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Some aspects of abelian and nonabelian T-duality and the gauge/gravity correspondence

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to Jeane & David

"The fundament upon which all our knowledge and learning rests is the inexplicable."

Arthur Schopenhauer

Acknowledgements

Abstract

In this thesis we study properties of type II supergravity solutions generated by abelian and nonabelian T-duality. Also we determine, through the gauge/gravity conjecture, some aspects of the field theory dual to the supergravity solutions obtained by T-dualization. We consider three distinct types of backgrounds solutions, namely, backgrounds that are dual to confining field theories, backgrounds dual to conformal field theories and those dual to nonrelativistic field theories. We conclude this thesis with an analysis of Wilson loops on backgrounds with nonrelativistic symmetries.

Keywords: AdS/CFT; gauge/gravity; supergravity; string theory.

Areas: High Energy Physics - Theory.

Resumo

Nessa tese estudamos propriedades de soluções de supergravidade tipo II obtidas através da dualidade T abeliana e não abeliana. Também determinamos, através da conjectura gauge/gravidade, aspectos da teoria de campos dual a essas soluções obtidas por dualidade T. Consideramos três tipos distintos de soluções: duais a teorias de campos que confinam, duais a teoria de campos conformes e duais a teoria de campos não-relativistas. Concluimos essa tese com uma análise dos laços de Wilson em soluções com simetria não relativista.

Palavras Chaves: AdS/CFT; gauge/gravidade; supergravidade; teoria de cordas.

Áreas do conhecimento: Física de Altas Energias - Teoria.

Disclaimer

Chapters $3, 4, 5$ and 6 of this thesis are based, respectively, on the following research papers

1. **Araujo, Thiago R. and Nastase, Horatiu** *Comments on the T-dual of the gravity dual of D5-branes on S^3*, JHEP, 04, 2015, 081, doi=10.1007/JHEP04(2015)081, [hep-th/1409.7333];

2. **Araujo, Thiago R. and Nastase, Horatiu** $\mathcal{N} = 1$ *SUSY backgrounds with an AdS factor from non-Abelian T duality*, Physical Review, D91, 2015, 12, doi=10.1103/PhysRevD.91.126015, [hep-th/1503.00553];

3. **Araujo, Thiago R. and Nastase, Horatiu**, *Non-Abelian T-duality for non-relativistic holographic duals*, JHEP, 1511, 2015, 203, doi=10.1007/JHEP11(2015)203, [hep-th/1508.06568];

4. **Araujo, Thiago R.**, *Revisiting Wilson loops for non-relativistic backgrounds*, Physical Review, D92, 2015, 12, doi=10.1103/PhysRevD.92.126007, [hep-th/1509.02011].

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Chapter 1

Introduction

N a speech titled "Nineteenth-Century Clouds over the Dynamical Theory of Heat and Light", presented in 1900, Lord Kelvin declared: *"The beauty and clearness of the dynamical theory, which asserts heat and light to be modes of motion, is at present obscured by two clouds"*. The two clouds were the *ultraviolet catastrophe* and the failure to detect the *Luminiferous æther*. From these two clouds emerged the two columns of modern physics, general relativity and quantum mechanics.

One hundred years later, in the age of the LHC and the Planck satellite, we physicists could – just in principle – describe all phenomena we can access through our experimental apparatus using the standard model of elementary particles and the standard model of cosmology. On the other hand, it is very embarrassing that the two frameworks we use to describe the very small and the very large cannot live together. In other words, we do not know how to study objects that are very massive and infinitely small, as a black hole, because quantum mechanics and gravity are incompatible.

In the quantum gravity program of research, we assume a pragmatic viewpoint of what a scientific theory means, since the phenomena relevant to its study lives beyond any experiment humans can probe. Obviously this does not mean that we can neglect the proof of a quantum gravity theory through the experimental tests, this just means that in order to prove a theory of this species, we need to see its low

energy effects. String theory is a conservative (yes, despite the criticism) attempt to merge quantum mechanics and gravity in an unified framework.

The search for a unified description of nature is one of the oldest quests of mankind, and goes back to the ancient Greeks motivated by theological and metaphysical propositions and to Isaac Newton, James Clerk Maxwell, Steven Weinberg and several others physicists, motivated by theoretical consistency[1]. The important lesson is that we, modern physicists, are part of their cultural heritage and the subject of this thesis in a tiny portion of it.

In this work we try to understand the effects of quantum symmetries, called dualities, on some string theory backgrounds. In particular, we study the action of the Nonabelian T-duality on string theory solutions and how it changes the field theory dual to these transformed solutions. In other words, we study some properties of T-duality [2,3] and of the gauge/gravity duality, originally proposed in [4].

The general concept behind the gauge/gravity conjecture is that of a holographic principle, that states that a d-dimensional field theory can be equivalent to a gravity theory in $d + 1$ dimensions when the symmetries of the field theory are realized as isometries of the gravity side [4–6]. This innocent, but powerful, idea has driven the vanguard of physics for almost twenty years. One important feature of original gauge/gravity correspondence [4] is that the duality relates string theory and a conformal field theory with maximal supersymmetry, with all fields transforming in the adjoint representation.

To make contact with the real world – that is for phenomenological applications – we need to extend these ideas to nonconformal field theories with minimal supersymmetry, $\mathcal{N} = 1$ SUSY, as well as adding fields transforming in the fundamental representation.

In [7] it was found the gravity dual of a pure $\mathcal{N} = 1$ SYM in $d = 2+1$ dimensions, and this second solution is known as Maldacena-Nastase solution[2]. Furthermore, in [9] a deformation of the solution in [7] was considered and fields transforming in the fundamental representation were added.

In chapter 03 of this thesis, we apply the abelian T-duality on the deformed Maldacena-Nastase solution [9], which gives a type IIA solution and we lift this

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[1]And by a refined aesthetic sense, "Beauty is the first test: there is no permanent place in the world for ugly mathematics", as G.H. Hardy said [1].

[2]See also [8] where the authors considered the gravity dual of $\mathcal{N} = 1$ SYM in $d = 3+1$, coupled to extra modes that could not be decoupled while maintaining calculability.

solution to eleven dimensions. Also, following the prescription of the gauge/gravity duality, we probe the field theory dual to the backgrounds we obtain through the T-duality.

Besides the T-duality [2, 10–13] and the gauge gravity duality [4–6, 14, 15], many different dualities exist in string theory, for instance S-duality [16–19] and Mirror Symmetry [20, 21]. In this thesis we are also interested in the nonabelian T-duality, started by the paper [3], which is the generalization of the T-duality (also called abelian) for the case when the background has a nonabelian isometry group. Differently than its abelian cousin, the nonabelian T-duality has been poorly understood, and just recently the action of the transformation on the RR fields was found [22, 23].

Similarly to the abelian case, the nonabelian T-duality can also be used as a solution generating technique. Then, starting from a solution of supergravity, we can find another solution by a simple set of transformations rules, and we can investigate these solutions through the gauge/gravity correspondence. Roughly speaking, this is the general idea we perform in chapters 04 and 05.

In chapter 04 we are particularly interested in string theory solutions which the metric has a d-dimensional anti-de Sitter space as a factor, that is, solutions of the form $AdS_d \times \mathcal{M}^{10-d}$. There are numerous solutions of this form, for instance, in the best known example of the gauge/gravity duality [4] we consider a string theory solution of the form $AdS_5 \times \mathbb{S}^5$.

Another important solution of this form was considered in [24], called Klebanov-Witten solution, which consists of a space of the form $AdS_5 \times T^{1,1}$, where $T^{1,1}$ is the homogenous space $(SU(2) \times SU(2))/U(1)$. This solution is the gravity dual of a superconformal field theory with $\mathcal{N} = 1$ and gauge group given by $SU(n) \times SU(N)$. In fact, one of the first examples of the application of a nonabelian T-duality transformation in a background supporting a nontrivial RR field was in the Klebanov-Witten solution [23, 25].

We apply the nonabelian T-duality transformation on the solutions found by Jerome Gauntlett and his collaborators in [26, 27] in chapter 04. In order to understand the dual conformal four dimensional theory, we find conserved charges of the backgrounds. Finally, we will see the effect of the renormalization group (RG) flow on these backgrounds, in particular, the duality does not affect the flow.

Furthermore, a lot of the recent interest in the gauge/gravity correspondence has been focused on applications to condensed matter physics, specifically in the study

of strongly coupled systems described by relativistic and also nonrelativistic field theories. Since gauge/gravity duality relates strong coupling in field theory to weak coupling in gravity (and vice versa), we can analyze models that are otherwise very difficult to study. However, in these AdS/CMT cases we usually have no decoupled system of branes, only a phenomenological construction of a gravity dual, therefore, we have usually less control over the construction, and one degree of control is obtained by analyzing the symmetries.

The holography for nonrelativistic systems is at an incipient stage, and there are several unknown aspects that we need to understand, for example, in theories that exhibit a d-dimensional Schrödinger symmetry – the symmetry of the Schrödinger equation for the free particle [28–30] – their algebra cannot be organized as an isometry of a $(d+1)$-dimensional space as usual, but in a $(d+2)$-dimensional space, and the role of this extra dimension is still unclear.

As we mentioned earlier, another important aspect of the gauge/gravity duality is that it relates weak and strong coupling regimes of the field theory to the gravitational theory. As a result, the savage strongly coupled regime of the field theory can be mapped to a docile weakly coupled regime in the gravitational side and vice-versa [14, 31, 32].

We have a plethora of strongly coupled systems in condensed matter physics, then it is perfectly reasonable to look for a gravitational dual to theories which describe these condensed matter systems. In the study of strongly coupled condensed matter systems, we have a variety of numerical and theoretical tools from statistical physics and quantum field theory [33, 34], but they usually are hard to use.

Then, if we want to study condensed matter systems through this modern perspective, we have a new paradigm in the gauge/gravity conjecture, namely, since the field theories in condensed matter fields are nonrelativistic, their dual backgrounds have nonrelativistic isometries [35–40], see [30] for a review. In these nonrelativistic spacetimes, we first define the nonrelativistic algebra and then we try to realize it geometrically [35–40], and then, the generic tools of AdS/CFT are applied in the usual way.

In the fifth chapter of this thesis we use these facts to study the nonabelian T-duality on nonrelativistic backgrounds and we also study charges of the new backgrounds. In particular, we apply the transformation rules on backgrounds with Schrödinger and Lifzshitz symmetries – symmetries of Lifshitz fixed-points [30, 41,

42]. Also, in this chapter we start studying one important observable in field theories: *Wilson loops*.

Wilson loops are gauge invariant observables constructed from the connection of the gauge group, and are associated to the parallel transport of a particle moving through the gauge field [43, 44]. In the holographic context, the prescription for the calculation of Wilson loops in the gravity side was given by [31] and was applied in the $AdS_5 \times S^5$ solution of type IIB supergravity which is, as we already know, dual to a superconformal field theory. This prescription has been extended to backgrounds that are not anti-de Sitter, and by consequence, to backgrounds that are not dual to conformal field theories - although, they preserve Lorentz symmetry - see [45, 46] for excellent reviews.

Moreover, just recently the gravity duals of some of these relativistic systems have been embedded into string theory, see for instance [37, 47–50], and it is evident that the fundamental nature of the field theories is still a mystery. However, one may hope to be able to identify and elucidate aspects of the nonrelativistic dual field theories, just applying the holographic principle in the gravity side in the calculation of familiar physical quantities.

Wilson loops seem to be a good starting point, since it is related to a probe string moving just on the external space. This means that we can ignore, for a moment, the internal space (as well as additional fields, such as the dilaton and p-forms) which composes the supergravity solution.

In this sense, it is an observable which demands a small amount of information on the background where the string is moving in, but it gives us important information about the nature of the field theory; for instance, if the theory confines, if the theory has conformal symmetry and so on [45, 46]. Also, we can compute drag forces and the energy loss of charged particles moving in these backgrounds [51–54]. In order to complete the analysis we started in chapter 05, we study some of these aspects in chapter 06 of this thesis.

Before we start addressing all these points, let us review the important aspects of dualities in field and string theories.

Chapter 2

Dualities

2.1 Dualities in general

IN THE LATE 60s there were several puzzles concerning the strong nuclear interaction. In particular, the scattering amplitudes at high-energy have some oddities [55]. When we consider just spinless particles, the Bose statistics demands the symmetry $s \Leftrightarrow t$ in the scattering amplitude $A(s,t)$, where (s,t,u) are the Mandelstam variables.

On the other hand, by that time there were a large amount of strongly interacting particles and they seemed to have arbitrary spin j. As it is well known, it is difficult to construct a theory with higher spins interacting particles, since at high energies they exceed unitarity bounds.

Consider one simple example, the scattering of scalar fields ϕ mediated by force carriers $\sigma^{\mu_1 \cdots \mu_j}$ of spin j and mass M_j. The interaction is of the form

$$\sum_j g_j (\phi^* \partial_{\mu_1} \cdots \partial_{\mu_j} \phi) \sigma^{\mu_1 \cdots \mu_j} \tag{2.1}$$

and the t-channel contribution to the amplitude is

$$A(s,t) = -\sum_j \frac{g_j^2 (-s)^j}{t - M_j^2} \ . \tag{2.2}$$

It is conceivable to think of (2.2) as an infinite sum, since there was no reason to consider a maximum value for the spin j. As a consequence, the equation (2.2) is not necessarily an entire function of s. The oddity mentioned above is that in

physical processes we need to consider the s-channel due to its poles, but since (2.2) may give us poles for finite values of s, it is not obvious that we need to consider the s-channel now. Evidently we can construct the amplitude in terms of the s-channel

$$A'(s,t) = -\sum_j \frac{g_j^2(-t)^j}{s - M_j^2} \ , \tag{2.3}$$

and the analysis would be the same.

The duality hypothesis states that the s- and t-channel represent alternative descriptions of the same physics, and it motivated Gabriele Veneziano to postulate, in 1968, the following formula for the scattering amplitude

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} \ , \tag{2.4}$$

where Γ is the Euler gamma function and $\alpha(x) = \alpha(0) + \alpha'x$ is the Regge trajectory and the constant α' the Regge slope [55–57].

It is well known that in the early 70s an alternative theory for the strong interaction arose, *quantum chromodynamics*, and the original motivation for the Veneziano ideas disappeared. Despite that fact, the study of the Veneziano model — or dual resonance model — showed a rich framework, known as *string theory*, with results ranging from pure mathematics to quantum gravity [10–13, 55, 58]. Much more important is that the concept of duality symmetries — and we will see that there is a variety of them — is in the core of string theory.

2.2 Dualities in QFT and String Theory

Let us first try to understand what a duality really is. Roughly speaking, we may say that a duality is a *nontrivial isomorphism* [59]. Observe that this definition has one important aspect that we must understand, the precise meaning of *nontrivial*. Equivalently, we will see that a duality is an unexpected equivalence between two physical systems.

For instance, when we perform a Poincaré transformation or general coordinate transformation, in our physical systems, we are using just a plain symmetry of the theory. In other words, we use the fact that the coordinates of our system is just a mathematical artefact and the nature itself does not care about the directions you

prefer to call left and right, up or down. In other types of symmetries, we just have redundancies in our description, for instance, the gauge symmetry.

Mathematicians know very well the equivalences between two algebraic structures, that they call *isomorphisms*, above. In fact, they have fancy names for isomorphisms depending on the type of algebraic structures they are studying: *homeomorphisms* for topological spaces, *diffeomorphisms* for smooth manifolds, *holomorphisms* for complex manifolds and *isometries* for metric spaces for example.

Also, these symmetries preserve properties of the objects we are dealing with, such as the the dimension, algebra, curvature, topology, etc. This is important for physics, since it would not make sense a symmetry of a system that changes the number of dimension of the physical system. Therefore, these invariant objects is what allow us to study physics, and are directly related to the conservation of momentum, energy and so on.

Duality is a new beast and comprises several new ingredients. For example, now we can find an equivalence between different algebraic structures, that is, some dualities can related algebraic geometry to representation theory, and this is an idea that is not embraced by ordinary isomorphisms. In other words, dimensions, topologies and so on have no fundamental meaning in the definition of dualities.

The physical aspects of the problem — the observables — guarantee the equivalence of the physical systems, then, we may say the duality is a quantum equivalence. In this sense is quite difficult to prove mathematically that two physical systems are dual to one another, but one can find some mathematical insights on this issue [59, 60].

Consider a quantum theory $\mathcal{Q}$ characterized by a set of parameters $\{\lambda_i\}$ — denoted collectively by $\mathcal{M}$, and called moduli space of parameters — by an algebra $\mathcal{A}_\lambda$ of observables and by a functional map $\langle \, \cdot \, \rangle : \mathcal{A}_\lambda \to \mathcal{C}^\infty(\mathcal{M})$, where $\mathcal{C}^\infty(\mathcal{M})$ is the space of smooth functions defined on the moduli space $\mathcal{M}$. In physical terms, to each observable $\mathcal{O}_\lambda \in \mathcal{A}_\lambda$ we find its vacuum expectation value

$$\langle \mathcal{O}_\lambda \rangle = f(\lambda_1, \lambda_2, \dots) , \tag{2.5}$$

that depends on the parameters λ_i. We say that the quantum theory $(\mathcal{Q}, \mathcal{M}, \mathcal{A}_\lambda)$ is dual to another quantum theory $(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{M}}, \widetilde{\mathcal{A}}_{\tilde\lambda})$ if there exists a map between them that preserves (2.5).

Essentially, we have three different types of dualities, namely dualities between

two-quantum field theories, dualities between string theories, between field theory and string theory [61]. Let us understand qualitatively some examples.

2.2.1 Quantum field theory dualities

In two-dimensional field theories, particularly with conformal invariance, the holomorphic properties of the field implies severe constraints on the theory and this explains — in part — an equivalence between fermions and bosons. In a CFT, for example, the [10, 62], we can represent a free Dirac fermion ψ as

$$\psi(z) \sim e^{i\phi(z)} \ , \quad \bar{\psi}(\bar{z}) \sim e^{-i\phi^{\dagger}(\bar{z})} \ . \tag{2.6}$$

where $\phi(z)$ is a holomorphic scalar field with propagator

$$\langle \phi(z)\phi(0) \rangle \sim -\ln(z) \ . \tag{2.7}$$

On the other hand, corformal invariance is not a mandatory requirement, for instance in [63] Sidney Coleman showed that the massive Tirring model, defined by the action

$$S_{\psi} = \int \mathrm{d}^2 x \left(i\bar{\psi}\slashed{\partial}\psi - m\bar{\psi}\psi - \frac{g}{2}\bar{\psi}\gamma^{\mu}\psi\bar{\psi}\gamma_{\mu}\psi \right) \ , \tag{2.8}$$

is equivalent to the sine-Gordon model defined by

$$S_{\phi} = \int \mathrm{d}^2 x \left(-\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi + m\cos\beta\phi \right) \ . \tag{2.9}$$

This interesting example of duality is called *bosonization* and it is useful in the study of condensed matter systems [64, 65].

Another interesting example — and quite relevant to string theory — is related to the electromagnetic theory [61]. Consider the electromagnetic action in the absence of sources

$$S_{EM} = -\frac{1}{2e^2} \int F \wedge *F \ . \tag{2.10}$$

The generacting functional is

$$Z = \int \mathcal{D}A \, e^{iS_{EM}} \ . \tag{2.11}$$

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We may notice that the field F and its Hodge dual $*F$ satisfy the Maxwell equations

$$\mathrm{d}F = 0 \qquad \mathrm{d} * F = 0 \ , \tag{2.12}$$

and we can easily see that they are invariant under $F \leftrightarrow *F$, that is simply the usual electromagnetic duality $\vec{E} \mapsto -\vec{B}$ and $\vec{B} \mapsto \vec{E}$ using an antiquated language. To see the quantum aspects of this duality [61], we can try to write the measure of the path integral (2.11) in terms of F, that is, we consider a transformation

$$\mathcal{D}A \mapsto \mathcal{D}F \prod \delta(\mathrm{d} * F) \ , \tag{2.13}$$

and if we write $*F = \frac{1}{2}\widetilde{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, the functional delta function is

$$\prod \delta(\mathrm{d} * F) = \int \mathcal{D}V \exp\left(\frac{i}{2\pi} \int \mathrm{d}^4x V_\nu \partial_\mu \widetilde{F}^{\mu\nu}\right) \ , \tag{2.14}$$

this functional representation is a mimic of $2\pi\delta(y) = \int \mathrm{d}x \exp(ixy)$. Using these results, the path integral (2.11) becomes

$$Z = \int \mathcal{D}F\mathcal{D}V \exp\left\{-i \int \mathrm{d}^4x \left(\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\pi}(\partial_{[\mu}V_{\nu]})\widetilde{F}^{\mu\nu}\right) \ , \right\} \tag{2.15}$$

and performing the Gaussian integral we find the final result

$$Z = \mathcal{N} \int \mathcal{D}A \ e^{i\widetilde{S}_{EM}} \ . \tag{2.16}$$

where $\mathcal{N}$ is just a normalization constant and

$$\widetilde{S}_{EM} = -\frac{1}{2e'^2} \int \ G \wedge *G \ . \tag{2.17}$$

where $e' = 2\pi/e$ and $G_{\mu\nu} = -\frac{2\pi}{e^2}\widetilde{F}_{\mu\nu} = 2\partial_{[\mu}V_{\nu]}$. Then, we see that the magnetic part of $F_{\mu\nu}$ is related to the electric part of $G_{\mu\nu}$, and it is simply the electromagnetic duality stated above. In fact, we can add the θ-term

$$\frac{\theta}{16\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu} \ , \tag{2.18}$$

which is related to the instantons and if we write the coupling constant as

$$\tau = \frac{\theta}{2\pi} + i\frac{2\pi}{e^2} \ , \tag{2.19}$$

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one can show that the electromagnetic duality implies the following transformation

$$\tau' = -\frac{1}{\tau} \ . \tag{2.20}$$

It must be noticed that since we are in a free theory, we can easily absorb the constants e and e' by a redefinition of A_μ and V_μ respectively, but we consider this explicit form, since the relation (2.20) is really suggestive.

When we try to extend the EM-duality above to a case in the presence of sources, it is mandatory to conjecture the existence of a magnetic charge g, in such a way that the Maxwell equations become

$$*\mathrm{d} * F = j_e \ , \quad \mathrm{d}F = j_g \ . \tag{2.21}$$

In 1931, Dirac showed, in his groundbreaking paper [66], that this duality condition demands a relation between the electric and magnetic charges, the Dirac quantization condition

$$eg = 2\pi\hbar n \ , \quad n \in \mathbb{Z} \ , \tag{2.22}$$

but since the existence of magnetic monopoles is a conundrum in our community, and its introduction is an *ad hoc* hypothesis, it does not seem very natural to consider that this duality is relevant in the Maxwell theory. Also, the existence of the magnetic monopoles is related to the existence of a string-like singular region: the *Dirac string*. It can be visualized as an infinitely long and infinitely thin solenoid, in such a way that the magnetic monopole is precisely the magnetic field flowing out the string. In quantum mechanics, we require that a wave function describing a particle that turns around n times the solenoid is completely determined except for an arbitrary phase. This condition is the Dirac quantization.

Furthermore, in certain gauge theories there exists a class o dualities that relates the weak and strong coupling regimes of the theory. But we may notice that in the Maxwell theory, we consider magnetic and electric charge at the same time, but if we consider $e << 1$ we see that $g >> 1$ and vice-versa. Then we do not have a true weak-strong duality in this case.

In 1974, Gerard 't Hooft [67] and Alexander Polyakov [68] showed that non-abelian gauge theories — the Georgi-Glashow model in particular — admit magnetic mopopoles as solutions [69–73], but differently from the Dirac monopole, this 't Hooft-Polyakov monopole is completely regular. Moreover, these solutions are very different from the quanta of the fields, these particles are solitons whose stability

is guaranteed by topological reasons and are extremely heavy at the weak-coupling regime, since their masses are proportional to $M \sim 1/g$, where g is the coupling constant [73, 74].

Montonen and Olive conjectured in 1977 that nonabelian gauge theories possessing monopoles as solutions present the strong-weak duality [75, 76], and this conjecture was extended by Witten and Olive and by Osborn [77, 78], what culminated in the conjectured self-duality of the $\mathcal{N} = 4$ super-Yang Mills theory. Supersymmetry is fundamental in this context, since the simplest form of the Montonem-Olive conjecture can not be true due to the running of the coupling constant. But one can overcome this problem if the β-function of the theory vanishes, as in the $\mathcal{N} = 4$ SYM.

There are another important detail concerning this duality. The action of the gauge sector in the theory is

$$S[A] = \int \left(-\frac{1}{2e^2}\, F \wedge *F + \frac{\theta}{8\pi^2} F \wedge F \right) \tag{2.23}$$

but using that the angle θ is defined up to 2π, we have the additional symmetry $\tau \to \tau + 1$. Together, we may conjecture that the gauge theory is invariant under an $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$ group transformation, that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \,, \quad a, b, c, d, \in \mathbb{Z} \,, \quad ad - cb = 1 \,, \tag{2.24a}$$

then the theory is invariant under

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \,. \tag{2.24b}$$

Furthermore, Seiberg and Witten generalized this strong-weak dualities to theories with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetries. These dualities between strong and weak coupling regimes are known generically as S-duality [17, 79–87].

2.2.2 String theory dualities

By the 1990s, it was known that there are five consistent string theories: type I, type IIA, IIB and the heterotic $SO(32)$ and $E_8 \times E_8$ string theories. However, this is an embarrassing situation, since an unconstrained structure of a physical theory

would make it useless, because we could not predict new results. In other words, the theory would not be *falsifiable* [10, 11].

This puzzle was solved when it was realized that there are nontrivial equivalences among the string theories, now known as T- and S-dualities. In fact, T-duality relates the type IIA and IIB string theory and the two heterotic theories, whilst the S-duality relates the type I to the heterotic $SO(2)$ and the type IIB to itself. Together with the fact that the type I theory is obtained from the type IIB from a procedure called *orientifold projection*, and is equivalent to the SO(32) theory by an S-duality, we conclude that these five theories are in fact, the same theory, or better, there may exist some underlying theory that governs these aspects of string theory, this is the *M-theory*.

The string coupling constant, g_s, is given by the vacuum expectation value of $\exp \phi$, where ϕ is the dilaton field. The S-duality relates the coupling constant g_s to $1/g_s$, therefore, if we know the behaviour of string theory for $g_s \to 0$ we can get insights of the theory for large g_s. For instance, strongly coupled type I theory will be related to weakly coupled $SO(32)$ heterotic strings by S-duality. In the case of type IIB string theory, S-duality relates this theory to itself.

In fact, using S-duality we can understand the behaviour of three of the five string theories at strongly coupled regime, but we need to see how the type IIA and the $E_8 \times E_8$ work in this limit. The answer goes as follows: For g_s large enough, a new dimension of size $g_s \ell_s$ emerges in these theories, in such a way that for the type IIA this dimension is a circle and in the heterotic string we have an interval. The important point is that this new 11-dimensional quantum theory demands a new techniques, but we know that its low energy limit is governed by the $d = 11$ SUGRA [88–91].

For instance, the low energy action (in the string frame) of the bosonic sector in the type I string theory is [1]

$$S_I = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left\{ e^{-2\phi} \left(R + 4\partial_\mu \phi \partial^\mu \phi \right) - \frac{1}{2} |F_3|^2 - \frac{\kappa^2 e^{-\phi}}{g^2} tr \left(|F_2|^2 \right) \right\},$$
(2.25a)

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[1] The trace Tr is calculated using the 496-dimensional adjoint representation of the gauge group $SO(32)$, while the trace tr is calculated using the 32-dimensional fundamental representation of the gauge group [10, 11]. Considering the field strength F, we have the identity $30 tr (F \wedge F) = Tr (F \wedge F)$.

and the corresponding action in the heterotic string is

$$S_H = \frac{1}{2\kappa^2} \int \mathrm{d}^{10}x \sqrt{-G} e^{-2\phi} \left\{ R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2 - \frac{\kappa^2}{30g^2} Tr\left(|F_2|^2\right) \right\} \; , \quad (2.25\mathrm{b})$$

where the parameter g above is $g^2 = 4\pi(2\pi\ell_s)^6$. Also, the gauge group of the type I theory is $SO(32)$, which suggests a deep connection with the Heterotic $SO(32)$ theory. In fact, the map between the theories is simply

$$\begin{aligned} \phi &\to -\phi \\ G_{\mu\nu} &\to e^{-\phi} G_{\mu\nu} \; . \end{aligned} \qquad (2.26)$$

And since the coupling constant is $g_s = \langle e^\phi \rangle$ we see that

$$g_s^I = \frac{1}{g_s^{SO}} \; . \qquad (2.27)$$

The type IIB S-duality is dramatically different and we will see in the next section that it is closely related to the S-duality discussed in the $\mathcal{N} = 4$ SYM. We already know that in this theory we have a pair of two-forms $(B, A^{(2)})$ related, respectively, to the NS-NS and R-R sectors (see A.4.2 in the appendix A and references [10, 11]), and these fields transform as a doublet under $SL(2,\mathbb{R})$, therefore we write them as

$$\mathcal{B} = \begin{pmatrix} B \\ A^{(2)} \end{pmatrix} \; , \qquad (2.28)$$

with field strength $\mathcal{H} = \mathrm{d}\mathcal{B}$. In this notation, if we consider

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \; , \quad a, b, c, d \in \mathbb{R} \; , \quad ad - bc = 1 \; , \qquad (2.29)$$

the $\mathcal{B}$ field transforms as $\mathcal{B} \mapsto \Lambda\mathcal{B}$. In addition, we define the *axion-dilaton* field $\tau = A^{(0)} + ie^{-\phi}$ transforms as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \; . \qquad (2.30)$$

We define the $SL(2,\mathbb{R}$ matrix

$$\mathcal{M} = e^\phi \begin{pmatrix} |\tau|^2 & -A^{(0)} \\ -A^{(0)} & 1 \end{pmatrix} \; , \qquad (2.31)$$

under $SL(2,\mathbb{R}$, we have

$$\mathcal{M} \mapsto (\Lambda^{-1})^T \mathcal{M}\Lambda \; . \qquad (2.32)$$

All in all, we may write the type IIB action (A.32), in the Einstein frame, as

$$S_{IIB} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G} \left(R - \frac{1}{12}\mathcal{H}_{\mu\nu\rho}^T \mathcal{M}\mathcal{H}^{\mu\nu\rho} + \frac{1}{4}tr\left(\partial^\mu \mathcal{M}\partial_\mu \mathcal{M}^{-1}\right)\right)$$
$$- \frac{1}{8\kappa_{10}^2}\left(\int \mathrm{d}^{10}x\sqrt{-G}|F^{(5)}|^2 + \epsilon_{ij}\int A^{(4)}\wedge \mathcal{H}^i \wedge \mathcal{H}^j\right) , \tag{2.33}$$

and in this form, the $SL(2,\mathbb{R})$ symmetry is manifest.

This symmetry is not present in the full type IIB string theory, and it is broken to the subgroup $SL(2,\mathbb{Z})$ due to stringy and quantum effects. In particular, the transformation of the axion-dilaton field τ is

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} . \tag{2.34}$$

but with

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) , \quad a,b,c,d \in \mathbb{Z} , \quad ad - bc = 1 . \tag{2.35}$$

This is the S-duality of the type IIB string theory, which relates this theory to itself. We may notice now that this transformation is similar to the transformation in the $\mathcal{N} = 4$ SYM (2.24a – 2.24b), and we will see in the next section that it is not an accident.

Before, we must consider another important duality in the string theory framework, the *T-duality*.

T-duality

The bosonic string theory compactified on a circle $\mathbb{S}^1$ of radius R allows us to introduce an important symmetry of the theory. Considering that the closed string moves in a space of the form $\mathbb{R}^{24,1} \times \mathbb{S}^1$, the coordinate $X^{25}(\sigma,\tau)$ must satisfy

$$X^{25}(\sigma + \pi, \tau) = X^{25}(\sigma,\tau) + 2\pi R\omega , \tag{2.36}$$

where $\omega \in \mathbb{Z}$, that bears the name of *winding number*, counts the number of times that the string winds around the compact dimension. It can be shown [10, 11] that this coordinate splits as

$$X^{25}(\sigma,\tau) = X_L^{25}(\tau + \sigma) + X_R^{25}(\tau - \sigma) \tag{2.37}$$

where

$$X_R^{25}(\tau - \sigma) = \frac{1}{2}(x^{25} - \tilde{x}^{25}) + \left(\alpha' \frac{\kappa}{R} - \omega R\right)(\tau - \sigma) + \cdots \qquad (2.38a)$$

$$X_R^{25}(\tau + \sigma) = \frac{1}{2}(x^{25} + \tilde{x}^{25}) + \left(\alpha' \frac{\kappa}{R} + \omega R\right)(\tau + \sigma) + \cdots \qquad (2.38b)$$

where $\kappa \in \mathbb{Z}$, called *Kaluza-Klein excitation number*, comes from the quantization of the momentum $p^{25} = \kappa/R$. In addition, the mass formula is simply

$$M^2 = \left(\frac{\kappa}{R}\right)^2 + \left(\frac{\omega R}{\alpha'}\right)^2 + \frac{2}{\alpha'}[(N_L + N_R) - 2] , \qquad (2.39)$$

where N_L and N_R are, respectively, the number left- and right-moving waves and they satisfy $N_R - N_L = \omega \kappa$.

Now, the curious fact comes when we notice that the mass formula (2.39) is invariant under the transformation $R \mapsto \tilde{R} = \alpha'/R$, and the winding number becomes the Kaluza-Klein number, and vice-versa. This unexpected symmetry of the theory is called T-duality. Finally, in terms of the mode expansion, this symmetry means that

$$X_R^{25} \mapsto -X_R^{25} \quad \text{and} \quad X_L^{25} \mapsto X_L^{25} . \qquad (2.40)$$

In the type II superstring theory, we can repeat this analysis for the bosonic coordinates, in such a way that under T-duality, the mode $X^9(\sigma, \tau)$ transforms as

$$X_R^9 \mapsto -X_R^9 \quad \text{and} \quad X_L^9 \mapsto X_L^9 , \qquad (2.41)$$

and in the RNS formalism, the worldsheet supersymmetry demands that

$$\psi_R^9 \mapsto -\psi_R^9 \quad \text{and} \quad \psi_L^9 \mapsto \psi_L^9 , \qquad (2.42)$$

and one can show that this condition implies that under this transformation we exchange from the type IIA and type IIB theory.

Also, we may study the action of T-duality in the presence of background fields, for instance, the graviton $G_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$ and dilaton ϕ and the R-R p-forms. The general procedure follows the original idea of Buscher [2]: We start with a σ-model which supports an isometry such as $U(N)$. Then we gauge the isometry, but we need to impose a constraint by means of Lagrange multipliers

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which guarantees that the connection field strength remains equal to zero. This constraint enforces the condition that after gauging the isometry, the initial degrees of freedom remain unchanged.

The duality works as follows. On one hand, by solving the equation of motion for the Lagrange multipliers and replacing the solution into the action, we recover the original model. If instead we solve the equation of motion for the connection and we gauge fix, we find the dual σ-model. Let us see how it works.

Consider the following worldsheet action in the presence of background fields

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \left(\sqrt{-h}h^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu - \epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu \right) . \qquad (2.43)$$

If we consider that the coordinate X^9 is compactified on a circle, the action has an isometry in this coordinate, and we use this isometry to find the T-dual theory. Let us introduce a Lagrange multiplier $\widetilde{X}^9$ in this theory and write the action (2.43) as:

$$4\pi\alpha'S = \int \mathrm{d}^2\sigma \left[\sqrt{-h}h^{\alpha\beta} \left(-G_{99}V_\alpha V_\beta - 2G_{\mu 9}V_\alpha\partial_\beta X^\mu - G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\right) \right.$$
$$\left. +\epsilon^{\alpha\beta} \left(B_{9\mu}V_\alpha\partial_\beta X^\mu + B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\right) + \widetilde{X}^9\epsilon^{\alpha\beta}\partial_\alpha V_\beta \right] , \qquad (2.44)$$

and we can see that this action corresponds to (2.43) using the equation of motion for $\tilde{X}^9$, $\epsilon^{\alpha\beta}\partial_\alpha V_\beta = 0 \Rightarrow V_\alpha = \partial_\alpha X^9$, and inserting it in (2.44) we recover the original action. On the other hand, if we use V_α to eliminate the Lagrange multiplier $\widetilde{X}^9$, we find the dual action

$$\tilde{S} = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \left(\sqrt{-h}h^{\alpha\beta}\widetilde{G}_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu - \epsilon^{\alpha\beta}\widetilde{B}_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu \right) , \qquad (2.45)$$

where[2]

$$e^{2\tilde{\phi}} = \frac{e^{2\phi}}{|G_{99}|} \qquad \widetilde{G}_{99} = \frac{1}{G_{99}}$$

$$\widetilde{G}_{MN} = G_{MN} - \frac{G_{9M}G_{9N} - B_{9M}B_{9N}}{G_{99}} \qquad \widetilde{G}_{9M} = \frac{1}{G_{99}}B_{9M}$$

$$\widetilde{B}_{MN} = B_{MN} - 2\frac{B_{9[M}G_{N]9}}{G_{99}} \qquad \widetilde{B}_{M9} = -\frac{G_{M9}}{G_{99}}$$

---

[2]Including the transformation for the dilaton field that needs a different approach. In fact, we can compute its transformation as consistent conditions for the quantization of the dual theory [92]. An easy (and somewhat lousy) way to see this transformation is requiring the invariance of $\sqrt{-G}e^{-2\phi} \to \sqrt{-\widetilde{G}}e^{-2\tilde{\phi}}$ under T-duality.

These transformations are known as *Buscher's rules* [2, 93–95].

We can study the behaviour of the R-R forms under T-duality using different approaches, e.g. [94, 96, 97] from a spacetime perspective, [98, 99] from a worldsheet viewpoint, [100] using pure spinors and finally [101].

In the approach of [94] (see also [93] for a detailed presentation) we consider a dimensional reduction of the type IIA theory on a circle $\mathbb{S}^1$ of radius R_x and of the type IIB on a circle of radius R_y in such a way that Kaluza-Klein scalar is independent of the theory we started, then $k^2 = G_{99} = 1/\widetilde{G}_{99}$, then we find

$$\frac{R_x}{\ell_s} = \frac{\ell_s}{R_y} \ . \tag{2.46}$$

Making this identification, one can show that the nine dimensional theories we find by dimensional reduction of the type IIA and type IIB theories are the same and that the R-R fields are related by

$$\begin{aligned} C^{(n)}_{M_1\ldots M_n} &= C^{(n)}_{M_1\ldots M_n 9} + n B_{[M_1|9|}C^{(n-1)}_{M_2\ldots M_n]} \\ &\quad + n(n-1)B_{[M_1|9|}G_{M_2|9|}C^{(n-1)}_{M_3\ldots M_n]9}/G_{99} \end{aligned} \tag{2.47a}$$

$$C^{(n)}_{M_1\ldots M_{n-1}9} = C^{(n-1)}_{M_1\ldots M_{n-1}} - (n-1)G_{[M_1|9|}C^{(n-1)}_{M_2\ldots M_{n-1}]9}/G_{99} \ , \tag{2.47b}$$

and these are the T-duality rules for the R-R sector.

In fact, the T-duality is a particular case of a deeper conjectured duality called *mirror symmetry* [20, 21, 102–104], that associates to each Calabi-Yau manifold $\mathcal{M}$ a mirror Calabi-Yau manifold $\mathcal{W}$ is such a way that type IIA string theory compactified on $\mathcal{M}$ is equivalent to the type IIB string theory compactified on $\mathcal{W}$.

In order to consider realistic compactifications in the string theory framework, we may split the ten-dimensional space where the string is defined as $\mathbb{R}^{1,3} \times \mathcal{M}^6$ where $\mathbb{R}^{1,3}$ is the four dimensional Minkowski space and $\mathcal{M}^6$ is an internal compact manifold, and consistence requires that this manifold is compact Kähler, Ricci-flat manifold with holonomy group $SU(3)$, which is precisely the definition [3] of a three-dimensional Calabi-Yau manifold, also called Calabi-Yau 3-fold [10, 55].

---

[3]In fact, there are many different definitions of a Calabi-Yau n-fold [105–107].

The simplest Calabi-Yau manifold, the 2-Torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$, where the circles have radii R_1 and R_2. If we regard the torus as the lattice $\mathbb{C}/\{\mathbb{Z} \oplus \tau\mathbb{Z}\}$ we can easily conclude that it is flat, since $\mathbb{C}$ is flat, then it is also Ricci flat and has trivial holonomy group. This space can be characterized by two numbers – the moduli space is two-dimensional – and obviously we can consider simply the radii (R_1, R_2), but it turns out that the it is more convenient to define the moduli space through

$$\varsigma = iR_1R_2 \, , \quad \vartheta = i\frac{R_2}{R_1} \, . \tag{2.48}$$

As we know, string theory is invariant under T-duality. Therefore, the transformation $R_1 \leftrightarrow 1/R_1$, for $\alpha' = 1$, implies that string theory is invariant under $\varsigma \leftrightarrow \vartheta$. Observe that this is a profound result, since from the classical viewpoint, two torus with different complex structures, ϑ and ϑ', are not holomorphically equivalent. Obviously, the mirror symmetry for Calabi-Yau 3-folds are much more difficult, but generically, the idea is the same.

Putting all these facts together, we conclude that the structure of the string theory is very constrained and the theories, in the duality sense, are tied. So we have the web of dualities depicted in the figure 2.1.

Now we want to turn our attention to a deeper and unexpected duality, that one that relates string theory to quantum field theories, known as *gauge/gravity correspondence*.

