XLI Brazilian Meeting on Nuclear Physics (RTFNB) IOP Publishing
IOP Conf. Series: Journal of Physics: Conf. Series **1291** (2019) 012012 doi:10.1088/1742-6596/1291/1/012012

Using pulsar's braking indices to estimate changes in their moments of inertia with age-related considerations

H O Oliveira [1], N S Magalhaes [1,2], R M Marinho, Jr [1], G A Carvalho [1,3,4] and C Frajuca [5],

[1]Graduate Program in Physics, Technological Institute of Aeronautics, Praça Marechal Eduardo Gomes 50, Sao Jose dos Campos, SP 12228-900, Brazil.

E-mail: heitoroliveiradeoliveira@gmail.com
[2]Federal University of Sao Paulo, Department of Physics, Rua Sao Nicolau 210, Diadema, SP 09913-030, Brazil.

E-mail: nadja.magalhaes@unifesp.br
[3]Dipartimento di Fisica and ICRA, Sapienza Università di Roma, P.le Aldo Moro 5, I–00185 Rome, Italy.
[4]ICRANet, P.zza della Repubblica 10, I–65122 Pescara, Italy.
[5]Federal Institute of Education, Science and Technology of Sao Paulo, R. Pedro Vicente 625, Sao Paulo, SP 01109-010, Brazil.

Abstract. Pulsars are modeled as neutron stars originated from the collapse of a progenitor one. In the canonical model they are described by spherical magnetized dipoles that rotate with the magnetic axis usually misaligned relative to the rotation axis, and such misalignment would explain the observation of radiation emitted in pulses in a certain direction rendering the typical observational characteristic of this kind of star. The frequency of such pulses decays with time and it can be quantified by the *braking index* (n). In the canonical model $n = 3$ for all pulsars but observational data show that $n \neq 3$. In this work we present a model for the understanding of the frequency decay of the rotation of a pulsar adapting the canonical one. We consider the pulsar a star that rotates in vacuum and has a strong magnetic field but, in contrast to the canonical model, we assume that its moment of inertia changes in time due to a uniform variation of a displacement parameter in time. We found that the old pulsars that present high values of the braking index tend to present smaller internal displacements of mass, in particular the superfluid neutron matter in the core. We relate this trend to neutron vortices' creep in rotating superfluids, indicating a possible reason for this coincidence.

1. Introduction

Pulsars are considered neutron stars that emit electromagnetic radiation in well-defined time intervals, rotate rapidly and are highly magnetized. The observed magnetic radiation is generated from its magnetosphere and is emitted due to the misalignment of the axis of rotation with respect to the magnetic axis of the star in the pattern from a rotating beacon [1].

The model to explain the frequency of the neutron stars' pulses [2], which we will call the canonical model, predicts a gradual deceleration of the rotation of these stars, quantified by a dimensionless parameter known as braking index, represented by "n". In that model this parameter has a theoretical value equal to 3 [3], but results derived from the observation are different from that predicted in the theoretical model, indicating that the canonical model needs improvement.

To this end the pulsar wind model [4] was presented in recent times yielding $n = 1$ such that, when combined with the contribution due to the magnetic dipole ($n = 3$), has presented interesting results although insufficient. A similar reasoning is followed in the work by [5].

In another investigation a phenomenological function was proposed in the energy conservation formula with the introduction of parameters which, although unrelated to known physical variables, allowed the prediction of ranges for braking indices [6]. A different approach by [7] proposed an increase in the angle between the magnetic moment and rotation axis as the cause of the evolution of the torque, while[8] investigated the possibility of an effective force acting on the star which varies with the first power of the tangential velocity of the crust of the pulsar.

In this work we focus on the internal dynamics of the pulsar: the interior of these stars present a significant amount of matter in the form of superfluid neutrons [9] and it is also expected that the coupling and decoupling of this matter alter the long-term dynamics that leads to the measurement of the braking index and therefore producing changes in the pure dipole model or deviations in the "normal" dipolar deceleration [10].

We present an approach to investigate the decay of the rotational frequency of pulsars that leads to the star's core and its evolution modifying an assumption of the canonical model. We introduce a time-varying parameter that accounts for an increasing moment of inertia in the pulsar core and relate it to the motion of superfluid vortices. Details on the results reported here can be found elsewhere, as a longer paper was accepted for publication during the preparation of this article [11].

