Branching rules for Weyl group orbits involving the Lie algebra $U(1)$

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Abstract. The orbits of Weyl groups $W(L)$ of all simple Lie algebras L are reduced to the union of orbits of the Weyl groups of maximal reductive non-semisimple subalgebras L' of L, whenever such subalgebras are present. Matrices transforming points of the orbits of $W(L)$ into points of $W(L')$ orbits are listed for the infinite series of algebra-subalgebra pairs $A_n \supset A_{n-k-1} \times A_k \times U(1)$, $B_n \supset B_{n-1} \times U(1)$, $C_n \supset A_{n-1} \times U(1)$, $D_n \supset A_{n-1} \times U(1)$, $D_n \supset D_{n-1} \times U(1)$, and for $E_6 \supset D_5 \times U(1)$ and $E_7 \supset E_6 \times U(1)$. Examples of branching rules are shown for all cases.

1. Introduction

Finite groups generated by reflections of a finite-dimensional Euclidean space are commonly known as finite Coxeter groups [1, 2]. They are of two types, crystallographic and non-crystallographic. Finite crystallographic Coxeter groups are the Weyl groups of semisimple Lie groups or Lie algebras. They are the symmetry groups of lattices of $\mathbb{R}^n$. They fall in four families – A_n, B_n, C_n, and D_n – with five exceptions E_6, E_7, E_8, F_4, and G_2. The index n, 6, 7, 8, 4 or 2, is the rank of the corresponding Lie algebra and the dimension of the finite Euclidean space in which the reflections take place.

We consider the orbits of Weyl groups $W(L)$ of semisimple Lie algebras L. An orbit of $W(L)$ is a finite set of points of $\mathbb{R}^n$ obtained from the action of $W(L)$ on a single point of $\mathbb{R}^n$. The number of points in the orbit is at most the order of the Weyl group. Since the reflections take place in mirrors that all go through the origin, all the points of an orbit are equidistant from the origin. Geometrically, points of an orbit can be seen as vertices of a polytope. A method has been developed in [3] to describe the faces of the polytopes using recursive decoration of the Dynkin diagrams.

Weyl group orbits are closely related to weight systems of finite-dimensional irreducible representations of semisimple Lie algebras. Indeed a weight system consists of many orbits of the corresponding Weyl group, a specific orbit often appearing more than once. Which orbits a particular representation is comprised of is well known, and extensive tables of multiplicities of dominant weights can be found in [4].

Reduction of weight systems of representations of simple Lie algebras to weight systems of representations of their maximal reductive subalgebras has been addressed several times in the literature [5, 6, 7, 8]. In physics that problem is often referred to as computation of branching rules. We consider here a related problem for individual orbits of Weyl groups. In

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both problems transformation of orbit points (weights in the case of weight systems) is involved. However, considering the reduction of individual orbits rather than of entire weight systems offers some advantages. Firstly, while the number weights of a weight system grows without limits with the dimension of the representation, the number of points of an individual orbit is at most the order of the corresponding Weyl group. When dealing with large-scale computation for representations, one often needs to break down the problem into smaller ones for individual orbits. Secondly, while weights of a weight system must be on a weight lattice of the Lie algebra, points of an individual orbit can be anywhere in $\mathbb{R}^n$. That allows orbits to be as close to one another as desired, as it is discussed in [9].

The list of possible reductions of orbits of $W(L)$ is the result of a major classification carried out more than sixty years ago, when the maximal reductive subalgebras of simple Lie algebras were determined [10]. We will only consider maximal reductive subalgebras that are not semisimple. Those are exactly the subalgebras that contain the one-parameter Lie algebra $U(1)$. We will discuss all the cases when such a subalgebra is present :

$$\begin{aligned}
&A_n \supset A_{n-1} \times U(1)\,, && n \geq 2\,, \\
&A_n \supset A_{n-k-1} \times A_k \times U(1)\,, && n \geq 3\,, \quad 1 \leq k \leq [\tfrac{n-1}{2}]\,, \\
&B_n \supset B_{n-1} \times U(1)\,, && n \geq 3\,, \\
&C_n \supset A_{n-1} \times U(1)\,, && n \geq 2\,, \\
&D_n \supset A_{n-1} \times U(1)\,, && n \geq 4\,, \\
&D_n \supset D_{n-1} \times U(1)\,, && n \geq 5\,, \\
&E_6 \supset D_5 \times U(1)\,, && \\
&E_7 \supset E_6 \times U(1)\,. &&
\end{aligned}$$