2.2.3   String-QFT duality

In the outstanding paper [108], 't Hooft studied the large N expansion of gauge theories and found that in this regime we have a deep connection between gauge theories and string theory. Naively, the Yang-Mills theory with gauge group $U(N_c)$, described by the Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{g_{YM}^2}Tr\left(F_{\mu\nu}F^{\mu\nu}\right) \, , \tag{2.49}$$

does not have a good dimensionless parameter that we can consider a perturbation expansion, since the coupling constant g_{YM} will be related to the scale Λ by dimensional transmutation [109]. On the other hand, 't Hooft noticed that we have one
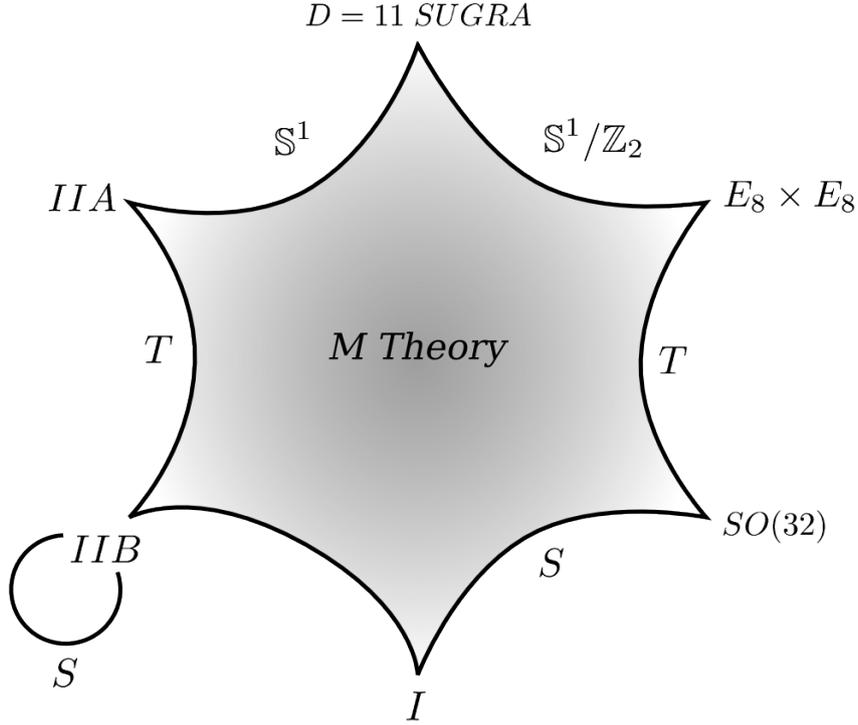
Figure 2.1: Web of dualities.

more dimensionless parameter, the number of colours N_c. When we consider a large number of colours N_c we could make a expansion in $1/N_c$ [14, 110].

If we consider a general theory with fields Φ_μ^a, where a is an index that labels the adjoint representation of $SU(N_c)$ and μ labels generic quantum numbers. The Lagrangian we want to consider is

$$\mathcal{L} \sim Tr\left(\partial\Phi_\mu\partial\Phi_\nu\right) + g_{YM}f^{\mu\nu\lambda}Tr\left(\Phi_\mu\Phi_\nu\Phi_\lambda\right) + g_{YM}^2 h^{\mu\nu\lambda\sigma}Tr\left(\Phi_\mu\Phi_\nu\Phi_\lambda\Phi_\sigma\right) \ , \quad (2.50)$$

where $f^{\mu\nu\lambda}$ and $h^{\mu\nu\lambda\sigma}$ are arbitrary constants. If we rescale the fields as $\Phi \mapsto g_{YM}^{-1}\Phi$, we find

$$\mathcal{L} \sim \frac{1}{g_{YM}^2}\left[Tr\left(\partial\Phi_\mu\partial\Phi_\nu\right) + f^{\mu\nu\lambda}Tr\left(\Phi_\mu\Phi_\nu\Phi_\lambda\right) + h^{\mu\nu\lambda\sigma}Tr\left(\Phi_\mu\Phi_\nu\Phi_\lambda\Phi_\sigma\right)\right] \ . \quad (2.51)$$

We may define the *'t Hooft coupling* $\lambda := g_{YM}^2 N_c$, and the limit when $N_c \to \infty$ and λ remains fixed is known as *'t Hooft limit*.

Using the double line notation, in which a field in the adjoint representation is the product two fields in the fundamental and antifundamental representations, then we may consider that $(\Phi^a)^i{}_j \equiv \phi^i \phi_j$. In this notation, the propagator is

$$\langle \Phi^i{}_j \Phi^k{}_l \rangle \propto \frac{\lambda}{N_c} \left(\delta^i_l \delta^k_j - \frac{1}{N_c} \delta^i_j \delta^k_l \right) \; , \tag{2.52}$$

and in the large N_c we can safely ignore the second term in the propagator, so that it is proportional to λ/N_c. Furthermore, from the Lagrangian (2.51) we easily see that the vertices are proportional to N_c/λ. Using the quarks and gluons as our prototypical example of fields, we can draw the Feynman diagrams as double lines an in the figure (2.2), where the orientation is taken from fundamental to antifundamental indices. All in all, the Feynman diagrams have the power in N_c
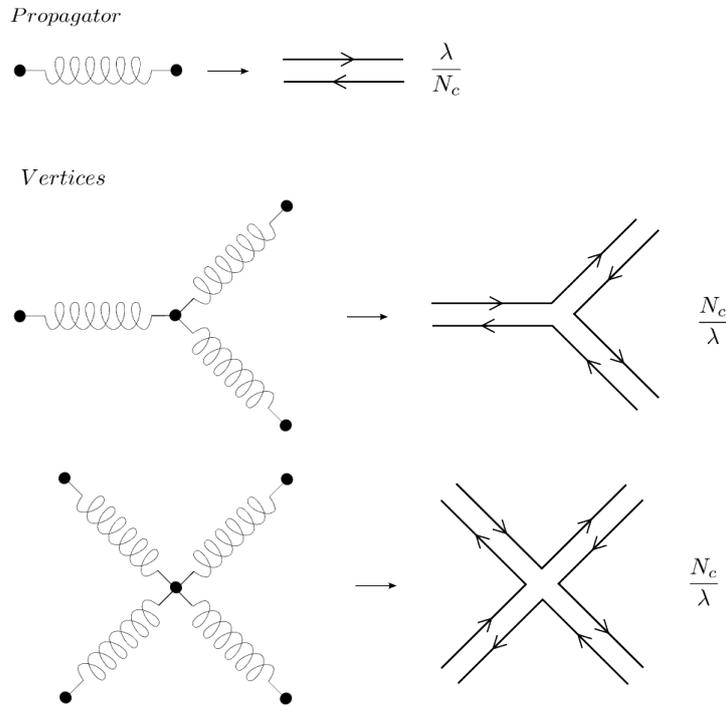
Figure 2.2: Double diagrams for gluons.

and λ

$$N_c^{V-E+F} \lambda^{E-V} \; , \tag{2.53}$$

where

$$V = \#\text{Vertices}$$

$$E = \#\text{Edges (propagators)}$$

$$F = \#\text{Faces (loops)}$$

The Euler number ξ defined by $\xi = V - E + F$ is a topological invariant and for closed surfaces it is given by $\xi = 2 - 2g$, where g is the number of handles of the surface, and it is called *genus*. Therefore, the perturbative expansion of the theory is simply

$$\sum_{g=0}^{\infty} N_c^{2-2g} f(\lambda) \, , \tag{2.54}$$

where $f(\lambda)$ is some polynomial in λ. We have an expansion in terms of Riemann surfaces of different genus in in the figure (2.3).
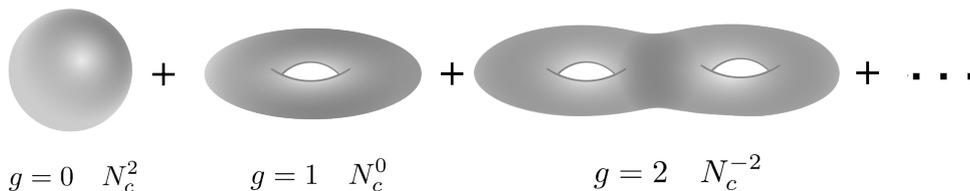
Figure 2.3: Perturbative expansion.

In particular, when we consider the large N_c limit, the surfaces with $g = 0$ are dominant, and the double line diagrams associated to these surfaces are called *planar diagrams*, since we can draw them in the surface of the sphere, as in the figure (2.4).

When we make the identification $g_s \sim 1/N_c$, the expansion above is the same expansion we find in the perturbative expansion of closed strings [10, 12], so this fact is an initial motivation to suppose that quantum field theories and string theory may be related. The revival of the interest in such a connection is mainly due the work of Maldacena in [4], and this duality known as gauge/gravity correspondence. It has been used to explore many aspects of gauge theories which cannot be studied using usual perturbation theory techniques.
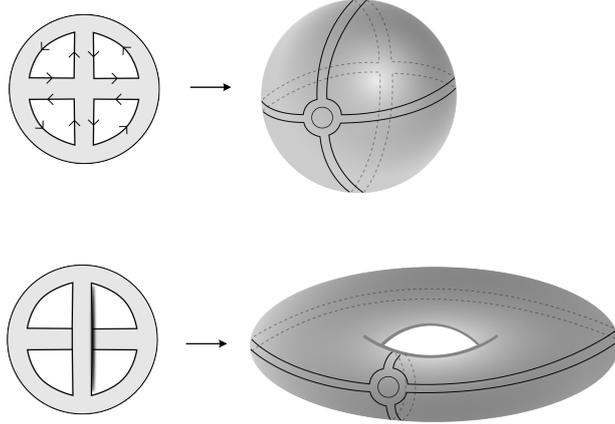
Figure 2.4: Planar and nonplanar diagrams.

The fundamental concept of the gauge/gravity duality is that the symmetries of the field theory are realized geometrically as isometries in the gravity dual side [4, 6, 14, 32, 111–116]. Let us see how this duality works using the original example. We need to consider two sides of this duality, the field theory side and the gravity side. In the field side we have a Yang-Mills theory with gauge group $U(N_c)$ and $\mathcal{N} = 4$ supersymmetries.

The field content of this theory consists of a gauge field A_μ, four fermions χ_α^i and their complex conjugates $\bar{\chi}_{\dot\alpha}^{\bar{i}}$ and six scalar fields ϕ^I, where i, $\bar{i} = 1, \ldots, 4$ describe, respectively, the fundamental $\mathbf{4}$ and antifundamental $\bar{\mathbf{4}}$ representations of the R-symmetry group $SU(4) \simeq SO(6)$ and $I = 1, \ldots, 6$ describes the fundamental representation $SO(6)$, in addition α and $\dot\alpha$ are chiral indices. Furthermore, all these fields transform in the adjoint representation of the gauge group $U(N_c)$. The Lagrangian of the theory is

$$
\begin{aligned}
\mathcal{L}_{\mathcal{N}=4} = Tr \Bigg\{ & -\frac{1}{g_{YM}^2} F \wedge \star F - D_\mu \phi^I D^\mu \phi^I + \frac{g_{YM}^2}{2} \sum_{IJ} [\phi^I, \phi^J]^2 + \theta_{YM} F \wedge F \\
& + \quad -\bar{\chi} \slashed{D} \chi + g_{YM} \left(C_I^{ij} \chi^i [\phi^I, \chi^j] + \bar{C}_I^{\bar{i}\bar{j}} \bar{\chi}^{\bar{i}} [\phi^I, \bar{\chi}^{\bar{j}}] \right) \Bigg\} \,,
\end{aligned}
\tag{2.55}
$$

where the 4×4 matrices C_I are related to the Dirac matrices of the group $SU(4)$, see these matrices explicitly in [91].

The other side of the duality rests in a type IIB string theory solution. When we consider N_c coincident BPS D-branes, we obtain, as the worldvolume theory, a

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maximally supersymmetric $U(N_c)$ gauge theory. In the low energy limit, the $U(1)$ subgroup of $U(N_c)$ decouples from the effective action on the Dp-brane, in such a way that the gauge theory is actually $SU(N_c)$.

The D3-brane solution of the type IIB theory is given by

$$ds^2_{IIB} = \frac{1}{\sqrt{H(r)}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{H(r)}\left(dr^2 + r^2 d\Omega_5^2\right) \,,$$

$$F = (1 + *)dH^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \,,$$

$$\tag{2.56}$$

where

$$H(r) = 1 + \frac{L^4}{r^4} \,, \quad L^4 = 4\pi\alpha'^2 g_s N_c \,, \tag{2.57}$$

and $d\Omega_5^2$ is the $SO(5)$-invariant metric of $\mathbb{S}^5$. In the near horizon limit, $r \to 0$, the metric approaches to

$$ds^2_{IIB} = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2 \,, \tag{2.58}$$

and we recognize it as the metric of the direct product space $AdS_5 \times \mathbb{S}^5$.

The isometry group of the gravity dual is $SO(4,2) \times SO(6)$, matching with the $\mathcal{N} = 4$ SYM considered above. While the $SO(4,2)$ symmetry of the anti-de Sitter space is reinterpreted as the conformal group in $3+1$ dimensions of the Yang-Mills theory, and the group $SO(6) \simeq SU(4)$ can be identified with the R-symmetry group of the conformal theory. Let us see how this duality works.

In the Kaluza-Klein (KK) analysis of fields in in the $AdS_5 \times \mathbb{S}^5$ background, the fields are expressed in terms of spherical harmonics of $\mathbb{S}^5$. For instance, a scalar field is expanded as

$$\phi(x^\mu, r, \theta) = \sum_k \phi_k(x^\mu, r)Y^k(y) \,, \tag{2.59}$$

where $Y^k(y)$ are spherical harmonics on $\mathbb{S}^5$, satisfying

$$\Box_{\mathbb{S}^5} Y^k = \frac{k(k+4)}{L^2}Y^k \,, \tag{2.60}$$

and $\phi_k(x^\mu, r)$ are scalar fields on the anti-de Sitter space. The dynamics of the four dimensional gauge theory is then encoded into the five dimensional AdS_5 space, then the gauge/gravity duality is a holographic duality, since a gravity theory in $(d+1)$-dimensions is dual to the a gauge theory in d-dimensions. It turns out that the radial coordinate r is related to the energy scale E of the field theory $E \sim r$,

therefore, if we consider that the field theory includes all degrees of freedom, we take $E \to \infty$, which corresponds to $r \to \infty$, and in this sense we may say that the field theory is located at the boundary of the anti-de Sitter space [14, 116].

Putting all these facts together, one can show that given generic fields Φ^i on the gravity side, we can regard the boundary values ϕ_0^i as sources of operators $\mathcal{O}^i$ on the field theory side, then

$$Z_{string} = \int_{\phi_0^i} \mathcal{D}\Phi^i e^{-S[\Phi^i]} = \left\langle e^{\int d^4 x \phi_0^i \mathcal{O}^i} \right\rangle = Z_{CFT} \; . \tag{2.61}$$

In [32] Joe Polchinski tells us an interesting history of a reader poll to determine the *greatest equation* of all time. His personal choice would be the equation (2.61), since it includes quantum field theory, general relativity, string theory, supersymmetry, extra dimensions and so on at once. Can we disagree? I don't think so.

Although there is not a precise mathematical proof of the gauge/gravity duality, there are several tests of the correspondence, see [14, 32, 111–116] for further details.

2.3 Nonabelian T-Duality

The motivation for dualities is now well motivated. In this section we return to the problem of the dualities in string theory. We may notice that the T-duality procedure is determined using the isometry group $U(1)$ of the compact manifold $\mathbb{S}^1$, but one natural problem is the generalization of T-duality to other group isometries. The first possibility is the isometry group of the torus T^n, but this case is trivially generalized, since its isometry group is just the product $U(1)^n$. Another possibility is the nonabelian generalization of T-duality, that is, we consider that the background we compactify the string theory supports a nonabelian abelian group G as the isometry group [3, 22, 23, 117–121]. Here we consider $G = SU(2)$.

We write the metric in the form

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu + 2G_{\mu i}(x)dx^\mu L^i + g_{ij}(x)L^i L^j \tag{2.62}$$

where $\mu, \nu = 1, \ldots, 7$, and L^i are the Maurer-Cartan forms for $SU(2)$. In general we also have nontrivial Kalb-Ramond two-forms

$$B = B_{\mu\nu}dx^\mu \wedge dx^\nu + B_{\mu i}dx^\mu \wedge L^i + \frac{1}{2}b_{ij}L^i \wedge L^j, \tag{2.63}$$

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and a dilaton $\Phi = \Phi(x)$. The important point here is that all dependence on the $SU(2)$ Euler angles (θ, ψ, ϕ) is contained in the one-forms L^i.

Next, define the vielbeins

$$\mathfrak{e}^A = e_\mu^A dx^\mu$$
$$\mathfrak{e}^a = \kappa^a{}_j L^j + \lambda_\mu^a dx^\mu, \tag{2.64}$$

with $A = 1, \ldots, 7$ and $a = 1, 2, 3$. Imposing

$$ds^2 = \eta_{AB}\mathfrak{e}^A\mathfrak{e}^B + \mathfrak{e}^a\mathfrak{e}^a, \tag{2.65}$$

by direct comparison with (2.62) we have

$$G_{\mu\nu} = \eta_{AB}\mathfrak{e}^A\mathfrak{e}^B + K_{\mu\nu}, \quad \kappa^a{}_i \kappa^a{}_j = g_{ij}, \quad \kappa^a{}_i \lambda_\mu^a = G_{\mu i} , \tag{2.66}$$

where we defined $\lambda_\mu^a \lambda_\nu^a = K_{\mu\nu}$.

If we combine the metric and B field into Q and E by

$$Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}, \quad Q_{\mu i} = G_{\mu i} + B_{\mu i}$$
$$Q_{i\mu} = G_{i\mu} + B_{i\mu}, \quad E_{ij} = g_{ij} + b_{ij} , \tag{2.67}$$

the Lagrangian density is

$$\mathcal{L} = Q_{\mu\nu}\partial_+ X^\mu \partial_- X^\nu + Q_{\mu i}\partial_+ X^\mu L_-^i + Q_{i\mu}L_+^i \partial_- X^\mu + E_{ij}L_+^i L_-^j , \tag{2.68}$$

where $L_\pm^i = -iTr\left(\tau^i g^{-1}\partial_\pm g\right)$, and $g \in SU(2)$. In order to find the nonabelian T-dual, we gauge the isometry $SU(2)$ by making the replacement

$$\partial_\pm g \to D_\pm g = \partial_\pm g - A_\pm g , \tag{2.69a}$$

and adding the Lagrange multiplier

$$-iv^i F_\pm^i , \quad F_\pm^i = \partial_+ A_-^i - \partial_- A_+^i - [A_+, A_-]^i . \tag{2.69b}$$

One can show that the nonabelian T-dual background is

$$\widehat{Q}_{\mu\nu} = Q_{\mu\nu} - Q_{\mu i}M_{ij}^{-1}Q_{j\nu}, \quad \widehat{E}_{ij} = M_{ij}^{-1}$$
$$\widehat{Q}_{\mu i} = Q_{\mu j}M_{ji}^{-1}, \quad \widehat{Q}_{i\mu} = -M_{ij}^{-1}Q_{j\mu}, \tag{2.70}$$

where the matrix M is defined by

$$M_{ij} = E_{ij} + \alpha' f_{ij}{}^k v_k. \tag{2.71}$$

Here $f_{ij}{}^k = \sqrt{2}\epsilon_{ijk}$ are the structure constants of the group $SU(2)$ and v_i are originally Lagrange multipliers, now dual coordinates. We can make the scaling $v_i \to \frac{1}{\sqrt{2}}v_i$, so that the dual fields are written as

$$d\hat{s}^2 = \widehat{G}_{\mu\nu}(x)dx^\mu dx^\nu + \frac{2}{\sqrt{2}}\widehat{G}_{\mu i}(x)dx^\mu dv^i + \frac{1}{2}\hat{g}_{ij}(x)dv^i dv^j \tag{2.72}$$

and

$$\widehat{B} = \widehat{B}_{\mu\nu}dx^\mu \wedge dx^\nu + \frac{1}{\sqrt{2}}\widehat{B}_{\mu i}dx^\mu \wedge dv^i + \frac{1}{4}\hat{b}_{ij}dv^i \wedge dv^j. \tag{2.73}$$

and the dilaton (transformed at the quantum level as usual)

$$\hat{\phi} = \phi - \frac{1}{2}\ln\left(\frac{\det M}{\alpha'^3}\right). \tag{2.74}$$

Besides the spectator fields x^μ, the dual theory depends on θ, ψ, ϕ, v^i, so we have too many degrees of freedom. We need to impose a gauge fixing in order to remove three of these variables, usually taken to be $\theta = \psi = \phi = 0$. Then one finds

$$(M^{-1})^{ij} = \frac{1}{\det M}\left(\det g g^{ij} + y^i y^j - \epsilon^{ijk}g_{kl}y^l\right) \tag{2.75}$$

where we have defined $b_{ij} = \epsilon_{ijk}b_k$ and $y_i = b_i + \alpha' v_i$. For a gauge fixing different than $\theta = \psi = \phi = 0$, one defines $\hat{v}_i = D_{ji}v^j$, where

$$D^{ij} = \frac{1}{2}\mathrm{Tr}\left(\tau^i g\tau^j g^{-1}\right), \quad g = e^{\frac{i}{2}\phi\tau_3}e^{\frac{i}{2}\theta\tau_2}e^{\frac{i}{2}\psi\tau_3} \tag{2.76}$$

(τ_i are the Pauli matrices) and replaces everywhere v_i by $\hat{v}_i$.

The dualization acts differently on the left- and the right-movers

$$L_+^i = -(M^{-1})_{ji}\left(\partial_+ v_j + Q_{\mu j}\partial_+ X^\mu\right) \tag{2.77a}$$

$$L_-^i = M_{ij}^{-1}\left(\partial_- v_j - Q_{j\mu}\partial_- X^\mu\right) \tag{2.77b}$$

and it produces two different sets of frames $\hat{\mathfrak{e}}_+^i$ and $\hat{\mathfrak{e}}_-^i$

$$\hat{\mathfrak{e}}_+ = -\kappa M^{-T}\left(\mathrm{d}v + Q^T \mathrm{d}X\right) + \lambda \mathrm{d}X \tag{2.78a}$$

$$\hat{\mathfrak{e}}_- = \kappa M^{-1}\left(\mathrm{d}v - Q\mathrm{d}X\right) + \lambda \mathrm{d}X \tag{2.78b}$$

that are related by a Lorentz transformation $\hat{\mathfrak{e}}_+^a = \Lambda^a{}_b\hat{\mathfrak{e}}_-^b$. The action on the spinor representation of the Lorentz group is given by

$$\Omega^{-1}\Gamma^a\Omega = \Lambda^a{}_b\Gamma^b. \tag{2.79}$$

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2.3.1 Nonabelian T-duality action on RR fields

One useful way to represent the fields in the R-R sector is as a product of spinors, that is as bispinors, see appendix (A). For instance,

$$F_{\mu_1\cdots\mu_n} = \bar{\psi}_{\mp}^L \Gamma^{\mu_1\cdots\mu_n} \psi_+^R \quad \begin{cases} -+\ IIA \\ ++\ IIB \end{cases}. \tag{2.80}$$

Taking an n-dimensional vector space V with basis $\{\gamma_1,\ldots,\gamma_n\}$, we can find an isomorphism between the Clifford $\mathcal{C}\ell(V)$ and the exterior algebra $\bigwedge V$, that is

$$\Lambda : \mathcal{C}\ell(V) \to \bigwedge V$$
$$\mathbb{1} \mapsto 1\ ,\quad \Gamma_i \mapsto \gamma_i\ ,\quad \Gamma_{i_1\cdots i_p} \mapsto \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_p}\ , \tag{2.81}$$

and we can use this fact, associated to the transformation that the nonabelian T-duality induces on the spinors (2.79), to find the action of this transformation in the R-R sector.

Considering the RR sector in the democratic formalism [122] (we consider the fluxes and their Hodges dual as well), we define the polyforms in type II supergravity

$$\textbf{IIB:}\ \ P = \frac{e^\phi}{2} \sum_{n=0}^{4} \slashed{F}_{2n+1}\ ,\quad \textbf{IIA:}\ \ \widehat{P} = \frac{e^{\hat\phi}}{2} \sum_{n=0}^{5} \widehat{\slashed{F}}_{2n} \tag{2.82}$$

Then the nonabelian T-dual forms are obtained by the transformation (applied to the nonabelian case by [22], following the work in the abelian case by [96])

$$\widehat{P} = P \cdot \Omega^{-1}. \tag{2.83}$$

We first write the p-form field strengths in the form

$$F_p = G_p^{(0)} + G_{p-1}^a \wedge \mathfrak{e}^a + \frac{1}{2} G_{p-2}^{ab} \wedge \mathfrak{e}^a \wedge \mathfrak{e}^b + G_{p-3}^{(3)} \wedge \mathfrak{e}^1 \wedge \mathfrak{e}^2 \wedge \mathfrak{e}^3. \tag{2.84}$$

Using a similar decomposition for the T-dual p-forms $\hat{F}_p$ in terms of the T-dual vielbeins $\mathfrak{e}'$,

$$\hat{F}_p = \hat{G}_p^{(0)} + \hat{G}_{p-1}^a \wedge \mathfrak{e}'^a + \frac{1}{2} \hat{G}_{p-2}^{ab} \wedge \mathfrak{e}'^a \wedge \mathfrak{e}'^b + \hat{G}_{p-3}^{(3)} \wedge \mathfrak{e}'^1 \wedge \mathfrak{e}'^2 \wedge \mathfrak{e}'^3\ , \tag{2.85}$$

we have the transformation rules

$$\hat{G}_p^{(0)} = e^{\phi-\hat\phi}(-A_0 G_p^{(3)} + A_a G_p^a)$$

$$\hat{G}^a_{p-1} = e^{\phi - \hat{\phi}}\left(-\frac{A_0}{2}\epsilon^{abc}G^{bc}_{p-1} + A_b G^{ab}_{p-1} + A_a G^{(0)}_{p-1}\right)$$

$$\hat{G}^{ab}_{p-2} = e^{\phi - \hat{\phi}}\left[\epsilon^{abc}(A_c G^{(3)}_{p-2} + A_0 G^c_{p-2}) - (A_a G^b_{p-2} - A_b G^a_{p-2})\right]$$

$$\hat{G}^{(3)}_{p-3} = e^{\phi - \hat{\phi}}\left(\frac{A_a}{2}\epsilon^{abc}G^{bc}_{p-3} + A_0 G^{(0)}_{p-3}\right). \tag{2.86}$$

Here, defining $y_i = b_i + \alpha' v_i$ as before and

$$z^i = \frac{y^i}{\sqrt{\det g}}$$

$$\zeta^a = \kappa^a{}_i z^i = \kappa^a{}_i \frac{y^i}{\sqrt{\det g}} \tag{2.87}$$

the coefficients of the transformation rules are

$$A_0 = \frac{1}{\sqrt{1 + \zeta^2}} = \frac{\sqrt{\det g}}{\sqrt{\det g + (\kappa^a{}_i y^i)^2}}$$

$$A_a = \frac{\zeta_a}{\sqrt{1 + \zeta^2}} = \frac{\kappa^a{}_i y^i}{\sqrt{\det g + (\kappa^a{}_i y^i)^2}}. \tag{2.88}$$

In the next chapter we start using the knowledge we have drawn here to study new string backgrounds and their dual field theories.

Chapter 3

D5-branes on $\mathbb{S}^3$

I N THIS CHAPTER we consider a T-duality on the Maldacena-Nastase solution [7] that defines also a dual field theory. We start with a review of the solution due to Canoura et al. [9], which contains the original solution [7] as a special case. It is a type-IIB supergravity solution that consists of D5-branes wrapping a 3-cycle in a manifold that supports a G_2-structure and in the IR limit, this theory is dual to $\mathcal{N} = 1$ SYM in three dimensions.

In [7,123,124], it was found a solution of 5-dimensional supergravity which can be lifted to 7 dimensions and then to 10 dimensions. In this case we have a gravitational solution that holographically describes D5-branes wrapping a three-cycle inside a G_2 manifold. In the IR limit, the theory living in the worldvolume of these branes was identified as being dual to $\mathcal{N} = 1$ $SU(N_c)$ SYM in three-dimensions with Chern-Simons level $\kappa = N_c/2$.

In this particular solution, we start with N_c D5-branes, where the field theory living on the worldvolume of these branes carries 16 supercharges, and we wrap them on a sphere $\mathbb{S}^3$, what in general breaks supersymmetry. In order to preserve some fraction of the original supersymmetries, we twist the fields in such a way that we have four supercharges [125, 126], equivalent to $\mathcal{N} = 1$ supersymmetry in $2 + 1$ dimensions.

In [9] the ansatz of [7, 123] was generalized and this allowed one to find a new class of solutions in which in the UV limit the metric is a product of a G_2 cone and a three dimensional Minkowski space, and a constant dilaton, in contrast to the original behaviour of the Maldacena-Nastase solution, where the dilaton diverges

as the holographic coordinate goes to infinity. It is important to realize that this solution corresponds to D5-branes wrapped on a three-cycle of a G_2 cone in which the near-horizon effects of the branes on the metric become negligible in the UV limit.

As we already mentioned, realistic theories require fields transforming in the fundamental representation. To address this, one considers flavor branes in the gravity side, which is equivalent to adding an open string sector [127]. One can start by studying the *quenched* approximation, that is when probe branes are used in a way that the number of flavor branes N_f is negligible compared to the number N_c of color branes. Then, the next natural step is to consider the unquenched case, that is the case in which the number of flavor branes is of the same order as the number color branes [128–130].

Canoura, Merlatti and Ramallo [9] added massless fundamental flavors to the Maldacena-Nastase (hereafter MNa) solution in the unquenched case. The authors found that this system with $N_f \geq 2N_c$ dramatically differs from $N_f < 2N_c$. Massive fundamental flavors were added to the MNa solution in [131] and the author showed that is is possible to find a solution which interpolates between the deformed unflavored MNa background and the massless flavored background.

As pointed in [132,133], we can obtain the UV completion of this solution considering a G_2-structure rotation [134] which is a solution generating technique analogous to the U-duality. The rotation procedure is implemented in a type IIA solution with $\mathcal{N} = 1$ SUSY and gives a more general type IIA solution. The important point is that in this rotation procedure, we have an extra warp factor in the metric and this term ensures the finiteness of the cycle along the energy scale.

The gauge theory analysis of the rotated MNa solution was performed in [132], and the author showed that the dual field theory is confining and that in the IR limit, the Chern-Simons term dominates the dynamics of the theory.

In [133], the nonabelian T-duality has been considered along the $SU(2)$ isometry of the deformed MNa solution [9], and this gave a massive type IIA solution, with no trivial field in the RR sector. The author showed that the generated solution is dual to a confining Chern-Simons gauge theory and using the gauge/gravity correspondence he studied several holographic properties of the dual field theory.

In this chapter we perform the abelian T-duality on MNa solution along an $U(1)$ isometry in the D5-brane solution, which gives a D4-brane solution wrapping

a two-cycle. Then we compute Maxwell and Page charges associated to this new solution.

Moreover, we consider some aspects of the dual gauge theory, defining it in the process. In section 3.4.1 we find the quark-antiquark potential and we see that the requirements for confinement are satisfied. In such a case we are able to compute the string tension. Next we follow considering the gauge coupling and the entanglement entropy, which has been used as a probe of confinement. Finally, we study some conditions in which we can treat the wrapped D4-branes as a domain wall, so that we induce a Chern-Simons term in the gauge theory.

3.1 Wrapped fivebranes on a three-cycle

In general, when we put a supersymmetric field theory on a curved manifold Ω, we break SUSY since we do not have a killing spinor satisfying $(\partial_\mu + \omega_\mu)\epsilon = 0$. On the other hand, if the theory has an R-symmetry, we can consider that the spin connection is equal to the gauge connection arising from the R-symmetry group, that is $\omega_\mu = A_\mu$, in such a way that now we can find a Killing spinor satisfying $(\partial_\mu + \omega_\mu - A_\mu)\epsilon = \partial_\mu\epsilon = 0$. This resourceful way of preserving supersymmetry is exactly the way that branes wrapping cycles in string/M theory operate to do it [125, 126], and theories satisfying this condition are called twisted theories.

Therefore, in order to preserve some fraction of supersymmetry of a type IIB (string theory) configuration, which consists of NS 5-branes wrapping a 3-sphere, we need to consider a twisting. The R-symmetry group is simply the rotation group $SO(4) \simeq SU(2)_L \times SU(2)_R$ and the spin connection lives in the Lie algebra $su(2)$ (tangent space of the space $\mathbb{S}^3 \simeq SU(2)$). In this case we embed the spin connection into $SU(2)_L$ and it can be checked to be enough to preserve the $\mathcal{N} = 1$ SUSY in three dimensions [7].

At low energies, compared to the inverse radius of $\mathbb{S}^3$, we have pure $\mathcal{N} = 1$ SYM theory in three dimensions with gauge group $U(N)$. Additionally, if we add a flux of the NS-NS sector H on the worldvolume $\mathbb{S}^3$ we induce a Chern-Simons coupling

in three-dimensions. In the S-dual description we have, on the D5-brane, the term

$$
\frac{1}{16\pi^3} \int_{\Sigma_6} B^{RR} \wedge Tr\left(F \wedge F\right) = -\frac{1}{16\pi^3} \int_{\Sigma_6} \mathrm{d}B^{RR} \wedge Tr\left(A \wedge \mathrm{d}A + \frac{2}{3}A^3\right)
$$
$$
= -\frac{\tilde{\kappa}}{4\pi} \int_{\Sigma_3} Tr\left(A \wedge \mathrm{d}A + \frac{2}{3}A^3\right) ,
$$

(3.1)

where the parameter $\tilde{\kappa}$ is related to the three dimensional Chern-Simons level κ by $\tilde{\kappa} = \kappa + \frac{N}{2}$ and this extra term appears when we integrate out all the six dimensional KK model.

In [7], the authors considered a supergravity solution that describes the system of branes wrapping an $\mathbb{S}^3$ given by

$$
\mathrm{d}s^2 = \mathrm{d}s_7^2 + \frac{\alpha'N}{4}\left(\tilde{\omega}^a - A^a\right)^2
$$

(3.2a)

$$
F = N\left[-\frac{1}{4}\left(\tilde{\omega}^1 - A^1\right) \wedge \left(\tilde{\omega}^2 - A^2\right) \wedge \left(\tilde{\omega}^3 - A^3\right) + \frac{1}{4}F^a \wedge \left(\tilde{\omega}^a - A^a\right)\right] + H ,
$$

(3.2b)

where

$$
\mathrm{d}s_7^2 = \mathrm{d}x_{2,1}^2 + \alpha'N\left(\mathrm{d}\rho^2 + R(\rho)^2\mathrm{d}\Omega_3^2\right)
$$

(3.3a)

$$
A^a = \frac{1 + w(\rho)}{2}\omega_L^a
$$

(3.3b)

$$
H = \frac{N}{16}\left(w^3(\rho) - 3w(\rho) + 2\right)\omega^1 \wedge \omega^2 \wedge \omega^3 ,
$$

(3.3c)

ω^a are the Maurer-Cartan forms of $SU(2)$ and this background has a nontrivial dilaton[1] φ. No analytic solution for this ansatz is known, but its asymptotic behaviour for large ρ is simply

$$
R^2(\rho) \sim 2\rho , \quad w(\rho) \sim 1/4\rho , \quad \varphi \sim -\rho + (3\ln\rho)/8 ,
$$

(3.4a)

while, for small ρ we find

$$
R^2(\rho) \sim \rho^2 + \mathcal{O}(\rho^4) , \quad w(\rho) \sim 1 + \mathcal{O}(\rho^2)\rho , \quad \varphi \sim \varphi_0 + \mathcal{O}(\rho^2) .
$$

(3.4b)

The topology of the seven dimensional space spanned by $(\rho, \omega^a, \tilde{\omega}^a)$ is asymptotically that of a cone whose base is $\mathbb{S}^3 \times \mathbb{S}^3$. The above solution bears the name *Maldacena-Nastase solution*.

---

[1]In this chapter we write the dilaton as φ and we use the symbol ϕ to denote angular coordinates, including the coordinate X^9 along which we perform the duality.

3.2 Deformed Maldacena-Nastase solution

In [9] the ansatz was generalized and this solution has the original solution of the previous section (3.1) as a special case. The string frame metric is given by

$$ds_{st}^2 = e^\varphi \left(dx_{1,2}^2 + ds_7^2 \right),\tag{3.5}$$

and the internal part of the metric, which describes the manifold supporting a G_2-structure, is

$$ds_7^2 = N_c \left[e^{2g} dr^2 + \frac{e^{2h}}{4}(\sigma^i)^2 + \frac{e^{2g}}{4} \left(\omega^i - \frac{1}{2}(1+w)\sigma^i \right)^2 \right],\tag{3.6}$$

where we are using an optimum holographic coordinate defined in [132]. Also, σ^i and ω^i are two sets of $SU(2)$ Maurer-Cartan forms satisfying

$$d\lambda_a^i = -\frac{1}{2}\epsilon_{ijk}\lambda_a^j \wedge \lambda_a^k,\tag{3.7}$$

where $\lambda_1^i = \sigma^i$ and $\lambda_2^i = \omega^i$ for $i = 1, 2, 3$. These forms can be represented in terms of Euler angles as

$$\lambda_a^1 = \cos\psi_a d\theta_a + \sin\psi_a \sin\theta_a d\phi_a\tag{3.8}$$

$$\lambda_a^2 = -\sin\psi_a d\theta_a + \cos\psi_a \sin\theta_a d\phi_a\tag{3.9}$$

$$\lambda_a^3 = d\psi_a + \cos\theta_a d\phi_a,\tag{3.10}$$

for $0 \le \theta_a \le \pi$, $0 \le \phi_a < 2\pi$, $0 \le \psi_a < 4\pi$.

Also, the MNa solution has a nontrivial RR 3-form

$$F_3 = \frac{N_c}{4} \left\{ (\sigma^1 \wedge \sigma^2 \wedge \sigma^3 - \omega^1 \wedge \omega^2 \wedge \omega^3) + \frac{\gamma'}{2} dr \wedge \sigma^i \wedge \omega^i - \right.$$
$$\left. - \frac{(1+\gamma)}{4}\epsilon_{ijk}[\sigma^i \wedge \sigma^j \wedge \omega^k - \omega^i \wedge \omega^j \wedge \sigma^k] \right\}.\tag{3.11}$$

One can easily show that this field strength is generated by the following two-form

potential

$$C^{(2)} = \left(-\frac{1}{4}N_c \cos\theta_1\right)d\phi_1 \wedge d\psi_1 + \left(\frac{1}{4}N_c \cos\theta_2\right)d\phi_2 \wedge d\psi_2+$$
$$+\frac{N_c(1+\gamma)}{8}d\psi_1 \wedge d\psi_2 + \frac{N_c(1+\gamma)}{8}\sin\theta_1 \sin(\psi_1 - \psi_2)d\phi_1 \wedge d\theta_2+$$
$$+\frac{N_c(1+\gamma)}{8}(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\psi_1 - \psi_2))d\phi_1 \wedge d\phi_2 \qquad (3.12)$$
$$+\frac{N_c(1+\gamma)}{8}\cos\theta_1 d\phi_1 \wedge d\psi_2 + \frac{N_c(1+\gamma)}{8}\cos(\psi_1 - \psi_2)d\theta_1 \wedge d\theta_2$$
$$+\left(-\frac{N_c(1+\gamma)}{8}\sin(\psi_1 - \psi_2)\sin\theta_2\right)d\theta_1 \wedge d\phi_2 + \frac{N_c(1+\gamma)}{8}\cos\theta_2 d\psi_1 \wedge d\phi_2 \ ,$$

so that $F_3 = dC^{(2)}$.

