



Characterization of Bianchi Type-I Space-Time Coupled with Bulk Viscosity and Magnetic field in Bimetric Theory

Yanlin Li¹ · Sunil Kumar Yadav² · Uday Chand De³ · Krishnendu De⁴

Received: 12 December 2025 / Accepted: 24 March 2026
© The Author(s) 2026

Abstract

The paper's main goal is to investigate the impact of h -almost conformal η -Ricci-Bourguignon soliton in Bianchi type-I space-time coupled with bulk viscosity and magnetic field in Rosen's bimetric theory of gravitation. Additionally, we demonstrate that some specific physical properties of a Bianchi type-I space-time that permit the inclusion of bulk viscosity and magnetic field in Rosen's bimetric theory with a conformal vector field, where the metric satisfies h -almost conformal η -Ricci-Bourguignon soliton. Moreover, we illustrate some physical relevance of conformal pressure \tilde{p} in terms of h -almost conformal η -Ricci-Bourguignon soliton in Rosen's bimetric theory. Within this ongoing work, using such solitons, we analyze the various energy conditions, some black holes criteria, and Penrose's singularity theorem in Bianchi type-I space-time coupled with bulk viscosity and magnetic field in Rosen's bimetric theory of gravitation. We further investigate the generalized Liouville and Poisson equations associated with the h -almost conformal η -Ricci-Bourguignon soliton on a Bianchi type-I space-time. Finally, in the context of Rosen's bimetric theory attached with bulk viscosity and magnetic field, we explore the harmonic aspects of h -almost conformal η -Ricci-Bourguignon soliton on such a space-time.

Keywords Bimetric theory · Bulk viscous · Magnetic field · Bianchi type-I space-time · h -almost conformal η -Ricci-Bourguignon solitons

Y. Li, S. K. Yadav, U. C. De, and K. De are contributed equally to this work.

Extended author information available on the last page of the article

1 Introduction

The theory of gravity made up of two metrics is known as Rosen's bimetric theory [1]. First, the gravitational potential is described by the fundamental metric \tilde{g}_{ij} , and the flat space-time is expressed by the second metric \tilde{d}_{ij} , which defines the inertial forces related to the acceleration of the frame of reference. The Riemannian geometry of the curved space-time is determined by the metric tensor \tilde{g}_{ij} , which interacts with matter and performs a similar function as stated in Einstein's general relativity. \tilde{d}_{ij} is a background metric that describes the inertial forces and the geometry of the empty cosmos, which has no substance other than gravity. Although it occurs in the field equations, the metric tensor \tilde{d}_{ij} has no direct physical meaning. As a result, it interacts with \tilde{g}_{ij} but not with matter directly. Without matter one would get $\tilde{g}_{ij} = \tilde{d}_{ij}$. Additionally, the bimetric theory supports general relativity by satisfying the covariance and equivalency principles. For more details see ([2–4]). Therefore, at every point of space-time, there are two metrics given by

$$ds^2 = \tilde{g}_{ij} dx^i dx^j, \quad d\eta^2 = \tilde{d}_{ij} dx^i dx^j. \quad (1)$$

According to Rosen[1], the field equations in bimetric theory of gravity is written by

$$E_i^j = F_i^j - \frac{1}{2} F \delta_i^j = -8\pi\kappa T_i^j, \quad (2)$$

where $F_i^j = \frac{1}{2} \tilde{d}^{pr} (\tilde{g}^{sj} \tilde{g}_{si|p})_{|r}$, $F = F_i^i$, $\kappa = \sqrt{\frac{g}{d}}$, $g = \det(\tilde{g}_{ij})$, $d = \det(\tilde{d}_{ij})$, T_i^j is the energy momentum tensor of matter fields and symbol $(|)$ stands for d -covariant derivative.

The underlying metric tensor \tilde{d}_{ij} in bimetric theory should be selected based on cosmological considerations rather than being interpreted as depicting an empty universe. According to Rosen's proposal, the metric \tilde{d}_{ij} may therefore be interpreted as the metric tensor of the universe where the perfect cosmological principle is true. Thus, a space-time with constant curvature is described by \tilde{d}_{ij} .

In this sequel, several researchers have examined Bianchi type I and V string cosmological models under various physical circumstances in general relativity (see, [5–7]). An electric current generated along the x -axis is the cause of the magnetic field. The Bianchi Type I bulk viscous fluid string dust magnetized cosmological model in general relativity was examined in [8]. Furthermore, a number of cosmologists created the most practical Bianchi type string models in general relativity as well as models in the field of bulk viscous fluid solutions [9].

For stars smaller than neutron stars, the degenerate pressure of quark matter has been used to support the idea of a quark star or a compact star. According to [10] and [11], few neutron stars could be exotic stars composed completely of strange materials. The characteristics of weird quark stars were examined by Cheng et al. [12]. Quarks are seen in this concept as a degenerate Fermi gas that is confined to the

spatial region that the vacuum energy density occupies. Electrons, massless quarks (u), heavy quarks (s), and quarks (d) make up the quark content in this model.

As per bag model [13], quarks are non interacting and massless. Quark pressure is written by

$$3p_Q = \rho_Q, \quad (3)$$

where p_Q , ρ_Q are the pressure and energy density of the quark. Also, the total pressure p_M and total energy density ρ_M are given by [13]

$$\rho_M = \rho_Q + \chi_B, \quad p_M = p_Q - \chi_B, \quad (4)$$

here χ_B is the vacuum energy density. Finally, quark matter is produced using the equation of state (briefly, EoS) [10]

$$p_M = \frac{1}{3}[\rho_M - 4\chi_B]. \quad (5)$$

The stability of strange quark matter is improved when the magnetic field exceeds a certain critical intensity, which uses a phenomenological bag model to study quark matter in a strong magnetic field. The existence of a magnetic field causes pressure anisotropy, as literature references [14] point out. According to [15], the bag model is thought to be the best method for studying magnetized strange quark matter.

In this work, instead of the particles in the string cloud, we concentrate on quarks. So from [16], we have

$$\rho = \rho_Q + \beta + \chi_B, \quad (6)$$

where χ_B vacuum energy density.

In the context of space-times, they are specifically referred to as perfect fluid space-times [17, 18]. The (0, 2) type Ricci tensor Ric is given by:

$$\text{Ric} = \gamma_1 \tilde{g} + \gamma_2 u \otimes u, \quad (7)$$

where γ_1 and γ_2 are scalars, and u is a 1-form corresponding to the velocity vector ξ , such that $u(\xi) = -1$, that is, the velocity vector ξ is a unit time-like vector.

A non-flat Riemannian or pseudo-Riemannian manifold $(\widetilde{M}^n, \tilde{g})$, where $n > 2$, is named a generalized quasi-Einstein manifold (briefly, $G(\text{QE})_n$) if its non-zero Ricci tensor obeys [19]:

$$\text{Ric} = \gamma_1 g + \gamma_2 u \otimes u + \gamma_3 \theta \otimes \theta, \quad (8)$$

where γ_1 , γ_2 , and γ_3 are non-zero scalars and u and θ are two non-zero 1-forms defined by:

$$g(Y_1, \xi) = u(Y_1), \quad g(Y_1, \zeta) = \theta(Y_1),$$

for all vector fields $Y_1 \in \mathfrak{X}(\widetilde{M}^n, \widetilde{g})$. ξ and ζ , are metrically equivalent to the 1-forms η and θ and $g(\xi, \zeta) = 0$.