The branching rules for $W(L) \supset W(L')$, where L' is a maximal reductive non-semisimple subalgebra of L, is a linear transformation between Euclidean spaces $\mathbb{R}^n \to \mathbb{R}^m$, where n and m are the ranks of L and L' respectively. The branching rules are unique, unlike transformations of individual orbit points, which depend on the relative choice of bases. We provide the linear transformation in the form of an $m \times n$ matrix, the 'projection matrix'. A suitable choice of bases allows one to obtain integer matrix elements in all the projection matrices listed here. Note that we use Dynkin notations and numberings for roots, weights and diagrams.

The method we use to compute the branching rules of orbits of $W(L)$ is an extension of the one used in [5, 6, 7, 8] for branching rules of representations of Lie algebras. Orbit-orbit branching rules have been discussed for one of the first times in the literature in [8]. They were then addressed in [11, 12, 13], where specific methods were developed for different algebra-subalgebra pairs. The main advantage of the projection matrix method is its uniformity, as it can be used for any algebra-subalgebra pair. The present paper can be viewed as a continuation of [14], where orbit-orbit branching rules are computed with the projection matrix method for orbits of $W(A_n)$.

2. The projection matrix method

Consider the pair $W(L) \supset W(L')$, where L' is a maximal reductive non-semisimple subalgebra of the simple Lie algebra L. The orbit reduction problem is solved when the $m \times n$ matrix P is found with the property that points of any orbit of the Weyl group $W(L)$ are projected by P into points of the corresponding orbits of $W(L')$. Computation of the branching rule for a specific orbit of $W(L)$ amounts to applying P to the points of the orbit, and to sorting out the projected points into a sum (union) of orbits of $W(L')$.

The projection matrix P is calculated from one known branching rule. The classification of maximal reductive subalgebras of simple Lie algebras [10] provides the information to find that branching rule. The projection matrix is then obtained using the weight systems of the representations, by requiring that weights of L be transformed by P to weights of L'. Since any ordering of the weights is admissible, the projection matrix is not unique. We choose the natural lexicographical ordering of the weights. The projection matrix obtained can then be used to project points of any orbit of $W(L)$ into points of orbits of $W(L')$. An example of the construction of a projection matrix is presented in [14] for the case $W(A_3) \supset W(C_2)$.

To compute the branching rule for a specific orbit of $W(L)$, one has to list all the points of that orbit and to multiply them by the projection matrix. A standard method to calculate points of an orbit of any finite Coxeter group is given in [9], where the points are given in the corresponding basis of fundamental weights, the so-called ω-basis. All the orbits appearing here will be given in that ω-basis, linked to the basis of simple roots by the Cartan matrix of the group. Since every orbit contains precisely one point with nonnegative coordinates in the ω-basis, we can identify the orbit by that point, called the dominant point of the orbit. Hence when referring to an orbit, one does not have to list all the points it contains.

The Weyl group of the one-parameter Lie algebra $U(1)$ is trivial, consisting of the identity element only. Its irreducible representations are all 1-dimensional, hence its orbits consist of one element. They are labeled by integers, which can also take negative values. The symbol (k) may stand for either the orbit $\{k, -k\}$ of $W(A_1)$, or for the $W(U(1))$ orbit of one point $\{k\}$. Distinction should be made from the context. For example, the orbit $(p)(q)$, where $p \in \mathbb{Z}^{>0}$, $q \in \mathbb{Z}$, of $W(A_1 \times U(1))$, has two elements, $\{(p)(q), (-p)(q)\}$. Since we are working with orbits of the Weyl group of $U(1)$ and the compactness of the Lie group is of no interest to us here, we can allow the orbits of $W(U(1))$ to take real values.

Reductions in this paper are all of the form $W(L) \supset W(L'') \times U(1)$, where the ranks of L and L'' are n and $n-1$ respectively. Hence reductions will all be to orbits of the same dimension. Geometrically, the orbit points are not displaced in this case; rather, they are relabeled by the coordinates given in the standard basis of the subgroup.

To alleviate notation, we will simply write L instead of $W(L)$ to refer to the Weyl group of the Lie algebra L, and λ instead of W_λ to refer to the orbit of the dominant point λ of the Weyl group W.

