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Christoph Fritz, BSc MSc

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Supervisor

Associate Prof. Dr.techn. Grumiller Daniel

E136 - Institute of Theoretical Physics

Head: Univ.Prof. Dipl.-Ing. Dr.techn. Rotter Stefan

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Abstract

In this thesis, we determine the Carroll z-scale invariance of Carroll(-invariant) scalar and vector field theories where z is an anisotropy parameter of Lifschitz type and $z = 1$ describes the isotropic (or "standard") scale. We mainly focus on Carroll swiftons, namely the bi-scalar, multi-scalar, electromagnetic and $2d$ dilaton gravity swifton model, recently introduced in the preprint [1] where we determine the scaling exponents for the scalar and vector fields, interaction terms, possible potentials and coupling functions of the field theories. We see that the bi-scalar, multi-scalar and electromagnetic model introduce the same restriction on the scale z which makes it possible to combine different interaction terms as well as spin-0 and spin-1 Carroll field theories. Furthermore, we show that the Carroll swifton scalar field action cannot be extended to the Carroll extremal surface if one wants Carrol z-scale invariance.

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1 Introduction

Until about a decade ago, Carrollian physics was the overlooked younger sibling of Galilean physics. However, over the past ten years the omnipresence of Carrollian structures was recognized and Carrollian physics became a rapidly developing research topic in a multitude of fields. Originally published in French in a journal quite far away from mainstream physics (see also the interesting paper of Lévy-Leblond [2] on the fate of (his) scientific ideas), the paper of Lévy-Leblond [3] as well as the paper by SenGupta [4], published nearly at the same time, marked the beginning of Carroll physics .

Mathematically, Carroll symmetries are obtained formally by taking the vanishing speed of light limit from Poincáre symmetries. The resulting group of this limit was coined Carroll group by Lévy-Leblond due to its seemingly paradoxical physical properties, one of them being that space is absolute but time is relative and the lightcone collapses. It took the physics community nearly half a century to realize that the consequences of this limit actually play a vital role in many different contexts.

One of the most prominent realizations was that Minkowski spacetime, crucial for quantum field theories, exhibits a Carroll structure at null infinity. Furthermore, the asymptotic symmetries of asymptotically flat spacetimes known as Bondi, van der Burgh, Metzner, and Sachs (BMS) symmetries [5], [6] align precisely with conformal Carroll symmetries [7]–[9]. In addition, it was realized that generic null hypersurfaces, ubiquitous in general relativity, have a Carroll structure. Therefore, Carrollian symmetries emerge in both pillars of theoretical physics, quantum field theories and general relativity, which paved the way for one of the most prominent applications in both contexts: the

Carrollian approach to flat space holography in three and four dimensions.

For the remainder of the introduction note that it would be impossible to cite every relevant paper in this rapidly developing field. We refer the interested reader to the some of the newer articles and preprints like [1], [10]–[12] for references and further resources.

Remarkably, flat space holography is not the only application of Carrollian physics as there are many more works in quantum gravity, tachyon condensates, the fluid/gravity correspondence, tensionless strings, cosmology, current-current deformations, Hall effects, fractons, flat bands, Bjorken flow, supersymmetry and supergravity, and black holes. Moreover, it is natural to gauge the Carroll algebra, and establish Carroll gravity theories, which may exhibit Carroll black hole solutions.

As mentioned above, one important aspect of (conformal) Carrollian field theories is that they might be dual to quantum gravity in asymptotically flat spacetime. Therefore, to better undstand them it is necessary to construct non-trivial examples of such field theories as was done in [1]. This preprint serves as the main resource that we base our investigations of the Carroll z-scale invariance on. We investigate for which scale z, where z is an anisotropy parameter of Lifschitz type and $z = 1$ describes the isotropic (or "standard") scale, and which dimensions d the Carrollian scalar and vector field theories, coined "swiftons" in [1], are invariant under the transformations of time and space coordinates. The transformations are given as

$$t \to \lambda^z t \tag{1.1}$$

$$x \to \lambda x. \tag{1.2}$$

Since we want this thesis to be as self-contained as possible we do not only recall basics in Carrollian geometry but also recall the constructions of Carroll-invariant theories made in [1] since they are the basis of our work. However, this chapter can be skipped if one is only interested in the results obtained for the Carroll z-scale invariance. The thesis is now organized as follows.

In chapter 2 we provide some motivation for why to study Carroll physics, quickly introduce the mathematical framework and notation needed for working in Carrollian physics and a rough derivation of the “electric” and “magnetic” scalar field theories.

In chapter 3 we recall the construction of the Carroll-invariant theories with fields propagating outside the Carroll lightcone (“Carroll swiftons”) and their properties from [1]. Contrary to the “electric” and “magnetic” Carrollian scalar field theories, the Carroll-invariant scalar/vector field theories constructed by the authors allow propagation at a non-vanishing velocity in arbitrary dimension, both with and without (Carroll) gravity, i.e. they do not remain at the same spatial location.

In chapter 4 we present the main body of work of this thesis and show for which scale z and dimension d the “electric” and “magnetic” Carroll scalar field theories as well as the Carroll-invariant scalar/vector field theories of [1] are Carroll z-scale invariant.

In Chapeer 5 we summarize our work in this thesis and give a short outlook in what directions this work could be extended as well as open questions.

2 Carrollian Geometry Basics

In this chapter we give some motivation on why to study Carrollian physics and a quick introduction to the mathematics of it.

The main motivation for studying systems with Carroll symmetry comes from the expectation that conformal Carroll field theories might be dual to quantum gravity in asymptotically flat spacetime. As already mentioned in the introduction, this is due to the fact that the asymptotic BMS symmetry group [5], [6] of flat spacetimes aligns with the conformal Carroll structure living on its null boundary [7]–[9]. There is also evidence that conformal Carrollian field theories play a role in the celestial holography approach to flat space holography, see e.g. [13]–[15].

Additionally, if Carrollian field theories are dual to quantum gravity in flat space, their thermal properties should say something about black holes (see [10] for an investigation of Carroll black holes and their thermal properties). Since black holes do not have well-defined partition functions at non-zero temperature in flat space, it may well be the case that one can expect a similar problem with defining partition functions of Carroll field theories. Therefore, the (at first glance pathological) properties of Carroll quantum field theories may actually be consequences of flat space holography.

A closely related aspect of Carollian physics is Carroll gravity which can be obtained by considering ultra-local (the small speed of light) limit of General Relativity. This was first first considered by Henneaux in [16]. Recently Hansen, Obers, Oling and Søgaard [17] looked at this ultra-local expansion of General Relativity by using a modern perspective of non-Lorentzian geometry, i.e. they used the fact that Carroll geometry arises from Lorentzian geometry when taking the limit $c \to 0$.

2.1 Mathematical Aspects of Carrollian Physics

In contrast to Lorentzian spacetimes in $n = d + 1$ dimensions which are equipped with indefinite metrics of signature $(-, +, \ldots, +)$ with d pluses, Carroll spacetimes in $n = d + 1$ dimensions are equipped with degenerate metrics of signature $(0, +, \ldots, +)$ with d pluses with one-dimensional kernel. As a simple example of such a spacetime we can consider the limit of the Minkowski metric where the speed of light vanishes

$$\lim_{c \to 0}(-c^2 dt^2 + \delta_{ij} dx^i dx^j) = \delta_{ij} dx^i dx^j. \tag{2.1}$$

This example clearly shows that, geometrically, the Carrollian signature of the metric collapses the lightcone and that Carrollian time is relative and Carrollian space is absolute. This is in stark contrast to the case where the metric has Lorentzian signature and Lorentzian time and Lorentzian space are relative.

In order to fully characterize a Carroll spacetime, we need a Carroll metric $h_{\mu\nu}$ and a Carroll vector v^μ that lies in the kernel of the metric, i.e. $v^\mu h_{\mu\nu} = 0$. It was shown in [16] that this is equivalent to endowing the manifold with a non-vanishing volume element Ω. In the above example of the limit of the Minkowski metric the vector field is $v = v^\mu \partial_\mu = \partial_t$ and the Carroll metric is $h_{\mu\nu} = \delta_{ij} \delta^i{}_\mu \delta^j{}_\nu$ which implies the unit volume $\Omega = 1$.

It is important to note that the metric $h_{\mu\nu} = \delta_{ij} \delta^i_\mu \delta^j_\nu$ is invariant under all Carroll transformations. However, its "inverse metric" $h^{\mu\nu} \equiv \delta^{ij} \delta^\mu_i \delta^\nu_j$ is not invariant under Carroll boosts. Therefore, in contrast to Lorentzian case, one cannot raise tensor indices in an invariant way. However, this difficulty can be overcome by using "transverse" or "spacelice" co-vectors θ_μ which are defined to be orthogonal to v^μ, i.e. $v^\mu \theta_\mu = 0$. In this way the norm squared $\theta_\mu \theta^\mu$ is well-defined and positive and this construction holds true for general covariant tensors $\theta_{\mu_1 \mu_2 \ldots \mu_k}$ transverse on all their indices.

