rho = \sigma = 0$ in (4.5) (i.e. the centre of the ring) requires that $\sqrt{k}l_s >> R$, which is the first constraint we find on the parameters k, l_s and R. Later on we will use numerical techniques to arrive at a form of the potential $V(T)$ which will use the exact form of the harmonic function as calculated in [20]. In principle we can then relax the throat approximation which leads to the cosine potential in (4.10) so that the previous inequality may not be needed.

We see that the Geometrical Tachyon potential is symmetric about the origin, which arises as a consequence of the background geometry. It should be noted that this mapping is non-trivial in the sense that we began by probing a gravitational background, and have ended up with a solution in flat Minkowski space. This tells us that there are two equivalent ways of visualizing the theory. Firstly there is the bulk viewpoint, where there is actually a ring of $NS5$-branes and the solitary probe brane universe moving in the throat geometry. Alternatively, we could view the problem as a single brane moving in flat space-time with a highly non-trivial field condensing on its world volume. In what follows we will find it useful to switch between these two pictures in order to better understand the physics.

Our earlier work showed that the Geometrical Tachyon takes values between $\pm\pi\sqrt{kl_s^2}/2$ in contrast to the usual open string tachyon which is valued between $\pm\infty$. This is simply due to the probe brane being confined *inside* the ring. Expanding the potential about the unstable vacuum yields a tachyonic mass of $M_T^2 = -1/kl_s^2$. For sufficiently large k this can be made much smaller than the open string tachyonic mass $M^2 = -1/2$ (in units where the string length is unity). It is this different mass scale and profile of the potential, which is similar to that of Natural Inflation [101], which suggest that the Geometrical Tachyon may be useful as an inflaton candidate.

In order for us to consider realistic cosmological solutions we must now specialise to the case of $p = 3$, and therefore we are in type IIB string theory. The energy momentum tensor can be calculated by variation of the action, and has non-zero components

$$\begin{aligned} T_{00} &= \frac{V(T)}{\sqrt{1-\dot{T}^2}} \\ T_{ij} &= -\delta_{ij} V(T)\sqrt{1-\dot{T}^2} \end{aligned} \tag{4.11}$$

from which we see that the pressure of the tachyon fluid tends to zero as the tachyon rolls toward the zero of $V(T)$. For the open string tachyon this implies that the vacuum is a gas of heavy closed string modes as discussed in [28, 80] In a cosmological context the condensing tachyon will generate a gravitational field on the D-brane and therefore we must include this minimal coupling in the action [80]. We will also assume that there is no coupling to any other string mode in order to keep the analysis simple, however we should be aware that there is no reason why other modes should not be included. Our Lagrangian density can thus be written

$$\mathcal{L} = \sqrt{-g}\left(\frac{R}{16\pi G} - V(T)\sqrt{1+g^{\mu\nu}\partial_\mu T \partial_\nu T}\right), \tag{4.12}$$

where $g^{\mu\nu}$ is the metric and R is the usual scalar curvature. For simplicity we will assume that there is a FRW metric of the form

$$ds^2 = -dt^2 + a(t)^2 dx_i^2, \tag{4.13}$$

with i running over the spatial directions. The effect of the scale factor is to modify the energy density, u for the flat background such that it is no longer conserved in time, which prevents us from obtaining an exact solution for the tachyon in the presence of the gravitational field in the usual manner.

From this we can determine the late time behaviour of the tachyon condensate. If we assume that u is constant, then the pressure will vary as $p = -V(T)^2 u$ and will tend to zero as $V(T)$ reaches its minimum. Using the standard equation of state $p = \omega u$, we find that $\omega = -(1-\dot{T}^2)$ which implies $-1 \leq \omega \leq 0$. From the Lagrangian density, we can also obtain the Friedman and Raychaudhuri equations for the tachyon condensate

$$\begin{aligned} H^2 &= \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2 V(T)}{3\sqrt{1-\dot{T}^2}} \\ \frac{\ddot{a}}{a} &= \frac{\kappa^2 V(T)}{3\sqrt{1-\dot{T}^2}}\left(1 - \frac{3\dot{T}^2}{2}\right), \end{aligned} \tag{4.14}$$

where $\kappa^2 = 8\pi G = M_p^{-2}$, $M_p = 2.2 \times 10^{18} GeV$ and the cosmological constant term is set to zero.

Before discussing the cosmological evolution of our scalar field we must first comment on the coupling to four dimensional Einstein gravity. In some ways the Geometrical Tachyon is reminiscent of the Mirage Cosmology scenario [102]. In fact recent work has used this approach to study inflation [103]. Mirage Cosmology requires us to re-write the induced metric from the supergravity background on the $D3$-brane world-volume in a FRW form. The universe will automatically be flat or closed if we choose the D-brane to be flat or spherical respectively. The problem however, is that there is no natural way to couple gravity to the brane action and therefore we must insert it by hand, however the cosmological dynamics are expected to be reliable virtually all the way to the string scale. A second way to think about the solution is to simply start with the Geometrical Tachyon. The tachyon mapping in this case is only concerned with time-dependent quantities, and in particular only with the temporal component of the Minkowski metric. Therefore we can choose to include a scale factor component in the spatial directions. This means that we have essentially the 'old' model of tachyon inflation [80], but with a rolling Geometrical Tachyon. As argued by Sen, this should generate a gravitational field on the brane and therefore a coupling to Einstein gravity. We can then compactify *this* solution down on a six-dimensional manifold and investigate the four dimensional cosmology. Of course the problem now is that there are α' corrections to the DBI action, which could become large near the tachyon vacuum. The final approach would be to perform a direct compactification of the full $NS5$-brane background down to four dimensions [25]. In order to do this we would first need to truncate the background to ensure the space is compact. In our case the ring can naturally impose a cut-off in the planar direction, however we must still impose some constraint in the transverse direction to the ring plane. Our solution simplifies somewhat if we can consider the $R \to 0$ limit, or equivalently the $\sigma \gg 1$ limit, as the background will appear point like. Smoothly gluing the truncated space to a proper compact manifold will now automatically include an Einstein-Hilbert term in the effective action. However, although we now have a natural coupling to gravity, the compactification itself is far from trivial as we also need to wrap two of the world-volume directions of the $NS5$-branes on a compact cycle. In order to proceed we must first uplift the full solution to M-theory [3], where we now have a ring of $M5$-branes magnetically charged under the three-form $C_{(3)}$. Compactification demands that the magnetic directions of the three-form are wrapped on toroidal cycles, which is further complicated by the ring geometry and will generally result in large corrections to the potential once reduced down to four-dimensions. So although we have a natural gravitational coupling we may have large corrections to the theory. The complete description of this compactification is interesting, but well beyond the scope of this work and should be tackled as a future problem. However we could also assume a large volume toroidal compactification, where again all the relevant moduli have been stabilised. Provided we introduce some 'sink' for the five-brane charge, located at the some distant point in the compact space, and also only concentrate on the region close to the branes so that the harmonic function remains valid we will have an induced gravitational coupling in the low energy theory. The corrections to the scalar potential in this region of moduli space may well be sub-leading with respect to the scalar field dynamics and thus we can treat our model as the leading order solution.

The approach we have in mind in this chapter is the second option, which treats the Geometrical Tachyon as being the fundamental object of interest. In fact we know that this object can be related to the open string tachyon through the S-type tachyon and therefore may be more than just a mathematical construct. We will assume that the theory is compactified on a six-torus T^6 for simplicity. Obviously this could be generalised to manifolds with $SU(3)$ structure[4], which could lead to a more phenomenologically realistic model. But this is left for future endeavour. However since the original picture of a brane moving in the ring background is intuitive, we will often make reference to the ring background in order to clarify the physics of the tachyon condensation.

Upon compactification on a T^6, we find that we can write the four-dimensional Planck

[3]This was discussed by Ghodsi in [21]. We refer the interested reader there for more details.

[4]See Grana [108] for example.

mass in terms of the string scale via the following relationship [80]

$$M_p^2 = \frac{vM_s^2}{g_s^2}, \tag{4.15}$$

where $M_s = l_s^{-1}$ is the fundamental string scale and the quantity v is given by

$$v = \frac{(M_s r)^d}{\pi}, \tag{4.16}$$

with r, d being the radius and number of compactified dimensions respectively (we will be using $d = 6$ in this chapter), and v being the volume of the torus. In a realistic toroidal compactification we must also stabilise the moduli of the torii so that $v >> 1$, in order to trust our effective action. We will assume that this can be done without altering the tachyon solution, although it would be useful to perform this explicitly.

The evolution of the universe is effectively determined by the Raychaudhuri equation which shows that inflation will cease when $\dot{T}^2 = 2/3$ and the universe will then decelerate as the tachyon velocity increases. Upon variation of the action, we find the equation of motion for the tachyon field can be written

$$\frac{V(T)\ddot{T}}{1-\dot{T}^2} + 3HV(T)\dot{T} + V'(T) = 0, \tag{4.17}$$

where a prime denotes differentiation with respect to T, and H is the Hubble parameter. Note that in deriving this equation we must also use the conservation of entropy of the tachyon fluid. We see that $3HV(T)\dot{T}$ acts as a friction term, in much the same way as in standard inflationary models, except that this term may vanish for the open string tachyon as the field rolls to $\pm\infty$ where its potential vanishes. For a scalar field to be a candidate for the inflaton field it must typically satisfy a set of slow roll parameters, as well as providing enough e-foldings during rolling. In contrast to the open string tachyon, our Geometrical Tachyon has a cosine potential and therefore, with appropriate tuning, can be flat near the origin. This should allow for sufficient inflation before velocity effects become important. In this analysis we use the conventions employed in the earliest papers in [80] to parameterise the slow roll parameters in the Hamilton-Jacobi formalism

$$\begin{aligned} \epsilon(T) &= \frac{2}{3}\left(\frac{H'(T)}{H^2(T)}\right)^2 \\ \eta(T) &= \frac{1}{3}\left(\frac{H''(T)}{H^3(T)}\right). \end{aligned} \tag{4.18}$$

Where we assume that the acceleration (and also the velocity) is negligible. By definition inflation ends when the slow roll parameters become unity. The number of e-folds produced between T_o and T_e is given by the standard expression

$$N(T_o, T_e) = \int_{T_o}^{T_e} dT \frac{H}{\dot{T}}, \tag{4.19}$$

which must satisfy $N \sim 60$ to agree with observational data. T_o is the value of the field N e-folds before the end of inflation, whilst T_e is the value of the field at the end of inflation.

We first consider the Geometrical Tachyon starting very close to the top of its potential with a small initial velocity to ensure that it will roll. Note that if $T = 0$ then spontaneous symmetry breaking will cause the universe to fragment into small domains which will each have differing values of the tachyon field. As usual in tachyon cosmology, inflation can occur only if $H^2 >> |M_T^2|$ near the top of the potential[5], which translates into the constraint

$$\frac{T_3 R}{3M_p^2} >> \frac{1}{\sqrt{kl_s^2}}. \tag{4.20}$$

Where T_3 is the tension of a *stable* $D3$-brane. Since we are considering the large k limit, the RHS is very small and so we find that this condition is satisfied [6]. Furthermore this also suggests that the effective theory for the Geometrical Tachyon is valid because we can clearly see that

$$(kl_s^2)^{1/2} >> H^{-1}, \tag{4.21}$$

and so the de-Sitter horizon may be larger than the string length for large k. This is in contrast to the open string scenario where we find that the horizon is smaller than the string length, and thus should not be described by an effective theory.

At this stage we must try and get a handle on the size of g_s which is the asymptotic string coupling constant. This is important since this is what rules out most models of tachyon inflation. Typically we find that in order to satisfy $H^2 >> |M_T^2|$ at the top of the potential, we are forced to tune [80]

$$g_s >> 260v. \tag{4.22}$$

However the effective field theory is only valid when $v >> 1$, which implies g_s is extremely large and well outside the realm of perturbation theory. Additionally we see that because the tension of the brane goes like $1/g_s$ this constraint on the coupling implies our effective action is breaking down leading to brane fragmentation. By contrast a similar calculation for the Geometrical Tachyon implies

$$g_s >> \frac{24\pi^3 v}{\sqrt{k} R M_s} \approx 744 \frac{v}{\sqrt{k} R M_s}, \tag{4.23}$$

[5] In our case this means that the Hubble scale could also be small.

[6] In the finite k case we will have to be careful to ensure that this constraint is fulfilled.

thus by fixing appropriate values for k and R we may ensure that $v >> 1$ and also that $g_s << 1$. Earlier we introduced the throat approximation $\sqrt{k}l_s >> R$, which means that $\sqrt{k}RM_s << k$. Thus in fact it is large values for k that will essentially allow a satisfactory solution to (4.23). For example, assuming $v = 10$ a value of $k \approx 10^5$ would allow for perturbative g_s to solve (4.23). Relaxing the throat approximation may allow much smaller values of k. This is interesting because we see that the weak coupling arises naturally because of the choice of the background parameters, whereas the reason for the weak coupling of the inflaton in standard field theory is unknown.

Despite this apparent success we may be concerned that the effective theory may still not be a valid description at the top of the potential. In order to check this we should compare the effective tension of the unstable brane to the Planck scale. After some algebra, and using the equation for weak coupling we find

$$\frac{T_3^u}{M_p^4} \sim \frac{3}{k^2}\left(\frac{24\pi^3}{RM_s}\right)^2 v. \tag{4.24}$$

Again we see that for a certain range of background parameters (and assuming $RM_s >> 1$), the effective tension need not be Super-Planckian and therefore the DBI can still be a good approximation to 4D gravity.

As there is an obvious similarity with the potential arising in Natural Inflation we could demand that the height of the potential to be of the order of M_{GUT}^4 (where $M_{GUT} \sim 10^{16}$ GeV) in order to generate inflation, however we will try to keep this scale arbitrary for the moment. Using the potential we immediately see that the slow roll conditions become

$$\begin{aligned} \epsilon &= \frac{M_p^2}{2T_3R\sqrt{kl_s^2}} \frac{\tan^2(T/\sqrt{kl_s^2})}{\cos(T/\sqrt{kl_s^2})} \\ \eta &= \frac{-M_p^2}{4T_3R\sqrt{kl_s^2}} \frac{\left(1+\cos^2(T/\sqrt{kl_s^2})\right)}{\cos^3(T/\sqrt{kl_s^2})}. \end{aligned} \tag{4.25}$$

Slow roll will only be a valid approximation when the tachyon is near the top of its potential and thus we should Taylor expand the trig functions to determine the analytic behaviour. In fact for small T we see that ϵ is already extremely small. Dropping the numerical factors gives us the primary constraint for slow roll;

$$M_p^2 << T_3^u R\sqrt{kl_s^2}, \tag{4.26}$$

which must be satisfied by both equations. Given that the reduced Planck mass in string theory is typically of the order of 2.4×10^{18} GeV, this means that k and R/l_s must be large. Generally we see that the slow roll conditions will be satisfied due to the mass scale of the geometrical tachyon. In the open string models the larger mass implies that the tachyon may only have been involved in some pre-inflationary phase. Of course, the analysis in both cases is classical and quantum corrections may well prove to be important is determining the exact behaviour near the top of the potential.

We can estimate the number of e-foldings using (4.19), however this turns out to be sensitive to the value of the tachyon velocity near the top of the potential. To remedy this we use the equations of motion and the slow roll approximation, which allows us to re-write this equation in terms of the potential and its derivative. In fact this is the method most commonly used in standard inflationary analysis. After some algebra we obtain

$$\begin{aligned} N_e &= -3\int dT \frac{H^2 V(T)}{V'(T)} \\ &= \frac{T_3 R\sqrt{kl_s^2}}{M_p^2}\left\{\cos\left(\frac{T_e}{\sqrt{kl_s^2}}\right) - \cos\left(\frac{T_o}{\sqrt{kl_s^2}}\right) + \ln\left(\frac{\tan(T_e/2\sqrt{kl_s^2})}{\tan(T_o/2\sqrt{kl_s^2})}\right)\right\} \end{aligned} \tag{4.27}$$

Using the constraint from the slow-roll equations we see that the leading term must be large. If we demand that T_o and T_e are reasonably close, then the contribution from the other terms will be small, and so the number of e-foldings will depend on the ratio

$$\nu = \frac{T_3 R\sqrt{kl_s^2}}{M_p^2}, \tag{4.28}$$

where $\nu \geq 60$ in order for there to be enough inflation. However if we don't impose this restriction, but allow inflation to begin near the top of the potential and end near the bottom, then there can be significant contribution to the number of e-foldings from the additional terms. This has the effect of reducing the value of ν - however it must still satisfy the slow roll constraint of being larger that unity [7]. We can write the unstable brane tension in terms of the string coupling, string mass scale and the parameter ν, thus we have the height of the potential given by

$$M_{infl}^4 = \frac{M_p^2 M_s^2 \nu}{k}, \tag{4.29}$$

which defines our effective inflation scale M_{infl}. The exact value of M_s depends on the particular string model but it is usually assumed to lie in the range 1 Tev - 10^{16} GeV. So as an example, if $\nu \sim 60$, $M_s \sim 10^{16}$ GeV and $k \sim 10^5$ we find $M_{infl} \sim 10^{16}$ GeV.

Numerical Analysis.

We can also check the consistency of our analytic solutions by numerically solving for the Hubble parameter. We choose to employ the Hamilton-Jacobi formalism [76], where the Hubble equation is written as a function of T rather than time (since the tachyon field is monotonic with respect to time), and then using the Friedmann equation we obtain the

[7]See Fairbairn and Tytgat in [80]for a more detailed inflationary analysis of the model presented here.

following first order differential equation

$$H'^2(T) - \frac{9}{4}H^4(T) + \frac{1}{4M_p^4}V(T)^2 = 0, \tag{4.30}$$

where a prime denotes differentiation with respect to T. Solving this for the Hubble term gives us a constraint on the velocity of the tachyon field

$$\dot{T}^2 = 1 - \left(\frac{V(T)}{3M_p^2 H(T)^2}\right)^2. \tag{4.31}$$

It will be convenient to work with dimensionless variables in our numerical analysis, so we define the dimensionless tachyon field and Hubble parameter as follows,

$$y = \frac{T}{\sqrt{kl_s^2}}, \qquad h(y) = \sqrt{kl_s^2}H(y). \tag{4.32}$$

We can solve (4.30) to obtain $h(y)$ and then substitute this into (4.31) to determine the velocity of the tachyon field. We choose the initial velocity of the field to be zero, and the initial value of T_0 to be very small - but non-zero. As in Fairbairn [80] the general behaviour is dependent upon the dimensionless ratio X_0, where $X_0^2 = \nu$. Some results are plotted in Figure 4.1. We find that the velocity (strictly speaking this is the square of the velocity) of the tachyon field is very small over a large range, only becoming large as it nears the bottom of the potential. In inflationary terms this implies that universe will be inflating for almost the entire duration of the rolling of the field. For increasing values of X_0, inflation ends at larger values of T. However even for the case of $X_0 = 2$, which barely satisfies the slow roll constraints, we expect inflation to end reasonably close to the bottom of the potential.
We can also make a numerical check on the smallness of the slow roll parameters ϵ, η (see

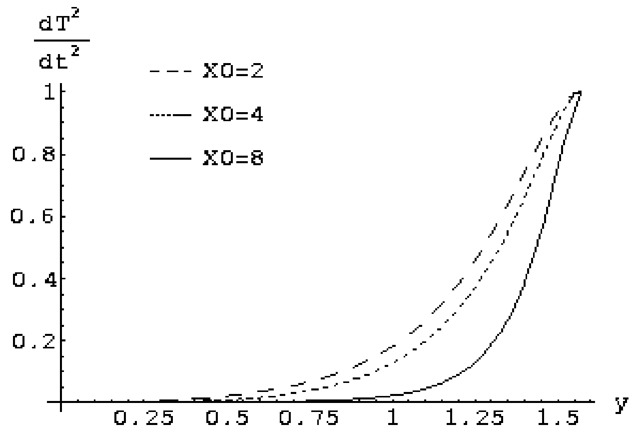

Figure 4.1: Velocity of tachyon field for differing values of X_0, with an initial velocity of zero.

Figs 4.2 and 4.3). Using our numerical solution for h we can also determine the amount

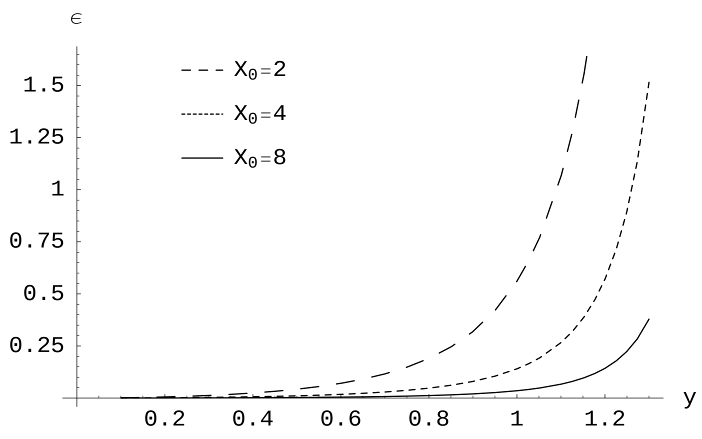

Figure 4.2: Value of the slow-roll parameter ϵ for various values of X_0

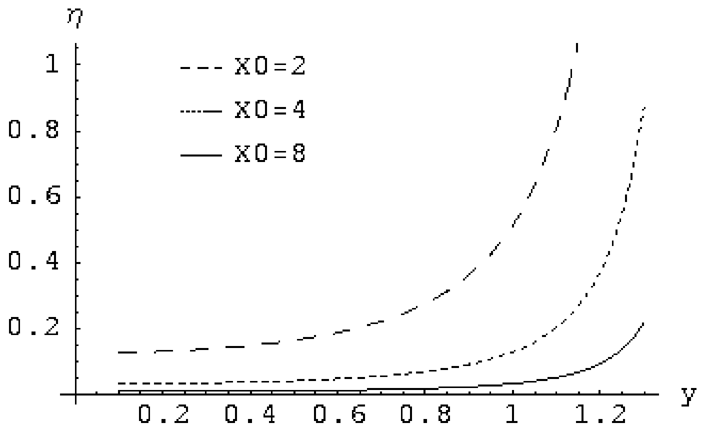

Figure 4.3: Value of the slow-roll parameter $|\eta|$ for various values of X_0

of e-foldings during inflation via (4.27). It turns out that in order to generate at least 60 e-foldings we only need to take $\nu \sim 30$

Finally we can also use numerical techniques to try and reconstruct the complete tachyon potential by using the full form of the ring harmonic function as derived in [20] without assuming the approximation that lead to the cosine potential (4.10). Recall that this approximation was that the $NS5$ branes were unresolvable as point sources arranged uniformly around the ring. As our tachyon field rolls from near the top of the cosine potential down towards the value $T/\sqrt{k}l_s = \pi/2$, the geometric picture of this process is that we start from near the centre of the ring at $\rho = 0$ and move towards the ring located at $\rho = R$. As $T/\sqrt{k}l_s$ nears $\pi/2$, even for large k, the approximation of a continuous distribution of $NS5$ branes around the ring will break down and individual sources will be resolvable. It is at this point that we expect the true potential $V(T)$ to deviate from the cosine form. Fig 4.4 shows the shape of the potential one obtains for the case $k = 1000$ by numerically implementing the

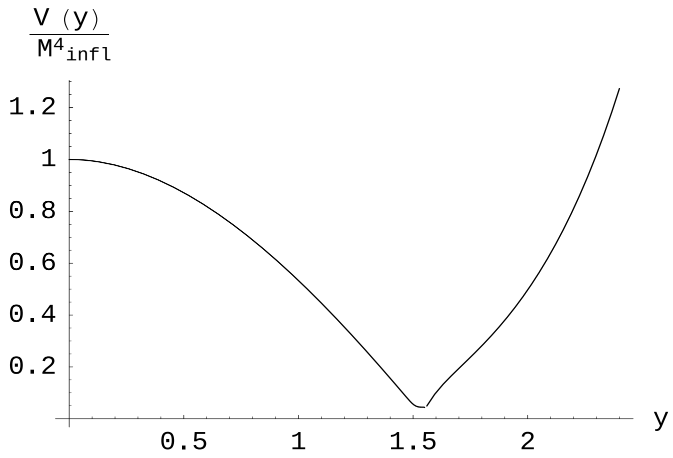

Figure 4.4: Profile of the tachyon potential taking $k = 1000$. The solutions from each region are matched onto each other at $T = \pi/2$ in dimensionless units.

tachyon map discussed earlier and using the exact form of the ring harmonic function. In this plot we have chosen the angular variable θ that appears in the exact form of the harmonic function to be fixed at $\pi/2k$ for simplicity. Details of the harmonic function relevant to fully resolvable $NS5$ branes are given in a later section. What is perhaps most apparent about this potential is the existence of a minimum very close to the ring location. It also turns out that our previous cosine potential is an excellent approximation to this numerical plot for values of T to the left of the minimum. Later on we shall see how analytic methods can be used to verify the existence of this minimum.