Hamilton developed the idea of a Ricci flow in the early 1980s after being inspired by Eells and Sampson's research on harmonic map heat flow, which is provided by [20]

$$\frac{\partial}{\partial t} \widetilde{g}_{ij} = -2\mathcal{R}ic. \quad (9)$$

A Ricci soliton, is defined on a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ by

$$\mathcal{R}ic + \frac{1}{2} \mathfrak{L}_{Y_1} \widetilde{g} = \pi_1 \widetilde{g}, \quad (10)$$

where \mathfrak{L}_{Y_1} indicates the Lie derivative, π_1 stands for a real constant and $\mathcal{R}ic$ denotes the Ricci tensor of $(\widetilde{M}, \widetilde{g})$. If $Y_1 = \nabla \psi$ for any function ψ on $(\widetilde{M}, \widetilde{g})$, the Ricci soliton becomes a gradient Ricci soliton. A soliton is called shrinking, steady or expanding when $\pi_1 > 0$, $\pi_1 = 0$ or $\pi_1 < 0$, respectively. According to [21] the idea of conformal Ricci soliton is described by

$$\mathcal{R}ic + \frac{1}{2} \mathfrak{L}_{Y_1} \widetilde{g} + [\pi_1 - \frac{1}{2}(\widetilde{p} + \frac{2}{n})] \widetilde{g} = 0, \quad (11)$$

where \widetilde{p} being the scalar non-dynamical field (time dependent scalar field), and π_1 is a constant.

For metrics on $(\widetilde{M}, \widetilde{g})$, the Ricci-Bourguignon flow is an evolution equation provided by [22].

$$\frac{\partial}{\partial t} \widetilde{g}_{ij} = -2(\mathcal{R}ic - \tau \omega \widetilde{g}_{ij}), \quad (12)$$

where $\omega \in \mathbb{R}$ denotes a constant and τ stands for the scalar curvature of the Riemannian metric \widetilde{g}_{ij} . As per [23], for special values of ω , equation 12 has different types of solitons as

(i) $\omega = \frac{1}{2}$, the Einstein soliton, that is $\mathcal{R}ic - \frac{\tau}{2} \widetilde{g}_{ij}$, (ii) $\omega = \frac{1}{n}$, the traceless Ricci soliton, that is, $\mathcal{R}ic - \frac{\tau}{n} \widetilde{g}_{ij}$, (iii) $\omega = \frac{1}{2(n-1)}$, the Schouten soliton, that is, $\mathcal{R}ic - \frac{\tau}{2(n-1)} \widetilde{g}_{ij}$, (iv) $\omega = 0$, the Ricci soliton.

A Riemannian or semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is named a Ricci-Bourguignon soliton if a smooth vector field Y_1 exists fulfilling the equation [23]

$$\mathcal{R}ic + \frac{1}{2}\mathfrak{L}_{Y_1}\tilde{g} = (\pi_1 + \tau\omega)\tilde{g}, \quad (13)$$

in which π_1 is a real constant. If $Y_1 = \nabla\Psi$ for some function Ψ on (\tilde{M}, \tilde{g}) , then \tilde{g} is named a gradient Ricci-Bourguignon soliton and the equation (13) turns up

$$\mathcal{R}ic + \nabla\nabla\Psi = (\pi_1 + \tau\omega)\tilde{g}. \quad (14)$$

So, combining the equations (11) and (13), we provide:

A metric \tilde{g}_{ij} on (\tilde{M}, \tilde{g}) is called an h -almost conformal Ricci-Bourguignon soliton, if there exists a smooth vector field Y_1 satisfies

$$\mathcal{R}ic + \frac{h}{2}\mathfrak{L}_{Y_1}\tilde{g} = [\pi_1 - \frac{1}{2}(\tilde{p} + \frac{2}{n}) + \tau\omega]\tilde{g}_{ij}, \quad (15)$$

In particular, if $Y_1 = \nabla\Psi$ for some function Ψ on (\tilde{M}, \tilde{g}) , then equation (15) is called an h -almost gradient conformal Ricci-Bourguignon soliton and defined as

$$\mathcal{R}ic + h\nabla\nabla\Psi = [\pi_1 - \frac{1}{2}(\tilde{p} + \frac{2}{n}) + \tau\omega]\tilde{g}_{ij}. \quad (16)$$

Likewise (15) and (16) we recall the notion of an h -almost conformal η -Ricci-Bourguignon soliton and gradient h -almost conformal η -Ricci-Bourguignon soliton on (\tilde{M}, \tilde{g}) as

$$\mathcal{R}ic + \frac{h}{2}\mathfrak{L}_{Y_1}\tilde{g} = [\pi_1 - \frac{1}{2}(\tilde{p} + \frac{2}{n}) + \tau\omega]\tilde{g}_{ij} + \pi_2 u \otimes u, \quad (17)$$

$$\mathcal{R}ic + h\nabla\nabla\Psi = [\pi_1 - \frac{1}{2}(\tilde{p} + \frac{2}{n}) + \tau\omega]\tilde{g}_{ij} + \pi_2 u \otimes u, \quad (18)$$

respectively, in which h and π_1 are smooth functions, π_2 indicates a real constant and u denotes a 1-form.

To address this situation, a number of mathematicians and physicists have investigated the geometrical and physical features of space-time in terms of solitons and other objects (see, [24–33]). Thus inspired by the above studies and explorations of results, we analyze the properties of cosmic string in Bianchi type-I space-time coupled with viscosity and magnetic field in bimetric relativity if the fundamental metric are h -almost conformal Ricci-Bourguignon soliton and h -almost gradient conformal Ricci-Bourguignon soliton.

2 Metric and Field Equations

Bianchi type-I space-time is considered here and its metric is given by

$$ds^2 = U^2 dx^2 + V^2 dy^2 + W^2 dz^2 - dt^2, \quad (19)$$

where U , V and W are solely functions of t . Thus the flat metric corresponding to (19) is defined as

$$d\eta^2 = dx^2 + dy^2 + dz^2 - dt^2. \quad (20)$$

The energy momentum tensor \mathcal{T}_i^j for a cloud of string with bulk viscosity and magnetic field is [34]:

$$\mathcal{T}_i^j = \rho u_i u^j - \beta \theta_i \theta^j - \mathcal{B}_v \Gamma (g_i^j + u_i u^j) + \mathbb{H}_i^j, \quad (21)$$

which satisfy the standard relation

$$u_i u^i = -\theta_i \theta^i = -1, \quad u^i \theta_i = 0, \quad (22)$$

where $\rho = \rho_0 + \beta$ is the energy density for a cloud of strings, ρ_0 is the rest particle density, β is the tension of cloud strings, $\Gamma = u_{,l}^l$ is the scalar expansion, θ^i is the direction of strings and \mathcal{B}_v is the coefficient of bulk viscosity.

According to Lichnerowicz [35] the electromagnetic field \mathbb{H}_{ij} is given by

$$\mathbb{H}_{ij} = \varrho \left[h^2 (u_i u_j + \frac{1}{2} g_{ij}) - \theta_i \theta_j \right], \quad (23)$$

where u_i is four velocity vector such that $g_{ij} u^i u^j = -1$ and ϱ is the magnetic permeability and the flux vector θ_i is written by

$$\theta_i = \frac{\sqrt{-g}}{2\psi} \mathcal{E}_{ijkl} \mathbb{K}^{kl} u^j, \quad (24)$$

where \mathbb{K}_{kl} denotes the electromagnetic field tensor and \mathcal{E}_{ijkl} indicates the Levi Civita tensor density.