We now take a quick look at Carroll symmetries which emerge as the $c \to 0$ limit of Poincaré symmetries. We denote the temporal translations $H = \partial_t$, spatial translations $P_i = \partial_i$, and rotations $J_{ij} = x_i\partial_j - x_j\partial_i$ which are all unaffected by this limit. The only generators changing are the boosts

$$B_i = -c^2 t\partial_i - x_i\partial_t \to B_i = -x_i\partial_t. \tag{2.2}$$

This also means that the only commutators that change compared to the ones in the Poincaré algebra are those involving the Carroll boosts:

$$[B_i, H] = -x_i\partial_t\partial_t + x_i\partial_t\partial_t = 0 \tag{2.3}$$
$$[B_i, B_j] = x_i x_j\partial_t\partial_t - x_i x_j\partial_t\partial_t = 0 \tag{2.4}$$
$$[B_i, P_j] = -x_i\partial_t\partial_j + \partial_j x_i\partial_t = \delta_{ij}H \tag{2.5}$$
$$[B_k, J_{ij}] = -x_k\partial_t(x_i\partial_j - x_j\partial_i) + (x_i\partial_j - x_j\partial_i)x_k\partial_t = \delta_{ik}B_j - \delta_{jk}B_i. \tag{2.6}$$

We see that the Hamiltonian H is a central element of the Carroll algebra, in contrast to the Poincaré algebra since there the Hamiltonian does not commute with Lorentzian boosts. Furthermore, there is no Carrollian analogue of Thomas precession since two Carroll boosts always commute. The third commutator and the fact that H commutes with the remaining Carroll generators show that H and, therefore, the energy/mass is an important invariant. This is again in contrast to the Poincaré energy.

Looking at the finite boosts, which are generated by some spatial co-vector b_i, we see that they leave invariant space but transform time like

$$t' \to t - b_i x^i \tag{2.7}$$
$$x^{i\prime} \to x^i. \tag{2.8}$$

Therefore, there is an absolute notion of space in Carrollian spacetimes.

2.2 Carroll Algebras with scale invariance

The precursor of the Carroll algebra (presented in the previous section with its commutator relations) is the Aristotelian algebra, which only consists of translations H, P_i and spatial rotations J_{ij}. The representation through differential operators is the same as for the Carroll algebra above and the commutator relations are

$$[P_k, J_{ij}] = \delta_{ik}P_j - \delta_{jk}P_i = 2\delta_{k[i}P_{j]} \tag{2.9}$$

$$[J_{ij}, J_{kl}] = (x_i\partial_j - x_j\partial_i)(x_k\partial_l - x_l\partial_k) = 4J_{[i[k}\delta_{j]l]} \tag{2.10}$$

$$[H, P_i] = [P_i, P_j] = [H, J_{ij}] = 0. \tag{2.11}$$

The extension of the Carroll algebra that we are interested in is the so-called finite z-Carroll algebra. This algebra additionally contains (for $z \neq 1$ anisotropic) dilatations D and includes the special cases $z = 1$ and $z = 0$ where one of the commutators vanishes. It is central for Carrollian theories that are scale but not conformally invariant. The dilatation is denoted as the differential operator $D = zt\partial_t + x_i\partial_i$ and the commutator relations are given by

$$[H, D] = \partial_t(zt\partial_t + x_i\partial_i) - (zt\partial_t + x_i\partial_i)\partial_t = zH \tag{2.12}$$

$$[P_i, D] = \partial_i(zt\partial_t + x_i\partial_i) - (zt\partial_t + x_i\partial_i)\partial_i = P_i \tag{2.13}$$

$$[B_i, D] = -x_i\partial_t(zt\partial_t + x_i\partial_i) + (zt\partial_t + x_i\partial_i) = (z-1)B_i \tag{2.14}$$

$$[J_{ij}, D] = (x_i\partial_j - x_j\partial_i)(zt\partial_t + x_i\partial_i) - (zt\partial_t + x_i\partial_i)(x_i\partial_j - x_j\partial_i) = 0. \tag{2.15}$$

This algebra can be further extended to contain temporal special transformations (SCTs) K, and spatial special transformations (SCTs) K_i. We only state them for completeness for the interested reader and as a motivation for further investigations. The temporal special transformation is denoted by $K = x_ix_i\partial_t$ and yields the extension of the finite z-Carroll algebra by temporal SCTs. For $z = 2$ we see that the commutator of the z-dilatation and the temporal SCTs vanishes.

$$[D, K] = (zt\partial_t + x_i\partial_i)x_ix_i\partial_t - x_ix_i\partial_t(zt\partial_t + x_i\partial_i) = (2-z)K \tag{2.16}$$

$$[P_i, K] = \partial_i(x_ix_i\partial_t) - x_ix_i\partial_t\partial_i = -2B_i \tag{2.17}$$

$$[H, K] = [B_i, K] = [J_{ij}, K] = 0. \tag{2.18}$$

The spatial special transformation is denoted by $K_i = x_j x_j \partial_i - 2x_i(zt\partial_t + x_j\partial_j)$ and yields the extension to the finite conformal Carroll algebra by spatial SCTs.

$$[H, K_i] = 2zB_i \tag{2.19}$$
$$[D, K_i] = K_i \tag{2.20}$$
$$[P_i, K_j] = 2J_{ij} - 2\delta_{ij}D \tag{2.21}$$
$$[K_k, J_{ij}] = 2\delta_{k[i}K_{j]}. \tag{2.22}$$

If we set $z = 1$, we also have

$$[K, K_i] = [K_i, K_j] = 0 \tag{2.23}$$
$$[B_i, K_j] = \delta_{ij}K. \tag{2.24}$$

2.3 The "electric" and "magnetic" scalar field model

Since we are looking at Carrollian scalar field theories, we include a short introduction and derivation of the "electric" and "magnetic" model for completeness sake. We follow the exposition [12] for a $2d$ massless Carrollian scalar field ϕ with conformal coupling very closely. Note that all results in this section are background independent.

We start from the Lorentzian action on a manifold M given by

$$I = -\frac{1}{2}\int_M d^2x\sqrt{-g}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) \tag{2.25}$$

and introduce the pre-ultralocal variables as in [17] by

$$V^\mu T_\mu = -1 \tag{2.26}$$
$$T_\mu E^\mu = 0 \tag{2.27}$$
$$V^\mu E_\mu = 0 \tag{2.28}$$
$$E^\mu E_\nu = \delta^\mu_\nu + V^\mu T_\nu \tag{2.29}$$

such that the metric is given by

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + E_\mu E_\nu \tag{2.30}$$

and the Lorentzian volume form is $cT \wedge E$.

If we expand the frame fields in powers of c^2 we get

$$V^\mu = v^\mu + \mathcal{O}(c^2) \tag{2.31}$$

$$T_\mu = \tau_\mu + \mathcal{O}(c^2) \tag{2.32}$$

$$E_\mu = e_\mu + \mathcal{O}(c^2). \tag{2.33}$$

Furthermore, we have the local Carroll boosts parametrized by $\lambda(x)$ acting as

$$\delta_\lambda e_\mu = 0 \quad \delta_\lambda \tau_\mu = -\lambda e_\mu \quad \delta_\lambda v^\mu = 0 \quad \delta_\lambda e^\mu = -\lambda v^\mu. \tag{2.34}$$

as well as the Weyl rescalings [18] parametrized by $\rho(x)$ acting on the frame fields as

$$\delta_\rho e_\mu = \rho e_\mu \quad \delta_\rho \tau_\mu = \rho \tau_\mu \quad \delta_\rho v^\mu = -\rho v^\mu \quad \delta_\rho e^\mu = -\rho e^\mu. \tag{2.35}$$

If we switch to a Hamiltonian formulation by defining the pre-ultralocal momentum

$$\Pi = \frac{c}{\sqrt{-c}} \frac{\delta L}{\delta(V^\mu \partial_\mu \phi)} = \pi + \mathcal{O}(c^2) \tag{2.36}$$

and insert the pre-ultralocal variables into the Lorentzian action above, we get

$$I = \int_M T \wedge E \left(\Pi V^\mu \partial_\mu \phi - \frac{1}{2}\Pi^2 - \frac{c^2}{2}(E^\mu \partial_\mu \phi)^2 \right). \tag{2.37}$$

Now we are ready to obtain the two possible actions for a Carroll invariant scalar field [19].

First, we look at the timelike (electric) scalar field which is obtained by directly sending $c \to 0$, replacing all fields by their leading order expressions, and integrating out the leading order momentum π,

$$I_e = \frac{1}{2} \int_M \tau \wedge e (v^\mu \partial_\mu \phi)^2. \tag{2.38}$$

Second, we look at the spacelike (magnetic) scalar field. In contrast to the direct limit $c \to 0$, there is a second possibility to contract the Hamiltonian action where the fields are rescaled as $\Pi \to c\Pi, \phi \to \frac{1}{c}\phi$. This rescaling preserves the symplectic form $\delta\Pi \wedge \delta\phi$ on field space and the leading order action

$$I_M = \int_M t \wedge e \left(\pi v^\mu \partial_\mu \phi - \frac{1}{2} (e^\mu \partial_\mu \phi)^2 \right) \tag{2.39}$$

does not permit integrating out the momentum π since its quadratic term cancels in the contraction. Instead, π acts as a Lagrange multiplier enforcing time-independence of the scalar field.

3 Carroll-invariant scalar and vector field theories ("Carroll swiftons")

Carroll-invariant scalar and vector field theories with fields propagating outside the Carroll lightcone were introduced in [1]. The authors construct and discuss examples of Carroll-invariant actions for (interacting) fields allowing propagation at a non-vanishing velocity in arbitrary dimensions, both with and without (Carroll) gravity but they restrict themselves to scalar and electromagnetic fields.

The term "Carroll swiftons" was chosen to distinguish the tachyon-like particles from the Lorentz tachyons, which usually come with pathologies associated with the unboundedness of their lower energy. In the preprint, they explicitly show for their theories that the energy is bounded from below which suggests that these models are free from the standard Lorentz tachyonic instabilities.