Perturbations.

Obtaining the minimal number of e-foldings tells us that inflation is possible, but we must also examine the perturbations of the field near the end of inflation [76]. The perturbations are directly relevant for observation, and are therefore highly constrained. One of the generic difficulties associated with open-string tachyonic inflation is the fact that the tension of the $D3$-brane must be significantly larger than the Planck mass. This indicates that the effective action cannot adequately describe 4D gravity, as it will have metric fluctuations that are always too large. This is not necessarily the case for our Geometrical Tachyon, as we seen there are additional scales in the theory which can reduce the overall effect of these fluctuations. There are two main perturbations to consider, the scalar, and the gravitational (tensor) ones which we will denote by $\mathcal{P}_{\mathcal{T}}$ and $\mathcal{P}_{\mathcal{G}}$ respectively. (Strictly speaking, $\mathcal{P}$ corresponds to the amplitude of the perturbation). Constraints from observational data imply the relation (at its absolute limit)[8]

$$|\mathcal{P}_T| + |\mathcal{P}_\mathcal{G}| \leq 10^{-5}. \tag{4.33}$$

During inflation gravitational waves are produced whose amplitude is given by the ratio $\mathcal{P}_\mathcal{G} \sim \frac{H}{M_p}$, but observational data of the anisotropy of the CMB [88] implies that at the end of inflation

$$\frac{H_{end}}{M_p} \leq 3.6 \times 10^{-5}, \tag{4.34}$$

and we must ensure that this condition is consistently satisfied in order for us to consider the Geometrical Tachyon as a viable inflaton candidate. In order to verify this we will first consider the scalar perturbation and use the solution from that to determine our mass scales for the tensor perturbations, as it is generally more important to see whether the tachyon action allows for small tensor fluctuations. For simplicity we will assume that inflation ends when the following constraint is satisfied

$$H \sim |M_T| \sim \frac{M_s}{\sqrt{k}}, \tag{4.35}$$

and we will also assume that the tachyon velocity at this time is given by $\dot{T} = \sqrt{2/3}$. The scalar perturbations are determined in the usual manner using [76, 106]

$$|\frac{\delta\rho}{\rho}| \sim H\frac{\delta T}{\dot{T}}, \tag{4.36}$$

where δT satisfies the following constraint near the top of the potential

$$\delta T \sim \frac{H^2}{2\pi\sqrt{V(T)}}. \tag{4.37}$$

Combining the last two equations we write the amplitude for the scalar perturbation as

$$\mathcal{P}_T \sim \frac{H^2}{2\pi\dot{T}\sqrt{V(T)}} \leq 10^{-5}. \tag{4.38}$$

(We should actually calculate the values of H and $\dot{T}$ during inflation in order to determine the ratio of the perturbations, however since we expect T to be a slowly varying field (4.38) should remain constant over a large range of wavelengths.) If we assume that T is small then the cosine part of the potential is close to unity, and upon substitution of the Hubble term we find

$$\mathcal{P}_T \sim \frac{M_s}{M_p}\sqrt{\frac{3}{8\pi^2 k\nu}} \leq 10^{-5}, \tag{4.39}$$

[8]Note that these are very old normalisations, and the current WMAP normalisation [91] is far more stringent than the one implied here.

We can use this to determine a constraint upon the string scale/Planck scale ratio as follows

$$\frac{M_s}{M_p} \leq \sqrt{\frac{8\pi^2 k\nu}{3}} \times 10^{-5}. \tag{4.40}$$

As an example, for $k \sim 10^3$ and $\nu \sim 28$ (4.40) implies $M_s \leq 10^{16} GeV$. However as we noted, the most current observational limit restricts the scalar mode to be less than 10^{-9} and so we would expect the string scale to be significantly smaller as a result.

Solving the equation for the tensor perturbation leaves us with

$$\mathcal{P}_G \sim \frac{H}{M_p} \sim \frac{M_s}{M_p\sqrt{k}} \leq 3.6 \times 10^{-5} \tag{4.41}$$

which is explicitly dependent upon this ratio. We can establish the absolute upper bound on the perturbation using (4.40)

$$\mathcal{P}_G \leq 2\pi \times 10^{-5}\sqrt{\frac{2\nu}{3}}. \tag{4.42}$$

If ν is $\mathcal{O}(30)$ then this implies the maximum perturbation will be of the order of 10^{-4} which is slightly too large. However in general we may expect that the metric perturbations will be acceptably small by assuming a smaller string scale than the one that saturates (4.40) for given k. This is encouraging since the open string tachyon always admits large metric fluctuations, and therefore cannot be responsible for the last 60 e-foldings of inflation. In our case these fluctuations can be suppressed when k is sufficiently large, and we can find inflationary behaviour leading to the correct amount of structure formation. This was confirmed in the work of [83], who performed a more detailed analysis.

Resolving the minimum.

Perhaps the most problematic aspect of tachyon inflation is the shape of the potential itself. The open string tachyon potential is exponentially decaying at large field values with its minimum at asymptotic infinity. Thus even if it were possible to satisfy all the inflationary conditions, the lack of minimum makes this model difficult to reconcile with phenomenology as there will be no reheating, in the classical sense of scalar field oscillation. Gravitational reheating, although possible, is far too weak in these models to account for the particle abundance we see today, although tachyonic pre-heating may still occur [81, 99]. It is also certainly possible that the potential vanishes for finite T, leading to small oscillations about a minimum which could provide a mechanism for reheating. The bottom line, however, is that there is no simple mechanism for particle creation.

Our Geometrical Tachyon is no exception to these criticisms. Although the minimum is not at infinity, the effective theory breaks down when the tachyon rolls to its maximum value and we are unable to proceed. In the 10D gravitational picture this is due to the probe brane hitting the ring of smeared fivebranes. However even with the simple form of the DBI action in this instance, we see that outside the ring the potential is approximately exponential and it is suggestive that it smoothly maps onto the cosine at $\rho = R$. One may well enquire what happens if we consider a case where the fivebranes are not smeared around the ring, rather that they appear resolvable thus allowing a probe brane to pass between them. In this case we would not expect the DBI action to break down allowing us to find corrections to the truncated cosine potential and thus obtain a minimum. This is exactly what we found following the numerical analysis presented earlier. Let us now see how the existence of such a minimum can be seen analytically. In order to proceed, we refer the reader back to the full harmonic function describing k branes at arbitrary points on the circle, with the interbrane distance, x, given by

$$x = \frac{2\pi R}{k}. \tag{4.43}$$

The full form of the function in the throat region is given by

$$H \sim \frac{kl_s^2}{2R\rho \sinh(y)} \frac{\sinh(ky)}{(\cosh(ky) - \cos(k\theta))}, \tag{4.44}$$

where ρ, θ parameterize the coordinates in the ring plane, and the factor y is given by

$$\cosh(y) = \frac{R^2 + \rho^2}{2R\rho}. \tag{4.45}$$

We clearly see that as $k \to \infty$ we recover the expression for the smeared harmonic function which we used in the previous sections to derive the tachyon potential. Furthermore we see that when $\rho = R$ the function reduces to

$$H \sim \frac{k^2 l_s^2}{2R^2} \frac{1}{(1 - \cos(k\theta))} \tag{4.46}$$

which is clearly finite provided that $\theta \neq 2n\pi/k$, which are the locations of the $NS5$ branes. In order to look for a minima we must expand about the point $\rho = R$ using $\rho = R + \xi$, where ξ is a small parameter which can be positive or negative. Using the expansion properties of hyperbolic functions we power expand the harmonic function for an arbitrary fixed angle θ, and we find to leading order

$$H \sim \frac{k^2 l_s^2}{2R^2} \frac{1}{1 - \cos(k\theta)} \left(1 - \left|\frac{\xi}{R}\right| + \left(\frac{5}{6} - k^2 \frac{2 + \cos(k\theta)}{6(1 - \cos(k\theta))}\right) \frac{\xi^2}{R^2} + \ldots\right), \tag{4.47}$$

where we have used the fact that $k\xi << R$ and have neglected any higher order correction terms. Note that the inter brane distance is given by $2\pi R/k$ and so our expansion will only be valid for distances much smaller than the brane separation. Of course we must be careful not to take k to be too small since our effective action for the Geometrical Tachyon will be invalid. We can clearly see that if the trajectory is at an angle $\theta = (2n+1)\pi/2k$, then the harmonic function will reduce to the form (again to leading order in large k)

$$H \sim \frac{k^2 l_s^2}{2R^2}\left(1 - \frac{k^2\xi^2}{3R^2} + \dots\right). \tag{4.48}$$

We now perform the tachyon map to determine the value of the tachyon as a function of ξ. Note that we expect this field to now have positive mass squared, since the potential has a minimum. Up to constants we find that

$$T(\xi) \sim \sqrt{\frac{k^2 l_s^2}{2(1-\cos(k\theta))}}\left(\frac{\xi}{R} - \frac{\xi^2}{2R^2} + \dots\right) \tag{4.49}$$

If we assume that the ξ^2 term is negligible then we can invert our solution and calculate the perturbed tachyon potential. Note that keeping higher order terms here does not lead to a simple analytic solution, and so we would hope that a numerical analysis would be more appropriate. After much manipulation we find

$$\begin{aligned} V(T) \quad &\sim \quad \frac{T_3}{kl_s}\left(2R^2(1-\cos(k\theta))\right)^{1/2}[1 + \frac{T}{2kl_s}\sqrt{2(1-\cos(k\theta))} + \\ & T^2\left(\frac{2+\cos(k\theta)}{6l_s^2} - \frac{(1-\cos(k\theta)\,)}{12k^2l_s^2}\right) + \dots] \end{aligned} \tag{4.50}$$

which shows that the potential is approximately linear around the minimum as it interpolates between the cosine and the exponential functions, however this linear term is suppressed by a factor of $1/k$ and we would expect it be negligible in the large k limit, thus we can see that there is an approximately *quadratic minimum*. We see that the minimum of the potential in the tachyonic direction will be

$$V(T(\xi=0)) = \frac{T_3 R}{kl_s}\sqrt{2(1-\cos(k\theta))} \tag{4.51}$$

which can obviously be made small in the large k limit, and will clearly go to zero when the $D3$-brane trajectory is such that it hits one of the $NS5$-branes. The local maximum will occur at the bisection angle $\theta = \pi/k$, which we suspect will be an unstable point. All this fits nicely with our earlier numerical analysis. Recall that Figure 4.4 showed the result of numerical methods used to plot the potential using the exact form of the ring harmonic function. Numerical solutions to the tachyon map inside and outside the ring were matched together to obtain this plot. The minimum can be seen to be quadratic for small distances before mapping onto an exponential function outside the ring as expected from the work in Chapter two. This is because the numerical analysis includes all the higher order correction terms, which produces a curved potential at the minimum.

The condensing tachyon field may oscillate about the minimum of this potential, assuming that the energy of the tachyon is such that it will not overshoot and return back up the potential toward $T = 0$. This assumption seems to be valid because as we have just seen the potential no longer has to vanish at $\rho = R$, so the friction term in (4.17) will not vanish as the tachyon condenses. However in order for reheating to occur we must ensure that this term sufficiently damps the motion, confining the field to very small oscillations about this minimum.

From standard inflationary models we know that the oscillations about the minimum can be thought of as being a condensate of zero momentum particles of $(\text{mass})^2 = \text{V}''(\text{T})$ [76]. The decay of the oscillations leads to the creation of new fields coupled to the tachyon condensate via the reheating mechanism [99]. The temperature of this reheating can be approximated as the difference between the maximum and minimum of the potential, and so we find

$$T_{RH}^4 \sim M_{infl}^4 \left(1 - \frac{1}{\sqrt{k}} \sqrt{2(1 - \cos(k\theta))} \right) \tag{4.52}$$

and so if we assume that the conversion of the tachyon energy is almost perfectly efficient then we will have an upper bound for the reheating temperature given by the effective inflation scale M_{infl}.

We must now consider the more general case where we perturb θ away from its bisection value of π/k. Since we are assuming that the $NS5$-branes are somehow resolvable we must also be aware that a single brane does not form an infinite throat. As such a passing probe brane will feel the gravitational effect of the fivebranes, but because we expect it to be moving relativistically we expect that its trajectory will only suffer a slight deflection. In this instance the perturbed harmonic function at $\rho = R$ reduces to

$$H \sim \frac{k^2 l_s^2}{2R^2} \frac{1}{(1 + \cos(k\delta))}, \tag{4.53}$$

where δ represents the angular perturbation. Now, we know that the harmonic function becomes singular when our probe brane hits a fivebrane so the function needs to be minimized to ensure a stable trajectory, This is clearly accomplished by sending $\delta \to 0$. So the value π/k corresponds to an unstable *maximum* from the viewpoint of the tachyon potential. Of course, we could also see this directly from (4.51) by considering perturbations about the bisection angle. For small angular deflection we may write

$$H \sim \frac{k^2 l_s^2}{4R^2} \left(1 + \frac{k^2 \delta^2}{4} + \ldots \right), \tag{4.54}$$

and so we see that provided $k\delta << 1$ the trajectory of the probe will be relatively unaffected by the presence of the fivebranes and therefore we may expect that it will pass between them. On the other hand, for larger values of $k\delta$, this will not be true and the probe brane may be pulled into the fivebranes. In any case we expect that the analysis of the Geometrical Tachyon will be invalid in this instance.

The analysis will also be true for a $D3$-brane in a ring $D5$-brane background, the only difference will be to replace

$$kl_s^2 \to 2g_s kl_s^2, \tag{4.55}$$

where g_s is the string coupling and we again consider k branes on the ring. The overall effect of switching to the $D5$-brane background is to allow for a weaker coupling at the top of the potential. In fact the analogue of (4.23) in this picture becomes

$$g_s >> \left(\frac{24\pi^3 v}{R\sqrt{2k}M_s}\right)^{2/3} \tag{4.56}$$

The situation is made slightly more complicated due to the presence of background RR charge, but this can be neglected when the tachyon is purely time dependent. Thus we would expect similar results to those obtained in the last two sections. Of course, we should remember that fundamental strings can end on the $D5$-branes and consequently there can be additional open string tachyons in the theory thus complicating the analysis. We will see a similar situation in a later section.

4.2.2 Cosmology in the plane transverse to the ring

Let us now consider the cosmology when the probe brane lies in the centre of the ring but is shifted a little from the plane. In this case the probe brane will exhibit transverse motion through the ring.

Recall that the tachyon map gave us the following solutions for the field and potential

$$\begin{aligned} T(\sigma) &= \int_0^\sigma \sqrt{H(\sigma')}d\sigma' \\ &= \sqrt{kl_s^2}\text{arcsinh}\left(\frac{\sigma}{R}\right) \\ V(T) &= \frac{T_3 R}{\sqrt{kl_s^2}}\cosh\left(\frac{T}{\sqrt{kl_s^2}}\right). \end{aligned} \tag{4.57}$$

Clearly we see that $T \to \pm\infty$ as $\sigma \to \pm\infty$, and that $T = 0$ at the minimum of the potential [9]. The shape of the potential suggests that the field is *massive*, with $(\text{mass})^2$ given by $1/kl_s^2$. However we will still refer to this as a Geometrical Tachyon in order to make its origins clear, One may ask if there is a known string mode exhibiting this profile. In fact the fluctuations of a massive scalar were computed in [82] using a similar approach to the construction of the open string tachyon mode in boundary conformal field theory. This field was then used in as a candidate for the inflaton living on a $\bar{D}3$-brane in the KKLT scenario [108]. The potential for the scalar is known to fourth order and was assumed to be exponential in profile.

[9] We must bear in mind that our approximation of the harmonic function prevents us from taking the $\sigma \to \infty$ limit.

We can now analyse our four dimensional minimally coupled action, where we find the same solutions to the Einstein equations as we did for the radial mode. We can now proceed with the analysis of our theory as in the previous section. It must be noted that this model corresponds to large field inflation, where the initial value of the scalar field must satisfy the following constraint

$$T_0 \ll \sqrt{kl_s^2}\text{arccosh}\left(\frac{\sqrt{kl_s^2}}{R}\right), \tag{4.58}$$

according to our truncation of the harmonic function. Using the slow-roll approximation, $H^2 \simeq V(T)/3M_p^2$ and $3H\dot{T} \simeq -V'/V$, the e-folding expression becomes

$$\begin{aligned} N &= \frac{T_3 R\sqrt{kl_s^2}}{M_p^2}\int_{x(T_f)}^{x(T)} \frac{\cosh^2 x}{\sinh x}dx \\ &= s\left[-\cosh(x_f) + \cosh(x) - \ln\left(\frac{\tanh(x_f/2)}{\tanh(x/2)}\right)\right]. \end{aligned} \tag{4.59}$$

Where we have introduced the dimensionless quantities $x = T/\sqrt{kl_s^2}$ and $s = T_3 R\sqrt{kl_s^2}/M_p^2$ for simplicity. Further defining the new quantity $y \equiv \cosh x$, we can write the number of e-folds as follows:

$$N = s\left[-y_f + y - \frac{1}{2}\ln\left(\frac{(y_f-1)(y+1)}{(y_f+1)(y-1)}\right)\right] \tag{4.60}$$

Now the relevant slow-roll parameter is more generally defined to be $\epsilon \equiv -\dot{H}/H$, which for our solution gives us

$$\epsilon = \frac{(y^2-1)}{2sy^3}. \tag{4.61}$$

Note that our model is explicitly non-supersymmetric and therefore we don't need to calculate the second slow roll parameter η since we anticipate that this will be trivially satisfied provided that ϵ is.

At the end of inflation $\epsilon = 1$, then $y_f \equiv f(s)$ is given by the root of above equation, setting $\epsilon = 1$

$$f(s) = \frac{1}{6s}\left[g(s) + \frac{1}{g(s)} + 1\right] \tag{4.62}$$

where we have defined $g(s) = \left(-54s^2 + 1 + 6s\sqrt{3(27s^2-1)}\right)^{1/3}$ for simplicity. From (4.60) the equation for y can be seen to satisfy

$$\ln\left(\frac{y+1}{y-1}\right) - 2y = -\frac{2N}{s} - 2f(s) - \ln\left(\frac{f(s)-1}{f(s)+1}\right)$$

Clearly for $s > 1$ and as $y_{\min} = 1$, ϵ always remains less than one leading to an ever accelerating universe. Thus in this case the Geometrical scalar field in the present setting is not suitable to describe inflation but could become a possible candidate for dark energy. However if T_3 is small enough so that $s < 1$, then we will find that inflation is possible as the slow roll parameter will naturally tend toward unity. In fact there is a critical bound $s \leq 1/(3\sqrt{3})$, which must be satisfied if we are to consider inflation in this context. In the context of the large k approximation, this constraint can be satisfied by again assuming a small string scale.

Inflationary Constraints.

To know the observational constraint on s we have to calculate the density perturbations. In the slow-roll approximation, the power spectrum of curvature perturbation is effectively given by [106]:

$$\begin{aligned} P_S &= \frac{1}{12\pi^2 M_p^6}\left(\frac{V^2}{V_T}\right)^2 \\ &= \frac{T_3^2 R^2}{12\pi^2 M_p^6}\left(\frac{\cosh^2(T/\sqrt{kl_s^2})}{\sinh(T/\sqrt{kl_s^2})}\right)^2 \end{aligned} \tag{4.63}$$

We will use the more stringent bound that $P_S \simeq 2 \times 10^{-9}$ for modes which crossed $N = 60$ e-foldings before the end of inflation [88–91], which gives the following constraint:

$$k(l_s M_p)^2 \simeq \frac{10^9}{12\pi^2}\frac{s^2\cosh^4(T/\sqrt{kl_s^2})}{\cosh^2(T/\sqrt{kl_s^2}) - 1} \tag{4.64}$$

From the numerics using (4.63), we find that

$$k(l_s M_p)^2 \geq 3 \times 10^{10} \tag{4.65}$$

which corresponds to a specified value of $s \sim 10^{-3}$ when we impose the constraints $T_3 = 10^{-10} M_p^4$ and $R = 10^2/M_p$ which we regard as being typical values. The constraint on the tension in fact implies the following relationship

$$\frac{M_p}{M_s} \sim \frac{10^2}{g_s^{1/4}}, \tag{4.66}$$

which we need to be consistently satisfied. However note that because of our basic assumptions about the theory, we will generally obtain the bound

$$\frac{T_3 R}{M_p^3} \leq \frac{1}{9 \times 10^5}. \tag{4.67}$$

If we write the tension of the brane in terms of fundamental parameters we can estimate the relationship between the string and Planck scales using the fact that we require $R > M_s^{-1}$

for the action to be valid

$$\frac{M_p}{M_s} \geq \frac{15}{g_s^{1/4}}, \tag{4.68}$$

where g_s is the string coupling constant. Note that a smaller string scale allows this constraint to be satisfied for a larger range of string couplings.

If we carefully consider the limiting case where s is small, making sure our effective action remains valid, then we may write

$$k(l_s M_p^2)^2 \simeq \frac{10^9}{48\pi^2}(2N+1)^2 \tag{4.69}$$

which corresponds to $s \sim 10^{-5}(2N+1)$ and $y \sim \frac{(2N+1)}{2s}$, when we choose $T_3 = 10^{-10}M_p^4$ and $R = 10^2/M_p$. More generally, however, we find the following upper limit on the solution

$$s \leq 10^{-3}(2N+1), \tag{4.70}$$

which is easily satisfied by our typical values. Having shown that the scalar amplitude can be normalised to the data, we must check that the spectral index of the amplitude is also satisfied [10]. The scalar index can be defined as follows [76, 106]

$$\begin{aligned} n_S - 1 &\equiv -4\frac{M_p^2 V_T^2}{V^3} + 2\frac{M_p^2 V_{TT}}{V^2} \\ &= \frac{2}{s}\left(\frac{2-y^2}{y^3}\right) \end{aligned} \tag{4.71}$$

Whilst the spectral index for the tensor perturbations is defined as:

$$\begin{aligned} n_T &= -\frac{M_p^2 V_T}{V^3} \\ &= -\frac{1}{s}\left(\frac{y^2-1}{y^3}\right) \end{aligned} \tag{4.72}$$

And the tensor-to-scalar ratio is:

$$\begin{aligned} r &\equiv 8\frac{M_p^2 V_T^2}{V^3} \\ &= \frac{8}{s}\left(\frac{y^2-1}{y^3}\right) \end{aligned} \tag{4.73}$$

With the asymptotic limit of $s \to 0$ we find

$$n_S = 1 - \tfrac{4}{(2N+1)}, \quad n_T = -\tfrac{2}{(2N+1)}, \quad r = \tfrac{16}{(2N+1)} \tag{4.74}$$

For $N = 60$ we get $n_S = 0.96694$ and $r = 0.13223$; for $N = 50$, we get $n_S = 0.96040$ and

[10] See Appendix for a more detailed description.

$r = 0.15842$. We know from observations that the constraint on the tensor-to-scalar ratio is $r < 0.24$ at the 95% confidence level, and so our model appears to be well within this bound [88–91]. Moreover the scalar spectral index is localised around $n_s \sim 0.96$, putting it at the lower range when we consider the WMAP three-year data set. However our assumptions here have relied heavily on the fact that s is small. In general this will not be the case, and so we will find an ever accelerating universe. We turn to this now, as it may correspond to a model for dark energy.

Dark energy

What are the implications of our model for dark energy [77, 90]? It is well known that the non-linear form of the DBI action admits an unusual equation of state, which is of the form

$$\begin{aligned} \omega &= \frac{P}{\rho} \\ &= \dot{T}^2 - 1 \end{aligned} \tag{4.75}$$

where P and ρ are the pressure and energy densities respectively. In tachyon models the field is moving relativistically near the vacuum and the equation of state will tend to $\omega \sim 0$, which is problematic for reheating. However our model has significantly different late time behaviour because our scalar field will oscillate about the minimum of its potential, eventually coming to a halt at the minimum. This assumption is based upon the fact that in the ten dimensional picture we know that the probe brane carries RR charge, which will be radiated during its oscillatory cycle. The explicit analysis of this emission remains to be calculated, however we expect that the corresponding Geometrical Tachyon behaviour would be that of a damped harmonic oscillator Therefore we expect the equation of state to become $\omega \sim -1$, corresponding to the vacuum energy of the universe. This motivates us to analyse our system as a potential candidate for dark energy. The corresponding evolution equations of interest are:

$$\begin{aligned} \frac{\ddot{T}}{1-\dot{T}^2} + 3H\dot{T} + \frac{V_T}{V} &= 0 \\ \dot{H} + \frac{V(T)\dot{T}^2}{2M_p^2\sqrt{1-\dot{T}^2}} + \frac{\gamma\rho_B}{2M_p^2} &= 0 \end{aligned} \tag{4.76}$$

where we have included the contribution from a barotropic fluid in the second equation [80, 82]. Defining the following dimensionless quantities:

$$\begin{aligned} Y_1 &= \frac{T}{\sqrt{kl_s^2}} \\ Y_2 &= \dot{T}, \end{aligned} \tag{4.77}$$

and combining the last two sets of equations we find the following

$$\begin{aligned} Y_1' &= \frac{1}{\sqrt{kl_s^2}H} Y_2 \\ Y_2' &= -\left(1-Y_2^2\right)\left(3Y_2 + \frac{1}{H}\frac{dY_3}{dY_1}\right) \end{aligned} \tag{4.78}$$

Where we have switched to using the number of e-folds as the time parameter, and now primes denote derivatives with respect to N. The final expressions we require can be read off as follows

$$\begin{aligned} Y_3 &= \ln\left(\frac{V(T)}{3M_p^2}\right) \\ H^2 &= \frac{e^{Y_3}}{\sqrt{1-Y_2^2}} + \frac{\rho_B}{3M_p^2}. \end{aligned} \tag{4.79}$$

Simple analysis shows us that critical points are at $Y_1 = 0$ and $Y_2 = 0$ which is a global attractor as can be seen in Fig 4.5. This agrees with our physical intuition since it implies the probe brane will slow down, eventually coming to rest at the origin of the transverse space. In terms of our critical ratios we find

$$\begin{aligned} \Omega_T &= \frac{e^{Y_3}}{e^{Y_3} + \frac{\rho_B}{3M_p^2}\sqrt{1-Y_2^2}} \\ \Omega_B &= \frac{\rho_B}{\frac{3M_p^2 e^{Y_3}}{\sqrt{1-Y_2^2}} + \rho_B} \end{aligned} \tag{4.80}$$

Note that they are constrained by $\Omega_T + \Omega_B = 1$. We also have $\Omega_B = \Omega_M + \Omega_R$, where M and R denote matter and radiation respectively. From the plots in Fig 4.6 we see that the Ω_T goes to 0.7, Ω_M goes to 0.3 and Ω_R goes to 0 in the present epoch.