Using (23) in (21), we yield

$$\mathcal{T}_{ij} = (\alpha_0 - \mathcal{B}_v \Gamma + \varrho h^2) u_i u_j + \left(\frac{\varrho h^2}{2} - \mathcal{B}_v \Gamma \right) g_{ij} - (\beta + \varrho) \theta_i \theta_j. \quad (25)$$

In view of (2), and (25), the Ricci tensor in bimetric theory of gravitation has the form

$$Ric_{ij} = \gamma_1 g_{ij} + \gamma_2 u_i u_j + \gamma_3 \theta_i \theta_j, \quad (26)$$

where $\gamma_1 = 4\pi\kappa(\mathcal{R} + \varrho h^2 - 2\mathcal{B}_v \Gamma)$, $\gamma_2 = 8\pi\kappa(\rho_0 - \mathcal{B}_v \Gamma + \varrho h^2)$, $\gamma_3 = -8\pi\kappa(\beta + \varrho)$.

This means that Bianchi Type-I space-time coupled with bulk viscosity and magnetic field in bimetric relativity is a $G(QE)_4$ -space-time.

Theorem 1 *A Bianchi-I space-time coupled with bulk viscosity and magnetic field in bimetric relativity is a $G(QE)_4$ -space-time.*

On contracting (26), we get

$$\mathcal{R} = \frac{-8\pi\kappa}{(1-16\pi\kappa)} [\rho_0 + \beta + \varrho - \varrho h^2 + 3\mathcal{B}_v\Gamma]. \quad (27)$$

Corollary 1 *The scalar curvature on a Bianchi-I space-time coupled with bulk viscosity and magnetic field in bimetric relativity is written by*

$$\mathcal{R} = \frac{-8\pi\kappa}{(1-16\pi\kappa)} [\rho_0 + \beta + \varrho - \varrho h^2 + 3\mathcal{B}_v\Gamma].$$

As per the result state in [36] and combining the Theorem 1 and Corollary 1, we reflect the following

Corollary 2 *A Bianchi-I space-time with bulk viscosity and magnetic field in bimetric relativity is a bulk viscous fluid space-time.*

Also, from (27), one can calculate the coefficient of bulk viscosity \mathcal{B}_v as

$$\mathcal{B}_v = \frac{1}{3\Gamma} \left[\frac{\mathcal{R}(1-16\pi\kappa)}{-8\pi\kappa} - \rho_0 - \varrho - \beta + \varrho h^2 \right]. \quad (28)$$

So, we state the following result

Corollary 3 *The coefficient of bulk viscosity on a Bianchi-I space-time coupled with bulk and magnetic field in bimetric relativity is determined by*

$$\mathcal{B}_v = \frac{1}{3\Gamma} \left[\frac{\mathcal{R}(1-16\pi\kappa)}{-8\pi\kappa} - \rho_0 - \varrho - \beta + \varrho h^2 \right].$$

Again, in light of (27) the value of scalar expansion Γ and tension of cloud string β is given by

$$\Gamma = \frac{1}{3\mathcal{B}_v} \left[\frac{\mathcal{R}(1-16\pi\kappa)}{-8\pi\kappa} - \rho_0 - \varrho - \beta + \varrho h^2 \right], \quad (29)$$

$$\beta = \left[\varrho h^2 - 3\mathcal{B}_v\Gamma - \rho_0 - \varrho - \frac{\mathcal{R}(1-16\pi\kappa)}{8\pi\kappa} \right]. \quad (30)$$

Thus we write:

Corollary 4 *The scalar expansion and the tension of cloud string on a Bianchi-I space-time coupled with bulk and magnetic field in in the Rosen's bimetric theory are determined by*

$$\Gamma = \frac{1}{3\mathcal{B}_v} \left[\frac{\mathcal{R}(1 - 16\pi\kappa)}{-8\pi\kappa} - \rho_0 - \varrho - \beta + \varrho h^2 \right],$$

$$\beta = \left[\varrho h^2 - 3\mathcal{B}_v\Gamma - \rho_0 - \varrho - \frac{\mathcal{R}(1 - 16\pi\kappa)}{8\pi\kappa} \right]$$

respectively.

Moreover, from (27), (4) and (6), we can find that

Corollary 5 *In the bag model, if Bianchi-I space-time along with bulk viscosity and magnetic field obeys Rosen's bimetric theory, then the quark energy density ρ_Q and quark pressure p_Q are given by*

$$\rho_Q = \left[-\frac{\mathcal{R}(1 - 16\pi\kappa)}{8\pi\kappa} - (\beta + \chi_B + \varrho - \varrho h^2 + 3\mathcal{B}_v\Gamma) \right],$$

$$p_Q = \left[-\frac{\mathcal{R}(1 - 16\pi\kappa)}{24\pi\kappa} - \frac{1}{3}(\beta + \chi_B + \varrho - \varrho h^2 + 3\mathcal{B}_v\Gamma) \right],$$

respectively.

Referring to equation (4) and the Corollary 5, we yield the next result

Corollary 6 *In the bag model, if Bianchi-I space-time along with bulk viscosity and magnetic field obeys Rosen's bimetric theory then the total energy density ρ_M and total pressure p_M are determined by*

$$\rho_M = \left[-\frac{\mathcal{R}(1 - 16\pi\kappa)}{8\pi\kappa} - (\beta + \varrho - \varrho h^2 + \mathcal{B}_v\Gamma) \right],$$

$$p_M = \left[-\frac{\mathcal{R}(1 - 16\pi\kappa)}{24\pi\kappa} - (\beta + 2\chi_B + \varrho - \varrho h^2 + \mathcal{B}_v\Gamma) \right]$$

respectively.

Now, using (5) and the Corollary 6, we yield the EoS

$$p_M = \left[-\frac{\mathcal{R}(1 - 16\pi\kappa)}{24\pi\kappa} - \frac{1}{3}(\beta + 2\chi_B + \varrho - \varrho h^2 + \mathcal{B}_v\Gamma + 4\chi_B) \right]. \quad (31)$$

Corollary 7 *If the matter of a Bianchi-I space-time in bimetric relativity subjected to bulk viscosity and magnetic field, then EoS is given by (31).*

According to [37], $p_M = \rho_M + \xi(r)$ is the EoS for dark energy, where t is the cosmic time and $\xi(r)$ being a function of the scale factor r and proved that $\omega < -1$ and $\omega > -1$ represent a shift from phantom to non-phantom, where as $\omega = \frac{p_M}{\rho_M} = -1$ yields a phantom barrier.

Thus, for phantom barrier, $\rho_M = -p_M = \left[\frac{\mathcal{R}(1-16\pi\kappa)}{24\pi\kappa} + (\beta + 2\chi_B + \varrho - \varrho h^2 + \mathcal{B}_v\Gamma) \right]$. Therefore, we infer

Corollary 8 *If source of matter is a phantom barrier type which is used in a Bianchi-I space-time coupled with bulk viscosity and magnetic field obeys Rosen’s bimetric theory, then the total energy density ρ_M and total pressure p_M are calculated as*

$$\rho_M = -p_M = \left[\frac{\mathcal{R}(1 - 16\pi\kappa)}{24\pi\kappa} + (\beta + 2\chi_B + \varrho - \varrho h^2 + \mathcal{B}_v\Gamma) \right]. \tag{32}$$

3 *h*-Almost Conformal η -Ricci-Bourguignon Soliton

In this section, we will investigate *h*-almost conformal η -Ricci-Bourguignon soliton of a Bianchi type-I space-time coupled with bulk viscosity and magnetic field whose conformal vector field ξ is the time-like velocity vector field. Thus we will recall the useful definition to prove the results.