Many examples of projection matrices and branching rules can be found in [14] for the lowest dimensional cases for orbits of the Weyl group of A_n. We give here an example for orbits of the Weyl group of B_3.

Example. *Consider the case* $B_3 \supset B_2 \times U(1)$. *The projection matrix* P *for that case is constructed using the known branching rule* $(1,0,0) \supset (1,0)(0) + (0,0)(2) + (0,0)(-2)$, *and we obtain the* 3×3 *matrix*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} . \qquad (1)$$

Let us illustrate the actual computation of branching rules for the B_3 *orbit* $(0,1,0)$ *containing 12 points. We write the coordinates of the points (given in the* ω*-basis of* B_3*) as column vectors:*

$$\begin{gathered} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}. \end{gathered} \qquad (2)$$

Multiplying each of the points of (2) *by the matrix* (1), *one gets the points of the* $B_2 \times U(1)$ *orbits written as column vectors. Rewriting them in the horizontal form and keeping in mind that the first two coordinates belong to* B_2 *orbits, the third one belonging to orbits of* $U(1)$, *we have the set of projected points. It remains to distribute the points into individual orbits. Practically it* *suffices to select just the dominant ones (no negative coordinates, except for the orbits of U(1)) because they represent the orbits that are present. The result is given by* (3)*, for* $b = 1$*.*

We give the branching rules for orbits of B_3 *of size 6, 12, 8 and 48 respectively:*

$$\begin{aligned}(a,0,0) &\supset (a,0)(0) + (0,0)(2a) + (0,0)(-2a)\,,\\ (0,b,0) &\supset (0,2b)(0) + (b,0)(2b) + (b,0)(-2b)\,,\\ (0,0,c) &\supset (0,c)(c) + (0,c)(-c)\,,\\ (a,b,c) &\supset (a,2b{+}c)(c) + (a{+}b,c)(2b{+}c)\\ &\quad + (a,2b{+}c)(-c) + (b,c)(2a{+}2b{+}c)\\ &\quad + (a{+}b,c)(-2b{-}c) + (b,c)(-2a{-}2b{-}c)\,,\end{aligned} \tag{3}$$

where $a, b, c \in \mathbb{R}^{>0}$*.*

3. Infinite series of classical cases

For each case involving a classical Lie algebra and the Lie algebra $U(1)$, we give the projection matrix for a general rank n and list branching rules for 3 different orbits of $\mathbb{R}^n$. The orbit of the highest weight of the adjoint representation of the classical Lie algebra is included as one of the special cases.

3.1. $A_n \supset A_{n-1} \times U(1), \quad n \geq 2$

The projection matrix is

$$\left(\begin{array}{ccccccc|c} & & & I_{n-1} & & & & \mathbf{0} \\ \hline 1 & 2 & 3 & \dots & n{-}2 & n{-}1 & & n \end{array}\right).$$

Note that, here and everywhere below, I_k denotes the $k \times k$ identity matrix and $\mathbf{0}$ represents the zero matrix of appropriate dimensions.

We give branching rules for this case for orbits of A_n of order $n{+}1$, $(n^2{+}n)/2$ and $n^2{+}n$ respectively:

$$\begin{aligned}(a,0,0,\dots,0) &\supset (a,0,0,\dots,0)(a) + (0,\dots,0)(-na)\,,\\ (0,b,0,\dots,0) &\supset (0,b,0,\dots,0)(2b) + (b,0,0,\dots,0)(-(n{-}1)b)\,,\\ (c,0,\dots,0,d) &\supset (c,0,0,\dots,0)(c{+}nd) + (c,0,0,\dots,0,d)(c{-}d)\\ &\quad + (0,0,\dots,0,d)(-nc{-}d)\,.\end{aligned}$$

Note that, here and everywhere below, $a, b, c, d \in \mathbb{R}^{>0}$.

3.2. $A_n \supset A_{n-k-1} \times A_k \times U(1), \quad 1 \leq k \leq [\frac{n-1}{2}]$

The previous case can be viewed as a special case of the present one where $k = 0$. Here we are considering the cases with general rank $n \geq 3$ and $1 \leq\ k \leq\ [\frac{n-1}{2}]$.