We see that at late times, the field settles at the potential minimum leading to de-Sitter solution with energy scale $V_0 = T_3 R/\sqrt{kl_s^2}$. Using the numerical data from the preceding sections we can write this an upper bound on the energy density as follows

$$V_0 \leq 10^{-12} M_p^4. \tag{4.81}$$

Although this is several orders of magnitude higher than the observed value [90], we note that this value is heavily dependent on the scales in the theory, and with appropriate tuning could be substantially smaller. Since there exists no realistic scaling solution (which could mimic matter/radiation), the model also requires fine tuning of the initial value of the field. It should remain sub-dominant for most of the cosmic evolution and become comparable to the background at late times. It would then evolve to dominate the background energy density ultimately settling down in the de-Sitter phase.

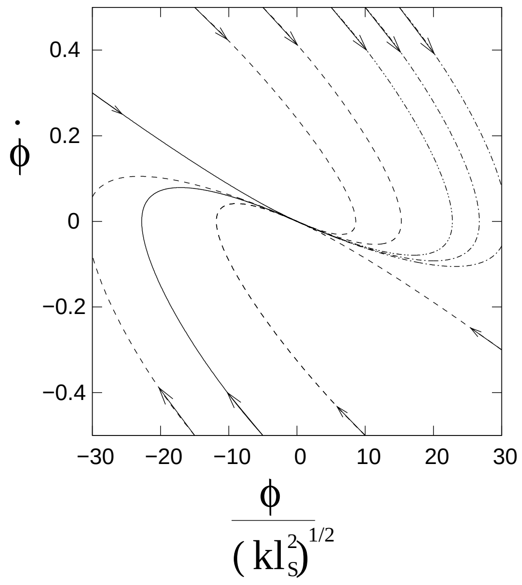

Figure 4.5: Plot of the phase space solution for the Geometrical Tachyon (here denoted ϕ) with a variety of initial conditions. Here we see the presence of global attractor at $(\phi = 0, \dot{\phi} = 0)$

However recall from the bulk picture that the point $\sigma = 0, \rho = 0$ will be gravitationally unstable and the probe brane will eventually be attracted toward the ring. In terms of our cosmological theory we see that this de-Sitter point will actually be only metastable and that a tachyonic field will eventually condense forcing the vacuum energy down toward zero. This suggests that the vacuum energy will not be constant, but will be slowly varying. Furthermore our equation of state should be modified to incorporate the dynamics of this additional field. It is trivial to see that the inflationary phase will terminate and give way to a dark energy phase where $\omega \sim -1$. Once the tachyon field starts to roll, ω will increase toward zero from below giving rise to a phase of quintessence. Eventually we will begin to probe the strong coupling regime and our effective action will break down unless we can fine tune the trajectory of the probe brane.

Let us return to the geometric picture to understand this in more detail. We introduce a complex field $\xi = \rho + i\sigma$ which can actually be globally defined in the target space. The harmonic function factorises in this coordinate system into holomorphic and anti-holomorphic parts $F(\xi, \bar{\xi}) = f(\xi)f(\bar{\xi})$. Thus the tachyon map will also split accordingly

$$\partial_t T = f(\xi)\partial_t \xi, \qquad \partial_t \bar{T} = f(\bar{\xi})\partial_t \bar{\xi}. \tag{4.82}$$

These expressions are exactly solvable provided we continue them into the complex plane. If we now re-construct the potential for these fields in terms of our holographic theory we

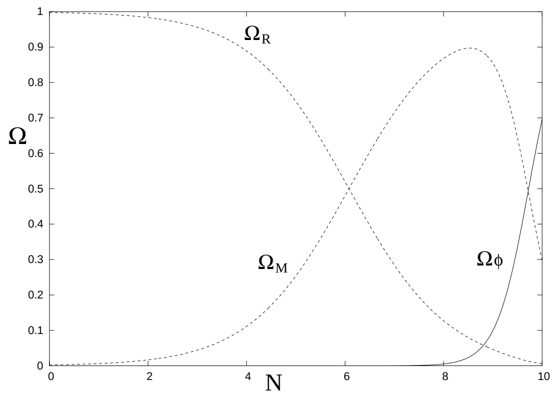

Figure 4.6: Here we have taken $\rho_m^0 = 4.58 \times 10^6$, $\rho_R^0 = 10^{10}$ and $V_0 = 10^{-6}$. The dark line is for Ω_R, dotted line is for Ω_ϕ and light line is for Ω_m

obtain the general solution

$$V(T,\bar{T}) = \frac{RT_3}{\sqrt{kl_s^2}} \left[\cos\left(\frac{T}{\sqrt{kl_s^2}} \right) \cos\left(\frac{\bar{T}}{kl_s^2} \right) \right]^{1/2}. \tag{4.83}$$

Clearly when T is real we recover our *cosine* potential, whilst if it is purely imaginary we recover the *cosh* solution. These correspond to motion inside the ring and motion transverse to the ring respectively. The tachyonic instability forces the field from the false vacuum state toward the true ground state. Therefore we expect the general dark energy potential to be

$$V(T,\bar{T}) \sim \frac{RT_3}{\sqrt{kl_s^2}} \cos\left(\frac{T}{\sqrt{kl_s^2}} \right), \tag{4.84}$$

and so the true minimum will occur when $V \sim 0$ at $T = \pm\pi\sqrt{kl_s^2}/2$ corresponding to the location of the ring in the bulk picture. The cosmological dynamics in this particular phase are well described by the results presented earlier (and also in [83]), where it was shown to be possible for the true vacuum to be non-zero, provided the trajectory of the probe brane is sufficiently fine tuned.

We finally comment on the instability for the field fluctuations for a potential with a minimum. In a flat FRW background each Fourier mode of T satisfies the following equation [80]

$$\frac{\delta\ddot{T}_{\tilde{k}}}{1-\dot{T}^2} + \left[3H + \frac{2\dot{T}\ddot{T}}{(1-\dot{T}^2)^2} \right] \delta\dot{T}_{\tilde{k}} + \Big[\frac{\tilde{k}^2}{a^2} + (\ln V)_{T,T} \Big] \delta T_{\tilde{k}} = 0 \tag{4.85}$$

Where $\tilde{k}$ is the comoving wavenumber. We can now compute the second derivatives of the potential and obtain

$$(\ln V)_{T,T} = \frac{1}{kl_s^2}\left(1 - \tanh\left[\frac{T}{\sqrt{kl_s^2}}\right]\right). \tag{4.86}$$

Here we see that $(\ln V)_{T,T}$ is never divergent for any value of T, and is always non-negative i.e that $(\ln V)_{T,T} \in [0,1]$. Thus we do not have any instability associated with the perturbation $\delta\phi_k$ with our potential. This is to be contrasted with the result obtained for the open string tachyon, which undergoes rapid fluctuations and instabilities during its evolution.

4.3 $D3$-brane dynamics in the $D5$-brane background

Using the fact that the $NS5$-brane is S-dual to the $D5$-brane, we could also consider inflationary solutions emerging in $D5$-brane backgrounds using the Geometrical Tachyon. We would anticipate that this gives rise to hybrid inflation because the fundamental strings stretched between the two different branes will become tachyonic at very late times. Therefore at early times we can treat this mode as being the inflaton, and inflation will end once this mode becomes tachyonic.

For simplicity we will consider the background to be generated by a stack of coincident and static $D5$-branes, rather than the $D5$-ring solution. The important difference between this solution and the $NS5$-brane solution is that there exists a non-zero RR charge, and that the solution is weakly coupled when we are deep in the throat geometry. The background fields, namely the metric, the dilaton ϕ and the RR field $C_{(6)}$ for a system of k coincident D5-branes are given by [10, 11, 23]

$$\begin{aligned} g_{\alpha\beta} &= F^{-1/2}\eta_{\mu\nu}, \quad g_{mn} = F^{1/2}\delta_{mn}, \\ e^{2\phi} &= F^{-1} = C_{0...5}, \quad F = 1 + \frac{kg_s l_s^2}{r^2}, \end{aligned} \tag{4.87}$$

where $\mu, \nu = 0,..,5;\ m,n = 6,...,9$ denote the indices for the world volume and the transverse directions respectively and F is now the harmonic function describing the position of the k D5-branes and satisfying the Green function equation in the transverse four dimensional space with $SO(4)$ symmetry.

The DBI action for the $D3$-brane in this background can be written as

$$\mathcal{S}_0 = -T_3 \int d^4\xi F^{-1/2}\sqrt{1 + F\partial_\alpha R \partial^\alpha R}, \tag{4.88}$$

where T_3 is again the tension of the 3-brane. Here the motion of the probe brane is restricted to be purely radial, denoted by the mode R, along the common four dimensional transverse space. We now map this action to that of a Geometrical Tachyon field, T, through the usual definition of the tachyon map

$$\frac{\mathrm{d}T}{\mathrm{d}R} = \sqrt{F(R)} = \sqrt{1 + L^2/R^2}\,, \tag{4.89}$$

where

$$L \equiv \sqrt{kg_s} l_s\,.$$

In terms of this field the potential for such a Geometrical Tachyon is given by

$$V \to \frac{T_3}{\sqrt{F(R)}} = \frac{T_3}{\sqrt{1 + L^2/R^2}}\,. \tag{4.90}$$

We have kept the full form of the harmonic function in order to show that this mapping is exactly solvable. In fact using (4.89) we can solve for $T(R)$ and find it to be a monotonically increasing function:

$$T(R) = \sqrt{L^2 + R^2} + \frac{L}{2}\ln\left(\frac{\sqrt{L^2+R^2} - L}{\sqrt{L^2+R^2} + L}\right) \tag{4.91}$$

This function is non-invertible but can be simplified by exploring limits of the field space solution. As $R \to 0$ we have $T(R) \to -\infty$ with dependence

$$T(R \to 0) \simeq L \ln\left(\frac{R}{L}\right)\,. \tag{4.92}$$

Whilst as $R \to \infty$ we have $T(R) \to \infty$with

$$T(R \to \infty) \simeq R\,. \tag{4.93}$$

The effective potential in these two asymptotic regions is given by:

$$\begin{aligned} \frac{V(T)}{T_3} &\simeq \exp\left(\frac{T}{L}\right) \qquad \text{for} \quad T \to -\infty\,, \\ &\simeq 1 - \frac{1}{2}\frac{L^2}{T^2} \qquad \text{for} \quad T \to +\infty\,. \end{aligned} \tag{4.94}$$

Thus in the limit $T \to -\infty$, corresponding to $R \to 0$, one observes that the potential goes to zero exponentially as shown in Fig 4.7. This is consistent with the late time behavior for the open string tachyon potential in the rolling tachyon solutions and leads to exponential decrease of the pressure at late times. At large distances the DBI action interpolates smoothly between standard gravitational attraction among the probe and the background branes and a "radion matter" phase when the probe brane is close to the five branes. The transition between the two behaviors occurs at $R \sim L$.

It is important to note that when the probe brane is within the distance $R \sim l_s$, the above description in terms of the closed string background is inappropriate and the system

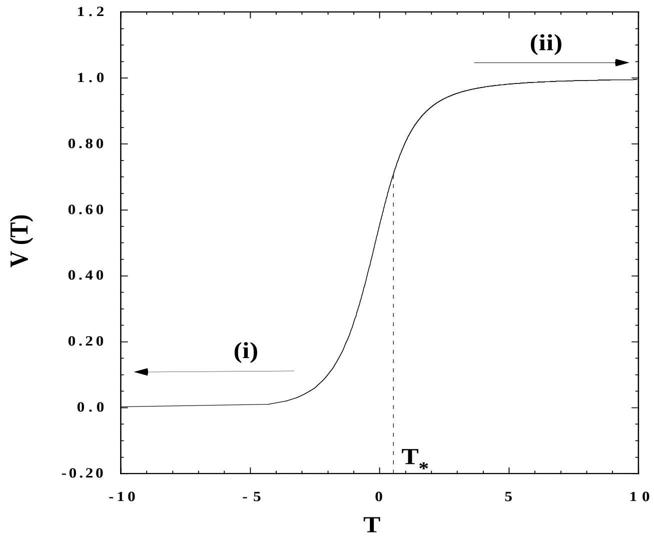

Figure 4.7: The potential of the field T. The value $T_* = [\sqrt{2} + \ln(\sqrt{2}-1)]L$ is determined by the condition $R = L$. The potential of the region (i) is approximately given by $V(T) = T_3 \exp(T/L)$, whereas $V(T) = T_3(1 - L^2/2T^2)$ in the region (ii).

should be studied using open strings stretched between the probe brane and the five branes. To be more precise, when the probe brane comes to within a distance l_s from the D5-branes, a tachyon appears in the open string spectrum and in principle the dynamics of the system will be governed by its condensation from that point on.

Thus the full dynamics can be divided into two regimes. When the distance R between the D3-brane and the D5-branes is much smaller than L but larger than l_s, we can describe the dynamics of the radial mode $R(x^\mu)$ with the non-BPS DBI action [31] with an exponentially decaying potential given by (4.94) (note that T is going toward $-\infty$). On the other hand when R is of the order of l_s, the dynamics would be be governed by the conventional Lagrangian describing the complex tachyonic scalar field χ present in the open string stretched between the D3-brane and the k D5-branes. The potential for such open string tachyon field has already been calculated and we will simply state its form without comment. Thus the dynamics of χ is described by the canonical action:

$$\mathcal{S}_2 = \int \mathrm{d}^4x[-\partial_\alpha\chi\partial^\alpha\chi^* - U(\chi,\chi^*)]\,, \tag{4.95}$$

where the potential, up to quartic order, is given by [84]:

$$U(\chi,\chi^*) = \frac{1}{4\pi^4 l_s^4 g_s k}\left[\pi(k+1)(\chi\chi^*)^2 - v\chi\chi^*\right]\,. \tag{4.96}$$

Note that χ and v are dimensionless quantities. Here v is a small parameter ($v \ll k$) corresponding to the volume of a two-torus. This arises as we are toroidally compactifying the directions transverse to the D3-brane, but parallel to the D5-branes, in order to describe the dynamics of the open string tachyon.

When we compactify our Geometrical Tachyon solution, we will neglect any string winding modes arising from this torus. Furthermore it can be seen that our fully compactified theory is actually not T^6 but the product space $T^4 \times T^2$, but for simplicity we shall assume that the relevant radii are approximately equal [80].

Let us briefly recapitulate and consider the bulk dynamics in more detail. At distances larger than the string length we know that the DBI action provides a good description of the low energy physics for a probe brane in the background geometry. As mentioned already several times, the probe brane is much lighter than the coincident D5-branes and so we can neglect the back reaction upon the geometry. Furthermore the SUGRA solution indicates that the string coupling tends to be zero as we probe smaller distances, providing a suitable background for perturbative string theory and implying that we can trust our description down to small distances without requiring a bound on the energy.

Because of the dimensionalities of the branes in the problem there is no coupling of the D3-brane to the bulk RR six form. This is because the only possible Chern-Simons interaction between the probe brane and the background can be through the self dual field strength $\tilde{F} = \mathrm{d}\tilde{C}_{(4)}$. However this field strength *must* be the Hodge dual of the background field strength - which is given here by $F = \mathrm{d}C_{(6)}$ for $D5$-branes - clearly this inconsistency implies that the coupling must vanish. For a more detailed explanation of the more general case we refer the reader to [23], however the basic result for our purpose is that there is only a non-zero interaction term when either the dimensionality of probe and background branes are the same, or they add up to six.

The energy-momentum tensor density of the probe brane in the background can be determined by variation of the DBI action

$$T_{ab} = \frac{T_3}{\sqrt{F}}\left(\frac{F\partial_a R \partial_b R}{\sqrt{1+F\eta^{cd}\partial_c R\partial_d R}} - \eta_{ab}\sqrt{1+F\eta^{cd}\partial_c R\partial_d R}\right), \tag{4.97}$$

where the roman indices are directions on the world-volume. Since we are only interested in homogeneous scalar fields, we find that this expression simplifies to

$$\begin{aligned} T_{00} &= \frac{T_3}{\sqrt{F}\sqrt{1-F\dot{R}^2}}, \\ T_{ij} &= -\frac{T_3\delta_{ij}\sqrt{1-F\dot{R}^2}}{\sqrt{F}}, \end{aligned} \tag{4.98}$$

where i, j are now the spatial directions on the D3-brane.

Using the energy conservation we can obtain the equation of motion for the probe brane in our background and estimate its velocity. By imposing the initial condition that the velocity is zero at the point $R = R_0$ we find that the expression for the velocity reduces to

$$\dot{R}^2 = \frac{R^2 L^2}{(R^2 + L^2)^2} \left(1 - \frac{R^2}{R_0^2}\right), \tag{4.99}$$

which is obviously valid for $R \leq R_0$ and in fact as expected it vanishes identically at $R = R_0$. We typically would expect R_0 to be extremely large. Note that in the two asymptotic regions of small and large R the velocity is tending to zero. This is understood because the throat geometry acts as a gravitational red-shift, giving rise to "D-cceleration phenomenon" [92]. It should be emphasised that the asymptotic limit $R \to 0$ is unphysical because the DBI is not valid once we reach energies of the order of string mass M_s, and so it is not strictly correct to say that the velocity goes to zero in the small R approximation. However note that when $R \to l_s$ we have $\dot{R}^2 \sim l_s^2/L^2 = 1/kg_s$ which is also negligibly small for large k. From our perspective this implies that the kinetic energy of the scalar field become sub-dominant at small distances. It is essentially frozen out and the dynamics of the open string tachyonic modes come to dominate. Once the probe brane reaches distances comparable with the string length our closed string description is no longer valid. Instead we must switch over to an open string description of the tachyonic modes χ described by the action.

It is worth pointing out that our discussion so far seems to suggest that the radionic mode and the open string tachyonic mode which are being described by two different action functionals have nothing in common, and can be described independent of each other. This is not strictly true. Firstly the number of background branes have to be same. Secondly, unlike the open string tachyon on the world volume of a non-BPS brane or a brane/anti-brane pair, the dynamics of the tachyon on the open string connecting a BPS Dp-brane and a BPS D$(p+2)$-brane is not described by a DBI type action. If this was the case, the above two fields could have been combined together, keeping in mind their individual regimes of validity.

However even in the present context we can combine the two actions by introducing an interaction term like $\lambda T^2 \chi^2$, where the coupling λ will be zero for values of the field T corresponding to R greater than l_s. Provided that inflation ends for $R > l_s$, this term does not affect the dynamics of inflation and for simplicity we have ignored it in the action functional. However such a term may play an important role in a possible reheating phase. We can now proceed with our analysis of inflation using the full form of the harmonic function - which specifies the scalar field potential in terms of the Geometrical Tachyon field rather than the radion field.

Inflation and observational constraints from CMB

Let us introduce the dimensionless quantity $x \equiv R/L$, the full potential (4.90) of the field T can be written as

$$V = \frac{xT_3}{\sqrt{1+x^2}}\,, \tag{4.100}$$

where $\tilde{T} \equiv T/L$ is related to x via

$$\frac{\tilde{T}}{x} = \frac{\sqrt{1+x^2}}{x} = \frac{1}{\tilde{V}}\,, \tag{4.101}$$

and $\tilde{V} \equiv V/T_3$. For the action to be valid we require R to be larger than l_s, which translates into the condition $x > 1/\sqrt{kg_s}$. In the flat FRW background, the field equations are once again

$$\begin{aligned} H^2 &= \frac{1}{3M_p^2}\frac{V(T)}{\sqrt{1-\dot{T}^2}}\,, \\ \frac{\ddot{T}}{1-\dot{T}^2} &+ 3H\dot{T} + \frac{V_T}{V} = 0\,, \end{aligned} \tag{4.102}$$

where $V_T \equiv \mathrm{d}V/\mathrm{d}T$ as usual. Combining both terms in (4.102) gives us the relation $\dot{H}/H^2 = -3\dot{T}^2/2$, and then the slow-roll parameter is given by

$$\begin{aligned} \epsilon &\equiv -\frac{\dot{H}}{H^2} = \frac{3}{2}\dot{T}^2 \simeq \frac{M_p^2}{2}\frac{V_T^2}{V^3} \\ &= \frac{1}{2s}\frac{\tilde{V}_x^2}{\tilde{V}} = \frac{1}{2s}\frac{1}{x(1+x^2)^{5/2}}\,, \end{aligned} \tag{4.103}$$

where the parameter s is defined by

$$s \equiv \frac{L^2 T_3}{M_p^2}\,.$$

In deriving the slow-roll parameter we used the usual slow-roll approximations $\dot{T}^2 \ll 1$ and $|\ddot{T}| \ll 3H|\dot{T}|$ in (4.102). Furthermore (4.103) shows that ϵ is a decreasing function as x increases, thus ϵ increases as the field evolves from the large R region to the small R region, marking the end of inflation at $\epsilon = 1$.

The number of e-foldings from the end of inflation is given by

$$\begin{aligned} N &\equiv \int_t^{t_f} H\mathrm{d}t \simeq \int_{T_f}^{T} \frac{V^2}{M_p^2 V_T}\mathrm{d}T \\ &= s\int_{x_f}^{x} (x^2+1)^{3/2}\mathrm{d}x\,. \end{aligned} \tag{4.104}$$

Which integrates to give

$$N = s[f(x) - f(x_f)]\,, \tag{4.105}$$

where we have defined

$$\begin{aligned} f(x) &= \frac{1}{4}x(x^2+1)^{3/2} + \frac{3}{8}x\sqrt{x^2+1} \\ &\quad + \frac{3}{8}\ln\left|x + \sqrt{x^2+1}\right|\,. \end{aligned} \tag{4.106}$$

The function $f(x)$ grows monotonically from $f(0) = 0$ to $f(\infty) = \infty$ with the increase of x. In principle we can obtain a sufficient amount of inflation to satisfy $N > 60$ if either s or x is large.