According to [38], a conformal vector field Y_1 on $(\widetilde{M}^4, \widetilde{g})$ is defined by

$$\mathfrak{L}_{Y_1}\widetilde{g} = 2\Pi \widetilde{g}, \tag{33}$$

where Π is a smooth function on \widetilde{M}^4 . The vector Z_1 is Killing if $\Pi = 0$ (briefly, KVF), and homothetic when Π is constant. So, for fix $Y_1 = \xi$, equation (17) become

$$Ric(D_1, D_2) + \frac{h}{2}\mathfrak{L}_\xi\widetilde{g}(D_1, D_2) = [\pi_1 - \frac{1}{2}(\widetilde{p} + \frac{2}{n}) + \tau\omega]\widetilde{g}(D_1, D_2) + \pi_2u(D_1)u(D_2). \tag{34}$$

In view of (33), we gain

$$Ric(D_1, D_2) = [\pi_1 - \frac{1}{2}(\widetilde{p} + \frac{2}{n}) + \tau\omega - h\Pi]\widetilde{g}(D_1, D_2) + \pi_2u(D_1)u(D_2). \tag{35}$$

Putting (26) in the above equation, we get

$$[\gamma_1 - \pi_1 + \frac{1}{2}(\widetilde{p} + \frac{1}{2}) - \tau\omega + h\Pi]\widetilde{g}(D_1, D_2) + (\gamma_2 - \pi_2)u(D_1)u(D_2) + \gamma_3\theta \otimes \theta = 0. \tag{36}$$

Now putting $D_1 = D_2 = \xi$ in (36), we have

$$\pi_1 = \gamma_1 + \frac{1}{2}(\tilde{p} + \frac{1}{2}) - \tau\omega + h\Pi - \gamma_2 + \pi_2. \quad (37)$$

So, referring to equations (7) and (35), we state the next outcome as

Theorem 2 *A Bianchi-I space-time coupled with bulk viscosity and magnetic field admitting an h -almost conformal η -Ricci-Bourguignon soliton with conformal vector field ξ in Rosen's bimetric theory is a perfect fluid spacetime.*

Again, in view of (37) we can acquire the result:

Theorem 3 *If a Bianchi-I space-time coupled with bulk viscosity and magnetic field admits an h -almost conformal η -Ricci-Bourguignon soliton with conformal vector field ξ in Rosen's bimetric theory, then the solitons is expanding, steady or shrinking as*

- (i) $\frac{1}{2}(\tilde{p} + \frac{1}{2}) > \tau\omega - h\Pi - \gamma_1 + \gamma_2 - \pi_2$,
- (ii) $\frac{1}{2}(\tilde{p} + \frac{1}{2}) = \tau\omega - h\Pi - \gamma_1 + \gamma_2 - \pi_2$, and
- (iii) $\frac{1}{2}(\tilde{p} + \frac{1}{2}) < \tau\omega - h\Pi - \gamma_1 + \gamma_2 - \pi_2$, respectively.

Corollary 9 *If a Bianchi-I space-time coupled with bulk viscosity and magnetic field admits an h -almost conformal η -Ricci-Bourguignon soliton with KVF ξ in Rosen's bimetric theory, then the solitons is expanding, steady or shrinking as*

- (i) $\frac{1}{2}(\tilde{p} + \frac{1}{2}) > \tau\omega - \gamma_1 + \gamma_2 - \pi_2$,
- (ii) $\frac{1}{2}(\tilde{p} + \frac{1}{2}) = \tau\omega - \gamma_1 + \gamma_2 - \pi_2$, and
- (iii) $\frac{1}{2}(\tilde{p} + \frac{1}{2}) < \tau\omega - \gamma_1 + \gamma_2 - \pi_2$, respectively.

3.1 Physical Significance of Conformal Pressure

In fluid mechanics, the real physical pressure is what maintains the fluids' incompressibility, and the time-dependent scalar field \tilde{p} described in equation (11) is referred to as the conformal pressure. According to [21], outside of an equilibrium point the conformal pressure \tilde{p} is negative and zero inside. Thus from (37), one can compute the value of conformal pressure. Therefore we state the findings:

Theorem 4 *If a Bianchi-I space-time coupled with bulk viscosity and magnetic field admits an h -almost conformal η -Ricci-Bourguignon soliton such that ξ is a conformal vector field in Rosen's bimetric theory, then the conformal pressure is*

$$\tilde{p} = 2[\tau\omega - h\Pi - \gamma_1 + \gamma_2 + \pi_1 - \pi_2 - \frac{1}{4}].$$

Since inside the equilibrium point $\tilde{p} = 0$, we reflect the outcome

Corollary 10 *If a Bianchi-I space-time attached with bulk viscosity and magnetic field admits an h -almost conformal η -Ricci-Bourguignon soliton such that ξ is a conformal vector field in Rosen's bimetric theory, then the metric \tilde{g} is an equilibrium point or Einstein, if and only if*

$$4\pi\kappa(2\rho_0 + \rho h^2 - \mathcal{R}) = h\Pi - \tau\omega + \pi_2 - \pi_1 + \frac{1}{4}.$$

Additionally, the metric \tilde{g} provides a nonlinear restoring force and is an equilibrium point or Einstein [39]. Then we entail the following result regarding the dynamical system.

Corollary 11 *Let a Bianchi-I space-time attached with bulk viscosity and magnetic field admits an h -almost conformal η -Ricci-Bourguignon soliton in Rosen's bimetric theory. If the unit time-like vector field ξ is a conformal vector field, then the metric \tilde{g} is an equilibrium point and provides a nonlinear restoring force.*

4 Energy Conditions in Bianchi-I Space-Time

In this section, with reference to [40], let the Ricci tensor $\mathcal{R}ic$ in the Bianchi-I space-time obey the restriction

$$\mathcal{R}ic(\xi, \xi) > 0, \quad (38)$$

where $\xi \in \chi(\tilde{M}^4)$. In such a case (38) is called the time-like convergence condition (briefly, TCC) and a spacetime fulfills the strong energy condition (briefly, SEC) if $\mathcal{R}ic(\xi, \xi) \geq 0$ holds for all time-like vectors ξ . The geometric criterion $\mathcal{T}(\xi, \xi) > 0$ is said to be the null convergence condition (briefly, NCC) where as $\mathcal{T}(\xi, \xi) \geq 0$ is called the null energy condition (briefly, NEC), where \mathcal{T} is the energy momentum tensor. Physically, TCC means that the matter content of the spacetime produces attractive gravity. SEC physically signifies that gravity is always attractive, causing spacetime to curve inwards and nearby geodesics converge, rather than diverge. NCC is a fundamental assumption in general relativity. It is a physical energy condition that dictates how gravity behaves toward light-like trajectories. The NEC is a fundamental constraint in general relativity stating that the energy density of matter, measured by any observer moving at the speed of light, is non-negative. It ensures that light rays focus due to gravity, preventing exotic phenomena like traversable wormholes.