Unfortunately, the solution for these equations is known just semi-analytically in the IR and UV limits. In the IR limit, that is, $r \sim 0$ we have

$$e^{2g} = g_0 + \frac{(g_0 - 1)(9g_0 + 5)}{12g_0}r^2 + \dots \qquad (3.13a)$$

$$e^{2h} = g_0 r^2 - \frac{3g_0^2 - 4g_0 + 4)}{18g_0}r^4 + \dots \qquad (3.13b)$$

$$w = 1 - \frac{3g_0 - 2}{3g_0}r^2 + \dots \qquad (3.13c)$$

$$\gamma = 1 - \frac{1}{3}r^2 + \dots \qquad (3.13d)$$

$$\phi = \phi_0 + \frac{7}{24g_0^2}r^2. \qquad (3.13e)$$

On the other hand, in the UV limit, where $r \sim \infty$, we have

$$e^{2g} = c_1 e^{4r/3} - 1 + \frac{33}{4c_1}e^{-4r/3} \qquad (3.14a)$$

$$e^{2h} = \frac{3c_1}{4}e^{4r/3} + \frac{9}{4} - \frac{77}{16c_1}e^{-4r/3} \qquad (3.14b)$$

$$w = \frac{2}{c_1}e^{-4r/3} + \dots \qquad (3.14c)$$

$$\gamma = \frac{1}{3} + \dots \qquad (3.14d)$$

$$\phi = \phi_\infty + \frac{2}{c_1^2}e^{-8r/3}. \qquad (3.14e)$$

We write the whole set of components of the string frame metric as

$$x^M = \{x^\mu, x^A\}; \quad \{(\mu = 0, 1, 2); (A = r, \tilde{\alpha}, \alpha)\},$$

where
$$\{x^r \equiv r; x^{\tilde{\alpha}} \equiv \theta_1, \phi_1, \psi_1; x^{\alpha} = \theta_2, \phi_2, \psi_2\}.$$

Now we have
$$(\lambda_a^i)^2 = d\theta_a^2 + d\phi_a^2 + d\psi_a^2 + 2\cos\theta_a d\psi_a d\phi_a \qquad (3.15)$$

and
$$\begin{aligned}
\omega^i \sigma^i =\ & \cos(\psi_1 - \psi_2)d\theta_1 d\theta_2 - \sin(\psi_1 - \psi_2)\sin\theta_2 d\theta_1 d\phi_2 \\
& + \sin(\psi_1 - \psi_2)\sin\theta_1 d\phi_1 d\theta_2 + [\cos(\psi_1 - \psi_2)\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2]d\phi_1 d\phi_2
\end{aligned}$$
$$(3.16)$$
$$+ \cos\theta_1 d\phi_1 d\psi_2 + \cos\theta_2 d\psi_1 d\phi_2 + d\psi_1 d\psi_2.$$

Then, we write the string-frame metric as
$$ds_{st}^2 = g_{MN}dx^M dx^N = e^{\varphi}dx_{1,2}^2 + \Delta dr^2 + \Sigma(\sigma^i)^2 + \Omega(\omega^i)^2 + 2\Xi\omega^i\sigma^i\ , \qquad (3.17)$$

where we define
$$\Delta = e^{\varphi+2g}N_c \qquad (3.18a)$$

$$\Sigma = \frac{e^{\varphi}}{4}N_c\left(e^{2h} + \frac{e^{2g}}{4}(1+w)^2\right) \equiv e^{\varphi}\widetilde{\Sigma} \qquad (3.18b)$$

$$\Omega = \frac{e^{\varphi+2g}}{4}N_c \equiv \frac{\Delta}{4} \qquad (3.18c)$$

$$\Xi = -\frac{e^{\varphi+2g}}{8}(1+w)N_c \equiv -\frac{\Omega}{2}(1+w) \qquad (3.18d)$$

for later convenience. Finally, using that $M = \{\mu, A\}$ we find the components of the metric matrix
$$(g_{MN}) = \begin{pmatrix} g_{\mu\nu} = e^{\varphi}\eta_{\mu\nu} & g_{\mu A} = 0 \\ g_{A\mu} = 0 & g_{AB} \end{pmatrix}.$$

Obviously, we need to find just the components g_{AB} and these are

$g_{rr} = \Delta$	$g_{r\mu} = 0$	$g_{r\tilde{\alpha}} = g_{r\alpha} = 0$

$g_{\tilde{\theta}\tilde{\theta}} = g_{\tilde{\phi}\tilde{\phi}} = g_{\tilde{\psi}\tilde{\psi}} = \Sigma$	$g_{\tilde{\phi}\tilde{\psi}} = \Sigma\cos\theta_1$
$g_{\theta\theta} = g_{\phi\phi} = g_{\psi\psi} = \Omega$	$g_{\phi\psi} = \Omega\cos\theta_2$

$g_{\theta\tilde{\theta}} = \Xi\cos(\psi_1 - \psi_2)$	$g_{\phi\tilde{\theta}} = -\Xi\sin(\psi_1 - \psi_2)\sin\theta_2$
$g_{\theta\tilde{\phi}} = \Xi\sin(\psi_1 - \psi_2)\sin\theta_1$	$g_{\phi\tilde{\phi}} = \Xi[\sin\theta_1\sin\theta_2\cos(\psi_1 - \psi_2) + \cos\theta_1\cos\theta_2]$
$g_{\theta\tilde{\psi}} = 0$	$g_{\phi\tilde{\psi}} = \Xi\cos\theta_2$

$g_{\psi\tilde{\theta}} = 0$
$g_{\psi\tilde{\phi}} = \Xi \cos\theta_1$
$g_{\psi\tilde{\psi}} = \Xi$

3.3   D4-brane solution

Now we perform a T-duality transformation in a direction along the brane, namely, the $x^9 \equiv x^{\tilde{\phi}} = \phi_1$ direction. If we consider the type-IIA solution with NS-NS sector given by $\{\tilde{\varphi},\ \tilde{g}_{MN},\ \mathcal{B}_{MN}\}$, the Buscher's rules [2,93–95] (see section 2.2.2 and the appendix A) are

$$
e^{2\tilde{\varphi}} = \frac{e^{2\varphi}}{|g_{\tilde{\phi}\tilde{\phi}}|} \qquad \tilde{g}_{\tilde{\phi}\tilde{\phi}} = \frac{1}{g_{\tilde{\phi}\tilde{\phi}}}
$$

$$
\tilde{g}_{MN} = g_{MN} - \frac{g_{\tilde{\phi}M}g_{\tilde{\phi}N} - B_{\tilde{\phi}M}B_{\tilde{\phi}N}}{g_{\tilde{\phi}\tilde{\phi}}} \qquad \tilde{g}_{\tilde{\phi}M} = \frac{1}{g_{\tilde{\phi}\tilde{\phi}}}B_{\tilde{\phi}M}
$$

$$
\mathcal{B}_{MN} = B_{MN} - 2\frac{B_{\tilde{\phi}[M}g_{N]\tilde{\phi}}}{g_{\tilde{\phi}\tilde{\phi}}} \qquad \mathcal{B}_{M\tilde{\phi}} = -\frac{g_{M\tilde{\phi}}}{g_{\tilde{\phi}\tilde{\phi}}}
$$

3.3.1   NS-NS sector

Using the transformation rules above, the dilaton is

$$
e^{2\tilde{\varphi}} = \frac{1}{\Sigma}e^{2\varphi} = \frac{1}{\widetilde{\Sigma}}e^{\varphi} \tag{3.19}
$$

and the dual metric is

$$
\begin{aligned}
d\tilde{s}^2_{st} =& e^{2\tilde{\varphi}}\widetilde{\Sigma}dx^2_{1,2} + \Delta dr^2 + \frac{1}{\Sigma}d\phi_1^2 + \Sigma(d\theta_1^2 + \sin^2\theta_1 d\psi_1^2) \\
& + 2\Xi\left[(\cos\psi_1\omega^1 - \sin\psi_1\omega^2)d\theta_1 - \sin\theta_1\cos\theta_1(\sin\psi_1\omega^1 + \cos\psi_1\omega^2)d\psi_1\right. \\
& \left. + \sin^2\theta_1\omega^3 d\psi_1\right] + \Omega(\omega^i)^2 \\
& - \frac{\Xi^2}{\Sigma}\left[\sin^2\psi_1\sin^2\theta_1(\omega^1)^2 + 2\sin\psi_1\cos\psi_1\sin^2\theta_1\omega^1\omega^2\right. \\
& + 2\sin\psi_1\cos\theta_1\sin\theta_1\omega^1\omega^3 + \cos^2\psi_1\sin^2\theta_1(\omega^2)^2 + 2\cos\psi_1\sin\theta_1\cos\theta_1\omega^2\omega^3 \\
& \left. + \cos^2\theta_1(\omega^3)^2\right]\ ,
\end{aligned} \tag{3.20}
$$

where we can rewrite the coefficients in terms of the type-IIA dilaton $\tilde{\varphi}$

$$\Delta = e^{2\tilde{\varphi}+2g}N_c\widetilde{\Sigma}$$
$$\Sigma = e^{2\tilde{\varphi}}\widetilde{\Sigma}^2$$
$$\Omega = \frac{e^{2\tilde{\varphi}+2g}}{4}N_c\widetilde{\Sigma}$$
$$\Xi = -\frac{e^{2\tilde{\varphi}+2g}}{8}(1+w)N_c\widetilde{\Sigma}. \tag{3.21}$$

Also, we define a first rotation

$$\tilde{\omega}^1 = \cos\psi_1\omega^1 - \sin\psi_1\omega^2 = \cos(\psi_2-\psi_1)d\theta_2 + \sin(\psi_2-\psi_1)\sin\theta_2 d\phi_2$$
$$\tilde{\omega}^2 = \sin\psi_1\omega^1 + \cos\psi_1\omega^2 = -\sin(\psi_2-\psi_1)d\theta_2 + \cos(\psi_2-\psi_1)\sin\theta_2 d\phi_2$$
$$\tilde{\omega}^3 = \omega^3 = d\psi_2 + \cos\theta_2 d\phi_2$$
$$\tilde{\sigma}^1 = \cos\psi_1\sigma^1 - \sin\psi_1\sigma^2$$
$$\tilde{\sigma}^2 = \sin\psi_1\sigma^1 + \cos\psi_1\sigma^2$$
$$\tilde{\sigma}^3 = \sigma^3 \; . \tag{3.22}$$

We then consider a second rotation

$$\hat{\omega}^1 = \tilde{\omega}^1$$
$$\hat{\omega}^2 = \cos\theta_1\tilde{\omega}^2 - \sin\theta_1\tilde{\omega}^3$$
$$\hat{\omega}^3 = \sin\theta_1\tilde{\omega}^2 + \cos\theta_1\tilde{\omega}^3$$
$$\hat{\sigma}^1 = \tilde{\sigma}^1$$
$$\hat{\sigma}^2 = \cos\theta_1\tilde{\sigma}^2 - \sin\theta_1\tilde{\sigma}^3$$
$$\hat{\sigma}^3 = \sin\theta_1\tilde{\sigma}^2 + \cos\theta_1\tilde{\sigma}^3 \; , \tag{3.23}$$

obtaining the metric

$$d\tilde{s}_{st}^2 = \frac{N_c}{4}e^{2\tilde{\varphi}}\left(e^{2h} + \frac{e^{2g}}{4}(1+w)^2\right)dx_{1,2}^2 + \Delta dr^2 + \frac{1}{\Sigma}d\phi_1^2 + \Sigma(d\theta_1^2 + \sin^2\theta_1 d\psi_1^2)$$
$$+ 2\Xi[\tilde{\omega}^1 d\theta_1 - \sin\theta_1\cos\theta_1\tilde{\omega}^2 d\psi_1 + \sin^2\theta_1\tilde{\omega}^3 d\psi_1] + \Omega(\tilde{\omega}^i)^2$$
$$- \frac{\Xi^2}{\Sigma}[\sin\theta_1\tilde{\omega}^2 + \cos\theta_1\tilde{\omega}^3]^2 \; ,$$

$$\tag{3.24}$$

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or reorganizing

$$
\begin{aligned}
d\tilde{s}_{st}^2 =& \frac{N_c}{4} e^{2\tilde{\varphi}} \left(e^{2h} + \frac{e^{2g}}{4}(1+w)^2 \right) dx_{1,2}^2 + \Delta dr^2 + \frac{1}{\Sigma} d\phi_1^2 + \\
& + \left(\Sigma - e^{2\tilde{\varphi}+2g} \frac{(1+w)^2}{4^2} N_c \widetilde{\Sigma} \right) (d\theta_1^2 + \sin^2\theta_1 d\psi_1^2) \\
& + e^{2\tilde{\varphi}+2g} \frac{N_c}{4} \widetilde{\Sigma} \left[\left(\hat{\omega}^1 - \frac{1}{2}(1+w)d\theta_1 \right)^2 + \left(\hat{\omega}^2 + \frac{1}{2}(1+w)\sin\theta_1 d\psi_1 \right)^2 \right] \\
& + \left(\Omega - \frac{\Xi^2}{\Sigma} \right) (\hat{\omega}^3)^2.
\end{aligned}
\tag{3.25}
$$

Finally, the 2-form field, which vanishes in the original solution, is nontrivial after the T-duality and one can write in the following form

$$
\begin{aligned}
\mathcal{B} = -& \left\{ \cos\theta_1 d\psi_1 \wedge d\phi_1 + \frac{\Xi}{\Sigma} \sin(\psi_1 - \psi_2) \sin\theta_1 d\theta_2 \wedge d\phi_1 \right. \\
& \left. + \frac{\Xi}{\Sigma}[\sin\theta_1 \sin\theta_2 \cos(\psi_1 - \psi_2) + \cos\theta_1 \cos\theta_2] d\phi_2 \wedge d\phi_1 + \frac{\Xi}{\Sigma} \cos\theta_1 d\psi_2 \wedge d\phi_1 \right\}.
\end{aligned}
\tag{3.26}
$$

One important cycle in this background is

$$
\mathcal{C}_2 = \{\theta_1 = \theta_2 \equiv \theta; \psi_1 = \psi_2 \equiv \psi | \phi_1, \phi_2, r, x_1, x_2 = \text{const.}\}
\tag{3.27}
$$

which is the cycle where the metric is wrapped.[2] The induced metric is given by

$$
ds_{\mathcal{C}_2}^2 = (\Sigma + 2\Xi + \Omega) d\theta^2 + \left(\Omega + \Sigma\sin^2\theta + 2\Xi\sin^2\theta - \frac{\Xi^2}{\Sigma}\cos^2\theta \right) d\psi^2 ,
\tag{3.28}
$$

and vanishes in the IR limit. The B field vanishes on this cycle.

3.3.2 R-R sector

Remember that the RR-sector for the type IIA supergravity is $\{C^{(1)}, C^{(3)}\}$ while the RR-sector for type IIB supergravity is $\{C^{(0)}, C^{(2)}, C^{(4)}\}$ and in the present case, the

---

[2]This cycle is a restriction for $\phi_1 = \phi_2 =$const. of the Σ_3 cycle $\{\sigma^i = \omega^i\}$ on which D5-branes are wrapped. Since ϕ_1 is the T-duality direction, after it, the D4-branes are wrapped on $\mathcal{C}_2$. Moreover, since supersymmetry was preserved before the T-duality, it should be preserved afterwards, making it likely the cycle is supersymmetric.

nontrivial field is just $C^{(2)}$, whose field strength is given by the F_3 in (3.11),

$$F_3 = \frac{N_c}{4} \left\{ (\sigma^1 \wedge \sigma^2 \wedge \sigma^3 - \omega^1 \wedge \omega^2 \wedge \omega^3) + \frac{\gamma'}{2} dr \wedge \sigma^i \wedge \omega^i - \right.$$
$$\left. - \frac{(1+\gamma)}{4} \epsilon_{ijk}[\sigma^i \wedge \sigma^j \wedge \omega^k - \omega^i \wedge \omega^j \wedge \sigma^k]. \right\} \tag{3.29}$$

Given the T-duality rules for going from type-IIB to type-IIA supergravity,

$$C^{(2n+1)}_{M_1...M_{2n+1}} = C^{(2n+2)}_{M_1...M_{2n+1}\tilde{\phi}} + (2n+1)B_{[M_1|\tilde{\phi}|}C^{(2n)}_{M_2...M_{2n+1}]}$$
$$+ 2n(2n+1)B_{[M_1|\tilde{\phi}|}g_{M_2|\tilde{\phi}|}C^{(2n)}_{M_3...M_{2n+1}]\tilde{\phi}}/g_{\tilde{\phi}\tilde{\phi}} \tag{3.30a}$$

$$C^{(2n+1)}_{M_1...M_{2n}\tilde{\phi}} = C^{(2n)}_{M_1...M_{2n}} - 2ng_{[M_1|\tilde{\phi}|}C^{(2n)}_{M_2...M_{2n}]\tilde{\phi}}/g_{\tilde{\phi}\tilde{\phi}} , \tag{3.30b}$$

we can use (3.12) and find the RR potential forms of the type IIA-solution

n=0
In this case, we have the following components of the dual theory

$$C^{(1)}_{M_1} = C^{(2)}_{M_1\tilde{\phi}}$$
$$C^{(1)}_{\tilde{\phi}} = C^{(0)} = 0 , \tag{3.31}$$

so we obtain the potential

$$C^{(1)} = - \left\{ \frac{N_c(1+\gamma)}{8} [\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\psi_1 - \psi_2)]d\phi_2 - \frac{N_c}{4} \cos\theta_1 d\psi_1 \right.$$
$$\left. + \frac{N_c(1+\gamma)}{8} \sin(\psi_1 - \psi_2) \sin\theta_1 d\theta_2 + \frac{N_c}{8}(1+\gamma) \cos\theta_1 d\psi_2 \right\} . $$
$$\tag{3.32}$$

n=1
In this case, we have

$$C^{(3)}_{M_1M_2M_3} = C^{(4)}_{M_1M_2M_3\tilde{\phi}} = 0$$
$$C^{(3)}_{M_1M_2\tilde{\phi}} = C^{(2)}_{M_1M_2} - \frac{1}{g_{\tilde{\phi}\tilde{\phi}}}(g_{M_1\tilde{\phi}}C^{(2)}_{M_2\tilde{\phi}} - g_{M_2\tilde{\phi}}C^{(2)}_{M_1\tilde{\phi}}). \tag{3.33}$$

Therefore we obtain the three-form potential

$$
\begin{aligned}
C^{(3)} = &-\frac{N_c(1+\gamma)}{8}\cos(\psi_1-\psi_2)d\theta_1\wedge d\phi_1\wedge d\theta_2\\
&+\frac{N_c(1+\gamma)}{8}\sin\theta_2\sin(\psi_1-\psi_2)d\theta_1\wedge d\phi_1\wedge d\phi_2\\
&-\frac{N_c}{8\Sigma}\cos\theta_1\sin\theta_1\sin(\psi_1-\psi_2)\left[2\Xi+\Sigma(1+\gamma)\right]d\psi_1\wedge d\phi_1\wedge d\theta_2\\
&-\left[\frac{N_c(1+\gamma)}{8}\cos\theta_2\right.\\
&\left.+\frac{N_c}{4\Sigma}\left(\Xi+\frac{(1+\gamma)}{2}\Sigma\right)\cos\theta_1[\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\psi_1-\psi_2)]\right]\times\\
&\times d\psi_1\wedge d\phi_1\wedge d\phi_2\\
&-\left[\frac{N_c(1+\gamma)}{8}+\frac{N_c}{4\Sigma}\left(\Xi+\frac{\Sigma}{2}(1+\gamma)\right)\cos^2\theta_1\right]d\psi_1\wedge d\phi_1\wedge d\psi_2\\
&+\frac{N_c}{4}\cos\theta_2 d\phi_1\wedge d\phi_2\wedge d\psi_2.
\end{aligned}
\tag{3.34}
$$

We have generated a type IIA-solution of supergravity wich consists of N_c D4-branes wrapping a two-cycle and with a perpendicular S^1 manifold. This solution has nontrivial RR 2 and 4-forms defined by $F_2 = dC^{(1)}$ and $F_4 = dC^{(3)}$.

For completeness, starting from a solution of supergravity in eleven dimensions, one can consider a dimensional reduction on a circle S^1 to a type-IIA solution. Conversely, given a solution of the type-IIA supergravity, we can lift it to a solution of eleven dimensional supergravity. In fact, the eleven dimensional fields corresponding to the type IIA ones are written as

$$
g_{MN}^{(11)} = e^{-2\tilde\varphi/3}\tilde g_{MN} + e^{4\tilde\varphi/3}C_M^{(1)}C_N^{(1)} \qquad (C^{(3)})_{MNP}^{11} = C_{MNP}^{(3)}
$$

$$
g_{M,11}^{(11)} = e^{4\tilde\varphi/3}C_M^{(1)} \qquad (C^{(3)})_{MN,11}^{11} = \mathcal{B}_{MN}
$$

$$
g_{11,11}^{(11)} = e^{4\tilde\varphi/3}
$$

Rewriting the dual metric (3.20) as

$$d\tilde{s}_{st}^2 = \frac{N_c}{4}e^{2(\tilde{\varphi})}\left(e^{2h} + \frac{e^{2g}}{4}(1+w)^2\right)dx_{1,2}^2 + \Delta dr^2 + \frac{1}{\Sigma}d\phi_1^2 + \Sigma(d\theta_1^2 + \sin^2\theta_1 d\psi_1^2)$$

$$+ 2\Xi[\cos(\psi_1 - \psi_2)d\theta_1 d\theta_2 - \sin(\psi_1 - \psi_2)\sin\theta_2 d\theta_1 d\phi_2$$

$$- \sin(\psi_1 - \psi_2)\sin\theta_1\cos\theta_1 d\psi_1 d\theta_2$$

$$+ (\cos\theta_2\sin^2\theta_1 - \cos\theta_1\sin\theta_1\sin\theta_2\cos(\psi_1 - \psi_2))d\psi_1 d\phi_2 + \sin^2\theta_1 d\psi_1 d\psi_2]$$

$$+ \left(\Omega - \frac{\Xi^2}{\Sigma}\sin^2(\psi_1 - \psi_2)\sin^2\theta_1\right)d\theta_2^2$$

$$+ \left(\Omega - \frac{\Xi^2}{\Sigma}[\sin\theta_1\sin\theta_2\cos(\psi_1 - \psi_2) + \cos\theta_1\cos\theta_2]^2\right)d\phi_2^2$$

$$+ \left(\Omega - \frac{\Xi^2}{\Sigma}\cos^2\theta_1\right)d\psi_2^2 +$$

$$+ 2\left(\Omega\cos\theta_2 - \frac{\Xi^2}{\Sigma}\cos\theta_1[\sin\theta_1\sin\theta_2\cos(\psi_1 - \psi_2) + \cos\theta_1\cos\theta_2]\right)d\phi_2 d\psi_2$$

$$- 2\frac{\Xi^2}{\Sigma}[\sin\theta_1\sin\theta_2\cos(\psi_1 - \psi_2) + \cos\theta_1\cos\theta_2]\sin(\psi_1 - \psi_2)\sin\theta_1 d\theta_2 d\phi_2$$

$$- 2\frac{\Xi^2}{\Sigma}\sin(\psi_1 - \psi_2)\sin\theta_1\cos\theta_1 d\theta_2 d\psi_2 , \tag{3.35}$$

the eleven dimensional metric becomes

$$ds_{(11)}^2 = e^{-2\tilde{\varphi}/3}d\tilde{s}_{st}^2$$

$$+ e^{4\tilde{\varphi}/3}\left(C_{\tilde{\psi}}^{(1)}C_{\tilde{\psi}}^{(1)} + C_{\tilde{\psi}}^{(1)}C_{\phi}^{(1)} + C_{\tilde{\psi}}^{(1)}C_{\psi}^{(1)} + C_{\tilde{\psi}}^{(1)}C_{\theta}^{(1)}\right.$$

$$\left. + C_{\phi}^{(1)}C_{\phi}^{(1)} + C_{\phi}^{(1)}C_{\psi}^{(1)} + C_{\phi}^{(1)}C_{\theta}^{(1)} + C_{\psi}^{(1)}C_{\psi}^{(1)} + C_{\psi}^{(1)}C_{\theta}^{(1)} + C_{\theta}^{(1)}C_{\theta}^{(1)}\right)$$

$$e^{4\tilde{\varphi}/3}\left(C_{\tilde{\psi}}^{(1)} + C_{\phi}^{(1)} + C_{\psi}^{(1)} + C_{\theta}^{(1)} + dx^{10}\right)dx^{10}. \tag{3.36}$$

3.3.3   Brane charges

Superstring theories have massless p-form potentials which may be regarded as generalizations of the electromagnetic gauge field. The Maxwell equations for the gauge field of electrodynamics $A^{(1)} = A_\mu dx^\mu$ in the presence of sources are

$$dF_2 = \star\mathcal{J}_m , \qquad d\star F_2 = \star\mathcal{J}_e. \tag{3.37}$$

It follows that the electric and magnetic charges are given by

$$e = \int_{S^2}\star F_2 , \qquad g = \int_{S^2}F_2 , \tag{3.38}$$

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where S^2 is a two-sphere surrounding the charges.

In string theory in the presence of n-forms, we can define conserved charges associated to the gauge potentials and then find the stable branes of given electric charge. For instance, a Dp-brane in type II superstring theory couples to a $(p+1)$-form $C^{(p+1)}$ with field strength $F_{p+2} = dC^{(p+1)}$. The corresponding electric-type charge is

$$\mathcal{Q}_{Dp} = \int_{\Sigma^{8-p}} \star F_{p+2} \, , \tag{3.39}$$

where Σ^{8-p} is a cycle surrounding the charge.

As an explicit example, consider the original background reviewed in section 3.2. We know that this solution corresponds to N_c D5-branes on an S^3. Consider then the 3-cycle

$$\tilde{S}^3 = \{\omega^i | \sigma^i = 0\}, \tag{3.40}$$

and integrate the RR three form (3.11) on it, obtaining ($\int \omega^1 \wedge \omega^2 \wedge \omega^3 = 16\pi^2$)

$$\frac{1}{4\pi^2} \int_{\tilde{S}^3} F_3 = N_c, \tag{3.41}$$

which means that we have a quantization condition.

In [135], the author showed that there are different types of electric or magnetic charge associated with a gauge field. Here we collect the main results for D4-branes, which is the case we are interested in.

In the T-dual solution that we computed above, we have one non trivial RR 1-form $C^{(1)}$ and one 3-form $C^{(3)}$, and the Kalb-Ramond field $\mathcal{B}$ is also nonvanishing. The 4-form gauge field, which is invariant under the abelian gauge transformation $C^{(1)} \to C^{(1)} + d\xi_0$ and $C^{(3)} \to C^{(3)} - \mathcal{B} \wedge d\xi_0$, is

$$\widetilde{F}_4 := dC^{(3)} - C^{(1)} \wedge d\mathcal{B}. \tag{3.42}$$

The Bianchi identity reads now

$$d\widetilde{F}_4 = -dC^{(1)} \wedge d\mathcal{B}, \tag{3.43}$$

and if we regard the right hand side of this equation as a kind of Maxwell current $\star \mathcal{J}^{Maxwell}$, we are allowed to define a Maxwell charge, by integration of the 4-form field $\widetilde{F}_4$ on a four cycle. Another type of charge may be defined when we consider the Bianchi identity as an exterior derivative of a form, say

$$d(\widetilde{F}_4 + C^{(1)} \wedge d\mathcal{B}) = \star \mathcal{J}^{Page}, \tag{3.44}$$

and again we would define the conserved charge by integration. Comparing the two definitions, we have that

$$Q_{D4}^{Page} = Q_{D4}^{Maxwell} + \int_{\mathcal{C}^4} C^{(1)} \wedge d\mathcal{B}. \tag{3.45}$$

One important feature of these charges is that the Maxwell charge is not quantized, while the Page charge satisfies a quantization condition.

Considering a fixed point in the radial coordinate, the following cycle

$$\mathcal{C}^4 = \{\theta_2, \phi_1, \phi_2, \psi_2 | \psi_1 = \theta_1 = 0\} \tag{3.46}$$

is particularly smooth in studying the above quantities. Let us start with the Page charge for convenience. On this cycle, the equation simplifies to

$$\star \mathcal{J}_{D4} = dF_4 \,, \tag{3.47}$$

and the quantized Page charge is the integral of this current in the five dimensional space whose boundary is the cycle $\mathcal{C}^4$. Therefore, using the Stokes theorem and normalizing our result, we find ($C^{(3)}|_{\mathcal{C}^4} = N_c/4 \cos\theta_2 d\phi_1 \wedge d\phi_2 \wedge d\psi_2$)

$$Q_{D4}^{Page} = -\frac{1}{8\pi^3} \int_{\mathcal{C}^4} F_4 = N_c. \tag{3.48}$$

Also, we can define the Maxwell charge in this cycle as

$$Q_{D4}^{Maxwell} := Q_{D4} - \frac{1}{4\pi^3} \int_{\mathcal{C}^4} C^{(1)} \wedge d\mathcal{B} \,, \tag{3.49}$$

and using the RR forms that we computed, we have

$$\begin{aligned} -C^{(1)} \wedge dB &= \frac{N_c(1+\gamma)}{8} \frac{\Xi}{\Sigma} \sin\theta_2 d\theta_2 \wedge d\phi_2 \wedge d\psi_2 \wedge d\phi_1 \\ &= \frac{N_c(1+\gamma)}{8} \frac{\Xi}{\Sigma} \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge d\phi_1 \,, \end{aligned} \tag{3.50}$$

so

$$-\frac{1}{4\pi^3} \int_{\mathcal{C}^4} C^{(1)} \wedge d\mathcal{B} = \frac{\Xi}{\Sigma}(1+\gamma)N_c \,, \tag{3.51}$$

and we see that the Maxwell charge in not quantized, but it runs along the radial direction.

3.4   Field theory aspects

The original motivation of the MNa solution was from the gauge/gravity correspondence. Since we have just found a background by T-duality of the MNa solution, we want to study properties of the dual gauge field theory to this background.

3.4.1   Wilson loops

Wilson loop observables are given by (see, e.g., [14, 43, 46, 114] for more details)

$$W(\mathcal{C}) := \frac{1}{N_c} \mathrm{Tr}\, P \exp\left(i \oint A_\mu dx^\mu \right), \tag{3.52}$$

where the trace is usually taken over the fundamental representation. From the expectation value of the Wilson loop, we can compute the quark-antiquark $(Q\bar{Q})$ potential. Choosing a rectangular loop with sides of length $L_{Q\bar{Q}}$ in the spatial direction and T for the time direction, with $L_{Q\bar{Q}} << T$, as $T \to \infty$ we have the behaviour

$$\langle W(\mathcal{C}) \rangle \sim e^{-V_{Q\bar{Q}} T}, \tag{3.53}$$

where $V_{Q\bar{Q}}$ is the quark-antiquark potential.

In a confining theory, the potential behaves as

$$V_{Q\bar{Q}} \sim \sigma L_{Q\bar{Q}}, \tag{3.54}$$

where the constant σ is called the *QCD string tension*, so the expectation value of the Wilson loop (3.53) obeys the area law,

$$\langle W(\mathcal{C}) \rangle \sim e^{-\sigma S}, \tag{3.55}$$

for the rectangular region considered.

In the case of $\mathcal{N} = 4$ SYM, dual to $AdS_5 \times S^5$, we have a holographic prescription for a supersymmetric version of the Wilson loop,

$$W(\mathcal{C}) := \frac{1}{N_c} TrP \exp\left[\oint (iA_\mu \dot{x}^\mu + \theta^I X^I(x)\sqrt{\dot{x}^2}) d\tau \right], \tag{3.56}$$

where $x^\mu(\tau)$ parametrizes the loop and θ^I parametrizes the sphere S^5 and couples to the scalars X^I in $\mathcal{N} = 4$ SYM.

The holographic prescription for the Wilson loop VEV is [31,136],

$$\langle W(\mathcal{C}) \rangle \sim e^{-S}, \tag{3.57}$$

where S is the area of a string world-sheet which ends on a curve $\mathcal{C}$ at the boundary of the AdS_5 space. Since the area of the worldsheet is divergent, we need to subtract the area of the string going straight down from $U = \infty$ to $U = U_0$,

$$W(\mathcal{C}) \sim e^{-(S-\ell\Phi)}, \tag{3.58}$$

where ℓ is the perimeter of the Wilson loop contour $\mathcal{C}$ and $\Phi = U_\infty - U_0$. The area of the worldsheet can be computed using the Nambu-Goto action

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma (\det g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu)^{1/2}, \tag{3.59}$$

where $g_{\mu\nu}$ is the $AdS_5 \times S^5$ metric. In $AdS_5 \times S^5$, we find the behaviour $V_{Q\bar{Q}} \sim 1/L_{Q\bar{Q}}$ determined by conformal invariance, see [31,136].

We now consider a more general background,

$$ds^2 = -g_{tt}dt^2 + g_{xx}dx^2 + g_{\rho\rho}d\rho^2 + g_{ij}^{int}dy^i dy^j , \tag{3.60}$$

where we assume that the functions $(g_{tt}, g_{xx}, g_{\rho\rho})$ are functions of ρ only. We do not fix the internal space, since we consider a probe string that is not excited in these directions; so the internal space has no role in the present study.

As in AdS space, we consider a string whose ends are fixed at $x = 0$ and $x = L_{Q\bar{Q}}$ at the boundary of space, $\rho \to \infty$. In addition, we assume that it can extend in the bulk, so that the radial coordinate of the string assumes its minimum value at ρ_0, and that by symmetry this occurs at $x = L_{Q\bar{Q}}/2$.

We choose a configuration such that

$$t = \tau \quad x = x(\sigma) \quad \rho = \rho(\sigma) , \tag{3.61}$$

and we compute the Nambu-Goto action (3.59) with relation to the metric (3.60). The induced metric on the worldsheet is $G_{\alpha\beta} = g_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu$, where

$$G_{\tau\tau} = -g_{tt}, \quad G_{\sigma\sigma} = g_{xx} \left(\frac{dx}{d\sigma}\right)^2 + g_{\rho\rho} \left(\frac{d\rho}{d\sigma}\right)^2, \quad G_{\tau\sigma} = 0 , \tag{3.62}$$

and the determinant of the worldsheet is

$$det G_{\alpha\beta} = -g_{tt}g_{xx}(x')^2 - g_{tt}g_{\rho\rho}(\rho')^2$$
$$\equiv -f^2(x')^2 - g^2(\rho')^2 \ , \tag{3.63}$$

where we have defined the functions $f^2 = g_{tt}g_{xx}$ and $g^2 = g_{tt}g_{\rho\rho}$. Hence we write the Nambu-Goto action as

$$S = \frac{T}{2\pi\alpha'} \int_0^{2\pi} d\sigma \sqrt{f^2(x')^2 + g^2(\rho')^2} \equiv \frac{T}{2\pi\alpha'} \int_0^{2\pi} d\sigma L. \tag{3.64}$$

Its equations of motion give

$$\partial_\tau \left[\frac{1}{L}(f^2 x'^2 + g^2 \rho'^2) \right] = 0 \tag{3.65}$$

$$\partial_\sigma \left[\frac{1}{L} f^2 x' \right] = 0 \tag{3.66}$$

$$\partial_\sigma \left[\frac{1}{L} g^2 \rho' \right] = \frac{1}{L}(x'^2 f f' + \rho'^2 g g'). \tag{3.67}$$

The first of these equations is trivially satisfied since we assume our background time independent. The second, (3.66), is satisfied if we assume that the term inside brackets is a constant C_0. That means

$$\frac{1}{L} f^2 x' = C_0 \ \Rightarrow \ \frac{f^2 x'}{C_0} = (f^2 x'^2 + g^2 \rho'^2)^{1/2} \ , \tag{3.68}$$

which implies that

$$\frac{d\rho}{d\sigma} = \pm \frac{dx}{d\sigma} \frac{f}{C_0 g} \sqrt{f^2 - C_0^2} \equiv \pm \frac{dx}{d\sigma} W_{eff} \ , \tag{3.69}$$

thus we write

$$\frac{d\rho}{d\sigma} = \pm \frac{dx}{d\sigma} W_{eff} \Rightarrow \frac{d\rho}{dx} = \pm W_{eff}. \tag{3.70}$$

Here we wrote W_{eff} just for convenience and one can check that the third equation (3.67) is satisfied once we assume that the above equation is true.

From the sort of solution we are looking for, one can show that there are two distinct regions

$$x < L_{Q\bar{Q}}/2 \qquad \frac{d\rho}{dx} = -W_{eff} \tag{3.71}$$

$$x > L_{Q\bar{Q}}/2 \qquad \frac{d\rho}{dx} = W_{eff}, \tag{3.72}$$

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and we can formally integrate these equations, so that

$$\frac{d\rho}{dx} = -W_{eff} \Rightarrow \int_\infty^\rho \frac{d\rho}{W_{eff}} = -\int_0^x dx \Rightarrow x(\rho) = \int_\rho^\infty \frac{d\rho}{W_{eff}}, \quad x < L_{Q\bar{Q}}/2 \quad (3.73)$$

$$\frac{d\rho}{dx} = W_{eff} \Rightarrow \int_\rho^\infty \frac{d\rho}{W_{eff}} = \int_x^{L_{Q\bar{Q}}} dx \Rightarrow x(\rho) = L_{Q\bar{Q}} - \int_\rho^\infty \frac{d\rho}{W_{eff}}, \quad x > L_{Q\bar{Q}}/2.$$
$$(3.74)$$

The fact that the string must be fixed at $\rho \to \infty$ and we must have $x(\rho)$ finite implies that the following condition must be satisfied

$$\lim_{\rho \to \infty} W_{eff}(\rho) \to \infty. \quad (3.75)$$

Once this equation is satisfied, the string moves to smaller values of the radial coordinate down to a *turning point* ρ_0 where $\frac{d\rho}{dx}|_{\rho_0} = 0$, namely where $W_{eff}(\rho_0) = 0$. We restrict ourselves to turning points $C_0 = f(\rho_0)$.

Now we can compute the quark-antiquark separation pair and its potential energy. The separation is written as

$$L_{Q\bar{Q}}(\rho_0) = 2\int_0^{L_{Q\bar{Q}}/2} dx = 2\int_{\rho_0}^\infty \frac{d\rho}{W_{eff}}. \quad (3.76)$$

In order to compute the potential $V_{Q\bar{Q}}$ we need the Nambu-Goto action S_{NG}/T which diverges, but we subtract the W-boson mass given by a string going straight down on ρ at $x =$ constant, i.e.

$$M = \int_0^\pi \sqrt{g^2 \rho'^2} = \int_{\rho_0}^\infty g(\rho)d\rho, \quad (3.77)$$

so that the renormalized quark-antiquark potential is given by

$$2\pi\alpha' V_{Q\bar{Q}}(\rho_0) = f(\rho_0)L_{Q\bar{Q}}(\rho_0) + 2\int_{\rho_0}^\infty dz \frac{g(z)}{f(z)}\sqrt{f^2(z) - f^2(\rho_0)} - 2\int_{\rho_0}^\infty g(z)dz,$$
$$(3.78)$$

and one can show that

$$2\pi\alpha' \frac{dV_{Q\bar{Q}}}{dL_{Q\bar{Q}}} = f(\rho_0). \quad (3.79)$$

We can now compute the Wilson loops for the T-dual of the MNa solution. In this case, the solution of the set of equations is not exactly known, but remember

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that in the UV limit (where we consider the cutoff $r \sim \Lambda$) we have the asymptotic expansion (3.14a – 3.14e), so that

$$f^2 = g_{tt}g_{xx} \simeq e^{2\phi_\infty} \tag{3.80}$$

$$g^2 = g_{tt}g_{rr} \simeq e^{2\phi_\infty}N_c c_1 e^{4\Lambda/3}, \tag{3.81}$$

therefore, one may check the boundary condition to see that

$$\lim_{r \to \Lambda} W_{eff} \sim \frac{1}{f(r_0)e^{2\Lambda/3}N_c^{1/2}c_1^{1/2}}\sqrt{e^{2\phi_\infty} - f^2(r_0)}, \tag{3.82}$$

where we will take $r_0 \sim 0$, implying $f^2(r_0) = e^{2\phi_0}$. A similar situation occurred in [137], where the authors found a finite value for the boundary condition $\lim_{r \to \infty} W_{eff}$ and it was argued that the QFT needs to be UV-completed.[3] Under this condition, we can calculate the QCD string tension (see [45, 46, 139]) through

$$\sigma = \frac{1}{2\pi\alpha'}f(r_0)\Big|_{IR} = \frac{1}{2\pi\alpha'}e^{\phi_0} , \tag{3.83}$$

and therefore

$$2\pi\alpha'\frac{dV_{Q\bar{Q}}}{dL_{Q\bar{Q}}} = f(r_0) \Rightarrow V_{Q\bar{Q}} \simeq \frac{e^{\phi_0}}{2\pi\alpha'}L_{Q\bar{Q}} , \tag{3.84}$$

which means that this theory exhibits linear confinement.

3.4.2   Gauge coupling

We can consider now another important quantity, the gauge coupling. Consider the Dirac-Born-Infeld action for a generic probe Dp-brane, wrapping an n-cycle Σ, with induced metric

$$ds_{Dp}^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + ds_\Sigma^2 \tag{3.85}$$

and components given by $M = \{\mu, a\}$, where $\mu = 0, \ldots, p - n$ are indices in the Minkowski space and $a = 1, \ldots, n$ are indices of the cycle. We also take the gauge

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[3]One possibility is that the QFT is deformed by an irrelevant operator, modifying the UV, and perhaps one could remove it by using the solution in [134] as a starting point, as opposed to the one in [9].It was argued in [138] that the UV behaviour of the solution in [9] is improved this way. We thank the referee for this observation.

field and the Kalb-Ramond field with non vanishing components $F_{\mu\nu}$ and B_{ab}. Therefore, the DBI action reads

$$S_{DBI} = -T_{Dp} \int d^{p+1}\sigma e^{-\phi}\sqrt{-\det(G_{MN} + B_{MN} + 2\pi\alpha' F_{MN})}$$
$$= -T_{Dp} \int_{\mathcal{M}} d^{p+1-n}\vec{x}\sqrt{-\det(G_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} \int_{\Sigma^n} d^n\Sigma e^{-\phi}\sqrt{-\det(G_{ab} + B_{ab})},$$

$$(3.86)$$

where $\mathcal{M}$ stands for Minkowski space and $d = p + 1 - n$ is the dimension of the reduced field theory. Taking an expansion of the first integral in terms of α', we get

$$S_{DBI} = -T_{Dp} \int_{\Sigma^n} d^n\Sigma e^{-\phi}\sqrt{-\det(G_{ab} + B_{ab})} \times$$
$$\times \int_{\mathcal{M}} d^d\vec{x} e^{dA}\left(1 + \frac{(2\pi\alpha')^2 e^{-4A}}{4}F_{\mu\nu}F^{\mu\nu} + \cdots\right),$$

$$(3.87)$$

so that we can recognize the gauge coupling as

$$\frac{1}{g_{YM}^2} = T_{Dp}(2\pi\alpha')^2 \int_{\Sigma^n} d^n\Sigma e^{-(4-d)\phi - A}\sqrt{-\det(G_{ab} + B_{ab})}. \qquad (3.88)$$

Consider first the MNa solution. In this case, the induced metric on the brane is

$$ds_{ind}^2 = e^\varphi\left[dx_{1,2}^2 + \frac{N_c}{4}\left(e^{2h} + \frac{e^{2g}}{4}(1 - w)^2\right)(\sigma^i)^2\right], \qquad (3.89)$$

therefore neglecting numerical factors, the coupling constant is given by

$$\frac{1}{g_{YM}^2} \sim \left(e^{2h} + \frac{e^{2g}}{4}(1 - w)^2\right)^{3/2}, \qquad (3.90)$$

and using the asymptotic expansions for these functions, we see that in the IR limit, the coupling constant diverges $g_{YM} \to \infty$, whilst in the UV limit the coupling constant vanishes $g_{YM} \to 0$, and this fact is consistent with confinement and asymptotic freedom respectively, as it should be.

