Derivation of field equations and investigation of gravitational waves in the gravitational field of a condensing cosmogonical body based on the statistical theory

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Abstract. This paper considers the statistical theory of cosmogonical body formation in order to derive field equations and investigate gravitational waves in the gravitational field of a condensing cosmogonical body. The models and evolution equations of the statistical mechanics have been proposed in this theory, while well-known problems of gravitational condensation of infinitely spread cosmic media have been solved on the basis of the proposed statistical model of a spheroidal body. Within the framework of the statistical theory, the presence of an ethereal medium (or a dark matter) around gravitationally interacting masses serves as the basis for their gravitational interactions and gravitational wave propagation. First, equation of motion of a "liquid" particle in the "quasi-solid" approximation inside a spheroidal body in gravitational and inertial fields is considered. Using the vector gravimagnetic potential introduced the equation of moving solid particle with an ethereal surrounding in gravitational and gravimagnetic fields is obtained. Using both Lagrange function of a moving "liquid" gaseous–ethereal particle (with a "solid" core) in the gravitational/gravimagnetic field and Lagrange function of the same field, the expression of a total action for the gravitational/gravimagnetic field and matter is obtained. Applying the principle of least action and varying only the gravitational potentials together with the anti-diffusion velocity (but not the coordinates) the system of equations of the gravitational/gravimagnetic field of a condensing spheroidal body is derived. The equations of plane gravitational waves are obtained. This paper shows that Newtonian approximation (following from Einstein's GR) does not fully describe the gravitational field of gravitating masses while in the framework of the proposed statistical theory the equation of particle motion in an arbitrary non-inertial frame of reference is derived.

1. A statistical approach to investigate the formation of cosmogonical body

Recently, in the problem of understanding the phenomenon of gravitation, the hypothesis of the existence of a *dark matter* has become essential. The history of the dark matter problem can be traced back to at least the 1930s, but it was not until the early 1970s that the issue of "missing matter" was widely recognized as problematic [1]. In 1933, astronomer F. Zwicky found in his famous paper [2] that «the velocity dispersion of members of the Coma galaxy cluster to be so high that, to keep the system stable, the average mass density in the Coma system would have to be much higher than that deduced from observed visible matter» [1].

There are a number of hypotheses for the nature of dark matter [3]. Another point of view identifies dark matter with *ether*. The influence of unknown matter (dark matter or ether) both on the dynamics of particle motion in a gravitational field and on the formation of the gravitational field itself has been considered in a number of papers. Indeed, E. Nelson assumed that «particles in empty space, or let us say the ether, are subject to Brownian motion» [4], i.e. any particle of mass m_0 (for example, an atom or electron) «constantly undergoes a universal Brownian motion with diffusion coefficient inversely proportional to m_0" [4], [5]. A. Bashkirov and A. Vityazev have found that "a dark matter allowance in the system» of homogeneous Maxwellian gas of gravitating particles "gives rise to a significant decrease in the oscillation period" of a renormalized gravitational potential, moreover, it has stated that the observable period of these oscillations enables us to estimate the Jeans wave number for the dark matter [6].

This paper develops the statistical theory [7] – [14] in order to derive equations for the gravitational field of a condensing cosmogonical body. First of all, the statistical aspect of the problem results from the fact that initially the cosmogonical body, consisting of the gaseous matter and therefore representing a *molecular cloud*, is a system containing a *large number* of particles (molecules or atoms) interacting among themselves by oscillations in a cosmic vacuum or "let us say the ether" [4]. In *microphysics*, the cosmic vacuum represents a ground energetic state of quantum fields, and its experimental manifestation is the Casimir effect [15], [16]. Therefore, in this way we can interpret their local *oscillatory interactions*. Such oscillations modifying forms of molecule/atom trajectories have been considered by E. Nelson [4], [5] and later on by L. Nottale [17], [18]. The important point in Nelson's works [4], [5] is that any particle in the empty space, or let us say the ether, under the influence of any interaction field, is also subject to a universal Brownian motion based on the hypothesis on the quantum nature of space-time or fluctuations on cosmic scale [19], [20]. In *macrophysics*, it is alleged that the cosmological constant describes the cosmic vacuum [21] – [23], for example, its experimental manifestation on cosmic scales can be the fluctuations stipulated by the radial and the axial orbital oscillations of Alfvén–Arrhenius [24], [25].

As shown in Refs [7] – [13], as result of oscillatory interactions of particles, a forming cosmogonical body is considered by the model of a uniformly rotating spheroidal body whose the mass density function (in spherical frame of reference (r,θ,ε)) has the form:

$$\rho^{(m)}(r,\theta)=\rho_0^{(m)}(1-\varepsilon_0^2)e^{-\alpha r^2(1-\varepsilon_0^2\sin^2\theta)/2}, \tag{1.1}$$

where ε_0^2 is a squared eccentricity, $\rho_0^{(m)}=M\left(\alpha/2\pi\right)^{3/2}$ is a mass density in the center of a spheroidal body, and M is a mass of a spheroidal body. In the particular case of $\varepsilon_0^2\to 0$ the mass density function (1.1) for a slowly rotating spheroidal body (or sphere-like cosmogonical body) becomes:

$$\rho^{(m)}(r)=\rho_0^{(m)}\cdot e^{-\alpha r^2/2}, \tag{1.2}$$

where α is a parameter of gravitational condensation, r is a spatial (radial) variable.

The main task in understanding the statistical model is what mechanism of particle interactions leads to the slow-flowing process of an initial *gravitational condensation* of a spheroidal body when parameter $\alpha=\alpha(t)$ is *a positively defined monotonically increasing in time function* [8]. According to the statistical theory of cosmogonical body formation [7] – [10], let us consider the mass density $\rho^{(m)}$ as a function of two variables r and t. Since the dependence on t is expressed by a composite function, we find partial derivative $\rho^{(m)}$ with respect to t as follows:

$$\frac{\partial\rho^{(m)}}{\partial t}=\frac{\partial\rho^{(m)}}{\partial\alpha}\cdot\frac{d\alpha}{dt}. \tag{1.3}$$

Taking into account Eqs. (1.2) and (1.3) we can obtain the following equation [7], [8], [10]:

$$\frac{\partial\rho^{(m)}}{\partial\alpha}=-\frac{1}{2\alpha^2}\nabla^2\rho^{(m)}. \tag{1.4}$$

Substituting Eq. (1.4) into Eq. (1.3), we can write the basic equation of the *initial gravitational condensation* of a slowly rotating (at $\varepsilon_0 \to 0$) spheroidal body in the form [7] – [10]:

$$\frac{\partial \rho^{(m)}}{\partial t} = -\mathrm{G}(t)\nabla^2 \rho^{(m)}, \tag{1.5}$$

where $\mathrm{G}(t)$ is a *gravitational compression function* (GCF):

$$\mathrm{G}(t) = \frac{1}{2\alpha^2}\cdot\frac{d\alpha}{dt}. \tag{1.6}$$

Since $\alpha(t)$ is a monotonically increasing function of time, then $\mathrm{G}(t) > 0$. The relation obtained (1.5) in its form is an *anti-diffusion equation*: the sign $-$ in the right-hand part testifies that the initial density disturbance will not decrease (in contrast to the diffusion equation [26]) but rather increase [7] – [10]. So, according to Eq. (1.5) interactions of *oscillating* particles inside a spheroidal body lead to a *gravitational condensation* increasing with time. We note that under the condition of critical value the parameter of gravitational condensation α_c, the centrally symmetric *gravitational field arises* in such a spheroidal body [7] – [10].

Bearing in mind that GCF (1.6) does not depend on the spatial variable *r*, we can rewrite Eq. (1.5) in the form of equation of continuity:

$$\frac{\partial \rho^{(m)}}{\partial t} + \mathrm{div}(\rho^{(m)}\,\vec{\mathrm{u}}) = 0, \tag{1.7}$$

where $\vec{j}^{(m)}_{\vec{\mathrm{u}}} = \rho^{(m)}\,\vec{\mathrm{u}} = \mathrm{G}(t)\,\mathrm{grad}\,\rho^{(m)}$ is a *mass flow density* of a continuous medium arising at the gravitational condensation of a centrally symmetric spheroidal body [7] – [10] (like a *conductive flow* [26] – [28]). According to Eq. (1.7) there exist an *anti-diffusion mass flow density* as well as a conductive velocity for the anti-diffusion mass flow density or, more simply, the *anti-diffusion velocity* (as opposed to the usual hydrodynamic velocity $\vec{\mathrm{v}}$) for a slowly rotating spheroidal body [7]–[12]:

$$\vec{\mathrm{u}} = \mathrm{G}(t)\frac{\nabla\rho^{(m)}}{\rho^{(m)}} = \mathrm{G}(t)\,\mathrm{grad}\,\ln(\rho^{(m)}/\rho_0^{(m)}). \tag{1.8}$$

Taking into account the mass density function (1.2) we can estimate the anti-diffusion velocity (1.8) of particles inside a slowly rotating spheroidal body:

$$\vec{\mathrm{u}}(\vec{r},t) = \mathrm{G}(t)\nabla\left(-\alpha(t)\vec{r}^{\,2}/2\right) = -\mathrm{G}(t)\alpha(t)\vec{r}\,. \tag{1.9}$$

Using formula (1.9) and representing the substantial derivative $d\,\vec{\mathrm{u}}/dt$ as the sum of the local (quotient) $\partial\,\vec{\mathrm{u}}/\partial t$ and convective $\left(\vec{\mathrm{u}}\cdot\nabla\right)\vec{\mathrm{u}}$ [12] we find that

$$\frac{d\,\vec{\mathrm{u}}}{dt} = \frac{\partial\vec{\mathrm{u}}}{\partial t} + \left(\vec{\mathrm{u}}\cdot\nabla\right)\vec{\mathrm{u}} = \frac{\partial\vec{\mathrm{u}}}{\partial t} + \left(\vec{\mathrm{u}}\cdot\frac{\partial}{\partial\vec{r}}\right)\vec{\mathrm{u}} = \frac{d(-\mathrm{G}(t)\alpha(t))}{dt}\vec{r} + \mathrm{u}\frac{\partial}{\partial r}\vec{\mathrm{u}} = -(\dot{\mathrm{G}}(t)\alpha(t) + \mathrm{G}(t)\dot{\alpha}(t))\vec{r} - \mathrm{G}(t)\alpha(t)\,\vec{\mathrm{u}} =$$
$$= -(\dot{\mathrm{G}}(t)\alpha(t) + \mathrm{G}(t)\dot{\alpha}(t))\vec{r} - \mathrm{G}(t)\alpha(t)\,\vec{\mathrm{u}} = -\dot{\mathrm{G}}(t)\alpha(t)\vec{r} - \mathrm{G}(t)\dot{\alpha}(t))\vec{r} + \mathrm{G}^2(t)\alpha^2(t)\vec{r} =$$
$$= -\dot{\mathrm{G}}(t)\alpha(t)\vec{r} - 2\,\mathrm{G}^2(t)\alpha^2(t)\vec{r} + \mathrm{G}^2(t)\alpha^2(t)\vec{r} = -\mathrm{G}^2(t)\alpha^2(t)\vec{r} - \dot{\mathrm{G}}(t)\alpha(t)\vec{r}, \tag{1.10}$$

because in accordance with definition (1.6) we take into account that

$$\mathrm{G}^2(t)\alpha^2(t) = \frac{1}{4\alpha^2(t)}\cdot\left(\frac{d\alpha}{dt}\right)^2 = \frac{1}{2}\mathrm{G}(t)\dot{\alpha}(t). \tag{1.11}$$

This paper investigates equation of motion of a "liquid" particle in the "quasi-solid" approximation (as moving solid particle with an *ethereal* surrounding) inside a spheroidal body in gravitational and gravimagnetic fields, introducing the vector *gravimagnetic* potential. As a result, the system of equations of the gravitational/gravimagnetic field of a condensing spheroidal body is derived.

2. Motion of a particle inside a spheroidal body in gravitational and inertial fields

As shown in [10], [14], the well-known in mechanics equation of motion of a particle with a mass m_0 in a *non-inertial* frame of reference which moves relative to the inertial one translationally with a variable velocity $\vec{V}(t)$ and rotationally with a variable angular velocity $\vec{\Omega}(t)$ [29]:

$$m_0\frac{d\vec{\mathrm{v}}}{dt} = -m_0\,\mathrm{grad}\,\varphi_{\mathrm{g}} - m_0\frac{d\vec{V}}{dt} + m_0\left[\vec{r}\times\dot{\vec{\Omega}}\right] + 2m_0\left[\vec{\mathrm{v}}\times\vec{\Omega}\right] + m_0\left[\vec{\Omega}\times\left[\vec{r}\times\vec{\Omega}\right]\right], \tag{2.1}$$

completely coincides with equation of motion of a particle with a mass m_0 inside the peripheral region of a gravitating spheroidal body, where φ_{g} is a potential of the gravitational field of a spheroidal body with mass M, $\vec{V}(t)=\vec{\upsilon}_r$ and $\vec{\upsilon}_r$ is the radial velocity inside the spheroidal body, that is, $\vec{W} = d\vec{V}/dt$ is an acceleration of translational motion.