The goal of this chapter is to offer the reader a way to get familiar with the Carroll-invariant scalar and vector field theories studied in chapter 4. However, it is included to make the thesis as self contained as possible and only recalls important results from the preprint [1] but does not expand on it. Therefore, this chapter can be safely skipped if one wants to jump directly to the results and derivations of chapter 4.

The first key result of the preprint is the bi-scalar model which couples two scalar fields ϕ, χ with canonically normalized kinematic terms and coupling constant g to any Carroll background. The action is given by

$$I_M = \frac{1}{2} \int d^n x \Omega \left(\left(v^\mu \partial_\mu \phi \right)^2 + \left(v^\mu \partial_m \chi \right) + g B_\mu B^\mu \right) \tag{3.1}$$

where

$$B_\nu = v^\mu \left(\partial_\mu\phi\partial_\nu\chi - \partial_\mu\chi\partial_n u\phi\right) = 2v^\mu\partial_{[\mu}\phi\partial_{\nu]}\chi \tag{3.2}$$

is a manifestly transverse covariant vector.

The Hamiltonian for this model can be derived by rewriting the Lagrangian density as

$$\mathcal{L} = \frac{\sqrt{h}}{2N}H_{AB}\mathring{\phi}^A\mathring{\phi}^B \tag{3.3}$$

with

$$H_{AB} = \begin{pmatrix} 1+g(\partial\chi)^2 & -g\partial\phi\cdot\partial\chi \\ -g\partial\phi\cdot\partial\chi & 1+g(\partial\phi)^2 \end{pmatrix} \tag{3.4}$$

where $\phi^A \equiv (\phi,\chi)$, N the Carroll lapse, h the determinant of the spatial metric h_{mn} with inverse h^{mn}, $\partial\phi^A\cdot\partial\phi^B \equiv h^{mn}\partial_m\phi^A\partial_n\phi^B$ and $\mathring{\phi} \equiv \dot{\phi}^A - N^k\partial_k\phi^A$ with N^k the Carroll shift.

The inverse matrix is given by

$$H^{AB} = \frac{1}{D}\left(\delta^{AB} + g\partial\phi^A\cdot\partial\phi^B\right) \tag{3.5}$$

where

$$D = 1 + g(\partial\phi)^2 + g(\partial\chi)^2 + g^2\left((\partial\phi)^2(\partial\chi)^2 - (\partial\phi\cdot\partial\chi)^2\right). \tag{3.6}$$

Note that if $g \geq 0$, then $D \geq 1$ which implies that the field space metric H_{AB} has Euclidean signature.

The Hamiltonian is given as

$$N\mathcal{H} + N^k\mathcal{H}_k \tag{3.7}$$

where the momentum density is given by

$$\mathcal{H}_k = \pi_A\partial_k\phi^A \tag{3.8}$$

and the energy density

$$\mathcal{H} = \frac{1}{2\sqrt{h}}H^{AB}\pi_A\pi_B \tag{3.9}$$

is bilinear in the conjugate momenta π_A. Since the quadratic form $H^{AB}\pi_A\pi_B$ is positive definite, the energy density is bounded from below by zero.

Furthermore, the Poisson brackets of the energy densities at different spacelike points vanish, i.e. $\{\mathcal{H}(x), \mathcal{H}(x')\} = 0$ which agrees with the general argument made in [19] and ensures that the constraints $\mathcal{H}^T \approx 0, \mathcal{H}^T_k \approx 0$ of the dynamical gravity and matter system

$$I = \int d^n x \left(\pi^{ij}\dot{h}_{ij} + \pi_A \dot{\phi}^A - N\mathcal{H}^T - N^k \mathcal{H}^T_k \right) \tag{3.10}$$

are first class.

In the above action, the π^{ij} are the conjugate momenta to the spatial metric and $\mathcal{H}^T = \mathcal{H}^G + \mathcal{H}$ and $\mathcal{H}^T_k = \mathcal{H}^G_k + \mathcal{H}_k$ are the sums of the Carroll gravity and matter contributions to the Hamiltonian and momentum constraints.

By a perturbation method where one of the scalar fields

$$\chi = \chi_{BG} + \mathcal{O}(\epsilon^2) \tag{3.11}$$

is a background field in addition to the static geometric background and the other scalar field

$$\phi = \epsilon\phi, \quad \epsilon \ll 1 \tag{3.12}$$

is a small fluctuation on top, the authors showed that even for a negative coupling constant g, the energy density remains positive as long as

$$g > \frac{-1}{(\partial\phi)^2 + (\partial\chi)^2}. \tag{3.13}$$

Since both $(\partial\phi)^2$ and $(\partial\chi)^2$ are small in this perturbative context, this is only a very weak bound.

The authors also proposed a generalization to a multi-scalar theory which can be written as (here in the case of three scalar fields ϕ, χ, ψ

$$I_{M_3} = \frac{1}{2} \int d^n x \Omega \left(\sum_{A=1}^{3} \left(v^\mu \partial_\mu \phi^A \right)^2 + g B_{\mu\nu} B^{\mu\nu} \right) \tag{3.14}$$

where the transverse tensor $B_{\mu\nu}$ is given in the familiar form

$$B_{\mu\nu} = v^{\rho} B_{\mu\nu\rho}, \quad B_{\mu\nu\rho} = \partial_{[\mu}\phi\partial_{\nu}\chi\partial_{\rho]}\psi. \tag{3.15}$$

Another result in [1] is their construction of a non-trivially interacting electromagnetic model.

$$I_{EM} = \frac{1}{2}\int d^n x\Omega\left(\left(v^{\mu}F_{\mu nu}\right)^2 + gC_{\mu\nu\rho}C^{\mu\nu\rho}\right) \tag{3.16}$$

where they consider the transverse three-form

$$C_{\mu\nu\rho} = v^{\sigma}C_{\mu\nu\rho\sigma}, \quad C_{\mu\nu\rho\sigma} = F_{[\mu\nu}F_{\rho\sigma]} \tag{3.17}$$

with the electromagnetic field

$$F_{\mu}\nu = (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}). \tag{3.18}$$

In this case the energy density is given by the expression

$$\mathcal{H} = \frac{1}{2\sqrt{h}}H_{ij}\pi^i\pi^j \tag{3.19}$$

where H_{ij} is the inverse of

$$H^{ij} = h^{ij} + \frac{2g}{3}\left(h^{il}h^{mk}h^{nj} + \frac{1}{2}h^{ij}h^{lk}h^{mn}\right)F_{lm}F_{kn}. \tag{3.20}$$

In four dimensions the determinant of H^{ij} is given by

$$D = h^{-1}\left(1 + \frac{2gB^2}{3}\right), \quad B^2 = \frac{h^{ik}h^{jl}F_{kl}F_{ij}}{2}. \tag{3.21}$$

Therefore, $\mathcal{H}$ is again positive definite if

$$g > -\frac{3}{2B^2} \tag{3.22}$$

which allows again negative values for g in a perturbative context.

The last Carroll-invariant scalar field model constructed in [1] is the Carroll swifton coupled to gravity in $2d$ since all known Carroll black hole solutions are described by 2d models. The Carroll dilaton gravity in 2d was introduced in [20], [21] and is given by

$$I_{CDG} \sim \int (X d\omega + X_H(d\tau + \omega \wedge e) + X_P\, de + \tau \wedge e\mathcal{V}(X, X_H)). \qquad (3.23)$$

where the action depends on the temporal einbein τ, the spatial einbein e, the Carroll boost connection ω, the dilaton X, the Lagrange multiplier X_H for the torsion constraint, and the Lagrange multiplier X_P for the intrinsic torsion constraint.

In order to investigate a Hawking-like effect (see [12]) one needs to couple matter to Carroll black holes since otherwise, the theory has no local propagating degrees of freedom. This adds to the list of motivations on why to study these models in the first place. The second key result of [1] is the 2d Carroll swifton scalar field action given by

$$I_{2d} = \frac{1}{2} \int d^2x \Omega F \left(\dot{\phi}^2 + g(\hat{\partial}\phi)^2 + h\dot{\phi}\hat{\partial}\phi \right) \qquad (3.24)$$

where the coupling function F, g, h may depend on the dilatation X and the Carroll boost-invariant scalar X_H. The volume form in this case is given by $d^2\Omega = t \wedge e$ and the Carroll boost-invariant derivative is introduced as

$$\hat{\partial} = e^\mu \partial_\mu + \frac{X_P}{X_H} v^\mu \partial_\mu. \qquad (3.25)$$

The first two terms in the above equation generalize to higher dimensions but the last one does not. In [1] they stress that they added a term in $\hat{\partial}$ that transforms like a Stückelberg field [22], but using only fields that were there already in the gravity action.

Furthermore, the definition of the Carroll-boost invariant derivative introduces the restriction $X_H \neq 0$. This shows that one is not allowed to sit on a Carroll extremal surface, which was introduced in [10]. Therefore, if one needs to extend the 2d Carroll swifton scalar field action onto a Carroll extremal surface $X_H = 0$, the coupling function g needs to be chosen appropriately, e.g. $g \propto X_H^2$.