2. Pulsar as a rotating magnetized conducting sphere: summary of the canonical model

For a rotating sphere the kinetic energy of rotation is equal to [12]:

$$E_{rot} = \frac{1}{2} I \Omega^2, \tag{1}$$

whose time derivative is

$$\dot{E}_{rot} = I \Omega \dot{\Omega}, \tag{2}$$

where Ω is the star's angular velocity and I is its moment of inertia.

The star'a magnetic radiation energy is believed to originate from the rotating magnetic dipole [12]

$$\dot{E}_{mr} = \frac{2}{3c^3} |\ddot{m}|^2, \tag{3}$$

with the magnetic dipole moment being given by

$$\vec{m} = \frac{B_P R^3}{2} (\cos\alpha \hat{k} + \sin\alpha \cos(\Omega \cdot t)\hat{\imath} + \sin\alpha \sin(\Omega \cdot t)\hat{\jmath}), \tag{4}$$

where B_P is the magnetic dipole field in the pole, R is the radius of the pulsar and α is the angle between the magnetic dipole axis and the rotation axis.

The angular velocity of the pulsar varies with time, as shown in Table 1. This could reflect on the behavior of the magnetic field with time since in this model it has the following expression when $\sin\alpha = 1$ [3]:

$$B_P = \sqrt{\frac{12c^3 M}{5R_0^4}} \sqrt{\frac{-\dot{\Omega}}{\Omega^3}}. \tag{5}$$

Table 1. Rotation frequency (ν) and its first and second time derivatives for the sample of pulsars.

Name	J name	ν (s^{-1})	$\dot{\nu}$ ($\times 10^{-10}$ s^{-2})	$\ddot{\nu}$ ($\times 10^{-21}$ s^{-3})	n	Refs.
B 0531+21	J0534+2200	29.946923	-3.77535	11.147	2.342(1)	[13]
B 0540-69	J0540-6919	19.7746860321	-1.8727175	3.772	2.13(1)	[14]
B0833-45	J0835-4510	11.200	-0.15375	0.036	1.7(2)	[15]
J1119-6127	J1119-6127	2.4512027814	-0.2415507	0.6389	2.684(2)[a]	[16]
J1208-6238	J1208-6238	2.26968010518	-1.6842733	0.33	2.598(1)	[17]
B1509-58	J1513-5908	6.611515243850	-0.6694371307	1.9185594	2.832(3)	[18]
J1734-3333	J1734-3333	0.855182765	-0.0166702	0.0028	0.9(2)	[19]
J1833-1034	J1833-1034	16.15935711336	-0.52751130	0.3197	1.857(1)	[20]
J1846-0258	J1846-0258	3.059040903	-0.665131	3.17	2.64(1)[b]	[21]

Notes. Besides these references, information regarding associations and most rotational parameters were taken from the ATNF Pulsar catalogue (http://www.atnf.csiro.au/research/pulsar/psrcat/; Manchester et al. 2005). Uncertainties (1σ) on the last quoted digit are shown between parentheses.
[a] A possible reduction of about 15% is observed after a large glitch [22].
[b] The braking index was found to decrease to $n = 2.19$ after a large glitch [23, 24].

In the canonical model, its rotating energy changes into electromagnetic energy:

$$\dot{E}_{rot} = -\dot{E}_{mr}, \tag{6}$$

implying

$$\dot{\Omega} = -k\Omega^3, \tag{7}$$

where k a positive constant.

The canonical model predicts a gradual slowdown of the star's rotation, which can be quantified by a dimensionless parameter, the the braking index n, defined by:

$$n \equiv \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2}. \tag{8}$$

In the canonical model this parameter assumes a single value ($n = 3$) for all pulsars. However, all the observational values for n are different from the one given by the canonical model (see table 1).

3. Inside a neutron star: summary of the dynamics of superfluid cores

3.1. Motion of neutron vortices

In order to characterize the flow type of a fluid it is important to analyze the circulation, defined as the line integral along a path C that surrounds the vortex circulation [25]:

$$\kappa = \oint_C \vec{v}_s \cdot d\vec{l}, \tag{9}$$

where v_s is the fluid velocity and dl is the line element along κ.

The polar decomposition ansatz for the condensate wave function allows us to describe the general shape of the Cooper pair [26], from which the velocity of the fluid can be found:

$$\psi(\vec{R}) = |\psi|e^{i\theta(\vec{R})}, \tag{10}$$

where $|\psi|$ is a thermodynamic state variable and the phase θ is a scalar.