Indeed, following D. Greenberger, in accordance with the principle of *strong equivalence*, the motion of a particle in a gravitational field is equivalent to a particle at rest in an *accelerating* (non-inertial) coordinate system [30]: "The strong equivalence principle states that the motion of a particle in a gravitational field is equivalent to that of a particle at rest in an accelerating coordinate system. So the motion in an accelerated system is equivalent to being in free fall in the gravitational field, and again, the mass will drop out. So if you can solve the much simpler motion of a particle in an accelerated frame, you should know how the particle behaves in a gravitational field".

On the other hand, in the case of a spheroidal body [7], [10], the translational velocity $\vec{V}(t)$ in Eq. (2.1) can be considered as a radial velocity $\vec{\upsilon}_r$ in the form of *anti-diffusion* velocity $\vec{\mathrm{u}}$ in accordance with Eq. (1.9), that is,

$$\vec{V} = \vec{\mathrm{u}} = -\mathrm{G}(t)\alpha(t)\vec{r}. \tag{2.2}$$

Taking into account Eq. (2.2) equation (2.1) takes the form:

$$m_0\frac{d\vec{\mathrm{v}}}{dt} = -m_0\,\mathrm{grad}\,\varphi_{\mathrm{g}} - m_0\frac{d\vec{\mathrm{u}}}{dt} + m_0\left[\vec{r}\times\dot{\vec{\Omega}}\right] + 2m_0\left[\vec{\mathrm{v}}\times\vec{\Omega}\right] + m_0\left[\vec{\Omega}\times\left[\vec{r}\times\vec{\Omega}\right]\right], \tag{2.3}$$

where $\vec{\mathrm{v}} = \vec{\mathrm{v}}_0 - \vec{V}(t) - \left[\vec{\Omega}\times\vec{r}\right]$, $\vec{\mathrm{v}}_0$ is a velocity relative to the *inertial* frame of reference [29].

Introducing the vector potential of a *homogeneous* gravimagnetic field of the form [10], [14]:

$$\vec{A}_{\mathrm{g}}\left(\vec{r},t\right) = \left[\vec{\Omega}\times\vec{r}\right], \quad \vec{\Omega} = \vec{\Omega}(t), \tag{2.4a}$$

and also bearing in mind that

$$\mathrm{rot}\,\vec{A}_{\mathrm{g}} = \mathrm{rot}\left[\vec{\Omega}\times\vec{r}\right] = 2\vec{\Omega}, \tag{2.4b}$$

$$\frac{1}{2}\mathrm{grad}\,\vec{A}_{\mathrm{g}}^2 = \frac{1}{2}\mathrm{grad}\left[\vec{\Omega}\times\vec{r}\right]^2 = \frac{1}{2}\mathrm{grad}\left[\vec{r}\times\vec{\Omega}\right]^2 = \left[\vec{\Omega}\times\left[\vec{r}\times\vec{\Omega}\right]\right], \tag{2.4c}$$

let us rewrite Eq. (2.3) as follows:

$$m_0\frac{d\vec{\mathrm{v}}}{dt} = -m_0\,\mathrm{grad}\,\varphi_{\mathrm{g}} - m_0\frac{d\vec{\mathrm{u}}}{dt} - m_0\frac{\partial\vec{A}_{\mathrm{g}}}{\partial t} + m_0\left[\vec{\mathrm{v}}\times\mathrm{rot}\,\vec{A}_{\mathrm{g}}\right] + \frac{m_0}{2}\mathrm{grad}\,\vec{A}_{\mathrm{g}}^2. \tag{2.5}$$

By analogy with equation of motion of a particle in the electromagnetic field [31], we introduce the *strength of the gravitational field* as a specific force acting on a unit mass in the gravitational field of a spheroidal body [10]:

$$\vec{g} = -\frac{\partial \vec{A}_{\text{g}}}{\partial t} - \operatorname{grad}\varphi_{\text{g}} + \frac{1}{2}\operatorname{grad}\vec{A}_{\text{g}}^{2}, \tag{2.6}$$

as well as the *strength of a homogeneous gravimagnetic field* [10]:

$$\vec{h}_{\text{g}} = \operatorname{rot}\vec{A}_{\text{g}} = 2\vec{\Omega}. \tag{2.7}$$

Taking into account notations (2.6), (2.7) Eq. (2.3) takes the form:

$$m_0 \frac{d(\vec{\text{v}} + \vec{\text{u}})}{dt} = m_0\vec{g} + m_0\left[\vec{\text{v}} \times \vec{h}_{\text{g}}\right] = m_0\vec{g} + 2m_0\left[\vec{\text{v}} \times \vec{\Omega}\right]. \tag{2.8}$$

From expressions (2.6), (2.7) let us try to obtain equations containing only $\vec{g}$ and $\vec{h}_{\text{g}}$ [10]. To this end, we define $\operatorname{rot}\vec{g}$ from both parts of Eq. (2.6):

$$\operatorname{rot}\vec{g} = -\frac{\partial(\operatorname{rot}\vec{A}_{\text{g}})}{\partial t} - \operatorname{rot}\operatorname{grad}\varphi_{\text{g}} + \frac{1}{2}\operatorname{rot}\operatorname{grad}\vec{A}_{\text{g}}^{2} = -\frac{\partial \vec{h}_{\text{g}}}{\partial t}. \tag{2.9}$$

Then taking the divergence from both sides of Eq. (2.7), we get:

$$\operatorname{div}\vec{h}_{\text{g}} = \operatorname{div}\operatorname{rot}\vec{A}_{\text{g}} = 0. \tag{2.10}$$

Equations (2.9) and (2.10) constitute *the first pair* of equations of the gravitational field of a spheroidal body in a state of relative mechanical equilibrium [10], [14]. They were first obtained in the paper [14] for the particular case of a vector potential $\vec{A}_{\text{g}}$ of the simplest form (2.4a).

Let us note that equations of the first pair (2.9), (2.10) are obtained with a *degenerate* gravimagnetic field strength $\vec{h}_{\text{g}} = 2\vec{\Omega}(t)$ when $\vec{h}_{\text{g}} \neq \vec{h}_{\text{g}}(\vec{r})$ leading to the condition:

$$\operatorname{rot}\vec{h}_{\text{g}} \equiv 0, \tag{2.11}$$

which does not allow rigorously deriving the fourth equation of the second pair of the gravitational field of a spheroidal body. Consequently, the choice of the vector potential of a gravimagnetic field in the form (2.4a) is not fully satisfactory.

3. Motion of a "liquid" particle in the "quasi-solid" approximation inside a spheroidal body in gravitational and inertial fields

To consider a more general non-degenerate case when $\operatorname{rot}\vec{h}_{\text{g}} \neq 0$, we turn to the *hydrodynamic* description of the motion of a particle inside a gravitating spheroidal body. We suppose that within the framework of the hydrodynamic description, it is advisable to find equation of motion of a "liquid" particle in two stages (similar to the derivation of equation (2.1) of motion of a particle in a non-inertial frame of reference [29]). At the beginning, at the *first stage* we consider equation of motion of a "liquid" particle under the action of only the pressure p of an ethereal medium. Then the movement of gaseous (gas-dust) matter together with the ether obeys the Euler hydrodynamic equation [12]:

$$\frac{d\vec{\text{v}}}{dt} = \frac{\partial\vec{\text{v}}}{\partial t} + \left(\vec{\text{v}}\nabla\right)\vec{\text{v}} = -\frac{1}{\rho}\operatorname{grad}p, \tag{3.1}$$

which, taking into account the well-known vector analysis formula, can be rewritten in the Gromeka–Lamb form [32]:

$$\rho\frac{\partial\vec{\text{v}}}{\partial t} + \frac{\rho}{2}\operatorname{grad}\vec{\text{v}}^{2} - \rho\left[\vec{\text{v}} \times \operatorname{rot}\vec{\text{v}}\right] = -\operatorname{grad}p, \tag{3.2}$$

where $\rho = \rho^{(e)} + \sum_i m_{0i}\delta(\vec{r} - \vec{r}_{0i})$ is a density of a continuous medium (consisting of "liquid" ethereal particles with "solid" nuclei in the form of substance molecules m_{0i}), $\rho^{(e)}$ is a density of the ether, m_{0i} is a mass of i-th particle (gas molecule), $\delta(\vec{r})$ is the Dirac delta function, $\vec{\mathrm{v}}$ is a hydrodynamic velocity of the "liquid" ethereal particle, p is a pressure of the ethereal medium.

At the *second stage* we consider the accelerated motion of a continuous ethereal and gaseous medium, so that according to Newton 2nd law the motion of a "liquid" particle obeys the equation:

$$\rho\vec{a} = -\operatorname{grad} p - \rho \operatorname{grad} \varphi_{\mathrm{g}}, \tag{3.3}$$

where $\vec{a}$ is an acceleration of particle. Expressing the pressure gradient $-\operatorname{grad} p$ in terms of $\rho\vec{a}$ and $\rho \operatorname{grad} \varphi_{\mathrm{g}}$ in accordance with Eq. (3.3), we can rewrite Eq. (3.2) as follows [10]:

$$\rho\vec{a} = -\rho \operatorname{grad} \varphi_{\mathrm{g}} + \rho \frac{\partial \vec{\mathrm{v}}}{\partial t} + \frac{\rho}{2} \operatorname{grad} \vec{\mathrm{v}}^2 - \rho\left[\vec{\mathrm{v}} \times \vec{\omega}\right], \tag{3.4}$$

where $\vec{\omega} = \operatorname{rot} \vec{\mathrm{v}}$ is a vorticity of a continuous medium [32]. According to (3.4) equation of motion of a "liquid" particle in a small volume element (Fig. 1) of a moving ideal continuous gaseous and ethereal medium with a mass $d\mu = \rho dV$ has the form [10]:

$$d\mu\vec{a} = -d\mu \operatorname{grad} \varphi_{\mathrm{g}} + d\mu \frac{\partial \vec{\mathrm{v}}}{\partial t} + \frac{d\mu}{2} \operatorname{grad} \vec{\mathrm{v}}^2 - d\mu\left[\vec{\mathrm{v}} \times \operatorname{rot} \vec{\mathrm{v}}\right]. \tag{3.5}$$

So, a solid particle with a mass m_0 inside a "liquid" particle with a mass $d\mu$ (see Fig. 1) undergoes translational and rotational motion (excluding the "deformation" motion) in accordance with the model of "quasi-solid" motion (according to the 1st theorem of Helmholtz [32]) with angular velocity $\vec{\Omega}$ depending both on t and on $\vec{r}$:

$$\vec{\Omega} = \frac{1}{2} \operatorname{rot} \vec{\mathrm{v}} = \frac{1}{2} \vec{\omega}, \tag{3.6}$$

where $\vec{\omega} = \operatorname{rot} \vec{\mathrm{v}}$ is a vorticity vector of ethereal continuous medium that makes up a peripheral region of the "liquid" particle, i.e. $\vec{\omega} = \vec{\omega}(\vec{r}, t)$.

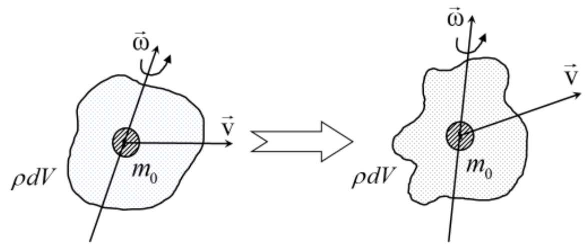

Figure 1. Motion of a "liquid" particle with a "solid" core being a molecule of matter

Let us note once again that despite the fact that a particle with a mass m_0 located inside an elementary volume dV of a continuous ethereal medium which in turn can experience a deformation motion [32] (see Fig. 1), we consider that an equation, which describes the motion "solid" particle of a gas molecule m_0, coincides *on average* with equation (3.5) of the motion of a "liquid" particle itself. If a motionless particle m_0 is singled out in a moving continuous ethereal and gaseous medium, and the moving continuous medium itself is associated with a moving coordinate system in which this continuous medium is at rest, then the particle m_0 will move relative to it with a velocity $-\vec{\mathrm{v}}$, so that equation of motion (3.12) for this particle in "quasi-solid" approximation (according to the Helmholtz 1st theorem [32]) will take the form:

$$m_0\vec{a} = -m_0\,\mathrm{grad}\,\varphi_{\mathrm{g}} - m_0\frac{\partial\vec{\mathrm{v}}}{\partial t} + \frac{m_0}{2}\mathrm{grad}\,\vec{\mathrm{v}}^2 + m_0\left[\vec{\mathrm{v}}\times\vec{\omega}\right]. \tag{3.7}$$

If now we take into account that the *continuous medium of a spheroidal body* is considered as a continuous gas–ethereal medium, then, as is known [7], [12], both the substantial (total) derivative (3.1) of the hydrodynamic velocity $d\,\vec{\mathrm{v}}/dt$ and the substantial (total) derivative (1.10) of the anti-diffusion velocity $d\,\vec{\mathrm{u}}/dt$ contribute themselves, i.e.

$$\vec{a} = \frac{d\,\vec{\mathrm{v}}}{dt} + \frac{d\,\vec{\mathrm{u}}}{dt} = \frac{\partial\vec{\mathrm{v}}}{\partial t} + \left(\vec{\mathrm{v}}\cdot\nabla\right)\vec{\mathrm{v}} + \frac{\partial\vec{\mathrm{u}}}{\partial t} + \left(\vec{\mathrm{u}}\cdot\nabla\right)\vec{\mathrm{u}}. \tag{3.8}$$