In order to test the robustness of our model with observations we need to consider the spectra of scalar and tensor perturbations that are generated during inflation [76, 106]. The power spectrum of scalar metric perturbations in this case is given by

$$\begin{aligned} \mathcal{P}_{\rm S} &= \frac{1}{12\pi^2 M_p^6}\left(\frac{V^2}{V_T}\right)^2 = \frac{T_3^2 L^2}{12\pi^2 M_p^6}\left(\frac{\tilde{V}}{\tilde{V}_x}\right)^2 \\ &= \frac{s^2}{12\pi^2 k g_s (l_s M_p)^2} x^2 (x^2+1)^2\,. \end{aligned} \tag{4.107}$$

The COBE normalization [88–91] corresponds to $\mathcal{P}_{\rm S} = 2 \times 10^{-9}$ around $N = 60$, which gives the condition

$$k g_s (l_s M_p)^2 = \frac{10^9}{24\pi^2} s^2 x_{60}^2 (x_{60}^2 + 1)^2\,. \tag{4.108}$$

where x_{60} is the value of x evaluated at $N = 60$. The spectral index of curvature perturbations is given by

$$\begin{aligned} n_{\rm S} - 1 &= -4\frac{M_p^2 V_T^2}{V^3} + 2\frac{M_p^2 V_{TT}}{V^2} \\ &= -\frac{2}{s}\frac{1+3x^2}{x(1+x^2)^{5/2}}\,, \end{aligned} \tag{4.109}$$

whereas the ratio of tensor to scalar perturbations is

$$r = 8\frac{V_T^2 M_p^2}{V^3} = \frac{8}{s}\frac{1}{x(x^2+1)^{5/2}}\,. \tag{4.110}$$

We shall study the case in which the end of inflation corresponds to the region with an exponential potential, i.e $x_f \ll 1$. When $s = 1$ we see that (4.103) shows that inflation ends around $x_f \sim 0.5$. Hence the approximation $x_f \ll 1$ is valid when s is larger than of order

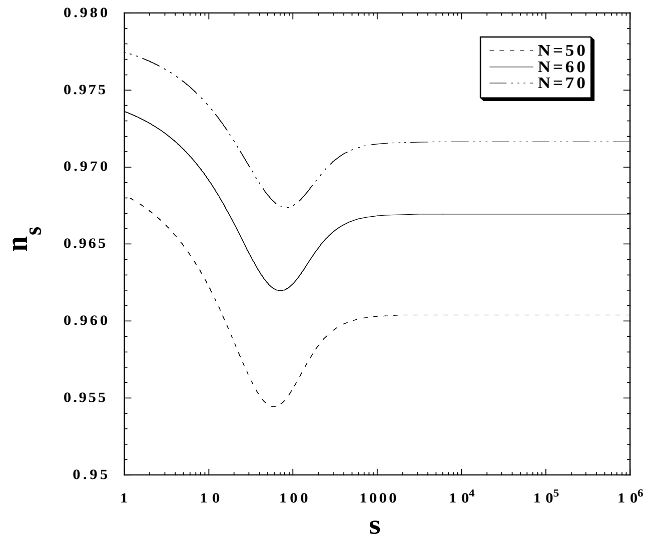

Figure 4.8: The spectral index n_S of scalar metric perturbations as a function of s with three different number of e-foldings ($N = 50, 60, 70$). This figure corresponds to the case in which inflation ends in the region $x_f \ll 1$.

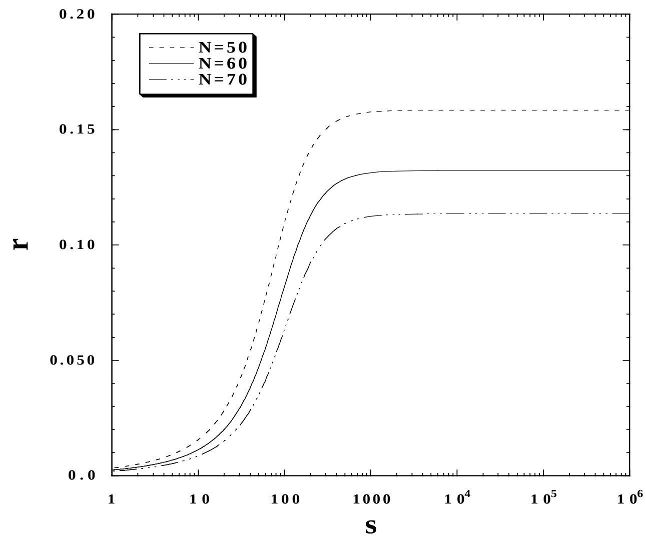

Figure 4.9: The tensor-to-scalar ratio r as a function of s with three different number of e-foldings ($N = 50, 60, 70$).

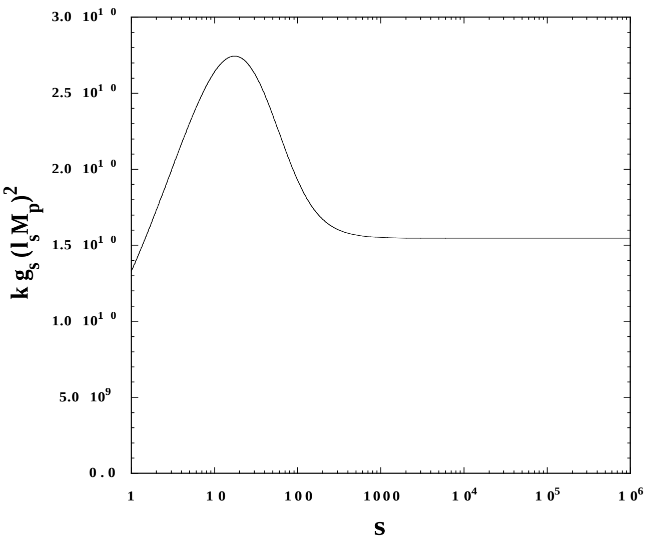

Figure 4.10: The quantity $kg_s(l_sM_p)^2$ as a function of s. This is derived by the COBE normalization at $N = 60$.

unity. In this case one has $x_f \simeq 1/2s$ from (4.103). Since $f(x) \simeq x$ for $x \ll 1$, we find

$$f(x) = \frac{1}{s}\left(N + \frac{1}{2}\right) . \tag{4.111}$$

Now in the regime of an exponential potential ($x \ll 1$) we see that this becomes $sx \simeq N + 1/2$. and (4.109, 4.110) give

$$\begin{aligned} n_{\rm S} - 1 &= -\frac{4}{2N+1} , \\ r &= \frac{16}{2N+1} . \end{aligned} \tag{4.112}$$

Hence $n_{\rm S}$ and r are dependent only on the number of e-foldings. Using (4.112) we find that the normalisation implies $n_{\rm S} = 0.9669$ and $r = 0.1322$ for $N = 60$. It was shown in that this case is well inside the 1σ contour bound coming from the observational constraints of WMAP, SDSS and 2dF [88–91].

Of course there is a situation in which cosmologically relevant scales ($55 \lesssim N \lesssim 65$) correspond to the region $x \gtrsim 1$. In Figs 4.8 and 4.9 we plot $n_{\rm S}$ and r as a function of s for three different values of N. For large s i.e $s \gg 1$, we find that the quantity x is much smaller than unity from the relation (4.111). Hence $n_{\rm S}$ and r are given by the formulas in (4.112). For smaller s the quantity x becomes larger than of order unity, which means that the results in (4.112) are no longer valid. From Fig 4.8 we see that the spectral index has a minimum around $s = 70$ for $N = 60$. This roughly corresponds to the region $x = R/L \sim 1$. As we see from Fig. 4.7, the potential becomes flatter for $x \gtrsim 1$ which leads to the increase of the spectral index toward $n_{\rm S} = 1$ with the decrease of s. Recent observations show that

$n_{\rm S} = 0.98 \pm 0.02$ at the 95% confidence level. As we find in Fig. 4.8 this condition is satisfied for $N \gtrsim 60$.

The tensor-to-scalar ratio given by (4.112) is valid for $s \gg 1$. For a fixed value of N this ratio gets smaller with the decrease of s. This is understandable since the potential becomes flatter as we enter the region $x \gtrsim 1$. The tensor-to-scalar ratio is constrained to be $r < 0.36$ at the 95% confidence level from recent observations. Hence our model satisfies this observational constraint.

When $x_{60} \ll 1$ the condition of the COBE normalization gives

$$kg_s(l_s M_p)^2 \simeq \frac{10^9}{24\pi^2}(60 + \frac{1}{2})^2 \simeq 1.55 \times 10^{10}\,, \tag{4.113}$$

which is independent of s. As we see from Fig. 4.10 the quantity $kg_s(l_s M_p)^2$ departs from the value (4.113) for smaller s. However $kg_s(l_s M_p)^2$ is of order 10^{10} for $s \gtrsim 1$. It is interesting to note that the COBE normalization uniquely fixes the value of the potential at the end of inflation, if it happens in the regime of an exponential potential, independently of the value where inflation started. In fact using (4.100) we see that

$$V_{\rm end} \simeq x_f T_3 = \frac{1}{2kg_s(l_s M_p)^2} M_p^4 \simeq 3.2 \times 10^{-11} M_p^4\,. \tag{4.114}$$

This sets the energy scale to be $V_{\rm end}^{1/4} \simeq 2.3 \times 10^{-3} M_p$.

The above discussion corresponds to the case in which inflation ends in the region $x_f \ll 1$. In order to understand the behavior of another asymptotic region let us consider a situation when inflation ends for $x_f \gg 1$. In this case the end of inflation is characterized by $x_f^6 \simeq 1/(2s)$. Since $x_f \gg 1$, we are considering a parameter range $s \ll 1$. When $x \gg 1$ the function $f(x)$ behaves as $f(x) \simeq x^4/4$, which gives the relation $x^4 \simeq 4N/s$. Hence we obtain the following

$$\begin{aligned} n_{\rm S} - 1 &= -\frac{3}{2N}\,, \\ r &= \frac{\sqrt{s}}{N^{3/2}}\,, \\ kg_s(l_s M_p)^2 &= \frac{10^9\sqrt{2}N^{3/2}}{6\pi^2}\sqrt{s}\,. \end{aligned} \tag{4.115}$$

While $n_{\rm S}$ is independent of s, both r and $kg_s(l_s M_p)^2$ are dependent on s and N. For example one has $n_{\rm S} = 0.975$, $r = 0.003\sqrt{s}$ and $kg_s(l_s M_p)^2 = 1.11 \times 10^{10}\sqrt{s}$ for $N = 60$. From Fig. 4.8 we find that $n_{\rm S}$ increases with the decrease of s in the region $1 \lesssim s \lesssim 50$ for a fixed value of N. This tendency persists for $s \lesssim 1$ and $n_{\rm S}$ approaches a constant value given by (4.115) as s decreases. We note that the spectral index $n_{\rm S}$ satisfies the observational constraint coming from recent observations. The tensor-to-scalar ratio is strongly suppressed in the region $s < 1$, which also satisfies the observational constraint, and the quantity $kg_s(l_sM_p)^2$ gets smaller with decreasing s.

We can estimate the the potential energy at the end of inflation in this regime via

$$\begin{aligned} V_{\text{end}} &\simeq T_3\,, \\ T_3 &= \frac{sM_p^4}{kg_s(l_sM_p)^2} \simeq 9.0\times 10^{-11}\sqrt{s}M_p^4\,. \end{aligned} \tag{4.116}$$

In this case V_{end} depends on the value of s. The order of the energy scale does not differ from (4.114) provided that s is not too much smaller than unity.

In summary we find that n_{S} and r in our model satisfy the CMB constraints for any values of s, which means that s is unconstrained. This is different from the case in the previous section, in which the spectral index n_{S} provides constraints on the model parameters. The only constraint in our model is the COBE normalization. If we demand that the value of R at the end of inflation is larger than l_s, this gives

$$k < 16\pi^6 g_s\left(\frac{M_p}{M_s}\right)^4\,, \tag{4.117}$$

where we used the standard form of the brane tension, $T_3 = M_s^4/(2\pi)^3 g_s$.

Combining this relation with the condition of the COBE normalization: $kg_s(l_sM_p)^2 \simeq 10^{10}$ for $s \gtrsim 1$, we find

$$g_s > \frac{10^5}{4\pi^3}\left(\frac{M_s}{M_p}\right)^3\,. \tag{4.118}$$

Since we require the condition $g_s \ll 1$ for the validity of the theory, this gives the constraint

$$M_s/M_p \ll 0.1, \tag{4.119}$$

thus favouring a smaller string scale.

After the field reaches the point $R = l_s$, we assume that the field T is frozen at this point, which is a reasonable assumption given what we understand from the bulk description of the dynamics. This gives us a positive cosmological constant in the system.

After the end of inflation

The first phase driven by the field T eventually gives way to a second phase driven by the field χ. Introducing new variables $\chi = \chi_1 + i\chi_2$, $X^2 = \chi_1^2 + \chi_2^2$, $\tilde{X} = M_pX$ and $\tilde{v} = M_p^2 v$, the potential (4.96) of the field X reduces to

$$U(\tilde{X}) = \frac{1}{4\pi^4 (l_s M_p)^4 g_s k} \left[\pi(k+1)\tilde{X}^4 - \tilde{v}\tilde{X}^2\right] \tag{4.120}$$

This potential has two local minima at $\tilde{X}_c = \pm\sqrt{\tilde{v}/(2\pi(k+1))}$ with corresponding negative energy

$$U(\tilde{X}_c) = -\frac{\tilde{v}^2}{16\pi^5 k(k+1)(l_s M_p)^4 g_s}\,. \tag{4.121}$$

One can cancel (or nearly cancel) this term by taking into account the energy of the field T at $R = l_s$. Since this is given by $V(R = l_s) = T_3/\sqrt{kg_s}$, the condition $V(R = l_s) + U(\tilde{X}_c) = 0$ leads to

$$\tilde{v}^2 = 16\pi^5 (k+1) T_3 (l_s M_p)^4 \sqrt{kg_s}\,. \tag{4.122}$$

Using the definition of the brane tension this can be written as follows

$$\tilde{v}^2 = \frac{2\pi^2\sqrt{k}(k+1)}{\sqrt{g_s}}\,. \tag{4.123}$$

Then the total potential for our system is

$$W = A\left(\tilde{X}^2 - \tilde{X}_c^2\right)^2\,, \tag{4.124}$$

where

$$A \equiv \frac{k+1}{4\pi^3 (l_s M_p)^4 g_s k}\,. \tag{4.125}$$

The mass of the potential at $\tilde{X} = 0$ is given by

$$m^2 \equiv \frac{\mathrm{d}^2 W}{\mathrm{d}\tilde{X}^2}_{(\tilde{X}=0)} = -4A\tilde{X}_c^2\,. \tag{4.126}$$

Meanwhile the square of the Hubble constant at $\tilde{X} = 0$ is

$$H_0^2 = \frac{A\tilde{v}^2}{12\pi^2 (k+1)^2 M_p^2}\,. \tag{4.127}$$

Then we obtain the following ratio

$$\frac{|m^2|}{H_0^2} = \frac{24\pi(k+1)}{v} = \frac{12\sqrt{2}(k+1)^{1/2}}{k^{1/4}} g_s^{1/4}\,, \tag{4.128}$$

where we used (4.123) in writing the second equality.

As we have just seen, the COBE normalization gives $kg_s(l_s M_p)^2 \simeq 10^{10}$ for $s \gtrsim 1$, and the ratio (4.128) can be estimated to be

$$\frac{|m^2|}{H_0^2} \simeq 5\times 10^3 \left(\frac{k+1}{k}\right)^{1/2} \left(\frac{M_s}{M_p}\right)^{1/2} \simeq 5\times 10^3 \left(\frac{M_s}{M_p}\right)^{1/2}, \tag{4.129}$$

using the large k limit. Thus we find that $|m^2| > H_0^2$ for

$$\frac{M_s}{M_p} > 4\times 10^{-8}. \tag{4.130}$$

This means that the second stage of inflation does not occur for the field χ provided that the string scale M_s satisfies the condition (4.130). When $4\times 10^{-8} < M_s/M_p \ll 10^{-1}$, inflation ends before the field T reaches the point $R = l_s$, which is triggered by a fast roll of the field χ. This situation is similar to the original hybrid inflation model [76].

When $M_s/M_p < 4\times 10^{-8}$, double inflation occurs even after the end of the first stage of inflation. In this case the CMB constraints for the inflationary model need to be modified. However the second stage of inflation is absent for a string scale close to the Planck scale.

We note that the vacuum expectation value of the field $\tilde{X}$ can be re-written as follows

$$\tilde{X}_c = 2\sqrt{3}\frac{H_0 M_p}{|m|}. \tag{4.131}$$

Therefore when $|m| \gtrsim H_0$ we find that $\tilde{X}_c$ is less than of order the Planck mass. When double inflation occurs ($|m| \lesssim H_0$), the amplitude of symmetry breaking takes a super-Planckian value $\tilde{X}_c \gtrsim M_p$. In this sense the latter case does not look natural compared to the case in which the second stage of inflation does not occur.

Since the field χ has a canonical kinetic term, reheating proceeds as in the case of potentials with spontaneous symmetry breaking [100]. This is in contrast to tachyon fields governed by the DBI action, in which the energy density of the tachyon overdominates the universe soon after the end of inflation. Thus the problem of reheating present in DBI tachyon models is absent in this instance. Since the potential of the field X has a negative mass given by (4.126), this leads to the exponential growth of quantum fluctuations of X with momenta $k < |m|$, i.e., $\delta X_k \propto \exp(\sqrt{|m^2| - k^2}\, t)$. This negative instability is so strong that one can not trust perturbation theory including the Hartree and $1/N$ approximations. We require lattice simulations in order to take into account rescattering of created particles and the production of topological defects.

It was further shown in [100] that symmetry breaking ends after one oscillation of the field distribution as the field evolves toward the potential minimum. This reflects the fact that gradient energies of all momentum modes do not return back to the original state at

$X = 0$ because of a very complicated field distribution after the violent growth of quantum fluctuations.

Finally we should mention that de-Sitter vacua can be obtained provided we tune the potential such that $V(R = l_s)$ does not exactly cancel the negative energy contribution $U(\tilde{X}_c)$. In order to match with the current energy scale of dark energy we require extreme fine tuning $V(R = l_s) + U(\tilde{X}_c) \simeq 10^{-123} M_p^4$. However this kind of fine tuning is a generic problem in all dark energy models, not just this one [77].

4.4 DBI Inflation in the IR.

In this section we will explore a specific inflationary model which is inherently 'stringy', that goes by the name of DBI inflation [92]. We have already used the DBI action to describe the dynamics of moving branes, and shown how the open string modes can play the role of the inflaton. However our analysis relied heavily upon the tachyon-radion correspondance to map the problem to a more field theoretic one. However it was shown in [92]that one can use the full DBI action to drive a period of inflation in a warped geometry, despite the fact that the brane moves relativistically. This model gives rise to specific inflationary signatures which may (or may not) be observable, and so it is useful to probe the full moduli space of solutions.

It was suggested that there were two kinds of DBI inflation. Firstly there is the so called UV model, where the brane is initially located far from the tip of a warped throat [92]. The potential generated by the fluxes in the throat attracts the probe $D3$-brane towards the tip of the throat, which can lead to inflation and a suitable amount of metric fluctuations. The alternate scenario is dubbed IR inflation [93]. In this case the probe $D3$-brane starts at the tip of the throat, and feels a potential generated from branes/fluxes in other throats which forces it up toward the internal space. Again this model can yield suitably small levels of metric perturbations during inflation.

However whilst one can argue that the UV model is reasonably generic, this cannot be said of the IR models. In this case the residual brane localised at the tip of the throat is assumed to be the remnant of brane/flux annihilation [96], which implies there must be some (additional) fine tuning. To see this consider a background where we have M units of D3-brane flux threading some three-cycle. If we insert N' $\bar{D}3$-branes into this background such that they fill the non-compact $3+1$ dimensional spacetime, they will feel an attractive force from the flux and roll down the throat to annihilate quantum mechanically. Then there will be precisely $N = M - N'$ D3-branes created after this annihilation process. The IR model assumes that $N = 1$, however the more generic scenario would in fact leave us with N branes at the tip. In fact we would expect the branes to be distributed randomly near the tip. If they somehow arranged themselves such that they were a distance $L > l_s$ apart, then we could mimic their cosmic evolution using a form of assisted inflation [97].

However on energetic grounds the branes would typically be coincident - therefore we need to describe their dynamics using the non-Abelian DBI (or Myers) action which was discussed in Chapter three.

One of the benefits of the single brane solution was that the backreaction of the $D3$-brane can be neglected in the analysis. Clearly this will not be the case when there are N-branes unless we assume that the background fluxes will dominate. This is not too bad for a non-compact model, however if this is compactified then we would expect higher order corrections to become important, and so we would lose control over the low energy theory. However recent developments have suggested a way to expand the Myers action in the *finite* N limit, which means that we can keep our non-Abelian degrees of freedom, whilst also keeping the back-reactive effects firmly under control [53]. This motivates us to consider a more general version of the IR scenario where we use multiple branes to drive inflation. This will have important consequences which we will discuss later.

For now let us assume that the ten dimensional type IIB spacetime metric factorises as follows

$$ds^2 = h^2 ds_4^2 + h^{-2}(d\rho^2 + ds_{X_5}^2), \tag{4.132}$$

where the four-dimensional metric is taken to be the usual FRW form characterised by the scale factor $a(t)$. There is also a throat region over some five-dimensional manifold X_5, with h being the corresponding warp factor. The internal space X_5 will be taken to be non-compact in this thesis. One may wonder about the validity of such an approximation with regard to modelling four-dimensional physics. However we will assume that most of the relevant physics occurs deep in the warped throat, far from the internal space which is where the model dependent effects typically come into play.

As we saw in Chapter three, the bosonic component of the Myers action can be written as follows

$$S = -T_3 \int d^4\xi \mathrm{STr}\left(\sqrt{-\det(\hat{E}_{ab} + \hat{E}_{ai}(Q^{-1}-\delta)^{ij}\hat{E}_{jb} + \lambda F_{ab})}\sqrt{\mathrm{det Q^i_j}}\right). \tag{4.133}$$

The fields $\hat{E}_{ab}$ are the non-Abelian pullback of the linear combination of closed string fields $E_{ab} = G_{ab} + B_{ab}$, while the matrix Q is determined as

$$Q^i_j = \delta^i_j + i\lambda[\phi^i, \phi^k]E_{kj}\,, \tag{4.134}$$

with the ϕ^i being scalar fields on the world-volume of the D-branes corresponding to the transverse fluctuations. As in the KS model [63], we will assume that the bulk B field is zero near the tip of the throat, and consider the case where the transverse coordinates define a fuzzy S^2 embedded within a three cycle in the X_5.

Now that we have restricted ourselves to the $SO(3) \sim SU(2)$ algebra, we expect the scalars to be proportional to its generators. The usual matrix ansatz that we take for our scalar fields is thus $\phi^i = R\alpha^i$ $(i = 1, 2, 3)$, where R is a variable of canonical mass dimension, and the α^i are the N-dimensional generators of the $SU(2)$ algebra.

In flat space we can find solutions where the radius of the fuzzy sphere grows without bound. However for most spaces there will be a maximum bound on the size of the fuzzy sphere, as illustrated by the famous example of the $\mathrm{AdS}_5 \times S^5$ metric where the fuzzy sphere radius is bounded by the radius of the five-sphere [57]. In the warped backgrounds where the length of the throat provides the cutoff scale, the throat is smoothly glued onto the five-dimensional internal space. We denote this as the UV cut-off, analogous to the UV brane in the Randall-Sundrum models [105]. As already discussed, provided inflation occurs far from this UV cutoff we should be able to trust our solution.

Let us insert the metric (4.132) into the Myers action (4.133). After expanding all the determinants, we find that the effective action for coincident D3-branes in this background becomes

$$S = -T_3 \int d^4\xi \mathrm{STr}\left(h^4 a^3 \sqrt{1 - h^{-4}\lambda^2 \alpha^i\alpha^i \dot{R}^2}\sqrt{1 + 4\lambda^2\alpha^i\alpha^i h^{-4} R^4} - a^3 h^4 \mathbf{1_N} + a^3 V(R)\mathbf{1_N}\right), \tag{4.135}$$

where the second term arises as the leading order contribution from the Chern-Simons coupling of the bulk RR fields, while the final term is a flux induced potential which comes from fluxes present in the background. Both terms are singlets under the trace, which is why they appear multiplied by the $N \times N$ identity matrix. The potential generated will depend on the topology of the internal space, and also the quantisation constraints of the fluxes. Here we have absorbed a factor of T_3^{-1} into the potential to make it dimensionless, and set the dilaton to be unity. This agrees with the supergravity background generated by $D3$-branes, and also the tip solution of the Klebanov-Strassler throat [63].

We now minimally couple the DBI action to the standard Einstein-Hilbert action. Fortunately the Einstein part of the action arises naturally in the warped backgrounds we are considering, therefore provided the branes don't move too far away from the tip - gravity should still be confined to their world-volumes. It is convenient to introduce the notation

$$W(R, \hat{C}) = \sqrt{1 + 4\lambda^2 h^{-4}\alpha^i\alpha^i R^4}, \tag{4.136}$$

which is essentially an additional potential induced by the fuzzy sphere geometry, which we will refer to as the fuzzy potential. For a single probe brane this contribution is always unity, since the corresponding matrix representation is commutative. This would also be the case if the branes are located at distances greater than l_s. We can now determine the non-zero components of the energy-momentum tensor on the world-volume. It gives the energy and pressure densities as follows

$$E = T_3 \mathrm{STr}\left(\frac{W(R,\hat{C})h^4}{\sqrt{1-h^{-4}\lambda^2\alpha^i\alpha^i\dot{R}^2}} - h^4 + V\right), \tag{4.137}$$

$$P = -T_3 \mathrm{STr}\left(h^4 W(R,\hat{C})\sqrt{1-h^{-4}\lambda^2\alpha^i\alpha^i\dot{R}^2} - h^4 + V\right). \tag{4.138}$$

These form the basis for all our analysis in the subsequent sections and we will return to them in due course.

One of the reasons that we can have DBI inflation in this background, despite the branes moving relativistically, is that the speed of sound is usually very small compared to unity. This is therefore a string theory motivated model of kinetic inflation [92, 106], using the warped nature of the spacetime to reduce the velocity scale. In the standard canonically normalised slow roll models, this factor is always unity since the scalar field moves slowly. The speed is given by the following expression

$$C_s^2 = \frac{\partial P/\partial \dot{R}}{\partial E/\partial \dot{R}}, \tag{4.139}$$

which is valid provided that the entropy corrections are negligible [76, 94].