4 Carroll z-scale invariance

We have already seen that in Carrollian geometry the lightcone collapses and space is absolute. This suggests fractonic behavior, i.e., "nothing can move". Therefore, it is not surprising that the two versions of a Carrollian scalar field theory looked at in the past exert this behavior. In the "electric" version, the variation of the action

$$I_e = \frac{1}{2}\int d^n x \Omega \left(v^\mu \partial_\mu \phi\right)^2 = \frac{1}{2}\int d^n x \left(\partial_t \phi\right)^2 \tag{4.1}$$

yields the ultra-local equation of motion $\partial_t^2 \phi = 0$. Therefore, the scalar field may depend on time but has no spatial derivatives. In the other "magnetic" version, the variation of the action

$$I_m = \int d^n x \left(\pi \partial_t \phi - \frac{1}{2}\delta^{ij}\partial_i \phi \partial_j \phi\right) \tag{4.2}$$

yields the time-independence constraint $\partial_t \phi = 0$ together with a Laplace equation of motion, $\delta^{ij}\partial_i \partial_j \phi = 0$. Neither of those leads to a scalar field propagating with finite, non-vanishing velocity.

Since Carroll causality forces information to stay within the lightcone, i.e. to stay at the same spatial location, propagation at a non-vanishing velocity $v > 0 = c_{\text{Carrol}}$ would define a tachyonic-like behavior.

As we have seen in the last section, the authors of [1] constructed and discussed examples of Carroll-invariant actions for interacting (scalar) fields which allow propagation at a non-vanishing velocity in arbitrary dimensions, both and without (Carroll) gravity. We will recall important results and equations in this chapter as necessary.

This chapter includes the main work of this thesis where we investigate the Carroll z-scale invariance of the Carroll-invariant actions for interacting scalar and vector fields presented in [1]. The calculations are performed rather explicitly so that the calculations are easy to follow and understand. The interested reader can skip to the last section of this chapter for a concise summary of all the results obtained.

Throughout this chapter, the scale parameter z defines an anisotropy parameter of Lifschitz type where $z = 1$ is the isotropic (or "standard") scale invariance.

4.1 Electric Model

Before we actually start our investigation of the Carroll-invariant actions for interacting scalar and vector fields from [1] we determine how a scalar field needs to transform to be Carroll z-scale invariant. The easiest Carrollian scalar field theory to look at in this case is the "electric" model given by

$$I_e = \frac{1}{2} \int d^n x \, (\partial_t \phi)^2 \,. \tag{4.3}$$

Generally there are two ways to determine the Carroll z-scale invariance by either looking at the transformations of the coordinates t, x or the vielbein τ, e, ω. We opted to use the transformations of the coordinates for all but the $2d$ Carroll dilaton gravity and $2d$ Carroll swifton scalar field model in the last section of this chapter.

The transformations are given as

$$t \to t\lambda^z \tag{4.4}$$

$$x \to x\lambda \tag{4.5}$$

and automatically determine the transformations of the partial derivatives as

$$\partial_t \to \lambda^{-z}\partial_t \tag{4.6}$$

$$\partial_i \to \lambda^{-1}\partial_i. \tag{4.7}$$

We now want to know how a scalar field needs to transform to be Carroll z-scale invariant and set

$$\phi \to \phi\lambda^{\alpha}. \tag{4.8}$$

where we need to determine the scaling exponent α. Substituting everything into the "electric" model we get

$$I_e = \frac{1}{2}\int dtdx_1\cdots dx_d(\partial_t\phi)^2 \to \frac{1}{2}\int d(t\lambda^z)d(x_1\lambda)\cdots d(x_d\lambda)(\lambda^{-z}\partial_t\lambda^{\alpha}\phi)^2 \tag{4.9}$$

$$= \frac{1}{2}\lambda^{z+d-2z+2\alpha}\int dtdx_1\cdots dx_d(\partial_t\phi)^2. \tag{4.10}$$

We specifically expanded $d^n x = dtdx_1\cdots dx_d$ to show the general transformation of the term

$$d^n x \to d^n x\lambda^{z+d} \tag{4.11}$$

where d is the number of spatial coordinates. Equation (4.10) gives us the necessary condition for the whole term to be Carroll z-scale invariant and a way to determine α

$$z + d - 2z + 2\alpha = 0 \Leftrightarrow \alpha = \frac{z-d}{2}. \tag{4.12}$$

Therefore, a scalar field needs to transform like

$$\phi \to \phi\lambda^{\alpha} = \phi\lambda^{\frac{z-d}{2}} \tag{4.13}$$

in order to be Carroll z-scale invariant. If we impose the restriction $z = 1$, i.e. isotropic scale invariance, we see that the scalar field would need to transform as

$$\phi \to \phi\lambda^{\alpha} = \phi\lambda^{\frac{1-d}{2}} \tag{4.14}$$

which can never be a positive integer for positive dimension d.

It is interesting to see if one could add a potential term depending on the scalar field ϕ and get a Carroll z-scale invariant expression. As in standard quantum field theory this should yield a specific integer exponent for the scalar field for different dimensions. We look at the full equation

$$I_e = \frac{1}{2}\int d^n x\left((\partial_t\phi)^2 - V(\phi)\right) \tag{4.15}$$

to see that

$$d^n x V(\phi)^n \to d^n x \lambda^{z+d} V(\phi)^n \lambda^{n\alpha} = \lambda^{z+d+n\alpha} d^n x V(\phi)^n. \tag{4.16}$$

Therefore, we can get the exponent of the potential by

$$z + d + n\alpha = z + d + n\frac{z-d}{2} = \frac{2z + 2d + n(z-d)}{2} \stackrel{!}{=} 0 \Leftrightarrow n = \frac{2(z+d)}{d-z}. \tag{4.17}$$

Let us first assume the isotropic case $z = 1$, i.e.

$$n = \frac{2(d+1)}{d-1} \Rightarrow \begin{cases} d = 1 : \text{does not exist} \\ d = 2 : n = 6 \\ d = 3 : n = 4 \\ d = 4 : \text{non-integer} \\ d = 5 : n = 3 \\ 6 \leq d < \infty : \text{non-integer} \\ d \to \infty : n = 2 \end{cases} \tag{4.18}$$

This shows that only certain positive dimensions can have integer exponents in the potential terms. The same analysis for $z = 0$ yields

$$n = \frac{2d}{d} = 2 \quad \forall d \in \mathbb{N}. \tag{4.19}$$

This case is interesting because of its relation to near horizon soft hair [23] and shows that any potential is compatible with $z = 0$ scale invariance.

Furthermore, if we let $z = 2$ we see that

$$n = \frac{2(d+2)}{d-2} \Rightarrow \begin{cases} d = 1 : \text{negative integer} \\ d = 2 : \text{does not exist} \\ d = 3 : n = 10 \\ d = 4 : n = 6 \\ d = 5 : \text{non-integer} \\ d = 6 : n = 4 \\ 7 \leq d \leq 9 : \text{non-integer} \\ d = 10 : n = 3 \\ d \geq 11 : \text{non-integer} \end{cases} \tag{4.20}$$

Of course, these special cases, which were motivated by the algebras and commutation relations presented in section 2.2, are not exhaustive since we can assume $d = z + 1$ and get

$$n = \frac{2(2z+1)}{1} = 4z + 2 \tag{4.21}$$

which yields even more examples with integer values for n. Furthermore, it would be possible for z to be a rational number, i.e. $z = \frac{p}{q}$. It turns out that this is a special case of the integer solution since

$$\frac{2(d + \frac{p}{q})}{d - \frac{p}{q}} = \frac{2(dq+p)}{dq-p} = \frac{2(\tilde{d} + \tilde{z})}{\tilde{d} - \tilde{z}} \tag{4.22}$$

where $\tilde{d} = dq$ and $\tilde{z} = p$.

4.2 Bi-Scalar Model

The bi-scalar Model introduced in [1] couples two scalar field ϕ, χ with canonically normalized kinetic terms and a coupling constant g. This is a generalized mode of the one introduced in [24] to any Carroll background. The action in covariant form is given as

$$I_M = \frac{1}{2}\int d^n x \Omega \left((v^\mu \partial_\mu \phi)^2 + (v^\mu \partial_m \chi) + g B_\mu B^\mu \right) \tag{4.23}$$

where

$$B_\nu = v^\mu \left(\partial_\mu \phi \partial_\nu \chi - \partial_\mu \chi \partial_\nu \phi \right) = 2 v^\mu \partial_{[\mu} \phi \partial_{\nu]} \chi \tag{4.24}$$

is a manifestly transverse covariant vector. Since B_ν involves simultaneously time and spatial derivatives, the model allows propagation off the Carroll lightcone (see [1]). The antisymmetry of the coefficient of v^μ is crucial for this transversality, which would not hold if $\phi = \chi$ since then B_ν would be identically zero.