Taking into account definition (3.8), Eq. (3.7) becomes the equation describing motion of a particle *inside a gravitating spheroidal body* in a "quasi-solid" approximation (according to the Helmholtz 1st theorem [32]) from the point of view of the strong equivalence principle of Greenberger [30]:

$$m_0\frac{d\,\vec{\mathrm{v}}}{dt} = -m_0\frac{d\,\vec{\mathrm{u}}}{dt} - m_0\,\mathrm{grad}\,\varphi_{\mathrm{g}} - m_0\frac{\partial\vec{\mathrm{v}}}{\partial t} + \frac{m_0}{2}\mathrm{grad}\,\vec{\mathrm{v}}^2 + m_0\left[\vec{\mathrm{v}}\times\vec{\omega}\right]. \tag{3.9}$$

In the *particular case* of rotational motion (3.6) with a time-varying angular velocity $\vec{\Omega}(t) = \frac{1}{2}\vec{\omega}(t)$ (see Section 2), we have (in accordance with Eqs (2.1) – (2.4a)) that

$$\vec{\mathrm{v}} = \left[\vec{\Omega}\times\vec{r}\right] \tag{3.10}$$

and, as consequence, Eq. (3.9) degenerates into Eq. (2.1) because

$$\vec{W} = \frac{d\vec{V}}{dt} = \frac{d\,\vec{\mathrm{u}}}{dt}; \tag{3.11a}$$

$$-\frac{\partial\vec{\mathrm{v}}}{\partial t} = -\left[\dot{\vec{\Omega}}\times\vec{r}\right] = \left[\vec{r}\times\dot{\vec{\Omega}}\right]; \tag{3.11b}$$

$$\vec{\omega} = 2\vec{\Omega}; \tag{3.11c}$$

$$\frac{1}{2}\mathrm{grad}\,\vec{\mathrm{v}}^2 = \frac{1}{2}\mathrm{grad}\left[\vec{\Omega}\times\vec{r}\right]^2 = \left[\vec{\Omega}\times\left[\vec{r}\times\vec{\Omega}\right]\right]; \tag{3.11d}$$

$$\left[\vec{\mathrm{v}}\times\vec{\omega}\right] = 2\left[\vec{\mathrm{v}}\times\vec{\Omega}\right]. \tag{3.11e}$$

The use of hydrodynamic velocity in Eqs (3.7) – (3.9) allows us to introduce a *vector hydrodynamic potential* $\vec{A}$ satisfying the following conditions:

$$\vec{\tilde{\mathrm{v}}} = \mathrm{rot}\,\vec{A}, \quad \mathrm{div}\,\vec{A} = 0, \tag{3.12}$$

where $\vec{\tilde{\mathrm{v}}}$ is, generally speaking, a *vortex velocity* [32]. According to Eq. (3.12) the velocity field implies the existence of a hydrodynamic singularity in the form of a *vortex tube* (or vortex line), i.e. we consider the velocity field around a given system of vortices $\vec{\omega}$ in a boundless continuous medium of a spheroidal body. In this case, it is necessary to solve the equation for $\vec{\mathrm{v}}$:

$$\mathrm{rot}\,\vec{\mathrm{v}} = \mathrm{rot}\,\vec{\tilde{\mathrm{v}}} = \vec{\omega}, \tag{3.13}$$

moreover, this solution should tend to zero as we move away from a region (occupied by vortices) to infinity [32].

Let us note that equation (3.9) already consider the motion of a particle in a "quasi-solid" representation [32], although the usage of vorticity $\vec{\omega}$ instead of the angular velocity $\vec{\Omega}$ of rotation (as a whole) of such particle indicates an accounting residual properties of the motion of a "liquid" particle (see the 1st and 2nd theorems of Helmholtz [32]). We emphasize that in a rigorous "quasi-solid" approximation we should put $\vec{\omega} = 2\vec{\Omega}(t)$ and $\vec{\mathrm{v}} = \left[\vec{\Omega}\times\vec{r}\right]$ in Eq. (3.9) (in accordance with formulas (3.6) and (3.10)) which in turn would allow us to come to the previously considered equation (2.1) in the framework of the classical mechanics of mass points moving in gravitational and inertial fields [29].

To simplify the calculations let us introduce a new *vector potential of the vortex velocity* in accordance with conditions (3.12):

$$\vec{B} = \vec{\tilde{\mathrm{v}}} = \operatorname{rot}\vec{A}. \tag{3.14}$$

Bearing in mind this notation (3.14) Eq. (3.9)) takes the form:

$$m_0 \frac{d\,\vec{\mathrm{v}}}{dt} = -m_0 \operatorname{grad}\varphi_{\mathrm{g}} - m_0 \frac{d\vec{\mathrm{u}}}{dt} - m_0 \frac{\partial \vec{B}}{\partial t} + \frac{m_0}{2}\operatorname{grad}\vec{B}^2 + m_0\left[\vec{\mathrm{v}}\times \operatorname{rot}\vec{B}\right]. \tag{3.15}$$

Let us note that equation obtained (3.15) can be directly derived on the basis of the Lagrange–Euler variational equation, starting from the Lagrange function of a moving particle in the gravitational and inertial fields of a spheroidal body:

$$L_{\mathrm{g}} = \frac{m_0\,\vec{\mathrm{v}}^2}{2} + m_0\vec{B}\cdot\vec{\mathrm{v}} + \frac{m_0}{2}\vec{B}^2 - \frac{m_0}{2}\vec{W}\cdot\vec{r} - m_0\,\varphi_{\mathrm{g}}, \tag{3.16}$$

where $\vec{W} = d\,\vec{\mathrm{u}}/dt = -\mathrm{G}^2(t)\alpha^2(t)\vec{r} - \dot{\mathrm{G}}(t)\alpha(t)\vec{r}$ in accordance with Eq. (1.10) (see also [7]).

4. Vector potentials of the hydrodynamic field and Biot–Savart–Laplace formula

Now let us try to find the form of the potential $\vec{B}$ in the general case of an *inhomogeneous* gravimagnetic field. Indeed, substituting (3.12) into (3.13) and taking into account (3.14) we obtain equation for the vector hydrodynamic potential $\vec{A}$ called also the *vector potential of ethereal vorticity field*:

$$\operatorname{rot}\operatorname{rot}\vec{A} = \vec{\omega}, \tag{4.1}$$

where $\vec{\omega}$ denotes a three-dimensional vorticity field in ethereal medium. Taking into account the well-known vector analysis formula [32]:

$$\operatorname{rot}\operatorname{rot}\vec{A} = \operatorname{grad}\left(\operatorname{div}\vec{A}\right) - \nabla^2\vec{A}$$

and bearing in mind the gauge condition of the vector hydrodynamic potential $\operatorname{div}\vec{A} = 0$, Eq. (4.1) becomes:

$$\nabla^2\vec{A} = -\vec{\omega}. \tag{4.2}$$

In Cartesian coordinates, vector equation (4.2) corresponds to 3 scalar equations of Poisson:

$$\nabla^2 A_x = -\omega_x; \tag{4.3a}$$

$$\nabla^2 A_y = -\omega_y; \tag{4.3b}$$

$$\nabla^2 A_z = -\omega_z. \tag{4.3c}$$

As known, the solution of Poisson equations (4.3a) – (4.3c) of the kind:

$$\nabla^2 A_{x_i} = -\omega_{x_i},\ \ i = 1,2,3 \quad (x_1 = x,\ x_2 = y,\ x_3 = z) \tag{4.4}$$

has the form [32]:

$$A_{x_i}\left(\vec{r}\right) = \frac{1}{4\pi}\iiint\limits_V \frac{\omega_{x_i}\left(\vec{r}'\right)}{\left|\vec{r}-\vec{r}'\right|}dV',\ \ i = 1,2,3, \tag{4.5}$$

where dV' is a volume element containing a vortex tube (line) section and located at a distance $\vec{r}'$ from the origin of frame of reference (see Fig. 2), $\vec{R} = \vec{r} - \vec{r}'$ is a distance vector from the considered volume element to the point for observation of vector hydrodynamic potential, $\vec{r}$ is a distance from the coordinate system origin to the point for measurement of the vector hydrodynamic potential $\vec{A}\left(\vec{r}\right)$ of ethereal vorticity field.

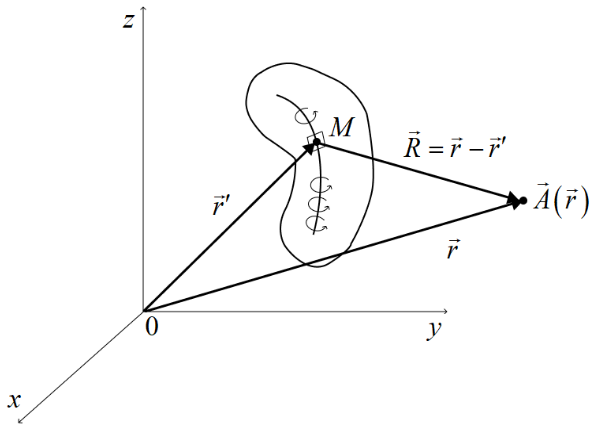

Figure 2. Scheme for calculating the vector hydrodynamic potential at a given point of measurement

Using the component solutions of Poisson equation (4.5) we can write the solution of Poisson vector equation (4.2):

$$\vec{A}(\vec{r}) = A_x\,\vec{\mathrm{i}} + A_y\,\vec{\mathrm{j}} + A_z\,\vec{\mathrm{k}} = \frac{1}{4\pi}\iiint_V \frac{\vec{\omega}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV', \tag{4.6}$$

where $\vec{\omega}(\vec{r}') = \omega_x(\vec{r}')\vec{\mathrm{i}} + \omega_y(\vec{r}')\vec{\mathrm{j}} + \omega_z(\vec{r}')\vec{\mathrm{k}}$. According to formulas (3.14) and (4.6) we now calculate the vector potential of the vortex velocities (it will be used later to construct the potential of inhomogeneous gravimagnetic field):

$$\vec{B}(\vec{r}) = \mathrm{rot}\,\vec{A}(\vec{r}) = \frac{1}{4\pi}\iiint_V \left[\nabla_{\vec{r}} \times \frac{\vec{\omega}(\vec{r}')}{R}\right]dV'. \tag{4.7}$$

Bearing in mind that ∇ is taken over $\vec{r}$ as well as the integrand $\vec{\omega}$ depends on $\vec{r}'$ we can see that differentiation of the integrand $\vec{\omega}(\vec{r}')/R(\vec{r})$ is carried out at a constant (fixed) $\vec{r}'$. In this connection, using the well-known vector analysis formula [32] we obtain that

$$\mathrm{rot}\left(\lambda\vec{\omega}(\vec{r}')\right) = \lambda\left[\nabla_{\vec{r}}\times\vec{\omega}(\vec{r}')\right] - \left[\vec{\omega}\times\mathrm{grad}\,\lambda(\vec{r})\right] = -\left[\vec{\omega}\times\mathrm{grad}\,\lambda\right], \tag{4.8}$$

where $\lambda = \dfrac{1}{R} = \dfrac{1}{|\vec{r}-\vec{r}'|}$ is a scalar function. According to Eq. (4.8) we need to find the gradient of this scalar function in spherical coordinates:

$$\mathrm{grad}\frac{1}{R} = \nabla_{\vec{r}}\frac{1}{R} = \frac{d}{dR}\left(\frac{1}{R}\right)\vec{e}_R = -\frac{\vec{e}_R}{R^2} = -\frac{\vec{R}}{R^3}, \tag{4.9}$$

since $\vec{\mathrm{e}}_R = \vec{R}/R$ is a radial unit vector in spherical frame of reference. Substituting Eqs (4.8), (4.9) into relation (4.7) we obtain:

$$\vec{B}(\vec{r}) = \frac{1}{4\pi}\iiint_V\left(-\left[\vec{\omega}\times\left(-\frac{\vec{R}}{R^3}\right)\right]dV'\right) = \frac{1}{4\pi}\iiint_V \frac{\left[\vec{\omega}\times\vec{R}\right]}{R^3}dV'. \tag{4.10}$$

Formula (4.10) describes the hydrodynamic analogue of the generalized *Biot–Savart–Laplace law* [32] since the Biot–Savart–Laplace law was originally formulated for linear conductors of electric current in electrodynamics [33]. Indeed, let us dwell in more detail on the case of an elementary vortex tube with a finite circulation Γ surrounding the vortex filament L (Fig. 3).