Consequently the superfluid velocity is [27]

$$\vec{v_s} = \frac{\hbar}{2m_n}\vec{\nabla}\theta, \tag{11}$$

with $2m_n$ being the mass of a pair of neutrons and $\hbar$ being Planck's constant divided by 2π.

A velocity field described by the gradient of a function is called the potential flow, and it is found that the flow of a superfluid is irrotational:

$$\vec{\nabla} \times \vec{v_s} = \frac{\hbar}{2m_n}\vec{\nabla} \times (\vec{\nabla}\theta) = 0. \tag{12}$$

This indicates that the condensate cannot withstand a circulation except at certain points (singularities) within the fluid [26], generating an isolated configuration of singularities known as vortex lines, where the circulation does not need to disappear [27]. In this configuration with non-zero circulation the vortex lines carry angular momentum.

3.2. Dynamics of fluid rotation inside neutron stars

The angular velocity Ω of a rotating superfluid is determined by distribution of quantized vortex lines in relation to an azimuthal symmetry about the rotation axis. In this case the linear velocity

$$v_s(r) \equiv r\Omega(r) \tag{13}$$

at distance r from the rotation axis is determined from equation 9 as

$$\oint \vec{v} \cdot d\vec{l} = 2\pi\Omega(R_n)R_n^2 = \kappa_0 \int_0^{R_n} 2\pi r' n(r') dr', \tag{14}$$

being κ_0 the vorticity quantum carried by each vortex line.

In fluid mechanics the mass conservation analog for vortices is called the vortex conservation law [27]:

$$\frac{\partial n_v}{\partial t} + \vec{\nabla} \cdot (n_v v_R \hat{r}) = 0, \tag{15}$$

where n_v is the density of vortices and v_R is the radial velocity in relation to the rotation axis of the neutron star. When this law is associated to other superfluid properties [26, 28] one finds an expression for the brake in the superfluid rotation, $\dot{\Omega}_s$:

$$\dot{\Omega}_s = -\frac{\kappa_0 n_v v_R}{R_n}. \tag{16}$$

Therefore, when the superfluid rotation decays in time the vortices move outward with velocity v_R. One can show that [29] $n_v \kappa_0 = 2\Omega_s$, allowing the above equation to be rewritten as

$$\frac{\dot{\Omega}_s}{\Omega_s} = -\frac{2v_R}{R_n}. \tag{17}$$

It is expected that the star's core and crust reach the same angular velocity at large time scales [30].On such scales the magnetic torque about the crust is then transmitted to the core, implying $\dot{\Omega}_s \equiv \dot{\Omega}$. Therefore the equation 17 can be rewriten as

$$\frac{\dot{\Omega}}{\Omega} = -\frac{2v_R}{R_n}. \tag{18}$$

We assume that the crust of a pulsar is thin, with the core occupying approximately 80% of its radius, allowing for 1.4 $M_\odot$ stars with physical equations of state. Then for a total radius of 10 km the core's radius is $R_n \approx 8$ km.

In the next section we will use this physical mechanism and the radiation due to the rotating magnetic dipole to present a new model to v_R that allows the calculation of the radial velocity of the flow of the superfluid vortices without using vortex conservation hypotheses. In particular, this variation is of the order of neutron vortices creep in rotating superfluids.

4. Magnetic dipole radiation and the displacement parameter

The possibility of the stretching of a pulsar in response to rotation [31] inspired this investigation of the behavior of the braking index in the presence of a variation in time of a displacement parameter, which we will argue that is the radial velocity flow discussed in the previous section. This flow of vortices would promote the transfer of the angular momentum of the star's core generating the variation of the moment of inertia between the core and the crust of the neutron star. This assumption differs from the canonical models approach [32].

As in the canonical model we assume that the pulsar changes its rotational energy (E_{rot}) into electromagnetic dipole radiation (E_{mr}) as in equation 6 and that it consists of a thin, solid crust with *constant* moment of inertia I_c. However, differently from that model we will consider that its large spherical core with total constant mass, M_n, made basically of superfluid neutrons, has moment of inertia given by

$$I_n(t) \equiv \lambda M_n R^2(t). \tag{19}$$

In this expression R is a *displacement parameter* that summarizes in its mathematical behavior all physical factors that influence the moment of inertia other than the core's total mass. We assume that the core's moment of inertia may change with time but not due to a change in its total mass or physical radius; instead, any change in I_n will be due to *internal displacements* of mass that are quantified by $R(t)$.