We have already seen how the scalar fields ϕ, χ need to transform to be Carroll z-scale invariant so we only need to check the transformation exponent of $B_\mu B^\mu$. It is instructive to go through the whole calculation once. Afterwards we will obtain the same result by arguing with the properties of B_μ being "transverse" or "spacelike" as introduced in chapter 2. Let us first expand the whole expression

$$B_\mu B^\mu = v^\mu v^\mu \left((\partial_\mu \phi)^2 (\partial_\nu \chi)^2 - 2 \partial_\mu \phi \partial_\mu \chi \partial_\nu \phi \partial_\nu \chi + (\partial_\mu \chi)^2 (\partial_\nu \phi)^2 \right) \tag{4.25}$$

$$= (\partial_t \phi)^2 (\partial_\nu \chi)^2 - 2 \partial_t \phi \partial_t \chi \partial_\nu \phi \partial_\nu \chi + (\partial_t \chi)^2 (\partial_\nu \phi)^2 \tag{4.26}$$

We need to unravel the remaining gradients $\partial_\nu \phi = (\partial_t \phi, \partial_i \phi)$ and $\partial_\nu \chi = (\partial_t \chi, \partial_i \chi)$ and see that

$$(\partial_\nu \phi)^2 = (\partial_t \phi)^2 + (\partial_i \phi)^2 \tag{4.27}$$

$$(\partial_\nu \chi)^2 = (\partial_t \chi)^2 + (\partial_i \chi)^2 \tag{4.28}$$

$$\partial_\nu \phi \partial_\nu \chi = \partial_t \phi \partial_t \chi + \partial_i \phi \partial_i \chi. \tag{4.29}$$

Substituting this back into equation (4.26) we get

$$B_\mu B^\mu = (\partial_t\phi)^2\left((\partial_t\chi)^2 + (\partial_i\chi)^2\right) - 2\partial_t\phi\partial_t\chi\left(\partial_t\phi\partial_t\chi + \partial_i\phi\partial_i\chi\right) \tag{4.30}$$

$$+ (\partial_t\chi)^2\left((\partial_t\phi)^2 + (\partial_i\phi)^2\right) \tag{4.31}$$

$$= (\partial_t\phi)^2(\partial_i\chi)^2 - 2\partial_t\phi\partial_t\chi\partial_i\phi\partial_i\chi + (\partial_t\chi)^2(\partial_t\phi)^2 \tag{4.32}$$

It is now straightforward to write down the transformation of each term

$$(\partial_t\phi)^2(\partial_i\chi)^2 \to \lambda^{-2z+2\alpha-2+2\alpha}(\partial_t\phi)^2(\partial_i\chi)^2 \tag{4.33}$$

$$(\partial_t\chi)^2(\partial_i\phi)^2 \to \lambda^{-2z+2\alpha-2+2\alpha}(\partial_t\chi)^2(\partial_i\phi)^2 \tag{4.34}$$

$$\partial_t\phi\partial_t\chi\partial_i\phi\partial_i\chi \to \lambda^{-2z+2\alpha-2+2\alpha}\partial_t\phi\partial_t\chi\partial_i\phi\partial_i\chi \tag{4.35}$$

The exponents of all terms are consistent and it is now possible to determine the transformation of the interaction term $B_\mu B^\mu$ by looking at the

$$d^n x g B_\mu B^\mu \to d^n x \lambda^{z+d} g B_\mu B^\mu \lambda^{-2z+2\alpha-2+2\alpha} \tag{4.36}$$

which yields

$$z + d - 2z + 4\alpha - 2 = -z + d + 4\frac{z-d}{2} - 2 = z - d - 2 = 0 \Leftrightarrow z = d + 2. \tag{4.37}$$

Finally, we get

$$B_\mu B^\mu \to B_\mu B^\mu \lambda^{z-d-2} \tag{4.38}$$

Another, easier and quicker, way to the same result is by using the fact that B_μ needs to be a "transverse" or "spacelike" co-vector defined as being orthonormal to v^μ, $v^\mu B_\mu = 0$. If we look at the definition of $B_\nu = v^\mu B_{\mu\nu}$ we see that

$$v^\nu B_\nu = v^\nu v^\mu B_{\mu\nu} = v^\nu B_{0\nu} = B_{00}. \tag{4.39}$$

Therefore, the only surviving term is

$$B_{0i} = (\partial_t\phi\partial_i\chi - \partial_i\phi\partial_t\chi) \to \left(\partial_t\lambda^{-z}\phi\lambda^\alpha\partial_i\lambda^{-1}\chi\lambda^\alpha - \partial_t\lambda^{-z}\chi\lambda^\alpha\partial_i\lambda^{-1}\phi\lambda^\alpha\right). \tag{4.40}$$

Collecting the terms we see that they are once again consistent (both terms have the same exponents) and yield the same expression as above

$$B_{0i} \to B_{0i}\lambda^{-z+2\alpha-1} \Rightarrow B_{0i}B^{0i} \to B_{0i}B^{0i}\lambda^{-2z+4\alpha-2}. \tag{4.41}$$

The full transformation can then be calculated again by looking at the full expression

$$d^n x B_{0i}B^{0i} \to d^n x \lambda^{z+d} B_{0i}B^{0i}\lambda^{-2z+4\alpha-2}. \tag{4.42}$$

This is the exact same expression as above and again yields the restriction

$$z - d - 2 = 0. \tag{4.43}$$

This simplifies the transformation exponents of the scalar fields to

$$\phi \to \phi\lambda^{\frac{z-d}{2}} = \phi\lambda^{\frac{d+2-d}{2}} = \phi\lambda^1. \tag{4.44}$$

We can again try to add a homogeneous polynomial as a potential term depending on the scalar fields. The full action is given as

$$I_M = \frac{1}{2}\int d^n x\Omega\left(\left(v^\mu\partial_\mu\phi\right)^2 + \left(v^\mu\partial_\mu\chi\right)^2 + gB_\mu B^\mu + V(\phi\chi)^n\right) \tag{4.45}$$

We want to see if there are any choices of exponents for positive integer dimension d. In contrast to the simple "electric" model, which does not lead to a scalar field propagating with finite, non-vanishing velocity, this is impossible. The transformation yields

$$d^n x V(\phi\chi)^n \to \lambda^{z+d} d^n x \lambda^{n\alpha} V(\phi\chi)^n = \lambda^{z+d+n\alpha} d^n x V(\phi\chi)^n \tag{4.46}$$

which results in

$$z + d + n\alpha = z + d + n\frac{z-d}{2} = 0 \Leftrightarrow n = \frac{-2(z+d)}{z-d} \tag{4.47}$$

This is the same expression as in the "electric" model but this time the parameter z is already determined by the interaction term as $z = d + 2$. Therefore, we see that

$$n = -2d - 2 \tag{4.48}$$

which is never positive for positive integer dimensions d.

4.3 Multi-Scalar Model

The bi-scalar model can easily be generalized to a multi-scalar model. The action with three scalar fields $\phi^A = (\phi, \chi, \psi)$ presented in [1] given by

$$I_{M_3} = \frac{1}{2} \int d^n x \Omega \left(\sum_{A=1}^{3} \left(v^\mu \partial_\mu \phi^A \right)^2 + g B_{\mu\nu} B^{\mu\nu} \right) \tag{4.49}$$

is the starting point of our investigations. Again, the transverse tensor $B_{\mu\nu}$ is given in the familiar form

$$B_{\mu\nu} = v^\rho B_{\mu\nu\rho}, \quad B_{\mu\nu\rho} = \partial_{[\mu} \phi \partial_\nu \chi \partial_{\rho]} \psi. \tag{4.50}$$

In the following it will be clear why the use of the transverse property of the interaction term is essential since the interaction term is already of order six in the derivatives.

For the multi-scalar model we start, again, from the definition of B_ν

$$B_\nu = v^\mu B_{\mu\nu} = B_{0\nu} \tag{4.51}$$

and then move forward in an iterative process. Since B_ν is a transverse form we get

$$v^\nu B_\nu = v^\nu v^\mu B_{\mu\nu} = v^\nu B_{0\nu} = B_{00} = 0 \tag{4.52}$$

as before with the only term surviving

$$B_{0i} = (\partial_t \phi \partial_i \chi - \partial_i \phi \partial_t \chi) \tag{4.53}$$

Similarly, for a three scalar model we look again at the transverse term to see

$$v^\nu B_{\mu\nu} = v^\nu v^\rho B_{\mu\nu\rho} = v^\nu B_{\mu\nu 0} = B_{\mu 00} = 0 \tag{4.54}$$

and see that the only surviving term is

$$B_{0ij} = (\partial_t \phi \partial_i \chi \partial_j \psi - \partial_t \phi \partial_j \chi \partial_i \psi + \partial_i \phi \partial_j \chi \partial_t \psi \tag{4.55}$$

$$- \partial_i \phi \partial_t \chi \partial_j \psi + \partial_j \phi \partial_t \chi \partial_i \psi - \partial_j \phi \partial_i \chi \partial_t \psi) \tag{4.56}$$

Each term consists of one time derivative and two spatial derivatives they all have the same transformation given by

$$\partial_t\phi\partial_i\chi\partial_j\psi \rightarrow \lambda^{-z}\partial_t\lambda^{\alpha}\phi\lambda^{-1}\partial_i\lambda^{\alpha}\chi\lambda^{-1}\partial_i\lambda^{\alpha}\psi = \lambda^{-z+3\alpha-2} \tag{4.57}$$

Substituting this into the full interaction term we get

$$d^n x B_{0ij}B^{0ij} \rightarrow d^n x\lambda^{z+d}B_{0ij}\lambda^{-z+3\alpha-2}B^{0ij}\lambda^{-z+3\alpha-2}\lambda^{z+d-2z+6\alpha-4}d^n x B_{0ij}B^{0ij} \tag{4.58}$$

and we see that $B_{\mu\nu}B^{\mu\nu}$ is Carroll z-scale invariant if

$$z+d-2z+6\alpha-4 = -z+d+6\frac{z-d}{2}-4 \tag{4.59}$$
$$= -z+d+3z-3d-4 \tag{4.60}$$
$$= 2z-2d-4 \stackrel{!}{=} 0 \Leftrightarrow z-d-2=0 \tag{4.61}$$

This is the exact same result as for the bi-scalar model which is not surprising since the interaction term always follows the same pattern. Due to the transverse property of the interaction term only one term survives with any number of scalar fields. As we have seen before, this term only has one time derivative and $m-1$ spatial derivatives where m is the number of scalar fields in the multi-scalar model. Concretely, for

$$B_{\mu_1\ldots\mu_{m-1}} = v^{\mu_m}B_{\mu_1\ldots\mu_m} \tag{4.62}$$

and due to the transverse property of $B_{\mu_1\ldots\mu_{m-1}}$

$$v^{\mu_{m-1}}B_{\mu_1\ldots\mu_{m-1}} = v^{\mu_{m-1}}v^{\mu_m}B_{\mu_1\ldots\mu_m} = B_{\mu_1\ldots\mu_{m-2}00} = 0 \tag{4.63}$$

the only surviving term is $B_{0i_2\ldots i_m}$.