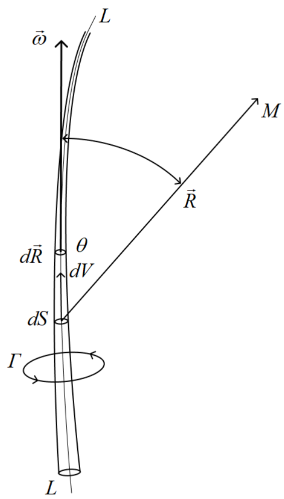

Figure 3. Scheme for calculating the vector potential of vortex velocities around a vortex filament *L* with circulation Γ

Let us denote by $d\vec{R}$ an element of filament oriented in the same direction as $\vec{\omega}$. Using the Helmholtz 2nd theorem as well as Stokes theorem (on the relationship between the intensity I of the vortex tube and the velocity circulation Γ along the enclosing contour [32]):

$$I = \iint_S \vec{\omega} d\vec{S} = \iint_S \operatorname{rot}\vec{v}\, d\vec{S} = \oint_C \vec{v} \cdot d\vec{l} = \Gamma(\vec{v}) \tag{4.11a}$$

and then carrying out changes of variables under integral (4.10) in accordance with the scheme:

$$\vec{\omega} dV = \vec{\omega} dS \cdot dR = \omega \vec{e}_\omega\, dSdR = \omega dS\, \vec{e}_\omega\, dR = \omega dS \cdot d\vec{R} = \Gamma\, d\vec{R}, \tag{4.11b}$$

we obtain, instead of integral (4.10), the following [32]:

$$\vec{B}(\vec{r}) = \frac{1}{4\pi} \int_L \frac{\left[\Gamma\, d\vec{R} \times \vec{R}\right]}{R^3} = \frac{\Gamma}{4\pi} \int_L \frac{\left[d\vec{R} \times \vec{R}\right]}{R^3}, \tag{4.12}$$

which fully corresponds to the *hydrodynamic analogue* of Biot–Savart–Laplace formula known in electromagnetism theory [33]. It describes the obtained solution of the problem relative to the field of vortex velocities $\tilde{\vec{v}} = \vec{B}$ around a given vortex filament L with a circulation Γ. The vortex filament can be considered as an axis of rotation in a rotating spheroidal body [7].

Let us show that the derived vector potential formula (4.10) is more general than the vector potential formula (2.4a) in the case of a homogeneous gravimagnetic field. Indeed, the volume integral on the right-hand side of Eq. (4.10) can be calculated on the basis of the mean-value theorem [34]:

$$\vec{B}(\vec{r}) = \frac{1}{4\pi} \cdot \overline{\frac{\left[\vec{\omega} \times \vec{R}\right]}{R^3}} \cdot V, \tag{4.13}$$

where the overbar in formula (4.13) means the averaging of the integrand. We assume that the volume average of the vorticity function $\vec{\omega}$ gives a tripled angular velocity $3\vec{\Omega}$ of the “quasi-solid” rotation of liquid particles of a spheroidal body (as a whole):

$$\vec{\Omega} = \frac{1}{3} \bar{\vec{\omega}}, \tag{4.14}$$

and the volume itself V can be estimated as the volume of a sphere with radius R:

$$V = \frac{4\pi}{3} R^3. \tag{4.15}$$

Substituting (4.14), (4.15) into (4.13) it is easy to come to formula (2.4a) when $\vec{B}=\left[\vec{\Omega}\times\vec{r}\right]$. Thus, in accordance with (4.10), (4.13) – (4.15) it follows from the above that the formula

$$\vec{B}(\vec{r})=\frac{1}{4\pi}\iiint_V \frac{\left[\vec{\omega}\times(\vec{r}-\vec{r}')\right]}{\left|\vec{r}-\vec{r}'\right|^3}dV', \tag{4.16}$$

goes into formula (2.4a) by means of *volume averaging* of the vorticity field $\vec{\omega}$, resulting in a uniform vortex velocity field and, as a consequence, a homogeneous gravimagnetic field. In this regard, as already mentioned, equation (3.9) of the motion of a "liquid" particle (in the "quasi-solid" approximation) in the gravitational and homogeneous gravimagnetic fields completely coincides with Eq. (2.1) after the change $\vec{B}(\vec{r},t)=\left[\vec{\Omega}(t)\times\vec{r}\right]$ in accordance with (4.13) and (4.16). It also coincides with the equation (3.15) which can be derived on the basis of the Lagrange–Euler variational principle when choosing the Lagrange function of the form (3.16), i.e.

$$L_{\rm g}=\frac{m_0\,\vec{\rm v}^2}{2}+m_0\vec{B}\cdot\vec{\rm v}+\frac{m_0}{2}\vec{B}^2-\frac{m_0}{2}\frac{d\,\vec{\rm u}}{dt}\cdot\vec{r}-m_0\,\varphi_{\rm g}, \tag{4.17}$$

where $d\,\vec{\rm u}/dt=\vec{W}$ is the radial acceleration (in Eq. (2.3)) induced by the anti-diffusion velocity $\vec{\rm u}$ in accordance with Eq. (1.10) under condensation of a spheroidal body [7].

5. Equation of motion of a "liquid" particle in gravitational and gravimagnetic fields and the first system of equations of the gravitational/gravimagnetic field

When considering Eq. (4.1) to determine the vector hydrodynamic potential $\vec{A}$, one usually does not ask the question about the causes of the occurrence in a continuous medium of a hydrodynamic singularity in the form of a vortex tube or vortex line leading to the appearance of *vorticity* $\vec{\omega}(\vec{r},t)$. However, in gravidynamics this question cannot be ignored, just as in electrodynamics (when considering a similar equation (4.1)) the answer lies in the presence of vortex currents $\vec{j}$ due to the circular motion of electrons in conductors [33], [35]. In this regard, when considering the gravimagnetic field, we first of all set a goal equation similar to Eq. (4.1) to determine the vector *gravimagnetic* potential:

$$\mathrm{rot\,rot}\,\vec{A}_{\rm g}=\frac{4\pi\gamma}{c_{\rm g}}\vec{j}^{(e)}, \tag{5.1}$$

where $\vec{j}^{(e)}$ is a vector value describing the vortex motion of ether particles, i.e. so-called ether current $\vec{j}^{(e)}=\rho^{(e)}\,\vec{\rm v}^{(e)}$, $c_{\rm g}$ is a gravimagnetic constant (which will be determined below).

Guided by the hydrodynamic generalization (3.15) of equation (2.1) (of the motion of a particle in a non-inertial frame of reference) and corresponding function of Lagrange (4.17) (or (3.16)), it is advisable to choose the vector of *gravimagnetic potential* $\vec{A}_{\rm g}$ equal (up to the gravimagnetic constant) to the vector potential $\vec{B}$ of the *vortex velocity* field:

$$\vec{A}_{\rm g}=c_{\rm g}\vec{B}=c_{\rm g}\,\mathrm{rot}\,\vec{A}, \tag{5.2}$$

expressing it in accordance with the transformed formulas (4.10) and (4.16) as follows:

$$\vec{A}_{\rm g}=\frac{c_{\rm g}}{4\pi}\iiint_V\frac{\left[\vec{\omega}\times\vec{R}\right]}{R^3}dV'=\frac{c_{\rm g}}{4\pi}\iiint_V\frac{\left[\vec{\omega}(\vec{r}')\times(\vec{r}-\vec{r}')\right]}{\left|\vec{r}-\vec{r}'\right|^3}dV'. \tag{5.3}$$

As already noted (regarding formula (4.13)), the application of the mean value theorem to the calculation of volume integral (5.3) gives formula for the vector gravimagnetic potential

$\vec{A}_{\text{g}} = c_{\text{g}}\left[\vec{\Omega}\times\vec{r}\right]$ in the case of a *homogeneous* gravimagnetic field where $\vec{\Omega} = \frac{1}{3}\bar{\vec{\omega}}$ in accordance with Eqs (4.14), (5.2).

To obtain important relations with respect to $\vec{A}_{\text{g}}$, let us use similar relations (4.2) – (4.5) with relative to the vector hydrodynamic potential $\vec{A}$. First of all, we transform Eq. (4.2), acting on both sides of it by the operator rot:

$$\nabla^2\left(\operatorname{rot}\vec{A}\right) = -\operatorname{rot}\vec{\omega}. \tag{5.4}$$

Taking into account definition (5.2) Eq. (5.4) becomes the following:

$$\nabla^2\vec{A}_{\text{g}} = -c_{\text{g}}\operatorname{rot}\vec{\omega}. \tag{5.5}$$

On the other hand, since according to Eq. (5.2) $\operatorname{div}\vec{A}_{\text{g}} \equiv 0$, then $\operatorname{rot}\operatorname{rot}\vec{A}_{\text{g}} = \operatorname{grad}\left(\operatorname{div}\vec{A}_{\text{g}}\right) - \nabla^2\vec{A}_{\text{g}} = -\nabla^2\vec{A}_{\text{g}}$, and Eq. (5.1) takes the form:

$$\nabla^2\vec{A}_{\text{g}} = -\frac{4\pi\gamma}{c_{\text{g}}}\vec{j}^{(e)}. \tag{5.6}$$

By comparing (5.5) with (5.6) we find that

$$\vec{j}^{(e)} = \frac{c_{\text{g}}^2}{4\pi\gamma}\operatorname{rot}\vec{\omega}. \tag{5.7}$$

Further, acting as an operator $c_{\text{g}}\operatorname{rot}$ on both parts of Eq. (4.6) we again obtain relation (5.3) in accordance with Eqs (4.7) – (4.10). However, there is another way for determining the vector gravimagnetic potential which directly follows from the solution of Poisson vector equation (5.6) (or (5.5)) and formula (5.7):

$$\vec{A}_{\text{g}} = \frac{1}{4\pi}\frac{4\pi\gamma}{c_{\text{g}}}\iiint_V \frac{\vec{j}^{(e)}\left(\vec{r}'\right)}{R}dV' = \gamma\iiint_V \frac{1}{c_{\text{g}}}\frac{\vec{j}^{(e)}\left(\vec{r}'\right)}{\left|\vec{r}-\vec{r}'\right|}dV' =$$

$$= \gamma\iiint_V \frac{c_{\text{g}}}{4\pi\gamma R}\operatorname{rot}\vec{\omega}\,dV' = \frac{c_{\text{g}}}{4\pi}\iiint_V \frac{\operatorname{rot}\vec{\omega}}{R}dV' = \frac{c_{\text{g}}}{4\pi}\iiint_V \frac{\operatorname{rot}\vec{\omega}\left(\vec{r}'\right)}{\left|\vec{r}-\vec{r}'\right|}dV'. \tag{5.8}$$

It is not difficult to show that relation (5.8) for the vector gravimagnetic potential $\vec{A}_{\text{g}}$ can be transformed to the form (5.3).

Finally, by analogy with the homogeneous case (2.7) we find the strength $\vec{h}_{\text{g}}$ of gravimagnetic field:

$$\vec{h}_{\text{g}} = \operatorname{rot}\vec{A}_{\text{g}}, \tag{5.9}$$

using relations (5.2), (5.8) and calculation scheme (4.7) – (4.10):

$$\vec{h}_{\text{g}} = \frac{c_{\text{g}}}{4\pi}\operatorname{rot}\iiint_V \frac{\operatorname{rot}\vec{\omega}}{R}dV' = \frac{c_{\text{g}}}{4\pi}\iiint_V \left[\nabla_{\vec{r}}\times\frac{\operatorname{rot}\vec{\omega}\left(\vec{r}'\right)}{R}\right]dV' = \frac{c_{\text{g}}}{4\pi}\iiint_V \left[-\operatorname{rot}\vec{\omega}\left(\vec{r}'\right)\times\left(-\frac{\vec{R}}{R^3}\right)\right]dV' =$$

$$= \frac{c_{\text{g}}}{4\pi}\iiint_V \frac{\left[\operatorname{rot}\vec{\omega}\times\vec{R}\right]}{R^3}dV' = \frac{c_{\text{g}}}{4\pi}\iiint_V \frac{\left[\operatorname{rot}\vec{\omega}\times\left(\vec{r}-\vec{r}'\right)\right]}{\left|\vec{r}-\vec{r}'\right|^3}dV'. \tag{5.10}$$

On the other hand, using the initial definition (5.3) of the vector gravimagnetic potential $\vec{A}_{\text{g}}$ and taking into account formula (4.8) with $\lambda = 1/R^3$, we can again calculate the strength of the gravimagnetic field:

$$\vec{h}_{\mathrm{g}} = \frac{c_{\mathrm{g}}}{4\pi}\,\mathrm{rot}\iiint\limits_V \frac{\left[\vec{\omega}\left(\vec{r}'\right)\times\vec{R}\right]}{R^3}dV' = \frac{c_{\mathrm{g}}}{4\pi}\iiint\limits_V \left[\nabla_{\vec{r}}\times\frac{\left[\vec{\omega}\left(\vec{r}'\right)\times\vec{R}\right]}{R^3}\right] =$$

$$= \frac{c_{\mathrm{g}}}{4\pi}\iiint\limits_V \left\{\frac{1}{R^3}\,\mathrm{rot}_{\vec{r}}\left[\vec{\omega}\left(\vec{r}'\right)\times\vec{R}\right] - \left[\left[\vec{\omega}\left(\vec{r}'\right)\times\vec{R}\right]\times\mathrm{grad}_{\vec{r}}\left(\frac{1}{R^3}\right)\right]\right\}dV'. \quad (5.11)$$

For the purpose of further transformation of Eq. (5.11), we use the well-known vector analysis formula [34] bearing in mind the fact that $\vec{\omega}$ is an integrand depending on $\vec{r}'$ only:

$$\begin{aligned}\mathrm{rot}_{\vec{r}}\left[\vec{\omega}\left(\vec{r}'\right)\times\vec{R}\right] &= \left(\vec{R}\cdot\nabla_{\vec{r}}\right)\vec{\omega} - \left(\vec{\omega}\cdot\nabla_{\vec{r}}\right)\vec{R} + \vec{\omega}\,\mathrm{div}_{\vec{r}}\,\vec{R} - \vec{R}\,\mathrm{div}_{\vec{r}}\,\vec{\omega} = \\ &= -\left(\omega_x\,\vec{\mathrm{i}} + \omega_y\,\vec{\mathrm{j}} + \omega_z\,\vec{\mathrm{k}}\right) + 3\vec{\omega} = 2\vec{\omega}\end{aligned} \quad (5.12a)$$

as well as we take into account that

$$\mathrm{grad}\left(\frac{1}{R^3}\right) = \frac{d}{dR}\left(\frac{1}{R^3}\right)\vec{\mathrm{e}}_R = -\frac{3}{R^4}\vec{\mathrm{e}}_R = -\frac{3\vec{R}}{R^5}. \quad (5.12b)$$

Then taking into account relations (5.12a), (5.12b) we rewrite Eq. (5.11) as follows:

$$\vec{h}_{\mathrm{g}} = \frac{c_{\mathrm{g}}}{4\pi}\iiint\limits_V \left\{\frac{2\vec{\omega}}{R^3} + \frac{3\left[\left[\vec{\omega}\times\vec{R}\right]\times\vec{R}\right]}{R^5}\right\}dV' = \frac{c_{\mathrm{g}}}{4\pi}\iiint\limits_V \left\{\frac{2\vec{\omega}}{R^3} - \frac{3\left[\vec{R}\times\left[\vec{\omega}\times\vec{R}\right]\right]}{R^5}\right\}dV' =$$

$$= \frac{c_{\mathrm{g}}}{4\pi}\iiint\limits_V \left\{\frac{2\vec{\omega}}{R^3} - \frac{3\vec{\omega}R^2 - 3\vec{R}(\vec{R}\cdot\vec{\omega})}{R^5}\right\}dV'. \quad (5.13)$$

Similarly to the established relationship between Eq. (5.3) and Eq. (5.8), one can also find a relationship between Eq. (5.10) and Eq. (5.13).