We can perform a Taylor expansion of the displacement parameter and we will assume that this expansion can be truncated after its second term due to negligible higher R derivatives:

$$R(t) \approx R_n + t\dot{R}. \tag{20}$$

Physically this implies that the displacement parameter varies at a nearly constant rate in time, $\dot{R}$. This constant with units of speed is expected to vary from pulsar to pulsar as it informs about the inner dynamics of the star.

Differentiating the expression for the rotational energy, equation (1), with respect to time, in view of the assumptions of our model results in

$$\dot{E}_{rot} = \frac{1}{2}\frac{d}{dt}(I_c\Omega_c^2 + I_n\Omega_n^2). \tag{21}$$

Since any changes in the angular velocity of the crust, Ω_c, are rapidly transmitted to the core, in practice the angular velocity of the latter, Ω_n, will be considered equal to $\Omega_c \equiv \Omega$. As the moment of inertia of the thin crust is expected to be much less than the moment of inertia of the large, heavy core, we approximate $I_c + I_n \approx I_n$. Similarly, we consider M_n practically equal to the total mass of the pulsar, M. We will further admit that for the duration of typical observational time intervals, τ, the condition $\tau\dot{R} << R_n$ holds such that equation (20) yields the typical value for R:

$$R = R_n\left(1 + \tau\frac{\dot{R}}{R_n}\right) \Rightarrow R \approx R_n. \tag{22}$$

IOP Conf. Series: Journal of Physics: Conf. Series **1291** (2019) 012012 doi:10.1088/1742-6596/1291/1/012012

Finally, assuming a core that occupies the vast majority of the pulsar's volume we have $R_n \approx R_0$ and the expression for the rotation power becomes

$$\dot{E}_{rot} = \lambda M R_0^2 \Omega^2 \left(\frac{\dot{\Omega}}{\Omega} + \frac{\dot{R}}{R_0} \right). \tag{23}$$

The above assumptions regarding our model can be used to find the following expression for the magnetic radiation power from equations (3) and (4)

$$\dot{E}_{mr} = \frac{\sin^2 \alpha B_P^2 R_0^6 \Omega^4 + 24 \sin^2 \alpha B_P^2 R_0^4 \dot{R}^2 \Omega^2 + 36 B_P^2 R_0^2 \dot{R}^4}{6c^3}. \tag{24}$$

Therefore, we are again assuming that the quantity $\dot{R}$, which does not correspond physically to a change in the star's radius, describes mathematically all unknown physical influences that may affect the magnetic radiation power.

Substituting equations (23) and (24) in equation (6) yields

$$\begin{aligned} \lambda M R^2 \Omega^2 \left(\frac{\dot{\Omega}}{\Omega} + \frac{\dot{R}}{R} \right) &= -\frac{\sin^2 \alpha B_P^2 R^6 \Omega^4}{6c^3} \\ &- \frac{24 \sin^2 \alpha B_P^2 R^4 \dot{R}^2 \Omega^2 + 36 B_P^2 R^2 \dot{R}^4}{6c^3}, \end{aligned} \tag{25}$$

where we dropped the sub index 0 in R_0 such that R henceforth corresponds to the typical star radius.

Solving this equation for the time variation of the angular velocity, $\dot{\Omega}$, we can obtain the braking index n using the definition (8):

$$n = \frac{(3 \sin^2 \alpha B_P^2 R^5 \Omega^2)}{(12 \lambda c^3 \dot{R} M + \sin^2 \alpha B_P^2 R^5 \Omega^2)}. \tag{26}$$

Solving this equation for $\dot{R}$ we find the expression for the time variation of the displacement parameter:

$$\dot{R} = -\frac{\sin^2 \alpha B_P^2 (n-3) R^5 \Omega^2}{12 \lambda c^3 n M}. \tag{27}$$

We will estimate the values of $\dot{R}$ assuming the following typical values, applied to the pulsars given in Table 1: star radius R = 10 km; total mass M = 1.4 $M_\odot$ (where $M_\odot$ denotes one solar mass). These values imply a moment of inertia $I_0 = 2MR^2/5 = 56\ M_\odot\ km^2$ when the pulsars were born.