4.4.1 The large N limit

Let us restrict ourselves to the large N limit when we ignore the backreaction[11]. The large N limit has proven to be useful for inflation in many other contexts such as assisted inflation [97] or N-flation [98] Now we can approximate the symmetrized trace by a trace once we neglect the contributions of terms of the order $1/N^2$ in the action. The resulting action simplifies to

$$S = -T_3 \int d^4\xi N a^3 \left(h^4\sqrt{1-h^{-4}\lambda^2\hat{C}\dot{R}^2}\sqrt{1+4\lambda^2\hat{C}h^{-4}R^4} - h^4 + V(R)\right) \tag{4.140}$$

where T_3 is the tension of the $D3$-branes given by the usual formula

$$T_3 = \frac{M_s^4}{8\pi^3 g_s} \tag{4.141}$$

with M_s being the mass scale for the open strings and g_s being the asymptotic string coupling which we take to be small to allow for perturbatively defined strings. In this chapter we will generally assume that the coupling is set to $g_s \sim 10^{-2}$ in order to make order of magnitude approximations.

[11]This amounts to a constraint on the energy of the D-branes which could lead to an unphysical solution upon compactification. However it can be interpreted as a metric constraint on the bulk fluxes.

The backreaction effect will be small provided that we ensure $MK >> N$ is also satisfied, in addition to the assumption of large N. It is instructive to make a redefinition of the scalar field in order to compare our model with the rest of the literature. Firstly we switch to 'physical' coordinates using the standard relationship $R^2 = r^2/(\lambda^2\hat{C})$ which parameterises the physical radius of the fuzzy sphere. Let us also further define a scalar field $\phi = r\sqrt{T_3}$ with canonical mass dimension, which we will use as the inflaton.

We have the standard relationship between the four-dimensional Planck scale and the ten dimensional one through the volume of the warped space V_6;

$$M_p^2 = \frac{V_6}{\kappa_{10}^2}. \tag{4.142}$$

Interestingly it was shown in Baumann [94]that with minimal assumptions about the volume of the throat, one can find the following bound on the maximal field variation:

$$\Delta\phi < \frac{2M_p}{\sqrt{MK}}, \tag{4.143}$$

where MK is the contribution from the background fluxes. In our case we demand that $MK >> N$ to neglect the back reaction upon the geometry, which restricts our field to move over very small distances in Planckian units.

We find from (4.133) the equation of motion of the ϕ field:

$$\begin{aligned} 0 &= W(\phi,\hat{C})\gamma^3\ddot{\phi} + 3H\dot{\phi}W(\phi,\hat{C})\gamma + \frac{8\gamma\phi^3}{T_3\lambda^2\hat{C}W(\phi,\hat{C})}\left(1-\frac{\phi h'}{h}\right) \\ &+ 2T_3h'W(\phi,\hat{C})\gamma\left(2h^3 - \frac{\gamma^2\dot{\phi}^2}{hT_3}\right) - 4T_3h^3h' + T_3V', \end{aligned} \tag{4.144}$$

where primes are derivatives with respect to ϕ and we have also introduced

$$\gamma = \frac{1}{\sqrt{1-h^{-4}T_3^{-1}\dot{\phi}^2}}, \tag{4.145}$$

for the analogue of the relativistic factor for the DBI action. This implies that the velocity of the brane is bounded as

$$\dot{\phi}^2 < h^4T_3. \tag{4.146}$$

These last two equations are exactly the same as in the case of a single brane. Recall from our discussion in the Chapter three that when we take the large N limit of the fuzzy S^2, we recover the classical S^2 with N units of charge. This suggests that the large N limit essentially behaves like a single object - which is why we should expect to find similarities with the $N = 1$ results.

Using the general expression (4.139), we calculate the speed of sound to be $C_s^2 = 1/\gamma^2$, in agreement with that of single brane inflation. As in that case we will now also assume that the scalar field is monotonic, at least for early times. This assumption allows us to switch again to the Hamilton-Jacobi formalism. We differentiate the Friedmann equation $H^2 = E/3M_P^2$ with respect to time, dropping terms proportional to $\ddot{\phi}$, and use the continuity equation, $\dot{E} = -3H(P+E)$ to get

$$\dot{\phi} = -\frac{2M_p^2 H'}{N\gamma W(\phi)}\,, \tag{4.147}$$

where the fuzzy potential W is now an explicit function of ϕ, and H' is the derivative of the Hubble parameter with respect to the inflaton. Substituting this $\dot{\phi}$ into (4.145), we obtain

$$\gamma(\phi) = \sqrt{1 + \frac{4M_p^4 H'^2}{N^2 W^2(\phi) h^4 T_3}}\,. \tag{4.148}$$

We can use (4.148) to write the velocity of the inflaton without reference to the relativistic factor γ:

$$\dot{\phi} = \frac{-2M_p^2 H'}{\sqrt{N^2 W^2 + 4M_p^4 H'^2 h^{-4} T_3^{-1}}}, \tag{4.149}$$

which allows us to write the corresponding Friedmann equation solely as a function of the inflaton:

$$\begin{aligned} H^2 &= \frac{NT_3}{3M_p^2}\left(W(\phi)h^4(\phi)\gamma(\phi) + V(\phi) - h^4(\phi)\right) \\ &= \frac{NT_3}{3M_p^2}\left(V(\phi) + h^4(\phi)\left\{\frac{1}{N}\sqrt{N^2W^2 + \frac{4M_p^4 H'^2}{h^4 T_3}} - 1\right\}\right), \end{aligned} \tag{4.150}$$

which is reminiscent of the equation found in for the Abelian DBI inflation model [92]. The main difference here is the presence of factors of N and the fuzzy sphere induced potential $W(\phi)$, the latter being an inherently non-Abelian feature.

We may be concerned that the DBI models of inflation do not exhibit standard attractor solutions for inflation, since we expect relativistic motion. This attractor behaviour is important as it implies that inflation will not be dependent upon the precise choice of initial conditions for the inflaton [76]. To check this let us suppose that $H_0(\phi)$ is *any* solution of (4.150), which can be either inflationary or non-inflationary. We add to this a linear homogeneous perturbation $\delta H(\phi)$. The attractor condition will be satisfied if it becomes small as ϕ increases. Upon substituting $H = H_0 + \delta H$ into (4.150) and linearizing the resultant expression, we find that the perturbation obeys

$$H_0'\delta H' = \frac{3NW\gamma}{2M_p^2} H_0 \delta H, \tag{4.151}$$

which has the general solution

$$\delta H(\phi) = \delta H(\phi_i) \exp\left[\frac{3N}{2M_p^2}\int_{\phi_i}^{\phi} d\phi W(\phi)\gamma(\phi)\frac{H_0(\phi)}{H_0'(\phi)}\right], \tag{4.152}$$

where $\delta H(\phi_i)$ is the value at some initial point ϕ_i, and $\gamma = \gamma(H_0)$. Because H_0' and $d\phi$ have opposite signs, the integrand within the exponential term is negative definite [12], and all linear perturbations indeed die away. This means that there is an attractor solution for this model regardless of the initial conditions and the velocity of the brane. This is also true for the single brane solutions with $N = 1 = W(\phi)$.

We note that the equation of state in this model is drastically different from the canonical field models. At large N we find that

$$\omega = -\frac{Wh^4\gamma^{-1} + V(\phi) - h^4}{Wh^4\gamma + V(\phi) - h^4}. \tag{4.153}$$

If the potential dominates all the other terms, we recover the usual de-Sitter solution with $\omega \sim -1$. However the DBI admits more interesting solutions due to its non-linear nature. For example, if we consider ultra-relativistic motion where $\gamma >> 1$, and demand that the h^4 terms are suppressed, we obtain

$$\omega \sim \frac{-V(\phi)}{Wh^4\gamma + V(\phi)}, \tag{4.154}$$

which can be very small depending on the scale of the fuzzy potential, and may give rise to a matter phase in the asymptotic velocity limit. This shows that we have a larger parameter space of solutions for ω than in the standard inflationary scenarios.

Our analysis thus far has been general. To make a more detailed investigation of the inflationary signature of this model, we must determine the background potential. Let us consider nonzero fluxes inducing the warped throat solution. The coincident branes localised at the bottom of the throat will feel a potential generated by branes/fluxes in other throats and will move towards them. We then expect a tachyonic potential of the form

$$V(\phi) \sim V_0 - \frac{V_2\phi^2}{2} + \ldots, \tag{4.155}$$

with higher order even powers of ϕ because of the $\mathbb{Z}_2$ symmetry of the throat. The various constants will be determined by the choice of geometry, fluxes, and also non-perturbative effects. The IR DBI inflation is thus a special case of small field inflation. The constant V_0 is the scale set by the fluxes, and need be large to be able to neglect back-reactive effects in our model. This is significantly different from the UV inflationary model. In this section we are mainly interested in the IR solution, and we consider that most of the dynamics will

[12] Assuming that γ is also positive definite.

take place in a region dominated by the potential energy.

Inflationary observables and constraints

In this subsection, we focus on inflationary solutions in two specific backgrounds. We show that inflation can be achieved in this model by allowing an appropriate tuning of the parameters in the theory.

We wish to consider inflation near the tip of a warped throat where ϕ is small. However we are also interested in solutions where the background is no longer approximately AdS but is constant. This is motivated by the work of Kecskemeti et al [94] and also the fact that a constant warp factor appears to be a generic feature of the IR end of warped throats. This is a phenomenologically motivated solution, because we expect the fluxes to back-react on the bulk geometry to form a warped throat. We expect that these throats will be of the Klebanov-Strassler (KS) variety, which have a finite cut-off at the origin. This cut-off is generically exponentially small due to its dependence on the three-form fluxes.

In order to mimic a constant warping in our non-compact theory we choose to put in 'by hand' a constant warp factor parameterised by a mass term μ, where we expect that μ is small at a scale set by the bulk fluxes [94]:

$$h(\phi) = \frac{\sqrt{\phi^2 + \mu^2}}{L}, \tag{4.156}$$

where we have used L to denote the background charge. It should be noted that this L will be different from that in the purely AdS-like backgrounds. When ϕ goes to zero, the warp factor remains finite. Strictly speaking away from the origin there may be a different ϕ dependence, but we use this form of the warp factor in the following sections.

The solutions we consider are
(i) the AdS type cases (where we set μ to 0), or
(ii) the mass gap cases (where we assume $\phi \sim 0$).

One further remark that we need to make is that in the AdS type solution we strictly need to introduce a cut-off for ϕ such that the solution is nowhere singular. The warp factor in this case would therefore be cut-off at the value h_C. In our analysis we will not write this explicitly, but we will always assume that $h_C << 1$. Let us assume that inflation occurs very close to the tip of the warped throat, in which case we expect the energy density to be dominated by the constant piece of the background potential. The other terms are suppressed by the square of the warp factor and will be small in this limit. It may appear that the warp factor can be easily vanishingly small, but more care is required in cases where h reduces to a constant, since in those backgrounds (as we shall see) the other parameters can *both* be large. Assuming $V_0 >> h^4(W(\phi)\gamma(\phi) - 1)$ is satisfied, the

Friedmann equation (4.150) can be approximated as

$$H^2 \sim \frac{NT_3V(\phi)}{3M_p^2}. \tag{4.157}$$

We then find that our solutions for the inflaton velocity and the gamma factor reduce to

$$\begin{aligned} \dot{\phi} &\sim -\frac{M_pV'}{\gamma W}\sqrt{\frac{T_3}{3NV}}, \\ \gamma(\phi) &\sim \sqrt{1+\frac{M_p^2}{3N}\left(\frac{V'^2}{h^4W^2V}\right)}, \end{aligned} \tag{4.158}$$

where we have expressed everything in terms of derivatives of the potential. Clearly these expressions are background dependent as they are functions of the warp factor h.

Inflation in AdS type backgrounds

Let us consider the locally AdS type backgrounds, such as those studied in the original IR inflation scenario [93]. The harmonic function can be approximated by $h \sim \phi/L$ in the near horizon region where L is essentially the charge of the background geometry given by[13] $L^4 = g_sMKT_3^2\lambda^2\pi^2/\mathrm{Vol}(\mathrm{X}_5)$, with M and K the corresponding quanta of flux and $\mathrm{Vol}(\mathrm{X}_5)$ corresponding to the dimensionless volume of the compact space. We expect to find that $\mathrm{Vol}(\mathrm{X}_5) = \mathrm{a}\pi^3$, where a is a topological parameter. For example we know that $a = 1$ for the five-sphere and $a = 16/27$ for the manifold $T^{1,1}$. In this situation the fuzzy sphere induced potential becomes constant which greatly simplifies the analysis. In fact the second term inside the square root is proportional to the ratio of the background fluxes and the number of coincident branes, and thus it is not clear a priori whether this term will be small or large. Under our assumption of no back reaction, we require the flux term to dominate over N, and so we may expect that $W >> 1$ which translates into the flux condition

$$\frac{L^2}{M_s^2} >> \frac{N}{8\pi^2 g_s} \quad \rightarrow \quad \sqrt{MKg_s} >> N \tag{4.159}$$

in the large N limit. For small values of the string coupling constant, we see that this requires the fluxes to be very large. This is to expected from out heuristic arguments regarding the backreaction. In the converse limit where $W \sim 1$ we see that the constraint becomes $\sqrt{g_sMK} << N$ - which although provides a bound on N is much harder to satisfy within the remit of our approximation.

[13] Note we have rescaled this quantity to ensure it has the correct dimensions. This corresponds to the radius of curvature for the AdS space scaled by the square of the brane tension.

This requirement also affects the definition of γ:

$$\gamma_{\rm AdS}(\phi) \to \sqrt{1+\frac{M_p^2V_2^2L^4}{3NW^2V_0\phi^2}\left(1+\frac{V_2\phi^2}{V_0}+\ldots\right)}. \tag{4.160}$$

As we are interested in DBI inflation, we should take $\gamma(\phi) >> 1$ for the speed of sound to be substantially reduced, and so the right hand side will dominate this expression. Furthermore since we are assuming that the background potential should dominate the energy density at this stage of the evolution, we should also assume that $V_0 >> V_2$, which implies that the new constant piece of γ will be subdominant.

In this regime we can approximately solve the inflaton equation of motion. In fact there is a cancellation between terms in the equation which implies that $\dot{\phi} \sim \mathcal{O}(\phi^2)+\mathcal{O}(\phi^6)$ and so for consistency we must drop all terms higher than quadratic in the fields. The resultant expression for the field is actually of the same functional form as the single brane case. Actually this is expected since we can view the fuzzy sphere as a classical sphere with N units of flux in the large N limit:

$$\phi \sim \frac{\phi_0 L^2}{L^2-\phi_0\sqrt{T_3}(t-t_0)}, \tag{4.161}$$

where the field is initially located at $\phi=\phi_0$ at $t=t_0$. This expression on the denominator will generally be much smaller than unity as time evolves, since we are assuming that $h_0 << 1$ which is the initial value of the warp factor.

Let us now compute the inflationary parameters in this large N limit. The non-linear form of the DBI action prevents us from using the traditional slow roll variables, and so we must establish new 'fast roll' variables. In reality the name 'fast roll' is somewhat of a misnomer because despite moving relativistically, the non-linear nature of the action allows for the brane to be held up on the potential for a significant amount of time as in slow roll scenarios. This has already been extensively discussed for the single brane models but the non-Abelian action requires us to modify these expressions. Suppose that the leading order term for the epsilon parameter expansion is given by

$$\frac{\ddot{a}}{a} = H^2(1-\varepsilon), \tag{4.162}$$

which yields the usual slow roll constraint $\varepsilon = -\dot{H}/H^2$. However now that we are working in the Hamilton-Jacobi formalism, we need derivatives with respect to the inflaton. This leads to the following modified expressions for the relevant slow roll parameter:

$$\varepsilon = \frac{2M_p^2}{N\gamma W(\phi,\hat{C})}\left(\frac{H'}{H}\right)^2. \tag{4.163}$$

This is clearly equivalent to the usual single brane slow roll conditions where we would have

$N = 1, W(\phi, \hat{C}) = 1$, although we cannot take the $N \to 1$ limit in our non-Abelian DBI description. Note that this slow roll parameter is suppressed not only by a factor of $1/\gamma$ as in the single brane inflation, but also by an additional factor of $1/N$. Intuitively we may have expected this since the coincident branes will tend to accelerate much more slowly than a single brane. Hence we would expect inflation to last for a longer period of time.

Using our approximate solutions we find for the AdS type backgrounds (and assuming $\gamma >> 1$)

$$\varepsilon \sim \frac{\sqrt{3} M_p V_2 \phi^3}{2L^2 V_0 \sqrt{N V_0}} + \ldots, \tag{4.164}$$

where we have neglected ϕ^5 terms. We must ensure that $\epsilon < 1$ for inflation to occur at all. We can also calculate the number of e-foldings for this model using

$$N_e = \int H dt. \tag{4.165}$$

We obtain

$$N_e \sim \sqrt{\frac{NV_0}{3}} \frac{L^2}{M_p} \int d\phi \frac{1}{\phi^2} \left(1 - \frac{V_2 \phi^2}{4V_0} + \mathcal{O}(\phi^4)\right), \tag{4.166}$$

where the integration should be between ϕ_0 and ϕ_f, the latter being determined though the fast roll parameter. Clearly for small ϕ the first term will dominate the integral and so we drop the higher order terms. The result is that the field value N_e e-folds before the end of inflation can be written as

$$\phi_0 \sim \frac{\phi_f L^2 \sqrt{N V_0}}{L^2 \sqrt{N V_0} + \sqrt{3} \phi_f N_e M_p}, \tag{4.167}$$

which we can use to determine the perturbation spectrum. For completeness, we write the fast roll parameter as a function of the number of e-foldings:

$$\varepsilon \sim \left(1 + N_e \left(\frac{6M_p^2}{NL^4 V_2}\right)^{1/3}\right)^{-3}. \tag{4.168}$$

Inflation in mass gap backgrounds

The equation of motion for the inflaton in the AdS background is basically the same as in the single brane case. Is this also true for the mass gap solution? In this instance the fuzzy sphere potential now has non-trivial field dependence which complicates the analysis.

Without loss of generality we will take the warp factor of the form $h \sim \mu/L$:

$$\begin{aligned}
W_{\rm mg}(\phi) &\sim \sqrt{1+\frac{4\phi^4 L^4}{\mu^4\lambda^2 N^2 T_3^2}}, \\
\gamma_{\rm mg}(\phi) &\sim \sqrt{1+\frac{M_p^2 L^4 V_2^2 \phi^2}{3N\mu^4 {W_{\rm mg}}^2 V_0}}.
\end{aligned} \tag{4.169}$$

Since we are mainly interested in the relativistic limits of the theory we should take γ to be large. This immediately imposes a constraint on the fuzzy potential - which is now an explicit function of the inflaton ϕ

$$\frac{M_p^2 V_2^2 \phi^2}{3V_0 N h^4} >> W^2. \tag{4.170}$$

The analytic form of W tells us that it is bounded from below by unity, but has no upper bound. Of course in reality we expect $W(\phi)$ to be a monotonically increasing function, however for analytical simplicity we will consider the two limits separately.

In the first instance let us assume that $W \sim 1$. From (4.170) this implies that we have an upper bound on the number of branes

$$N << \frac{M_p^2 V_2^2 \phi^2}{3V_0 h^4} \tag{4.171}$$

which can be satisfied by having a small enough warp factor. However we must also impose the constraint coming from the definition of the fuzzy potential which actually implies a lower bound on N through the relation

$$N >> \frac{2\phi^2}{h^2 \lambda T_3} \tag{4.172}$$

so upon combining these constraints we see that both assumptions are valid provided that

$$\frac{V_2^2}{V_0} >> \frac{h^2}{M_s^2 M_p^2} \tag{4.173}$$

We now ask about the constraints arising from the converse limit, when both γ and W are large. For the fuzzy potential to be large we must ensure that

$$N << \frac{7\times 10^{-1}\phi^2}{h^2 M_s^2}, \tag{4.174}$$

however the relativistic limit also requires N to be bounded from below

$$N >> \frac{\mathcal{O}(1)\phi^2 V_0}{V_2^2 M_s^4 M_p^2}. \tag{4.175}$$

Combining both of these constraints also results in (4.173). It is straight-forward to note that due to the constant warp factor near the tip of the throat, the relativistic limit implies that ϕ is a linear function of time. The overall scale is simply set by the warp factor and the brane tension, and is independent or our parameterisation of the fuzzy potential.

Let us now consider how inflation occurs in this model. Our starting point will once again be (4.163). Inserting the mass gap solution and also demanding relativistic motion is enough to ensure that W drops out of the analysis, thus the solution is independent of the fuzzy potential contribution. After integrating to find the number of e-folds we see that the slow roll parameter can be written in the form (N_e e-folds before the end of inflation)

$$\varepsilon \sim 1 - \frac{3N_e V_2 M_p^2 h^4}{2V_0 N} \tag{4.176}$$

We have seen that for a certain range of parameters, we can obtain the required number of e-foldings. The real signature of the model lies in the perturbation spectrum and the scale of the spectral indices, which we now address.

Cosmological Signatures.

The derivation of the various perturbation spectra for this model is presented in the Appendix, and we refer the reader there to see how the expressions arise [76, 106]. We will simply quote the important results in the following section. The main equations we need to calculate the perturbation spectra are (A.13) and (A.15) respectively. In terms of our standard notation employed in the rest of this section these translate into the following conditions when we substitute for the velocity equation in the Hamilton-Jacobi formalism, and re-insert the factors of the reduced Planck mass

$$\begin{aligned} \mathcal{A}_S^2 &\simeq \frac{H^2}{8\pi^2 M_p^2 \varepsilon C_s}, \\ \mathcal{A}_T^2 &\simeq 8\left(\frac{H}{2\pi M_p}\right)^2. \end{aligned} \tag{4.177}$$

The slow-rolling inflation generally predicts very low non-gaussianity since at leading order the quantum fluctuations are generated by free fields in the dS background. However in the DBI inflation, much larger non-gaussianity can be generated since the causality constraint in the kinetic term introduces non-linear interactions among different momentum modes of the scalar field [94, 95]. Recently it was shown that in the equilateral triangle limit, the leading-order contribution to the non-linearity parameter is given by

$$f_{NL} = \frac{35}{108}\left(\frac{1}{C_s^2} - 1\right) - \frac{5}{81}\left(\frac{1}{C_s^2} - 1 - 2\Lambda\right), \tag{4.178}$$

where, using the definitions in the appendix,

$$\Lambda \equiv \frac{X^2 p_{,XX} + \frac{2}{3}X^3 p_{,XXX}}{X p_{,X} + 2X^2 p_{,XX}}\,. \tag{4.179}$$

It can be shown that the second term on the right-hand side of (4.178) vanishes in the large N case, as it does in the single brane case. Therefore we have the usual non-gaussian parameter

$$f_{NL} \approx 0.32\gamma^2. \tag{4.180}$$

This once again emphasizes the similarity between the $N = 1$ and the $N >> 1$ descriptions. Current measurements indicate a rather weak bound on the level of non-gaussianities $f_{NL} \leq 100$, however the upcoming Planck mission aims to increase the sensitivity to probe down to regions where $f_{NL} \leq 5$ and will therefore provide a robust test of these predictions.

We will now discuss the expected level of perturbations in each of the two cases we have discussed so far.

- **Ads backgrounds.** Using the formulae derived in the appendix we find that the scalar amplitude at leading order becomes

$$\mathcal{A}_S^2 \sim \frac{NV_0}{12\pi^5 g_s}\frac{\gamma}{16\epsilon}\left(\frac{M_s}{M_p}\right)^4. \tag{4.181}$$

Now using the WMAP normalisation for the non-gaussianities we must ensure that $\gamma \leq 10\sqrt{3}$, which in turn fixes ϵ through the (weak) constraint that $r \leq 0.5$. Therefore we see that at horizon crossing we must ensure that

$$\varepsilon \leq \frac{5\sqrt{3}}{16} \qquad \rightarrow \qquad MKNV_2 \leq \frac{10^6 M_p^2}{M_s^4} \tag{4.182}$$

is satisfied where we assume that the string coupling is roughly 10^{-2}. This essentially means that the slow roll parameter must be less than one half. Using this constraint to fix the scalar amplitude we find the following condition needs to be satisfied

$$NV_0\left(\frac{M_s}{M_p}\right)^4 \leq 10^{-8}. \tag{4.183}$$

Clearly this is linear in NV_0, therefore requires that the string scale must be low in order for the WMAP normalisation to hold, assuming that the constant part of the potential is sub-Planckian. In addition we see that the scalar index can be expanded as a power series in ε, which at leading order becomes

$$n_s \sim 1 - 4\varepsilon - 4\varepsilon^{1/3}\delta n_s + \ldots \tag{4.184}$$

where we have defined the 'perturbation'

$$\delta n_s \sim \left(\frac{10^2 M_p}{M_s^2\sqrt{NV_2 g_s MK}}\right)^{2/3}. \tag{4.185}$$

Recall that the WMAP bound for this index is given by $n_s = 0.987^{+0.019}_{-0.037}$. Using the constraint in (4.182) we find that the δn_s term must be bounded from below by unity. Moreover we see that by constraining ε to be smaller - this in fact makes the δn_s term larger. Given this, one sees that the scalar index will always be large and negative in this instance and therefore incompatible with observation.