From (35), it gives

$$\mathcal{R}ic(\xi, \xi) = -\pi_1 + \frac{1}{2}\left(\tilde{p} + \frac{1}{2}\right) - \tau\omega + h\Pi + \pi_2. \quad (39)$$

The Bianchi-I space-time coupled with bulk viscosity and magnetic field admits an h -almost conformal Ricci-Bourguignon soliton with a conformal vector field ξ obey the TCC, if $\mathcal{R}ic(\xi, \xi) > 0$. So, from (39), we yield

$$\pi_1 < \frac{1}{2}(\tilde{p} + \frac{1}{2}) - \tau\omega + h\Pi + \pi_2, \quad (40)$$

In view of (37), equation (40) implies that

$$\mathcal{R} < \varrho h^2 + 2\rho_0. \quad (41)$$

Thus from (40), we state that

Theorem 5 *If a Bianchi-I space-time coupled with bulk viscosity and magnetic field admits an h -almost conformal η -Ricci-Bourguignon soliton with a conformal vector field ξ and satisfies TCC, then soliton is always shrinking.*

According to Hawking and Ellis [41], in cosmology the TCC implies the SEC, $NEC \Rightarrow NCC$, SEC infers the NEC. Consequently, $TCC \Rightarrow NCC$ as well. Thus together with this fact and Theorem 5 we reflect that

Theorem 6 *If a Bianchi-I space-time in Rosen's bimetric theory coupled with bulk viscosity and magnetic field admits a shrinking h -almost conformal η -Ricci-Bourguignon soliton with a conformal vector field ξ , if (40) holds then the space-time satisfies SEC and NCC.*

Remark 1 *If a Bianchi-I space-time in Rosen's bimetric theory coupled with bulk viscosity and magnetic field admits an h -almost conformal η -Ricci-Bourguignon soliton with a conformal vector field ξ and fulfills SEC, then the Ricci tensor $\mathcal{R}ic$ is of the second Segre type [42].*

4.1 Singularity Theorem Admitting an h -Almost Conformal η -Ricci-Bourguignon Soliton

Vilenkin and Wall [43] have demonstrated that the space-time obeys NCC, signifying the existence of black holes and a trapped surface beyond these black holes within the space-time, using Penrose's singularity theorem. Utilizing above facts and Theorem 5, Theorem 6, we conclude the upcoming result.

Theorem 7 *If a Bianchi-I space-time in Rosen's bimetric theory coupled with bulk viscosity and magnetic field admits an h -almost conformal η -Ricci-Bourguignon soliton with a conformal vector field ξ and satisfies NCC, then the space-time of bulk viscosity and magnetic field include black holes with a trapped surface, which is outside the black holes in the Rosen's bimetric theory.*

Remark 2 The above theorem also holds if we replace the conformal vector field ξ with a Killing vector field.

Remark 3 A conformal vector field is a mathematical generator of local transformation that preserve the conformal structure (angles, shapes) of a manifold. Physically, it represents symmetries that preserve shapes, such as dilations (scaling) and special conformal transformations. They generate transformations that maps Light cones to light cones in spacetime preserving the causal structure but not necessarily the specific distance.

Remark 4 A Killing vector field represents a direction of symmetry in a spacetime or manifold, where the metric tensor remains invariant. Physically, it acts as a generator of isometries, meaning moving along these fields (orbits) preserves distances and angles. They correspond to conserved quantities like energy or momentum for particles moving along geodesics.

5 Generalized Liouville Equation on Bianchi-I Space-Time

Let a Bianchi-I space-time attached with bulk viscosity and magnetic field admit an h -almost conformal η -Ricci-Bourguignon soliton. Then from (17) and (26), we get

$$\begin{aligned}
 &[\gamma_1 - \pi_1 + \frac{1}{2}(\tilde{p} + \frac{1}{2}) - \tau\omega]\tilde{g}(D_1, D_2) + (\gamma_2 + \pi_2)u(D_1)u(D_2) \\
 &+ \gamma_3\theta(D_1)\theta(D_2) + \frac{h}{2}[\tilde{g}(\nabla_{D_1}\xi, D_2) + \tilde{g}(D_1, \nabla_{D_2}\xi)],
 \end{aligned}
 \tag{42}$$

for any $D_1, D_2 \in \chi(\tilde{M}^4)$.

On contracting (42), and using the value of γ_1, γ_2 and γ_3 , we gain

$$Div(\xi) = 4\pi_1 + \pi_2 + 4\tau\omega - 2\tilde{p} - 1 + 8\pi\kappa[\rho_0 + 3B_v\Pi - \rho(h^2 - 1) - 2\mathcal{R}\beta] \tag{43}$$

Now, for a smooth function $\psi^* \in C^\infty(M^4, g)$ and a vector field D_1 , we have

$$Div(\psi^*D_1) = v(d\psi^*) + \psi^*Div(D_1), \tag{44}$$

the function ψ^* is referred to as last Lagrange multiplier of the vector field D_1 corresponding to \tilde{g} if $div(\psi^*D_1) = 0$. So the concerning equation gives

$$v(d \log \psi^*) = -Div(D_1), \tag{45}$$

is known as generalized Liouville equation of the vector D_1 with respect to \tilde{g} [44]. Therefore, from (43) and (45) we state

Theorem 8 *Let Bianchi-I space-time attached with bulk viscosity and magnetic field in Rosen’s bimetric theory admitting a h -almost conformal η -Ricci-Bourguignon soli-*

ton with a unit time-like vector field ξ and ψ^* is the last multiplier of ξ , and let u be the \tilde{g} -dual 1-form of the vector field ξ , then the generalized Liouville equation is given by

$$\xi(d \log \psi^*) = -4\pi_1 - \pi_2 - 4\tau\omega + 2\tilde{p} + 1 - 8\pi\kappa[\rho_0 + 3\mathcal{B}_v\Pi - \varrho(h^2 - 1) - 2\mathcal{R}\beta].$$

Remark 5 The Liouville equation physically signifies that the phase-space density of system points remains constant along their trajectories, describing an incompressible flow of probability density in Phase space. It represents the conservation of probability and states that a "cloud" of particles in phase space moves like an incompressible fluid, preserving its volume, even if its shape change.

If the vector field ξ is Killing or incompressible, then from (43), we have a result

Corollary 12 *In the above theorem, if the field ξ is Killing or incompressible, then the soliton is expanding, steady, and shrinking as*

(i)

$$\left(\frac{(2\tilde{p} + 1)}{4} \right) > \left[\frac{1}{4}(\pi_2 + 4\tau\omega) + 2\pi\kappa(\rho_0 + 3\mathcal{B}_v\Pi - \varrho(h^2 - 1) - 2\mathcal{R}\beta) \right],$$

(ii)

$$\left(\frac{(2\tilde{p} + 1)}{4} \right) = \left[\frac{1}{4}(\pi_2 + 4\tau\omega) + 2\pi\kappa(\rho_0 + 3\mathcal{B}_v\Pi - \varrho(h^2 - 1) - 2\mathcal{R}\beta) \right],$$

(iii)

$$\left(\frac{(2\tilde{p} + 1)}{4} \right) < \left[\frac{1}{4}(\pi_2 + 4\tau\omega) + 2\pi\kappa(\rho_0 + 3\mathcal{B}_v\Pi - \varrho(h^2 - 1) - 2\mathcal{R}\beta) \right].$$

Remark 6 Incompressible vector field has zero divergence everywhere, meaning it has no sources or sinks. Physically this represents a flow where the volume or density of the fluid element remains constant, as the amount of fluid flowing into any region exactly equals the amount flowing out.