We especially note that, by analogy with Eq. (4.13), integral (5.13) can be calculated on the basis of the mean value theorem (for $R \to \infty$):

$$\frac{1}{c_{\mathrm{g}}}\vec{h}_{\mathrm{g}} = \frac{1}{4\pi}\cdot\frac{2\bar{\vec{\omega}}}{R^3}\cdot V = \frac{2\cdot 3\vec{\Omega}}{4\pi R^3}\cdot V = 2\vec{\Omega}, \quad (5.14)$$

as well as the integral (5.3) which allows us to find:

$$\frac{1}{c_{\mathrm{g}}}\vec{A}_{\mathrm{g}} = \frac{1}{4\pi}\cdot\frac{\left[\bar{\vec{\omega}}\times\vec{R}\right]}{R^3}\cdot V = \frac{1}{4\pi R^3}\cdot\left[3\vec{\Omega}\times\vec{R}\right]\cdot V = \left[\vec{\Omega}\times\vec{R}\right]. \quad (5.15)$$

Since the expression on the right-hand side of Eq. (5.15) was chosen as the vector gravimagnetic potential (2.4a) in the case of a *homogeneous* gravimagnetic field, then when choosing the vector gravimagnetic potential of the general form (5.3) we can derive the equation of gas–ethereal particle motion in the "quasi-solid" approximation in a non-inertial frame of reference.

To do this, we take into account the relationship formula (5.2) for the values $\vec{B}$ and $\vec{A}_{\mathrm{g}}$ as well as use Lagrange function (4.17) of a moving gaseous–ethereal particle (in the "*quasi-solid*" approximation) in the inertial and gravitational fields of a spheroidal body:

$$L_{\mathrm{g}}^{(m)} = \frac{m_0}{2}\vec{\mathrm{v}}^2 + \frac{m_0}{c_{\mathrm{g}}}\vec{A}_{\mathrm{g}}\cdot\vec{\mathrm{v}} + \frac{m_0}{2c_{\mathrm{g}}^2}\vec{A}_{\mathrm{g}}^2 - \frac{m_0}{2}\frac{d\,\vec{\mathrm{u}}}{dt}\cdot\vec{r} - m_0\varphi_{\mathrm{g}}, \quad (5.16)$$

where $d\,\vec{\mathrm{u}}/dt = -\left\{\mathrm{G}^2(t)\alpha^2(t) + \dot{\mathrm{G}}(t)\alpha(t)\right\}\vec{r}$ is the radial acceleration (3.11a) induced by the anti-diffusion velocity $\vec{\mathrm{u}}$ under condensation of a spheroidal body in accordance with Eq. (1.10). Then, according to the Lagrange–Euler variational equation [31] we calculate the partial derivatives of $L_{\mathrm{g}}^{(m)}$ (5.16) with respect to $\vec{\mathrm{v}}$ and $\vec{r}$:

$$\frac{\partial L_{\mathrm{g}}^{(m)}}{\partial \vec{\mathrm{v}}} = m_0\,\vec{\mathrm{v}} + \frac{m_0}{c_{\mathrm{g}}}\vec{A}_{\mathrm{g}}; \tag{5.17a}$$

$$\frac{\partial L_{\mathrm{g}}^{(m)}}{\partial \vec{r}} \equiv \nabla L_{\mathrm{g}}^{(m)} = \frac{m_0}{c_{\mathrm{g}}}\operatorname{grad}\left(\vec{A}_{\mathrm{g}}\cdot\vec{\mathrm{v}}\right) + \frac{m_0}{2c_{\mathrm{g}}^2}\operatorname{grad}\vec{A}_{\mathrm{g}}^2 - \frac{m_0}{2}\operatorname{grad}\left(\frac{d\vec{\mathrm{u}}}{dt}\cdot\vec{r}\right) - m_0\operatorname{grad}\varphi_{\mathrm{g}}. \tag{5.17b}$$

Applying the well-known vector analysis formula [34]:

$$\operatorname{grad}\left(\vec{a}\cdot\vec{c}\right) = \left(\vec{a}\cdot\nabla\right)\vec{c} + \left(\vec{c}\cdot\nabla\right)\vec{a} + \left[\vec{c}\times\operatorname{rot}\vec{a}\right] + \left[\vec{a}\times\operatorname{rot}\vec{c}\right] \tag{5.17c}$$

to the calculation of $\operatorname{grad}\left(\vec{A}_{\mathrm{g}}\cdot\vec{\mathrm{v}}\right)$ and $\operatorname{grad}\left(\frac{d\vec{\mathrm{u}}}{dt}\cdot\vec{r}\right)$, we transform the right-hand side of Eq.(5.17b) (taking into account the fact that differentiation with respect to $\vec{r}$ is carried out at a *constant* $\vec{\mathrm{v}}$):

$$\frac{\partial L_{\mathrm{g}}^{(m)}}{\partial \vec{r}} = \frac{m_0}{c_{\mathrm{g}}}\left(\vec{\mathrm{v}}\cdot\nabla\right)\vec{A}_{\mathrm{g}} + \frac{m_0}{c_{\mathrm{g}}}\left[\vec{\mathrm{v}}\times\operatorname{rot}\vec{A}_{\mathrm{g}}\right] + \frac{m_0}{2c_{\mathrm{g}}^2}\operatorname{grad}\vec{A}_{\mathrm{g}}^2 - \frac{m_0}{2}2\left(-\mathrm{G}^2(t)\alpha^2(t) - \dot{\mathrm{G}}(t)\alpha(t)\right)\vec{r} - m_0\operatorname{grad}\varphi_{\mathrm{g}}. \tag{5.18}$$

Taking into account Eqs (1.10), (5.17a), (5.18) we can write Lagrange–Euler equation in the form:

$$\frac{d}{dt}\left(m_0\,\vec{\mathrm{v}} + \frac{m_0}{c_{\mathrm{g}}}\vec{A}_{\mathrm{g}}\right) = \frac{m_0}{c_{\mathrm{g}}}\left(\vec{\mathrm{v}}\cdot\nabla\right)\vec{A}_{\mathrm{g}} + \frac{m_0}{c_{\mathrm{g}}}\left[\vec{\mathrm{v}}\times\operatorname{rot}\vec{A}_{\mathrm{g}}\right] + \frac{m_0}{2c_{\mathrm{g}}^2}\operatorname{grad}\vec{A}_{\mathrm{g}}^2 - m_0\frac{d\vec{\mathrm{u}}}{dt} - m_0\operatorname{grad}\varphi_{\mathrm{g}}. \tag{5.19}$$

Using relation for the total derivative of the vector gravimagnetic potential $\vec{A}_g$ [31]:

$$\frac{d\vec{A}_{\mathrm{g}}}{dt} = \frac{\partial\vec{A}_{\mathrm{g}}}{\partial t} + \left(\vec{\mathrm{v}}\cdot\nabla\right)\vec{A}_{\mathrm{g}} \tag{5.20}$$

and substituting it into Eq. (5.19) we obtain the following equation:

$$m_0\frac{d\vec{\mathrm{v}}}{dt} = -m_0\operatorname{grad}\varphi_{\mathrm{g}} - m_0\frac{d\vec{\mathrm{u}}}{dt} - \frac{m_0}{c_{\mathrm{g}}}\cdot\frac{\partial\vec{A}_{\mathrm{g}}}{\partial t} + \frac{m_0}{c_{\mathrm{g}}}\left[\vec{\mathrm{v}}\times\operatorname{rot}\vec{A}_{\mathrm{g}}\right] + \frac{m_0}{2c_{\mathrm{g}}^2}\operatorname{grad}\vec{A}_{\mathrm{g}}^2, \tag{5.21}$$

which, as shown above, completely coincides with Eq. (3.15) taking into account notation (5.2), and also coincides with Eq. (2.1) under the assumption $(1/c_{\mathrm{g}})\vec{A}_{\mathrm{g}}\left(\vec{r},t\right) = \left[\vec{\Omega}\times\vec{r}\right]$.

Thus, to obtain the equation of motion (5.21) based on (5.16), it is necessary to use the principle of least action $S_{\mathrm{g}}^{(m)} = \int_{t_1}^{t_2} L_{\mathrm{g}}^{(m)}dt$ and vary only the coordinates $\left\{\vec{r},\vec{\mathrm{v}}\right\}$ but not the gravitational potentials $\left\{\varphi_{\mathrm{g}},\vec{A}_{\mathrm{g}}\right\}$. Similarly to the equation of motion of a particle in an electromagnetic field [31], the expression on the right-hand side of Eq. (5.21) for the motion of a solid particle m_0 with a "liquid" ethereal surrounding is the force acting on a point mass m_0 in the gravitational and gravimagnetic fields. Obviously, this force consists of two parts. The first part (which includes the first, second, third, and fifth terms in Eq. (5.21)) does not explicitly depend on the particle velocity. The second part (the fourth term in Eq. (5.21)) depends on this velocity: it is proportional to the velocity and perpendicular to it. The force of the first kind, rationed to the mass m_0 of this particle, is called the *generalized strength* of the gravitational field:

$$\vec{g} = -\operatorname{grad}\varphi_{\mathrm{g}} - \frac{d\vec{\mathrm{u}}}{dt} - \frac{1}{c_{\mathrm{g}}}\cdot\frac{\partial\vec{A}_{\mathrm{g}}}{\partial t} + \frac{1}{2c_{\mathrm{g}}^2}\operatorname{grad}\vec{A}_{\mathrm{g}}^2. \tag{5.22}$$

According to Eq. (5.9) the multiplier at the velocity $\vec{\mathrm{v}}$ in the force of the second kind on the right-hand side of Eq. (5.21), rationed to the mass m_0, is called the *strength of gravimagnetic field*:

$$\vec{h}_{\mathrm{g}} = \mathrm{rot}\,\vec{A}_{\mathrm{g}}. \tag{5.23}$$

Obviously, in the general case, the gravitational field of a spheroidal body is a superposition of fields characterized by both the polar vector of strength $\vec{g}$ and the axial vector of strength $\vec{h}_{\mathrm{g}}$.

Taking into account the notation (5.22), (5.23) equation (5.21) of motion of a particle with a mass m_0 in the gravitational and gravimagnetic fields of a spheroidal body can be written as:

$$m_0 \frac{d\vec{\mathrm{v}}}{dt} = m_0\vec{g} + \frac{m_0}{c_g}\left[\vec{\mathrm{v}}\times\vec{h}_{\mathrm{g}}\right]. \tag{5.24}$$

By analogy with Eqs (2.8) – (2.10), from expressions (5.22) and (5.23) it is also easy to obtain equations containing only $\vec{g}$ and $\vec{h}_{\mathrm{g}}$. Indeed, according to Eq. (5.22) we can define $\mathrm{rot}\,\vec{g}$:

$$\mathrm{rot}\,\vec{g} = -\mathrm{rot}\,\mathrm{grad}\,\varphi_{\mathrm{g}} + \mathrm{rot}\left(-\frac{d\vec{\mathrm{u}}}{dt}\right) - \frac{1}{c_{\mathrm{g}}}\mathrm{rot}\frac{\partial\vec{A}_{\mathrm{g}}}{\partial t} + \frac{1}{2c_{\mathrm{g}}^2}\mathrm{rot}\,\mathrm{grad}\,\vec{A}_{\mathrm{g}}^2 =$$

$$= \left\{\mathrm{G}^2(t)\alpha^2(t) + \dot{\mathrm{G}}(t)\alpha(t)\right\}\mathrm{rot}\,\vec{r} - \frac{1}{c_{\mathrm{g}}}\cdot\frac{\partial\left(\mathrm{rot}\,\vec{A}_{\mathrm{g}}\right)}{\partial t} = -\frac{1}{c_{\mathrm{g}}}\cdot\frac{\partial\vec{h}_{\mathrm{g}}}{\partial t},$$

whence

$$\mathrm{rot}\,\vec{g} = -\frac{1}{c_{\mathrm{g}}}\cdot\frac{\partial\vec{h}_{\mathrm{g}}}{\partial t}. \tag{5.25}$$

Further, taking the divergence operator div from both parts of Eq. (5.23) we get:

$$\mathrm{div}\,\vec{h}_{\mathrm{g}} = 0. \tag{5.26}$$

Eqs (5.25), (5.26) constitute the *first system* of equations for the gravitational/gravimagnetic field:

$$\begin{cases} \mathrm{rot}\,\vec{g} = -\dfrac{1}{c_{\mathrm{g}}}\dfrac{\partial\vec{h}_{\mathrm{g}}}{\partial t}; & (5.27a) \\ \mathrm{div}\,\vec{h}_{\mathrm{g}} = 0. & (5.27b) \end{cases}$$

Obviously, these two equations do not yet completely determine the properties of the gravitational/gravimagnetic field. Outwardly, they coincide with the corresponding Maxwell equations of electromagnetic field [31], [33].