We also see using the relation

$$\frac{dn_X}{d\ln k} \sim \frac{dn_X}{dN_e} \qquad\qquad X = S, T \tag{4.186}$$

evaluated at horizon crossing, that both tensor and scalar indices are positive running.

- **Mass Gap Backgrounds.** The form of the Hubble parameter in this instance is the same as in the AdS case, therefore we expect similar arguments to hold in this instance. Once again this favours a smaller string scale using (4.183). Assuming similar constraints on r and γ we see that the only real distinction between the two cases arises through the spectral indices, since we again have to ensure that (4.182) is satisfied. Explicit calculation of the scalar index in this case reveals that

$$n_s - 1 \sim -4\varepsilon\left(1 - \frac{H}{4H'}\left\{\frac{3H''}{H'} - \frac{2W'}{W}\right\}\right). \tag{4.187}$$

If we restrict ourselves to the case where $W \sim 1$ then this simplifies down to the following

$$n_s \sim 1 - 7\varepsilon - \frac{9M_p^2 V_2}{2V_0 N\varepsilon}. \tag{4.188}$$

The scalar index will clearly be sensitive to the magnitude of the last term, which we write as $-\delta n_s/\varepsilon$, and we see that

$$\delta n_s \geq \frac{1}{20h^4}\left(1 - \frac{5\sqrt{3}}{16}\right) \tag{4.189}$$

using the WMAP normalisation. Clearly for small values of the warp factor we will see that the δn_s term will be large, which implies that we cannot obtain the observed spectrum of scalar perturbations. The only way we can satisfy the experimental data in this instance is to assume a large warp factor - however this is contrary to all our assumptions thus far. So we must conclude that this particular model does not agree with the data.

Conversely in the limit where we take $W >> 1$ we find a cancellation between dangerous $1/\phi$ terms which gives the final result for the scalar tilt

$$n_s \sim 1 - 7\varepsilon, \tag{4.190}$$

which can be seen to arise as the limiting case of the $W \sim 1$ solution. Using the WMAP bound, this serves to fix a much tighter bound on ε at horizon crossing, which we can interpret as a upper bound on N

$$N \le \frac{V_2 10^2 M_p^2 h^4}{V_0}, \tag{4.191}$$

and therefore we can satisfy the experimental bounds on inflation. In both cases we see that the indices are positive definite, as in the Ads models.

Examples of other solutions

Let us consider more general solutions which arise from the master equation (4.150). We will consider both UV and IR inflationary solutions for generality and then make some comments about the general signals of inflation in the large N limit.

To distinguish between each solution branch, we note that on the UV side we have large field inflation. So there will generally be positive contributions to the potential, and the vacuum energy V_0 will be set to zero. In the IR branch, we have $V_0 \neq 0$ at all times and the potential will generally be taken to be tachyonic indicating that the branes move away from the tip of the throat. In the fuzzy sphere picture, the UV side corresponds to a collapsing sphere, while the IR side corresponds to an expanding sphere.

An interesting solution was analysed in [92] when there is no quadratic term in the potential i.e $V(\phi) = V_0 - V_4\phi^4$. Solving the master equation (4.150) for $H(\phi)$ and integrating back to find the time dependence of the inflaton, we obtain the following solutions for the AdS type backgrounds:

$$\begin{aligned} H(\phi) &\sim H_0 + H_4\phi^4 \\ \phi(t) &\sim \frac{1}{4M_p\sqrt{(t_f - t)}}\sqrt{\frac{NW_0}{H_4}}, \end{aligned} \tag{4.192}$$

where the terms in the Hubble parameter are calculated to be

$$H_0 = \sqrt{\frac{NT_3V_0}{3M_p^2}}, \qquad H_4 = \sqrt{\frac{NT_3}{3V_0}}\frac{1}{2M_p}\left(\frac{W_0 - 1}{L^4} \pm V_4\right). \tag{4.193}$$

Note that there is a potential sign ambiguity in the definition of H_4. This is because we can consider either the IR inflation (where we have a minus sign in the potential) or the UV inflation (where we have a plus sign). In both cases the equation of motion for the inflaton is the same. In the IR case this is the early time, small ϕ solution, whereas in the UV case it corresponds to the *late* time solution where the fuzzy sphere has collapsed to almost zero size. In both cases the form of the equation of motion implies that in the small field limit the inflaton is moving relativistically with large γ. The number of e-foldings can be determined as follows:

$$N_e \sim H_0(t_e - t_0) + \frac{NW_0}{16M_p^2}\left(\phi^2(t_e) - \phi^2(t_0)\right), \tag{4.194}$$

where t_0 and t_e are the times at the beginning and end of inflation, respectively. We must ensure that $t_f \geq t_e$, where t_f represents the time at which we can no longer trust our approximations.

Let us now reconsider the mass gap solution. Since the throat is finite and the warp factor is actually constant for small values of ϕ, there is no reason why the field is moving rapidly near the origin. Let us take a quadratic potential of the form $V(\phi) = V_0 \pm V_2\phi^2$, where the positive sign signifies UV type inflation. For small values of the inflaton, we have the Hubble parameter $H(\phi) = H_0 + H_2\phi^2$, where the coefficients are determined in a similar way to those in the AdS case. If we assume that the field is small (in both cases), the general solution can be written as

$$H_0 \sim \sqrt{\frac{NT_3V_0}{3M_p^2}}, \qquad H_2 \sim \frac{3H_0}{8M_p^2}\left(1 \pm \sqrt{1 + \frac{8NT_3V_2}{9H_0^2}}\right). \tag{4.195}$$

This solution is valid for the IR inflation. For the UV solution, one only need substitute a minus sign in front of the V_2 term inside the square root. Again this means that for the IR solution this is the early time evolution, while for the UV solution it is the late time evolution of the Hubble parameter. In both cases, the solution for the field can be written as

$$\phi(t) \sim \phi_0 e^{-\frac{4M_p^2H_2(t_f - t)}{N}}, \qquad t_f \geq t, \tag{4.196}$$

which means that the field is rolling non-relativistically because H_2 and ϕ are both small.

Let us focus initially on the IR solution. If we wish the inflaton to be increasing, corresponding to the branes moving away from the origin, we are forced to choose the minus sign in H_2. The other sign is a solution where the field is getting smaller. In the event that V_2 is zero, we find either a solution where the inflaton is at a constant value, or H_2 is positive definite and again the field is getting smaller. The former case corresponds to de-Sitter expansion since we can immediately integrate the solution for the scale factor to get $a = e^{H_0t}$. In the UV region we must also impose an additional reality constraint $9H_0^2 \geq 8NT_3V_2$. If this reality bound is saturated, the field is again rolling towards the origin. In both cases the evolution of the field is essentially determined by the vacuum energy V_0 which sets the overall scale for the Hubble parameter.

Using the solution to the equation of motion, we can calculate the number of e-foldings and write is as a function of the field. In the UV case the branes would be near the bottom of the throat, and so this correspond to late time evolution of the inflaton. As such we interpret this as the early stage evolution of the IR inflationary model. We obtain

$$N_e \sim H_0(t_e - t_0) + \frac{N}{8M_p^2}\left(\phi^2(t_e) - \phi^2(t_0)\right), \tag{4.197}$$

e-foldings in this regime and we must again ensure that $t_e \leq t_f$. Since we expect ϕ to be small over this time period in accordance with our approximation of the Hubble parameter, the dominant contribution to the number of e-foldings comes from the constant part of the potential. However since $N >> 1$ we may well see a sizable contribution to the number of e-foldings.

Let us consider what happens in more generality, although we will assume that the Hubble parameter is generically of the form shown in (4.157). Focusing our attention on the power spectra, we find that the amplitudes are given by

$$\begin{aligned} A_T^2 &= \frac{2H^2}{\pi^2 M_p^2}, \\ A_S^2 &= \frac{H^2}{8\pi^2 M_p^2 \varepsilon}\left(1 + \frac{4M_p^4 H'^2}{N^2 W^2 h^4 T_3}\right)^{1/2}, \\ r &= 16\varepsilon\left(1 + \frac{4M_P^4 H'^2}{N^2 W^2 h^4 T_3}\right)^{-1/2}, \end{aligned} \tag{4.198}$$

where we have included r as the ratio of the tensor/curvature amplitudes. Recall that each of these is to be evaluated at horizon crossing if we wish to normalise them to the WMAP data. Note that ε is expected to be small at horizon crossing. We can repeat the same analysis as before and consider limits of the term in parenthesis. (i) In the first case when we consider relativistic motion, the scalar amplitude and ratio reduce to

$$\begin{aligned} A_s^2 &\sim \frac{H^2|H'|}{4\pi^2\varepsilon N W h^2\sqrt{T_3}} \\ r &\sim \frac{8\varepsilon N W h^2\sqrt{T_3}}{M_p^2|H'|}. \end{aligned} \tag{4.199}$$

If we satisfy the condition that $r \leq 0.24$ then we find from the scalar amplitude, saturating the bound, that $8H^2 \sim 10^{-9}M_p^2$ or more concretely that

$$NV(\phi) \sim 10^{-10}\left(\frac{M_p}{M_s}\right)^4, \tag{4.200}$$

assuming that the string coupling is around 10^{-2}. This will be extremely difficult to satisfy under the assumption of large N, and also vacuum energy dominance, unless we are willing to postulate a low string scale

(ii) If we consider the non-relativistic limit, we find the solutions

$$\begin{aligned} A_s^2 &\sim \frac{H^2}{8\pi^2 M_p^2 \varepsilon}\left(1+\frac{2M_p^2 H'^2}{N^2W^2h^4T_3}+\dots\right), \\ r &\sim 16\varepsilon\left(1-\frac{M_p^2H'^2}{N^2W^2h^4T_3}+\dots\right). \end{aligned} \tag{4.201}$$

The first expression implies that $\varepsilon >> H^2$ at horizon crossing, which can be satisfied provided that the energy density of the inflaton is vanishingly small. In fact if this condition is met, the ratio will simultaneously be satisfied. A quick calculation shows that if we demand $r \leq 0.24$ (i.e using the strongest possible WMAP bound), then the energy density must satisfy $E \leq 10^{-7} M_P^4$ which is very small in Planck units.

Using the fact that $f_{NL} \leq 100$ we can obtain a bound on N for our theory. For an arbitrary warp factor we see that

$$N \geq \sqrt{\frac{32\pi g_s}{299}\frac{|H'|}{Wh^2}\left(\frac{M_p}{M_s}\right)^2}. \tag{4.202}$$

To an order of magnitude approximation the numerical factor is $\mathcal{O}(1)$. Now for mass gap backgrounds we may generally expect $W \sim 1$ or $W >> 1$ and h is a small constant which forces N to be large. This will be further enhanced by a smaller string scale unless the derivative of the Hubble parameter is vanishingly small. For the AdS scenarios we have competition between the W and h^2 terms in the denominator - so we would expect N to be set explicitly by the choice of background fluxes and the string scale.

The general form for the scalar spectral index can be calculated to give

$$\begin{aligned} n_S - 1 &= -2\varepsilon\left(2-\frac{H}{H'}\Delta\right) \qquad (4.203) \\ \Delta &= \frac{H''(\gamma^2+1)}{2H'\gamma^2}+\frac{(W^2-1)(2\gamma^2-1)}{W\gamma^2\phi}+\frac{h'}{h\gamma^2W}\left(W(\gamma^2-1)+(W^2-1)(1-2\gamma^2)\right) \end{aligned}$$

for an arbitrary warp factor. In principle the backreaction effects will appear through a redefinition of the harmonic function, and so this expression should be valid for all theories satisfying our assumptions. This equation simplifies once we assume relativistic motion, i.e $\gamma >> 1$

$$\Delta \sim \frac{H''}{2H'}+\frac{2(W^2-1)}{W\phi}+\frac{h'}{Wh}\left(W-2(W^2-1)\right). \tag{4.204}$$

For AdS type solutions the fuzzy potential is constant, however we can see that the limiting solutions are

$$\begin{aligned} \Delta &\sim \frac{H''}{2H'} + \frac{h'}{h} && W \sim 1 \\ \Delta &\sim \frac{H''}{2H'} + 2W\left(\frac{1}{\phi} - \frac{h'}{h}\right) && W >> 1. \end{aligned} \tag{4.205}$$

Clearly in our simplest case analyzed in the previous section we see that for $W >> 1$ the second term will be identically zero thus cancelling out any dangerous $1/\phi$ dependence. For the mass gap solution we find similar expressions to those presented above except that the h' terms will be zero - at least to leading order. Unless the backreaction dramatically alters the solution, we should expect inflation to favour the $W >> 1$ regime, in which case the scalar index is essentially only a function of the potential and its derivatives - the overall scale being set by the ε term.

In general we expect the number of e-foldings to be enhanced by factors of N thus making the universe generically very flat. However the factors of N tend to increase the level of scalar and tensor perturbations, making it difficult to satisfy observational bounds without imposing restrictive fine tuning. Very recently it was shown that this model is in fact the only DBI inflation model which gives testable predictions for the gravitational wave spectrum [109]. The other models have almost vanishingly small tensor fluctuations, well below the proposed sensitivity of Plank.

4.4.2 Inflation at finite N

In this section, we investigate cosmic inflation due to a small number of coincident branes. We begin with some general remarks about the finite N formalism before specialising to two simple cases, namely $N = 2$ and $N = 3$. In these cases the action is highly non-linear and gives an expression for the speed of sound and the inflationary parameters in certain regions of the phase space. They are very different from those in the single brane models.

General remarks and motivations

We will now switch to the finite N formulation of the non-Abelian Myers action, using the prescription for the symmetrized trace as given in [53]. We believe this prescription to be correct, however a concrete proof remains an outstanding problem. In most models of brane inflation, the bulk fluxes are tuned so that only a single brane is left after the brane-flux annihilation process [93]. In the context of the landscape, this is a very special case and the general expectation is that there remain several residual branes which will tend to coincide to minimise their energy in a warped throat. In the first part of this section we looked at the large N limit, which has many problems due to the large back-reactive effects on the geometry although it is a more general solution than the single brane cases. In the remainder of the section we will look at the solution when there are a handful of residual branes at the tip of a warped throat. This means that we can effectively neglect back-reactive effects as in the single brane models, while still retaining the enhanced non-Abelian world-volume symmetry.

Let us rewrite the expressions for the energy and pressure of the coincident branes in a more suitable manner for a finite N analysis:

$$\begin{aligned} E &= T_3 \mathrm{STr}\left(h^4 \sum_{k,p=0}^{\infty} (-X\dot{R}^2)^k Y^p (\alpha^i\alpha^i)^{k+p}(1-2k) \binom{1/2}{k}\binom{1/2}{p} + V - h^4 \right) \\ P &= -T_3 \mathrm{STr}\left(h^4 \sum_{k,p=0}^{\infty} (-X\dot{R}^2)^k Y^p (\alpha^i\alpha^i)^{k+p} \binom{1/2}{k}\binom{1/2}{p} + V - h^4 \right), \end{aligned} \tag{4.206}$$

where we have used the following definitions

$$X = \lambda^2 h^{-4}, \qquad Y = 4\lambda^2 R^4 h^{-4}, \qquad \binom{1/2}{m} = \frac{\Gamma(3/2)}{\Gamma(3/2-m)\Gamma(m+1)}, \tag{4.207}$$

and the fact that the scalar potential is a singlet under the trace. We employ the symmetrization procedure in [53]. The basic formulas we need are then once again

$$\begin{aligned} \mathrm{STr}[(\alpha^i\alpha^i)^m] &= 2(2m+1)\sum_{i=1}^{(n+1)/2}(2i-1)^m \\ &= 2(2m+1)\sum_{i=1}^{n/2}(2i)^m. \end{aligned} \tag{4.208}$$

The first line corresponds to odd $n = N-1$, and the second line to even n. Note that we move from working with the N-dimensional representation to the spin representation with $n = 2J$. It is important to consider what we mean by a physical radius in this context. We use the definition

$$r^2 = \lambda^2 R^2 \mathrm{Lim}_{m\to\infty}\left(\frac{STr(\alpha^i\alpha^i)^{m+1}}{STr(\alpha^j\alpha^j)^m}\right) = \lambda^2 R^2 n^2, \tag{4.209}$$

which implies that the Lagrangian will converge for velocities from 0 to 1, and moreover that the radius of this convergence will be unity. This definition is consistent with what we know about the solution in the large N limit.

To illustrate the additional complexity arising from the finite N solution, let us calculate the speed of sound in two examples using (4.139). For $N = 2$, we find

$$C_s^2(N=2) = \frac{(1-X\dot{R}^2)(3+4Y-X\dot{R}^2[2+3Y])}{3+Y(4-X\dot{R}^2)}, \tag{4.210}$$

where $R^2 = r^2/\lambda^2$. This is obviously far more complicated than the large N expression which is $C_s^2 = 1 - X\dot{R}^2$. For $N = 3$, we obtain

$$C_s^2(N=3) = \frac{(1 - 4X\dot{R}^2)(3 + 16Y - 8X\dot{R}^2[1 + 6Y])}{3 + 16Y(1 - X\dot{R}^2)}, \tag{4.211}$$

where now $R^2 = r^2/(4\lambda^2)$. Note that in both cases, we recover the usual result that $C_s^2 = 1$ when the velocity of the branes is zero.

4.4.3 Two brane inflation.

In this subsection, we consider inflation driven by two coincident branes moving in the warped throat. This gives rise to a $U(2)$ symmetry on the world-volume. The relevant expressions for the energy and pressure can be calculated using (4.206) and (4.208):

$$\begin{aligned} E_2 &= 2T_3\left(\frac{h^4(1 + 2Y - XY\dot{R}^2)}{\sqrt{1+Y}(1 - X\dot{R}^2)^{3/2}} + V - h^4\right) \\ P_2 &= -2T_3\left(\frac{h^4(1 + 2Y - X\dot{R}^2[2 + 3Y])}{\sqrt{1+Y}\sqrt{1 - X\dot{R}^2}} + V - h^4\right). \end{aligned} \tag{4.212}$$

Note that in order to keep the energy finite, we should impose the constraint $\dot{\phi}^2 \leq h^4 T_3$. This also ensures that the contribution coming from the DBI part of the action will be non-negative, as can be seen from the first term in the numerator of the energy equation.

In general it is difficult to get solutions due to the complicated form of the energy density. So let us make the approximation that the inflaton is rolling ultra relativistically. We can define the relativistic factor γ much as we did in the large N solution by $\gamma = (1 - X\dot{R}^2)^{-1/2}$.

We now write the energy and pressure as functions of γ, and then take the large γ limit. By utilising the conservation equation and dropping all acceleration terms, we can find the solution

$$\gamma^3 \sim \mp\frac{M_p^2 H'}{\sqrt{T_3(1+Y)}h^2}, \tag{4.213}$$

where H' is the derivative of the Hubble parameter with respect to ϕ. The sign in (4.213) corresponds to the two choices $\dot{\phi} = \pm\sqrt{T_3}h^2 + ...$ in the expansion of the velocity $\dot{\phi}$ about its saturation value. The $-$ sign is for the choice $\dot{\phi} > 0$ while the $+$ sign for $\dot{\phi} < 0$. Note that in order to have $\gamma > 0$, we must demand that $H' < 0$ for $\dot{\phi} > 0$ and $H' > 0$ for $\dot{\phi} < 0$. The choice of sign here is vital to obtaining the correct solution branch for inflation.

We can rewrite the speed of sound as a function of γ and Y as follows:

$$C_s^2 = \frac{1}{\gamma^2}\left(\frac{\gamma^2(1+Y) + 2 + 3Y}{3\gamma^2(1+Y) + Y}\right), \tag{4.214}$$

which shows the finite N corrections to the equation in this limit. Without recourse to a specific background, we can make the following observation: If we consider limiting solutions for Y, i.e that it is either $>> 1$ or $<< 1$, all Y dependence drops out of the expression and the equation reduces to $C_s^2 \sim 1/(3\gamma^2)$. The sound speed is thus only a third of that in the large N limit. This appears to be the attractor point for the velocity with this action regardless of background choice. In this case we find (4.179) is given by

$$\Lambda = \frac{X\dot{R}^2(5 + 6Y - XY\dot{R}^2)}{2(1 - X\dot{R}^2)(3 + 4Y - XY\dot{R}^2)}. \tag{4.215}$$

In the $\gamma \gg 1$ case, the non-linearity parameter is

$$f_{NL} \approx 0.24\gamma^2, \tag{4.216}$$

which is a little smaller than the large N (and single brane) solution and effectively means that we can satisfy the observational bounds whilst considering larger velocities than in the single brane scenario.

The corresponding Friedmann equation in this case becomes

$$H^2 = \frac{E_2}{3M_p^2}, \tag{4.217}$$

where we are using the energy as defined in (4.212). Substitute our expression for γ into this equation, we find that the Hamilton Jacobi equation for $N = 2$ (in the large γ limit) becomes

$$H^2(\phi) \sim \frac{2T_3}{3M_p^2}\left(V(\phi) - h^4 \mp \frac{h^2 M_p^2 H'}{\sqrt{T_3}}\right), \tag{4.218}$$

for an arbitrary flux induced potential. At this stage we could either specify the form of the Hubble parameter and then consider how this modifies the potential, or we could specify the form of the potential and then solve for H. We use this latter approach as this appears to be more within the spirit of the Hamilton-Jacobi formalism we have employed thus far.

Inflation in AdS type backgrounds

Let us first consider a solution where the potential is dominated by a constant, $V \sim V_0$. Let us also assume that the background is approximately AdS. In the small field limit, we expect the h^4 term is negligible compared to the remaining terms. So as a first approximation we ignore its contribution. Solving this differential equation, with an appropriate constant of integration $\tilde{C}$, we obtain

$$H^2(\phi) \sim \frac{2T_3V_0}{3M_p^2}\tanh^2\left(\sqrt{\frac{3V_0}{2M_p^2}}\frac{L^2}{\phi}(-1 + \tilde{C}\phi)\right). \tag{4.219}$$

Since $H' < 0$, we have assumed $\dot{\phi} > 0$ in obtaining this solution. In the limit of very small ϕ, we can approximate the solution by $H^2 \sim 2T_3V_0/(3M_p^2)$, which is of the same functional form as in the large N case. However this expression is not consistent with our approximation that $\gamma >> 1$ since the ratio H'/ϕ^2 is approximately zero for vanishingly small values of ϕ. Therefore we must be careful to choose a regime of validity where this solution is valid. Careful inspection shows that the function γ^3 has a turning point in the small field limit, with a maximal value given by

$$\gamma^3_{\max} \sim \frac{256e^{-4}}{9}\frac{M_p^4}{L^4V_0^{3/2}\sqrt{1+Y}}, \tag{4.220}$$

where Y is constant in the AdS background, and the value of the field at this point is given by $\phi = \phi_{\max} \sim L^2\sqrt{6V_0}/(4M_p)$. Thus to consider inflation in this region, we need some 'extreme fine tuning' to set up the initial value of the field.

Rather than proceeding this way, we make a Taylor series expansion of the Hubble parameter for small ϕ but without dropping the h^4 term in (4.218). This means we must include quartic terms in the expansion of the Hubble parameter and so we need

$$H(\phi) \sim \sum_{i=0}^{4} H_i\phi^i, \tag{4.221}$$

We will also keep quartic terms in the inflationary potential for consistency, $V(\phi) \sim V_0 - V_2\phi^2/2 - V_4\phi^4/4 + \ldots$. Equating the various coefficients, we find that the linear term in the Hubble parameter actually vanishes, leaving us with the residual terms

$$\begin{aligned} H_0 &= \sqrt{\frac{2T_3V_0}{3M_p^2}}, \\ H_2 &= -\frac{T_3V_2}{6M_p^2H_0}, \\ H_3 &= \frac{2\sqrt{T_3}H_2}{3L^2H_0}, \\ H_4 &= -\frac{1}{12M_p^2L^4H_0\sqrt{T_3}}\left(V_4L^4T_3^{3/2} + 4T_3^{3/2} - 12M_p^2H_3L^2T_3 + 6M_p^2L^4\sqrt{T_3}H_2^2\right). \end{aligned} \tag{4.222}$$

We find that the constant piece of the potential dominates the Hubble parameter when the field is vanishingly small. The sign of the last term is potentially ambiguous which can lead to interesting cosmological behaviour. It turns out that the Hubble term is extremised at the usual $\phi = 0$ solution (which is a local maximum), and there exists a non-trivial solution given by

$$\phi_{min} = \frac{1}{8H_4}\left(-3H_3 \pm \sqrt{9H_3^2 - 32H_2H_4}\right), \tag{4.223}$$

where we must require the term inside the square root to be non-negative. We can rewrite this reality constraint as

$$\frac{H_2}{H_4} \geq \frac{8H_0^2 L^4}{T_3}. \tag{4.224}$$

We now find the possibility of a 'cosmic turnaround' because H_2 will be negative definite for those regions of phase space where H_4 is also negative. If we concentrate on regions of ϕ near the origin, $H' < 0$ so we necessarily have $\dot{\phi} > 0$ in order for $\gamma > 0$ in our approximation (4.213). Of course, we have implicitly assumed that the field is monotonic so we cannot say anything about the reality of such a bounce solution within the current framework.