Remark 7 Steady solitons represent a "steady state" in the evolution of geometry, analogous to a standing wave that does not disperse or dissipate over time. The most well known examples include Hamilton's cigar soliton in two dimensions and the 3D steady solitons are often described having "cigar-paraboloid" asymptotes, essentially acting as finite-width, permanent structures at infinity. Expanding solitons represent a space that is stretching or growing under the flow, similar to an expanding universe or wave that spreads out while maintaining its relative profile. Both types of solitons serve as singularity models for the flow. Shrinking solitons evolve only by rescaling

(shrinking) and diffeomorphisms, acting as self-similar models for infinite singularities. They represent shapes that become smaller and more intensely curves, ultimately vanishing at a singular point. Flow models, the time-evolution of space, serving as a key tool to study initial data sets, black hole entropy and potential approaches to quantum gravity by providing a rigorous framework to handle singularities.

6 Harmonic Characteristics of h -Almost Conformal η -Ricci-Bourguignon Soliton

Here, we demonstrate h -almost conformal η -Ricci-Bourguignon soliton on Bianchi-I space-time coupled with bulk viscosity and magnetic field under some specific condition if the \tilde{g} -gual form ξ , the 1-form u is a harmonic or Schrödinger-Ricci harmonic.

Let u be a \tilde{g} -dual 1-form of ξ such that $\tilde{g}(D_1, \xi) = u(D_1)$ and $\tilde{g}(\xi, \xi) = -1$, then ξ is named a solution of the Schrödinger-Ricci equation if it obeys

$$\text{Div}(\mathcal{L}_\xi \tilde{g}) = 0, \quad (46)$$

where $(\mathcal{L}_\xi \tilde{g})$ is Lie derivative for the vector field ξ . According to [45], the Lie derivative's divergence is written by

$$\text{Div}(\mathcal{L}_\xi \tilde{g}) = (\Theta + \mathcal{R}ic)(\xi) + d(\text{Div}(\xi)), \quad (47)$$

where Θ is the Laplace-Hodge operator with respect to the metric \tilde{g} and $\mathcal{R}ic$ is the Ricci curvature tensor field. So, from (17) we have

$$\mathcal{R}ic + \frac{h}{2} \mathcal{L}_\xi \tilde{g} - [\pi_1 - \frac{1}{2}(\tilde{p} + \frac{2}{n}) + \tau\omega] \tilde{g} - \pi_2 u \otimes u = 0. \quad (48)$$

Taking trace of (48), we get

$$h \text{Div}(\xi) + \mathcal{R} - [4\pi_1 - 2\tilde{p} - 1 + 4\tau\omega] - \pi_2 |\xi|^2 = 0. \quad (49)$$

By direct calculation, we have

$$\text{Div}(u \otimes u) = \text{Div}(\xi)u + \nabla_\xi u. \quad (50)$$

After taking the divergence of (48) and making use of (50), we acquire

$$h \text{Div}(\mathcal{L}_\xi \tilde{g}) + (1 + \omega)d(\mathcal{R}) - 2\pi_2[\text{Div}(\xi)u + \nabla_\xi u] = 0. \quad (51)$$

If the 1-form u is a solution of the Schrödinger-Ricci equation, then

$$(\Theta + \mathcal{R}ic)(u) + d(\text{Div}(u)) = 0.$$

Thus, we state:

Theorem 9 Let $(\tilde{g}, \xi, \pi_1, \pi_2)$ be an h -almost conformal η -Ricci-Bourguignon soliton in a Bianchi-I space-time attached with bulk viscosity and magnetic field where u the \tilde{g} -dual of the vector field ξ . Then u is the solution of Schrödinger-Ricci equation if and only if

$$d(\mathcal{R}) = \frac{2\pi_2}{(1+\omega)h} [\{(4\omega - 1)\mathcal{R} + 4\pi_1 - (2\tilde{p} + 1) - \pi_2\}u + h\nabla_\xi u]. \quad (52)$$

Proof With the help of equations (46),(47), (49),(51) and (27) it follows that u is a solution of the Schrödinger-Ricci equation if and only if equation (13) holds.

Also we turn up the following

Corollary 13 Let $(\tilde{g}, \xi, \pi_1, \pi_2)$ be an Einstein soliton, u -traceless soliton, u -Schouten soliton or a Ricci soliton in a Bianchi-I space-time attached with bulk viscosity and magnetic field where u the \tilde{g} -dual of the vector field ξ . Then u is the solution of Schrödinger-Ricci equation if and only if

$$d(\mathcal{R}) = \frac{4\pi_2}{3h} [\{\mathcal{R} + 4\pi_1 - (2\tilde{p} + 1) - \pi_2\}u + h\nabla_\xi u],$$

$$d(\mathcal{R}) = \frac{10\pi_2}{4h} [\{4\pi_1 - (2\tilde{p} + 1) - \pi_2\}u + h\nabla_\xi u],$$

$$d(\mathcal{R}) = \frac{14\pi_2}{6h} [\{-3\mathcal{R} + 4\pi_1 - (2\tilde{p} + 1) - \pi_2\}u + h\nabla_\xi u]$$

or

$$d(\mathcal{R}) = \frac{2\pi_2}{h} [\{-\mathcal{R} + 4\pi_1 - (2\tilde{p} + 1) - \pi_2\}u + h\nabla_\xi u],$$

respectively.

Finally, for Schrödinger-Ricci harmonic form, let the 1-form u be a Schrödinger-Ricci harmonic form if[46]

$$(\Gamma + Ric)(u) = 0.$$

Beside this if $\pi_2=0$, which yield the h -almost conformal Ricci-Bourguignon soliton, or

$$h\nabla_\xi u = -\{(4\omega - 1)\mathcal{R} + 4\pi_1 - (2\tilde{p} + 1)\}u.$$

Hence, we state the next result as

Theorem 10 Let $(\tilde{g}, \xi, \pi_1, \pi_2)$ be an h -almost conformal η -Ricci-Bourguignon soliton in a Bianchi-I space-time attached with bulk viscosity and magnetic field where u is the solution of Schrödinger-Ricci equation if and only $\pi_2=0$, which produce h -almost conformal Ricci-Bourguignon soliton or

$$h\nabla_{\xi}u = -\{(4\omega - 1)\mathcal{R} + 4\pi_1 - (2\tilde{p} + 1)\}u.$$

Remark 8 The Schrödinger-Ricci equation suggests a universe where geometry and probability are linked. Instead of a particle moving through space, the particle's existence changes the shape of space, which in turn dictates where the particle can go.