In conclusion, we estimate the physical dimensions of the introduced physical values. According to Eq. (5.24) we have that $\left[\vec{h}_{\mathrm{g}}\right] = \left[\vec{g}\right] = \mathrm{m/s^2}$ and, as consequence of Eq. (5.9), we find that $\left[\vec{A}_{\mathrm{g}}\right] = \left[\varphi_{\mathrm{g}}\right] = \mathrm{m^2/s^2}$ is the dimension of both vector gravimagnetic and scalar gravitational potentials. On the other hand, according to relation (5.15) we obtain that $\left[\vec{A}_{\mathrm{g}}\right] = \left[c_{\mathrm{g}}\right]\cdot\left[[\vec{\omega}\times\vec{R}]\right] = \mathrm{m^2/s^2}$, whence $\left[c_{\mathrm{g}}\right] = \mathrm{m/s}$ is the dimension of the *speed of gravitational interactions*.

6. The second system of equations for the gravitational/gravimagnetic field of a condensing spheroidal body

As indicated, Eq. (2.1) of motion of a particle in a gravitational field is generalized by the equation of motion (5.21) of a "liquid" (gaseous–ethereal) particle in the "quasi-solid" approximation [32] when the angular velocity of rotation $\vec{\Omega}=\vec{\Omega}(t)$ is generalized by the vorticity $\vec{\omega}=\vec{\omega}(\vec{r},t)$, i.e. the following transition is performed (see (5.15)):

$$\left[\vec{\Omega}\times\vec{r}\,\right]\rightarrow\frac{1}{c_{\mathrm{g}}}\vec{A}_{\mathrm{g}}(\vec{r},t)=\frac{1}{4\pi}\iiint\limits_{V}\frac{\left[\vec{\omega}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')\right]}{\left|\vec{r}-\vec{r}\,'\right|^{3}}dV', \tag{6.1}$$

but at the same time, the particle remains "quasi-solid", i.e. "liquid" particle with a "solid" core (in the form of gas molecules m_0) surrounded by a continuous ethereal shell. A more general approach, however, implies the equation of motion of a "liquid" particle with mass $dm=\rho dV$ (dV is an elementary volume of a "liquid" particle) in gravitational and gravimagnetic fields.

In this connection, let us represent Lagrange function (5.16) for a "quasi-solid" particle with mass m_0 moving in a non-inertial frame of reference into a *more general form* for a moving "liquid" particle with mass $dm=\rho dV$ in gravitational and gravimagnetic fields. That is why to derive the second system of equations for the gravitational/gravimagnetic field, we are going to use the version of Lagrange function for a moving "liquid" particle:

$$L_{\mathrm{g}}^{(m)}=\iiint\limits_{V}\rho\left(\frac{\vec{\mathrm{v}}^{2}}{2}+\frac{\vec{A}_{\mathrm{g}}\cdot\vec{\mathrm{v}}}{c_{\mathrm{g}}}+\frac{\vec{A}_{\mathrm{g}}^{2}}{2c_{\mathrm{g}}^{2}}+\frac{1}{2\,\mathrm{G}(t)\alpha(t)}\frac{\partial}{\partial t}\left(\frac{\vec{\mathrm{u}}^{2}}{2}\right)-\varphi_{\mathrm{g}}\right)dV. \tag{6.2}$$

Moreover, to use the principle of least action $S_{\mathrm{g}}^{(m)}=\int_{t_1}^{t_2}L_{\mathrm{g}}^{(m)}dt$ based on function (6.2) now we must vary only the gravitational potentials $\{\varphi_{\mathrm{g}},\vec{A}_{\mathrm{g}}\}$ but not the coordinates $\{\vec{r},\vec{\mathrm{v}}\}$ which in turn completely excludes the use of convective (spatial) derivatives of $\vec{\mathrm{u}}$ (like Eqs (1.10), (5.16)).

Similarly to Lagrange function (6.2) for a moving "liquid" (gaseous–ethereal) particle, let us construct Lagrange function of the gravitational and gravimagnetic fields:

$$L_{\mathrm{g}}^{(f)}=-\frac{1}{8\pi\gamma}\iiint\limits_{V}\left(\vec{g}^{2}-\vec{h}_{\mathrm{g}}^{\,2}\right)dV, \tag{6.3}$$

where γ is Newton's gravitational constant. From (6.3) it follows that the weak gravitational field, like the electromagnetic one [31], [33] obeys the *principle of superposition*: the field created by a system of masses (particles) is the result of an addition of fields that are created by each of the masses separately. Then, by varying the corresponding action $S_{\mathrm{g}}^{(f)}=\int_{t_1}^{t_2}L_{\mathrm{g}}^{(f)}dt$, one can obtain equations describing the gravitational and gravimagnetic fields.

Let us note that to obtain the *second system* of equations, it is necessary to use the *principle of least action* and vary only the gravitational potentials $\{\varphi_{\mathrm{g}},\vec{A}_{\mathrm{g}}\}$ as well as $\vec{\mathrm{u}}$ (since $\varphi_{\mathrm{g}}=\varphi_{\mathrm{g}}(\vec{\mathrm{u}})$[7]) and not the coordinates $\{\vec{r},\vec{\mathrm{v}}\}$, as in the case of the first system (see Eqs (5.27a), (5.27b)). Using both Lagrange function (6.2) of a moving "liquid" gaseous–ethereal particle with a "solid" core in the gravitational and gravimagnetic fields and Lagrange function (6.3) of the gravitational and gravimagnetic fields, we can compose the expression for a *total action* S_{g} for the gravitational/gravimagnetic field and matter of a spheroidal body containing the "liquid" particles with masses ρdV:

$$S_g = S_g^{(m)} + S_g^{(f)} = \int_{t_1}^{t_2}\iiint_V \left(\frac{\rho \vec{v}^2}{2} + \frac{\rho \vec{v}\cdot \vec{A}_g}{c_g} + \frac{\rho \vec{A}_g^2}{2c_g^2} + \frac{\rho}{2G(t)\alpha(t)}\frac{\partial}{\partial t}\left(\frac{\vec{u}^2}{2}\right) - \rho\varphi_g \right) dVdt - \frac{1}{8\pi\gamma}\int_{t_1}^{t_2}\iiint_V \left(\vec{g}^2 - \vec{h}_g^{\,2}\right) dVdt. \quad (6.4)$$

As mentioned above, when finding the field equations from the principle of least action (in contrast to the procedure (5.17a, b) – (5.21) for finding the equations of motion, it is necessary to vary only the gravitational potentials $\{\varphi_g, \vec{u}, \vec{A}_g\}$ but not coordinates $\{\vec{r}, \vec{v}\}$ [31]. In this regard, the variation of the first term in the first integral in Eq. (6.4) is now equal to zero, the mass flow density $\rho\vec{v}$ should not vary in the second term, so that the variation of the total action S_g is equal to:

$$\delta S_g = \int_{t_1}^{t_2}\iiint_V \left(\frac{\rho}{c_g}\vec{v}\cdot\delta\vec{A}_g + \frac{\rho}{2c_g^2}2\vec{A}_g\cdot\delta\vec{A}_g + \frac{\rho}{2G(t)\alpha(t)}\frac{\partial}{\partial t}\left(\frac{2\vec{u}\cdot\delta\vec{u}}{2}\right) - \rho\delta\varphi_g \right) dVdt -$$

$$-\frac{1}{4\pi\gamma}\int_{t_1}^{t_2}\iiint_V \left(\vec{g}\cdot\delta\vec{g} - \vec{h}_g\cdot\delta\vec{h}_g\right) dVdt. \quad (6.5)$$

According to Eqs (5.22), (5.23) and taking into account the mentioned remark about not using the convective derivative of $\vec{u}$, i.e. $\frac{d\vec{u}}{dt} = \frac{\partial\vec{u}}{\partial t}$, we calculate the variations $\delta\vec{g}$ and $\delta\vec{h}_g$:

$$\delta\vec{g} = -\mathrm{grad}(\delta\varphi_g) - \frac{\partial}{\partial t}(\delta\vec{u}) - \frac{1}{c_g}\cdot\frac{\partial}{\partial t}(\delta\vec{A}_g) + \frac{1}{c_g^2}\mathrm{grad}(\vec{A}_g\cdot\delta\vec{A}_g), \quad (6.6a)$$

$$\delta\vec{h}_g = \mathrm{rot}(\delta\vec{A}_g). \quad (6.6b)$$

Substituting Eqs (6.6a), (6.6b) into Eq. (6.5) we obtain:

$$\delta S_g = \frac{1}{c_g}\int_{t_1}^{t_2}\iiint_V \rho\vec{v}\cdot\delta\vec{A}_g dVdt + \frac{1}{c_g^2}\int_{t_1}^{t_2}\iiint_V \rho\vec{A}_g\cdot\delta\vec{A}_g dVdt + \int_{t_1}^{t_2}\iiint_V \frac{\rho}{2G(t)\alpha(t)}\cdot\frac{\partial}{\partial t}(\vec{u}\cdot\delta\vec{u}) dVdt -$$

$$-\int_{t_1}^{t_2}\iiint_V \rho\delta\varphi_g dVdt + \frac{1}{4\pi\gamma}\int_{t_1}^{t_2}\iiint_V \vec{g}\cdot\nabla(\delta\varphi_g) dVdt + \frac{1}{4\pi\gamma}\int_{t_1}^{t_2}\iiint_V \vec{g}\cdot\frac{\partial}{\partial t}(\delta\vec{u}) dVdt +$$

$$+\frac{1}{4\pi\gamma c_g}\int_{t_1}^{t_2}\iiint_V \vec{g}\cdot\frac{\partial}{\partial t}(\delta\vec{A}_g) dVdt - \frac{1}{4\pi\gamma c_g^2}\int_{t_1}^{t_2}\iiint_V \vec{g}\cdot\nabla(\vec{A}_g\cdot\delta\vec{A}_g) dVdt + \frac{1}{4\pi\gamma}\int_{t_1}^{t_2}\iiint_V \vec{h}_g\cdot\mathrm{rot}(\delta\vec{A}_g) dVdt. \quad (6.7)$$

Carrying out integration by parts in Eq. (6.7) as well as using the Gauss–Ostrogradsky theorem we find:

$$\delta S_g = \int_{t_1}^{t_2}\iiint_V \left(-\rho - \frac{1}{4\pi\gamma}\nabla\vec{g}\right)\delta\varphi_g dVdt + \int_{t_1}^{t_2}\iiint_V \left(-\frac{d}{dt}\left(\frac{1}{G(t)\alpha(t)}\right)\frac{\rho\vec{u}}{2} - \frac{1}{4\pi\gamma}\frac{\partial\vec{g}}{\partial t}\right)\delta\vec{u} dVdt +$$

$$+\int_{t_1}^{t_2}\iiint_V \left(\frac{1}{c_g}\rho\vec{v} + \frac{1}{c_g^2}\rho\vec{A}_g - \frac{1}{4\pi\gamma c_g}\cdot\frac{\partial\vec{g}}{\partial t} + \frac{1}{4\pi\gamma c_g^2}\nabla\vec{g}\vec{A}_g + \frac{1}{4\pi\gamma}\mathrm{rot}\,\vec{h}_g\right)\delta\vec{A}_g dVdt = 0. \quad (6.8)$$

In view of the fact that, according to the principle of least action the variations $\delta\varphi_g, \delta\vec{u}, \delta\vec{A}_g$ are arbitrary, therefore the terms at $\delta\varphi_g, \delta\vec{u}, \delta\vec{A}_g$ in Eq. (6.8) must equal zero, i.e. the following equations are valid:

$$-\rho - \frac{1}{4\pi\gamma}\nabla\vec{g} = 0; \quad (6.9a)$$

$$-\frac{1}{2}\frac{d}{dt}\left(\frac{1}{G(t)\alpha(t)}\right)\rho\vec{u} - \frac{1}{4\pi\gamma}\frac{\partial\vec{g}}{\partial t} = 0; \quad (6.9b)$$

$$\frac{1}{c_{\rm g}}\rho\vec{\rm v}+\frac{1}{c_{\rm g}^2}\rho\vec{A}_{\rm g}-\frac{1}{4\pi\gamma c_{\rm g}}\cdot\frac{\partial\vec{g}}{\partial t}+\frac{1}{4\pi\gamma c_{\rm g}^2}\nabla\vec{g}\vec{A}_{\rm g}+\frac{1}{4\pi\gamma}\operatorname{rot}\vec{h}_{\rm g}=0\cdot \tag{6.9c}$$