We define

$$m = \text{ number of scalar fields} \tag{4.64}$$
$$l = \text{ number of spatial derivatives} \tag{4.65}$$

which are related by $l = m - 1$. Therefore, we can write down the most general transformation for a multi-scalar model with m scalar fields as

$$d^n x B_{0i_1\dots i_m} B^{0i_2\dots i_m} \to d^n x \lambda^{z+d} B_{0i_2\dots i_m} \lambda^{-z+m\alpha-l} B^{0i_2\dots i_m} \lambda^{-z+m\alpha-l} \tag{4.66}$$

$$= \lambda^{z+d-2z+2m\alpha-2l} d^n x B_{0i_2\dots i_m} B^{0i_2\dots i_m} \tag{4.67}$$

Therefore, for $B_{\mu_1\dots\mu_{m-1}}$ to be Carroll z-scale invariant it needs to transform as follows

$$z + d - 2z + 2m\alpha - 2l = -z + d + m(z-d) - 2l \tag{4.68}$$

$$= -z + d + mz - md - 2(m-1) \tag{4.69}$$

$$= z(m-1) - d(m-1) - 2(m-1) \tag{4.70}$$

$$= zm - dm - 2m \stackrel{!}{=} 0 \Leftrightarrow \begin{cases} m = 1 \\ z - d - 2 = 0 \end{cases} \tag{4.71}$$

Since $m = 1$ means that there is only one scalar field we cannot have any interaction term at all so this case in not interesting. However, the other case $z - d - 2 = 0$ is the same as for the bi-/tri-scalar model which shows that one can couple different interaction terms together to yield a new action. Again, this simplifies the transformation exponent of the scalar fields to

$$\phi \to \phi\lambda^{\frac{z-d}{2}} = \phi\lambda^{\frac{d+2-d}{2}} = \phi\lambda^1. \tag{4.72}$$

If we try to add a homogeneous polynomial as a potential term depending on multiple scalar fields again we get the same result as for the bi-scalar model due to the parameter z which is determined by the interaction term. Therefore, it is also not possible to add a potential term to this swifton action since

$$n = -2d - 2. \tag{4.73}$$

is never positive for positive integer dimensions d.

However, since the restriction $z = d + 2$ is the same for all multi-scalar models it is now possible to combine the interaction terms of an arbitrary amount of scalar fields in the action, i.e. actions of the form

$$I_{M_3} = \frac{1}{2}\int d^n x \Omega \left(\sum_{A=1}^{3} \left(v^\mu \partial_\mu \phi^A \right)^2 + g(B_\mu B^\mu + B_\nu B^\nu + B_\rho B^\rho + \dots) \right) \tag{4.74}$$

where

$$B_\mu = 2v^\nu \partial_{[\mu}\phi \partial_{\nu]}\chi \tag{4.75}$$
$$B_\nu = 2v^\rho \partial_{[\nu}\chi \partial_{\rho]}\psi \tag{4.76}$$
$$B_\rho = 2v^\mu \partial_{[\rho}\psi \partial_{\mu]}\phi \tag{4.77}$$
$$\dots \tag{4.78}$$

are the interaction terms with the scalar fields switched. We will see that the Carroll multi-scalar swifton actions can even be combined with the spin 1 Carroll swifton theories in the next section.

4.4 Electromagnetic Model

The same ideas as before apply to a non-trivially interaction electromagnetic model which is constructed as

$$I_{EM} = \frac{1}{2}\int d^n x \Omega \left(\left(v^\mu F_{\mu\nu} \right)^2 + g C_{\mu\nu\rho} C^{\mu\nu\rho} \right) \tag{4.79}$$

where we consider the transverse three-form

$$C_{\mu\nu\rho} = v^\sigma C_{\mu\nu\rho\sigma}, \quad C_{\mu\nu\rho\sigma} = F_{[\mu\nu} F_{\rho\sigma]} \tag{4.80}$$

with the electromagnetic field

$$F_\mu \nu = (\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{4.81}$$

A priori the vector A_μ transforms with the unknown exponents a, b as

$$A_t \rightarrow A_t \lambda^a \tag{4.82}$$
$$A_i \rightarrow A_i \lambda^b \tag{4.83}$$

We first look at the transformation restriction for the electromagnetic field given in the first term

$$v^{\mu} F_{\mu\nu} = F_{0i} \tag{4.84}$$

to get

$$F_{0i} = (\partial_t A_i - \partial_i A_t) \to (\lambda^{-z}\partial_t \lambda^b A_i - \lambda^{-1}\partial_i \lambda^a A_t) = (\lambda^{-z+b}\partial_t A_i - \lambda^{-1+a}\partial_i A_t). \tag{4.85}$$

This time the individual terms do not transform consistently so when we square the whole expression we get three different exponents

$$-2z + 2b, \quad -2 + 2a, \quad -z + a + b - 1 \tag{4.86}$$

and in the full term

$$d^n x \left(v^{\mu} F_{\mu\nu}\right)^2 = d^n x (F_{0i})^2 \tag{4.87}$$

we then get

$$z + d - 2z + 2b = -z + d + 2b \overset{!}{=} 0 \tag{4.88}$$

$$z + d - 2 + 2a = z + d + 2a - 2 \overset{!}{=} 0 \tag{4.89}$$

$$z + d - z + a + b - 1 = d + a + b - 1 \overset{!}{=} 0 \tag{4.90}$$

In order for the electromagnetic field to be Carroll z-scale invariant all three equations have to be fulfilled. The first equation yields

$$b = \frac{z-d}{2} \tag{4.91}$$

The second equation yields

$$a = \frac{2-z-d}{2} \tag{4.92}$$

Finally, the third equation is consistent with the solutions for the exponents a, b

$$d + \frac{2-z-d}{2} + \frac{z-d}{2} - 1 = \frac{2d + 2 - z - d + z - d - 2}{2} = 0 \tag{4.93}$$

Therefore, for the vector A_μ to be Carroll z-scale invariant it needs to transform as

$$A_t \to A_t \lambda^{\frac{2-z-d}{2}} \tag{4.94}$$

$$A_i \to A_i \lambda^{\frac{z-d}{2}} \tag{4.95}$$

Since the interaction term is a transverse three-form we can use the ideas from the previous sections again to see that

$$C_{\mu\nu\rho} v^\rho = C_{\mu\nu\rho\sigma} v^\sigma v^\rho = C_{\mu\nu 00} = 0 \tag{4.96}$$

Therefore, the only term surviving is C_{0ijk} given by

$$\begin{aligned} C_{0ijk} &= F_{0i} F_{jk} && (4.97)\\ &= (\partial_t A_i - \partial_i A_t)(\partial_j A_k - \partial_k A_j) && (4.98)\\ &= \partial_t A_i \partial_j A_k - \partial_t A_i \partial_k A_j - \partial_i A_t \partial_j A_k - \partial_i A_t \partial_k A_j && (4.99) \end{aligned}$$

Similarly to the case of the electromagnetic field the terms do not transform consistently but yield two different exponents

$$\partial_t A_i \partial_j A_k \to \lambda^{-z} \partial_t \lambda^b A_i \lambda^{-1} \partial_i \lambda^b A_k = \lambda^{-z+2b-1} \partial_t A_i \partial_j A_k \tag{4.100}$$

$$\partial_i A_t \partial_j A_k \to \lambda^{-1} \partial_t \lambda^a A_i \lambda^{-1} \partial_i \lambda^b A_k = \lambda^{a+b-2} \partial_i A_t \partial_j A_k \tag{4.101}$$

So when we square the expression we get three different terms again

$$-2z + 4b - 2 = 0, \quad 2a + 2b - 4 = 0, \quad -z + a + 3b - 3 = 0 \tag{4.102}$$

For the full interaction term to be Carroll z-scale invariant we then get

$$z + d - 2z + 4b - 2 = -z + d + 4z - 2 \overset{!}{=} 0 \tag{4.103}$$

$$z + d + 2a + 2b - 4 \overset{!}{=} 0. \tag{4.104}$$

$$z + d - z + a + 3b - 3 = d + a + 3b \overset{!}{=} 0 \tag{4.105}$$

Since we already know the values of a, b a short calculation shows that all three equations are fulfilled if

$$z = d + 2. \tag{4.106}$$

This simplifies the transformation exponents of the components of A_μ to

$$A_t \to A_t \lambda^{\frac{2-z-d}{2}} = A_t \lambda^{\frac{2-d-2-d}{2}} = A_t \lambda^{-d} \tag{4.107}$$

$$A_i \to A_i \lambda^{\frac{z-d}{2}} = A_i \lambda^{\frac{d+2-d}{2}} = A_i \lambda^{1} \tag{4.108}$$

As with the multi-scalar model, if we try to add a homogeneous polynomial as a potential term depending on multiple scalar fields again we get the same result

$$n = -2d - 2. \tag{4.109}$$

Therefore, it is also not possible to add a potential term to this swifton action since it is never positive for positive integer dimensions d.