The expression for the magnetic field for pulsars in our model is biven by

$$B_P = \sqrt{\frac{6 \lambda c^3 M}{R^4 sin^2(\alpha)}} \sqrt{\frac{-\dot{\Omega}}{\Omega^3} - \frac{\dot{R}}{R \Omega^2}}. \tag{28}$$

In this expression, when $\dot{R} = 0$ and $\sin \alpha = 1$ the canonical expression (5) is recovered. The second term under the second square root of this equation, which has the contribution of the $\dot{R}$, will be negligible when

$$|\dot{R}| \ll |\frac{\dot{\Omega}}{\Omega} R|. \tag{29}$$

Table 2. Time variation of the radius ($\dot{R}$) and magnetic field at the pole according to our model (B_P) for our sample of pulsars.

Pulsar	$\dot{R}$ (cm s^{-1})	B_P (G)
B0531+21	1.2×10^{-6}	1.1×10^{13}
B0540-69	1.3×10^{-6}	1.4×10^{13}
B0833-45	3.0×10^{-7}	9.0×10^{12}
J1119-6127	4.4×10^{-7}	1.2×10^{14}
J1208-6238	3.8×10^{-7}	1.2×10^{14}
B1509-58	2.4×10^{-7}	4.8×10^{13}
J1734-3333	8.6×10^{-7}	1.1×10^{14}
J1833-1034	6.1×10^{-7}	9.8×10^{12}
J1846-0258	2.7×10^{-6}	1.4×10^{14}

Substituting (28) in (26) yields

$$n = 3\frac{\dot{\Omega}/\Omega + \dot{R}/R}{\dot{\Omega}/\Omega - \dot{R}/R}. \tag{30}$$

This equation can be inverted, yielding an expression for the variation in time of the displacement parameter:

$$\dot{R} = \frac{\dot{\Omega}}{\Omega} R \frac{n-3}{n+3}. \tag{31}$$

We used this equation to obtain the values of $\dot{R}$ presented in Table 2, which show that for this sample of pulsars the condition (29) is not completely fulfilled. The values of the magnetic field given by the canonical model have the same order of magnitude of the values obtained with our model from equation (28). Nevertheless, canonical values of the magnetic field should not be used in equations (26) and (27), as they would yield canonical results. The small difference between the values of the magnetic field in the two models is essential to yield observational braking indices.

4.1. When angular momentum is conserved

From the angular momentum definition, $L = I\Omega$, when angular momentum is conserved then $\dot{L} = \dot{I}\Omega + I\dot{\Omega} = 0$. In our case the moment inertia of the core (I_n) is changing in time and is much larger than the crust's moment of inertia (I_c, which is constant). Therefore, angular momentum conservation implies:

$$\dot{I}_n\Omega = -I_n\dot{\Omega}. \tag{32}$$

As I_n is given by equation 19 we can rewrite this equation as:

$$\frac{\dot{\Omega}}{\Omega} = \frac{-2\dot{R}}{R_n}, \tag{33}$$

which allows us to identify: $\dot{R} = v_R$.

5. Inferring age from the displacement parameter

A study that may in the future lead to an age calculation for pulsars with a nucleus of superfluid matter different from the characteristic age may be: to estimate the order of the displacement parameter according to the possible age of the pulsar. As there is an estimated rotation frequency at birth of the pulsar [15] it may be possible to calculate a pulsars age from the superfluid displacement parameter and the braking index. We will find the range of the displacement parameter according to the age of the pulsars between 0.5 and 100 thousand years.

From the equation 31 and of the characteristic age (t_c): $t_c = -\Omega/2\dot{\Omega}$, we can write:

$$\dot{R} = \frac{(3-n)}{(3+n)}\frac{R}{2t_c}, \tag{34}$$

from this equation we made the figures 1, 2 and 3.