7 Gradient h -Almost Conformal η -Ricci-Bourguignon Soliton on Bianchi-I Space-Time

Let the soliton vector $Y_1=\mathcal{D}\Psi$, where Ψ be a smooth function and \mathcal{D} denotes gradient operator of \tilde{g} on Bianchi-I space-time attached with bulk viscosity and magnetic field. So from (18) we have

$$\mathcal{R}ic + h\nabla\mathcal{D}\Psi = [\pi_1 - \frac{1}{2}(\tilde{p} + \frac{1}{2}) + \tau\omega]\tilde{g} + \pi_2u \otimes u,$$

which is equivalent to,

$$h\nabla\mathcal{D}\Psi = [\pi_1 - \frac{1}{2}(\tilde{p} + \frac{1}{2}) + \tau\omega]I - \mathcal{Q} + \pi_2u \otimes \xi. \quad (53)$$

After, contracting (53), we yield

$$\nabla\Psi = \frac{1}{h}[4\pi_1 - 2(\tilde{p} + 1) + (4\omega - 1)\mathcal{R} - \pi_2]. \quad (54)$$

Hence, we state the outcome

Theorem 11 If a Bianchi-I space-time in Rosen's bimetric theory coupled with bulk viscosity and magnetic field admits gradient h -almost conformal η -Ricci-Bourguignon soliton, then the potential function Ψ of the soliton satisfies Poisson's equation

$$\nabla\Psi = \frac{1}{h}[4\pi_1 - 2(\tilde{p} + 1) + (4\omega - 1)\mathcal{R} - \pi_2].$$

Also from (54), we conclude the following corollary

Corollary 14 If a Bianchi-I space-time in Rosen's bimetric theory coupled with bulk viscosity and magnetic field admits gradient h -almost conformal η - traceless Ricci-

Bourguignon solitons or η -Schouten-Bourguignon solitons, then the potential function Ψ of the soliton satisfies Poisson's equation

$$\nabla\Psi = \frac{1}{h}[4\pi_1 - 2(\tilde{p} + 1) - \pi_2],$$

or

$$\nabla\Psi = \frac{1}{3h}[12\pi_1 - 6(\tilde{p} + 1) - \mathcal{R} - 3\pi_2].$$

Remark 9 Poisson's equation physically relates the spatial variation (curvature) of a potential field to the density of its source at any given point. It describes how fields like electric potential or gravity are generated and shaped by internal charges or masses.

A smooth function χ on a Riemannian manifold \widetilde{M} of $\dim \widetilde{M} = n \geq 3$ is said to be harmonic, strictly super-harmonic and strictly sub-harmonic if $\nabla\chi = 0$, $\nabla\chi < 0$ and $\nabla\chi > 0$. So from above facts and (54), we reveal the corollary

Corollary 15 *Let a Bianchi-I space-time in Rosen's bimetric theory coupled with bulk viscosity and magnetic field admit a gradient h -almost conformal η -Schouten-Bourguignon soliton. Then the solitons function Ψ is*

- (i) harmonic if $\pi_1 = \frac{1}{2}(\tilde{p} + 1) - (\omega - \frac{1}{4})\mathcal{R} + \frac{\pi_2}{4}$,
- (ii) strictly super-harmonic if $\pi_1 < \frac{1}{2}(\tilde{p} + 1) - (\omega - \frac{1}{4})\mathcal{R} + \frac{\pi_2}{4}$,
- (iii) strictly sub-harmonic if $\pi_1 > \frac{1}{2}(\tilde{p} + 1) - (\omega - \frac{1}{4})\mathcal{R} + \frac{\pi_2}{4}$.

8 Conclusions

This research focuses on identifying significant results and properties that emerge from the interaction of a Bianchi-I space-time with bulk viscosity and magnetic field in Rosen's bimetric theory of gravitation. A key aspect of this study is the demonstration that physical characteristics of Bianchi-I space-time allow for the integration of bulk viscosity and magnetic field within Rosen's bimetric theory.

In particular, the space-time under h -almost conformal η -Ricci-Bourguignon soliton provides insight into the underlying geometry and the dynamic behavior of the space-time, contributing to our understanding of the space-times physical and mathematical properties. Apart from this, We have explored that the space-time in Rosen's bimetric theory attached bulk viscosity and magnetic field satisfied the SEC and the NCC in terms of h -almost conformal η -Ricci-Bourguignon soliton. Also, this study demonstrated the implications of singularity theorem and the occurrence of a black hole of space-time, admitting the such type soliton in Rosen's bimetric theory coupled to bulk viscosity and magnetic field.

Finally, we extend our analysis by examining harmonic, the modified versions of the Liouville and Poisson equations in the context of such solitons on Bianchi-I space-time. These equations play a critical role in defining the geometric and physical behavior of the system, and our findings help bridge the gap between Rosen's bimetric theory and general theory of relativity.

Acknowledgements We gratefully acknowledge the constructive comments from the editor and the anonymous referees to improve the overall quality of the paper.

Author Contributions Conceptualization, Yanlin Li, Sunil Kumar Yadav, Uday Chand De, Krishnendu De; methodology, Yanlin Li, Sunil Kumar Yadav, Uday Chand De, Krishnendu De; investigation, Yanlin Li, Sunil Kumar Yadav, Uday Chand De, Krishnendu De; writing—original draft preparation, Yanlin Li, Sunil Kumar Yadav, Uday Chand De, Krishnendu De; writing—review and editing, Yanlin Li, Sunil Kumar Yadav, Uday Chand De, Krishnendu De. All authors have read and agreed to the published version of the manuscript.

Funding Not applicable.

Data Availability No datasets were generated or analysed during the current study.

Code Availability Not applicable.

Declarations

Competing interests The authors declare no competing interests.

Ethics approval Not applicable.

Consent to participate Not applicable.

Consent for publication: Not applicable.

Open Access This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License, which permits any non-commercial use, sharing, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if you modified the licensed material. You do not have permission under this licence to share adapted material derived from this article or parts of it. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by-nc-nd/4.0/>.

References

1. Rosen, N.: A Theory of Gravitation. *Annl. of Phys.* **84**, 455–473 (1974)
2. Karade, T.M.: Spherically symmetric space times in bimetric relativity theory I. *Indian J. Pure-appl. Math.* **11**(9), 1202–1209 (1980)
3. Katore, S.D., Rane, R.S.: Magnetized cosmological models in bimetric theory of gravitation. *Pramana J. Phys.* **67**(2), 237–227 (2006)