However, since the electromagnetic model has the same restriction $z = d + 2$ from the interaction term as the multi-scalar model it is possible to combine spin 0 and spin 1 theories for Carroll swiftons.

4.5 Carroll Dilaton Gravity in 2D

For the coupling of swiftons to gravity we focus on two spacetime dimensions. As described in [1] the reason for this is that all known Carroll black hole solutions are described by 2d models (intrinsically or by dimensional reduction). The Carroll dilaton gravity in 2d was introduced in [20], [21] and its action given by

$$I_{CDG} \sim \int \left(X d\omega + X_H(d\tau + \omega \wedge e) + X_P\, de + \tau \wedge e\mathcal{V}(X, X_H)\right) \tag{4.110}$$

depends on the temporal einbein τ, the spatial einbein e, the Carroll boost connection ω, the dilaton X, the Lagrange multiplier X_H for the torsion constraint, and the Lagrange multiplier X_P for the intrinsic torsion constraint.

Like before we want to find the transformation exponents for X, X_H, X_P and $\mathcal{V}(X, X_H)$ so that the whole expression is Carroll z-scale invariant. As quickly mentioned in the first section, we will now look at the transformations of the vielbein instead of the transformations of the coordinates themselves. The transformations are given as

$$e \rightarrow e\lambda \tag{4.111}$$

$$\tau \rightarrow \tau\lambda^z \tag{4.112}$$

$$\omega \rightarrow \omega\lambda^{z-1} \tag{4.113}$$

whereas, for the unknown exponents of X, X_H, X_P and $\mathcal{V}(X, X_H)$, we just set

$$X \rightarrow X\lambda^a \tag{4.114}$$

$$X_H \rightarrow X_H\lambda^b \tag{4.115}$$

$$X_P \rightarrow X_P\lambda^c \tag{4.116}$$

$$\mathcal{V}(X, X_H) \rightarrow \mathcal{V}(X, X_H)\lambda^d \tag{4.117}$$

The transformation of the first term of the Carroll dilaton gravity

$$Xd\omega \rightarrow X\lambda^a d\omega\lambda^{z-1} = \lambda^{a+z-1}Xd\omega \tag{4.118}$$

already gives us the transformation for X as

$$a = 1 - z \tag{4.119}$$

Now all the other terms need to be consistent with this transformation restriction so we see that the transformation for X_H from the second term must be given by

$$X_H d\tau \rightarrow X_H\lambda^b X_H\lambda^z d\tau = \lambda^{b+z}X_H d\tau \tag{4.120}$$

where we can substitute $z = 1 - a$ to get

$$b = a - 1 = -z \tag{4.121}$$

Furthermore, this needs to be consistent with the term

$$X_H\ \omega \wedge e \rightarrow X_H\lambda^b\omega\lambda^{z-1} \wedge e\lambda = \lambda^{b+z-1+1}X_H\ \omega \wedge e \tag{4.122}$$

which yields the same equation $b = -z$. We only get one equation for X_P

$$X_P\ de \to X_P \lambda^c\ de\lambda = \lambda^{c+1} X_P\ de \tag{4.123}$$

which simplifies down to

$$c = -1 = a + z - 2. \tag{4.124}$$

Finally, for the potential $\mathcal{V}(X, X_H)$ we get

$$\tau \wedge e\mathcal{V}(X, X_H) \to \tau\lambda^z \wedge e\lambda\mathcal{V}(X, X_H)\lambda^d \tag{4.125}$$

which yields

$$d = a - 2. \tag{4.126}$$

Therefore, if we set $z = 1$ to be the isotropic scaling we get

$$\begin{aligned} I_{CDG} \sim & \int (X d\omega + X_H(d\tau + \omega \wedge e) + X_P\ de + \tau \wedge e\mathcal{V}(X, X_H)) \to \\ & \int (X d\omega + \lambda^{-1} X_H(d\tau + \omega \wedge e) + \lambda^{-1} X_P\ de + \tau \wedge e\lambda^{-2}\mathcal{V}(X, X_H)) \end{aligned} \tag{4.127}$$

We are now able to state a $2d$ Carroll swifton scalar field action

$$I_{2d} = \frac{1}{2}\int d^2x \Omega F \left(\dot{\phi}^2 + g(\hat{\partial}\phi)^2 + h\dot{\phi}\hat{\partial}\phi\right) \tag{4.128}$$

where the coupling function F, g, h may depend on the dilatation X and the Carroll boost-invariant scalar X_H. The volume form in this case is given by $d^2\Omega = t \wedge e$. The most general Carroll invariant second order equation above combines non-trivially time- and space-derivatives of the scalar field ϕ. Furthermore, it does not introduce any extra structure besides the Carroll background. The Carroll boost-invariant derivative is defined as

$$\hat{\partial} = e^\mu \partial_\mu + \frac{X_P}{X_H} v^\mu \partial_\mu. \tag{4.129}$$

The first two terms in equation (4.128) generalize to higher dimensions but the last one does not. In [1] the authors stress that they added a term in $\hat{\partial}$ that transforms like a Stückelberg field [22], but using only fields that were there already in the gravity action.

Furthermore, the definition of the Carroll-boost invariant derivative introduces the restriction $X_H \neq 0$. This shows that one is not allowed to sit on a Carroll extremal surface introduced in [10]. Therefore, if one needs to extend the $2d$ Carroll swifton scalar field action onto a Carroll extremal surface $X_H = 0$, the coupling function g needs to be chosen appropriately, e.g. $g \propto X_H^2$. However, we will see that this is not possible if we want the whole action to be Carroll z-invariant.

We state the transformations of X, X_H and X_P in terms of z again since it is easier to see what the coupling functions need to be proportional to.

$$X \to X\lambda^a = X\lambda^{1-z} \tag{4.130}$$

$$X_H \to X_H\lambda^b = X_H\lambda^{-z} \tag{4.131}$$

$$X_P \to X_P\lambda^c = X_P\lambda^{-1} \tag{4.132}$$

Since we know how X and X_H need to transform we can quickly see that

$$X \to \lambda^{1-z}X = \left(\lambda^{-z}\right)^{\frac{z-1}{z}} X \Leftrightarrow \lambda^{-z}X_H^{\frac{z-1}{z}} \tag{4.133}$$

$$X_H \to \lambda^{-z}X_H = \left(\lambda^{1-z}\right)^{\frac{\frac{z}{z}}{z-1}} X \Leftrightarrow \lambda^{1-z}X^{\frac{z}{z-1}} \tag{4.134}$$

Furthermore, we set the transformation of the coupling functions F, g and h as

$$F \to F\lambda^p \tag{4.135}$$

$$g \to g\lambda^q \tag{4.136}$$

$$h \to h\lambda^r \tag{4.137}$$

with the scaling exponents p, q, r to be calculated.

We start with the first term to calculate the transformation of the coupling function F

$$d^2xF\dot{\phi}^2 \to \lambda^{z+1}d^2xF\lambda^p(\dot{\phi})^2\lambda^{2(\alpha-z)} = \lambda^{z+1+p+2\alpha-2z}d^2xF\dot{\phi}^2. \tag{4.138}$$

This yields

$$p = 0 \tag{4.139}$$

which means that F has to be constant for the term to be Carroll z-scale invariant but can be proportional to $X_H^{\frac{z-1}{z}}/X$.

Before looking at the second term and the second coupling function g we first see how $\hat{\partial}\phi$ transforms. We start by unraveling the definition

$$\hat{\partial}\phi = e^{\mu}\partial_{\mu}\phi + \frac{X_P}{X_H}v^{\mu}\partial_{\mu}\phi = \partial_i\phi + \frac{X_P}{X_H}\partial_t\phi \tag{4.140}$$

where we can then apply the transformations we already know

$$\partial_i\phi + \frac{X_P}{X_H}\partial_t\phi \to \partial_i\phi\lambda^{\alpha-1} + \frac{X_P}{X_H}\lambda^{-1+z}\partial_t\phi\lambda^{\alpha-z} \tag{4.141}$$

$$= \lambda^{\alpha-1}\partial_i\phi + \lambda^{\alpha-1}\frac{X_P}{X_H}\partial_t\phi \tag{4.142}$$

We can now calculate the scaling exponent for the second coupling function g by

$$d^2xFg(\hat{\partial}\phi)^2 \to d^2x\lambda^{z+1}Fg\lambda^q(\hat{\partial}\phi)^2\lambda^{2(\alpha-1)} = \lambda^{z+1+q+2\alpha-2}d^2xFg(\hat{\partial}\phi)^2 \tag{4.143}$$

which yields

$$q = 2 - 2z. \tag{4.144}$$

Therefore, we see that for the whole action to be Carroll z-scale invariant the coupling function g cannot be proportional to X_H^2 but needs to be proportional to X^2 or $X_H^{\frac{2(z-1)}{z}}$. This shows that a Carroll z-scale invariant action of the form equation (4.128) cannot be extended onto a Carroll extremal surface $X_H = 0$.

Lastly, we look at the transformation exponent for the coupling function h

$$d^2xFh\dot{\phi}\hat{\partial}\phi \to d^2x\lambda^{z+1}Fh\lambda^r\dot{\phi}\lambda^{\alpha-z}\hat{\partial}\phi\lambda^{\alpha-1} = \lambda^{z+1+r+\alpha-z+\alpha-1}d^2xFh\dot{\phi}\hat{\partial}\phi \tag{4.145}$$

Therefore, we get

$$r = 1 - z. \tag{4.146}$$

and the coupling function h is proportional to X or $X_H^{\frac{z-1}{z}}$. This shows a peculiar feature of the $2d$-Carroll dilaton gravity action that all coupling functions F, g, h are different.