Using Eq. (1.11) let us make some transformation in Eq. (6.9b):

$$-\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\mathrm{G}(t)\alpha(t)}\right)=-\frac{1}{2}\left(-\frac{\dot{\mathrm{G}}(t)\alpha(t)+\mathrm{G}(t)\dot{\alpha}(t)}{\mathrm{G}^2(t)\alpha^2(t)}\right)=\frac{1}{2}\frac{\dot{\mathrm{G}}(t)\alpha(t)+\mathrm{G}(t)\dot{\alpha}(t)}{\mathrm{G}^2(t)\alpha^2(t)}=$$

$$=\frac{1}{2}\frac{\dot{\mathrm{G}}(t)\alpha(t)+\mathrm{G}(t)\dot{\alpha}(t)}{\frac{1}{2}\mathrm{G}(t)\dot{\alpha}(t)}=1+\frac{\dot{\mathrm{G}}(t)}{\mathrm{G}(t)}\cdot\frac{\alpha(t)}{\dot{\alpha}(t)}. \tag{6.10}$$

We can note that in the case of *equilibrium condensation* of a spheroidal body $\mathrm{G}(t)=\mathrm{G}_{\rm s}=\mathrm{const}$ (see [7]), so that relation (6.10) takes the form:

$$1+\frac{\dot{\mathrm{G}}(t)}{\mathrm{G}(t)}\cdot\frac{\alpha(t)}{\dot{\alpha}(t)}\equiv 1 \text{ at } \mathrm{G}(t)=\mathrm{G}_{\rm s}, \tag{6.11a}$$

and respectively Eq. (6.9b) (with regard to Eq. (1.8)) becomes the following:

$$\frac{1}{4\pi\gamma}\frac{\partial\vec{g}}{\partial t}=\rho\vec{\rm u}=\mathrm{G}_{\rm s}\nabla\rho\cdot \tag{6.11b}$$

Taking into account Eq. (1.8), (6.10) we rewrite the basic equations (6.9a)–(6.9c) as follows:

$$\operatorname{div}\vec{g}=-4\pi\gamma\rho; \tag{6.12a}$$

$$\frac{1}{4\pi\gamma}\frac{\partial\vec{g}}{\partial t}=\left(1+\frac{\dot{\mathrm{G}}(t)}{\mathrm{G}(t)}\cdot\frac{\alpha(t)}{\dot{\alpha}(t)}\right)\rho\vec{\rm u}=\rho\vec{\rm u}+\dot{\mathrm{G}}(t)\frac{\alpha(t)}{\dot{\alpha}(t)}\nabla\rho=\left[\mathrm{G}(t)+\delta\mathrm{G}^{\sim}(t)\right]\nabla\rho=\mathscr{G}(t)\nabla\rho; \tag{6.12b}$$

$$\operatorname{rot}\vec{h}_{\rm g}=-\frac{4\pi\gamma}{c_{\rm g}}\rho\vec{\rm v}-\frac{4\pi\gamma}{c_{\rm g}^2}\rho\vec{A}_{\rm g}+\frac{1}{c_{\rm g}}\cdot\frac{\partial\vec{g}}{\partial t}-\frac{1}{c_{\rm g}^2}\nabla\vec{g}\vec{A}_{\rm g}, \tag{6.12c}$$

where $\delta\mathrm{G}^{\sim}(t)=\dot{\mathrm{G}}(t)\frac{\alpha(t)}{\dot{\alpha}(t)}$ is a fluctuation (oscillatory) increment of GCF (1.6), $\mathscr{G}(t)=\mathrm{G}(t)+\delta\mathrm{G}^{\sim}(t)$.

Then substituting the first equation (6.12a) into the third (6.12c), we obtain:

$$\operatorname{rot}\vec{h}_{\rm g}=-\frac{4\pi\gamma}{c_{\rm g}}\rho\vec{\rm v}-\frac{4\pi\gamma}{c_{\rm g}^2}\rho\vec{A}_{\rm g}+\frac{1}{c_{\rm g}}\cdot\frac{\partial\vec{g}}{\partial t}-\frac{1}{c_{\rm g}^2}(-4\pi\gamma\rho)\vec{A}_{\rm g}=-\frac{4\pi\gamma}{c_{\rm g}}\rho\vec{\rm v}+\frac{1}{c_{\rm g}}\cdot\frac{\partial\vec{g}}{\partial t},$$

whence it follows that the three basic equations (6.12a) – (6.12c) take the form:

$$\operatorname{div}\vec{g}=-4\pi\gamma\rho; \tag{6.13a}$$

$$\frac{1}{4\pi\gamma}\frac{\partial\vec{g}}{\partial t}=\rho\vec{\rm u}+\delta\mathrm{G}^{\sim}(t)\nabla\rho; \tag{6.13b}$$

$$\operatorname{rot}\vec{h}_{\rm g}=-\frac{4\pi\gamma}{c_{\rm g}}\rho\vec{\rm v}+\frac{1}{c_{\rm g}}\cdot\frac{\partial\vec{g}}{\partial t}. \tag{6.13c}$$

Equations (6.13a) – (6.13c) represent the *second system* of equations for the gravitational and gravimagnetic fields of a spheroidal body. In the state of *equilibrium condensation* of a spheroidal body (see Eqs (6.11a), (6.11b)) equation (6.13b) of the gravitational condensation of a spheroidal body is noticeably simplified, so that the second system of equations (6.13a) – (6.13c) of the gravitational and gravimagnetic fields of a spheroidal body takes the form:

$$\begin{cases} \operatorname{div}\vec{g} = -4\pi\gamma\rho; & (6.16a) \\ \dfrac{\partial\vec{g}}{\partial t} = 4\pi\gamma\rho\,\vec{\mathrm{u}}; & (6.16b) \\ \operatorname{rot}\vec{h}_{\mathrm{g}} = -\dfrac{4\pi\gamma}{c_{\mathrm{g}}}\rho\,\vec{\mathrm{v}} + \dfrac{1}{c_{\mathrm{g}}}\cdot\dfrac{\partial\vec{g}}{\partial t}. & (6.16c) \end{cases}$$

7. Gravitational waves in the gravitational field of a condensing cosmogonical body

Using the equations of the gravitational/gravimagnetic field (5.27a), (6.16c) we multiply both parts of Eq. (6.16c) by $\vec{g}$, and both parts of Eq. (5.27a) by $\vec{h}_{\mathrm{g}}$ and then subtract the resulting equations term by term:

$$\vec{g}\operatorname{rot}\vec{h}_{\mathrm{g}} - \vec{h}_{\mathrm{g}}\operatorname{rot}\vec{g} = -\frac{4\pi\gamma}{c_{\mathrm{g}}}\left(\vec{g}\cdot\rho\,\vec{\mathrm{v}}\right) + \frac{1}{c_{\mathrm{g}}}\vec{g}\cdot\frac{\partial\vec{g}}{\partial t} + \frac{1}{c_{\mathrm{g}}}\vec{h}_{\mathrm{g}}\cdot\frac{\partial\vec{h}_{\mathrm{g}}}{\partial t} \tag{7.1}$$

Then using the well-known vector analysis formula [34]:

$$\operatorname{div}\left[\vec{h}_{\mathrm{g}}\times\vec{g}\right] = \vec{g}\operatorname{rot}\vec{h}_{\mathrm{g}} - \vec{h}_{\mathrm{g}}\operatorname{rot}\vec{g},$$

we can rewrite relation (7.1) in the form:

$$\operatorname{div}\left[\vec{h}_{\mathrm{g}}\times\vec{g}\right] = -\frac{4\pi\gamma}{c_{\mathrm{g}}}\left(\rho\,\vec{\mathrm{v}}\cdot\vec{g}\right) + \frac{1}{2c_{\mathrm{g}}}\cdot\frac{\partial}{\partial t}\left(\vec{g}^{\,2} + \vec{h}_{\mathrm{g}}^{\,2}\right),$$

whence

$$\frac{\partial w}{\partial t} = \rho\left(\vec{\mathrm{v}}\cdot\vec{g}\right) - \operatorname{div}\vec{s}_{\mathrm{g}}, \tag{7.2}$$

where w is an energy density:

$$w = -\frac{g^2 + h_{\mathrm{g}}^2}{8\pi\gamma} \tag{7.3a}$$

and a vector

$$\vec{s}_{\mathrm{g}} = \frac{c_{\mathrm{g}}}{4\pi\gamma}\left[\vec{g}\times\vec{h}_{\mathrm{g}}\right] \tag{7.3b}$$

is called the *Poynting vector* [31].

The gravitational field in a space filled with ether is determined by the derived Eqs (5.27a), (5.27b), (6.16a), (6.16c) under condition $\rho = 0$ and $\vec{j}_{\mathrm{v}} = \rho\,\vec{\mathrm{v}} = 0$. Taking this into account these equations become:

$$\operatorname{rot}\vec{g} = -\frac{1}{c_{\mathrm{g}}}\cdot\frac{\partial\vec{h}_{\mathrm{g}}}{\partial t}; \quad \operatorname{div}\vec{h}_{\mathrm{g}} = 0; \tag{7.4}$$

$$\operatorname{rot}\vec{h}_{\mathrm{g}} = \frac{1}{c_{\mathrm{g}}}\cdot\frac{\partial\vec{g}}{\partial t}; \quad \operatorname{div}\vec{g} = 0. \tag{7.5}$$

Then acting on first Eq. (7.4) by rotor operator and taking into account first Eq. (7.5) we obtain:

$$\operatorname{rot}\operatorname{rot}\vec{g} = -\frac{1}{c_{\mathrm{g}}}\cdot\frac{\partial}{\partial t}\operatorname{rot}\vec{h}_{\mathrm{g}} = -\frac{1}{c_{\mathrm{g}}}\cdot\frac{\partial}{\partial t}\left(\frac{1}{c_{\mathrm{g}}}\cdot\frac{\partial\vec{g}}{\partial t}\right) = -\frac{1}{c_{\mathrm{g}}^2}\frac{\partial^2\vec{g}}{\partial t^2},$$

whence

$$-\nabla^2 \vec{g} + \mathrm{grad}(\mathrm{div}\,\vec{g}) = -\frac{1}{c_{\mathrm{g}}^2} \cdot \frac{\partial^2 \vec{g}}{\partial t^2}.$$

Taking into account second Eq. (7.5) this equation takes the form of D'Alembert wave equation for the strength of gravitation field:

$$\nabla^2 \vec{g} - \frac{1}{c_{\mathrm{g}}^2} \cdot \frac{\partial^2 \vec{g}}{\partial t^2} = 0. \tag{7.6}$$

Analogously, acting on second Eq. (7.5) by rotor operator and taking into account first Eq. (7.4) D'Alembert wave equation for the strength of gravimagnetic field can be derived:

$$\nabla^2 \vec{h}_{\mathrm{g}} - \frac{1}{c_{\mathrm{g}}^2} \cdot \frac{\partial^2 \vec{h}_{\mathrm{g}}}{\partial t^2} = 0. \tag{7.7}$$

Following D'Alembert the general solution of wave equation (7.6) (and wave Eq. (7.7)) has the form:

$$\vec{g} = \vec{g}^{(1)}\left(t - \frac{\vec{r} \cdot \vec{n}}{c_{\mathrm{g}}}\right) + \vec{g}^{(2)}\left(t + \frac{\vec{r} \cdot \vec{n}}{c_{\mathrm{g}}}\right), \tag{7.8}$$

where $\vec{n}$ is a unit vector in the direction of wave propagation. Since there are non-zero solutions (7.8) then a gravitational field can exist even in the absence of any masses of material particles, but not ether! Gravitational fields that exist in ether space in the absence of masses of material bodies are called *gravitational waves*.

In the case of *plane* gravitational waves, the gravitational $\vec{g}$ and gravimagnetic $\vec{h}_{\mathrm{g}}$ fields of a plane wave are directed perpendicular to the direction $\vec{n}$ of wave propagation [31]:

$$\vec{h}_{\mathrm{g}} = \left[\vec{n} \times \vec{g}\right]. \tag{7.9}$$

According to Eq. (7.3b) and Eq. (7.9) the energy flow in a plane gravitational wave is equal to

$$\vec{s}_{\mathrm{g}} = \frac{c_{\mathrm{g}}}{4\pi\gamma}\left[\vec{g} \times \vec{h}_{\mathrm{g}}\right] = \frac{c_{\mathrm{g}}}{4\pi\gamma}\left[\vec{g} \times \left[\vec{n} \times \vec{g}\right]\right] = \frac{c_{\mathrm{g}}}{4\pi\gamma}\left\{\vec{n}\left(\vec{g} \cdot \vec{g}\right) - \vec{g}\left(\vec{g} \cdot \vec{n}\right)\right\} =$$

$$= \frac{c_{\mathrm{g}}}{4\pi\gamma}\vec{n}g^2 = \frac{c_{\mathrm{g}}}{4\pi\gamma}g^2\vec{n} = \frac{c_{\mathrm{g}}}{4\pi\gamma}h_{\mathrm{g}}^2\vec{n}, \tag{7.10}$$

because $\vec{g} \cdot \vec{n} = 0$. Thus, the energy flow is directed along the direction of wave propagation.