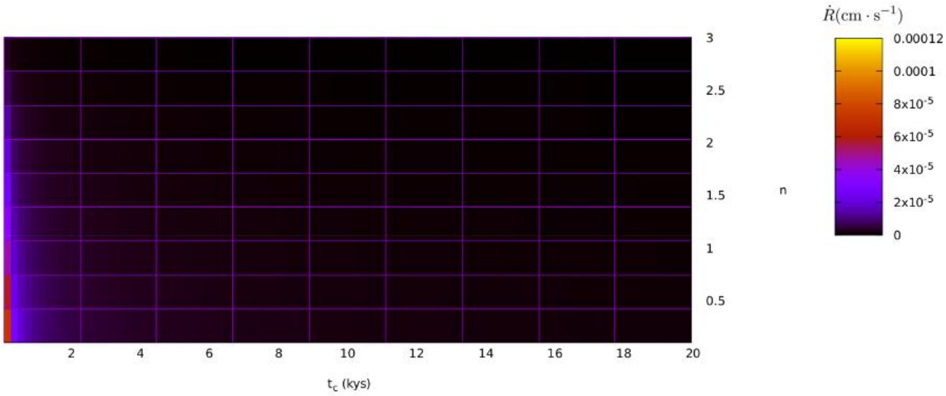

Figure 1. In the figure the displacement parameter is the axis represented by the right bar in cms^{-1}, while the braking index is represented by the vertical axis with "n" from 0 to 3, and the age is on the horizontal axis of 0.5 to 20 thousand years.

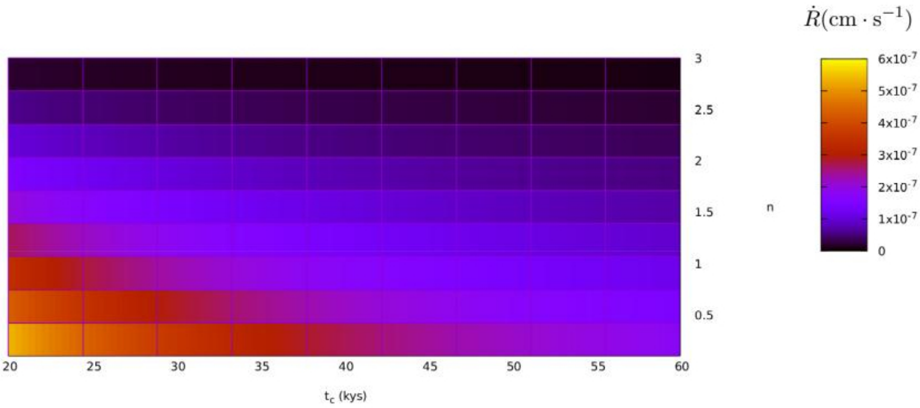

Figure 2. In the figure the displacement parameter is the axis represented by the right bar in cms^{-1}, while the braking index is represented by the vertical axis with "n" from 0 to 3, and the age is on the horizontal axis of 20 to 60 thousand years.

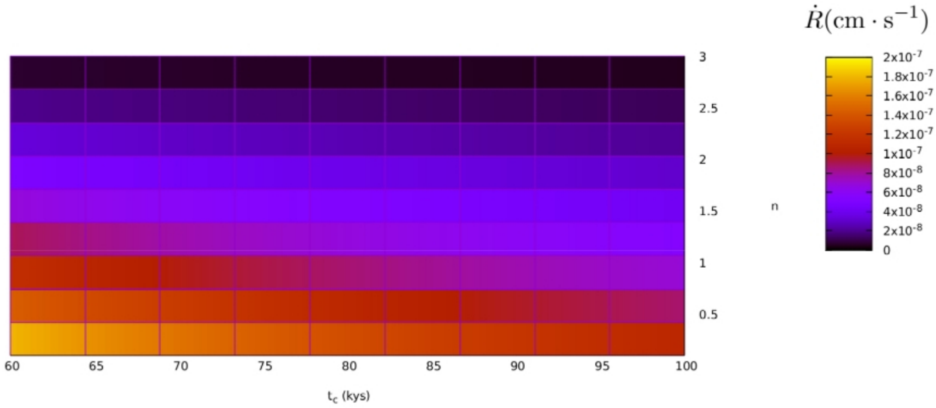

Figure 3. In the figure the displacement parameter is the axis represented by the right bar in cms^{-1}, while the braking index is represented by the vertical axis with "n" from 0 to 3, and the age is on the horizontal axis of 60 to 100 thousand years.

The figures represent the same but for different age ranges for better visualization. We observe from the figures that when the braking index approaches the value of 3 the variation of the parameter decreases tending to zero. We note that the greater age the values of the displacement parameter become lower. In addition to this trend note that the higher the age and the braking index the lower the displacement parameter values.

6. Conclusions

In this work we modified the canonical model for pulsars including changes in moment of inertia, expecting to provide a better explanation for pulsars' braking indices. The moment of inertia would change due to mass motions inside the star, quantified by a displacement parameter. We found that the displacement parameter relates to the velocity of superfluid neutron vortices when n=1. Our model assumes that the a time-varying moment of inertia changes uniformly in the radial direction which coincides with the direction of motion of neutron superfluid vortex lines. In this work we introduced the displacement parameter $\dot{R}$ and its estimates were found based on observational data.