Again, if we set $z = 1$ to be the isotropic (or "standard") scale again, we see that

$$F \to F\lambda^0 \tag{4.147}$$

$$g \to g\lambda^2 \tag{4.148}$$

$$h \to h\lambda^1 \tag{4.149}$$

4.6 Summary

In this section we summarize the results of the whole chapter such that all the information is easily accessible at a glance.

We started out by investigating the transformation of the "electric" model

$$I_e = \frac{1}{2} \int d^n x \left((\partial_t \phi)^2 - V(\phi) \right) \tag{4.150}$$

and saw that the scalar field needs to transform like

$$\phi \to \lambda^{\frac{z-d}{2}} \phi \tag{4.151}$$

to be z-scale invariant. If one wants to add an additional potential (depending on the scalar field ϕ) the scaling exponent of the potential term needs to be

$$n = \frac{2(z+d)}{d-z}. \tag{4.152}$$

From this equation it is easy to see that only for specific values of the scaling parameter z and (positive) dimension d the scaling exponent n is an integer, i.e. for $z = 1$ we get

$$n = \frac{2(d+1)}{d-1} \Rightarrow \begin{cases} d = 2 : n = 6 \\ d = 3 : n = 4 \\ d = 5 : n = 3 \\ d \to \infty : n = 2 \end{cases} \tag{4.153}$$

In the bi-scalar model

$$I_M = \frac{1}{2}\int d^n x \Omega \left((v^\mu \partial_\mu \phi)^2 + (v^\mu \partial_\mu \chi)^2 + g B_\mu B^\mu + V(\phi\chi)^n \right) \tag{4.154}$$

the terms for the scalar fields have not changed, hence they transform the same way as before. However, due to the additional interaction term we get a restriction on the values on the scaling exponents since it has to transform like

$$B_\mu B^\mu = \lambda^{z-d-2} B_\mu B^\mu. \tag{4.155}$$

Therefore, in the bi-scalar model, the scalar fields need to transform like

$$\phi \to \lambda\phi \tag{4.156}$$

in order to be z-scale invariant. In contrast to the "electric" model, this Carroll-invariant scalar field theory cannot have a positive scaling exponent n for the potential term for positive dimension d since

$$n = -2d - 2. \tag{4.157}$$

Surprisingly, the same multi-scalar model (here with three scalar fields)

$$I_{M_3} = \frac{1}{2}\int d^n x \Omega \left(\sum_{A=1}^{3} \left(v^\mu \partial_\mu \phi^A \right)^2 + g B_{\mu\nu} B^{\mu\nu} + V(\phi\chi\psi)^n \right) \tag{4.158}$$

has the exact same scaling exponents for the scalar fields and the interaction term as the bi-scalar model. On the one hand, this means that it is still not possible to add a potential term with positive scaling exponent. On the other hand, this shows that it is possible to combine the interaction terms of an arbitrary amount of scalar fields in the action, e.g.

$$I_{M_3} = \frac{1}{2}\int d^n x \Omega \left(\sum_{A=1}^{3} \left(v^\mu \partial_\mu \phi^A \right)^2 + g (B_\mu B^\mu + B_\nu B^\nu + B_\rho B^\rho + \dots) \right). \tag{4.159}$$

The next Carroll-invariant field theory we looked at was the electromagnetic model

$$I_{EM} = \frac{1}{2}\int d^n x \Omega \left((v^\mu F_{\mu\nu})^2 + g C_{\mu\nu\rho} C^{\mu\nu\rho} \right) \tag{4.160}$$

with the electromagnetic field

$$F_{\mu}\nu = (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}). \tag{4.161}$$

For the vector A_{μ} we found that it has to transform like

$$A_t \rightarrow A_t \lambda^{\frac{2-z-d}{2}} \tag{4.162}$$
$$A_i \rightarrow A_i \lambda^{\frac{z-d}{2}} \tag{4.163}$$

to be z-scale invariant. However, the interaction term also imposes restriction on the values of the scaling exponents which, again, turns out to be

$$C_{\mu\nu\rho} C^{\mu\nu\rho} \rightarrow \lambda^{z-d-2} C_{\mu\nu\rho} C^{\mu\nu\rho}. \tag{4.164}$$

Therefore, the transformation exponents of the vector A_{μ} reduce to

$$A_t \rightarrow A_t \lambda^{-d} \tag{4.165}$$
$$A_i \rightarrow A_i \lambda. \tag{4.166}$$

The last model we looked at was the Carroll dilaton gravity in two dimensions for the coupling of swiftons to gravity. The Carroll dilaton gravity in $2d$ is given by

$$I_{CDG} \sim \int (X d\omega + X_H (d\tau + \omega \wedge e) + X_P \, de + \tau \wedge e \mathcal{V}(X, X_H)) \tag{4.167}$$

and is z-scale invariant if

$$X \rightarrow \lambda^{1-z} X \tag{4.168}$$
$$X_H \rightarrow \lambda^{-z} X_H \tag{4.169}$$
$$X_P \rightarrow \lambda^{-1} X_P \tag{4.170}$$
$$\mathcal{V}(X, X_H) \rightarrow \lambda^{-z-1} \mathcal{V}(X, X_H). \tag{4.171}$$

A quick calculation also shows that

$$X \rightarrow \lambda^{-z} X_H^{\frac{z-1}{z}} \tag{4.172}$$
$$X_H \rightarrow \lambda^{1-z} X^{\frac{z}{z-1}}. \tag{4.173}$$

Finally, the $2d$ Carroll swifton scalar field action is given as

$$I_{2d} = \frac{1}{2}\int d^2x\Omega F\left(\dot{\phi}^2 + g(\hat{\partial}\phi)^2 + h\dot{\phi}\hat{\partial}\phi\right) \tag{4.174}$$

where the coupling function F, g, h may depend on the dilatation X and the Carroll boost-invariant scalar X_H and the Carroll boost-invariant derivative is introduced as

$$\hat{\partial} = e^{\mu}\partial_{\mu} + \frac{X_P}{X_H}v^{\mu}\partial_{\mu}. \tag{4.175}$$

The scaling exponents of the coupling functions are then given by

$$F \to \lambda^0 F = F \tag{4.176}$$

$$g \to \lambda^{2-2z} g \tag{4.177}$$

$$h \to \lambda^{1-z} h \tag{4.178}$$

which shows the peculiar feature of the $2d$-Carroll dilaton gravity action that all coupling functions are different. Furthermore, the coupling constant g is proportional to X^2 or $X_H^{\frac{2(z-1)}{z}}$ which shows that the above Carroll z-scale invariant action cannot be extended to a Carroll extremal surface $X_H = 0$.

5 Conclusion

In this thesis, we first introduced the general mathematical framework of Carrollian physics as well as the derivation of the "electric" and "magnetic" Carrollian scalar field theories. We also gave a motivation on why the study of Carroll symmetries might be fruitful as conformal Carroll field theories are expected to be dual to quantum gravity in asymptotically flat spacetimes. Furthermore, different examples of the Carroll groups/algebras were introduced.

In chapter 3, we recalled the construction and properties of Carroll-invariant scalar and vector field theories with fields propagating outside the Carroll lightcone, i.e. at a speed strictly greater than zero. The authors of [1] coined those "Carroll swiftons" to distinguish them from the standard Lorentzian "tachyons" since they also propagate outside the lightcone but do not possess the usual pathologies that come from the standard Lorentz tachyonic instabilities (like the unboundedness from below of the energy). Specifically, we recalled that the bi-scalar, multi-scalar and electromagnetic model had indeed a lower bound for the energy density and the model for Carroll swiftons coupled with $2d$ dilaton gravity.

The next step and main part of this thesis was then to check for which scaling exponent z and which dimension d these Carroll-invariant scalar and vector field theories are Carroll z-scale invariant. Here, scaling exponent z represents an anisotropy parameter of Lifschitz type where $z = 1$ is the isotropy (or "standard") scale. We have seen that, analogously to classical quantum field theories, only a certain exponent for a potential term is allowed for a certain dimension d and scale z. However, we have also shown that, for any of the constructed "Carroll swiftons" in [1], the exponent of the additional potential term can not be positive opposed to the "electric" model.

From this point on the Carroll z-scale invariance of the remaining Carroll-invariant field theories was studied and it turned out that the bi-scalar, multi-scalar and electromagnetic model have the same restriction on the scale z, namely

$$z = d + 2. \tag{5.1}$$

Therefore, it is possible to combine interaction terms from the bi-scalar and multi-scalar models not limited to the ones introduced [1]. Furthermore, it is possible to combine the spin 0 and spin 1 Carroll-invariant scalar/vector field theories and their interaction terms.

Lastly, we investigated the Carroll z-scale invariance of the Carroll dilaton gravity in $2d$ introduced in [20], [21] were we used the equivalent transformations of the Cartan variables e, τ, ω instead of the coordinates t, x to determine for which scale z and dimension d the action is Carroll z-scale invariant. We found the scaling exponent of the dilaton X, the Lagrange multiplier X_H for the torsion constraint, the Lagrange multiplier X_P for the intrinsic torsion constraint and the potential $\mathcal{V}(X, X_H)$. Consequently, we then found the transformations for the coupling functions F, g and h of the $2d$ Carroll swifton scalar field action and showed that, in contrast to the the Lorentzian case, the coupling functions all transform differently and that this model cannot be extended to the Carroll extremal surface $X_H = 0$.

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