4. Rosen, N.: Bimetric Theory of Gravitation. In: De Sabbata, V., Weber, J. (eds) Topics in Theoretical and Experimental Gravitation Physics. NATO Advanced Study Institutes Series, vol 27. Springer, Boston, 271–294 (1977)
5. Bali, R., Upadhaya, R.D.: LRS Bianchi type I string dust magnetized cosmological models. *Astrophys. Space Sci.* **283**, 97–108 (2003)
6. Gaikwad, N.P., et al.: Bianchi type-I massive string magnetized barotropic perfect fluid cosmological model in the bimetric theory of gravitation. *Chinese Phys. Lett.* 089803 (2011). <https://doi.org/10.1088/0256-307x/28/8/089803>
7. Bali, R., Singh, D.: Bianchi type-V bulk viscous fluid string dust cosmological model in general relativity. *Astrophys. Space Sci.* **300**, 387–394 (2005)
8. Bali, R., Anjali.: Bianchi type-I bulk viscous fluid string dust magnetized cosmological model in general relativity. *Pramana-J. of Physics* **63**(3), 481–490 (2004)
9. Wang, X.X.: Bianchi type-III string cosmological models with bulk viscosity and magnetic field. *Chinese Phys. Lett.* **23**, 1702–1704 (2006)
10. Alcock, C., Farhi, E., Olinto, A.: Strange Stars. *Astrophys. J.* **310**, 261–272 (1986)
11. Haensel, P., Zdunik, J.L., Schaffer, R.: Strange quark stars. *Astron. Astrophys.* **160**, 121–128 (1986)
12. Cheng, K.S., Dai, Z.G., Lu, T.: Strange stars and related astrophysical phenomena. *Int. J. Mod. Phys., D* **7**, 139–176 (1998)
13. Johnson, K.: The M. I. T bag model. *Acta Phys. Pol., B* **6**, 865–892 (1975)
14. Sotani, H., Kohri, K., Harada, T.: Restriction quark matter models by gravitational wave observation. *Phys. Rev. D* **69**, 084008 (2004)
15. Perez Martinez, A., Perez Rojas, H., Mosquera Cuesta, H.J., Boligan, M., Orsaria, M.G.: Quark stars and Quantum-Magnetically induced collapse. *Int. J. Mod. Phys. D* **14**, 1959–1969 (2005)
16. Liu, Z.H., Niu, J.C.: Vibrational energy flow model for functionally graded beams. *Compos. Struct.* **186**, 17–28 (2018)
17. De, K., De, U.C., Velimirovic, L.: Some curvature properties of perfect fluid spacetimes. *Quaest. Math.* **47**(4), 751–764 (2024)
18. Guler, S., Demirbag, S.A.: Study of generalized quasi-Einstein space-times with applications in general relativity. *Int. J. Theor. Phys.* **55**, 548–562 (2016)
19. De, U.C., Ghosh, G.C.: On generalized quasi Einstein manifolds. *Kyungpook Math. J.* **44**, 607–615 (2004)
20. Hamilton, R.: Three manifolds with positive Ricci curvature. *J. Differential Geo.* **17**, 255–306 (1982)
21. Basu, N., Bhattacharyya, A.: Conformal Ricci soliton in Kenmotsu manifold. *Glob. J. Adv. Res. Class. Mod. Geom.* **4**, 15–21 (2015)
22. Bourguignon, J.P.: Ricci curvature and Einstein metrics. In: Ferus, D., et al. (eds.) Global differential geometry and global analysis, 1979 Conference Proceedings, pp. 42–63. Springer, Berlin (1981)
23. Dwivedi, S.: Some results on Ricci-Bourguignon solitons and almost solitons. *Canad. Math. Bull.* **64**(2021), 591–604 (2021)
24. De, K., De, U.C., Gezer, A.: Perfect fluid space-times and k -almost Yamabe solitons. *Turk. J. Math.* **47**, 1236–1246 (2023)
25. Blaga, A.M., Özgür, C.: Almost η -Ricci and almost η -Yamabe solitons with torse-forming potential vector field. *Quaest. Math.* **5**(1), 143–163 (2022)
26. De, K., De, U.C.: Investigations on solitons in $f(R)$ -gravity. *Eur. Phys. J. Plus* **137**, 180 (2022)
27. Azami, S.: Some results on h-almost Ricci-Bourguignon soliton, *Afr. Mat.* **33**, 8 pp (2022)
28. De, K., Khan, M.N., De, U.C.: Characterizations of GRW space-times concerning gradient solitons. *Heliyon* (2024). <https://doi.org/10.1016/j.heliyon.2024.e25702>
29. De, K., De, U.C.: Investigation on gradient solitons in perfect fluid space-times, *Reports on Math. Phys.* **91**, 277–289 (2023)
30. Li, Y., De, U.C., De, K.: Weakly Ricci-Symmetric Space-Times and $f(R, G)$ Gravity. *Mathematics* **13**, 943 (2025). <https://doi.org/10.3390/math13060943>
31. Li, Y., Xie, Y., Gupta, M., Sharma, S.: On projective Ricci curvature of cubic metrics. *AIMS Math.* **10**(5), 11305–11315 (2025)
32. Li, Y., Azami, S.: Generalized $*$ -Ricci soliton on Kenmotsu manifolds. *AIMS Math.* **10**(3), 7144–7153 (2025)
33. Li, Y., Alharbi, F., El-Ahmady, A.E.: Singularities of Fuzzy Friedmann-Lemaître-Robertson-Walker Space. *Axioms* **14**, 591 (2025). <https://doi.org/10.3390/axioms14080591>
34. Melvin, M.A.: Homogeneous Axial Cosmologies with Electromagnetic Fields and Dust. *Ann. N. Y. Acad. Sci.* **262**(2), 253–274 (1975)

35. Lichnerowicz, A.: Relativistic hydrodynamics and magneto-hydrodynamics, p. 13. Benjamin, New York (1967)
36. Satish, J., Venkateswarlu, R.: Bulk viscous fluid cosmological models in $f(\mathcal{R}, \mathcal{T})$ -gravity, Chinese. J. Phys. **54**, 830–838 (2016)
37. Srivastava, S.K.: Scale factor dependent equation of state for curvature inspired dark energy, phantom barrier and late cosmic acceleration. Phys. Lett. B **646**, 1–4 (2006)
38. Kuhnel, W., Rademacher, B.H.: Conformal vector fields on pseudo-Riemannian spaces. Differ. Geom. Appl. **7**, 237–250 (1997)
39. Fischer, A.E.: An introduction to conformal Ricci flow. Class. Quantum Gravity **21**, 171–218 (2004)
40. Sachs, R.K., Hu, W.: General Relativity for Mathematician. Springer, New York, NY, USA (1997)
41. Hawking, S.W., Ellis, G.F.R.: The Large scale structure of spacetime. Cambridge University Press, UK (1973)
42. Chaturvedi, B.B., Bhagat, P., Islam Khan, M.N.: Novel theorems for a Bochner flat Lorentzian Kaehler space-time manifold with η -Ricci-Yamabe solitons. Chaos Solitons Fractals: X **11**, 100097 (2023)
43. Vilenkin, A., Wall, A.C.: Cosmological singularity theorems and black holes. Phys. Rev. D **89**, 064035 (2024)
44. Popov, A.G.: Exact formula for constructing solutions of the Liouville equation $\Delta_2 x = e^x$ from solution of the Laplace equations $\Delta_2 y = 0$. Dokl. Akad. Nauk **333**, 440–441 (1993)
45. Chow, B., Chu, S.C., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., et al.: The Ricci flow: techniques and applications, part I: geometric aspects. American Mathematical Society, Rhode Island (2007). <https://doi.org/10.1090/surv/163>
46. Blaga, A.M.: Harmonic aspects in an η -Ricci soliton. Int. Electron. J. Geom. **13**, 41–49 (2020). <http://doi.org/10.36890/iejg.573919>

Authors and Affiliations

Yanlin Li¹ · Sunil Kumar Yadav² · Uday Chand De³ · Krishnendu De⁴

✉ Yanlin Li
liy1@hznu.edu.cn

Sunil Kumar Yadav
prof_sky16@yahoo.com

Uday Chand De
uc_de@yahoo.com

Krishnendu De
krishnendu.de@outlook.in

- ¹ School of Mathematics, Hangzhou Normal University, Hangzhou 311121, Zhejiang, China
- ² Department of Applied Science and Humanities, United Collage of Engineering & Research, Uttar Pradesh, India
- ³ Department of Pure Mathematics, University of Calcutta, 712221 West Bengal, India
- ⁴ Department of Mathematics, Kabi Sukanta Mahavidyalaya, The University of Burdwan, West Bengal 712221, India