The projection matrix is

$$\left(\begin{array}{c|c} I_{n-k-1} & \mathbf{0} \\ \hline \mathbf{0} & I_k \\ \hline \underbrace{k{+}1\ \ 2(k{+}1) \cdots (n{-}k{-}1)(k{+}1)}_{n-k-1} \ \ \underbrace{(n{-}k)(k{+}1)}_{1} & \underbrace{k(n{-}k) \cdots 2(n{-}k)\ \ n{-}k}_{k} \end{array}\right) \begin{array}{l} \}\, n-k-1 \\ \}\, k \\ \}\, 1 \end{array}$$

We give branching rules for this case for orbits of A_n of order $n{+}1$, $(n^2{+}n)/2$ and $n^2{+}n$ respectively:

$$\begin{aligned} (a,0,0,\dots,0) &\supset (a,0,\dots,0)(0,\dots,0)((k{+}1)a) + (0,\dots,0)(a,0,\dots,0)((k{-}n)a)\,, \\ (0,b,0,\dots,0) &\supset (0,b,0,\dots,0)(0,\dots,0)(2(k{+}1)b) + (b,0,\dots,0)(b,0,\dots,0)((2k{+}1{-}n)b) \\ &\quad + (0,\dots,0)(0,b,0,\dots,0)(2(k{-}n)b)\,, \\ (c,0,\dots,0,d) &\supset (c,0,\dots,0)(0,\dots,0,d)((k{+}1)c{+}(n{-}k)d) \\ &\quad + (c,0,\dots,0,d)(0,\dots,0)((k{+}1)(c{-}d)) \\ &\quad + (0,\dots,0)(c,0,\dots,0,d)((n{-}k)(d{-}c)) \\ &\quad + (0,\dots,0,d)(c,0,\dots,0)((-k{-}1)d{+}(k{-}n)c)\,. \end{aligned}$$

3.3. $B_n \supset B_{n-1} \times U(1), \quad n \geq 3$

The projection matrix is

$$\left(\begin{array}{c|cc} I_{n-2} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \mathbf{0} & \begin{smallmatrix} 2 & 1 \\ \cdot & 1 \end{smallmatrix} \end{array}\right).$$

Note that, here and everywhere below, a dot in a matrix denotes a zero matrix element.

We give branching rules for this case for orbits of B_n of order $2n$, $2n(n{-}1)$ and 2^n respectively:

$$\begin{aligned} (a,0,0,\dots,0) &\supset (a,0,\dots,0)(0) + (0,\dots,0)(2a) + (0,\dots,0)(-2a)\,, \\ (0,b,0,\dots,0) &\supset (0,b,0,\dots,0)(0) + (b,0,\dots,0)(2b) + (b,0,\dots,0)(-2b)\,, \\ (0,0,\dots,0,c) &\supset (0,\dots,0,c)(c) + (0,\dots,0,c)(-c)\,. \end{aligned}$$

Note that in the case of B_2, the orbit $(0,b,0,\dots,0)$ is $(0,2b)$.

3.4. $C_{2n} \supset A_{2n-1} \times U(1), \quad n \geq 1$

The projection matrix is

$$\begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \vdots & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & 1 & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ & & & & & & \vdots & & & & & & \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & 1 & \cdot & & 1 & \cdot & 1 & \cdot & 1 & \cdot \end{pmatrix}.$$

We give branching rules for this case for orbits of C_{2n} of order $4n$, $4n(2n-1)$ and 2^{2n} respectively:

$$
\begin{aligned}
(a,0,0,\dots,0) &\supset (a,0,\dots,0)(a)+(0,\dots,0,a)(-a)\,,\\
(0,b,0,\dots,0) &\supset (b,0,\dots,0,b)(0)+(0,b,0,\dots,0)(2b)+(0,\dots,0,b,0)(-2b)\,,\\
(0,0,\dots,0,c) &\supset (\underbrace{0,\dots,0}_{n-1},2c,\underbrace{0,\dots,0}_{n-1})(0)+(\underbrace{0,\dots,0}_{n},2c,\underbrace{0,\dots,0}_{n-2})(2c)\\
&\quad+(\underbrace{0,\dots,0}_{n-2},2c,\underbrace{0,\dots,0}_{n})(-2c)+(\underbrace{0,\dots,0}_{n+1},2c,\underbrace{0,\dots,0}_{n-3})(4c)\\
&\quad+(\underbrace{0,\dots,0}_{n-3},2c,\underbrace{0,\dots,0}_{n+1})(-4c)+\dots+(0,\dots,0,2c)((2n-2)c)\\
&\quad+(2c,0,\dots,0)(-(2n-2)c)+(0,\dots,0)(2nc)+(0,\dots,0)(-2nc)\,.
\end{aligned}
$$