Now, we need to consider the case for the T-dual solution of the MNa. As we know, we need to consider first the case of the D4-brane wrapping a 2-cycle defined by

$$\mathcal{C}^2 = \{\psi_1 = \psi_2 \equiv \psi; \theta_1 = \theta_2 \equiv \theta\}, \qquad (3.91)$$

with ϕ_1 and ϕ_2 fixed. Therefore, the induced metric is given by

$$d\tilde{s}^2_{ind} = e^{2\tilde{\varphi}}\widetilde{\Sigma}dx^2_{1,2} + (\Sigma + 2\Xi + \Omega)d\theta^2 + \left(\Sigma\sin^2\theta + 2\Xi\sin^2\theta + \Omega - \frac{\Xi^2}{\Sigma}\cos^2\theta\right)d\psi^2 ,$$
(3.92)

and since the Kalb-Ramond field vanishes in this cycle, we can compute the determinant of the induced metric easily. In fact, up to numerical factors the gauge coupling is

$$\frac{1}{g^2_{YM}} \sim \sqrt{\widetilde{\Sigma}}e^{-\phi}(\Sigma + 2\Xi + \Omega)^{1/2} \int_{S^2}\left(\Sigma\sin^2\theta + 2\Xi\sin^2\theta + \Omega - \frac{\Xi^2}{\Sigma}\cos^2\theta\right)^{1/2} ,$$
(3.93)

and the bracket inside the integral can be written as

$$\Omega - \frac{\Xi^2}{\Sigma} + \sin^2\theta\left(\Sigma + 2\Xi + \frac{\Xi^2}{\Sigma}\right) ,$$
(3.94)

whereas

$$\Sigma + 2\Xi + \Omega = \Sigma - w\Omega.$$
(3.95)

All terms, $\sqrt{\widetilde{\Sigma}}e^{-\phi}$, $(\Sigma + 2\Xi + \Omega)^{1/2}$, $\Omega - \Xi^2/\Sigma$ and $\Sigma + 2\Xi + \Xi^2/\Sigma$, go to infinity at $r \to \infty$, so $1/g^2_{YM} \to \infty$. At $r \to 0$, $\sqrt{\widetilde{\Sigma}}e^{-\phi}$ goes to a constant, whereas $\Sigma - w\Omega$, $\Omega - \Xi^2/\Sigma$ and $\Sigma + 2\Xi + \Xi^2/\Sigma$ go to 0 as r^2, so $1/g^2_{YM} \to 0$. Therefore we again have confinement ($g^2_{YM} \to \infty$ as $r \to 0$) and asymptotic freedom ($g^2_{YM} \to 0$ as $r \to \infty$).[4]

3.4.3 Nonlocality and entanglement entropy

Another useful quantity is the entanglement entropy (EE), which can be defined as the von Neumann entropy for a reduced system, in a sense that we will explain below.

Consider a quantum mechanical system (we closely follow the formalisms presented in [140–142]), described by a pure ground state $|\Psi\rangle$. The density matrix is

$$\rho_{tot} = |\Psi\rangle\langle\Psi|$$
(3.96)

and it is easy to see that the von Neumann entropy

$$S_{tot} := -\text{Tr}\left(\rho_{tot}\ln\rho_{tot}\right)$$

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[4]Of course, as usual one would need to see whether other couplings (to KK modes, for instance) go to zero as well, in order to have real asymptotic freedom.

vanishes. By an imaginary process, we can divide the total systems into two subsystems A and B, so that, the total Hilbert space is given by the direct product of the corresponding subsystems Hilbert spaces, that is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

We may think of the EE as the entropy felt by an observer who has access only to the subsystem A. Such observer will think that the system is described by the reduced density matrix

$$\rho_A = \text{Tr}\,_B \rho_{tot}, \tag{3.97}$$

where we have smeared out the information of the subsystem B, by taking the trace over the Hilbert space $\mathcal{H}_B$. Then the entanglement entropy is defined as the von Neumann entropy for the reduced system A, that is

$$S_A := -\text{Tr}\,(\rho_A \ln \rho_A).$$

In a $(d+1)-$dimensional QFT, it has been proved that the entanglement entropy diverges, but after introducing an ultraviolet cut-off ε, the divergence behaves as

$$S_A \propto \frac{Area(\partial A)}{\varepsilon^{d-1}} + subleading\ terms\,, \tag{3.98}$$

since the entanglement between the subsystems A and B is more severe at the boundary ∂A.

For our purposes, we can take the QFT defined on $\mathbb{R}^{d+1}$ with the following intervals [5] [137, 138, 142],

$$A = \mathbb{R}^{d-1} \times \mathcal{I}_\ell$$
$$B = \mathbb{R}^{d-1} \times \mathbb{R}\backslash\mathcal{I}_\ell \tag{3.99}$$

where $\mathcal{I}_\ell$ is a line segment of length ℓ. In such a case, the entanglement entropy is

$$S_A \propto \frac{Vol(\mathbb{R}^{d-1})}{\varepsilon^{d-1}}, \tag{3.100}$$

where $Vol(\mathbb{R}^{d-1})$ is the volume of the space $\mathbb{R}^{d-1}$, since the boundary of the d-dimensional region A are two copies of the space $\mathbb{R}^{d-1}$ with separation ℓ.

The computation of the EE in a QFT is not an easy task for an arbitrary region A, even if we consider a free theory. If we consider a theory with a gravity dual, we can compute the EE using the holographic prescription of [140]. In a large N_c

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[5]At fixed time, $t = t_0$.

$(d+1)$−dimensional CFT, we find the minimal area of the d−dimensional surface γ in the $(d+2)$-dimensional AdS space at $t = t_0$, whose boundary of γ coincides with the boundary of the region A, that is $\partial\gamma = \partial A$.

The holographic entanglement entropy is given by the area of this surface

$$S_A = \frac{1}{4G_N^{(d+2)}} \int_\gamma d^d\sigma \sqrt{G_{ind}^{(d)}} \, , \qquad (3.101)$$

where the $G_{ind}^{(d)}$ is the induced string frame metric on the surface γ. Considering a ten-dimensional metric, we need to take into account the fact that in nonconformal theories the dilaton and the volume of the internal space are not constant, therefore a natural generalization is the prescription

$$S_A = \frac{1}{4G_N^{(10)}} \int_\gamma d^8\sigma e^{-2\phi} \sqrt{G_{ind}^{(8)}} \, . \qquad (3.102)$$

The entropy is obtained by minimizing the action (3.102) above, over all surfaces that approach the boundary of the entangling region A. Klebanov, Kutasov and Murugan found in [142] that in a confining background there are two surfaces minimizing the action, the first one is *disconnected* which consists of two cigars descending straight down to the IR cut-off r_0, separated by a distance ℓ, and the second is a *connected* surface, in which the cigars are connected by a tube with the width depending on ℓ.

Consider a gravitational background in the string frame of the form [142]

$$ds^2 = \alpha(r)[\beta(r)dr^2 + \eta_{\mu\nu}dx^\mu dx^\nu] + g_{ij}^{int}dy^i dy^j \qquad (3.103)$$

where x^μ $(\mu = 0, 1, \ldots, d)$ parametrize the flat space $\mathbb{R}^{d+1}$, r is the radial coordinate and θ^i $(i = d+2, \ldots, 9)$ are internal coordinates. The volume of the internal manifold is

$$V_{int} = \int d^6y \sqrt{\det[g_{ij}^{int}]}, \qquad (3.104)$$

and if we plug the background (3.103), into the prescription (3.102), we get

$$\begin{aligned} S_A &= \frac{1}{4G_N^{(10)}} \int_{\mathbb{R}^{d-1}} d^{d-1}x \int d^6y \sqrt{\det[g_{ij}^{int}]} \int_{-\ell/2}^{+\ell/2} dx e^{-2\phi}\alpha(r)^{d/2}\sqrt{1 + \beta(r)(\partial_x r)^2} \\ &= \frac{1}{4G_N^{(10)}} Vol(\mathbb{R}^{d-1}) \int_{-\ell/2}^{+\ell/2} dx \; e^{-2\phi} V_{int}\alpha(r)^{d/2}\sqrt{1 + \beta(r)(\partial_x r)^2} \\ &= \frac{1}{4G_N^{(10)}} Vol(\mathbb{R}^{d-1}) \int_{-\ell/2}^{+\ell/2} dx \sqrt{H(r)}\sqrt{1 + \beta(r)(\partial_x r)^2} \, , \end{aligned} \qquad (3.105)$$

where we have denoted by x the direction along which the interval $\mathcal{I}_\ell$ lies, and also we have defined the useful quantity

$$H = e^{-4\phi} V_{int}^2 \alpha^d \ . \tag{3.106}$$

We need to find the solution for the equation of motion in the integral (3.105). Since this integral does not depend explicitly on x, we argue that the "energy" defined with respect to it is conserved [143], that is, if we take $\mathcal{L} = \sqrt{H + H\beta(r')^2}$, then

$$\frac{d}{dx}\left(\frac{d\mathcal{L}}{dr'}r' - \mathcal{L}\right) = 0$$

implies that

$$\frac{d}{dx}\left(\frac{H(r)}{\sqrt{H + H\beta(r')^2}}\right) = 0 \ , \tag{3.107}$$

and after fixing the constant at the minimum value of the radial coordinate r^*, we have the solution

$$\frac{dr}{dx} = \frac{1}{\sqrt{\beta(r)}}\left(\frac{H(r)}{H(r^*)} - 1\right)^{1/2} \ , \tag{3.108}$$

and integrating between r^* and infinity, we obtain

$$\frac{\ell(r^*)}{2} = \sqrt{H(r^*)}\int_{r^*}^{r_\infty} dr\left(\frac{\beta(r)}{H(r) - H(r^*)}\right)^{1/2} \ . \tag{3.109}$$

Finally, we insert equation (3.108) into (3.105), and we get the entropy density for the connected solution,

$$\frac{S_A}{Vol(\mathbb{R}^{d-1})} = \frac{1}{2G_N^{(10)}}\int_{r^*}^{r_\infty} dr\frac{\sqrt{\beta(r)}H(r)}{\sqrt{H(r) - H(r^*)}} \ , \tag{3.110}$$

where we write the UV cut-off r_∞. As we already know, the entanglement entropy generally is UV divergent, but KKM found that the difference between the EE of the connected and disconnected solutions is finite, and is easily seen to be given by

$$\frac{2G_N^{(10)}}{Vol(\mathbb{R}^{d-1})}\left(S^{(c)} - S^{(d)}\right) = \int_{r_*}^{\infty} dr\frac{\sqrt{\beta H}}{\sqrt{1 - H(r_*)/H(r)}} - \int_{r_0}^{\infty} dr\sqrt{\beta H} \ . \tag{3.111}$$

The EE can be used as an order parameter for the confinement/deconfinement phase transition in a confining theory. In fact, a similar phase transition was found by KKM in [142], where they showed that depending on the value of ℓ, the relevant

solutions can be either the connected or the disconnected solutions and the phase transition between these two solutions is a characteristic of confining theories.

Moreover, in [138], it was proved that a sufficient condition for the existence of phase transitions is that the length $\ell(r_*)$ has an upper bound, and the nonexistence of this maximum correlates with the absence of the phase transition.

We note that the quantity (3.106) is related to the warp factor we get after a dimensional reduction on the $(8 - d)$-dimensional compact manifold.

In our particular case, the metric (3.20) can be written as

$$d\tilde{s}_{st}^2 = e^{2\tilde{\varphi}}\widetilde{\Sigma}dx_{1,2}^2 + e^{2\tilde{\varphi}}\widetilde{\Sigma}(e^{2g}N_c)dr^2 + \tilde{g}_{ij}^{int}dy^i dy^j, \tag{3.112}$$

so that we can compute the volume of the internal manifold (3.104) and the warp factor (3.106) and find $H = \widetilde{\Sigma}\sqrt{\tilde{g}_{int}}$, as well as $\beta = e^{2g}N_c$.

Using the metrics presented in the section (3.3.1) we can find

$$l(r_*) = 2\sqrt{N_c H(r_*)} \int_{r_*}^{\infty} dr \frac{e^g}{\sqrt{H(r) - H(r_*)}} \tag{3.113}$$

$$\frac{2G_N^{(10)}}{Vol(\mathbb{R}^{d-1})} \left(S^{(conn)} - S^{(disconn)} \right) \sim N_c \int_{r_*}^{\infty} dr e^g \sqrt{H} \left(\frac{1}{\sqrt{1 - H(r_*)/H(r)}} - 1 \right)$$
$$- N_c \int_{r_0}^{r_*} dr e^g \sqrt{H}. \tag{3.114}$$

One could in principle compute the volume of the internal manifold (3.104), but this gives us a very complicated equation. We then would need to do the following: firstly, evaluate the determinant of the internal metric and then solve the integral.

But we cannot solve analytically the integral, since we just have asymptotic solutions. We can nevertheless find the behavior of V_{int}.

The asymptotic behavior of the determinant is important, so we need to know - at least qualitatively - its expression. In fact, the metric of the internal manifold is of the form

$$[\tilde{g}^{int}] = \begin{pmatrix} \tilde{g}_{\tilde{\theta}\tilde{\theta}} & \tilde{g}_{\tilde{\theta}\tilde{\phi}} & \tilde{g}_{\tilde{\theta}\tilde{\psi}} & \tilde{g}_{\tilde{\theta}\theta} & \tilde{g}_{\tilde{\theta}\phi} & \tilde{g}_{\tilde{\theta}\psi} \\ \tilde{g}_{\tilde{\phi}\tilde{\theta}} & \tilde{g}_{\tilde{\phi}\tilde{\phi}} & \tilde{g}_{\tilde{\phi}\tilde{\psi}} & \tilde{g}_{\tilde{\phi}\theta} & \tilde{g}_{\tilde{\phi}\phi} & \tilde{g}_{\tilde{\phi}\psi} \\ \tilde{g}_{\tilde{\psi}\tilde{\theta}} & \tilde{g}_{\tilde{\psi}\tilde{\phi}} & \tilde{g}_{\tilde{\psi}\tilde{\psi}} & \tilde{g}_{\tilde{\psi}\theta} & \tilde{g}_{\tilde{\psi}\phi} & \tilde{g}_{\tilde{\psi}\psi} \\ \tilde{g}_{\theta\tilde{\theta}} & \tilde{g}_{\theta\tilde{\phi}} & \tilde{g}_{\theta\tilde{\psi}} & \tilde{g}_{\theta\theta} & \tilde{g}_{\theta\phi} & \tilde{g}_{\theta\psi} \\ \tilde{g}_{\phi\tilde{\theta}} & \tilde{g}_{\phi\tilde{\phi}} & \tilde{g}_{\phi\tilde{\psi}} & \tilde{g}_{\phi\theta} & \tilde{g}_{\phi\phi} & \tilde{g}_{\phi\psi} \\ \tilde{g}_{\psi\tilde{\theta}} & \tilde{g}_{\psi\tilde{\phi}} & \tilde{g}_{\psi\tilde{\psi}} & \tilde{g}_{\psi\theta} & \tilde{g}_{\psi\phi} & \tilde{g}_{\psi\psi} \end{pmatrix}, \tag{3.115}$$

where the nonvanishing components are

$\tilde{g}_{\tilde{\theta}\tilde{\theta}} = \Sigma$	$\tilde{g}_{\tilde{\phi}\tilde{\phi}} = \Sigma^{-1}$	$\tilde{g}_{\tilde{\psi}\tilde{\psi}} = \Sigma \sin^2 \theta_1$
$\tilde{g}_{\theta\theta} = \Omega - \frac{1}{\Sigma}\Xi^2 \sin^2(\psi_1 - \psi_2)\sin^2 \theta_1$		
$\tilde{g}_{\theta\phi} = -\frac{1}{\Sigma}\Xi^2 \sin(\psi_1 - \psi_2)\sin\theta_1[\sin\theta_1\sin\theta_2\cos(\psi_1 - \psi_2) + \cos\theta_1\cos\theta_2]$		
$\tilde{g}_{\theta\psi} = -\frac{1}{\Sigma}\Xi^2 \sin(\psi_1 - \psi_2)\sin\theta_1\cos\theta_1$		
$\tilde{g}_{\phi\phi} = \Omega - \frac{1}{\Sigma}\Xi^2[\sin\theta_1\sin\theta_2\cos(\psi_1 - \psi_2) + \cos\theta_1\cos\theta_2]^2$		
$\tilde{g}_{\phi\psi} = \Omega\cos\theta_2 - \frac{1}{\Sigma}\Xi^2\cos\theta_1[\sin\theta_1\sin\theta_2\cos(\psi_1 - \psi_2) + \cos\theta_1\cos\theta_2]$		
$\tilde{g}_{\psi\psi} = \Omega - \frac{1}{\Sigma}\Xi^2\cos^2\theta_1$		
$\tilde{g}_{\tilde{\theta}\theta} = \Xi\cos(\psi_1 - \psi_2)$	$\tilde{g}_{\tilde{\theta}\phi} = -\Xi\sin(\psi_1 - \psi_2)\sin\theta_2$	$\tilde{g}_{\tilde{\psi}\theta} = -\Xi\sin\theta_1\cos\theta_1\sin(\psi_1 - \psi_2)$
$\tilde{g}_{\tilde{\psi}\phi} = \Xi[\cos\theta_2\sin^2\theta_1 - \cos\theta_1\sin\theta_1\sin\theta_2\cos(\psi_1 - \psi_2)]$		
$\tilde{g}_{\tilde{\psi}\psi} = \Xi\sin^2\theta_1$		

The determinant of this matrix is really laborious to calculate. However, the volume element acts just in the angular directions, $0 \leq \theta_a \leq \pi$, $0 \leq \phi_a < 2\pi$, $0 \leq \psi_a < 4\pi$. So, we can ignore the expression of the angular directions, since it only gives us numerical factors, which in the asymptotic limit are not important at all. We are mainly interested in the radial direction.

In the UV limit $r \to \infty$, the determinant is a function of the form

$$\det[\tilde{g}^{int}] \sim e^{16r/3}A + \text{subleading} , \qquad (3.116)$$

where A is a function of the angular directions only, so V_{int} diverges at $r \to \infty$. We also then find $H \sim e^{16r/3}$ and $e^g \sim e^{2r/3}$, so $l(r^*)$ in (3.113) and $S^{(conn)} - S^{(disconn)}$ in (3.114) are actually convergent at $r \to \infty$.

We also obtain that, modulo possible cancellations, $\det[\tilde{g}_{int}]$ is finite at $r \to 0$, therefore both H and β remain finite at $r \to 0$.

Then from (3.113), as $r^* \to 0$, $l(r^*)$ goes to a constant, whereas at $r^* \to \infty$,

$$l(r^*) \sim e^{8r_*/3} \int_{r^*}^{\infty} dr \frac{e^{2r/3}}{\sqrt{e^{16r/3} - e^{16r^*/3}}} = (\tilde{r}^*)^4 \int_{\tilde{r}^*}^{\infty} \frac{d\tilde{r}}{\sqrt{\tilde{r}^8 - (\tilde{r}^*)^8}} = \tilde{r}^* \int_1^{\infty} \frac{dz}{\sqrt{z^8 - 1}} ,$$
$$(3.117)$$

where $\tilde{r} = e^{2r/3}$ and $z = \tilde{r}/\tilde{r}^*$, so $l(r^*)$ goes to infinity. This behaviour ($l(r^*)$ increasing to infinity) already suggests there is no phase transition. Indeed, as was pointed in [137, 138], the absence of a maximum value for $l(r_*)$ suggests the absence of a first order phase transition in the entanglement entropy (in the cases

with phase transition in the entanglement entropy, we have a maximum for $l(r^*)$: l increases to a maximum, then decreases with r^*). To verify this, we check the sign of $S^{(conn)} - S^{(disconn)}$ at zero and infinity. At $r^* \to 0$,

$$\Delta S|_{r^* \to 0} \sim \int_{r^* \to 0}^{\infty} dr e^g(r) \sqrt{H(r)} \left(\frac{1}{\sqrt{1 - H(r^*)/H(r)}} - 1 \right) > 0 , \qquad (3.118)$$

since the integrand is positive. At $r^* \to \infty$,

$$
\begin{aligned}
\Delta S|_{r^* \to \infty} \quad &\sim \quad \int_{r^* \to \infty}^{\infty} dr e^g(r) \sqrt{H(r)} \left(\frac{1}{\sqrt{1 - H(r^*)/H(r)}} - 1 \right) - \int_0^{r^* \to \infty} dr e^g(r) H(r) \\
&\sim \quad \int_{r^* \to \infty}^{\infty} dr e^{\frac{10r}{3}} \left(\frac{1}{\sqrt{1 - e^{\frac{8(r^*-r)}{3}}}} - 1 \right) - \int_0^{r^* \to \infty} dr e^{\frac{10r}{3}} \\
&= \quad \frac{3}{2} (\tilde{r}^*)^5 \left[\int_1^{\infty} dz \, z^4 \left(\frac{1}{\sqrt{1 - z^{-8}}} - 1 \right) - \int_0^1 dz \, z^4 \right] \\
&= \quad \frac{3}{2} (\tilde{r}^*)^5 \frac{\sqrt{\pi} \Gamma \left(\frac{3}{8} \right)}{40 \Gamma \left(\frac{7}{8} \right)} \to +\infty ,
\end{aligned}
\qquad (3.119)
$$

so is not only positive, but goes to infinity. If nothing strange happens in between (at finite r^*), it means that the disconnected solution has always the lower entropy, implying that there is no phase transition. It is worth mentioning that this behavior is consistent with [138], where a detailed study of entanglement entropy as a probe of confinement was considered. In fact, they showed that the UV completion done in [134] provides a consistent model with phase transitions.

3.4.4 Domain walls

Our configuration consists of a D4-Brane wrapping a two-cycle defined by $\mathcal{C}^2 = \{\theta_1 = \theta_2, \psi_1 = \psi_2\}$ and for $\phi_1 =$const., this cycle vanishes in the IR limit.

We may think of probe D4 branes that wrap the cycle $S^2 = \{\theta_1, \psi_1\}$ at $r \to 0$ and the remaining angular directions are fixed. This configuration can act as a domain wall if it has finite tension. This is an useful observable, since even in the presence of singularities, the tension of the domain wall remains finite. Taking the cycle S^2, the induced metric is

$$d\tilde{s}^2_{S^2} = \frac{N_c e^{2\tilde{\varphi}}}{4} \left(e^{2h} + \frac{e^{2g}}{4} (1+w)^2 \right) dx^2_{1,2} + \Sigma (d\theta_1^2 + \sin^2 \theta_1 d\psi_1^2). \qquad (3.120)$$

The tension of the domain wall can be computed from the DBI action of the $D4$-brane

$$S = -T_{D4} \int d\theta_1 d\psi_1 \int d^3 x e^{-\tilde{\varphi}} \sqrt{|\tilde{g}|} \equiv -T_{eff} \int d^3 x, \qquad (3.121)$$

so that the tension in the IR,

$$T_{eff} = 4\pi e^{-\phi/2} \left(\frac{N_c}{4}\right)^2 \left(e^{2h} + \frac{e^{2g}}{4}(1+w)^2\right)^2 \Sigma T_{D4} \simeq 4\pi e^{\phi_0/2} \left(\frac{N_c}{4}\right)^3 g_0^3 T_{D4}$$

$$(3.122)$$

is finite. We can follow the formalism of [144] (see also [137]) and add a gauge field A_1, with field strength $G_2 = dA_1$ in the Minkowski part of the world volume of the brane, in such a way that we induce a Wess-Zumino term of the form

$$S_{WZ} = T_{D4} \int C^{(1)} \wedge G_2 \wedge G_2 \equiv -T_{D4} \int F_2 \wedge G_2 \wedge A_1, \qquad (3.123)$$

where $C^{(1)}$ is the one-form that we found above, and $F_2 = dC^{(1)}$ its field strength. Using the cycle S^2, in which the field strength is

$$F_2 \bigg|_{S^2} = -\frac{N_c}{4} \sin\theta_1 d\theta_1 \wedge d\psi_1, \qquad (3.124)$$

we can perform the integral

$$\int_{S^2} F_2 = -2\pi N_c,$$

and we insert this integral into the Wess-Zumino action (3.123) above, so that

$$S = 2\pi N_c T_{D4} \int G_2 \wedge A_1. \qquad (3.125)$$

We see that we have induced a Chern-Simons term in the $2+1$ gauge theory, on the domain wall.

Chapter 4

$\mathcal{N} = 1$ AdS Backgrounds

I N the last chapter we studied the action of an abelian T-duality on a background dual to a nonconformal field theory. In this chapter we want to see how a nonabelian T-duality acts on a backgroud with an AdS factor. The study of nonabelian T-duality of AdS backgrounds was initiated in [23,145], and recently, [146] reported a large class of new solutions with AdS_5 factors and made the analysis of the field theory [1], following [119], that performed the nonabelian T-duality in a type IIB solution of the type $AdS_5 \times X^5$ obtained in [26] after a dimensional reduction of the warped solution $AdS_5 \times_w \mathcal{M}_6$ of $D = 11$ supergravity, followed by an abelian T-duality.

In particular, we explore the nonabelian T-duality on the type IIA supergravity solution (that is, before the abelian T-duality which gives $AdS_5 \times X^5$) of the form $AdS_5 \times_w \mathcal{M}_5$, where the internal manifold is obtained after a dimensional reduction of a space that consists of a 2-sphere bundle over $S^2 \times T^2$ [26].

Another application considered relates to the background found in [27]. It consists of a domain wall with non trivial fluxes in the NS-NS and RR sectors. This domain wall solution flows to the background $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$ in the IR limit, and in the UV to $AdS_5 \times T^{1,1}$. We study the T-dual of this domain wall and see that it has as limits the T-dual of $AdS_5 \times T^{1,1}$ and $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$. We then study the implication of nonabelian T-duality for the dual conformal field theories, through a calculation of central charges.

The chapter is organized as follows. In section 4.1 we apply the nonabelian

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[1]Nonabelian T-duality on solutions with AdS factors was considered also in [147–149].

T-duality to the warped AdS_5 solution. In section 4.2 we consider the T-dual of the domain wall solution and in section 4.3 we consider dual conformal field theory aspects of the T-dual solution and calculate central charges.

4.1 Warped AdS_5 solution

Supersymmetric solutions of $D = 11$ supergravity of the form $AdS_5 \times_w \mathcal{M}_6$, with nontrivial four form flux living in the internal Riemann manifold were considered in [26]. The authors found that the six dimensional Riemannian manifold always admits a Killing vector, and that locally, the five-dimensional space orthogonal to the Killing vector is a warped product of a one dimensional space parametrized by the coordinate y and a four-dimensional complex space $\mathcal{M}_4$.

Also, the authors found a large class of regular solutions. One of this solutions, namely $\mathcal{M}_4 = S^2 \times T^2$ is peculiar. Firstly we can reduce on an S^1 direction in the torus T^2 so that we can obtain a regular solution of type IIA solution of the form, $AdS_5 \times X'_5$. Moreover, after a T-duality on the other S^1 we get a type IIB solution of the form $AdS_5 \times X_5$, where X_5 is a family of Sasaki-Einstein manifolds, and the global aspects of these spaces was studied in [150, 151].

The type IIA solution of [26] is of the form

$$\frac{1}{R^2}ds^2 = \underbrace{ds^2(AdS_5) + \alpha_1(y)dy^2 + \alpha_2(y)dx^2}_{G_{\mu\nu}(x)dx^\mu dx^\nu} + \underbrace{\beta_1(y)(L_1^2 + L_2^2) + \beta_2(y)L_3^2}_{g_{ij}(x)L_iL_j}, \quad (4.1\text{a})$$

$$\frac{1}{R^2}B = \underbrace{\gamma(y)dx \wedge L_3}_{B_{\mu i}dx^\mu \wedge L_i} \quad (4.1\text{b})$$

$$\phi = \phi(y) \quad (4.1\text{c})$$

$$\frac{1}{R^3}F_4^{(RR)} = \eta(y)dy \wedge Vol(S^2) \wedge L_3 \quad (4.1\text{d})$$

where $L_i = \sigma_i/\sqrt{2}$, with $i = 1, 2, 3$ are the Maurer-Cartan forms of the group $SU(2)$, satisfying

$$dL_i = -\frac{1}{2}\sqrt{2}\epsilon_{ijk}L_j \wedge L_k, \quad (4.2)$$

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with the left invariant forms

$$\sigma_1 = \cos\psi d\theta + \sin\psi\sin\theta d\phi$$
$$\sigma_2 = -\sin\psi d\theta + \cos\psi\sin\theta d\phi \tag{4.3}$$
$$\sigma_3 = d\psi + \cos\theta d\phi.$$

The coefficients of this solution are given by

$$\alpha_1(y) = e^{-6\lambda}\sec^2\zeta, \quad \alpha_2(y) = e^{-6\lambda}, \quad \beta_1(y) = \frac{1-cy}{3}, \quad \beta_2(y) = \frac{2\cos^2\zeta}{9},$$
$$\gamma(y) = -\frac{\sqrt{2}(ca+cy^2-2y)}{6(a-y^2)} \quad \text{and} \quad \eta(y) = -\frac{2\sqrt{2}(1-cy)}{9} = -\frac{2\sqrt{2}}{3}\beta_1, \tag{4.4}$$

so that the metric is

$$ds^2 = R^2 ds^2(AdS_5) + R^2 e^{-6\lambda}\sec^2\zeta dy^2 + R^2 e^{-6\lambda}dx^2 + R^2\frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2)$$
$$+ \frac{R^2}{9}\cos^2\zeta(d\psi + \cos\theta d\phi)^2, \tag{4.5a}$$

where x parametrizes the circle S^1 of length $2\pi\alpha'/(lR^2)$, with[2]

$$l = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}, \tag{4.5b}$$

(θ,ϕ) are the polar and azimuthal angles in S^2, $y \in (y_1, y_2)$ and $0 \le \psi \le 2\pi$ (note that in our conventions, x and y are dimensionless, i.e. are written in units of R). The angle ζ is defined by $\sin\zeta = 2ye^{-3\lambda}$ and $e^{6\lambda} = 2(a-y^2)/(1-cy)$ and a, c are constants such that, if $c \ne 0$ then $0 < a < 1$, and if $c = 0$ then $a \ne 0$, and if $c \ne 0$ one can set it to 1 and find

$$a = \frac{1}{2} + \frac{3q^2 - p^2}{4p^3}\sqrt{4p^2 - 3q^2}, \tag{4.5c}$$

where $p, q \in \mathbb{Z}$.

The dilaton is

$$\phi = -3\lambda \tag{4.5d}$$

---

[2]At the level of the supergravity action, the periodicity of x is arbitrary [26]. But it is T-dual to a IIB solution involving Sasaki-Einstein spaces, for which there is a geometric constraint on the periodicity [150].

and the Kalb-Ramond field is

$$B = R^2 \frac{(ca + cy^2 - 2y)}{6(a - y^2)}(d\psi + \cos\theta d\phi) \wedge dx. \tag{4.5e}$$

In the RR sector, we have only a nonzero four-form field

$$F_4 = -R^3 \frac{2(1 - cy)}{9} dy \wedge (d\psi + \cos\theta d\phi) \wedge Vol(S^2). \tag{4.6}$$

In what follows, it is convenient to use the frame fields

$$\mathfrak{i}^a = e_{\hat{a}}^a dx^{\hat{a}} \quad AdS_5 \text{ directions}$$
$$\mathfrak{e}^x = R\alpha_1^{1/2} dx, \quad \mathfrak{e}^y = R\alpha_2^{1/2} dy \tag{4.7}$$
$$\mathfrak{e}^1 = R\beta_1^{1/2} L_1, \quad \mathfrak{e}^2 = R\beta_1^{1/2} L_2, \quad \mathfrak{e}^3 = R\beta_2^{1/2} L_3,$$

so that we have the matrix $\kappa^a{}_j$ given by

$$\kappa = \begin{pmatrix} R\beta_1^{1/2} & 0 & 0 \\ 0 & R\beta_1^{1/2} & 0 \\ 0 & 0 & R\beta_2^{1/2} \end{pmatrix}. \tag{4.8}$$

4.1.1 Nonabelian T-dual model

We want to T-dualize the solution of the previous section with respect to the $SU(2)$. In fact, due to the AdS/CFT duality, there are a significant number of research concerning the nonabelian T-duality of solutions with metric with a AdS_5 factor, for instance, in [22, 25, 119, 152–155].

As in section 2.3, we form the matrix M_{ij}, given by $M_{ij} = g_{ij} + b_{ij} + \alpha' \epsilon_{ijk} \hat{v}_k$, so ($b_{ij} = 0$, $g_{ij} = \kappa^a{}_i \kappa^a{}_j$),

$$M = \begin{pmatrix} R^2\beta_1 & \alpha\hat{v}_3 & -\alpha'\hat{v}_2 \\ -\alpha'\hat{v}_3 & R^2\beta_1 & \alpha'\hat{v}_1 \\ \alpha'\hat{v}_2 & -\alpha'\hat{v}_1 & R^2\beta_2 \end{pmatrix}. \tag{4.9a}$$

We pick a gauge where $\theta = \phi = v_2 = 0$, so that $\hat{v} = (\cos\psi v_1, \sin\psi v_1, v_3)$. This gauge is useful when the vector ∂_ψ is a Killing vector as the present case (see [23], for further possible choices). Therefore, the matrix M in this gauge is

$$M = \begin{pmatrix} R^2\beta_1 & \alpha' v_3 & -\alpha'\sin\psi v_1 \\ -\alpha' v_3 & R^2\beta_1 & \alpha'\cos\psi v_1 \\ \alpha'\sin\psi v_1 & -\alpha'\cos\psi v_1 & R^2\beta_2 \end{pmatrix}. \tag{4.9b}$$

The dilaton in the dual theory is given by

$$\widehat{\phi} = \phi - \frac{1}{2} \ln \left(\frac{\Delta}{\alpha'^3} \right),$$ (4.10)

where $\Delta \equiv \det M = R^2[(R^4 \beta_1^2 + \alpha'^2 v_3^2)\beta_2 + \alpha'^2 v_1^2 \beta_1]$.

To simplify the notation, from now on we absorb R^2 in β_1, β_2, α' in v_1, v_3, as well as R^2 in $\alpha_1, \alpha_2, \gamma$.

The inverse of the matrix M is then

$$(M^{-1})^T = \frac{1}{\Delta} \begin{pmatrix} \beta_1\beta_2 + v_1^2 \cos^2 \psi & v_3\beta_2 + v_1^2 \cos\psi\sin\psi & v_1 v_3 \cos\psi - v_1\beta_1\sin\psi \\ -v_3\beta_2 + v_1^2 \cos\psi\sin\psi & \beta_1\beta_2 + v_1^2 \sin^2\psi & v_1\beta_1\cos\psi + v_1 v_3 \sin\psi \\ v_1 v_3 \cos\psi + v_1\beta_1\sin\psi & -v_1\beta_1\cos\psi + v_1 v_3 \sin\psi & v_3^2 + \beta_1^2 \end{pmatrix}.$$ (4.11)

Finally, taking the symmetric and skew-symmetric part of (2.70), we get the following T-dual fields

$$\widehat{G}_{\mu\nu} = G_{\mu\nu} - \frac{1}{2}\left(Q_{\mu i}M_{ij}^{-1}Q_{j\nu} + Q_{\nu i}M_{ij}^{-1}Q_{j\mu}\right)$$

$$\widehat{G}_{\mu i} = \frac{1}{2}\left(Q_{\mu j}M_{ji}^{-1} - Q_{j\mu}M_{ij}^{-1}\right)$$

$$\hat{g}_{ij} = \frac{1}{2}\left(M_{ij}^{-1} + M_{ji}^{-1}\right)$$

$$\widehat{B}_{\mu\nu} = B_{\mu\nu} - \frac{1}{2}\left(Q_{\mu i}M_{ij}^{-1}Q_{j\nu} - Q_{\nu i}M_{ij}^{-1}Q_{j\mu}\right)$$

$$\widehat{B}_{\mu i} = \frac{1}{2}\left(Q_{\mu j}M_{ji}^{-1} + Q_{j\mu}M_{ij}^{-1}\right)$$

$$\hat{b}_{ij} = \frac{1}{2}\left(M_{ij}^{-1} - M_{ji}^{-1}\right)$$

For the solution (4.1a - 4.1d), where $x^\mu = \{x, y, AdS_5 \text{ coordinates}\}$ and $i = 1, 2, 3$, we consider just the terms which will be affected by the nonabelian T-duality, namely, Q_{xx}, Q_{xi} and Q_{ij}, giving

$Q_{xx} = G_{xx} = \alpha_2(y)$	$Q_{x3} = B_{x3} = \gamma(y)$
$Q_{11} = Q_{22} = g_{11} = \beta_1(y)$	$Q_{33} = g_{33} = \beta_2(y)$

For the metric, we obtain $\widehat{G}_{\mu\nu} = G_{\mu\nu}$, $\widehat{G}_{\mu i} = 0 \,\forall\, \mu, \nu \neq x$. Moreover, we have the diagonal component

$$\widehat{G}_{xx} = \alpha_2(y) + \frac{1}{\Delta}(v_3^2 + \beta_1^2)\gamma^2, \tag{4.12}$$

the crossed terms

$$\widehat{G}_{x1} = \frac{1}{\Delta}\gamma v_1 v_3 \cos\psi$$

$$\widehat{G}_{x2} = \frac{1}{\Delta}\gamma v_1 v_3 \sin\psi \tag{4.13}$$

$$\widehat{G}_{x3} = \frac{1}{\Delta}\gamma(v_3^2 + \beta_1^2) \,,$$

and the g_{ij} components

$$\hat{g}_{11} = \frac{1}{\Delta}(\beta_1\beta_2 + v_1^2 \cos^2\psi), \quad \hat{g}_{12} = \frac{1}{\Delta}v_1^2 \cos\psi\sin\psi, \quad \hat{g}_{13} = \frac{1}{\Delta}v_1 v_3 \cos\psi$$

$$\hat{g}_{21} = \frac{1}{\Delta}v_1^2 \cos\psi\sin\psi, \quad \hat{g}_{22} = \frac{1}{\Delta}(\beta_1\beta_2 + v_1^2 \sin^2\psi), \quad \hat{g}_{23} = \frac{1}{\Delta}v_1 v_3 \sin\psi$$

$$\tag{4.14}$$

$$\hat{g}_{31} = \frac{1}{\Delta}v_1 v_3 \cos\psi, \quad \hat{g}_{32} = \frac{1}{\Delta}v_1 v_3 \sin\psi, \quad \hat{g}_{33} = \frac{1}{\Delta}(v_3^2 + \beta_1^2).$$

All in all, we have the type IIB metric

$$d\hat{s}^2 = d\tilde{s}^2 + \frac{1}{\Delta}d\Sigma^2, \tag{4.15}$$

where

$$d\tilde{s}^2 = ds_{AdS}^2 + \alpha_1(y)dy^2 + \alpha_2(y)dx^2 \tag{4.16a}$$

and

$$d\Sigma^2 = \gamma^2(v_3^2 + \beta_1^2)dx^2 + \frac{2\gamma}{\sqrt{2}}dx\left[v_1 v_3(\cos\psi d\hat{v}_1 + \sin\psi d\hat{v}_2) + (v_3^2 + \beta_1^2)d\hat{v}_3\right]$$

$$+ \frac{1}{2}\left[(\beta_1\beta_2 + v_1^2 \cos^2\psi)d\hat{v}_1^2 + (\beta_1\beta_2 + v_1^2 \sin^2\psi)d\hat{v}_2^2 + 2v_1^2 \cos\psi\sin\psi d\hat{v}_1 d\hat{v}_2\right.$$

$$\tag{4.16b}$$

$$\left. + 2v_1 v_3 \cos\psi d\hat{v}_1 d\hat{v}_3 + 2v_1 v_3 \sin\psi d\hat{v}_2 d\hat{v}_3 + (v_3^2 + \beta_1^2)d\hat{v}_3^2\right].$$

Remembering that $\hat{v} = (v_1 \cos\psi, v_1 \sin\psi, v_3)$, we rewrite it as

$$d\Sigma^2 = \gamma^2(v_3^2 + \beta_1^2)dx^2 + \frac{2\gamma}{\sqrt{2}}dx\left(v_1 v_3 dv_1 + (v_3^2 + \beta_1^2)dv_3\right) + \frac{1}{2}\beta_1\beta_2 v_1^2 d\psi^2 +$$

$$\tag{4.16c}$$

$$+ \frac{1}{2}(\beta_1\beta_2 + v_1^2)dv_1^2 + v_1 v_3 dv_1 dv_3 + \frac{1}{2}(v_3^2 + \beta_1^2)dv_3^2.$$

For later use, we calculate $\sqrt{\det g_{int}}$ for this metric, where g_{int} refers to the internal, non-AdS, part of the metric. Writing explicitly the factors of R and α', we obtain

$$\sqrt{g_{int}} = \frac{1}{\Delta^2} R^3 \alpha'^3 \sqrt{\alpha_1} \sqrt{\beta_1 \beta_2} \frac{v_1}{\sqrt{2}} \sqrt{\det \tilde{M}} \, , \tag{4.17}$$

where $\tilde{M}$ is the matrix

$$\tilde{M} = \begin{pmatrix} \Delta R^2 \alpha_2 + \gamma^2 R^4 (\alpha'^2 v_3^2 + \beta_1^2 R^4) & \frac{\gamma}{\sqrt{2}} R^2 \alpha'^2 v_1 v_3 & \frac{\gamma}{\sqrt{2}} R^2 (\alpha'^2 v_3^2 + \beta_1^2 R^4) \\ \frac{\gamma}{\sqrt{2}} R^2 \alpha'^2 v_1 v_3 & \frac{\beta_1 \beta_2 R^4 + \alpha'^2 v_1^2}{2} & \alpha'^2 \frac{v_1 v_3}{2} \\ \frac{\gamma}{\sqrt{2}} R^2 (\alpha'^2 v_3^2 + \beta_1^2 R^4) & \alpha'^2 \frac{v_1 v_3}{2} & \frac{\alpha'^2 v_3^2 + \beta_1^2 R^4}{2} \end{pmatrix} \tag{4.18}$$

and we find

$$\det \tilde{M} = \frac{\alpha_2 \beta_1 R^4}{4} \Delta^2 \Rightarrow \sqrt{\det g_{int}} = \frac{R^5 \alpha'^3}{\Delta} \sqrt{\alpha_1 \alpha_2} \beta_1 \sqrt{\beta_2} \frac{v_1}{2\sqrt{2}}. \tag{4.19}$$

Finally, the T-dual Kalb-Ramond field is given by

$$\begin{aligned} \widehat{B} &= \frac{\gamma v_1 \beta_1}{\sqrt{2}\Delta} dx \wedge (-\sin\psi d\hat{v}_1 + \cos\psi d\hat{v}_2) \\ &\quad + \frac{1}{2\Delta}(-v_3 \beta_2 d\hat{v}_1 \wedge d\hat{v}_2 + v_1 \beta_1 \sin\psi d\hat{v}_1 \wedge d\hat{v}_3 - v_1 \beta_1 \cos\psi d\hat{v}_2 \wedge d\hat{v}_3) \\ &= \frac{1}{\Delta}\left[\frac{v_1^2 \beta_1}{\sqrt{2}}\left(\gamma dx + \frac{1}{\sqrt{2}}dv_3\right) - \frac{1}{2}v_1 v_3 \beta_2 dv_1\right] \wedge d\psi. \end{aligned} \tag{4.20}$$

The T-dual vielbeins are [3]

$$\hat{\mathfrak{e}}'_1 = -\frac{\sqrt{\beta_1}}{\sqrt{2}\Delta}\left(v_1 v_3 \beta_2 d\psi + (v_1^2 + \beta_1 \beta_2)dv_1 + v_1 v_3 dv_3\right) - \frac{\gamma\sqrt{\beta_1}}{\Delta} v_1 v_3 dx \tag{4.21a}$$

$$\hat{\mathfrak{e}}'_2 = -\frac{\sqrt{\beta_1}}{\sqrt{2}\Delta}\left(v_1 \beta_1 \beta_2 d\psi - \beta_2 v_3 dv_1 + v_1 \beta_1 dv_3\right) - \frac{\gamma\sqrt{\beta_1}}{\Delta} v_1 \beta_1 dx \tag{4.21b}$$

$$\hat{\mathfrak{e}}_3 = -\frac{\sqrt{\beta_2}}{\sqrt{2}\Delta}\left(-v_1^2 \beta_1 d\psi + v_1 v_3 dv_1 + (v_3^2 + \beta_1^2)dv_3\right) - \frac{\gamma\sqrt{\beta_2}}{\Delta}(v_3^2 + \beta_1^2)dx, \tag{4.21c}$$

where we have defined the rotated vielbeins

$$\begin{pmatrix} \hat{\mathfrak{e}}'_1 \\ \hat{\mathfrak{e}}'_2 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \hat{\mathfrak{e}}_1 \\ \hat{\mathfrak{e}}_2 \end{pmatrix}. \tag{4.21d}$$

---

[3]In fact, we have two different sets of dual frame fields related by a Lorentz transformation, that is, $\hat{\mathfrak{e}}_+ = \Lambda\hat{\mathfrak{e}}_-$, as a result of the different transformation rules of the left- and the right- movers in the sigma model [23]. For simplicity, in this letter we consider just the $\hat{\mathfrak{e}}_+$ terms.