It is easy to solve the equation of motion in this limit for relativistic motion. As in the case of a single brane, and for the large N solution, we obtain the following term for the inflaton equation of motion

$$\phi \sim \frac{\phi_0}{1 - \phi_0 \sqrt{T_3}(t - t_0)/L^2}, \tag{4.225}$$

where again we define ϕ_0 as the field value at time $t = t_0$, and it can be seen that $\dot{\phi} > 0$.

To see the implications of this for inflation, we must first determine which are the relevant parameters in this finite N formulation. The modified 'fast roll' parameter in this case can be written as

$$\varepsilon \sim \pm \frac{\phi^2 \sqrt{T_3}}{L^2} \frac{H'}{H^2}, \tag{4.226}$$

where the sign is related to the sign for $\dot{\phi}$ and we need to demand that inflation ends when $\varepsilon = 1$ as usual. Recall that for small ϕ, $H' < 0$ and we need to choose the minus sign in this equation (coming from the choice $\dot{\phi} > 0$). Inserting our expression into the Hubble parameter, we see that ε can be expanded in powers of the inflaton. Keeping only the leading order term (which amounts to dropping $\mathcal{O}(\phi^4)$ contributions), we see that inflation will end around

$$\phi_e \sim \left(\frac{L^2 H_0^2}{2|H_2|\sqrt{T_3}} \right)^{1/3}. \tag{4.227}$$

The corresponding number of e-foldings given by this Hubble parameter is generically a power series in ϕ. We expect the dominant contribution to arise from the constant piece H_0 as in the standard inflationary scenario. Integrating over the field, we find the expressions for the inflaton as a function of e-folding number:

$$\phi_0 \sim \phi_e \left(1 + \frac{N_e \sqrt{T_3} \phi_e}{L^2 H_0} \right)^{-1}. \tag{4.228}$$

Inserting this back into the 'fast roll' parameter (4.226), we find that the dependence on the number of e-foldings is of the same functional form as in the large N case:

$$\varepsilon \sim \left(1 + N_e \left(\frac{3M_p^2}{L^4 V_2} \right)^{1/3} \right)^{-3} \tag{4.229}$$

For the perturbation amplitudes we can use the general results developed in the appendix. We find that the gravitational wave amplitude will be constant to leading order, and given by

$$\mathcal{A}_T^2 \sim \frac{4T_3V_0}{3\pi^2 M_p^2} = \frac{V_0}{6\pi^5 g_s}\left(\frac{M_s}{M_p}\right)^4 \tag{4.230}$$

the corresponding expression for the scalar amplitude is given by

$$\mathcal{A}_S^2 \sim \frac{V_0\gamma}{32\sqrt{3}\pi^5 g_s}\left(\frac{M_s}{M_p}\right)^4 \left(1+60\left\{\frac{3M_p^2}{L^4V_2}\right\}^{1/3}\right)^3 \tag{4.231}$$

We choose to re-write this in terms of the tensor amplitude as

$$\mathcal{A}_S^2 \sim 10^{-1}\gamma\mathcal{A}_T^2\left(1+60\left\{\frac{3M_p^2}{L^4V_2}\right\}^{1/3}\right)^3 \tag{4.232}$$

Now the non-gaussianity condition implies that the bound $1 << \gamma^2 << 400$ must be satisfied to comply with observation. The spectral indices for this model at large γ are

$$\begin{aligned} n_T &\sim \frac{2H'\sqrt{T_3}\phi^2}{H^2L^2} \\ n_S - 1 &\sim -\frac{4X_2}{(1+60X_2)} + \dots \end{aligned} \tag{4.233}$$

where the first line is understood to be evaluated at horizon crossing, and we have written $X_2 = (3M_p^2/L^4V_2)^{1/3}$ as a dimensionless parameter. Note that the tensor index is negative, but suppressed by the Hubble parameter. Clearly the scalar index is bounded from above by unity, and so normalising to WMAP data implies that $0 \le X_2 \le 0.05$, or more concretely that

$$\frac{L^4V_2}{2.4\times 10^4} \ge M_p^2. \tag{4.234}$$

Using this constraint in (4.232) we see that the term in brackets varies between 1 and 64. The relationship between the two amplitudes is characterised by the parameter r and so we recover the anticipated DBI relation

$$r \sim \frac{1}{\gamma}. \tag{4.235}$$

The fact that $1 << \gamma \le 20$ in this model implies that r will generically be small and thus well within the WMAP confidence bounds. [91] Furthermore this implies that the tensor amplitude will be smaller in magnitude than the scalar one, something like 10^{-10} for a range of γ. Using the normalisation for the scalar amplitude, namely that it satisfies $\mathcal{A}_S^2 \sim 10^{-9}$ at horizon crossing, we see that this constrains the potential in terms of the string scale. For small X_2 we find

$$\frac{V_0\mathcal{O}(10^3)}{g_s}\left(\frac{M_s}{M_p}\right)^4 \ge 1 \tag{4.236}$$

where we are interested in an order of magnitude approximation. Whilst for the maximal value of X_2 we recover the same constraint but with an additional factor of 10^3 in the numerator. Clearly both solutions are sensitive to the mass splitting between Planck and string scales and imply that generic inflation prefers the string scale to be close to the Planck scale in order not to have super-Planckian scalar potentials. For example if we have $M_s \sim 10^{-1}M_p$ then the potential constraint becomes $\mathcal{O}(10^1 - 10^4)V_0 \geq 1$. What about the constraint in (4.234)? Upon substituting for the background parameters we see that this equation can be written as

$$V_2 \geq \frac{\mathcal{O}(10^5)}{MK}\frac{M_p^2}{M_s^4} \tag{4.237}$$

which gives us a constraint on the inflaton mass scale.

Inflation in mass gap backgrounds

If we repeat the analysis for the mass gap backgrounds, (assuming that the constant part of the potential dominates), we find the following solution for the Hubble parameter:

$$H(\phi) = \pm\sqrt{\frac{2T_3}{2M_p^2}(V_0 - h^4)}\,\tanh\left(\sqrt{\frac{3(V_0-h^4)}{2M_p^2}}\frac{\phi+\tilde{C}}{h^2}\right) \tag{4.238}$$

where we have used the fact that the warp factor is constant to write the solution as a function of h.

Substituting this into the gamma constraint, we must require that the solution is larger than unity even when ϕ is vanishingly small. We make a Taylor series expansion of the resultant function, and find that $\gamma^3 \propto \mathrm{sech}^2(\mathrm{F}(\tilde{\mathrm{C}}))$, where the amplitude of the function is determined by the ratio of the potential and the warp factor. Now the hyperbolic trigonometric function is a decreasing function of its argument, which forces us to take the limit $\tilde{C}^2 << h^4/V_0$ in order for the large velocity expansion to hold.

Let us assume that we can in fact take this limit and consider the implications for inflation. Calculation of the fast roll parameter ε yields the following

$$\varepsilon \sim \frac{3}{2}\left(\mathrm{Cosh}^2\left(\frac{\sqrt{6(\mathrm{V}_0-\mathrm{h}^4)}\phi}{2\mathrm{M_p}\mathrm{h}^2}\right)-1\right)^{-1}, \tag{4.239}$$

which is a decreasing function of the inflaton field. Thus after some critical field value ϕ_c, we will find a solution where inflation never ends. It may appear that this is an artifact due to the neglect of higher order terms in the potential. However if we consider quartic terms in $V(\phi)$, and also up to the same order in a Taylor expansion of $H(\phi)$, we find the same result that ε is a decreasing function of the inflaton. We conclude that in the relativistic limit that inflation (once started) never ends[14] unless we turn on extra effects, such as nonzero gauge fields. This result is not anything that we expect from the results of the single brane case, or the large N limit discussed in the previous sections and appears to be a distinctly finite N effect. Of course, it may well be that standard inflation can occur for moderate values of γ. However this would almost certainly require the use of numerical methods.

Of course the mass gap background will eventually give way to something similar to the Ads solution, where the harmonic function will have explicit dependence upon the inflaton field. Therefore we expect inflation to end in this regime. The fact that the mass gap solution has finite warping means that it will be relatively easy to produce the necessary 60 e-folds of expansion.

Non-relativistic limit

Let us restrict ourselves to the non-relativistic regime in order to see the consequences for brane inflation [15]. It would of course be more preferable to obtain an interpolating solution between these two extremes, however it is analytically challenging and would be better suited to a numerical analysis. After performing a series expansion of the continuity equation, we find the following solution for the velocity of the field in the Hamilton-Jacobi formalism:

$$\dot{\phi} = -\frac{\sqrt{1+Y}M_p^2 H'}{(3+4Y)}, \tag{4.240}$$

which means that the corresponding Friedmann equation reduces to

$$\frac{3M_p^2 H^2}{2T_3} \sim \frac{h^4}{\sqrt{1+Y}}\left(1+2Y+\frac{Z_2 M_p^4 H'^2}{2T_3 h^4}\right) + V(\phi) - h^4, \tag{4.241}$$

where $Z_2 = (1+Y)/(3+4Y)$. Let us solve this equation by considering a standard quadratic potential. There are two solution branches, one of which has an imaginary component of H. We ignore this solution as it appears unphysical. The other real solution can be parameterised by a quadratic Hubble parameter with nonzero components given by

$$H_0 = \sqrt{\frac{2T_3 V_0}{3M_p^2}} \qquad H_2 = \frac{9H_0}{M_p^2}\left(1 \pm \sqrt{1+\frac{V_2 M_p^2}{18V_0}}\right). \tag{4.242}$$

For inflation to occur we must ensure that we take the minus sign in the solution for H_2.

[14]Technically this is no longer true once the branes reach the gluing region. However the effective action is no longer expected to be a good description of the physics in this region.

[15]This has recently been examined in [104].

The speed of sound in this instance reduces to

$$C_s^2 \sim 1 - \frac{15\dot{\phi}^2}{h^4 T_3(3+4Y)}\left(1 + \frac{5Y}{4} + \ldots\right) \tag{4.243}$$

where we can substitute $\dot{\phi}$ for derivatives of the Hubble parameter.

The non-relativistic assumption means that we can find inflationary solutions even in the mass gap backgrounds. Using the definition of the ε parameter we find that the leading order contribution yields

$$\varepsilon \sim e^{2\beta N_e} \tag{4.244}$$

at N_e e-folds before the end of inflation. We have introduced the dimensionless ratio $\beta = M_p^2 H_2/H_0$ for simplicity. For inflation to occur we must ensure $\beta < 0$, which implies that $H_2 < 0$. However the fact that the slow roll parameter is now exponential implies that the level of scalar perturbations will now be enhanced by this exponential term. Our assumption is that ϕ is small, so the speed of sound is essentially unity in this instance. It transpires that the simplest equation to study is the tensor to scalar ratio r, which is now given by $r \sim 16\varepsilon$. If we demand that $r \leq 1/4$ to satisfy the more stringent bound, then after some algebra we find the following constraint

$$V_2 \geq \frac{0.1 V_0}{M_p^2} \tag{4.245}$$

where we have explicitly left in the numerical value. If this bound is not satisfied then we find H_2 to be very small which suppresses the number of e-foldings and sufficient inflation is generically difficult to achieve.

For Ads type backgrounds we have the following solution for the Hubble parameter

$$H_0 = \sqrt{\frac{2T_3V_0}{3M_p^2}} \qquad H_2 = \frac{3H_0(3+4Y)}{M_p^2\sqrt{1+Y}}\left(1 \pm \sqrt{1 + \frac{T_3V_2\sqrt{1+Y}}{9H_0^2(3+4Y)}}\right) \tag{4.246}$$

where we must again take the minus sign for an inflationary solution. Interestingly for $Y << 1$ we see that the solution becomes exactly the same as the mass gap one. We can follow the same procedure and obtain a similar result for the number of e-folds. The difference is of course due to the constant nature of the factor Y.

$$\varepsilon \sim \exp\left(\frac{2\delta N_e\sqrt{1+Y}}{3+4Y}\right) \tag{4.247}$$

where δ is defined in a similar way to β except that the Hubble parameters are different in this case.

The observed constraint on the amplitude ratio can now be written as

$$V_2 \geq \frac{0.1V_0}{M_p^2}F(Y) \qquad F(Y) = \frac{3+4Y}{\sqrt{1+Y}} \tag{4.248}$$

which is slightly different from the mass gap solutions. The function $F(Y)$ acts to increase the RHS of the expression above, ranging from $F(Y) \sim 3 \to 4\sqrt{Y}$ depending upon our choice of fluxes. Larger values of Y clearly impose tighter constraints on the parameter V_2 and so one would anticipate that smaller values are more preferential.

In both instances we see that in order to satisfy the scalar curvature constraints, we require the dominant term in the potential to satisfy the following

$$V_0 \leq 3 \times 10^{-7}\left(\frac{M_p}{M_s}\right)^4, \tag{4.249}$$

which can be combined with the expressions for V_2 to yield a constraint purely on that variable in terms of the Planck and string scales (and also the fluxes for the AdS case). We have again assumed a string coupling of $g_s \sim 10^{-2}$ in the above expression.

4.4.4 Three brane inflation

Let us now move to the case where there are three coincident branes, giving rise to a $U(3)$ world-volume symmetry. We have the energy and pressure:

$$\begin{aligned} E &= 2T_3\left(\frac{h^4(1+4Y[1+X\dot{R}^2])}{\sqrt{1+2Y}(1-2X\dot{R}^2)^{3/2}} + \frac{3V}{2} - \frac{3h^4}{2}\right), \\ P &= -2T_3\left(\frac{h^4(1+4Y-4X\dot{R}^2[1+3Y])}{\sqrt{1-2X\dot{R}^2}\sqrt{1+2Y}} + \frac{3V}{2} - \frac{3h^4}{2}\right). \end{aligned} \tag{4.250}$$

The symmetry breaking induced by a gauge field in this case will be $U(3) \to SU(3) \times U(1)$. Let us again consider the large γ solution for the fast rolling action. It is convenient to define

$$\gamma = \frac{1}{\sqrt{1-2X\dot{R}^2}}, \tag{4.251}$$

which allows us to write the energy and pressure in (4.250) as explicit functions of γ.

However the exact solution for the speed of sound can be written as a function of γ

$$C_s^2 = \left(\frac{\gamma^2-2}{\gamma^2}\right)\left(\frac{\gamma^2(1+8Y)-4(1+6Y)}{\gamma^2(3+8Y)+8Y}\right), \tag{4.252}$$

which, unlike the other solutions studied so far, allows for the possibility that the sound speed is zero. This is the case if either of the following critical conditions are satisfied:

$$\gamma_c^2 = 2 \qquad \text{or,} \qquad \gamma_c^2 = \frac{4(1+6\mathrm{Y})}{(1+8\mathrm{Y})}. \tag{4.253}$$

The first condition corresponds to $\dot{\phi}^2/h^4 = T_3/4$. The second condition is a little more difficult to deal with due to the potential ϕ-dependence of Y. Now we can consider the two simplifying limits. (i) In the limit where $Y \to 0$, we see that the constraint on the velocity becomes $\dot{\phi}^2/h^4 = 3T_3/8$, while (ii) in the converse limit (where Y is dominant), we see that $\dot{\phi}^2/h^4 = T_3/3$. All of these conditions are allowed because they satisfy the causality constraint on the velocity. We know that fluctuation modes exit the horizon at the reduced scale $kC_s = aH$ in these models, so a zero speed of sound tells us that the modes will never exit the horizon. In order to consider an inflationary epoch, we have to ensure the velocity is either much smaller than either of the critical bounds (corresponding to non-relativistic motion), or much higher corresponding to ultra relativistic motion. Thus unlike the case of $N = 2$, we are lead to selecting a specific velocity range. Even for $Y \sim \mathcal{O}(1)$, we find that C_s rapidly tends towards unity as in normal models of scalar field inflation.

In order to consider inflationary solution, we start with the continuity equation. Taking the large velocity limit we find the general result

$$\gamma^3 = \frac{-(\pm 1)M_p^2 H' \sqrt{2(1+2Y)}}{h^2\sqrt{T_3}(1+6Y)}, \tag{4.254}$$

where we have made use of the fact that $\dot{\phi} = \pm\sqrt{2T_3}h^2 + ..$ in this limit. The sign ambiguity here can be resolved by demanding γ to be positive. Since we are interested in solutions where $H' < 0$, we take the $+$ sign in the definition of the velocity. Substituting our expression back into the Friedmann equation (4.250) yields the Hamilton-Jacobi equation

$$\frac{M_p^2 H^2}{T_3} = V(\phi) - h^4 - \frac{h^2 M_p^2 H'}{3}\sqrt{\frac{8}{T_3}}, \tag{4.255}$$

which can again be integrated to solve for H once we specify the background potential. The level of non gaussianities arising from this action can be parameterised by

$$f_{NL} \sim \frac{162}{85(1+8Y)}\left(1 + \frac{10(1+8Y)}{51}\gamma^2\right) \tag{4.256}$$

which clearly has non-trivial dependence on the inflaton field for the mass gap backgrounds (since Y is constant for the AdS solutions). Let us explore the possible solution space here. For the AdS case we find that

$$Y = \frac{KMg_s}{4a\pi} \tag{4.257}$$

and so can be small with appropriate tuning of the fluxes and the string coupling. If we assume $Y << 1$ then we see that the non-gaussianities are (up to $\mathcal{O}(Y^2)$ terms - and dropping the constant piece)

$$f_{NL} \sim 0.37\gamma^2 + \dots, \tag{4.258}$$

whilst if we assume that Y is large (corresponding to large fluxes) we find the following

$$f_{NL} \sim 2.99\gamma^2 + \mathcal{O}\left(\frac{1}{Y}\right). \tag{4.259}$$

The latter condition is much larger than anything encountered before, and severely restricts the relativistic approximation we have been making. In fact if we have $Y \sim \mathcal{O}(1)$ we find a similar condition. However for small Y we see that the non-gaussianities are roughly the same as in the previous sections, and would appear to be the more favourable regime for inflation. This tells us that we require $g_s << 1/(MK)$.

Inflation in AdS type backgrounds

It is generically difficult to find inflationary solution for AdS backgrounds. To proceed with our small ϕ, but large gamma solution, we again turn to a Taylor series approach to the Hubble parameter. Let us take the same form for the expansion as in the last section, with a similar expression for the inflaton potential. Again we find that there is no linear dependence in this limit, but the non zero coefficients can be seen to be

$$\begin{aligned} H_0 &= \sqrt{\frac{V_0 T_3}{M_p^2}}, \\ H_2 &= -\frac{V_2 T_3}{4M_p^2 H_0}, \\ H_3 &= -\frac{\sqrt{8T_3}H_2}{3L^2 H_0}, \\ H_4 &= -\frac{1}{8M_p^2 L^4 H_0}\left(V_4 L^4 T_3 + 4T_3 + 8M_p^2 H_3 L^2 \sqrt{2T_3} + 4M_p^2 L^4 H_2^2\right). \end{aligned} \tag{4.260}$$

The conditions for inflation are basically the same as in the previous section, which is to be expected since we are assuming that inflation is essentially driven by the constant part of the Hubble parameter. The slow roll parameter is shifted only slightly by the extra brane because the velocity in this case is increased by an extra factor of $\sqrt{2}$. Solving for the inflaton at the end of inflation, we find

$$\phi_e \sim \left(\frac{L^2 H_0^2}{\sqrt{8T_3}|H_2|}\right)^{1/3}, \tag{4.261}$$

where we have absorbed the minus sign into the definition of $|H_2|$ to make the solution manifestly positive. This is only slightly different from that obtained in the $N = 2$ case. By integrating the Hubble term, we can invert again the resulting expression to obtain the inflaton as a function of the number of e-foldings. The result is the same as for $N = 2$

except that now the tension is doubled. Finally we obtain ε as a function of the number of e-foldings

$$\varepsilon \sim \left(1 + N_e \left(\frac{4M_p^2}{L^4 V_2}\right)^{1/3}\right)^{-3} \tag{4.262}$$

which represents only a slight numerical shift with regard to the expression in the previous section for $N = 2$. We can once again calculate the relevant signals for this model, and the analysis proceeds much as in the case of $N = 2$, except that we are now forced to restrict ourselves to the $\gamma >> 1$ solution. The tensor amplitude at leading order becomes

$$\mathcal{A}_T^2 \sim \frac{V_0}{4\pi^4 g_s} \left(\frac{M_s}{M_p}\right)^4 \tag{4.263}$$

which is a factor of 3/2 larger than the amplitude in the $N = 2$ case (4.230). Whilst the scalar amplitude can again be written solely in terms of the tensor amplitude divided by the parameter r.

The tensor spectral index is relatively suppressed as in the $N = 2$ case, however for the scalar index we find

$$n_S - 1 \sim -\frac{4X_3}{1 + 60X_3} \tag{4.264}$$

which is identical in form to the $N = 2$ solution in (4.233) under the replacement by $X_3 = (4M_p^2/L^4 V_2)^{1/3}$. The same remarks apply here except the physical constraint is slightly tighter than before

$$\frac{L^4 V_2}{3.2 \times 10^4} \geq M_p^2. \tag{4.265}$$

Using the small Y constraint in order to suppress the non-gaussianities we can write this constraint purely in terms of the potential term

$$V_2 >> \frac{g_s \times 10^9 M_p^2}{M_s^4} \qquad (Y << 1) \tag{4.266}$$

Let us consider the two limiting solutions, bearing in mind that we expect that the $Y >> 1$ case will lead to extremely large non-gaussianities. For small Y we see that the sound speed becomes

$$C_s(Y << 1) \sim \frac{\sqrt{\gamma^2 - 4}}{3\gamma} \tag{4.267}$$

and so if we also assume that $\gamma^2 >> 4$ then we see that this becomes 1/3. This is unlike all the other DBI models studies so far. Interestingly if we take the limit where $Y >> 1$ we also see that it drops out of the analysis

$$C_s(Y >> 1) \sim \frac{\sqrt{\gamma^2 - 28/8}}{\gamma} \tag{4.268}$$

and in fact we find that $C_s \sim 1$ as in standard slow roll models of inflation. Following the same procedure as in the $N = 2$ case we can constrain the potential using the scalar amplitude. We find that the range for V_0 is

$$\begin{aligned} V_0 &\sim \mathcal{O}(10^{-10} - 10^{-9}) \left(\frac{M_p}{M_s}\right)^4 \qquad Y << 1 \\ V_0 &\sim \mathcal{O}(10^{-9} - 10^{-7}) \left(\frac{M_p}{M_s}\right)^4 \qquad Y >> 1 \end{aligned} \tag{4.269}$$

which once again indicates the sensitivity of inflation to the string scale.

Inflation in mass gap backgrounds.

Let us now restrict our analysis to the mass gap backgrounds. We can again solve the master equation assuming that the constant part of the potential dominates the solution. The result is

$$H(\phi) = \frac{\sqrt{T_3(V_0 - h^4)}}{M_p} \tanh\left(\frac{3(\phi + \tilde{\mathrm{C}})}{\sqrt{8}\mathrm{M_p h^2}}\sqrt{\mathrm{V_0 - h^4}}\right), \tag{4.270}$$

which should be valid for small values of the inflaton field, and we have left the mass gap warp factor as an arbitrary constant. We must ensure that this expression is consistent with our demand that the γ factor is large. This requires us firstly to take the minus sign in the velocity term, and secondly to examine the behaviour of the function for small values of the inflaton. Differentiating this function and then performing a Taylor series expansion of (4.254) for small ϕ yields the constraint (valid up to terms of $\mathcal{O}(\phi^4)$)

$$3Q >> \left\{1 + \cosh\left(\frac{3\tilde{C}}{M_p}\sqrt{\frac{Q}{2}}\right)\right\}\left(1 + \frac{7\phi^4 L^4}{4\mu^4\lambda^2 T_3^2}\right), \tag{4.271}$$

where we have introduced the simplifying notation $Q = V_0/h^4 - 1$. Clearly to satisfy this condition we must require that the term $\tilde{C}$ arising from the boundary condition be very small in Planck units. Neglecting the ϕ^4 terms, we can again use a Taylor series expansion, this time for $\tilde{C} \sim 0$. At leading order we must satisfy the following condition on the parameter Q:

$$Q >> \frac{2}{3}\left(1 + \frac{3\tilde{C}^2}{4M_p^2} + \dots\right). \tag{4.272}$$

However the fact that we require $\tilde{C}$ to be small has an effect on the amount of inflation we can have in this fast rolling regime. To see this, let us calculate the fast rolling parameter ε, making use of our near relativistic approximation. A short calculation shows that

$$\varepsilon \sim -\frac{3}{2}\mathrm{csch}^2\left(\frac{3(\phi + \tilde{C})}{M_p}\sqrt{\frac{Q}{8}}\right). \tag{4.273}$$

For small values of $\tilde{C}$ and the inflaton field, we see that the real part of this function is divergent. In fact ε is a decreasing function of ϕ which suggests that inflation will only be possible once the field reaches a critical point given by $\phi_c \sim \frac{M_p}{3}\sqrt{\frac{8}{Q}}\text{Arccsch}(\sqrt{\frac{2}{3}}) - \tilde{C}$, after which we enter a phase of eternal inflation which will not end within the bounds set by our theory.