3.5. $C_{2n+1}\supset A_{2n}\times U(1),\quad n\geq 1$
The projection matrix is

$$
\begin{pmatrix}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
 & & & & & & \vdots & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot\\
 & & & & & & \vdots & & & & & & & \\
\cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
1 & \cdot & 1 & \cdot & 1 & \cdot & & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1
\end{pmatrix}.
$$

We give branching rules for this case for orbits of C_{2n+1} of order $2(2n+1)$, $4n(2n+1)$ and 2^{2n+1} respectively:

$$
\begin{aligned}
(a,0,0,\dots,0) &\supset (a,0,\dots,0)(a)+(0,\dots,0,a)(-a)\,,\\
(0,b,0,\dots,0) &\supset (b,0,\dots,0,b)(0)+(0,b,0,\dots,0)(2b)+(0,\dots,0,b,0)(-2b)\,,\\
(0,0,\dots,0,c) &\supset (\underbrace{0,\dots,0}_{n},2c,\underbrace{0,\dots,0}_{n-1})(c)+(\underbrace{0,\dots,0}_{n-1},2c,\underbrace{0,\dots,0}_{n})(-c)\\
&\quad+(\underbrace{0,\dots,0}_{n+1},2c,\underbrace{0,\dots,0}_{n-2})(3c)+(\underbrace{0,\dots,0}_{n-2},2c,\underbrace{0,\dots,0}_{n+1})(-3c)\\
&\quad+\dots+(0,\dots,0,2c)((2n-1)c)+(2c,0,\dots,0)(-(2n-1)c)\\
&\quad+(0,\dots,0)((2n+1)c)+(0,\dots,0)(-(2n+1)c)\,.
\end{aligned}
$$

3.6. $D_{2n}\supset A_{2n-1}\times U(1),\quad n\geq 2$
The projection matrix is

$$
\begin{pmatrix}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
 & & & & & & \vdots & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & 1 & 1 & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & 1\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 1 & 1 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot & \cdot\\
 & & & & & & \vdots & & & & & & \\
\cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
1 & \cdot & 1 & \cdot & 1 & \cdot & & 1 & \cdot & 1 & \cdot & 1 & \cdot
\end{pmatrix}.
$$

We give branching rules for this case for orbits of D_{2n} of order $4n$, $4n(2n{-}1)$ and 2^{2n-1} respectively:

$$
\begin{aligned}
(a,0,0,\dots,0) &\supset (a,0,\dots,0)(a) + (0,\dots,0,a)(-a)\,,\\
(0,b,0,\dots,0) &\supset (b,0,\dots,0,b)(0) + (0,b,0,\dots,0)(2b) + (0,\dots,0,b,0)(-2b)\,,
\end{aligned}
$$

$$
(0,0,\dots,0,c) \supset
\begin{cases}
(\underbrace{0,\dots,0}_{n-1},c,\underbrace{0,\dots,0}_{n-1})(0) + (\underbrace{0,\dots,0}_{n+1},c,\underbrace{0,\dots,0}_{n-3})(2c) \\
+(\underbrace{0,\dots,0}_{n-3},c,\underbrace{0,\dots,0}_{n+1})(-2c) + (\underbrace{0,\dots,0}_{n+3},c,\underbrace{0,\dots,0}_{n-5})(4c) \\
+(\underbrace{0,\dots,0}_{n-5},c,\underbrace{0,\dots,0}_{n+3})(-4c) + ... + (0,\dots,0,c,0)((n{-}2)c) \\
+(0,c,0,\dots,0)(-(n{-}2)c) + (0,\dots,0)(nc) + (0,\dots,0)(-nc) & n \text{ even} \\
\\
(\underbrace{0,\dots,0}_{n-1},c,\underbrace{0,\dots,0}_{n-1})(0) + (\underbrace{0,\dots,0}_{n+1},c,\underbrace{0,\dots,0}_{n-3})(2c) \\
+(\underbrace{0,\dots,0}_{n-3},c,\underbrace{0,\dots,0}_{n+1})(-2c) + (\underbrace{0,\dots,0}_{n+3},c,\underbrace{0,\dots,0}_{n-5})(4c) \\
+(\underbrace{0,\dots,0}_{n-5},c,\underbrace{0,\dots,0}_{n+3})(-4c) + ... + (0,\dots,0,c)((n{-}1)c) \\
+(c,0,\dots,0)(-(n{-}1)c) & n \text{ odd}
\end{cases}
$$