In term of this basis we write the Kalb-Ramond field (4.20) as

$$-\frac{\widehat{B}}{2} = -\frac{v_3}{\beta_1}\hat{\mathfrak{e}}_1' \wedge \hat{\mathfrak{e}}_2' + \frac{v_1}{(\beta_1\beta_2)^{1/2}}\hat{\mathfrak{e}}_3 \wedge \hat{\mathfrak{e}}_2'. \tag{4.22}$$

Using these results, we are able to find the RR forms in this type IIB background. We write the four-form (4.1d) as ($\mathfrak{e}_1 \wedge \mathfrak{e}_2 \wedge \mathfrak{e}_3 = \frac{\beta_1\sqrt{\beta_2}}{2}vol(S^2) \wedge L_3$, remembering that β_i contain R^2)

$$F_4 = \Xi_0 \, dy \wedge \mathfrak{e}_1 \wedge \mathfrak{e}_2 \wedge \mathfrak{e}_3 \equiv G_1^{(3)} \wedge \mathfrak{e}_1 \wedge \mathfrak{e}_2 \wedge \mathfrak{e}_3 \ , \tag{4.23}$$

where $G_1^{(3)} = \Xi_0 dy$ with $\Xi_0 = -4\sqrt{2}R/(3\beta_2^{1/2}) = 4\sqrt{2}/\sqrt{3(1-cy)}$. In this way we have written the RR 4-form in the way suited to apply the nonabelian T-duality as described in the Appendix.

Using these rules, we find $\hat{F}_4 = \hat{F}_2 = 0$ and (reintroducing all factors of R and α')

$$
\begin{aligned}
\widehat{F}_1 &= -e^{\phi-\hat{\phi}}A_0 G_1^{(3)} = d\widehat{C}_0 = \frac{R^3}{\alpha'^{3/2}}\frac{4\sqrt{2}}{3}\beta_1 dy \\[6pt]
\widehat{F}_3 &= d\widehat{C}_2 - \widehat{C}_0 d\widehat{B} = \frac{1}{2}e^{\phi-\hat{\phi}}G_1^{(3)} \wedge \epsilon^{abc}A_c\hat{\mathfrak{e}}_a \wedge \hat{\mathfrak{e}}_b \\[6pt]
&= \Xi_0\frac{1}{2}\epsilon^{abc}\mathcal{A}^a dy \wedge \hat{\mathfrak{e}}_b \wedge \hat{\mathfrak{e}}_c \\[6pt]
&= R^5\sqrt{\alpha'}\frac{4\sqrt{2}}{3\Delta}\beta_1 dy \wedge \left[\frac{v_1^2\beta_1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}dv_3 + R^2\gamma dx\right) - \frac{v_1 v_3 \beta_2}{2}dv_1\right] \wedge d\psi \\[6pt]
&= -\frac{1}{\alpha'^{3/2}}\frac{4\sqrt{2}}{3}\frac{1}{\beta_2^{1/2}}dy \wedge \left(\beta_2^{1/2}v_3 \mathfrak{e}_1' \wedge \mathfrak{e}_2' + \beta_1^{1/2}v_1 \mathfrak{e}_2' \wedge \mathfrak{e}_3\right) \\[6pt]
&= \widehat{B} \wedge \widehat{F}_1 \ ,
\end{aligned}
\tag{4.24}
$$
$$\tag{4.25}$$

where the coefficients from the appendix are

$$A_a = \frac{1}{\Delta^{1/2}}\mathcal{A}_a, \tag{4.26}$$

and $\mathcal{A}_a = \kappa^a{}_i\hat{v}^i = R\alpha'(\beta_1^{1/2}v_1\cos\psi, \beta_1^{1/2}v_1\sin\psi, \beta_2^{1/2}v_3)$. This background is supplemented by the forms $\widehat{F}_9 = \star\widehat{F}_1$ and $\widehat{F}_7 = -\star\widehat{F}_3$. Using these expressions it is straightforward to verify that the Bianchi identities $dF_1 = 0$ and $dF_3 = H \wedge F_1$ are satisfied. Moreover, $B \wedge F_3 = 0$.

For later use, we also compute the Page charges in this geometry. The quantized Page charges in this background are given by [4]

$$
\begin{aligned}
\mathcal{Q}_{D3}^{Page} &= \frac{1}{2\kappa_{10}^2 T_{D3}} \int_{\Sigma_5} (\hat{F}_5 - \hat{B} \wedge \hat{F}_3) = 0 \\
\mathcal{Q}_{D5}^{Page} &= \frac{1}{2\kappa_{10}^2 T_{D5}} \int_{\Sigma_3} (\hat{F}_3 - \hat{B} \wedge \hat{F}_1) = 0 \\
\mathcal{Q}_{D7}^{Page} &= \frac{1}{2\kappa_{10}^2 T_{D7}} \int_{y_1}^{y_2} \widehat{F}_1 = \frac{R^3}{\alpha'^{3/2}} \frac{4\sqrt{2}}{9} (y_2 - y_1) \left(1 - \frac{c(y_1 + y_2)}{2}\right) = N_{D7}
\end{aligned}
\tag{4.27}
$$

where, since after an abelian T-duality along the x-direction on the solution (4.1a-4.1d) we get a the Sasaki-Einstein manifold, we have [150, 156]

$$
\begin{aligned}
y_1 &= \frac{1}{4p}(2p - 3q - \sqrt{4p^2 - 3q^2}) \\
y_2 &= \frac{1}{4p}(2p + 3q - \sqrt{4p^2 - 3q^2}) ,
\end{aligned}
\tag{4.28}
$$

the solutions to $\cos^2 \zeta = 0$, and $p, q \in \mathbb{N}$ with $(p, q) = 1$ for $p > q$. One may verify that this new background has $\mathcal{N} = 1$ supersymmetry, under the criteria of [23]. In fact, in [121] the authors have proved that the vanishing of the Kosmann derivative in the dualized directions of the Killing spinors means supersymmetry is preserved.[5] In the present case, the derivative trivially vanishes, because the Killing spinors are independent of the dualized directions. Moreover, in [121] a proof was given for the formula (2.83), with closed expressions for the dual p-form potentials, that can be applied more easily to specific cases.

Note that we could have considered the same calculation with a different gauge fixing for the Lagrange multipliers. Consider that the matrix M is instead

$$
M = \begin{pmatrix} \beta_1 & v_3 & -v_2 \\ -v_3 & \beta_1 & v_1 \\ v_2 & -v_1 & \beta_2 \end{pmatrix},
\tag{4.29}
$$

with $v = (\rho \cos \zeta \sin \chi, \rho \sin \zeta \sin \chi, \rho \cos \chi)$. In this coordinate system, we have that $\Delta = \beta_2(\beta_1^2 + \rho^2 \cos^2 \chi) + \beta_1^2 \rho^2 \sin^2 \chi$. The inverse of the matrix M gives equation

---

[4]Note that $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$ and $T_{Dp} = (2\pi)^{-9}\alpha'^{-\frac{p+1}{2}}$, so $2\kappa_{10}^2 T_{Dp} = (2\pi l_s)^{7-p}$.

[5]The supersymmetry preservation under nonabelian T-duality was discussed before in [147] and [133].

(4.11), but with the replacements

$$\psi \rightsquigarrow \zeta, \quad v_1 \rightsquigarrow \rho \sin \chi, \quad v_3 \rightsquigarrow \rho \cos \chi. \tag{4.30}$$

4.2 Flowing from AdS_5 to AdS_3

In a recent paper [27], the authors considered the construction of a supersymmetric domain wall that approaches $AdS_5 \times T^{1,1}$ in the UV limit, and $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$ in the IR limit. In this section we consider the nonabelian T-dual solution of the domain wall ansatz and see that it has as its limit the nonabelian T-dual of the $AdS_5 \times T^{1,1}$ and $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$ in the UV and IR respectively.

In fact, the nonabelian T-dual solution of $AdS_5 \times T^{1,1}$ is already known from [23]. We therefore start with a short review of this solution. We consider the conventions of [27]. Then the type IIB solution is

$$\frac{1}{R^2} ds^2_{AdS_5 \times T^{1,1}} = ds^2_{AdS_5} + \frac{1}{6}(ds_1^2 + ds_2^2) + \frac{1}{9}(d\psi + P)^2 \tag{4.31a}$$

$$\frac{1}{R^4} F_5 = 4(vol_{AdS_5} + vol_{T^{1,1}}), \tag{4.31b}$$

and $B = 0$, $\phi =$ constant, where $ds_i^2 = d\theta_i^2 + \sin^2 \theta_i d\phi_i^2$ and $P = \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2$ and we make the replacements $v_1 \rightsquigarrow 2y_1$ and $v_3 \rightsquigarrow 2y_2$. The NS-NS sector of the T-dual background is given by

$$ds^2_{T(AdS_5 \times T^{1,1})} = ds^2_{AdS_5} + \lambda_0^2 ds_1^2 + \frac{\lambda_0^2 \lambda^2}{\Delta} y_1^2 \sigma_{\hat{3}}^2$$
$$+ \frac{1}{\Delta} \left[(y_1^2 + \lambda^2 \lambda_0^2) dy_1^2 + (y_2^2 + \lambda_0^4) dy_2^2 + 2y_1 y_2 dy_1 dy_2 \right] \tag{4.32a}$$

$$\widehat{B} = -\frac{\lambda^2}{\Delta} \left[y_1 y_2 dy_1 + (y_2^2 + \lambda_0^4) dy_2 \right] \wedge \sigma_{\hat{3}}, \tag{4.32b}$$

$$e^{-2\hat{\phi}} = 8\Delta \alpha'^{-3/2}, \tag{4.32c}$$

where $\lambda_0^2 = 1/6$, $\lambda^2 = 1/9$, $\sigma_{\hat{3}} = d\psi + \cos\theta_1 d\phi_1$, and

$$\begin{aligned} \hat{\Delta} &\equiv \det M = 8\Delta = 8[\lambda_0^2 y_1^2 + \lambda^2(y_2^2 + \lambda_0^4)] \\ &= \beta_1 v_1^2 + \beta_2(v_3^2 + \beta_1^2). \end{aligned} \tag{4.33}$$

Here $\beta_1 = 2\lambda_0^2$, $\beta_2 = 2\lambda^2$, $v_1 = 2y_1$ and $v_3 = 2y_2$, and as in section 2, we have absorbed a factor of R^2 in β_1, β_2, and a factor of α' in v_1, v_3. The RR-sector is given

by

$$\alpha'^{3/2} R \widehat{F}_2 = 8\sqrt{2}\lambda_0^4 \lambda \sin\theta_1 d\phi_1 \wedge d\theta_1$$

$$\alpha'^{3/2} R \widehat{F}_4 = -8\sqrt{2}\lambda_0^4 \lambda \frac{y_1}{\Delta} \sin\theta_1 d\phi_1 \wedge d\theta_1 \wedge \sigma_{\hat{3}} \wedge (\lambda_0^2 y_1 dy_2 - \lambda^2 y_2 dy_1). \tag{4.34}$$

For completeness, the T-dual vielbeins are given by

$$\hat{\mathfrak{e}}_1' = -\frac{\lambda_0}{\Delta}\left[(y_1^2 + \lambda^2\lambda_0^2)dy_1 + y_1 y_2(dy_2 + \lambda^2\sigma_{\hat{3}})\right] \tag{4.35a}$$

$$\hat{\mathfrak{e}}_2' = \frac{\lambda_0}{\Delta}\left[\lambda^2 y_2 dy_1 - \lambda_0^2 y_1(dy_2 + \lambda^2\sigma_{\hat{3}})\right] \tag{4.35b}$$

$$\hat{\mathfrak{e}}_3 = -\frac{\lambda}{\Delta}\left[y_1 y_2 dy_1 + (y_2^2 + \lambda_0^4)dy_2 - \lambda_0^2 y_1^2 \sigma_{\hat{3}}\right], \tag{4.35c}$$

and as before, we defined the rotated vielbeins

$$\begin{pmatrix} \hat{\mathfrak{e}}_1' \\ \hat{\mathfrak{e}}_2' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \hat{\mathfrak{e}}_1 \\ \hat{\mathfrak{e}}_2 \end{pmatrix}. \tag{4.36}$$

This completes the type IIA background T-dual to $AdS_5 \times T^{(1,1)}$ in type IIB supergravity.

4.2.1 AdS_3 solution and its nonabelian T-dual

The solution with metric $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$ is given by

$$\frac{1}{R^2}ds^2_{AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3} = \frac{1}{3\sqrt{3}}\left(2ds^2_{AdS_3} + dz_1^2 + dz_2^2 + ds_1^2 + ds_2^2 + \frac{1}{2}(d\psi + P)^2\right) \tag{4.37a}$$

$$\frac{1}{R^2}B = \frac{-\tau}{6\sqrt{6}}z_1(vol_1 - vol_2) \equiv \frac{-\tau\tilde{\beta}_2}{2\sqrt{2}R^2}z_1(vol_1 - vol_2) \tag{4.37b}$$

$$\frac{1}{R^2}F_3 = \frac{\tau}{6\sqrt{6}}dz_2 \wedge (vol_1 - vol_2) \tag{4.37c}$$

$$\frac{1}{R^4}F_5 = \frac{1}{27}\left\{vol_{AdS_3} \wedge \left[4dz_1 \wedge dz_2 + \frac{\tau^2}{2}(vol_1 + vol_2)\right] \right.$$
$$\left. + (d\psi + P) \wedge \left[vol_1 \wedge vol_2 + \frac{\tau^2}{8}dz_1 \wedge dz_2 \wedge (vol_1 + vol_2)\right]\right\}, \tag{4.37d}$$

where τ is a constant.

In order to find its T-dual, we consider the Maurer-Cartan forms

$$L_1 = \frac{1}{\sqrt{2}}(\cos\psi d\theta_2 + \sin\psi\sin\theta_2 d\phi_2)$$

$$L_2 = \frac{1}{\sqrt{2}}(-\sin\psi d\theta_2 + \cos\psi\sin\theta_2 d\phi_2) \qquad (4.38)$$

$$L_3 = \frac{1}{\sqrt{2}}(d\psi + \cos\theta_2 d\phi_2),$$

such that $vol_2 = 2L_1 \wedge L_2$. Using the set-up of section 2.3, the vielbeins related to the directions to be T-dualized are

$$\mathfrak{e}^1 = \tilde{\beta}_1^{1/2} L_1 \qquad (4.39a)$$

$$\mathfrak{e}^2 = \tilde{\beta}_1^{1/2} L_2 \qquad (4.39b)$$

$$\mathfrak{e}^3 = \tilde{\beta}_2^{1/2}(L_3 + 1/\sqrt{2}\cos\theta_1 d\phi_1). \qquad (4.39c)$$

where we have defined $\tilde{\beta}_1 = \frac{2}{3\sqrt{3}}$ and $\tilde{\beta}_2 = \frac{1}{3\sqrt{3}}$, absorbing the factors of R^2 in them for simplicity.

With these definitions, we may write the metric as

$$ds^2 = \tilde{\beta}_2(2ds^2_{AdS_3} + ds_1^2 + ds_2^2 + dz_1^2 + dz_2^2) + (\mathfrak{e}^1)^2 + (\mathfrak{e}^2)^2 + (\mathfrak{e}^3)^2 \qquad (4.40)$$

and the RR-forms as $(vol_2 = \frac{2}{\tilde{\beta}_1}\mathfrak{e}^1 \wedge \mathfrak{e}^2, \ d\psi + P = \frac{\sqrt{2}}{\sqrt{\tilde{\beta}_2}}\mathfrak{e}^3)$

$$\frac{1}{R^2}F_3 = \frac{\tau}{6\sqrt{6}}dz_2 \wedge vol_1 - \frac{\tau}{\sqrt{2}}dz_2 \wedge \mathfrak{e}^1 \wedge \mathfrak{e}^2 \qquad (4.41a)$$

$$\frac{1}{R^4}F_5 = \frac{1}{27}\left\{vol_{AdS_3} \wedge \left[4dz_1 \wedge dz_2 + \frac{\tau^2}{2}\left(vol_1 + \frac{2}{\tilde{\beta}_1}\mathfrak{e}^1 \wedge \mathfrak{e}^2\right)\right]\right.$$

$$\left. + \frac{\sqrt{2}}{\sqrt{\tilde{\beta}_2}}\mathfrak{e}^3 \wedge \left[vol_1 \wedge \frac{2}{\tilde{\beta}_1}\mathfrak{e}^1 \wedge \mathfrak{e}^2 + \frac{\tau^2}{8}dz_1 \wedge dz_2 \wedge \left(vol_1 + \frac{2}{\tilde{\beta}_1}\mathfrak{e}^1 \wedge \mathfrak{e}^2\right)\right]\right\},$$

$$(4.41b)$$

or as

$$F_3 = G_3^{(0)} + G_1^{12} \wedge \mathfrak{e}^1 \wedge \mathfrak{e}^2 \qquad (4.41c)$$

$$F_5 = G_5^{(0)} + G_4^3 \wedge \mathfrak{e}^3 + G_3^{12} \wedge \mathfrak{e}^1 \wedge \mathfrak{e}^2 + G_2^{(3)} \wedge \mathfrak{e}^1 \wedge \mathfrak{e}^2 \wedge \mathfrak{e}^3, \qquad (4.41d)$$

where

$$\frac{1}{R^4}G_5^{(0)} = \frac{1}{27}vol_{AdS_3} \wedge \left[4dz_1 \wedge dz_2 + \frac{\tau^2}{2}vol_1\right]$$

$$\frac{1}{R^4}G_4^3 = \frac{\sqrt{2}\tau^2}{216\tilde{\beta}_2^{1/2}}dz_1 \wedge dz_2 \wedge vol_1,$$

$$\frac{1}{R^2}G_3^{(0)} = \frac{\tau}{6\sqrt{6}}dz_2 \wedge vol_1, \qquad \frac{1}{R^4}G_3^{12} = \frac{\tau^2}{27\tilde{\beta}_1}vol_{AdS_3}$$

$$\frac{1}{R^4}G_2^{(3)} = \frac{4}{27\sqrt{2}}\frac{1}{\tilde{\beta}_2^{1/2}\tilde{\beta}_1}\left(vol_1 + \frac{\tau^2}{8}dz_1 \wedge dz_2\right)$$

$$\frac{1}{R^2}G_1^{12} = -\frac{\tau}{\sqrt{2}}dz_2. \tag{4.42}$$

The matrix M is given by $M_{ij} = g_{ij} + b_{ij} + \alpha'\epsilon_{ijk}\hat{v}_k$, so (after absorbing α' factors in $\hat{v}_i$)

$$M = \begin{pmatrix} \tilde{\beta}_1 & \frac{\tau z_1}{\sqrt{2}}\tilde{\beta}_2 + \hat{v}_3 & -\hat{v}_2 \\ -\frac{\tau z_1}{\sqrt{2}}\tilde{\beta}_2 - \hat{v}_3 & \tilde{\beta}_1 & \hat{v}_1 \\ \hat{v}_2 & -\hat{v}_1 & \tilde{\beta}_2 \end{pmatrix}, \tag{4.43}$$

As before, we consider the gauge fixing $\theta = \phi = v_2 = 0$, so that the coordinates become $\hat{v} = (\cos\psi v_1, \sin\psi v_1, v_3)$, and for simplicity we define $\tilde{v}_3 = \frac{\tau z_1}{\sqrt{2}}\tilde{\beta}_2 + \hat{v}_3$, in such a way that the inverse of M is (4.11), with the replacement $v_3 \rightsquigarrow \tilde{v}_3$, that is,

$$(M^{-1})^T = \frac{1}{\tilde{\Delta}}\begin{pmatrix} \tilde{\beta}_1\tilde{\beta}_2 + v_1^2\cos^2\psi & \tilde{v}_3\tilde{\beta}_2 + v_1^2\cos\psi\sin\psi & v_1\tilde{v}_3\cos\psi - v_1\tilde{\beta}_1\sin\psi \\ -\tilde{v}_3\tilde{\beta}_2 + v_1^2\cos\psi\sin\psi & \tilde{\beta}_1\tilde{\beta}_2 + v_1^2\sin^2\psi & v_1\tilde{\beta}_1\cos\psi + v_1\tilde{v}_3\sin\psi \\ v_1\tilde{v}_3\cos\psi + v_1\tilde{\beta}_1\sin\psi & -v_1\tilde{\beta}_1\cos\psi + v_1\tilde{v}_3\sin\psi & \tilde{v}_3^2 + \tilde{\beta}_1^2 \end{pmatrix},$$
$$\tag{4.44}$$

where the determinant $\det M$ is $\tilde{\Delta} \equiv \det M = (\tilde{\beta}_1^2 + \tilde{v}_3^2)\tilde{\beta}_2 + v_1^2\tilde{\beta}_1$.

Under these definitions, we must apply the duality on the following fields[6]

$Q_{\phi\phi} = G_{\phi\phi} = \tilde{\beta}_2\left(\sin^2\theta_1 + \frac{1}{2}\cos^2\theta_1\right)$	$Q_{\phi3} = G_{\phi3} = Q_{3\phi} = \frac{\sqrt{2}}{2}\tilde{\beta}_2\cos\theta_1$
$Q_{\theta\theta} = G_{\theta\theta} = \tilde{\beta}_2$	
$Q_{\theta\phi} = B_{\theta\phi} = -\frac{\tau}{2\sqrt{2}}z_1\tilde{\beta}_2\sin\theta_1$	$E_{12} = b_{12} = \frac{2\tau}{\sqrt{2}}\tilde{\beta}_2 z_1$
$E_{11} = E_{22} = g_{11} = \tilde{\beta}_1$	$E_{33} = g_{33} = \tilde{\beta}_2$

---

[6]Note that since the dependence on the angular coordinates (ϕ_2, θ_2) is encapsulated into the Maurer-Cartan forms L_i, in what follows the subscript (ϕ, θ) refers logically to (ϕ_1, θ_1).

Using these results and the same procedure as in section 3, we find that the dual metric, dilaton and B field are

$$
\begin{aligned}
d\hat{s}^2_{AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3} = {} & \tilde{\beta}_2 \left(2ds^2_{AdS_3} + dz_1^2 + dz_2^2 + ds_1^2\right) + \frac{1}{2\tilde{\Delta}}\tilde{\beta}_1\tilde{\beta}_2 v_1^2(d\psi + \cos\theta_1 d\phi_1)^2 \\
& + \frac{1}{2\tilde{\Delta}}\left[(\tilde{\beta}_1\tilde{\beta}_2 + v_1^2)dv_1^2 + (\tilde{v}_3^2 + \tilde{\beta}_1^2)dv_3^2 + 2v_1\tilde{v}_3 dv_1 dv_3\right] \qquad (4.45)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\phi} = {} & \phi - \frac{1}{2}\ln\frac{\tilde{\Delta}}{\alpha'^3} \\
\widehat{B} = {} & -\frac{v_1}{2\tilde{\Delta}}(\tilde{v}_3\tilde{\beta}_2 dv_1 - v_1\tilde{\beta}_1 d\hat{v}_3) \wedge d\psi \\
& -\frac{\tilde{\beta}_2 \tau z_1}{2\sqrt{2}}\sin\theta_1 d\theta_1 \wedge d\phi_1 + \frac{\tilde{\beta}_2}{2\tilde{\Delta}}v_1\tilde{v}_3 \cos\theta_1 d\phi_1 \wedge dv_1 \\
& +\frac{\tilde{\beta}_2}{2\tilde{\Delta}}\cos\theta_1(\tilde{v}_3^2 + \tilde{\beta}_1^2)d\phi_1 \wedge d\hat{v}_3 \\
= {} & -\frac{\tau R z_1}{6\sqrt{6}}vol_1 + \frac{\tilde{\beta}_2}{2\tilde{\Delta}}\sigma_{\hat{3}} \wedge (v_1\tilde{v}_3 dv_1 + (\tilde{v}_3^2 + \tilde{\beta}_1^2)dv_3). \qquad (4.46)
\end{aligned}
$$

For later use, the $\sqrt{\det g_{int}}$ for this metric (g_{int} is as before the internal, i.e. non-AdS, part of the metric) is

$$
\sqrt{\det g_{int}} = \alpha'^3 \frac{\sin\theta_1}{2\sqrt{2}}\frac{\tilde{\beta}_1\tilde{\beta}_2^{5/2}}{\tilde{\Delta}}v_1. \qquad (4.47)
$$

With F_3 and F_5 written as in (4.41c) and (4.41d), we can apply the formulas in the appendix, reintroduce the factors of α' in (4.42), (4.45) and (4.46) and obtain the RR-sector T-dual forms $\widehat{F}_1 = \widehat{F}_3 = \widehat{F}_5 = 0$ and ($\hat{F}_6$ and $\hat{F}_8$ would be redundant, as we consider their Poincaré duals $\hat{F}_4$ and $\hat{F}_2$)

$$
\begin{aligned}
\widehat{F}_2 = {} & e^{\phi-\hat{\phi}}\left\{-A_0 G_2^{(3)} + G_1^{12} \wedge (A_2\hat{\mathfrak{e}}^1 - A_1\hat{\mathfrak{e}}^2 - A_0\hat{\mathfrak{e}}^3)\right\} \\
\widehat{F}_4 = {} & e^{\phi-\hat{\phi}}\left\{A_3 G_4^3 + G_3^{12} \wedge (A_2\hat{\mathfrak{e}}^1 - A_1\hat{\mathfrak{e}}^2 - A_0\hat{\mathfrak{e}}^3) + G_3^{(0)}(A_1\hat{\mathfrak{e}}^1 + A_2\hat{\mathfrak{e}}^2 + A_3\hat{\mathfrak{e}}^3)\right. \\
& \left. + G_2^{(3)} \wedge (A_3\hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 + A_1\hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3 + A_2\hat{\mathfrak{e}}^3 \wedge \hat{\mathfrak{e}}^1) + A_3 G_1^{12}\hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3\right\} , (4.48)
\end{aligned}
$$

where as before, $e^{\phi-\hat{\phi}} = \sqrt{\tilde{\Delta}}\alpha'^{-3/2}$, $\alpha'^{3/2}e^{\phi-\hat{\phi}}A_0 = \tilde{\beta}_1\sqrt{\tilde{\beta}_2}$ and $\alpha'^{3/2}e^{\phi-\hat{\phi}}A_a = \mathcal{A}_a$, and the dual vielbeins are

$$
\hat{\mathfrak{e}}'^1_{AdS_3} = -\frac{\tilde{\beta}_1^{1/2}}{\sqrt{2}\tilde{\Delta}}\left[(\tilde{\beta}_1\tilde{\beta}_2 + v_1^2)dv_1 + v_1\tilde{v}_3 dv_3 + v_1\tilde{v}_3\tilde{\beta}_2(d\psi + \cos\theta_1 d\phi_1)\right] \qquad (4.49a)
$$

$$
\hat{\mathfrak{e}}'^2_{AdS_3} = \frac{\tilde{\beta}_1^{1/2}}{\sqrt{2}\tilde{\Delta}}\left[\tilde{\beta}_2\tilde{v}_3 dv_1 - v_1\tilde{\beta}_1 dv_3 - v_1\tilde{\beta}_1\tilde{\beta}_2(d\psi + \cos\theta_1 d\phi_1)\right] \qquad (4.49b)
$$

$$
\hat{\mathfrak{e}}^3_{AdS_3} = -\frac{\tilde{\beta}_2^{1/2}}{\sqrt{2}\tilde{\Delta}}\left[v_1\tilde{v}_3 dv_1 + (\tilde{v}_3^2 + \tilde{\beta}_1^2)dv_3 - v_1^2\tilde{\beta}_1(d\psi + \cos\theta_1 d\phi_1)\right]. \qquad (4.49c)
$$

4.2.2 Domain Wall and its nonabelian T-dual

The Domain Wall solution which has as limits the above AdS_3 and AdS_5 solution is given by

$$\frac{1}{R^2}ds^2_{DW} = e^{2A}(-dt^2 + dx^2) + e^{2B}(dx_1^2 + dx_2^2) + d\rho^2$$

$$+ \frac{1}{6}e^{2U}(ds_1^2 + ds_2^2) + \frac{1}{9}e^{2V}(\sqrt{2}L_3 + \cos\theta_1 d\phi_1)^2 \tag{4.50a}$$

$$\frac{1}{R^2}B = \frac{-\tau}{6}x_1(vol_1 - vol_2) \tag{4.50b}$$

$$\frac{1}{R^2}F_3 = \frac{\tau}{6}dx_2 \wedge (vol_1 - vol_2) \tag{4.50c}$$

$$\frac{1}{R^4}F_5 = 4e^{2A+2B-V-4U}dt \wedge dx \wedge dx_1 \wedge dx_2 \wedge d\rho + \frac{1}{27}(\sqrt{2}L_3 + \cos\theta_1 d\phi_1) \wedge vol_1 \wedge vol_2$$

$$+ \frac{\tau^2}{36}dx_1 \wedge dx_2 \wedge (\sqrt{2}L_3 + \cos\theta_1 d\phi_1) \wedge (vol_1 + vol_2) \tag{4.50d}$$

$$+ \frac{\tau^2}{12}e^{2A-2B-V}dt \wedge dx \wedge d\rho \wedge (vol_1 + vol_2).$$

Here τ is a constant and A, B, U, V are functions of the radial coordinate ρ. From this solution, we see that we can recover $AdS_5 \times T^{(1,1)}$ by setting the constant $\tau = 0$ and $A = B = \rho$ and $U = V = 0$. On the other hand, to recover the $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$ solution, we set

$$A = \frac{3^{3/4}}{\sqrt{2}}\rho, \qquad B = U = -V = \frac{1}{4}\ln\left(\frac{4}{3}\right), \tag{4.51}$$

and change variables by $x_i \rightsquigarrow z_i/\sqrt{6}$.

As before, the T-dual model is given by

$$
\begin{aligned}
d\hat{s}^2_{DW} &= R^2 e^{2A}(-dt^2 + dx^2) + R^2 e^{2B}(dx_1^2 + dx_2^2) + R^2 d\rho^2 \\
&\quad + \frac{R^2}{6}e^{2U}ds_1^2 + \frac{1}{2\bar{\Delta}}\bar{\beta}_1\bar{\beta}_2 v_1^2(d\psi + \cos\theta_1 d\phi_1)^2 \\
&\quad + \frac{1}{2\bar{\Delta}}\left\{(\bar{\beta}_1\bar{\beta}_2 + v_1^2)dv_1^2 + (\bar{\beta}_1^2 + \bar{v}_3^2)dv_3^2 + 2v_1\bar{v}_3 dv_1 dv_3\right\} \\
\widehat{B} &= -\frac{\tau R x_1}{6}vol_1 + \frac{\bar{\beta}_2}{2\bar{\Delta}}\sigma_{\hat{3}} \wedge (v_1\bar{v}_3 dv_1 + (\bar{v}_3^2 + \bar{\beta}_1^2)dv_3) \\
\hat{\phi} &= \phi - \frac{1}{2}\ln\frac{\bar{\Delta}}{\alpha'^3} ,
\end{aligned} \tag{4.52}
$$

where we have defined

$$\bar{\beta}_1 = \frac{1}{3}e^{2U}, \quad \bar{\beta}_2 = \frac{2}{9}e^{2V}, \quad \bar{v}_3 = \frac{\tau}{3}x_1 + \hat{v}_3, \quad \bar{\Delta} = (\bar{\beta}_1^2 + \bar{v}_3^2)\bar{\beta}_2 + v_1^2\bar{\beta}_1 , \tag{4.53}$$

and as before we absorbed R^2 factors in $\bar{\beta}_i$ and α' in v_i.

We can easily see that we can obtain the correct limits in the NS-NS sector. The UV and IR limits of the T-dual solution to the domain wall are the nonabelian T-duals of the $AdS_5 \times T^{(1,1)}$ and the $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$ solutions, respectively.

In the RR sector, we could verify term by term that the equality holds, but alternatively, one can find the RR-forms components in the same way as in (4.42). In the present case, we obtain

$$\frac{1}{R^4}G_5^{(0)} = dt \wedge dx \wedge d\rho \wedge \left(4e^{2A+2B-V-4U}dx_1 \wedge dx_2 + \frac{\tau^2}{12}e^{2A-2B-V}vol_1 \right) \quad (4.54)$$

$$\frac{1}{R^4}G_4^3 = \frac{\sqrt{2}\tau^2}{36\bar{\beta}_2^{1/2}}dx_1 \wedge dx_2 \wedge vol_1 \quad (4.55)$$

$$\frac{1}{R^2}G_3^{(0)} = \frac{\tau}{6}dx_2 \wedge vol_1 \,, \quad \frac{1}{R^2}G_3^{12} = \frac{\tau^2}{6\bar{\beta}_1}e^{2A-2B-V}dt \wedge dx \wedge d\rho \quad (4.56)$$

$$\frac{1}{R^4}G_2^{(3)} = \frac{2\sqrt{2}}{27\bar{\beta}_1\bar{\beta}_2^{1/2}}vol_1 + \frac{2\sqrt{2}\tau^2}{36\bar{\beta}_1\bar{\beta}_2^{1/2}}dx_1 \wedge dx_2 \quad (4.57)$$

$$\frac{1}{R^2}G_1^{12} = -\frac{\tau}{3\bar{\beta}_1}dx_2 \,, \quad (4.58)$$

Then the T-dual RR-forms are as in (4.48), i.e.,

$$\begin{aligned}
\widehat{F}_2 &= e^{-\hat{\phi}}\left\{ -A_0 G_2^{(3)} + G_1^{12} \wedge (A_2\hat{\mathfrak{e}}^1 - A_1\hat{\mathfrak{e}}^2 - A_0\hat{\mathfrak{e}}^3) \right\} \\
\widehat{F}_4 &= e^{-\hat{\phi}}\left\{ A_3 G_4^3 + G_3^{12} \wedge (A_2\hat{\mathfrak{e}}^1 - A_1\hat{\mathfrak{e}}^2 - A_0\hat{\mathfrak{e}}^3) + G_3^{(0)} \wedge (A_1\hat{\mathfrak{e}}^1 + A_2\hat{\mathfrak{e}}^2 + A_3\hat{\mathfrak{e}}^3) \right. \\
&\quad \left. + G_2^{(3)} \wedge (A_3\hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 + A_2\hat{\mathfrak{e}}^3 \wedge \hat{\mathfrak{e}}^1 + A_1\hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3) + A_3 G_1^{12}\hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3 \right\} \,. \quad (4.59)
\end{aligned}$$

Finally, we can also compute the vielbeins and see that they have the correct limits, therefore the RR-sector also has the correct limits. For instance, the frame field $\mathfrak{e}^3$ of the Domain Wall is

$$\mathfrak{e}_{AdS(DW)}^3 = -\frac{\bar{\beta}_2^{1/2}}{\sqrt{2}\bar{\Delta}}\left[v_1\bar{v}_3 dv_1 + (\bar{v}_3^2 + \bar{\beta}_1^2)dv_3 - v_1^2\bar{\beta}_1(d\psi - \cos\theta_1 d\phi_1) \right] \,, \quad (4.60a)$$

and we can easily verify that the UV and IR limits are the frame field $\mathfrak{e}^3$ in the AdS_5, AdS_3

$$\mathfrak{e}_{AdS_5}^3 = -\frac{\beta_2^{1/2}}{\sqrt{2}\bar{\Delta}}\left[v_1 v_3 dv_1 + (v_3^2 + \beta_1^2)dv_3 - v_1^2\beta_1(d\psi + \cos\theta_1 d\phi_1) \right] \quad (4.60b)$$

$$\mathfrak{e}_{AdS_3}^3 = -\frac{\tilde{\beta}_2^{1/2}}{\sqrt{2}\bar{\Delta}}\left[v_1\tilde{v}_3 dv_1 + (\tilde{v}_3^2 + \tilde{\beta}_1^2)dv_3 - v_1^2\tilde{\beta}_1(d\psi + \cos\theta_1 d\phi_1) \right] \quad (4.60c)$$

respectively.

4.3 Dual conformal field theories, central charges and RG flow

An interesting question is, what happens to the conformal field theories dual to the gravity backgrounds with AdS factor under nonabelian T-duality on the extra dimensional space? The answer is not obvious. Abelian T-duality on a direction transverse to a Dp-brane turns it into a D$(p+1)$-brane, but if the original direction is infinite in extent, the T-dual direction is infinitesimal in extent. However, this discussion makes sense only in the region far from the region where AdS/CFT is relevant, the core of the D-brane.

Naively, abelian T-duality on the transverse part of a gravity dual should increase the dimensionality of the brane, therefore of the field theory dual to the background. But if we perform a nonabelian T-duality on a space with an AdS factor, in such a way that the AdS factor is not affected, and moreover the T-duality does not introduce a new AdS direction, then it seems that the dimensionality of the dual conformal field theory is unaffected. And yet since the gravity dual is modified, it is logical to assume that the conformal field theory is modified as well.

To understand the effect of nonabelian T-duality on the conformal field theory, we need some probes of the transverse space in AdS/CFT. Such probes are for instance wrapped branes, dual to solitonic states in the field theory, like the example of the 5-brane wrapped on S^5 in $AdS_5 \times S^5$, giving the baryon vertex operator [157].[7] But a more relevant probe was considered in [146], namely the central charge of the dual field theory as a function of the number of branes.

One can calculate Page charges in a gravitational background, and identify those with the number of branes that generate the geometry. For the central charge of the dual conformal field theory, a slight generalization of the usual formula was provided in [146]. For a metric on $M^D = AdS_{d+2} \times X^n$, of the type

$$ds_D^2 = A \, d\vec{z}_{(1,d)}^2 + AB \, dr^2 + g_{ij}d\theta^i d\theta^j \, , \tag{4.61}$$

with a dilaton ϕ, define the modified internal volume as

$$\hat{V}_{int} = \int d\vec{\theta}\sqrt{e^{-4\phi} \det[g_{int}]A^d} \tag{4.62}$$

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[7]Baryon vertex probes in this context, but in other dimensions have been considered in [145] and [148].

and then $\hat{H} = \hat{V}_{int}^2$. Then the central charge is given by

$$\mathcal{C} = d^d \frac{B^{d/2} \hat{H}^{\frac{2d+1}{2}}}{G_N (\hat{H}')^d} \tag{4.63}$$

where $G_N = (\alpha')^{\frac{D}{2}-1}$ is the Newton constant in D dimensions and prime denotes the derivative with respect to r.

The expectation of increase in dimensionality through T-duality affects the D-brane charges of the gravity background. For a geometry with an AdS_5 factor in type IIB, generated only by D3-branes (with only D3-brane Page charges), after T-duality we expect the geometry to be generated by D4- and D6-branes only, i.e. to have only D4- and D6-brane Page charges

$$\begin{aligned} \mathcal{Q}_{D4}^{Page} &= \frac{1}{2\kappa_{10}^2 T_{D4}} \int_{\Sigma_4} (\hat{F}_4 - \hat{B} \wedge \hat{F}_2) \\ \mathcal{Q}_{D6}^{Page} &= \frac{1}{2\kappa_{10}^2 T_{D6}} \int_{\Sigma_2} \hat{F}_2. \end{aligned} \tag{4.64}$$

For an abelian T-duality, we would expect only D4-brane charge, but for nonabelian T-duality (in some sense a T-duality on 3 coordinates), the expectation, confirmed by a calculation, is that only D6-brane charges appear. One can calculate the central charges and express them as a function of the Page charges. In the $AdS_5 \times S^5$ case, we find that $\mathcal{C} = 32\pi^3 R^8 \alpha'^{-4} = 2\pi^5 N_{D3}^2$ before, and $\mathcal{C} = (8\pi^5/3) R^8 \alpha'^{-4} = (2\pi^5/24) N_{D6}^2$ after the nonabelian T-duality, leading to the relation[8]

$$\frac{\mathcal{C}_{before}}{\mathcal{C}_{after}} = \frac{24 N_{D3}^2}{N_{D6}^2} , \tag{4.65}$$

which is found to be satisfied also in other cases of nonabelian T-duality on type IIB geometries generated by D3-branes.