Since, according to formula (7.3a) $w = -\frac{g^2 + h_{\mathrm{g}}^2}{8\pi\gamma} = -\frac{g^2}{4\pi\gamma}$ is the energy density of gravitational wave, then relation (7.10) takes the form:

$$\vec{s}_{\mathrm{g}} = -c_{\mathrm{g}} w \vec{n} \tag{7.11}$$

in accordance with the fact that the gravitational wave propagates with the speed c_{g}.

An important special case of plane gravitational waves is *monochromatic* gravitational waves in which the field is a simple periodic function of time. All components of the fields and potentials in a monochromatic wave depend on time by means of trigonometric functions, so if a plane gravitational wave is monochromatic, then its field is described by a trigonometric function of $t - (\vec{n} \cdot \vec{r})/c_{\mathrm{g}}$:

$$\vec{g} = \vec{g}_0^{(1)} \cos\omega\left(t - \frac{\vec{n} \cdot \vec{r}}{c_{\mathrm{g}}}\right) + \vec{g}_0^{(2)} \sin\omega\left(t - \frac{\vec{n} \cdot \vec{r}}{c_{\mathrm{g}}}\right) =$$

$$= \vec{g}_0^{(1)} \cos\left(-\omega t + \vec{k} \cdot \vec{r}\right) - \vec{g}_0^{(2)} \sin\left(-\omega t + \vec{k} \cdot \vec{r}\right), \tag{7.12}$$

where $\vec{k} = (\omega / c_{\mathrm{g}})\vec{n}$ is a wave vector. The strength $\vec{g}$ in such a wave is most conveniently written as the real part of a complex expression:

$$\vec{g} = \mathrm{Re}\left\{\vec{g}_0 e^{-\mathrm{i}\omega(t-\vec{n}\cdot\vec{r}/c_{\mathrm{g}})}\right\} = \mathrm{Re}\left\{\vec{g}_0 e^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)}\right\}, \tag{7.13a}$$

where $\vec{g}_0$ is some constant complex vector. Obviously, the strength of gravimagnetic field $\vec{h}_{\mathrm{g}}$ of such a wave has a similar form:

$$\vec{h}_{\mathrm{g}} = \mathrm{Re}\left\{\vec{h}_{\mathrm{g}_0} e^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)}\right\} \tag{7.13b}$$

8. Discussion on the obtained results and concluding remarks

Albert Einstein in his work "The problems of cosmology and the general theory of relativity" [21], pointed out the same difficulties inherent in both Newton's theory of gravitation and general relativity (GR), as applied to an *infinitely spread* cosmic matter. Here is what, in particular, he wrote about this [21]: "It is known that that Poisson differential equation[1]

$$\nabla^2 \varphi_{\mathrm{g}} = 4\pi\gamma\rho \tag{8.1}$$

in conjunction with the equation of motion of a mass point cannot completely replace the theory of long-range action of Newton. It is necessary to add the condition that the potential φ_{g} in spatial infinity tends to a certain limit. The situation is similar in the theory of gravity resulting from the general principle of relativity; here also the boundary conditions for spatial infinity should be added to the differential equations if we really consider the world as infinitely extended in space.

…These difficulties, apparently, cannot be overcome by remaining within the framework of Newton's theory. The question arises whether it is possible to overcome them by modifying Newton's theory. To this end, we first point out a path that should not be taken too seriously, as it only serves to better understand the subsequent reasoning. Instead of Poisson equation, we write

$$\nabla^2 \varphi_{\mathrm{g}} - \lambda\varphi_{\mathrm{g}} = 4\pi\gamma\rho, \tag{8.2}$$

where λ is some universal constant".

As is known in astrophysics, the problem of gravitational condensation of an infinitely spread spatial matter is closely related to the well-known Jeans criterion and the problem of gravitational instability [36] which is successfully overcome in the statistical theory of formation of gravitating cosmogonical bodies [7]. Indeed, in the framework of Einstein's GR (in the Newtonian approximation) the total action S_{g} for the gravitational field together with the masses distributed in space with mass density ρ is determined by the following formula [31 (p.433)]:

$$S_{\mathrm{g}} = \int_{t_1}^{t_2}\iiint_V \left[\frac{\rho\vec{\mathrm{v}}^2}{2} - \rho\varphi_{\mathrm{g}} - \frac{1}{8\pi\gamma}\left(\nabla\varphi_{\mathrm{g}}\right)^2\right] dVdt, \tag{8.3a}$$

whereas within the framework of the proposed statistical theory this total action for the gravitational field together with the masses distributed in space with mass density ρ is described by relation (6.4):

$$S_{\mathrm{g}} = \int_{t_1}^{t_2}\iiint_V \left[\frac{\rho\vec{\mathrm{v}}^2}{2} + \frac{\rho}{c_{\mathrm{g}}}\vec{\mathrm{v}}\cdot\vec{A}_{\mathrm{g}} + \frac{\rho}{2c_{\mathrm{g}}^2}\vec{A}_{\mathrm{g}}^2 + \frac{\rho}{2\mathrm{G}(t)\alpha(t)}\frac{\partial}{\partial t}\left(\frac{\vec{\mathrm{u}}^2}{2}\right) - \rho\varphi_{\mathrm{g}} - \frac{1}{8\pi\gamma}\left(\vec{g}^2 - \vec{h}_{\mathrm{g}}^2\right)\right] dVdt =$$

[1]With author's designations of values

$$=\int_{t_1}^{t_2}\iiint_V\left[\frac{\rho\left(\vec{\mathrm{v}}+\vec{A}_{\mathrm{g}}/c_{\mathrm{g}}\right)^2}{2}-\rho\varphi_{\mathrm{g}}+\frac{\rho}{2\mathrm{G}(t)\alpha(t)}\frac{\partial}{\partial t}\left(\frac{\vec{\mathrm{u}}^2}{2}\right)-\frac{1}{8\pi\gamma}\left(\vec{g}^2-\vec{h}_{\mathrm{g}}^{\,2}\right)\right]dVdt. \tag{8.3b}$$

However, in contrast to Eq. (8.3b), using the action (8.3a) it is impossible to derive equation (5.21) of the motion of a particle in a gravitational field in the "quasi-solid approximation" as well as equation (2.1) known in mechanics for the motion of a particle in an arbitrary non-inertial frame of reference [29]. Consequently, the so-called "Newtonian approximation" [31] following from Einstein's GR *does not fully describe the gravitational field* of gravitating masses. On the contrary, in the framework of the statistical theory of gravitating cosmogonical bodies formation [7], using the full action (8.3b) it can be to derive equation (5.21) and, as a consequence, the above equation (2.1) of particle motion in an arbitrary non-inertial frame of reference.

We also note that according to Einstein's GR the total energy of matter and field in the Newtonian approximation is expressed by the formula (see p.433 in [31]):

$$E=\iiint_V\left\{\frac{\rho\vec{\mathrm{v}}^2}{2}-\frac{1}{8\pi\gamma}\left(\nabla\varphi_{\mathrm{g}}\right)^2\right\}dV, \tag{8.4a}$$

which is a special case of formula derived within the framework of the statistical theory of gravitating cosmogonical bodies formation:

$$E=\iiint_V\left\{\frac{\rho\left(\vec{\mathrm{v}}+\vec{A}_{\mathrm{g}}/c_{\mathrm{g}}\right)^2}{2}-\rho\vec{\mathrm{u}}^2-\frac{\vec{g}^2+\vec{h}_{\mathrm{g}}^2}{8\pi\gamma}\right\}dV. \tag{8.4b}$$

As follows from (8.4a) and (8.4b), the energy density of the gravitational field [31 (p.433)] in the Newtonian approximation of GR theory, respectively, is:

$$w=-\frac{\left(\nabla\varphi_{\mathrm{g}}\right)^2}{8\pi\gamma}, \tag{8.5a}$$

which is a special case of formula (7.3a) obtained in the framework of the statistical theory of gravitating cosmogonical bodies formation:

$$w=-\frac{\vec{g}^2+\vec{h}_{\mathrm{g}}^2}{8\pi\gamma}\approx-\frac{\left(\nabla\varphi_{\mathrm{g}}\right)^2+\left[\nabla\times\vec{A}_{\mathrm{g}}\right]^2}{8\pi\gamma}. \tag{8.5b}$$

However, as just noted, the total energy of matter and gravitational field in the form (8.4a) and the corresponding action (8.3a) do not allow us to derive equation (5.21) for the motion of a particle in a gravitational field within the framework of Newtonian theory. Consequently, Poisson equation (8.1), (which can be obtained by varying the action (8.3a) with respect to φ_{g}) does not fully describe Newton's theory of gravitation [37] which was rightly assumed by Einstein when introducing the cosmological constant λ in Eq. (8.2) [21]. On the contrary, the action (8.3b) in the statistical theory of gravitating bodies (by means of varying with $\vec{r}$ and $\vec{\mathrm{v}}$) gives us equation of motion (5.21) in the Newtonian theory and, moreover, by varying with φ_{g}, $\vec{\mathrm{u}}$ and $\vec{A}_{\mathrm{g}}$ it leads to equations of gravitational and gravimagnetic fields (6.16a) – (6.16c). It looks very much like that the system of first two equations (6.16a) – (6.16b):

$$\begin{cases}\operatorname{div}\vec{g}=-4\pi\gamma\rho; & (8.6a)\\ \dfrac{\partial\vec{g}}{\partial t}=4\pi\gamma\rho\,\vec{\mathrm{u}} & (8.6b)\end{cases}$$

can quite correspond to Einstein's equation (8.2).

Nevertheless, as S. Weinberg pointed out [22], [38], due to Einstein sought a cosmological model satisfying requirements of homogeneous, isotropic, and static, he was forced (as Weinberg claimed) to "disfigure" his equations by introducing the cosmological constant λ which "spoiled the elegance of the original theory but could serve to balance the force of gravitational attraction at large distances" [38 (p.37)]. Moreover, L.D. Landau and E.M. Lifschitz [31] as well as several other scientists also held a similar negative opinion regarding the introduction of Einstein's cosmological constant. In particular, in [31 (p. 544)] it is indicated that "the introduction of a constant term into the density of the Lagrangian function, which is completely independent of the field state, would mean ascribing to space-time a fundamentally unremovable curvature that is connected neither matter nor gravitational waves".

However, in support of Einstein, it should be said that when creating GR, he was not guided at all by concern for the outward elegance of his theory, but rather by concern for consistency with other established theories (in particular, statistical mechanics [21]) and observational facts in astronomy (for example, the fact that the average mass density in the Coma system would have to be much higher than that deduced from observed visible matter [2], indicating the presence of dark matter). Indeed, modern knowledge indicates that "clusters of galaxies represent the highest-density peaks of the matter distribution in the Universe. Forming at the intersection of cosmic filaments, they grow hierarchically through continuous accretion of material. Composed of dark matter (DM; 85%), ionised hot gas in the intracluster medium (ICM; 12%), and stars (∼3%), their matter content reflects that of the Universe" [39]. Now astronomers have compared how dark matter models based on heavy and light particles (WIMPs and axions) cope with reproducing the features produced by gravitational lensing. In the paper [40], the study showed that the wave nature of axions, could explain various fluctuations and anomalies in the images: "Leading candidates for dark matter are weakly interacting massive particles or ultralight bosons (axions)…Quantum interference between dark matter axions is manifested as *waves* (ψDM)… The ability of ψDM to resolve lensing anomalies even in demanding cases such as HS 0810+2554, together with its success in reproducing other astrophysical observations, tilt the balance toward new physics invoking axions." In this regard, within the framework of the statistical theory of gravitating bodies formation, the presence of the ethereal medium (or the dark matter) around gravitationally interacting masses serves as the basis for gravitational interactions and gravitational wave propagation. In particular, an appearance of vorticity $\vec{\omega}(\vec{r},t)$ in ethereal substance determines the vector of gravimagnetic potential $\vec{A}_{\text{g}}$ as well as the vector of ether current $\vec{j}^{(e)}$ describing the vortex (wave) motion of ether.

Conclusion. As shown in this work, the presence of a dark matter (or an ethereal medium) around gravitationally interacting masses serves as the basis for their gravitational interactions and gravitational wave propagation. First, equation of motion of a "liquid" particle in the "quasi-solid" approximation inside a spheroidal body in gravitational and inertial fields is considered. Using the vector gravimagnetic potential introduced the equation of moving solid particle with an ethereal surrounding in gravitational and gravimagnetic fields is obtained. Taking into account this equation the first system of equations of the gravitational/gravimagnetic field is derived. Using both Lagrange function of a moving "liquid" gaseous–ethereal particle (with a "solid" core) in the gravitational/gravimagnetic field and Lagrange function of the same field, the expression of a total action for the gravitational/gravimagnetic field and matter is obtained. Applying the principle of least action and varying only the gravitational potentials together with the anti-diffusion velocity (but not the coordinates) the second system of equations of the gravitational/gravimagnetic field of a condensing spheroidal body is derived. The equations of plane gravitational waves are obtained. This paper shows that the Newtonian approximation (following from Einstein's GR) does not fully describe the gravitational field of gravitating masses while within the framework of the proposed statistical theory the equation of particle motion in an arbitrary non-inertial frame of reference is derived.

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