3.7. $D_{2n+1} \supset A_{2n} \times U(1), \quad n \geq 2$
The projection matrix is

$$
\left(\begin{array}{cccccccccccccc}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \vdots & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \vdots & & & & & & & \\
\cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & 2 & \cdot & 2 & \cdot & & 2 & \cdot & 2 & \cdot & 2 & -1 & 1
\end{array}\right).
$$

We give branching rules for this case for orbits of D_{2n+1} of order $2(2n{+}1)$, $4n(2n{+}1)$ and 2^{2n} respectively:

$$
\begin{aligned}
(a,0,0,\dots,0) &\supset (a,0,\dots,0)(2a) + (0,\dots,0,a)(-2a)\,,\\
(0,b,0,\dots,0) &\supset (b,0,\dots,0,b)(0) + (0,b,0,\dots,0)(4b) + (0,\dots,0,b,0)(-4b)\,,
\end{aligned}
$$

$$
(0,0,\dots,0,c) \supset
\begin{cases}
(\underbrace{0,\dots,0}_{n},c,\underbrace{0,\dots,0}_{n-1})(c) + (\underbrace{0,\dots,0}_{n-2},c,\underbrace{0,\dots,0}_{n+1})(-3c) \\
+(\underbrace{0,\dots,0}_{n+2},c,\underbrace{0,\dots,0}_{n-3})(5c) + (\underbrace{0,\dots,0}_{n-4},c,\underbrace{0,\dots,0}_{n+3})(-7c) \\
+... + (c,0,\dots,0)(-(2n{-}1)c) + (0,\dots,0)((2n{+}1)c) & n \text{ even} \\
\\
(\underbrace{0,\dots,0}_{n},c,\underbrace{0,\dots,0}_{n-1})(c) + (\underbrace{0,\dots,0}_{n-2},c,\underbrace{0,\dots,0}_{n+1})(-3c) \\
+(\underbrace{0,\dots,0}_{n+2},c,\underbrace{0,\dots,0}_{n-3})(5c) + (\underbrace{0,\dots,0}_{n-4},c,\underbrace{0,\dots,0}_{n+3})(-7c) \\
+... + (0,\dots,0,c)((2n{-}1)c) + (0,\dots,0)(-(2n{+}1)c) & n \text{ odd}
\end{cases}
$$

3.8. $D_n \supset D_{n-1} \times U(1), \quad n \geq 5$
The projection matrix is

$$\left(\begin{array}{c|c} I_{n-3} & \mathbf{0} \\ \hline \mathbf{0} & \begin{smallmatrix} 1 & \cdot & \cdot \\ 1 & 1 & 1 \\ \cdot & 1 & -1 \end{smallmatrix} \end{array}\right).$$

We give branching rules for this case for orbits of D_n of order $2n$, $2n(n-1)$ and 2^{n-1} respectively:

$$\begin{aligned}
(a,0,0,\dots,0) &\supset (a,0,\dots,0)(0) + (0,\dots,0)(2a) + (0,\dots,0)(-2a)\,,\\
(0,b,0,\dots,0) &\supset (0,b,0,\dots,0)(0) + (b,0,\dots,0)(2b) + (b,0,\dots,0)(-2b)\,,\\
(0,0,\dots,0,c) &\supset (0,\dots,0,c,0)(c) + (0,\dots,0,c)(-c)\,.
\end{aligned}$$

4. Exceptional cases

For each exceptional case involving the Lie algebra $U(1)$, we give the projection matrix and list branching rules for different orbits of $\mathbb{R}^n$. The orbit of the highest weight of the adjoint representation of the exceptional Lie algebra is included as one of the special cases.