An interesting question which we will try to answer in this section is whether a similar formula is valid in more general contexts in the case of geometries with an AdS factor.

---

[8]The formula in [146] is actually with a factor of 3 instead of 24, since different conventions for T-duality were considered, with $L_i = \sigma_i$ instead of $L_i = \sigma_i/\sqrt{2}$, giving an extra $2\sqrt{2}$ in the quantization of the Page charges after T-duality.

4.3.1 Page charges

- In the case of section 4.1, the starting geometry is in type IIA, the reverse of the situation considered in [146]. Since $F_2 = 0$ in the background before T-duality, $\mathcal{Q}_{D6}^{Page} = 0$, and we only have a nonzero result for

$$
\begin{aligned}
N_{D4} = |\mathcal{Q}_{D4}^{Page}| &= \frac{R^3}{2\kappa_{10}^2 T_{D4}} \int_{y_1}^{y_2} \eta(y) dy \int_{X_3} vol(S^2) \wedge L_3 \\
&= \left(\frac{R}{2\pi\sqrt{\alpha'}}\right)^3 \frac{2\sqrt{2}}{9} (y_2 - y_1) \left(1 - c\frac{y_1 + y_2}{2}\right) 4\pi^2 \sqrt{2} \\
&\equiv \left(\frac{R}{l_s}\right)^3 \frac{2}{9\pi} K.
\end{aligned}
\tag{4.66}
$$

After the nonabelian T-duality, we have calculated in section 3 that $\mathcal{Q}_{D3}^{Page} = \mathcal{Q}_{D5}^{Page} = 0$ and

$$
\begin{aligned}
N_{D7} = |\mathcal{Q}_{D7}^{Page}| &= \frac{R^3}{\alpha'^{3/2}} \frac{4\sqrt{2}}{9} (y_2 - y_1) \left(1 - c\frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{R}{l_s}\right)^3 \frac{4\sqrt{2}}{9} K.
\end{aligned}
\tag{4.67}
$$

- In the case of section 4.2, the we have a Domain Wall solution that interpolates between an $AdS_5 \times T^{1,1}$ and an $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$. This can be also found in the $\mathcal{N} = 4$ D=5 gauged supergravity arising as a consistent KK truncation of type IIB on $T^{1,1}$ [27], and as such it can be interpreted as an RG flow between two fixed points in the dual field theory. A relevant question is then, is the ratio of the central charges before and after the nonabelian T-duality modified by the RG flow?

For $AdS_5 \times T^{1,1}$, the Page charges before and after the nonabelian T-duality were found in [146], $\mathcal{Q}_{D5}^{Page} = \mathcal{Q}_{D7}^{Page} = 0$ and $|\mathcal{Q}_{D3}^{Page}| = N_{D3}$ before, and $|\mathcal{Q}_{D6}^{Page}| = N_{D6}$, $\mathcal{Q}_{D4}^{Page} = 0$ after the T-duality, with (in our conventions)

$$
N_{D3} = \frac{4R^4}{27\pi\alpha'^2}, \quad N_{D6} = \frac{4\sqrt{2}}{27\alpha'^2} R^4.
\tag{4.68}
$$

For $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$, the Page charges before the T-duality were found in [27]. Assuming that $\mathbb{R}^2$ is compactified to a $T^2 = S_{(1)}^1 \times S_{(2)}^1$ with period

$2\pi R d_i \sqrt{6}$, and defining $s(S)$ as a homology 2-cycle generator in $S^2 \times S^3$, one has the integers

$$
\begin{aligned}
Q_{N5} &= \frac{1}{(2\pi l_s)^2} \int_{S^1_{(1)} \times s(S)} H \\
Q_{D5} &= \frac{1}{(2\pi l_s)^2} \int_{S^1_{(2)} \times s(S)} dC_2
\end{aligned}
\tag{4.69}
$$

and the (D3-brane) Page charge quantization condition is

$$
\frac{1}{(2\pi l_s)^4} \int_{\Sigma_5} (F_5 - B \wedge dC_2) \in \mathbb{Z}.
\tag{4.70}
$$

For $\Sigma_5 = S^2 \times S^3$, one obtains an integer

$$
N = \left(\frac{R}{l_s}\right)^4 \frac{vol(T^{1,1})}{4\pi^4},
\tag{4.71}
$$

and for $\Sigma_5 = T^2 \times M_3$, where M_3 is a homology 3-cycle generator in $S^2 \times S^3$, one obtains an integer

$$
\bar{N} = \left(\frac{R}{l_s}\right)^4 \frac{8 d_1 d_2}{9} = -\frac{1}{2} Q_{N5} Q_{D5}.
\tag{4.72}
$$

Moreover, the above flux quantization is actually valid over the whole domain wall solution.

After the T-duality, we have F_2 and F_4, so we need to consider the quantization of D4-brane Page charges

$$
\frac{1}{(2\pi l_s)^3} \int_{\Sigma_4} (F_4 - B \wedge F_2) \in \mathbb{Z}
\tag{4.73}
$$

and

$$
\frac{1}{(2\pi l_s)} \int_{\Sigma_2} F_2 \in \mathbb{Z}.
\tag{4.74}
$$

For $\Sigma_2 = T^2$, we obtain

$$
N_{D6} = -\frac{\tau^2 2\sqrt{2}}{216} 4\pi^2 \frac{6 d_1 d_2}{2\pi l_s} \frac{R^4}{l_s^3},
\tag{4.75}
$$

and for $\Sigma_2 = S^2$, we obtain

$$
\bar{N}_{D6} = -\frac{2\sqrt{2}}{27} \frac{4\pi}{2\pi l_s} \frac{R^4}{l_s^3}.
\tag{4.76}
$$

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4.3.2 Central charges

- For the case in section 4.1, the central charge before the T-duality is obtained using $A = R^2 r^2$, $B = r^{-4}$ and $d = 3$, leading to ($\int L_1 \wedge L_2 \wedge L_3 = 2\pi^2\sqrt{2}$)

$$\hat{V}_{int} = \frac{\alpha' r^3 R^6}{l}(2\pi)\frac{4\pi^2}{9}(y_2 - y_1)\left(1 - \frac{c(y_1 + y_2)}{2}\right) \equiv \frac{\alpha' r^3 R^6}{l}\frac{8\pi^3}{9}K \ , \quad (4.77)$$

and therefore

$$\mathcal{C}_{before} = \frac{R^6}{8\alpha'^3}\frac{8\pi^3}{9}\frac{K}{l} \ , \quad (4.78)$$

where the Page charge quantization condition (4.66) means that we can write $R^3/\alpha'^{3/2}$ as a function of N_{D4}, giving

$$\mathcal{C}_{before} = \frac{9\pi^5}{4}\frac{N_{D4}^2}{Kl}. \quad (4.79)$$

After the T-duality, the central charge is found using the same $A = R^2 r^2$, $B = r^{-4}$ and $d = 3$, leading to (also using the $\sqrt{\det g_{int}}$ calculated in (4.19))

$$\hat{V}_{int} = \frac{\alpha' R^6 r^3}{2l}(2\pi)^2\frac{K}{9}\int \frac{dv_1}{\alpha'}\frac{v_1}{\alpha'}\int\frac{dv_3}{\alpha'}. \quad (4.80)$$

To calculate the integral over the v_i, we can use as another gauge fixing, related to the previous coordinates by $v_1/\alpha' \rightsquigarrow \rho\cos\chi$ and $v_3/\alpha' \rightsquigarrow \rho\sin\chi$ with $\rho, \chi \in [0, \pi]$, leading to a value of $2\pi^3/3$ for the integral.[9] We then obtain

$$\mathcal{C}_{after} = \frac{\pi^5 K}{54l}\left(\frac{R}{l_s}\right)^6 \ , \quad (4.81)$$

and from the Page charge quantization condition (4.67) we can write $R^3/\alpha'^{3/2}$ as a function of N_{D7}, giving

$$\mathcal{C}_{after} = \frac{3\pi^5}{64Kl}N_{D7}^2 \quad (4.82)$$

We see that the ratio is

$$\frac{\mathcal{C}_{before}}{\mathcal{C}_{after}} = \frac{48N_{D4}^2}{N_{D7}^2} \ , \quad (4.83)$$

which is basically the same as in (4.65), with the obvious generalization to N_{Dp}^2/N_{Dp+3}^2, and an extra factor of 2 which is probably the effect of a different normalization.

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[9]The rangle of ρ was defined in [148], and was then used for the calculation of central charges in [146].

- For the case in section 4.2, on the $AdS_5 \times T^{1,1}$ side, the central charge before the T-duality was found to be [146]

$$\mathcal{C}^{(1)}_{before} = \frac{\pi^3 R^8}{27\alpha'^4} = \frac{27}{8}\pi^5 N_{D3}^2 \, , \tag{4.84}$$

and after the T-duality

$$\mathcal{C}^{(1)}_{after} = \frac{2R^8 \pi^5 \lambda \lambda_0^4}{3\alpha'^4} = \frac{9}{64}\pi^5 N_{D6}^2 \, , \tag{4.85}$$

leading to the ratio in (4.65). On the $AdS_3 \times T^2 \times S^2 \times S^3$ side, the central charge before the T-duality is [27]

$$
\begin{aligned}
\mathcal{C}^{(2)}_{before} &= \frac{3R_{AdS_3}}{2G_3} = \frac{3}{2}\left(\frac{R}{l_s}\right)^8 \frac{8 d_1 d_2}{9} \frac{vol(T^{1,1})}{4\pi^4} \\
&= \frac{3}{2}|N Q_{N5} Q_{D5}| = 3|N\bar{N}| = 3N_{D3}\bar{N}_{D3}.
\end{aligned}
\tag{4.86}
$$

Here G_3 is the effective Newton's constant, obtained from the dimensional reduction of the action in string frame, thus proportional to

$$G \propto (R/l_s)^7 vol(T^{1,1})(2\pi d_1)(2\pi d_2). \tag{4.87}$$

After the T-duality, using the $\sqrt{\det g_{int}}$ calculated in (4.47), and doing the integration over v_i in the same way as in the case in section 3, with result $2\pi^3/3$, we obtain

$$\hat{V}_{int} = \frac{R^8}{r} \frac{12(2\pi)^4 d_1 d_2}{\sqrt{2}} \left(\frac{1}{3\sqrt{3}}\right)^{7/2} \frac{2\pi^3}{3} \, , \tag{4.88}$$

leading to

$$\mathcal{C}_{after} = \frac{32\sqrt{2}\pi^7}{3^{21/4}} d_1 d_2 \frac{R^8}{l_s^8} = \frac{4}{\tau^2} 3^{-1/4} \sqrt{2}\pi^6 N_{D6}\bar{N}_{D6}. \tag{4.89}$$

The ratio of central charges before and after the T-duality can therefore be expressed as

$$\frac{\mathcal{C}^{(2)}_{before}}{\mathcal{C}^{(2)}_{after}} = \frac{3^{5/4}\sqrt{2}\tau^2}{8\pi^6} \frac{N_{D3}\bar{N}_{D3}}{N_{D6}\bar{N}_{D6}}. \tag{4.90}$$

Note that now we can fix τ such that the prefactor equals 24, obtaining

$$\frac{\mathcal{C}^{(2)}_{before}}{\mathcal{C}^{(2)}_{after}} = \frac{24 N_{D3}\bar{N}_{D3}}{N_{D6}\bar{N}_{D6}} \, , \tag{4.91}$$

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which is essentially the same formula (4.65) that was valid on the AdS_5 side of the domain wall. The factor τ is related to a redefinition of the fields, coupled to a rescaling of the x_i (or z_i) coordinates [27], which are the two coordinates that change from the AdS_5 on one side of the domain wall to a $AdS_3 \times T^2$ on the other. It is therefore not surprising that changing τ allows us to change the normalization of the central charge dual to AdS_3, with respect to the one dual to AdS_5.

Chapter 5

Nonabelian T-duality for nonrelativistic holographic duals

OW we want to move one step further and consider a nonabelian T-duality on backgrounds with nonrelativistic isometries. As we explained in the introduction, since the gauge/gravity duality is also a strong/weak duality, we may try to apply the its lessons in condensed matter systems, where the strong coupling regime is quite common.

As we know, the methodology of the gauge/gravity duality implies that the symmetries of the field theory are mapped in isometries of the gravity theory. There are two symmetry algebras that are relevant in the nonrelativistic case (see [30] and references therein). The first one, known as Lifshitz algebra, contains the generators for rotations $\{M_{ij}\}$, translations $\{P_i\}$, time translations $\{H\}$ and dilatations D, satisfying the standard commutation relations for $\{M_{ij}, P_i, H\}$ together with

$$[D, M_{ij}] = 0 , \quad [D, P_i] = iP_j , \quad [D, H] = izH , \tag{5.1a}$$

and in [41] the geometric realization of the above symmetry (which has been embedded in string theory in [48, 49]) was defined by the gravity dual

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{dx^i dx^i}{r^2} + \frac{dr^2}{r^2} \right) . \tag{5.1b}$$

As we can see, for $z = 1$ we recover Anti-de Sitter space.

A second relevant algebra is the conformal Galilean algebra which contains, besides the generators for rotations $\{M_{ij}\}$, translations $\{P_i\}$, time translations $\{H\}$

and dilatations D, also the "Galilean boosts", generated by K_i, with nontrivial commutators

$$[M_{ij}, K_k] = i(\delta_{ik}K_j - \delta_{jk}K_i) \ , \quad [P_i, K_j] = -i\delta_{ij}N \ , \quad [K_i, H] = -iP_i \ ,$$
$$[D, K_i] = i(1-z)K_i \ ,$$
(5.2a)

In the special case $z = 2$, the algebra is called the Schrödinger algebra. Here N is the number operator, which counts the number of particles with a given mass m, and in general has has only one nontrivial commutation relation, $[D, N] = i(2-z)N$, but in the $z = 2$ (Schrödinger) case we see that it becomes a central charge. Curiously, it is not possible to arrange the D-dimensional Schrödinger algebra as an isometry in $(D+1)$-dimensions, but in [36,37] it was realized that we can write a gravity dual as a $(D+2)$-dimensional space, with metric

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{-2dtd\xi + dx^i dx^i}{r^2} + \frac{dr^2}{r^2} \right) \ .$$
(5.2b)

Obtaining nonrelativistic gravity duals in string theory turns out to be difficult (see [38–40, 47–50, 158–163]).[1] In relativistic cases, several different techniques have been employed in order to generate supergravity solutions, see [165–167] for recent developments. One particularly interesting solution generating technique which has been applied extensively is T-duality. In the usual case, T-duality relates strings in a background with a compact direction, S^1 of radius R, with a background with an S^1 of radius α'/R. The physical spectrum of a string in the geometry is invariant under this transformation, see e.g. [2, 92, 168–170].

This usual duality (on S^1) is abelian ($U(1)$ group), but a nonabelian generalization for the group $SU(2)$, called nonabelian T-duality, was introduced in [3] and became an issue of recent interest [22, 23, 117, 121, 171–173]. This nonabelian T-duality (NATD) transformation has been used successfully as a solution generating technique [25, 133, 137, 145, 147–149, 152, 153, 174–177], although some issues concerning global properties of the dual manifold remain.

Considering the difficulties in constructing string theory gravity duals with nonrelativistic symmetries, in this chapter we consider NATD of known gravity dual solutions. In section 5.2, we apply this technique to the solutions with conformal

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[1]See [164] for the embedding of nonrelativistic string backgrounds via the use of abelian T-duality in the context of double field theory.

Galilean symmetry constructed in [162], and in section 5.3 to the solutions with Lifshitz symmetries constructed in [48].

In order to define the dual field theory, in section 5.4 we start by calculating the quantized Page charges of the spaces constructed in the section 5.2 and 5.3. In particular, we compare the charges of the Galilean solution constructed in section 5.2 with the charges calculated in [148]. We then define and study holographic Wilson loops in these backgrounds.

5.1   Nonabelian T duality revisited

In principle we could follow using the usual method employed in the last section to find the nonabelian T duality rules. On the other hand, here we will consider an alternative method to find the transformation rules [121], and obviously we must have the same results, but this alternative path has the advantage of giving a closed form for the fields in the RR sector[2]. Considering a spacetime metric and a Kalb-Ramond two-form given by

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu + 2G_{\mu i}(x)dx^\mu \tau_i + G_{ij}(x)\tau_i\tau_j \qquad (5.3)$$

$$B = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu + B_{\mu i}dx^\mu \wedge \tau_i + \frac{1}{2}B_{ij}\tau_i \wedge \tau_j, \qquad (5.4)$$

in such a way that $\mu, \nu = 1, \ldots, 7$ and all dependence on the $SU(2)$ angles θ, ψ, ϕ is contained in the Maurer-Cartan forms τ_i for $SU(2)$, which satisfy $d\tau_i = \frac{1}{2}\epsilon_{ijk}\tau_j \wedge \tau_k$. Furthermore, in general this background has a nontrivial dilaton $\Phi = \Phi(x)$.

If we define the field Q by its components

$$\begin{aligned} Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}, \quad & Q_{\mu i} = G_{\mu i} + B_{\mu i} \\ Q_{i\mu} = G_{i\mu} + B_{i\mu}, \quad & E_{ij} = G_{ij} + B_{ij} , \end{aligned} \qquad (5.5)$$

one can show that the nonabelian T-dual background is given by

$$\begin{aligned} \widehat{Q}_{\mu\nu} = Q_{\mu\nu} - Q_{\mu i}M_{ij}^{-1}Q_{j\nu}, \quad & \widehat{E}_{ij} = M_{ij}^{-1} \\ \widehat{Q}_{\mu i} = Q_{\mu j}M_{ji}^{-1}, \quad & \widehat{Q}_{i\mu} = -M_{ij}^{-1}Q_{j\mu}, \end{aligned} \qquad (5.6)$$

where the matrix M is defined by

$$M_{ij} = E_{ij} + \alpha'\epsilon_{ijk}v_k , \qquad (5.7)$$

_____

[2]We thank to the authors of [121] for clarifications.

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and ϵ_{ijk} are the structure constants of the group $SU(2)$ and v_i are Lagrange multipliers. Hereafter we absorb the factor of α' into v_i and we present all the correct factors in the final answers. All in all, the dual fields are written as

$$d\hat{s}^2 = \widehat{G}_{\mu\nu}(x)dx^\mu dx^\nu + 2\widehat{G}_{\mu i}(x)dx^\mu dv^i + \widehat{G}_{ij}(x)dv^i dv^j \tag{5.8a}$$

$$\widehat{B} = \frac{1}{2}\widehat{B}_{\mu\nu}dx^\mu \wedge dx^\nu + \widehat{B}_{\mu i}dx^\mu \wedge dv^i + \frac{1}{2}\widehat{B}_{ij}dv^i \wedge dv^j \ , \tag{5.8b}$$

and the one-loop contribution to the dilaton is given by

$$\hat{\phi} = \phi - \frac{1}{2}\ln\left(\frac{\Delta}{\alpha'^3}\right) \ , \tag{5.8c}$$

where $\Delta = \det M$. Besides the spectator fields, the dual theory depends on θ, ψ, ϕ, v^i, meaning that we have too many degrees of freedom and we need to impose a gauge fixing in order to remove three of these variables.

It is convenient to write the metric as

$$ds_{10}^2 = ds_7^2 + \sum_{i=1}^{3} e^{2C_i}(\tau_i + A^i)^2 \ , \tag{5.9}$$

where A^i are $SU(2)$-valued gauge fields and C_i are scalars. Moreover, we define the vielbeins $\{\mathfrak{e}^\mu, \mathfrak{e}^i\}$, such that

$$\mathfrak{e}^\mu \ \Rightarrow \ ds_7^2 = g_{\mu\nu}dx^\mu dx^\nu = \sum_{\mu=0}^{6}(\mathfrak{e}^\mu)^2$$

$$\mathfrak{e}^i = e^{C_i}(\tau_i + A^i) \ \Rightarrow \ \sum_{i=1}^{3} e^{2C_i}(\tau_i + A^i)^2 = \sum_{i=1}^{3}(\mathfrak{e}^i)^2, \tag{5.10}$$

implying that the components of the metric (5.3) are

$$G_{\mu\nu} = g_{\mu\nu} + \sum_{i=1}^{3} e^{2C_i}A_\mu^i A_\nu^i \ , \quad G_{\mu i} = e^{2C_i}A_\mu^i \ , \quad G_{ij} = e^{2C_i}\delta_{ij} \ . \tag{5.11}$$

In the same way, it is useful to write the Kalb-Ramond as

$$B = \frac{1}{2}b_{\mu\nu}dx^\mu \wedge dx^\nu + (\beta_i + db_i) \wedge \tau_i + \frac{1}{2}\epsilon_{ijk}b_k\tau_i \wedge \tau_j \ , \tag{5.12}$$

and the components of (5.4) are

$$B_{\mu\nu} = b_{\mu\nu} \ , \quad B_{\mu i} = \beta_{\mu i} + \partial_\mu b_i \ , \quad B_{ij} = \epsilon_{ijk}b_k \ . \tag{5.13}$$

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We next write the inverse of the matrix M_{ij},

$$M_{ij}^{-1} = \frac{1}{\Delta} \begin{pmatrix} e^{2(C_2+C_3)} + z_1^2 & z_1 z_2 - e^{2C_3} z_3 & z_1 z_3 + e^{2C_2} z_2 \\ z_1 z_2 + e^{2C_3} z_3 & e^{2(C_1+C_3)} + z_2^2 & z_2 z_3 - e^{2C_1} z_1 \\ z_1 z_3 - e^{2C_2} z_2 & z_2 z_3 + e^{2C_1} z_1 & e^{2(C_1+C_2)} + z_3^2 \end{pmatrix} , \tag{5.14}$$

where $\Delta = e^{2(C_1+C_2+C_3)} + e^{2C_1} z_1^2 + e^{2C_2} z_2^2 + e^{2C_3} z_3^2$ and $z_i = \alpha' v_i + b_i$. In fact, it is easy to see the general formula for the components of M^{-1} is

$$M_{ij}^{-1} = \frac{1}{\Delta} (z_i z_j + \delta_{ij} e^{2(C_1+C_2+C_3)} e^{-2C_i} - \epsilon_{ijk} e^{2C_k} z_k) . \tag{5.15}$$

Using all these equations, the authors of [121] were able to find a closed form for the dual metric and Kalb-Ramond field,

$$d\hat{s}^2 = ds_7^2 + \frac{1}{\Delta} \Big[(z_1 Dz_1 + z_2 Dz_2 + z_3 Dz_3)^2 + e^{2(C_2+C_3)} Dz_1^2$$
$$+ e^{2(C_1+C_3)} Dz_2^2 + e^{2(C_1+C_2)} Dz_3^2 \Big] \tag{5.16a}$$

$$\widehat{B} = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu - \frac{1}{\Delta} (e^{2C_1} z_1 Dz_2 \wedge Dz_3 + e^{2C_2} z_2 Dz_3 \wedge Dz_1 + e^{2C_3} z_3 Dz_1 \wedge Dz_2)$$
$$- Dz_1 \wedge A_1 - Dz_2 \wedge A_2 - Dz_3 \wedge A_3 - z_1 A_2 \wedge A_3 - z_2 A_3 \wedge A_1 - z_3 A_1 \wedge A_2 \tag{5.16b}$$

where

$$Dz_i = dz_i + \beta_i - \epsilon_{ijk} z_j A^k . \tag{5.16c}$$

For the RR sector the authors of [121] have shown explicit closed forms for the dual backgrounds. Considering first a (massive) type IIA sector with fields given by

$$F_0 = m \tag{5.17a}$$

$$F_2 = G_2 + J_1^i \wedge (\tau_i + A^i) + \frac{1}{2} \epsilon_{ijk} K_0^i (\tau_j + A^j) \wedge (\tau_k + A^k) \tag{5.17b}$$

$$F_4 = G_4 + L_3^i \wedge (\tau_i + A^i) + \frac{1}{2} \epsilon_{ijk} M_2^i \wedge (\tau_j + A^j) \wedge (\tau_k + A^k)$$
$$+ N_1 \wedge (\tau_1 + A^1) \wedge (\tau_2 + A^2) \wedge (\tau_3 + A^3) , \tag{5.17c}$$

one can find the dual type IIB RR fields as

$$\alpha'^{3/2}\widehat{F}_1 = m z_i e^{C_i}\hat{\mathfrak{e}}^i - z_i J_1^i - K_0^i e^{C_i}\hat{\mathfrak{e}}^i + \epsilon_{ijk} K_0^i z_j e^{-C_k}\hat{\mathfrak{e}}^k + N_1 \tag{5.18a}$$

$$\alpha'^{3/2}\widehat{F}_3 = m e^{C_1+C_2+C_3}\hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3 + e^{C_1+C_2+C_3} *_7 G_4 + G_2 \wedge z_i e^{C_i}\hat{\mathfrak{e}}^i$$

$$- \frac{1}{2}\epsilon_{ijk} J_1^i \wedge e^{C_j+C_k}\hat{\mathfrak{e}}^j \wedge \hat{\mathfrak{e}}^k + J_1^i \wedge e^{-C_i}\hat{\mathfrak{e}}^i \wedge z_j e^{C_j}\hat{\mathfrak{e}}^j$$

$$+ z_i K_0^i e^{2C_i} e^{-C_1-C_2-C_3}\hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3 \tag{5.18b}$$

$$- N_1 \wedge \frac{1}{2}\epsilon_{ijk} z_i e^{-C_b-C_c}\hat{\mathfrak{e}}^j \wedge \hat{\mathfrak{e}}^k - z_i L_3^i - M_2^i \wedge e^{C_i}\hat{\mathfrak{e}}^i + \epsilon_{ijk} M_2^i z_j \wedge e^{-C_k}\hat{\mathfrak{e}}^k$$

$$\alpha'^{3/2}\widehat{F}_5 = (1 + *)\left[G_4 \wedge z_i e^{C_i}\hat{\mathfrak{e}}^a i + e^{C_1+C_2+C_3} G_2 \wedge \hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3 \right.$$

$$- \frac{1}{2}\epsilon_{ijk} L_3^i \wedge e^{C_j+C_k}\hat{\mathfrak{e}}^j \wedge \hat{\mathfrak{e}}^k$$

$$\left. + L_3^i \wedge e^{-C_i}\hat{\mathfrak{e}}^i \wedge z_j e^{C_j}\hat{\mathfrak{e}}^j + z_i M_2^i e^{2C_i} \wedge e^{-C_1-C_2-C_3}\hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3 \right] . \tag{5.18c}$$

Reversely, starting from a type IIB solution, with RR-fields given by

$$F_1 = G_1 \tag{5.19a}$$

$$F_3 = G_3 + X_2^i \wedge (\tau_i + A^i) + \frac{1}{2}\epsilon_{ijk} Y_1^i \wedge (\tau_j + A^j) \wedge (\tau_k + A^k)$$

$$+ m(\tau_1 + A^1) \wedge (\tau_2 + A^2) \wedge (\tau_3 + A^3) \tag{5.19b}$$

$$F_5 = (1 + *)\left[Z_4^i \wedge (\tau_i + A^i) + G_2 \wedge (\tau_1 + A^1) \wedge (\tau_2 + A^2) \wedge (\tau_3 + A^3) \right] , \tag{5.19c}$$

we have the dual fields in IIA supergravity [3]

$$\widehat{F}_0 = -m \tag{5.19d}$$

$$\widehat{F}_2 = -e^{C_a} z_a G_1 \wedge \hat{\mathfrak{e}}^a + z_a X_2^a + Y_1^a \wedge (e^{C_a}\hat{\mathfrak{e}}^a) - \epsilon^{abc} Y_1^a \wedge (z_b e^{-C_c}\hat{\mathfrak{e}}^c)$$

$$+ \frac{m}{2} e^{-C_a-C_b}\epsilon^{abc} z_c \hat{\mathfrak{e}}^a \wedge \hat{\mathfrak{e}}^b - G_2 \tag{5.19e}$$

$$\widehat{F}_4 = e^{C_1+C_2+C_3} *_7 G_3 - z_a Z_4^a - e^{C_1+C_2+C_3} *_7 Z_4^a \wedge (e^{-C_a}\hat{\mathfrak{e}}^a) - \epsilon^{abc} e^{-C_a} *_7 Z_4^a \wedge z_b e^{C_b}\hat{\mathfrak{e}}^c$$

$$- e^{C_a} z_a G_3 \wedge \hat{\mathfrak{e}}^a + \frac{1}{2} G_2 \wedge (\epsilon^{abc} z_a e^{-C_b-C_c}\hat{\mathfrak{e}}^b \wedge \hat{\mathfrak{e}}^c) + \frac{1}{2}\epsilon^{abc} X_2^a \wedge (e^{C_b+C_c}\hat{\mathfrak{e}}^b \wedge \hat{\mathfrak{e}}^c)$$

$$- X_2^a \wedge (e^{-C_a}\hat{\mathfrak{e}}^a) \wedge (z_b e^{C_b}\hat{\mathfrak{e}}^b) - e^{C_1+C_2+C_3} G_1 \wedge \hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3$$

$$- z_a e^{2C_a} Y_1^a \wedge (e^{-C_1-C_2-C_3}\hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3) \tag{5.19f}$$

where the dual vielbeins are defined to be

$$\hat{\mathfrak{e}}^i = e^{C_i}\Delta^{-1}\left\{ z_i z_j Dz_j + e^{2\sum_{i\neq j} C_j} Dz_i + \epsilon_{ijk} z_j e^{2C_j} Dz_k \right\} . \tag{5.20}$$

---

[3]Observe that, compared with [121], we have some different signs in the dual RR-fields. We thank Eoin Ó Colgáin for letting us know about it.

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5.2   Galilean Solutions

In this section, we first give a short review of the solutions of [162], which are nonrelativistic generalizations of the gravity dual to ABJM [178] in type IIA string theory. We then perform nonabelian T-duality on them, obtaining new type IIB backgrounds. We consider the solutions in [162] because they have the nontrivial $z = 3$, even though we don't know much about their holographic dual field theory.

5.2.1   Galilean type-IIA solution and its NATD

The Galilean nonrelativistic solution of type IIA string theory of [162] has string frame metric[4]

$$ds^2_{IIA} = \underbrace{\frac{R^2}{4}\left(-\frac{\beta^2(dx^+)^2}{z^6} + \frac{dy^2 + dz^2 - 2dx^+dx^-}{z^2}\right)}_{ds^2_{Gal}} + R^2 \; ds^2_{\mathbb{CP}^3} \qquad (5.21)$$

where $R^2 = \sqrt{\tilde{R}^3/k}$ and the Fubini-Study metric for $\mathbb{CP}^3$ is (see, e.g. [179, 180])

$$ds^2_{\mathbb{CP}^3} = d\zeta^2 + \frac{1}{4}\cos^2\zeta(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{4}\sin^2\zeta(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2)$$
$$+ \frac{1}{4}\sin^2\zeta\cos^2\zeta(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2,$$
$$= d\zeta^2 + \frac{1}{4}\cos^2\zeta ds_1^2 + \frac{1}{4}\sin^2\zeta(\tau_1^2 + \tau_2^2) + \frac{1}{4}\sin^2\zeta\cos^2\zeta\left(\tau_3 + \cos\theta_1 d\phi_1\right)^2 \; . \tag{5.22}$$

Here $ds_1^2 = d\theta_1 + \sin^2\theta_1 d\phi_1^2$, $\zeta \in [0, \pi/2]$, $\theta_i \in [0, \pi]$, $\phi_i \in [0, 2\pi]$, $\psi \in [0, 4\pi]$ and τ_i are the Maurer-Cartan forms for the group $SU(2)$, namely

$$\begin{aligned} \tau_1 &= -\sin\psi d\theta_2 + \cos\psi\sin\theta_2 d\phi_2 \\ \tau_2 &= \cos\psi d\theta_2 + \sin\psi\sin\theta_2 d\phi_2 \\ \tau_3 &= d\psi + \cos\theta_2 d\phi_2 \; , \end{aligned} \tag{5.23}$$

with $d\tau_i = \frac{1}{2}\epsilon_{ijk}\tau_j \wedge \tau_k$. Considering the equation (5.9), we see that

$$ds_7^2 = ds^2_{Gal} + R^2 d\zeta^2 + \frac{R^2}{4}\cos^2\zeta ds_1^2 \; , \tag{5.24a}$$

---

[4]Using the fact that the constant β is arbitrary, we make the transformation $\beta \to \frac{1}{\sqrt{2}}\beta$, compared with [162]. Also, remember that $g^{str}_{\mu\nu} = e^{\frac{4}{D-2}\phi} g^E_{\mu\nu}$.

$$\sum_{i=1}^{3} e^{2C_i} (\tau_i + A^i)^2 = \frac{R^2}{4} \sin^2 \zeta (\tau_1^2 + \tau_2^2) + \frac{R^2}{4} \sin^2 \zeta \cos^2 \zeta (\tau_3 + \cos \theta_1 d\phi_1)^2 \;, \quad (5.24b)$$

that is,

$$
\begin{aligned}
e^{C_1} &= e^{C_2} = \frac{R}{2} \sin \zeta \equiv \beta_1^{1/2} \;, \quad e^{C_3} = \frac{R}{2} \sin \zeta \cos \zeta \equiv \beta_2^{1/2} \;, \\
A^1 &= A^2 = 0 \;, \quad A^3 = \cos \theta_1 d\phi_1 \;.
\end{aligned}
\quad (5.24c)
$$

We define the vielbeins associated to the Galilean metric

$$ds_{Gal}^2 = -\mathfrak{e}^+ \mathfrak{e}^+ + \mathfrak{e}^- \mathfrak{e}^- + \mathfrak{e}^y \mathfrak{e}^y + \mathfrak{e}^z \mathfrak{e}^z \;, \quad (5.25a)$$

as

$$
\begin{aligned}
\mathfrak{e}^+ &= \frac{R\beta}{2} \left(\frac{1}{z^3} dx^+ + \frac{z}{\beta^2} dx^- \right) \;, \quad \mathfrak{e}^- = \frac{Rz}{2\beta} dx^- \\
\mathfrak{e}^y &= \frac{R}{2z} dy \;, \quad \mathfrak{e}^z = \frac{R}{2z} dz \;.
\end{aligned}
\quad (5.25b)
$$

This solution is also supplemented with the following fields

$$
\begin{aligned}
e^\phi &= \frac{R}{k} \;, \quad B = \frac{\beta}{\sqrt{2}} \frac{R^2 p}{z^4} dx^+ \wedge dy \;, & (5.26) \\
C_{(1)} &= \frac{\beta}{\sqrt{2}} \frac{qk}{z^3} dx^+ + 2k\omega \;, \\
dC_{(3)} &= \frac{3R^2 k}{8z^4} dx^+ \wedge dx^- \wedge dy \wedge dz = \frac{6k}{R^2} e^+ \wedge e^- \wedge e^y \wedge e^z \;, & (5.27)
\end{aligned}
$$

where $q = 2p = \frac{1}{\sqrt{2}}$, $\mathcal{J} = d\omega$ is a Kähler 2-form on $\mathbb{CP}^3$ and the level k is the quantum of $dC_{(1)}$ on $\mathbb{CP}^1 \in \mathbb{CP}^3$, that is

$$\int_{\mathbb{CP}^1} dC_{(1)} = 2\pi k \;. \quad (5.28)$$

Considering that on the $\mathbb{CP}^1(\theta_1, \phi_1)$ with $\zeta = 0$ we have (e.g. [148])

$$
\begin{aligned}
\omega &= -\frac{1}{4} \sin^2 \zeta \left(\tau_3 + \cos \theta_1 d\phi_1 \right) + \frac{1}{4} \cos \theta_1 d\phi_1 \\
&= -\frac{1}{2} \tan \zeta \; \mathfrak{E}_3 + \frac{1}{4} \cos \theta_1 d\phi_1 \;,
\end{aligned}
\quad (5.29)
$$

we have

$$\mathcal{J} = \mathfrak{E}_3 \wedge \mathfrak{E}_\zeta + \mathfrak{E}_\phi \wedge \mathfrak{E}_\theta - \mathfrak{E}_1 \wedge \mathfrak{E}_2 \quad (5.30)$$

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where we have defined the following vielbeins with relation to the metric $ds^2_{\mathbb{CP}^3}$ in (5.22)

$$\mathfrak{E}_\zeta = d\zeta \;, \quad \mathfrak{E}_\theta = \frac{1}{2}\cos\zeta d\theta_1 \;, \quad \mathfrak{E}_\phi = \frac{1}{2}\cos\zeta \sin\theta_1 d\phi_1$$
$$\mathfrak{E}_1 = \frac{1}{2}\sin\zeta\tau_1 \;, \quad \mathfrak{E}_2 = \frac{1}{2}\sin\zeta\tau_2 \;, \quad \mathfrak{E}_3 = \frac{1}{2}\sin\zeta\cos\zeta(\tau_3 + \cos\theta_1 d\phi_1) \;. \tag{5.31}$$

With these definitions we can easily see that $vol(\mathbb{CP}^3) = \frac{1}{3!}\mathcal{J} \wedge \mathcal{J} \wedge \mathcal{J}$, that is

$$vol(\mathbb{CP}^3) = \mathfrak{E}_\zeta \wedge \mathfrak{E}_\phi \wedge \mathfrak{E}_\theta \wedge \mathfrak{E}_1 \wedge \mathfrak{E}_2 \wedge \mathfrak{E}_3$$
$$= \frac{1}{32}\cos^3\zeta\sin^3\zeta\sin\theta_1\sin\theta_2 d\zeta \wedge d\theta_1 \wedge d\phi_1 \wedge d\theta_2 \wedge d\phi_2 \wedge d\psi \;. \tag{5.32}$$

Therefore, using the quantization of the Page charge

$$\frac{1}{(2\pi\alpha'^{1/2})^{7-p}}\int_{\Sigma^{8-p}} \mathcal{F}_{8-p} = \mathcal{Q}_{Dp} \in \mathbb{Z} \;, \tag{5.33}$$

where $\mathcal{F} = F \wedge e^{-B}$, for some cycle Σ^{8-p}, we can see that

$$\frac{1}{(2\pi\alpha'^{1/2})^5}\int_{\mathbb{CP}^3} *\mathcal{F}_4 = N_{D2} \in \mathbb{Z} \;\Rightarrow\; R^4 = \frac{32\pi^2\alpha'^{5/2}}{k}N_{D2} \;, \tag{5.34}$$

and in the fourth section, we find the condition that the radius R must satisfy in terms of charges of the dual background. In particular, we will see that this condition is consistent with the results of [148].

The vielbeins with relation to the metric (5.21) for the internal space are defined as

$$\mathfrak{e}_\zeta = Rd\zeta \;, \quad \mathfrak{e}_\theta = \frac{R}{2}\cos\zeta d\theta_1 \;, \quad \mathfrak{e}_\phi = \frac{R}{2}\cos\zeta \sin\theta_1 d\phi_1 \;, \tag{5.35a}$$

$$\mathfrak{e}_1 = \frac{R}{2}\sin\zeta\tau_1 \equiv \beta_1^{1/2}\tau_1 \;, \quad \mathfrak{e}_2 = \frac{R}{2}\sin\zeta\tau_2 \equiv \beta_1^{1/2}\tau_2 \;, \tag{5.35b}$$

$$\mathfrak{e}_3 = \frac{R}{2}\sin\zeta\cos\zeta\,(\tau_3 + \cos\theta_1 d\phi_1) \equiv \beta_2^{1/2}(\tau_3 + A_3) \;. \tag{5.35c}$$

The relativistic limit of this solution, that is $AdS_4 \times \mathbb{CP}^3$, can be recovered by setting $\beta \to 0$.

Nonabelian T-dual of the Galilean background

Now we want to perform a T-duality transformation [23, 25, 120, 121] with respect to the $SU(2)$ isometry. We construct the matrix M_{ij}, defined by $M_{ij} = g_{ij} + b_{ij} +$

$\alpha' \epsilon_{ijk} \hat{v}_k$, obtaining (since $b_{ij} = 0$),

$$M = \begin{pmatrix} \beta_1 & \alpha' \hat{v}_3 & -\alpha' \hat{v}_2 \\ -\alpha' \hat{v}_3 & \beta_1 & \alpha' \hat{v}_1 \\ \alpha' \hat{v}_2 & -\alpha' \hat{v}_1 & \beta_2 \end{pmatrix} . \tag{5.36a}$$

We consider a gauge where $\theta_2 = \phi_2 = v_2 = 0$, so that $\hat{v} = (\cos \psi v_1, \sin \psi v_1, v_3)$, where $\psi \in [0, 2\pi]$. We can make connections to the gauge choice in [148] by making the transformation $(v_1 = \rho \sin \chi, v_3 = \rho \cos \chi)$ with $\chi \in [0, \pi]$ and the range of the coordinate ρ is not yet determined, but we argue that $\rho \in [n\pi, (n+1)\pi)$ as in [146, 148], (see [23] for other possible gauge choices).

Therefore, the matrix M in this gauge is

$$M = \begin{pmatrix} \beta_1 & \alpha' v_3 & -\alpha' \sin \psi v_1 \\ -\alpha' v_3 & \beta_1 & \alpha' \cos \psi v_1 \\ \alpha' \sin \psi v_1 & -\alpha' \cos \psi v_1 & \beta_2 \end{pmatrix} . \tag{5.36b}$$

The dilaton in the dual theory is given by

$$\widehat{\phi} = \phi - \frac{1}{2} \ln \left(\frac{\Delta}{\alpha'^3} \right) \Rightarrow e^{\widehat{\phi}} = \frac{R \alpha'^{3/2}}{k \Delta^{1/2}} , \tag{5.37}$$

where $\Delta \equiv \det M = [(\beta_1^2 + \alpha'^2 v_3^2)\beta_2 + \alpha'^2 v_1^2 \beta_1]$.