Non-relativistic limits

In this subsection, we examine the non-relativistic motion of the branes and compare with the results from the previous sections. From the continuity equation and the definition of the energy momentum tensor, we find the inflaton velocity

$$\dot{\phi} = -\frac{2M_p^2 H' \sqrt{1+2Y}}{3(1+4Y)}. \tag{4.274}$$

Upon substitution of this back into the Friedmann equation, we obtain

$$\frac{3M_p^2 H^2}{2T_3} \sim \frac{h^4}{\sqrt{1+2Y}}\left(1+4Y+\frac{Z_3 M_p^4 H'^2}{9T_3 h^4}\right) + \frac{3}{2}(V(\phi)-h^4), \tag{4.275}$$

where we have introduced another function Z_3:

$$Z_3 = \frac{(3+16Y)(1+2Y)}{(1+4Y)^2}. \tag{4.276}$$

The above equation (4.275) is difficult to solve analytically for either background. So we resort to the usual trick of Taylor expanding the Hubble term for a given potential.

Let us consider the mass gap backgrounds. The simplest analytic solutions are obtained when we keep only terms up to quadratic in the potential and Hubble parameter. It is easy to see that the coefficient H_1 is imaginary and so we drop it from the analysis. The results for the remaining components are

$$H_0 = \sqrt{\frac{T_3}{2M_p^2}\,(3(V_0-h^4)+2h^2)}, \tag{4.277}$$

$$H_2 = \frac{9H_0 h^2}{8M_p^2}\left(1 \mp \sqrt{1+\frac{4V_2 T_3}{9h^2 H_0^2}}\right). \tag{4.278}$$

The general result for the ε equation reduces to

$$\varepsilon \sim \frac{8M_p^2 H_2 \phi^2}{3H_0^2}\frac{\sqrt{1+2Y}}{(1+4Y)}, \tag{4.279}$$

for all backgrounds, where Y is a function of the inflaton for the mass gap solutions. For small values of ϕ we can expand this and obtain a value for the field at the end of inflation.

As a result we can write the slow roll parameter as an explicit function of the number of e-foldings:

$$\varepsilon \sim e^{\frac{8\beta}{3}N_e}, \tag{4.280}$$

where we have used the previous definition of β. For any inflation to occur we must have $\beta < 0 \rightarrow H_2 < 0$ therefore we must again choose the minus sign in the expression above.

Following the same line of reasoning as in the $N = 2$ case we see that the constraint on the potential contributions can be written as follows

$$V_2 \geq \frac{10^{-3}V_0}{M_p^2} \tag{4.281}$$

where we have neglected higher order contributions in h^2. This is smaller than the constraint in the two-brane solution due to the additional 'mass' coming from the extra brane. The inflaton is effectively weighted by this contribution thus making it roll more slowly. Of course this analysis is only valid for small velocities, which is more problematic in this instance as there are zeros for the speed of sound function which destroys any hope of obtaining an inflationary solution.

We can also obtain a simple analytic solution if the constant parts of the potential and the Hubble parameter are the dominant contribution. In this case we find

$$H_0 = \sqrt{\frac{T_3}{M_p^2}\left(V_0 - h^4 + \frac{2h^2}{3}\right)}, \tag{4.282}$$

where the warp factor contribution is subdominant. This implies that the velocity of the inflaton will be zero from the Hamilton-Jacobi expression.

As for the AdS type backgrounds, we again cannot obtain simple analytic solutions when we keep quartic terms in the Friedmann equation. So again we restrict our analysis to the purely quadratic pieces. As in the other cases, the linear term in H must vanish for consistency and so the physical solutions are

$$H_0 = \sqrt{\frac{3T_3V_0}{2M_p^2}} \tag{4.283}$$

$$H_2 = \frac{H_0\sqrt{1+2Y}}{4Z_3M_p^2}\left(1 \mp \sqrt{1 + \frac{6ZT_3V_2}{H_0^2\sqrt{1+2Y}}}\right). \tag{4.284}$$

The inflation in this limit is parameterised by the slow roll term

$$\varepsilon \sim \exp\left(\frac{4\delta N_e\sqrt{1+2Y}}{3(1+4Y)}\right), \tag{4.285}$$

where we have reintroduced the parameter δ as in the $N = 2$ section. The validity of the expression is determined by the background fluxes manifest in the Y terms.

Again we find a similar bound on V_2 as in the two-brane case, namely

$$V_2 \geq \frac{10^{-1} V_0}{M_p^2} F(Y) \tag{4.286}$$

where the function $F(Y)$ now ranges between $F(Y) = 1 \to 2\sqrt{2Y}$.

In both cases we see that in order to satisfy the observed scalar curvature bound, we require

$$V_0 \leq 1 \times 10^{-7} \left(\frac{M_p}{M_s}\right)^4 \tag{4.287}$$

which can again be used to constrain the maximal value of V_2. Once again we see that the inflationary scale is sensitive to the magnitude of the string scale.

4.5 Discussion

In this chapter we have investigated three different inflationary scenarios within a string theory context. The first was based upon the Geometrical Tachyon arising from the $NS5$-ring background. In this case we found a suitable inflationary phase was possible when we considered dynamics in the ring plane (and inside the ring), with sufficiently small metric perturbations. The effective potential also has a resolvable minimum in this instance which leads to the possibility of a reheating phase as in standard field theory models of inflation. We also studied the theory in the direction transverse to the ring, and found that the general solution was a never ending period of inflation. However our analysis was only valid as a leading order approximation. A more general analysis would show that the scalar field would be damped and therefore could be used as an initial phase of inflation. Moreover this suggested a phase of double expansion, whereby we consider an initial inflationary phase arising from motion transverse to the ring. As the field gets damped it comes to rest at the origin before quantum effects force the field to condense in another direction leading to another phase of accelerated expansion.

In the second model we developed a hybrid inflation scenario using the Geometrical Tachyon coupled to an open string tachyon field. The results in this case showed that inflation with appropriately small tensor perturbations was indeed possible. However recent work has suggested that the level of non-gaussian fluctuations generated near the end of inflation may be slightly too high.

Despite the attractive simplicity of both scenarios there remain several difficulties. Firstly there are certainly α' corrections which become important as the inflaton field nears its global vacuum. There is also the issue of the toroidal compactification and moduli stabilisation, as this was put in by some ad hoc mechanism. In addition there is also the issue about transmitting the massive (emitted) closed string sector to the Standard Model sector.

The final model was multiple brane DBI inflation in the IR end of a warped throat. This is the more phenomenologically viable model of the three, but significantly more complicated due to the non-Abelian nature of the effective action. We studied the large N limit, and found many similarities between this and the $N = 1$ case, however the physically relevant observables in this case are all weighted by factors of N, making it difficult to normalise them to the experimental data without a great deal of fine tuning. We then investigated two further cases where $N = 2$ and $N = 3$, which are more under control from a closed string perspective. However due to the non-linear nature of the action functionals we were only able to look at two limiting regimes, namely non-relativistic and highly relativistic motion. A more detailed analysis of these two cases would be welcome.

The results obtained here are surprisingly different from the usual IR models [93]. We saw that inflation was possible in the large N limit, and furthermore predicts relatively large levels of tensor perturbations [109]. For the $N = 2$ case we saw that inflation never ends in the mass gap backgrounds, however for AdS type solutions we require the string scale to be close to the Planck scale. The level of non-gaussian fluctuations in this case is smaller than those associated with the single brane models. The final case of three-brane inflation led to larger non-gaussian fluctuations. Furthermore inflation in the AdS solution appears to be extremely sensitive to the flux parameters, preferring small fluxes. As before we found that inflation in the mass gap backgrounds was never ending.

Despite the difficulties, this remains a robust inflationary proposal. The single brane models require additional fine tuning of the background fluxes, before inflation even occurs. Moreover once inflation ends in these models, much of the released energy will go into reheating the $U(1)$ gauge bosons in the inflationary throat, and not into reheating the Standard Model degrees of freedom in another throat. Our model doesn't have this problem, since it is a worldvolume theory, thus the universe lives on the moving stack of branes and therefore energy loss during reheating is no longer a problem. Of course to make the model more realistic we must include more branes so that the gauge group becomes large enough to contain the Standard Model. We must also ensure that there are intersections at angles between some of these branes in order to generate chiral fermions. These are certainly problems that need to be addressed in the future.

CHAPTER 5
CONCLUSIONS

We have studied three different aspects of D-brane dynamics in this thesis. Firstly we investgated the dynamics of a Dp-brane in a fivebrane background through the tachyon-radion correspondance. At its simplest level, the correspondance allows us to map the gravitational problem to that of a condensing open string mode in flat space. However as emphasised by Sen [27] this mapping may shed new light on the condensation of the open string tachyon, in particular the tachyonic vacuum.

It would be interesting to develop this duality further by mapping to a complex Geometrical Tachyon field. In fact this may be possible using the tools developed in Section 2.5, whereby we combine the open string tachyon with the Geometrical Tachyon. Alternatively we may be able to construct a purely geometrical, but complex solution. This would be useful for obtaining a better understanding of coincident D-$\bar{D}$ systems, which generally admit complex open string tachyons. The geometrical origin of such a construction would shed new light on the vacuum state which is currently only really understood from the CFT description.

The dynamics of a brane in this background are also relevant for a discussion of the string - black hole transition [32]. The ring solution can be exactly described in terms of a $SU(2)/U(1) \times SL(2,\mathbb{R})/U(1)$ coset [20], which allows for the construction of the hairpin brane [21] in this instance. It would also be possible to use the CFT to calculate the emission rate of the closed string modes, and thus allow us to learn more about the decay of the Dp-brane [30] in addition to the possible scattering of branes whose trajectory takes them into the ring. Finally we point out that the work in this section involved an approximation of the full harmonic function for this solution [20], and it would be useful to extend the work to the exact case.

In Chapter three we investigated the dynamics of multiple branes in various supergravity backgrounds. We interpreted this as the collapse or expansion of the fuzzy sphere in these geometries. The equations of motion in general are difficult to solve, and so we made various approximations of resorted to numerical solutions. In all cases, bar the $D6-D0$ solution, the fuzzy spheres collapse toward zero size. Using the fact that the equations of motion can be mapped to curves on Riemannian surfaces of varying genus, we suspect that there is a relationship between brane dynamics and algebraic geometry. It was noticeable that for the solutions in flat space [51], and those which corresponded to supersymmetric configurations [23], the underlying geometrical structure was of low genus - and typically solvable. Understanding this in more detail would shed more light on non-commutative geometries in curved space.

We also considered more general background metrics, and included the effects of $U(1)$ gauge fields. Again the solutions indicate that the fuzzy sphere will always collapse, but with the time of collapse dependent upon the strength of the $U(1)$ field. The large-small dualities that can be seen in flat space [51] are no-longer present in curved backgrounds, although parts of the dualities continue to exist. We then proceeded to construct both the macroscopic and microscopic models of the BIon spike on a $D3$-brane in an arbitrary background, using the gravitational Myers effect [55]. This proved to be easiest to solve in the $NS5$-brane background, but we found interesting behaviour in all cases.

In the final part of the chapter we investigated the tension spectrum for (p, q)-strings in the Warped Deformed Conifold [70]. This is highly relevant from a cosmological perspective, since the bound state gives us information about the tension of cosmic superstrings and thus opens up the possibility of experimental verification of string theory. It is also relevant from the gauge theory perspective because the dual field theory to the Warped Deformed Conifold is a confining $SU(M)$ theory, and therefore F-strings in this background are confining strings between quark-antiquark pairs [73]. This may lead to new insights into QCD type gauge theories. In particular the tension spectrum appears to be of a similar form to that proposed by Douglas and Shenkar [74], and is therefore inconsistent with the Casimir scaling hypothesis [73]. Recently it has also been proposed that the microscopic (p, q)-bound state may have relevance in the context of open string attractors [107].

There are many further things to investigate along these lines. Firstly our initial analysis neglected the NS two-form, and also higher order RR fields. Both should be included if we are to fully understand the dynamics of fuzzy spheres in general backgrounds. We also restricted the analysis to fuzzy even spheres, since they are simpler to deal with. It would be useful to consider the fuzzy odd spheres aswell [54], since this could be relevant for cosmology or black hole physics. It is also important to understand how the action behaves when one considers finite brane corrections. Recall that everything in the first sections of this chapter were only valid in the large N limit. There has been a recent proposal for the action of the symmetrized trace [53](for the case of $SO(3)$), which we utilised explicitly in section 3.4. A full proof of the validity of this proposal is still required.

The final chapter considered how brane dynamics could be useful for inflationary cosmology, with the open string mode playing the role of the inflaton. We presented three different scenarios along these lines. Firstly Geometrical Tachyon Inflation, then a hybrid inflation scenario and finally DBI inflation in the IR region of a warped throat.

The first model is a modification of the usual tachyonic inflation scenario, but with a cosine potential similar to those arising in discussions of natural inflation [101]. The difference between the two is that there is no fundamental scale to set the maximum height of the Geometrical Tachyon potential. As such we can tune the parameter space of k and R to obtain a sufficient amount of inflation, with suitable levels of metric perturbations. The resulting inflaton potential admits a minimum, and therefore classical reheating is possible [99] provided one imposes additional tuning on the brane trajectory. The scalar mode emerging from the transverse ring plane leads to an eternally accelerating solution unless the tension of the brane is sufficiently small, and is therefore better suited to a model of dark energy [77]. This model can be developed further, especially in the light of the Sen conjectures [27], and could still be phenomenologically relevant.

The second model used the Geometrical Tachyon to drive an initial (hybrid) inflationary phase, with the open string tachyon playing the role of the waterfall field.. Again we saw that with appropriate tuning one could find inflation with small levels of metric perturbations, although the non-gaussian fluctuations [95] may be too large in this model. It would be useful to try and embed this model into a more realistic compactification, since the torus is known not to be phenomenologically favoured [10]. One may also expect that there is a sigma model description of such a theory, which would shed valuable light onto the coupling of the Geometrical Tachyon to the open string tachyon. It also opens up the possibility that there may be a "landscape" of such geometrical hybrid inflation scenarios, and as such it is useful to understand them in as much detail as possible.

The final model was a modification of the usual IR inflation [93] scenario, where there were multiple coincident branes at the tip of a warped throat. This proved to be a generalisation of the DBI inflation scenario [92], sharing many of the features of those models [94]. Although severe fine tuning is required in order to satisfy the observational data, this model is phenomenologically robust. In this model the universe is contained on the worldvolume of the branes, as opposed to living in another throat as in the usual models [92]. This means that reheating after the end of inflation will be due to open string modes. As mentioned in Chapter four, the traditional DBI models rely on closed string interactions to reheat the standard model degrees of freedom but there is a residual $U(1)$ gauge boson in the inflationary throat which absorbs much of the reheating energy. Thus it is not clear how much we can learn about the coupling of the inflaton sector to the standard model in these cases. There remains much work to be done on this model in particular, in order to make it more phenomenologically viable. Firstly we need to arrange for the branes to intersect at an angle in order to obtain chiral fermions in the spectrum. We also need to extend the analysis to include a larger number of branes, in order to have a large enough gauge group which will include the standard model, whilst still being able to neglect the backreaction. It is also interesting to explore the relationship with the Randall-Sundrum model [105], since the gluing region acts as a UV cutoff for the theory. It would also be useful to consider a fully compactified version of the model using the ideas of flux compactifications [108] in type IIB string theory, and then determine how reheating and particle creation fit into the overall picture. This is important if we want to relate string theory to particle phenomenology.

APPENDIX A

COSMOLOGICAL PERTURBATIONS.

In this section we explicitly calculate the relevant perturbation amplitudes for the non-Abelian action The definitions of the parameters in this section differ from those in other sections in order to simplify the calculations as much as possible - and again we use units where $M_p = 1$. The action in the general case can is a non-linear function of the inflaton field and its time derivative, therefore it can be written in the following form - consistent with the general prescription described in [106]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{2} + p(\phi, X)\right] , \tag{A.1}$$

where

$$p = -NT_3\left[h^4(\phi)\sqrt{1 - 2h^{-4}(\phi)T_3^{-1}X}\sqrt{1 + C^{-1}h^{-4}(\phi)\phi^4} - h^4(\phi) + V(\phi)\right] . \tag{A.2}$$

with $X = \dot{\phi}^2/2$ and $C = \lambda^2\hat{C}T_3^2/4$. We will explicitly consider the case of large N in this appendix, however it is straightforward to show that the derived results also apply for the finite N case. The background equations following from this are

$$\begin{aligned}
&3H^2 = 2Xp_{,X} - p \equiv \rho\,, \\
&\dot{H} = -Xp_{,X}\,, \\
&\frac{1}{a^3}(a^3\dot{\phi}p_{,X})^{\cdot} - p_{,\phi} = 0\,.
\end{aligned} \tag{A.3}$$

Note that the energy density ρ is given here by

$$\rho = NT_3\left[\frac{h^4(\phi)W(\phi)}{\sqrt{1 - 2h^{-4}(\phi)T_3^{-1}X}} - h^4(\phi) + V(\phi)\right] , \tag{A.4}$$

where the fuzzy potential is

$$W(\phi) = \sqrt{1 + C^{-1}h^{-4}(\phi)\phi^4}\,. \tag{A.5}$$

We consider the following general perturbed metric about a FRW background

$$\begin{aligned} \mathrm{ds}^2 &= -(1+2A)\mathrm{dt}^2 + 2\mathrm{a}\partial_\mathrm{i}\mathrm{B}\mathrm{dx}^\mathrm{i}\mathrm{dt} \\ &\quad + a^2\left[(1+2\psi)\delta_{ij} + 2\partial_{ij}E + 2h_{ij}\right]\mathrm{dx}^\mathrm{i}\mathrm{dx}^\mathrm{j}\,, \end{aligned} \tag{A.6}$$

where ∂_i represents the spatial partial derivative $\partial/\partial x^i$ and $\partial_{ij} = \nabla_i\nabla_j - (1/3)\delta_{ij}\nabla^2$. Here A, B, ψ and E denote scalar metric perturbations, whereas h_{ij} represents tensor perturbations. Defining the so-called comoving perturbation

$$\mathcal{R} \equiv \psi - \frac{H}{\dot\phi}\delta\phi\,, \tag{A.7}$$

the Fourier modes of curvature perturbations satisfy the following expression

$$v'' + \left(c_S^2 k^2 - \frac{z''}{z}\right) v = 0\,, \tag{A.8}$$

where

$$\begin{aligned} z^2 &= \frac{a^2\dot\phi^2(p_{,X} + 2Xp_{,XX})}{H^2}\,, \\ v &= z\mathcal{R}\,, \\ c_S^2 &= \frac{p_{,X}}{\rho_{,X}} = \frac{p_{,X}}{p_{,X} + 2Xp_{,XX}}\,. \end{aligned} \tag{A.9}$$

Note that k is a comoving wavenumber and a prime represents a derivative with respect to a conformal time $\tau = \int a^{-1}\mathrm{d}t$. If the variable z has a time-dependence $z \propto |\tau|^q$, one has $z''/z = \gamma_S/\tau^2$ with $\gamma_S = q(q-1)$. As long as c_S^2 is a positive constant or a slowly varying positive function, the solution for (A.8) is given by

$$v = \frac{\sqrt{\pi|\tau|}}{2}\left[c_1(k)H_{\nu_S}^{(1)}(c_S k|\tau|) + c_2(k)H_{\nu_S}^{(2)}(c_S k|\tau|)\right]\,, \tag{A.10}$$

where $\nu_S = \sqrt{\gamma_S + 1/4} = |q - 1/2|$. The coefficients are chosen to be $c_1 = 0$ and $c_2 = 1$ to recover positive frequency solutions in a Minkowski vacuum in an asymptotic past.

Defining the spectrum of curvature perturbation as $\mathcal{P}_\mathcal{R} = k^3|\mathcal{R}|^2/2\pi^2$, we obtain

$$\begin{aligned} \mathcal{P}_\mathcal{R} &= \frac{a^2 c_S^{-2\nu_S}}{z^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{1}{aH|\tau|}\right)^2\left(\frac{\Gamma(\nu_S)}{\Gamma(3/2)}\right)^2\left(\frac{k|\tau|}{2}\right)^{3-2\nu_S} \\ &\equiv \mathcal{A}_S^2\left(\frac{k|\tau|}{2}\right)^{3-2\nu_S}\,, \end{aligned} \tag{A.11}$$

The spectral index of the power spectrum is

$$n_S - 1 = 3 - 2\nu_S = 3 - \sqrt{4\gamma_S + 1}\,, \tag{A.12}$$

which means that the scale-invariant spectrum corresponds to $\nu_S = 3/2$. About the de-Sitter background with $|\tau| = 1/aH$, the amplitude of the curvature perturbation is given by

$$\mathcal{A}_S^2 \simeq \frac{1}{p_{,X} c_S} \left(\frac{H^2}{2\pi \dot{\phi}} \right)^2 . \tag{A.13}$$

The tensor perturbations satisfy the same equation as in the case of standard slow-roll inflation. Taking into account polarization states of tensor modes, The power spectrum is given by

$$\begin{aligned} \mathcal{P}_T &= 8 \left(\frac{H}{2\pi} \right)^2 \left(\frac{1}{aH|\tau|} \right)^2 \left(\frac{\Gamma(\nu_T)}{\Gamma(3/2)} \right)^2 \left(\frac{k|\tau|}{2} \right)^{3-2\nu_T} \\ &\equiv \mathcal{A}_T^2 \left(\frac{k|\tau|}{2} \right)^{3-2\nu_T} , \end{aligned} \tag{A.14}$$

where $\nu_T = \sqrt{\gamma_T + 1/4}$ with $a''/a = \gamma_T/\tau^2$. Hence about the de-Sitter background the amplitude of the tensor perturbation is

$$\mathcal{A}_T^2 \simeq 8 \left(\frac{H}{2\pi} \right)^2 . \tag{A.15}$$

The spectral index of the power spectrum is

$$n_T = 3 - 2\nu_T = 3 - \sqrt{4\gamma_T + 1} . \tag{A.16}$$

The tensor to scalar ratio is

$$r = \frac{\mathcal{A}_T^2}{\mathcal{A}_S^2} = 8 \frac{\dot{\phi}^2}{H^2} p_{,X} c_S . \tag{A.17}$$

To study the running of the spectral indices we find it convenient to introduce the following parameters:

$$\epsilon_1 = -\frac{\dot{H}}{H^2} , \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}} , \quad \epsilon_3 = \frac{\dot{F}}{2HF} . \tag{A.18}$$

where $F \equiv p_{,X} + 2Xp_{,XX}$ and ϵ_1 is the same as the ε which we used in the latter part of Chapter four. If $\dot{\epsilon}_i = 0$, we can derive

$$\frac{z''}{z} = \frac{\gamma_{\mathcal{R}}}{\tau^2} , \quad \gamma_{\mathcal{R}} = \frac{(1 + \epsilon_1 + \epsilon_2 + \epsilon_3)(2 + \epsilon_2 + \epsilon_3)}{(1 - \epsilon_1)^2} . \tag{A.19}$$

Under the slow-roll approximation $|\epsilon_i| \ll 1$, we find that the spectral index of the curvature perturbation is given by

$$n_S - 1 = -2(2\epsilon_1 + \epsilon_2 + \epsilon_3)\,. \tag{A.20}$$

Similarly the spectral index of the tensor perturbation is

$$n_T = -2\epsilon_1\,. \tag{A.21}$$

By using the background equations we have $\epsilon_1 = \dot{\phi}^2 p_{,X}/(2H^2)$. This then shows that the tensor to scalar ratio (A.17) yields

$$r = 16\epsilon_1 c_S = -8c_S n_T\,. \tag{A.22}$$

Again this is the same expression as in the single brane case, and is a distinctive feature of DBI inflation. The WMAP normalisation we will employ in the latter section of this thesis is the following [88–91]

$$\begin{aligned}
{\mathcal{A}_{\mathcal{S}}}^2 &= 10^{-9} \\
r &= \frac{\mathcal{A}_T^2}{\mathcal{A}_S^2} \leq 0.55 \quad (\leq 0.24 \;\; \text{at} \;\; 0.95 \;\; \text{C.L}) \\
n_s &= 0.987^{+0.019}_{-0.037} \\
f_{NL} &\leq 100
\end{aligned} \tag{A.23}$$

which may differ slightly from normalisation used elsewhere.

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