4.1. $E_6 \supset D_5 \times U(1)$
The projection matrix is

$$\begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & -1 & \cdot \end{pmatrix}.$$

We give branching rules for this case for orbits of E_6 of order 27, 216, 720 and 72 respectively:

$$\begin{aligned}
(a,0,0,0,0,0) &\supset (0,0,0,0,a)(a) + (a,0,0,0,0)(-2a) + (0,0,0,0,0)(4a)\,,\\
(0,b,0,0,0,0) &\supset (b,0,0,0,b)(-b) + (0,0,b,0,0)(2b) + (0,b,0,0,0)(-4b)\\
&\quad + (0,0,0,0,b)(5b)\,,\\
(0,0,c,0,0,0) &\supset (c,0,c,0,0)(0) + (0,c,0,c,0)(3c) + (0,c,0,0,c)(-3c)\\
&\quad + (0,0,c,0,0)(-6c) + (0,0,c,0,0)(6c)\,,\\
(0,0,0,0,0,d) &\supset (0,d,0,0,0)(0) + (0,0,0,d,0)(3d) + (0,0,0,0,d)(-3d)\,.
\end{aligned}$$

Many more examples can be found in [8].

4.2. $E_7 \supset E_6 \times U(1)$
The projection matrix is

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \end{pmatrix}.$$

We give branching rules for this case for orbits of E_7 of order 126, 2016, 4032, 756, 56 and

576 respectively:

$$\begin{aligned}
(a,0,0,0,0,0,0) &\supset (0,0,0,0,0,a)(0) + (0,0,0,0,a,0)(2a) + (a,0,0,0,0,0)(-2a)\,,\\
(0,b,0,0,0,0,0) &\supset (0,0,b,0,0,0)(0) + (0,0,0,0,b,b)(2b) + (b,0,0,0,0,b)(-2b)\\
&\quad + (0,0,0,b,0,0)(4b) + (0,b,0,0,0,0)(-4b)\,,\\
(0,0,0,c,0,0,0) &\supset (0,c,0,0,c,0)(c) + (c,0,0,c,0,0)(-c) + (0,0,c,0,0,0)(3c)\\
&\quad + (0,0,c,0,0,0)(-3c) + (0,c,0,0,0,0)(5c) + (0,0,0,c,0,0)(-5c)\,,\\
(0,0,0,0,d,0,0) &\supset (d,0,0,0,d,0)(0) + (0,d,0,0,0,0)(2d) + (0,0,0,d,0,0)(-2d)\\
&\quad + (d,0,0,0,0,0)(4d) + (0,0,0,0,d,0)(-4d)\,,\\
(0,0,0,0,0,e,0) &\supset (e,0,0,0,0,0)(e) + (0,0,0,0,e,0)(-e) + (0,0,0,0,0,0)(3e)\\
&\quad + (0,0,0,0,0,0)(-3e)\,,\\
(0,0,0,0,0,0,f) &\supset (0,0,0,f,0,0)(f) + (0,f,0,0,0,0)(-f) + (0,0,0,0,0,f)(3f)\\
&\quad + (0,0,0,0,0,f)(-3f)\,,
\end{aligned}$$

where e and f are also in $\mathbb{R}^{>0}$. Many more examples can be found in [8].

5. Concluding remarks

- The pairs $W(L) \supset W(L')$ in this paper involve a maximal subalgebra L' in L. A chain of maximal subalgebras linking L and any of its reductive non-maximal subalgebras L'' can be found. Corresponding projection matrices combine, by the common matrix multiplication, into the projection matrix for $W(L) \supset W(L'')$.
- Projection matrices of $W(L) \supset W(L')$ when the ranks of L and L' are the same are square matrices with determinant different from zero. Hence they can be inverted and used in the opposite direction. The inverse matrix transforms an orbit of $W(L')$ into the linear combination of orbits of $W(L)$, where $L' \subset L$. The linear combination has integer coefficients of both signs in general. We know of no interpretation of such 'branching rules' in applied literature, although they have their place in the Grothendieck rings of representations.
- Curious and completely unexplored relations between pairs of maximal subalgebras, say L' and L'', of the same Lie algebra L can be found by combining the projection matrices $P(L \supset L')$ and $P(L \supset L'')$ as

$$P(L' \to L'') = P(L \supset L'')P^{-1}(L \supset L').$$

  Here L' must be of the same rank as L for $P(L \supset L')$ to be invertible. We write $L' \to L''$ instead of $L' \supset L''$ here because L'' is obviously not a subalgebra of L'.
- The index of a semisimple subalgebra in a simple Lie algebra is an invariant of all branching rules for a fixed algebra-subalgebra pair. It was introduced in [15], see Equation (2.26). It is also an invariant for any pair $W(L) \supset W(L')$.

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