Using the results of 2.3, the dual metric becomes

$$d\hat{s}_{IIB}^2 = ds_7^2 + \frac{1}{\Delta} d\Sigma^2 \tag{5.38a}$$

where

$$\begin{aligned} d\Sigma^2 &= (z_1 Dz_1 + z_2 Dz_2 + z_3 Dz_3)^2 + e^{2(C_2+C_3)} Dz_1^2 + e^{2(C_1+C_3)} Dz_2^2 + e^{2(C_1+C_2)} Dz_3^2 \\ &= \alpha'^2 v_1^2 \beta_1 \beta_2 \hat{\eta}^2 + \alpha'^2 \left\{ (\beta_1 \beta_2 + \alpha'^2 v_1^2) dv_1^2 + (\beta_1^2 + \alpha'^2 v_3^2) dv_3^2 + 2\alpha'^2 v_1 v_3 dv_1 dv_3 \right\} , \end{aligned} \tag{5.38b}$$

with $\hat{\eta} = d\psi + \cos \theta_1 d\phi_1$. Here we have used that $z_a = \alpha' \hat{v}_a$ and

$$\frac{1}{\alpha'} Dz_1 = d\hat{v}_1 - \hat{v}_2 A^3 , \quad \frac{1}{\alpha'} Dz_2 = d\hat{v}_2 + \hat{v}_1 A^3 , \quad \frac{1}{\alpha'} Dz_3 = d\hat{v}_3 . \tag{5.39}$$

The dual vielbeins are

$$\hat{\mathfrak{e}}_a = e^{C_a} \Delta^{-1} \left[z_a z_b Dz_b + e^{2 \sum_{b \neq a} C_b} Dz_a + \epsilon_{abc} z_b e^{2C_b} Dz_c \right] , \tag{5.40}$$

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such that

$$\hat{\mathfrak{e}}'_1 \equiv \cos\psi\hat{\mathfrak{e}}_1 + \sin\psi\hat{\mathfrak{e}}_2 = \frac{\alpha'\beta_1^{1/2}}{\Delta}\left[(\beta_1\beta_2 + \alpha'^2 v_1^2)dv_1 + \alpha'^2 v_1 v_3 dv_3 - \alpha'\beta_2 v_1 v_3 \hat{\eta}\right]$$
$$(5.41\text{a})$$

$$\hat{\mathfrak{e}}'_2 \equiv -\sin\psi\hat{\mathfrak{e}}_1 + \cos\psi\hat{\mathfrak{e}}_2 = \frac{\alpha'\beta_1^{1/2}}{\Delta}\left[\alpha'\beta_2 v_3 dv_1 - \alpha'\beta_1 v_1 dv_3 + \beta_1\beta_2 v_1 \hat{\eta}\right] \qquad (5.41\text{b})$$

$$\hat{\mathfrak{e}}_3 = \frac{\alpha'\beta_2^{1/2}}{\Delta}\left[\alpha'^2 v_1 v_3 dv_1 + (\beta_1^2 + \alpha'^2 v_3^2)dv_3 + \alpha'\beta_1 v_1^2 \hat{\eta}\right] \ . \qquad (5.41\text{c})$$

The Kalb-Ramond field is given by

$$\widehat{B} = \frac{\beta}{\sqrt{2}}\frac{R^2 p}{z^4}dx^+ \wedge dy$$

$$- \frac{1}{\Delta}\left\{e^{2C_1}z_1 Dz_2 \wedge Dz_3 + e^{2C_2}z_2 Dz_3 \wedge Dz_1 + e^{2C_3}z_3 Dz_1 \wedge Dz_2\right\} - Dz_3 \wedge A^3$$

$$= \frac{\beta}{\sqrt{2}}\frac{R^2 p}{z^4}dx^+ \wedge dy - \frac{\alpha'^3 v_1 v_3 \beta_2}{\Delta}dv_1 \wedge \hat{\eta} + \frac{\alpha'(\alpha'^2 v_1^2 \beta_1 - \Delta)}{\Delta}dv_3 \wedge (\cos\theta_1 d\phi_1)$$

$$+ \frac{\alpha'^3 v_1^2 \beta_1}{\Delta}dv_3 \wedge d\psi$$

$$= \frac{\beta}{\sqrt{2}}\frac{R^2 p}{z^4}dx^+ \wedge dy - \underbrace{\frac{\alpha'\beta_2}{\Delta}\left(\alpha'^2 v_1 v_3 dv_1 + (\alpha'^2 v_3^2 + \beta_1^2)dv_3\right) \wedge \hat{\eta}}_{=\sqrt{\beta_2}\,\hat{\mathfrak{e}}^3 \wedge \hat{\eta}} + \underbrace{\alpha' dv_3 \wedge d\psi}_{(closed\ form)}$$

$$(5.42\text{a})$$

or, using the spherical coordinates $(v_1, v_3) = (\rho\sin\chi, \rho\cos\chi)$,

$$\widehat{B} = \frac{\beta}{\sqrt{2}}\frac{R^2 p}{z^4}dx^+ \wedge dy - \frac{\alpha'\beta_2}{\Delta}\left((\beta_1^2 + \alpha'^2\rho^2)\cos\chi d\rho - \beta_1^2\rho\sin\chi d\chi\right) \wedge \hat{\eta}$$

$$= \frac{\beta}{\sqrt{2}}\frac{R^2 p}{z^4}dx^+ \wedge dy - \alpha'\frac{R^2\sin^2\zeta}{64\Delta}\left\{R^4\rho\cos^2\zeta\sin^4\zeta\cos\theta_1\sin\chi d\phi_1 \wedge d\chi\right.$$

$$+ \cos^2\zeta\cos\theta_1\cos\chi(R^4\sin^4\zeta + 16\alpha'^2\rho^2)d\rho \wedge d\phi_1 \qquad (5.42\text{b})$$

$$\left. - \rho R^4\sin^4\zeta\cos^2\zeta\sin\chi d\chi \wedge d\psi + (R^4\sin^4\zeta + 16\alpha'^2\rho^2)\cos^2\zeta\cos\chi d\rho \wedge d\psi\right\}$$

After a gauge transformation, we can write the B-field as

$$\widehat{B} = \frac{\beta}{\sqrt{2}}\frac{R^2 p}{z^4}dx^+ \wedge dy - \alpha'\frac{R^2 \sin^2\zeta}{64\Delta}\left\{ R^4\rho\cos^2\zeta\sin^4\zeta\cos\theta_1\sin\chi d\phi_1 \wedge d\chi \right.$$

$$+ \cos^2\zeta\cos\theta_1\cos\chi(R^4\sin^4\zeta + 16\alpha'^2\rho^2)d\rho \wedge d\phi_1 \qquad (5.42c)$$

$$+ 16\alpha'^2\rho^2\left[\rho(\sin^2\chi + \cos^2\chi\cos^2\zeta)\sin\chi d\chi \wedge d\psi - \sin^2\chi\sin^2\zeta\cos\chi d\rho \wedge d\psi\right]\bigg\}$$

$$- \alpha'd(\rho\cos\chi d\psi) \, ,$$

with the term on the last line being a pure gauge contribution. The $\widehat{B}$-field at the 2-cycle defined by $\tilde{S}^2 = (\phi_1 = const., x^+ = const., y = const.; \chi, \psi)$ is

$$\widehat{B}\bigg|_{\tilde{S}^2} = -\alpha'\rho\sin\chi d\chi \wedge d\psi \, , \qquad (5.43)$$

where we also have used that $\lim_{\zeta\to 0}\frac{\Delta}{\sin^2\zeta} = \frac{R^2\rho^2}{4}\alpha'^2$. Large gauge transformations are defined such that the holonomy of $\widehat{B}$ satisfies

$$b = \frac{1}{4\pi^2\alpha'}\left|\int_{\tilde{S}^2}\widehat{B}\right| \in [n, n+1), \qquad (5.44)$$

which justifies our choice $\rho \in [n\pi, (n+1)\pi)$.

In order to find the dual R–R fields, it is convenient to write the RR-fields before the T-duality as

$$F_2 = dC_{(1)} = -12\frac{qk}{R^2}\mathfrak{e}^z \wedge \mathfrak{e}^t + \frac{2k}{R^2}(\mathfrak{e}_3 \wedge \mathfrak{e}_\zeta + \mathfrak{e}_\phi \wedge \mathfrak{e}_\theta - \mathfrak{e}_1 \wedge \mathfrak{e}_2)$$

$$= \frac{2k}{R^2}\left(6q\,\mathfrak{e}^t \wedge \mathfrak{e}^z + \mathfrak{e}^\phi \wedge \mathfrak{e}^\theta\right) - \frac{2k\beta_2^{1/2}}{R^2}\mathfrak{e}_\zeta \wedge (\tau^3 + A_3) - \frac{2k\beta_1}{R^2}\tau_1 \wedge \tau_2$$

$$\equiv G_2 + J_1^3 \wedge (\tau^3 + A_3) + K_0^3\tau_1 \wedge \tau_2 \qquad (5.45a)$$

$$F_4 = dC_{(3)} - H \wedge C_{(1)}$$

$$= \frac{2k}{R^2}\mathfrak{e}^t \wedge \mathfrak{e}^y \wedge \mathfrak{e}^z \wedge \left(3\mathfrak{e}^- + 8p\frac{\cot\theta_1}{\cos\zeta}\mathfrak{e}_\phi\right) - \frac{16pk}{R^2}\tan\zeta\,\mathfrak{e}^t \wedge \mathfrak{e}^y \wedge \mathfrak{e}^z \wedge \mathfrak{e}_3$$

$$= \frac{2k}{R^2}\mathfrak{e}^t \wedge \mathfrak{e}^y \wedge \mathfrak{e}^z \wedge \left(3\mathfrak{e}^- + \frac{8p\cot\theta_1}{\cos\zeta}\mathfrak{e}_\phi\right) - \frac{16pk\beta_2^{1/2}}{R^2}\tan\zeta\,\mathfrak{e}^t \wedge \mathfrak{e}^y \wedge \mathfrak{e}^z \wedge (\tau^3 + A_3)$$

$$\equiv G_4 + L_3^3 \wedge (\tau^3 + A_3) \, , \qquad (5.45b)$$

with $\mathfrak{e}^t = (\mathfrak{e}^+ - \mathfrak{e}^-)/\sqrt{2}$, $F_6 = -*F_4$ and $F_8 = *F_2$. Using equations (5.17a–5.18c) we have that the dual fields in the RR sector are given by

$$
\begin{aligned}
\alpha'^{3/2}\widehat{F}_1 &= -\alpha' v_3\, J_1^3 - K_0^3 \beta_2^{1/2}\hat{\mathfrak{e}}_3 + \alpha' K_0^3 \beta_1^{-1/2} v_1 \left(\cos\psi\hat{\mathfrak{e}}_2 - \sin\psi\hat{\mathfrak{e}}_1\right) \\
&= -\alpha' v_3 J_1^3 - \alpha' K_0^3 dv_3 \qquad\qquad (5.46\text{a})
\end{aligned}
$$

$$
\begin{aligned}
\alpha'^{3/2}\widehat{F}_3 &= \beta_1\beta_2^{1/2} *_7 G_4 + \alpha' G_2 \wedge \left[v_1\beta_1^{1/2}(\cos\psi\hat{\mathfrak{e}}_1 + \sin\psi\hat{\mathfrak{e}}_2) + v_3\beta_2^{1/2}\hat{\mathfrak{e}}_3 \right] \\
&+ J_1^3 \wedge \left[-\beta_1\hat{\mathfrak{e}}_1 \wedge \hat{\mathfrak{e}}_2 + \frac{\alpha'\beta_1^{1/2}v_1}{\beta_2^{1/2}}\hat{\mathfrak{e}}_3 \wedge (\cos\psi\hat{\mathfrak{e}}_1 + \sin\psi\hat{\mathfrak{e}}_2) \right] \\
&+ \frac{\alpha' v_3 K_0^3 \beta_2^{1/2}}{\beta_1}\hat{\mathfrak{e}}_1 \wedge \hat{\mathfrak{e}}_2 \wedge \hat{\mathfrak{e}}_3 \\
&= \beta_1\beta_2^{1/2} *_7 G_4 + \alpha'^2 G_2 \wedge (v_1 dv_1 + v_3 dv_3) - \frac{\alpha'^2 v_1}{\Delta} J_1^3 \wedge \left[(\alpha'^2 v_1^2\beta_1 + \beta_1^2\beta_2)dv_1 \right. \\
&+ \left. v_1 v_3\alpha'^2\beta_1 dv_3 \right] \wedge \hat{\eta} - \frac{\alpha'^4 v_3 v_1}{\Delta} K_0^3\beta_2 dv_1 \wedge dv_3 \wedge \hat{\eta} \qquad (5.46\text{b})
\end{aligned}
$$

$$
\begin{aligned}
\alpha'^{3/2}\widehat{F}_5 &= (1+*)\left\{ \alpha' G_4 \wedge \left[v_1\beta_1^{1/2}(\cos\psi\hat{\mathfrak{e}}_1 + \sin\psi\hat{\mathfrak{e}}_2) + v_3\beta_2^{1/2}\hat{\mathfrak{e}}_3 \right] \right. \\
&+ \beta_1\beta_2^{1/2}G_2 \wedge \hat{\mathfrak{e}}_1 \wedge \hat{\mathfrak{e}}_2 \wedge \hat{\mathfrak{e}}_3 \\
&+ \left. L_3^3 \wedge \left[-\beta_1\hat{\mathfrak{e}}_1 \wedge \hat{\mathfrak{e}}_2 + \frac{\alpha'\beta_1^{1/2}v_1}{\beta_2^{1/2}}\hat{\mathfrak{e}}_3 \wedge (\cos\psi\hat{\mathfrak{e}}_1 + \sin\psi\hat{\mathfrak{e}}_2) \right] \right\} \\
&= -\beta_1\beta_2^{1/2} *_7 G_2 + \alpha' G_4 \wedge \left[v_1\beta_1^{1/2}\hat{\mathfrak{e}}_1' + v_3\beta_2^{1/2}\hat{\mathfrak{e}}_3 \right] \\
&+ *_7 L_3^3 \wedge \left[-\beta_1\hat{\mathfrak{e}}_3 + \frac{\alpha'\beta_1^{1/2}v_1}{\beta_2^{1/2}}\hat{\mathfrak{e}}_2' \right] + L_3^3 \wedge \left[-\beta_1\hat{\mathfrak{e}}_1 \wedge \hat{\mathfrak{e}}_2 + \frac{\alpha'\beta_1^{1/2}v_1}{\beta_2^{1/2}}\hat{\mathfrak{e}}_3 \wedge \hat{\mathfrak{e}}_1' \right] \\
&- \alpha' *_7 G_4 \wedge \left[v_1\beta_1^{1/2}\hat{\mathfrak{e}}_2' \wedge \hat{\mathfrak{e}}_3 + v_3\beta_2^{1/2}\hat{\mathfrak{e}}_1' \wedge \hat{\mathfrak{e}}_2' \right] \\
&+ \beta_1\beta_2^{1/2}G_2 \wedge \hat{\mathfrak{e}}_1 \wedge \hat{\mathfrak{e}}_2 \wedge \hat{\mathfrak{e}}_3 \\
&= -\beta_1\beta_2^{1/2} *_7 G_2 + \alpha'^2 G_4 \wedge [v_1 dv_1 + v_3 dv_2] - \frac{\alpha'\beta_1}{\beta_2^{1/2}}(*_7 L_3^3) \wedge dv_3 \\
&- \frac{v_1\alpha'^2}{\Delta}L_3^3 \wedge \left[(\alpha'^2 v_1^2\beta_1 + \beta_1^2\beta_2)dv_1 + \alpha'^2 v_1 v_3\beta_1 dv_3 \right] \wedge \hat{\eta} \qquad (5.46\text{c}) \\
&- \frac{\alpha'^3 v_1\beta_1\beta_2^{1/2}}{\Delta} *_7 G_4 \wedge [v_3\beta_2 dv_1 - v_1\beta_1 dv_3] \wedge \hat{\eta} \\
&- \frac{\alpha'^3 v_1\beta_1^2\beta_2}{\Delta}G_2 \wedge dv_1 \wedge dv_3 \wedge \hat{\eta}
\end{aligned}
$$

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or, using spherical coordinates,

$$\alpha'^{3/2}\widehat{F}_1 = \frac{k}{2}\left(-\rho\sin^2\zeta\sin\chi d\chi + \sin^2\zeta\cos\chi d\rho - \rho\sin 2\zeta\cos\chi d\zeta\right) \tag{5.46d}$$

$$\alpha'^{3/2}\widehat{F}_3 = \frac{kR}{4}\sin^3\zeta\cos\zeta\left(3\mathfrak{e}^\zeta\wedge\mathfrak{e}^\theta\wedge\mathfrak{e}^\phi + \frac{8p\cot\theta_1}{\cos\zeta}\mathfrak{e}^-\wedge\mathfrak{e}^\zeta\wedge\mathfrak{e}^\theta\right)$$

$$+ \frac{2k\alpha'^2}{R^2}\rho\left(6q\mathfrak{e}^t\wedge\mathfrak{e}^z + \mathfrak{e}^\phi\wedge\mathfrak{e}^\theta\right)\wedge d\rho + \frac{\alpha'^4 R^2}{4\Delta}\sin^2\zeta\cos^2\zeta\rho^3\cos\chi\sin\chi d\rho\wedge d\chi\wedge\hat\eta$$

$$+ \frac{\alpha'^2 kR^2}{4\Delta}\rho\sin^3\zeta\cos\zeta\sin\chi\, d\zeta\wedge\left[\frac{R^4}{16}\sin^4\zeta\cos^2\zeta d(\rho\sin\chi) + \alpha'^2\rho^2\,\sin\chi\, d\rho\right]\wedge\hat\eta$$

$$\tag{5.46e}$$

$$\alpha'^{3/2}\widehat{F}_5 = (1+*)\left\{\frac{2k\alpha'^2}{R^2}\rho\,\mathfrak{e}^t\wedge\mathfrak{e}^y\wedge\mathfrak{e}^z\wedge\left(3\mathfrak{e}^- + \frac{8p\cot\theta_1}{\cos\zeta}\mathfrak{e}^\phi\right)\wedge d\rho\right.$$

$$+ \frac{\alpha'^3 kR^4}{32\Delta}\sin^6\zeta\cos^2\zeta\,\rho^2\sin\chi\left(6q\,\mathfrak{e}^t\wedge\mathfrak{e}^z + \mathfrak{e}^\phi\wedge\mathfrak{e}^\theta\right)\wedge d\rho\wedge d\chi\wedge\hat\eta$$

$$+ \frac{2pkR\alpha'^2}{\Delta}\sin^4\zeta\,\rho\sin\chi\,\mathfrak{e}^t\wedge\mathfrak{e}^y\wedge\mathfrak{e}^z\wedge$$

$$\left.\wedge\left[\frac{R^4}{16}\sin^4\zeta\cos^2\zeta d(\rho\sin\chi) + \alpha'^2\rho^2\,\sin\chi\, d\rho\right]\wedge\hat\eta\right\}\ . \tag{5.46f}$$

5.2.2 Galilean solution in massive type-IIA and its NATD

We also have the following background, in the string frame [162]

$$ds^2_{mIIA} = \underbrace{a_0\left(-\frac{\beta^2(dx^+)^2}{z^6} + \frac{dy^2 + dz^2 - 2dx^+dx^-}{z^2}\right)}_{ds^2_{mGal}} + \frac{5a_0}{2}ds^2_{\mathbb{CP}^3} \tag{5.47a}$$

where $a_0 f^2 = 2$, $L^2 = a_0 e^{-\phi/2}$, $f^{5/2} L^2 = 2\, m_0^{1/2}$ and m_0 is the Romans' mass. In this background we also have the nontrivial fields

$$e^{2\phi} = \frac{f^2}{m_0^2}\ , \quad B = \frac{2\beta}{f^2}\frac{1}{z^4}dx^+\wedge dy\ ,$$

$$C_{(1)} = \frac{2\sqrt{5}\beta m_0}{3f^2}\frac{1}{z^3}dx^+\ , \quad dC_{(3)} = 4\frac{\sqrt{5}m_0}{f^4 z^4}dx^+\wedge dx^-\wedge dy\wedge dz\ . \tag{5.47b}$$

Note that the solution is similar to the one in the previous subsection, but there are subtle differences. This solution does not preserve any supersymmetry even in the relativistic case $\beta = 0$. We write the metric (5.47a) in a similar manner to the

previous subsection, as

$$ds^2_{mIIA} = d\tilde{s}^2_7 + \sum_{i=1}^{3} e^{2\tilde{C}_i}(\sigma^i + A^i)^2 \tag{5.48a}$$

where

$$d\tilde{s}^2_7 = ds^2_{mGal} + \frac{5a_0}{2}\left(d\zeta^2 + \frac{1}{4}\cos^2\zeta\, ds^2_1\right) \tag{5.48b}$$

and

$$\sum_{i=1}^{3} e^{2\tilde{C}_i}(\sigma^i + A^i)^2 = \frac{5a_0}{8}\left[\sin^2\zeta(\tau_1^2 + \tau_2^2) + \sin^2\zeta\cos^2\zeta(\tau_3 + \cos\theta_1 d\phi_1)^2\right] \ , \tag{5.48c}$$

such that

$$e^{\tilde{C}_1} = e^{\tilde{C}_2} = \frac{\sqrt{5a_0}}{2\sqrt{2}}\sin\zeta \equiv \tilde{\beta}_1^{1/2} \ , \quad e^{\tilde{C}_3} = \frac{\sqrt{5a_0}}{2\sqrt{2}}\sin\zeta\cos\zeta \equiv \tilde{\beta}_2^{1/2} \ ,$$
$$A^1 = A^2 = 0 \ , \quad A^3 = \cos\theta_1 d\phi_1 \ . \tag{5.48d}$$

Nonabelian T-dual of the massive Galilean background

The dual dilaton is given by

$$\widehat{\phi} = \phi - \frac{1}{2}\ln\left(\frac{\tilde{\Delta}}{\alpha'^3}\right) \ , \tag{5.49}$$

where $\tilde{\Delta} = [(\tilde{\beta}_1^2 + \alpha'^2 v_3^2)\tilde{\beta}_2 + \alpha'^2 v_1^2\tilde{\beta}_1]$. The nonabelian T-dual metric is

$$d\hat{s}^2_{mIIB} = ds^2_7 + \frac{1}{\tilde{\Delta}}d\tilde{\Sigma}^2 \tag{5.50a}$$

where

$$d\tilde{\Sigma}^2 = (z_1 Dz_1 + z_2 Dz_2 + z_3 Dz_3)^2 + e^{2(C_2+C_3)}Dz_1^2 + e^{2(C_1+C_3)}Dz_2^2 + e^{2(C_1+C_2)}Dz_3^2$$
$$= \alpha'^2 v_1^2\tilde{\beta}_1\tilde{\beta}_2\hat{\eta}^2 + \alpha'^2\left\{(\tilde{\beta}_1\tilde{\beta}_2 + \alpha'^2 v_1^2)dv_1^2 + (\tilde{\beta}_1^2 + \alpha'^2 v_3^2)dv_3^2 + 2\alpha'^2 v_1 v_3 dv_1 dv_3\right\},$$
$$\tag{5.50b}$$

with $\hat{\eta} = d\psi + \cos\theta_1 d\phi_1$. Here we have used that

$$\frac{1}{\alpha'}Dz_1 = d\hat{v}_1 - \hat{v}_2 A^3 \ , \quad \frac{1}{\alpha'}Dz_2 = d\hat{v}_2 + \hat{v}_1 A^3 \ , \quad \frac{1}{\alpha'}Dz_3 = d\hat{v}_3 \ . \tag{5.51}$$

The dual Kalb-Ramond field is given by

$$\widehat{B} = \frac{2\beta}{f^2 z^4} dx^+ \wedge dy$$

$$- \frac{1}{\tilde{\Delta}} \left\{ e^{2\tilde{C}_1} z_1 Dz_2 \wedge Dz_3 + e^{2\tilde{C}_2} z_2 Dz_3 \wedge Dz_1 + e^{2\tilde{C}_3} z_3 Dz_1 \wedge Dz_2 \right\} - Dz_3 \wedge A^3$$

$$= \frac{2\beta}{f^2 z^4} dx^+ \wedge dy - \frac{\alpha'^3 v_1 v_3 \tilde{\beta}_2}{\tilde{\Delta}} dv_1 \wedge \hat{\eta} + \frac{\alpha'(\alpha'^2 v_1^2 \tilde{\beta}_1 - \tilde{\Delta})}{\tilde{\Delta}} dv_3 \wedge (\cos\theta_1 d\phi_1)$$

$$+ \frac{\alpha'^3 v_1^2 \tilde{\beta}_1}{\tilde{\Delta}} dv_3 \wedge d\psi$$

$$= \frac{2\beta}{f^2 z^4} dx^+ \wedge dy - \frac{\alpha' \tilde{\beta}_2}{\tilde{\Delta}} \left(\alpha'^2 v_1 v_3 dv_1 + (\alpha'^2 v_3^2 + \tilde{\beta}_1^2) dv_3 \right) \wedge \hat{\eta} + \text{closed} . \quad (5.52)$$

Given that the original R-R fields are

$$F_0 = m_0 \tag{5.53a}$$

$$F_2 = dC_{(1)} + m_0 B = m_0 \, \mathfrak{e}_{m_0}^t \wedge \left(\sqrt{5} \, \mathfrak{e}_{m_0}^z + \mathfrak{e}_{m_0}^y \right) \tag{5.53b}$$

$$F_4 = dC_{(3)} - H \wedge C_{(1)} + \frac{m_0}{2} B \wedge B$$

$$= \sqrt{5} \, m_0 \, \mathfrak{e}_{m_0}^t \wedge \mathfrak{e}_{m_0}^- \wedge \mathfrak{e}_{m_0}^y \wedge \mathfrak{e}_{m_0}^z , \tag{5.53c}$$

where these vielbeins are related to the metric (5.47a), it folllows that the T-dual fields are given by

$$\alpha'^{3/2} \widehat{F}_1 = \alpha' m_0 \left[v_1 \tilde{\beta}_1^{1/2} (\cos\psi \hat{\mathfrak{e}}^1 + \sin\psi \hat{\mathfrak{e}}^2) + v_3 \tilde{\beta}_2^{1/2} \hat{\mathfrak{e}}^3 \right]$$

$$= m_0 \alpha'^2 (v_1 dv_1 + v_3 dv_3) \tag{5.54a}$$

$$\alpha'^{3/2} \widehat{F}_3 = m_0 \tilde{\beta}_1 \tilde{\beta}_2^{1/2} \hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3 + \tilde{\beta}_1 \tilde{\beta}_2^{1/2} *_7 F_4$$

$$+ \alpha' F_2 \wedge \left[v_1 \tilde{\beta}_1^{1/2} (\cos\psi \hat{\mathfrak{e}}^1 + \sin\psi \hat{\mathfrak{e}}^2) + v_3 \tilde{\beta}_2^{1/2} \hat{\mathfrak{e}}^3 \right] \tag{5.54b}$$

$$= -\frac{\alpha'^3 m_0 \tilde{\beta}_1^2 \tilde{\beta}_2}{\tilde{\Delta}} v_1 dv_1 \wedge dv_3 \wedge \hat{\eta} + \tilde{\beta}_1 \tilde{\beta}_2^{1/2} *_7 F_4 + \alpha'^2 F_2 \wedge (v_1 dv_1 + v_3 dv_3)$$

$$\alpha'^{3/2} \widehat{F}_5 = (1 + *) \left\{ \alpha' F_4 \wedge \left[v_1 \tilde{\beta}_1^{1/2} (\cos\psi \hat{\mathfrak{e}}^1 + \sin\psi \hat{\mathfrak{e}}^2) + v_3 \tilde{\beta}_2^{1/2} \hat{\mathfrak{e}}^3 \right] + \right.$$

$$\left. + \tilde{\beta}_1 \tilde{\beta}_2^{1/2} F_2 \wedge \hat{\mathfrak{e}}^1 \wedge \hat{\mathfrak{e}}^2 \wedge \hat{\mathfrak{e}}^3 \right\}$$

$$= -\tilde{\beta}_1 \tilde{\beta}_2^{1/2} *_7 F_2 + \alpha'^2 F_4 \wedge (v_1 dv_1 + v_3 dv_3) - \frac{v_1 \alpha'^3 \tilde{\beta}_1^2 \tilde{\beta}_2}{\tilde{\Delta}} F_2 \wedge dv_1 \wedge dv_3 \wedge \hat{\eta}$$

$$- \alpha'^3 \frac{v_1 \tilde{\beta}_1 \tilde{\beta}_2^{1/2}}{\tilde{\Delta}} (*_7 F_4) \wedge (v_3 \tilde{\beta}_2 dv_1 - v_1 \tilde{\beta}_1 dv_3) \wedge \hat{\eta} \tag{5.54c}$$

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or

$$\alpha'^{3/2}\widehat{F}_1 = \alpha'^2 m_0 \, \rho d\rho \qquad (5.54\text{d})$$

$$\alpha'^{3/2}\widehat{F}_3 = \frac{m_0\alpha'^3}{\Delta}\tilde{\beta}_1^2\tilde{\beta}_2\rho^2 d\rho \wedge vol_{\tilde{S}^2} + \tilde{\beta}_1\tilde{\beta}_2^{1/2} *_7 F_4 + \alpha'^2 \rho F_2 \wedge d\rho$$

$$\alpha'^{3/2}\widehat{F}_5 = \alpha'^2 \rho F_4 \wedge d\rho + \frac{\alpha'^3}{\Delta}\tilde{\beta}_1^2\tilde{\beta}_2\rho^2 F_2 \wedge d\rho \wedge vol_{\tilde{S}^2} - \tilde{\beta}_1\tilde{\beta}_2^{1/2} *_7 F_2$$

$$+ \frac{\alpha'^3}{\Delta}\tilde{\beta}_1\tilde{\beta}_2^{1/2} *_7 F_4 \wedge \left[-\rho^2 \sin^2\chi\cos\chi\left(\tilde{\beta}_1 - \tilde{\beta}_2\right)\hat{\eta}\wedge d\rho \right.$$

$$+ \left. \rho^3 \sin\chi\left(\tilde{\beta}_2\cos^2\chi + \tilde{\beta}_1\sin^2\chi\right)\hat{\eta}\wedge d\chi \right] \qquad (5.54\text{e})$$

where $vol_{\tilde{S}^2} = \sin\chi d\chi \wedge d\psi$ and

$$\hat{\mathfrak{e}}_a = e^{C_a}\tilde{\Delta}^{-1}\left[z_a z_b Dz_c + e^{2\sum_{b\neq a}C_b}Dz_a + \epsilon_{abc}z_b e^{2C_b}Dz_c \right] \ . \qquad (5.55)$$

5.3   Lifshitz Solutions

In [48,49], an infinite class of Lifshitz solutions of $D = 10$ and $D = 11$ supergravity with dynamical exponent $z = 2$ was considered. In this section we review some aspects of this class of solutions in [48], which has as a special limit the solutions of [49].

This type IIB supergravity solution has a metric of the form

$$ds^2 = ds_{Lif}^2 + L^2 ds_{E_5}^2 \qquad (5.56\text{a})$$

where

$$ds_{Lif}^2 = L^2\left(r^2(+2d\sigma dt + dx_1^2 + dx_2^2) + \frac{1}{r^2}dr^2 + fd\sigma^2 \right)$$

$$= L^2\left(-\frac{r^4}{f}dt^2 + r^2(dx_1^2 + dx_2^2) + \frac{1}{r^2}dr^2 + f\left(d\sigma + \frac{r^2}{f}dt \right)^2 \right), \qquad (5.56\text{b})$$

where f is a function of σ and of the coordinates of the Sasaki-Einstein manifold

E_5. This background has also the nontrivial fields[5]

$$F_5 = 4L^4(1 + *)Vol_{E_5} \ , \tag{5.56c}$$

$$G = d\sigma \wedge W \ , \tag{5.56d}$$

$$P = gd\sigma \ . \tag{5.56e}$$

We can recover the standard solution $AdS_5 \times E^5$ solution by $W = f = g = 0$ and the solution of [49] can be obtained in the special limit $W = 0$, $f = f(\sigma) > 0$, $g = g(\sigma) \in \mathbb{R}$. In addition, the coordinate σ is compact and parametrizes a circle S^1.

Another interesting class of solutions are those with constant f. When we set f to a constant, $f = 1$, the four dimensional noncompact part of this metric is precisely the metric with the Lifshitz symmetry for $z = 2$ and $r = \frac{1}{u}$, that is

$$ds^2 = L^2 \left(-\frac{dt^2}{u^4} + \frac{dx^i dx^i}{u^2} + \frac{du^2}{u^2} \right) . \tag{5.57}$$

Also, in [48], the authors showed that under certain conditions, we can consider the KK-reduction on $S^1 \times E_5$ and we get contact with the bottom-up construction of Lifshitz solutions.

5.3.1 Homogeneous Space $T^{(1,1)}$

We start considering the particular solution in which E_5 is the homogeneous space $(SU(2) \times SU(2))/U(1)$, that is, $E_5 = T^{(1,1)}$ with metric

$$ds^2_{T^{(1,1)}} = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2)$$

$$= \frac{1}{L^2}\left((\mathfrak{e}^1)^2 + (\mathfrak{e}^2)^2 + (\mathfrak{e}^3)^2 + (\mathfrak{e}^{\hat{1}})^2 + (\mathfrak{e}^{\hat{2}})^2 \right) \tag{5.58}$$

where

$$\mathfrak{e}^1 = \frac{L}{\sqrt{6}}\tau^1 \ , \quad \mathfrak{e}^2 = \frac{L}{\sqrt{6}}\tau^2 \ , \quad \mathfrak{e}^3 = \frac{L}{3}\left(\tau^3 + \cos\theta_1 d\phi_1\right)$$

$$\mathfrak{e}^{\hat{1}} = \frac{L}{\sqrt{6}}d\theta_1 \ , \quad \mathfrak{e}^{\hat{2}} = \frac{L}{\sqrt{6}}\sin\theta_1 d\phi_1 \tag{5.59}$$

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[5]For $\tau = C_0 + ie^{-\phi}$, $P = (i/2)e^\phi d\tau$ and $G = ie^{\phi/2}d\tau(\tau dB - dC_2)$, where the scalar C_0 is the axion and ϕ the dilaton, and the 2-forms in the NS-NS and the RR sectors are B and C_2, respectively.

and τ_i are given by (5.24) Using these results and the notation of [121], we see that

$$e^{C_1} = e^{C_2} = \frac{L}{\sqrt{6}} \equiv \bar{\beta}_1^{1/2} \,, \quad e^{C_3} = \frac{L}{3} \equiv \bar{\beta}_2^{1/2} \,,$$

$$A^1 = A^2 = 0 \,, \quad A^3 = \cos\theta_1 d\phi_1 \,. \tag{5.60}$$

In the NS-NS sector we have the Kalb-Ramond field B with field strength

$$H_3 = -\sqrt{2}k \, d\sigma \wedge \left(\mathfrak{e}^{\hat{1}} \wedge \mathfrak{e}^{\hat{2}} + \mathfrak{e}^1 \wedge \mathfrak{e}^2 \right) \,, \tag{5.61}$$

and this field strength can be generated by

$$B_2 = -\frac{kL^2}{3\sqrt{2}} \cos\theta_1 \, d\sigma \wedge d\phi_1 + \frac{kL^2}{3\sqrt{2}} d\sigma \wedge \tau_3 \,, \tag{5.62}$$

which means that $\beta_3 = \frac{kL^2}{3\sqrt{2}}d\sigma$. The dilaton and the axion are taken to be trivial.

In the R-R sector we just have the self-dual 5-form

$$\begin{aligned} F_5 &= 4 \, L^4 \left(r^3 \, d\sigma \wedge dt \wedge dr \wedge dx_1 \wedge dx_2 + Vol_{T^{(1,1)}} \right) \\ &= 4 \, L^4 (1 + *) \, Vol_{T^{(1,1)}} \,. \end{aligned} \tag{5.63}$$

Note that we can consider an ordinary T-duality and an uplift of this solution in order to find type IIA and $D = 11$ solutions (in this particular case we have $f = k$).

Nonabelian T-dual

The nonabelian T-duality with respect to the $SU(2)$ isometry parametrized by the (ψ, ϕ_2, θ_2) coordinates in the space $T^{(1,1)}$ has been considered in [25] and was reviewed in [23]. Here we consider a slight modification of [25], namely now we have a nonvanishing Kalb-Ramond field and obviously the noncompact space is not AdS$_5$. Then the T-dual space has metric

$$\begin{aligned} d\hat{s}^2 &= \underbrace{ds_{Lif}^2 + \frac{L^2}{6}ds_1^2}_{ds_7^2} + \frac{1}{\bar{\Delta}} \left[\left(\sum_{i=1}^3 z_i Dz_i \right)^2 + e^{2(C_1+C_2+C_3)} \sum_{i=1}^3 e^{-2C_i}(Dz_i)^2 \right] \\ &= ds_7^2 + \frac{1}{\bar{\Delta}} \left\{ \frac{\sqrt{2}L^2 k\alpha'}{3} \left(\alpha'^2 v_1 v_2 dv_1 + (\alpha'^2 v_3^2 + \bar{\beta}_1^2)dv_3 \right) d\sigma + \frac{k^2 L^4}{18}(\alpha'^2 v_3^2 + \bar{\beta}_1^2)d\sigma^2 \right\} \\ &\quad + \frac{\alpha'^2 v_1^2 \bar{\beta}_1 \bar{\beta}_2}{\bar{\Delta}}\hat{\eta}^2 + \frac{\alpha'^2}{\bar{\Delta}} \left\{ (\bar{\beta}_1 \bar{\beta}_2 + \alpha'^2 v_1^2)dv_1^2 + (\bar{\beta}_1^2 + \alpha'^2 v_3^2)dv_3^2 + 2\alpha'^2 v_1 v_3 dv_1 dv_3 \right\} \end{aligned} \tag{5.64}$$

where the coordinates are $\{z_1 = \alpha'\hat{v}_1, z_2 = \alpha'\hat{v}_2, z_3 = \alpha'\hat{v}_3\}$ and their "covariant" derivatives Dz_i are

$$\frac{1}{\alpha'}Dz_1 = d\hat{v}_1 - \hat{v}_2 A^3 \ , \quad \frac{1}{\alpha'}Dz_2 = d\hat{v}_2 + \hat{v}_1 A^3 \ , \quad \frac{1}{\alpha'}Dz_3 = d\hat{v}_3 + \frac{1}{\alpha'}\beta_3 \ , \quad (5.65)$$

and finally $\bar{\Delta} = [(\bar{\beta}_1^2 + \alpha'^2 v_3^2)\bar{\beta}_2 + \alpha'^2 v_1^2 \bar{\beta}_1]$, which is related to the dual dilaton $\hat{\phi}$ field by $\bar{\Delta} = \alpha'^3 e^{-2\hat{\phi}}$.

The T-dual NS–NS two-form is

$$\begin{aligned}
\widehat{B} &= -\frac{kL^2}{3\sqrt{2}}\cos\theta_1\, d\sigma \wedge d\phi_1 - \frac{1}{\bar{\Delta}}\left(\epsilon_{ijk}e^{2C_i}z_i Dz_j \wedge Dz_k\right) - \cos\theta_1 Dz_3 \wedge d\phi_1 \\
&= -\frac{kL^2}{3\sqrt{2}}\cos\theta_1\, d\sigma \wedge d\phi_1 - \frac{\alpha'^3}{\bar{\Delta}}\left(\bar{\beta}_2 v_1 v_3 dv_1 - \bar{\beta}_1 v_1^2 dv_3\right) \wedge \hat{\eta} \\
&\quad - \frac{kL^2}{3\sqrt{2}\bar{\Delta}}\bar{\beta}_2(\bar{\beta}_1^2 + \alpha'^2 v_3^2)d\sigma \wedge \hat{\eta} + \frac{kL^2}{3\sqrt{2}}d\sigma \wedge d\psi - \alpha'\cos\theta_1 dv_3 \wedge d\phi_1 \ . \quad (5.66)
\end{aligned}$$

Using the fact that the original self-dual five-form is given by

$$\begin{aligned}
F_5 &= \frac{2L^2}{9}(1+*)\, \mathfrak{e}^{\hat{1}} \wedge \mathfrak{e}^{\hat{2}} \wedge \tau_1 \wedge \tau_2 \wedge (\tau_3 + \cos\theta_1 d\phi_1) \\
&\equiv (1+*)\, G_2 \wedge \tau_1 \wedge \tau_2 \wedge (\tau_3 + \cos\theta_1 d\phi_1)\, , \quad (5.67)
\end{aligned}$$

the dual R–R sector is defined by

$$\alpha'^{3/2}\widehat{F}_2 = -G_2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.68a)$$

$$\begin{aligned}
\alpha'^{3/2}\widehat{F}_4 &= G_2 \wedge \left\{ \frac{\alpha' v_1}{(\bar{\beta}_1\bar{\beta}_2)^{1/2}}(\cos\psi\hat{\mathfrak{e}}_2 - \sin\psi\hat{\mathfrak{e}}_1) \wedge \hat{\mathfrak{e}}_3 + \frac{\alpha' v_3}{\bar{\beta}_1}\hat{\mathfrak{e}}_1 \wedge \hat{\mathfrak{e}}_2 \right\} \\
&= \frac{\alpha'^2}{\bar{\Delta}}G_2 \wedge \left\{ \alpha'\bar{\beta}_2 v_1 v_3 dv_1 - \bar{\beta}_1 v_1^2(\alpha' dv_3 + \beta_3) \right\} \wedge \hat{\eta}\ , \quad (5.68b)
\end{aligned}$$

where

$$\hat{\mathfrak{e}}_a = e^{C_a}\bar{\Delta}^{-1}\left[z_a z_b Dz_b + e^{2\sum_{b \neq a} C_b} Dz_a + \epsilon_{abc}z_b e^{2C_b} Dz_c \right]\ , \qquad (5.69)$$

which implies

$$\begin{aligned}
\hat{\mathfrak{e}}_1' &\equiv \cos\psi\hat{\mathfrak{e}}_1 + \sin\psi\hat{\mathfrak{e}}_2 \\
&= \frac{\bar{\beta}_1^{1/2}}{\bar{\Delta}}\left[\alpha'(\bar{\beta}_1\bar{\beta}_2 + \alpha'^2 v_1^2)dv_1 + \alpha'^2 v_1 v_3(\alpha' dv_3 + \beta_3) - \alpha'^2 \bar{\beta}_2 v_1 v_3\hat{\eta} \right] \quad (5.70a)
\end{aligned}$$

$$\begin{aligned}
\hat{\mathfrak{e}}_2' &\equiv -\sin\psi\hat{\mathfrak{e}}_1 + \cos\psi\hat{\mathfrak{e}}_2 \\
&= \frac{\bar{\beta}_1^{1/2}}{\bar{\Delta}}\left[\alpha'^2 \bar{\beta}_2 v_3 dv_1 - \alpha'\bar{\beta}_1 v_1(\alpha' dv_3 + \beta_3) + \alpha'\bar{\beta}_1\bar{\beta}_2 v_1\hat{\eta} \right] \quad (5.70b)
\end{aligned}$$

$$\hat{\mathfrak{e}}_3 = \frac{\bar{\beta}_2^{1/2}}{\bar{\Delta}}\left[\alpha'^3 v_1 v_3 dv_1 + (\bar{\beta}_1^2 + \alpha'^2 v_3^2)(\alpha' dv_3 + \beta_3) + \alpha'^2 \bar{\beta}_1 v_1^2\hat{\eta} \right]\ . \quad (5.70c)$$

Here $\beta_3 = \frac{kL^2}{3\sqrt{2}}d\sigma$.