We conjecture that the increase in moment of inertia in this model may be related to the dynamics of superfluid vortex lines in the pulsars core due to the coincidence between the estimated value for $\dot{R}$ and the approximate speed of travel of vortex lines in the core (less than the cm/day).

As consequences of this study other questions are unfolding, such as calculation of pulsars' ages and the relation between torque and the angle between magnetic moment and rotation axis, which are under investigation.

Acknowledgments

HOO thanks *Coordenação de Aperfeiçoamento de Pessoal de Nível Superior* (CAPES) for the financial support. NSM acknowledges the National Institute of Science and Technology in Astrophysics (INCT-A, Brazil; a joint CNPq and FAPESP project, FAPESP grant # 2008/57807-5) for support. GAC thanks CAPES for financial support under the process PDSE # 88881.188302/2018-01. CF thanks CPNQ for support, CNPQ grant # 309098/2017-3. The authors acknowledge FAPESP for support under the thematic project # 2013/26258-4.

References

[1] Gold T 1968 *Nature* **218** 731
[2] Gunn J and Ostriker J P 1969 *Nature* **221** 454
[3] Glendenning N K 2000 Compact stars: Nuclear physics, particle physics, and general relativity (New York: ed Springer)
[4] Xu R and Qiao G 2001 *Astrophys. J. Lett.* **561** L85
[5] Kou F and Tong H 2015 *Mon. Not. R. Astron. Soc.* **450** 1990
[6] Magalhaes N S *et al.* 2012 *Astrophys. J.* **755** 54
[7] Allen M P and Horvath J E 1997 *Mon. Not. R. Astron. Soc.* **287** 615
[8] Magalhaes N S *et al.* 2016 *Mon. Not. R. Astron. Soc.* **461** 3993
[9] Baym G *et al.* 1969 *Nature* **224** 673
[10] Ho W C G and Andersson N 2012 *Nature Phys.* **8** 787
[11] de Oliveira H O *et al.* 2018 *JCAP* 11 025. See also arXiv:1807.11474v3 [astro-ph.HE] 13 Aug 2018
[12] Shapiro S L and Teukolsky S A 2008 Black holes, white dwarfs and neutron neutron stars: the physics of compact objects, (New York: ed John Wiley & Sons)
[13] Lyne A *et al.* 2015 *Mon. Not. R. Astron. Soc.* **446** 857
[14] Ferdman R D *et al.* 2015 *Astrophys. J.* **812** 95
[15] Espinoza C M *et al.* 2016 *Mon. Not. R. Astron. Soc.* **466** 147
[16] Weltevrede P *et al.* 2011 *Mon. Not. R. Astron. Soc.* **411** 1917
[17] Clark C J *et al.* 2016 *Astrophys. J. Lett.* **832** L15
[18] Livingstone M A and Kaspi V M 2011 *Astrophys. J.* **742** 31
[19] Espinoza C M *et al.* 2011 *Astrophys. J.* **741** L13
[20] Roy J *et a.* 2012 *Mon. Not. R. Astron. Soc.* **424** 2213
[21] Livingstone M A *et al.* 2007 *Astrophys. Space Sci.* **308** 317
[22] Antonopoulou D *et al.* 2015 *Mon. Not. R. Astron. Soc.* **447** 3924
[23] Livingstone M A *et al.* 2011 *Astrophys. J.* **730** 66
[24] Archibald R F *et al.* 2015 *Astrophys. J.* **810** 67
[25] Landau L and Lifshitz E 1980 Course of theoretical physics Vol. 9: Statistical physics (Oxford: ed Pergamon Press)
[26] Sauls J A 1989 Superfluidity in the interiors of neutron stars In: Timing neutron stars (Dordrecht: ed Springer)
[27] Ghosh P 2007 Rotation and accretion powered pulsars, (Singapore: ed World Scientific)
[28] Alpar M A *et al.* 1984 *Astrophys. J.* **276** 325
[29] Hall H E 1960 *Adv. Phys.* **9** 89
[30] Paschalidis V and Stergioulas N 2017 *Living Rev. Relativ.* **20** 7
[31] Hartle J B Thorne K S 1968 *Astrophys. J.* **153** 807
[32] Ostriker J P and Gunn J 1969 *Astrophys. J.* **157** 1395