5.3.2 Sasaki-Einstein Space $Y^{p,q}$

Another possible choice is $E_5 = Y^{p,q}$, such that the Sasaki-Einstein metric is

$$ds^2_{Y^{p,q}} = w(y)\,(d\alpha + h(y)\tau_3)^2 + \frac{1-y}{6}(\tau_1^2 + \tau_2^2) + \frac{dy^2}{w(y)q(y)} + \frac{q(y)}{9}\tau_3^2$$

$$= g(y)\left(\tau_3 + \frac{w(y)h(y)}{g(y)}d\alpha\right)^2 + \frac{w(y)q(y)}{9g(y)}d\alpha^2 + \frac{1}{w(y)q(y)}dy^2 + \frac{1-y}{6}(\tau_1^2 + \tau_2^2)$$

(5.71a)

where $g(y) = q(y)/9 + w(y)h(y)^2$ and

$$h(y) = \frac{a - 2y + y^2}{6(a-y^2)} \;,\quad w(y) = \frac{2(a-y^2)}{1-y} \;,\quad q(y) = \frac{a - 3y^2 + 2y^3}{a - y^2} \;.$$

(5.71b)

Here a is a real constant. As studied in [150], in these manifolds there is a 2-sphere fibration parametrized by (y, ψ), with $y \in [y_1, y_2]$ over a 2-sphere parametrized by (θ, ϕ). Also, the coordinate α parametrizes a circle of length $2\pi l_\alpha$. In these spaces, we have

$$a = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\sqrt{4p^2 - 3q^2}$$

$$y_1 = \frac{1}{4p}\left(2p - 3q - \sqrt{4p^2 - 3q^2}\right)$$

(5.72)

$$y_2 = \frac{1}{4p}\left(2p + 3q - \sqrt{4p^2 - 3q^2}\right) \;,$$

where (p, q) are relative integers and $p > q > 0$.

We can set the axion and the dilaton to be zero, but, contrary to the previous solution, this condition does not imply that the function f is a constant. In fact, it satisfies the following equation

$$-4f + \frac{2}{1-y}\partial_y[(a - 3y^2 + 2y^3)\partial_y f] + \frac{1}{(1-y)^4} = 0.$$

(5.73)

We define the vielbeins

$$\mathfrak{e}^\alpha = \frac{L}{3}\sqrt{\frac{wq}{g}}d\alpha \;,\quad \mathfrak{e}^y = \frac{L}{\sqrt{wq}}dy$$

(5.74a)

$$\mathfrak{e}^1 = L\sqrt{\frac{(1-y)}{6}}\tau_1 \;,\quad \mathfrak{e}^2 = L\sqrt{\frac{(1-y)}{6}}\tau_2 \;,\quad \mathfrak{e}^3 = L\sqrt{g}\left(\tau_3 + \frac{wh}{g}d\alpha\right) \;.$$

(5.74b)

In this $Y^{p,q}$ background, one has

$$W = -\frac{L^2}{\sqrt{72}} d\left[\frac{1}{1-y}(6d\alpha + \tau_3)\right] = -\frac{L^2}{6\sqrt{2}(1-y)}\left[\frac{1}{(1-y)}dy \wedge (6d\alpha + \tau_3) + \tau_1 \wedge \tau_2\right]$$
$$= -\frac{L^2}{6\sqrt{2}(1-y)^2}\left\{dy \wedge \left[\left(6 - \frac{wh}{g}\right)d\alpha + \frac{1}{L\sqrt{g}}\mathfrak{e}^3\right] + \frac{6}{L^2}\,\mathfrak{e}^1 \wedge \mathfrak{e}^2\right\}\,,$$

$$(5.75)$$

which means that

$$H = -d\sigma \wedge W$$
$$= \frac{L^2}{6\sqrt{2}(1-y)^2}\left(6 - \frac{wh}{g}\right)d\sigma \wedge dy \wedge d\alpha + \frac{L}{6\sqrt{2}g(1-y)^2}d\sigma \wedge dy \wedge \mathfrak{e}^3 \quad (5.76)$$
$$+ \frac{1}{\sqrt{2}(1-y)^2}d\sigma \wedge \mathfrak{e}^1 \wedge \mathfrak{e}^2\,.$$

Using the notation of [121], we find

$$e^{C_1} = e^{C_2} = L\sqrt{\frac{1-y}{6}} = \breve{\beta}_1^{1/2}\,, \quad e^{C_3} = L\sqrt{g} = \breve{\beta}_2^{1/2}\,, \tag{5.77a}$$

$$A_1 = A_2 = 0\,, \quad A_3 = \frac{wh}{g}d\alpha\,, \tag{5.77b}$$

$$\beta_1 = \beta_2 = 0\,, \quad \beta_3 = -\frac{L^2}{6\sqrt{2}(1-y)}d\sigma\,, \quad D\beta_3 = d\beta_3 = -\frac{L^2}{6\sqrt{2}(1-y)^2}dy \wedge d\sigma$$

$$b_i = 0\,, \tag{5.77c}$$

therefore

$$B = -\frac{L^2}{\sqrt{2}(1-y)}d\sigma \wedge d\alpha - \frac{L^2}{6\sqrt{2}(1-y)}d\sigma \wedge \tau_3\,. \tag{5.78}$$

The R–R five-form for the solution is

$$F_5 = 4L^4\,(1+*)\,Vol_{Y^{p,q}} \tag{5.79}$$

$$= \frac{2L^2}{3}(1-y)\sqrt{g}\,(1+*)\,\mathfrak{e}^\alpha \wedge \mathfrak{e}^y \wedge \tau^1 \wedge \tau^2 \wedge \left(\tau^3 + \frac{wh}{g}d\alpha\right) \tag{5.80}$$

$$= (1+*)\,\breve{G}_2 \wedge \tau_1 \wedge \tau_2 \wedge \left(\tau^3 + \frac{wh}{g}d\alpha\right)\,. \tag{5.81}$$

Finally, the metric of the T-dual space becomes

$$d\hat{s}^2 = \underbrace{ds_{Lif}^2 + \frac{wqL^2}{9g}d\alpha^2 + \frac{L^2}{wq}dy^2}_{d\breve{s}_7^2} + \frac{1}{\breve{\Delta}}\left[\left(\sum_{i=1}^{3}z_i Dz_i\right)^2 + e^{2(C_1+C_2+C_3)}\sum_{i=1}^{3}e^{-2C_i}(Dz_i)^2\right]$$

$$= d\breve{s}_7^2 + \frac{1}{\breve{\Delta}}\left\{-\frac{\alpha'}{3\sqrt{2}(y-1)}\left(\alpha'^2 v_1 v_3 dv_1 + (\alpha'^2 v_3^2 + \breve{\beta}_1^2)dv_3\right)d\sigma\right.$$

$$\left.+\frac{(\alpha'^2 v_3^2 + \breve{\beta}_1^2)}{72(1-y)^2}L^4 d\sigma^2\right\} + \frac{\alpha'^2 v_1^2 \breve{\beta}_1 \breve{\beta}_2}{\breve{\Delta}}\hat{\gamma}^2$$

$$+\frac{\alpha'^2}{\breve{\Delta}}\left\{(\breve{\beta}_1\breve{\beta}_2 + \alpha'^2 v_1^2)dv_1^2 + (\breve{\beta}_1^2 + \alpha'^2 v_3^2)dv_3^2 + 2\alpha'^2 v_1 v_3 dv_1 dv_3\right\}\,, \tag{5.82}$$

where $\hat{\gamma} = d\psi + \frac{hw}{g}d\alpha$ and $\breve{\Delta} = (\breve{\beta}_1^2 + \alpha'^2 v_3^2)\breve{\beta}_2 + \alpha'^2 v_1^2 \breve{\beta}_1$. The NS–NS two-form is

$$\widehat{B} = -\frac{L^2}{\sqrt{2}(1-y)}d\sigma \wedge d\alpha - \frac{1}{\breve{\Delta}}\left(\epsilon_{ijk}e^{2C_i}z_i Dz_j \wedge Dz_k\right) - Dz_3 \wedge A_3$$

$$= \frac{L^2}{\sqrt{2}(1-y)}\left(1 - \frac{hw}{6g}\right)d\alpha \wedge d\sigma - \frac{\alpha'^2}{\breve{\Delta}}\left[\alpha'\breve{\beta}_2 v_1 v_3 dv_1 - \breve{\beta}_1 v_1^2(\alpha' dv_3 + \breve{\beta}_3)\right]\wedge\hat{\gamma}$$

$$+\frac{\alpha' hw}{g}d\alpha \wedge dv_3\,, \tag{5.83}$$

and the dilaton is

$$\hat{\phi} = -\frac{1}{2}\ln\left(\frac{\breve{\Delta}}{\alpha'^3}\right)\,. \tag{5.84}$$

The fields of the T-dual R–R sector are

$$\alpha'^{3/2}\widehat{F}_2 = -\breve{G}_2 \tag{5.85a}$$

$$\alpha'^{3/2}\widehat{F}_4 = \breve{G}_2 \wedge \left\{\frac{\alpha' v_1}{(\breve{\beta}_1\breve{\beta}_2)^{1/2}}(\cos\psi\hat{\mathbf{e}}_2 - \sin\psi\hat{\mathbf{e}}_1)\wedge\hat{\mathbf{e}}_3 + \frac{\alpha' v_3}{\breve{\beta}_1}\hat{\mathbf{e}}_1 \wedge \hat{\mathbf{e}}_2\right\}$$

$$= \frac{\alpha'^2}{\breve{\Delta}}\breve{G}_2 \wedge \left\{\alpha'\breve{\beta}_2 v_1 v_3 dv_1 - \breve{\beta}_1 v_1^2(\alpha' dv_3 + \breve{\beta}_3)\right\}\wedge\hat{\gamma}\,, \tag{5.85b}$$

with the vielbeins

$$\hat{\mathfrak{e}}'_1 \equiv \cos\psi\hat{\mathfrak{e}}_1 + \sin\psi\hat{\mathfrak{e}}_2$$

$$= \frac{\breve{\beta}_1^{1/2}}{\breve{\Delta}}\left[\alpha'(\breve{\beta}_1\breve{\beta}_2 + \alpha'^2 v_1^2)dv_1 + \alpha'^2 v_1 v_3(\alpha' dv_3 + \breve{\beta}_3) - \alpha'^2\breve{\beta}_2 v_1 v_3\hat{\gamma}\right] \qquad (5.86a)$$

$$\hat{\mathfrak{e}}'_2 \equiv -\sin\psi\hat{\mathfrak{e}}_1 + \cos\psi\hat{\mathfrak{e}}_2$$

$$= \frac{\breve{\beta}_1^{1/2}}{\breve{\Delta}}\left[\alpha'^2\breve{\beta}_2 v_3 dv_1 - \alpha'\breve{\beta}_1 v_1(\alpha' dv_3 + \breve{\beta}_3) + \alpha'\breve{\beta}_1\breve{\beta}_2 v_1\hat{\gamma}\right] \qquad (5.86b)$$

$$\hat{\mathfrak{e}}_3 = \frac{\breve{\beta}_2^{1/2}}{\breve{\Delta}}\left[\alpha'^3 v_1 v_3 dv_1 + (\breve{\beta}_1^2 + \alpha'^2 v_3^2)(\alpha' dv_3 + \breve{\beta}_3) + \alpha'^2\breve{\beta}_1 v_1^2\hat{\gamma}\right] \ . \qquad (5.86c)$$

Here $\breve{\beta}_3 = \frac{-L^2}{6\sqrt{2}(1-y)}d\sigma$.

5.4   Holographic Dual Field Theory

5.4.1   Quantized Charges for Galilean Solutions

In [148], the authors considered the dualization of the background holographic dual to the ABJM theory [178], which consists of a metric for $AdS_4 \times \mathbb{CP}^3$ in type IIA, together with two R–R fields, F_2 and F_4. They also calculated the conserved charges of the dual background.

Considering the effect of the nonrelativistic deformation of the ABJM background considered in [162], we compute the conserved charges of the background that we found in the last section. We compare our results with [148] in order to see the effect of the nonrelativistic deformation of the background [178]. We calculate the conserved charges of the solutions in sections 2.1 (massless type IIA) and 2.2 (massive type IIA) separately.

Massless type IIA

We start with a short review of the conserved charges of $AdS_4 \times \mathbb{CP}^3$ and its NATD solution. The solution has the metric of $AdS_4 \times \mathbb{CP}^3$, the dilaton $\phi = \ln(R/k)$ and the R–R forms [148]

$$dC_{(1)} = 2kd\omega \qquad (5.87a)$$

$$dC_{(3)} = \frac{3}{8}kL^2 Vol_{AdS_4} \qquad (5.87b)$$

in such a way that

$$\int_{\mathbb{CP}^1} dC_{(1)} = 2\pi k \quad \text{and} \quad \frac{1}{(2\pi\alpha'^{1/2})^5} \int_{\mathbb{CP}^3} *\mathcal{F}_4 = N \in \mathbb{Z} \Rightarrow L^4 = \frac{32\pi^2\alpha'^{5/2}}{k} N \ , \tag{5.88}$$

where we have used the $\mathbb{CP}^1$ defined by $(\xi = \pi/2, \theta_2, \phi_2)$, and $\int Vol_{\mathbb{CP}^3} = \frac{\pi^3}{6}$. We see that these quantization conditions agree perfectly with the quantization conditions (5.28-5.34) of the Galilean solution.

In [148], the authors calculated the charge N_{D5} in the dual field theory, which was found from an integration of the dual 3-form over the cycle defined by $\Sigma^3 = (\zeta, \theta_1, \phi_1)$, such that[6]

$$N_5 = \frac{kL^4}{64\pi\alpha'^{5/2}}. \tag{5.89}$$

In our case of the Galilean solution of massless type IIA, we must consider the same calculation for the dual $\widehat{\mathcal{F}}_3$. Using that

$$\widehat{\mathcal{F}}_3\bigg|_{\Sigma^3} = \frac{6\beta_1\beta_2^{1/2}k}{R^2\alpha'^{3/2}} \mathfrak{e}^\zeta \wedge \mathfrak{e}^\theta \wedge \mathfrak{e}^\phi = \frac{3kR^4}{16\alpha'^{3/2}} \sin^3\zeta \cos^3\zeta d\zeta \wedge vol_{S_1^2} \ , \tag{5.90}$$

where $vol_{S_1^2} = \sin\theta_1 d\theta_1 \wedge d\phi_1$, we compute the charge

$$\int_{\Sigma^3} \widehat{\mathcal{F}}_3 = \frac{k\pi R^4}{16\alpha'^{3/2}}. \tag{5.91}$$

Imposing the quantization condition for the Page charge,

$$\frac{1}{(2\pi\alpha'^{1/2})^2} \int_{\Sigma_3} \mathcal{F}_3 = \hat{Q}_{D5} \in \mathbb{Z} \ , \tag{5.92}$$

we obtain

$$\widehat{\mathcal{Q}}_{D5} = \frac{kR^4}{64\pi\alpha'^{5/2}}. \tag{5.93}$$

But since originally R^4 satisfied the relation $kR^4 = 32\pi^2\alpha'^{1/2}N$, the charge $\widehat{\mathcal{Q}}_{D5}$ cannot be an integer, and in this case, the radius R in the dual theory will be defined through new relations. The noninteger charge in the nonabelian T-dual theory is a generic feature which arises from the violation of the condition $T_{D(p-n)} = (2\pi)^n T_{Dp}$.

In the present case, we see that if we consider the 5-cycle $\Sigma^5 = (\zeta, \theta_1, \phi_1, \chi, \psi) \equiv (\zeta, \theta_1, \phi_1, v_1 = n\pi\sin\xi, v_3 = n\pi\cos\xi, \psi)$ in the T-dual background, we compute the

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[6]In their notation $\alpha' = 1$.

restriction

$$\alpha'^{3/2}\widehat{\mathcal{F}}_5 \bigg|_{\Sigma^5} = n\pi\alpha'\widehat{\mathcal{F}}_3 \wedge vol_{\tilde{S}^2} , \tag{5.94}$$

which is consistent with a large gauge transformation $\widehat{\mathcal{F}}_5 \to \widehat{\mathcal{F}}_5 + n\pi\alpha'\widehat{\mathcal{F}}_3 \wedge vol_{\tilde{S}^2}$ (with the volume form $vol_{\tilde{S}^2} = \sin\chi d\chi \wedge d\psi$), and therefore we find

$$\mathcal{Q}_{D3} = n\mathcal{Q}_{D5} . \tag{5.95}$$

The field theory on the boundary is a 2+1 dimensional CS gauge theory, as was the ABJM theory before the NATD. The CS gauge groups have levels, that should be possible to calculate from the gravity dual. As in [148], we can define the levels of the AdS/CFT dual field theory as

$$q_5 = \left| \frac{1}{(2\pi\alpha'^{1/2})^2} \int_{\tilde{\Sigma}^3} \widehat{\mathcal{F}}_3 \right| , \qquad q_3 = \left| \frac{1}{(2\pi\alpha'^{1/2})^4} \int_{\tilde{\Sigma}^5} \widehat{\mathcal{F}}_5 \right| , \tag{5.96}$$

where the integrations are performed on the cycles $\tilde{\Sigma}_3 = (\rho, \theta_1, \phi_1) \equiv (v_3, \theta_1, \phi_1)$ and $\tilde{\Sigma}_5 = (\rho, \theta_1, \phi_1, \chi, \psi) \equiv (\theta_1, \phi_1, v_1, v_3, \psi)$, respectively. In the presence of a large gauge transformation, one obtained in the case in [148]

$$q_5 = k\frac{(2n+1)\pi}{4\alpha'^{1/2}} , \qquad q_3 = k\frac{(3n+2)\pi}{12\alpha'^{1/2}} . \tag{5.97}$$

Using the same definitions, in our case we obtain from (5.46f)

$$\alpha'^{3/2}\widehat{F}_3 \bigg|_{\tilde{\Sigma}_3} = -\frac{\alpha'^2 k}{2}\rho\, vol_{S_1^2} \wedge d\rho \quad , \quad \alpha'^{3/2}\widehat{F}_1 \wedge \widehat{B} \bigg|_{\tilde{\Sigma}_3} = 0 , \tag{5.98a}$$

therefore we obtain in a similar manner to the above case

$$k_5 = k\frac{(2n+1)\pi}{4\alpha'^{1/2}} . \tag{5.98b}$$

We also find from (5.46f) that

$$\alpha'^{3/2}\widehat{\mathcal{F}}_5 \bigg|_{\tilde{\Sigma}_5} = \frac{k\alpha'^3}{2}\rho^2 d\rho \wedge vol_{S_1^2} \wedge vol_{\tilde{S}^2} , \tag{5.99a}$$

and given that

$$\alpha'^{3/2}F_3 \wedge vol_{\tilde{S}^2}\big|_{\tilde{\Sigma}_5} = -\frac{k\alpha'^2}{2}\rho d\rho \wedge vol_{S_1^2} \wedge vol_{\tilde{S}^2} \tag{5.99b}$$

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under a large gauge transformation $\widehat{\mathcal{F}}_5 \to \widehat{\mathcal{F}}_5 + n\pi\alpha'\widehat{F}_3 \wedge vol_{\tilde{S}^2}$, we have

$$\int_{\tilde{\Sigma}^5} (\widehat{\mathcal{F}}_5 + n\pi\alpha'\widehat{F}_3 \wedge vol_{\tilde{S}^2}) = k\frac{(4\pi)^2}{12\alpha'^{1/2}}(3n+2)\pi^3 \ . \tag{5.99c}$$

Then finally

$$k_3 = k\frac{(3n+2)\pi}{12\alpha'^{1/2}} \ , \tag{5.100}$$

such that $(3n+2)k_5 = 3(2n+1)k_3$. Using these relations, we find the relations between the radius R and the quantized charges of the background

$$R^4 k_5 = 32\pi^2\alpha'^2 \left(\widehat{\mathcal{Q}}_{D3} + \frac{1}{2}\widehat{\mathcal{Q}}_{D5} \right) \tag{5.101a}$$

$$R^4 k_3 = 16\pi^2\alpha'^2 \left(\widehat{\mathcal{Q}}_{D3} + \frac{2}{3}\widehat{\mathcal{Q}}_{D5} \right) . \tag{5.101b}$$

If we compare our results with [148] we can see that the nonrelativistic deformation does not change the quantization condition of the theory and of its nonabelian T-dual.

Massive type IIA

We now turn to the model in section 2.2. We first consider the model before the T-duality. We find

$$*_{10}F_4 = \frac{5^3\sqrt{5}m_0}{f^6}vol_{\mathbb{CP}^3} \ , \tag{5.102a}$$

giving

$$\mathcal{Q}_{D2}^{m_0} = \frac{5^3\sqrt{5}}{192\pi^2\alpha'^{5/2}}\frac{m_0}{f^6} \ . \tag{5.102b}$$

In [162], the author considered the compactification of the ordinary type IIA theory and of the massive type IIA theory to four dimensions.[7]

We calculate the D5-charge by using (5.53) to write

$$\widehat{\mathcal{F}}_3^{m_0}\bigg|_{\Sigma^3} = \frac{1}{\alpha'^{3/2}}\tilde{\beta}_1\tilde{\beta}_2^{1/2} *_7 F_4\bigg|_{\Sigma^3} = \frac{\sqrt{5}m_0}{\alpha'^{3/2}}\tilde{\beta}_1\tilde{\beta}_2^{1/2}\mathfrak{e}_{m_0}^{\zeta} \wedge \mathfrak{e}_{m_0}^{\theta} \wedge \mathfrak{e}_{m_0}^{\phi} \ , \tag{5.103}$$

---

[7]Since the only relationship between these two theories is Hull's duality [181], the author argued that the similarity between the 4D actions means that there is a mapping between the Romans' mass and the flux k. In that case, $f \propto 1/R$ and $m_0 \propto k/R^2$, so that, up to numerical constants, one could write $\mathcal{Q}_{D2}^{m_0} \propto \mathcal{Q}_{D2}$.

where $*_7$ is Poincaré duality in ds_7^2. We then calculate the magnetic D5-charge associated with this flux as

$$\widehat{\mathcal{Q}}_{D5}^{m_0} = \frac{5^3 \sqrt{5}}{384\pi\alpha'^{5/2}} \frac{m_0}{f^6} \; . \tag{5.104}$$

For the cycle Σ^5, we have $\widehat{\mathcal{F}}_5^{m_0}|_{\Sigma^5} = 0$, so now we obtain $\widehat{\mathcal{Q}}_{D3}^{m_0} = n\widehat{\mathcal{Q}}_{D5}^{m_0}$ when we consider a large gauge transformation.

On the other hand, $\widehat{\mathcal{F}}_3^{m_0}|_{\tilde{\Sigma}^3} = 0$, which remains equal to zero after a large gauge transformation.

We can define a third cycle $\check{\Sigma}^3 = (\rho, \chi, \psi)$, which gives

$$\left.\widehat{\mathcal{F}}_3^{m_0}\right|_{\check{\Sigma}^3} = 0, \tag{5.105}$$

but after a large gauge transformation $\widehat{\mathcal{F}}_3^{m_0} \to \widehat{\mathcal{F}}_3^{m_0} + n\pi\alpha'\widehat{\mathcal{F}}_1^{m_0} \wedge vol_{\tilde{S}^2}$, we find

$$k_5^{m_0} = \frac{m_0\pi^2\alpha'^{1/2}}{2}(2n+1) \; . \tag{5.106}$$

Finally, for the cycle $\tilde{\Sigma}^5 = (\theta_1, \phi_1, \rho, \chi, \psi)$, we have that $k_3 = 0$ even after a large gauge transformation.

5.4.2 Quantized Charges for Lifshitz Solutions

Homogeneous Space $SU(2) \times SU(2)/U(1)$

For the background (5.58) in section 3.1 we start with a 5-form

$$F_5 = \frac{2L^4}{9}(1+*)vol_{S_1^2} \wedge \tau_1 \wedge \tau_2 \wedge (\tau_3 + \cos\theta_1 d\phi_1) \; , \tag{5.107}$$

and using similar methods we find the quantized charge

$$N_{D3} = \frac{4}{27\pi}\frac{L^4}{\alpha'^2}. \tag{5.108}$$

After the T-duality we obtain the charges

$$L^4 = \frac{27\alpha'^2}{2}\widehat{\mathcal{Q}}_{D6} \; , \tag{5.109}$$

on the cycle $S^2 = (\theta_1, \phi_1)$. Using the fact that $\widehat{\mathcal{F}}_4 = 0$, after a large gauge transformation $\widehat{\mathcal{F}}_4 \to \widehat{\mathcal{F}}_4 + n\pi\alpha'\widehat{\mathcal{F}}_2 \wedge vol_{\tilde{S}_2}$, we find $\widehat{\mathcal{Q}}_{D4} = n\widehat{\mathcal{Q}}_{D6}$ on the cycle $\Sigma^4 = (\theta_1, \phi_1, \chi, \psi)$,.

Sasaki-Einstein Space

In the Sasaki-Einstein case in section 3.2, we have a similar situation. The quantized charge before the T-duality is

$$\check{N}_{D3} = \frac{1}{4\pi^4} \frac{L^4}{\alpha'^2} V_{Y^{p,q}} \,, \tag{5.110}$$

on the cycle $\Sigma^2 = (\alpha, y)$ and

$$V_{Y^{p,q}} = \int_{Y^{p,q}} Vol_{Y^{p,q}} = \frac{8\pi^3 l_\alpha}{3} \int_{y_1}^{y_2} dy(1-y) \,. \tag{5.111}$$

Repeating the previous analysis, we find

$$\widehat{\mathcal{Q}}_{D6} = \frac{L^4}{4\pi^3 \alpha'^2} V_{Y^{p,q}} \tag{5.112}$$

on the cycle $\Sigma^2 = (\alpha, y)$.

Again we can use the same arguments from the previous subsection to find $\widehat{\mathcal{F}}_4 = 0$ on $\Sigma^4 = (\alpha, y, \chi, \psi)$. If we take a large gauge transformation $\widehat{\mathcal{F}}_4 \to \widehat{\mathcal{F}}_4 + n\pi\alpha' \widehat{\mathcal{F}}_2 \wedge vol_{\tilde{S}_2}$, where $\tilde{S}_2 = (\chi, \psi)$, we also find $\widehat{\mathcal{Q}}_{D4} = n\widehat{\mathcal{Q}}_{D6}$.

5.4.3   Wilson Loops

One can in principle define a Wilson loop variable in the case of nonrelativistic gravity duals, even though it is not really clear what it would mean in the field theory. However, we can simply calculate the observable, and leave for later issues of interpretation.

One way to embed the Schrödinger algebra with $z = 2$ (a particular case of conformal Galilean algebra) into string theory is to consider a DLCQ of a known duality [30, 38–40]. The general conformal Galilean algebra is realized holographically through the metric

$$ds^2 = L^2 \left[-\frac{dt^2}{r^{2z}} + \frac{2dt d\xi + d\vec{x}^2}{r^2} + \frac{dr^2}{r^2} \right] . \tag{5.113}$$

For the Lifshitz case, gravity duals are instead usually of the type

$$ds^2 = L^2 \left[-\frac{dt^2}{u^{2z}} + \frac{d\vec{x}^2}{u^2} + \frac{du^2}{u^2} \right] . \tag{5.114}$$

However, in [48, 49] it was suggested that for $d = 4$ and $z = 2$, the case considered in section 3, we can consider the gravity dual

$$ds_{Lif}^2 = L^2 \left(r^2(-2d\sigma d\tau + dx_1^2 + dx_2^2) + \frac{1}{r^2}dr^2 + fd\sigma^2 \right) , \qquad (5.115)$$

and for $\sigma = x^+$ and $\tau = x^-$, σ must be a compact coordinate to obtain a 2+1 dimensional field theory dual with coordinates τ, x_1, x_2.

Note that compared with the Schrödinger case, the roles of x^+ and x^- are interchanged and x^+ is compact.

Wilson Loops in conformal Galilean spacetime

The general prescription for the calculation of Wilson lines in relativistic field theories is well known   [14, 45, 46, 114, 116, 182–184]. Recently, important hints in the identification of the dual field theory of nonrelativistic systems were studied in [185–189].

We want to consider the Wilson loops for the conformal Galilean gravity dual case in section 2. This formalism was also considered in [188].

Considering a probe string which is not excited in the internal space directions, our gravity dual manifold is of the general form (without the internal space)

$$ds^2 = \frac{R^2}{r^2} \left(-\frac{dt^2}{r^{2(z-1)}} + 2d\xi dt + d\vec{x} \cdot d\vec{x} \right) + \frac{R^2}{r^2}dr^2 , \qquad (5.116)$$

with ξ compact and null, for $z = 3$ (thus is not of the Schrödinger form, which would correspond to $z = 2$). We consider the following ansatz

$$t = \tau , \quad x = x(\sigma) , \quad r = r(\sigma) , \quad \xi = constant . \qquad (5.117)$$

Given that the induced metric on the world-sheet is $G_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$, the Nambu-Goto action becomes

$$S = -\frac{1}{2\pi\alpha'} \int_0^T d\tau \int d\sigma \sqrt{-\det G} = -\frac{TR^2}{2\pi\alpha'} \int d\sigma \sqrt{\frac{(\partial_\sigma r)^2 + (\partial_\sigma x)^2}{r^{2(z+1)}}} . \qquad (5.118)$$

As usual, the analysis of the differential equations (see [45, 46, 186]) shows that the separation between the endpoints of the ∩–shaped string that extends from the point $x = -\ell/2$ to the point $x = \ell/2$ at the boundary $r = 0$ is

$$\ell = 2 \int_0^{r_{max}} dr \frac{H(r_{max})}{\sqrt{H(r)^2 - H(r_{max})^2}} , \qquad (5.119a)$$

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with $H^2 = R^4/r^{2(z+1)}$. Therefore

$$\ell(r_{max}, z) = 2r_{max}\sqrt{\pi}\frac{\Gamma\left(\frac{z+2}{2z+2}\right)}{\Gamma\left(\frac{1}{2z+2}\right)}, \tag{5.119b}$$

and we can invert this expression, giving $r_{max} = r_{max}(\ell)$. For $z = 3$, we obtain

$$l = 2r_{max}\sqrt{\pi}\frac{\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{1}{8}\right)}. \tag{5.120}$$

The general formalism [45, 46] allows us to compute a would-be quark-antiquark potential, which gives

$$V_{q\bar{q}} = \frac{2R^2\sqrt{\pi}}{r_{max}^z(2z+2)}\frac{\Gamma\left(\frac{-z}{2z+2}\right)}{\Gamma\left(\frac{1}{2z+2}\right)} . \tag{5.121}$$

It is not clear what would be the interpretation of this quantity in the field theory, since it was defined for relativistic gauge theories. But we can continue with the assumption that it still gives the potential between external "quarks" introduced in the theory, and see what we can deduce from it.

Therefore, if we consider the solutions in the section 5.2, with $z = 3$, the potential is

$$V_{q\bar{q}} = -\frac{2R^2\sqrt{\pi}}{3r_{max}^3}\frac{\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{1}{8}\right)} , \tag{5.122}$$

which implies

$$\frac{dV}{d\ell} = \frac{R^2}{r_{max}^4} > 0 . \tag{5.123}$$

This means that the would-be quark-antiquark interaction is atractive everywhere [139, 188, 190]. We also have

$$\frac{d^2V}{d\ell^2} = -\frac{2R^2}{\sqrt{\pi}r_{max}^5}\frac{\Gamma\left(\frac{1}{8}\right)}{\Gamma\left(\frac{5}{8}\right)} < 0 , \tag{5.124}$$

and this condition means that the force is a monotonically nonincreasing function of their separation.

Wilson Loops in Lifshitz spacetime

Consider the spacetime metric[8] of the form in the section 5.3,

$$ds_{Lif}^2 = \frac{L^2}{r^2}(-2d\xi dt + dx^2 + dy^2) + \frac{L^2}{r^2}dr^2 + L^2 f(\xi)d\xi^2 , \tag{5.125}$$

---

[8]We changed the notation $\sigma \to \xi$, and we keep the symbol σ to the spacelike worldsheet coordinate. Also, we renamed $r \to \frac{1}{r}$.

where the coordinate ξ parametrizes the circle.

First, we notice that due to the absence of the component g_{tt} in the metric above, we cannot find a string configuration such that

$$t = \tau \,, \quad x = x(\sigma) \,, \quad r = r(\sigma) \,, \quad \xi = constant \,, \tag{5.126}$$

so one might consider an ansatz with the string moving also on the compact coordinate ξ, despite the fact that its physical meaning is rather uncertain [30].

We consider the following ansatz (see [186], for similar considerations in spacetimes with Schrödinger symmetry)

$$t = \tau \,, \quad \xi = \xi(\tau) \,, \quad x = x(\sigma) \,, \quad r = r(\sigma) \,. \tag{5.127}$$

Then the components of the induced metric are

$$G_{\tau\tau} = -\frac{2L^2}{r^2}\,\partial_\tau\xi + L^2 f(\xi)(\partial_\tau\xi)^2 \,, \quad G_{\sigma\sigma} = \frac{L^2}{r^2}\left((x')^2 + (r')^2\right) \,, \tag{5.128}$$

where $x' \equiv \partial_\sigma x$, $r' \equiv \partial_\sigma r$ and $G \equiv \det G_{\alpha\beta} = G_{\tau\tau}G_{\sigma\sigma}$. The Nambu-Goto action is given by

$$S = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{g^2(\sigma,\tau)\left((x')^2 + (r')^2\right)} \,, \tag{5.129}$$

where

$$g^2 = -G_{\tau\tau}L^2/r^2. \tag{5.130}$$

We consider the equation of motion for ξ,

$$\partial_\tau\left[G^{\tau\tau}\sqrt{-G}\left(-\frac{1}{r^2} + f(\xi)\partial_\tau\xi\right)\right] = 0 \ \Rightarrow \ \partial_\tau\left(\frac{G^{\tau\tau}\sqrt{-G}}{r^2}\right) = \partial_\tau\left(G^{\tau\tau}\sqrt{-G}\,f(\xi)\partial_\tau\xi\right) \,, \tag{5.131}$$

where $G^{\tau\tau} \equiv G_{\tau\tau}^{-1}$. From (5.127) we see that r is independent of τ and from [48], we already know that the function f does not have functional dependence on r. Therefore, both sides in (5.131) must vanish independently. The left-hand side of the equation (5.131), implies that $\frac{G^{\tau\tau}\sqrt{-G}}{r^2} = h_0(\sigma)$ therefore we take $\xi = v_\xi\tau$, where v_ξ is a constant. The right-hand side gives $\partial_\tau f(\xi) = 0$ and since f cannot be a function of $r = \sigma$, we conclude that f is a constant.

This means that the configuration (5.127) is allowed just for particular metrics (5.125) (as in [48]), namely those with f constant, which occurs for instance when the internal manifold is $T^{1,1}$, whereas this configuration is forbidden for the Sasaki-Einstein manifolds $Y^{p,q}$. It is rather curious that although we consider the string

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propagating just in the noncompact spacetime, the form of the internal manifold can determine physical aspects of the string propagation.

The equation of motion for $x = x(\sigma)$ is

$$\partial_\sigma \left(\frac{g^2}{\sqrt{g^2 \left((x')^2 + (r')^2 \right)}} \, \partial_\sigma x \right) = 0 \; \Rightarrow \; \partial_\sigma r = \pm V_{eff} \, \partial_\sigma x \, , \tag{5.132}$$

where

$$V_{eff} = \frac{1}{c_0} \sqrt{g^2(r) - c_0^2}, \tag{5.133}$$

and c_0 is just an integration constant. We consider a $\cap$–shaped string similar to the solution considered in the last section, namely a string which extends from $x = -\ell/2$ to $x = \ell/2$ and it reaches a maximum point r_{max} in the bulk space.

The boundary conditions for this configuration [46] imply that $\frac{dr}{dx}\big|_{r \to 0} \to \infty$. In our case, we can easily see that this condition is satisfied since $\lim_{r \to 0} V_{eff} \to \infty$.

The turning point, i.e. the maximum point in the r direction, is determined by the condition $\frac{dr}{dx}(r_{max}) = 0$, which gives

$$g^2(r_{max}) - c_0^2 = 0 \; \Rightarrow \; c_0^2 = \frac{2L^4}{r_{max}^4} v_\xi - \frac{L^4 f}{r_{max}^2} v_\xi^2 \, . \tag{5.134}$$

In order for c_0 to be real, we see that we need $v_\xi < 2/(f r_{max}^2)$.

Finally, the distance between the string endpoints is

$$\ell_{q\bar{q}}(r_{max}) = 2g(r_{max}) \int_0^{r_{max}} dr \frac{1}{\sqrt{g^2(r) - g^2(r_{max})}} \, , \tag{5.135}$$

and if we define $w = r/r_{max}$ we find

$$\begin{aligned}
\ell_{q\bar{q}}(r_{max}) &= \frac{2r_{max}^3}{L^2 \sqrt{v_\xi}} g(r_{max}) \int_0^1 dw \frac{w^2}{\sqrt{(f v_\xi r_{max}^2 - 2)w^4 - f v_\xi r_{max}^2 w^2 + 2}} \\
&\equiv \frac{2r_{max}^3}{L^2 \sqrt{v_\xi}} g(r_{max}) \, \mathcal{I}(r_{max}) \, .
\end{aligned} \tag{5.136}$$

In order to solve the integral, we write it as

$$\mathcal{I}(r_{max}) = \int_0^1 dw \frac{w^2}{\sqrt{[(f v_\xi r_{max}^2 - 2)w^2 - 2](w^2 - 1)}} \, , \tag{5.137a}$$

and performing the substitution $w = \sin u$, we find the elliptic integral

$$\mathcal{I}(r_{max}) = \frac{1}{\sqrt{2}} \int_0^{\pi/2} du \frac{\sin^2 u}{\sqrt{1 + \frac{(2 - f v_\xi r_{max}^2)}{2} \sin^2 u}} \, , \tag{5.137b}$$

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with $(2 - fv_\xi r_{max}^2) > 0$.

In terms of the complete elliptic integrals of first and second kind [191],

$$\mathbf{K}(k) = \int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2 \sin^2 u}} \tag{5.138a}$$

$$\mathbf{E}(k) = \int_0^{\pi/2} du \sqrt{1 - k^2 \sin^2 u} \tag{5.138b}$$

(The constant k is called elliptic modulus and it can take any complex or real value [9].) we can write (5.137b) as

$$\mathcal{I}(r_{max}) = \frac{\mathbf{K}(k) - \mathbf{E}(k)}{\sqrt{2}\; k^2} \;, \tag{5.139}$$

where $k^2 = (fv_\xi r_{max}^2 - 2)/2$. Then the distance between the string endpoints is given by

$$
\begin{aligned}
\ell_{q\bar{q}}(r_{max}) &= \frac{\sqrt{2} r_{max}^3}{L^2 \sqrt{v_\xi}} g(r_{max}) \frac{\mathbf{K}(k) - \mathbf{E}(k)}{k^2} \\
&= \frac{2\sqrt{2}}{\sqrt{fv_\xi}} \sqrt{-\frac{k^2 + 1}{k^2}} \left(\mathbf{K}(k) - \mathbf{E}(k)\right) \equiv \frac{2\sqrt{2}}{\sqrt{fv_\xi}} \Lambda(-k^2) \;.
\end{aligned}
\tag{5.140}
$$

Now observe that $-k^2 = (2 - fv_\xi r_{max}^2)/2 > 0$, since $fv_\xi r_{max}^2 < 2$, therefore $k^2 + 1 > 0$, which implies that $v_\xi > 0$. Therefore $v_\xi \in \left(0, \frac{2}{fr_{max}^2}\right)$, and $-k^2 \in (0, 1)$, see figure 5.1.
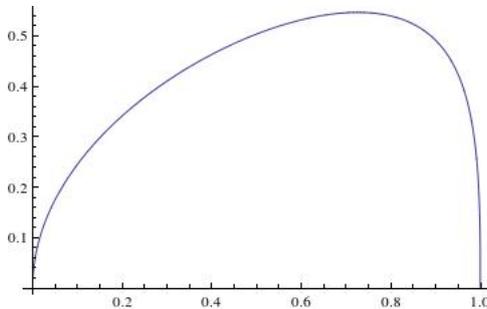
Figure 5.1: Graph of the function $\Lambda(-k^2)$.

Finally, following the standard calculation [45, 46], we compute the observable that would correspond to the energy of a $q\bar{q}$-pair (defined in relativistic gauge theories by introducing external quarks into the theory, and measuring their potential),

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[9]Generally in physics and engineering problems, the modulus k^2 is parametrized in such that $k^2 \in (0, 1)$, but it is not our case. See [192] for details.

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by subtracting from the string action the action of two 'rods' that would fall from the end of the space to the boundary. The renormalized energy is obtained to be

$$V_{q\bar{q}}(r_{max}) = 2 \int_0^{r_{max}} dr \frac{g^2(r)}{\sqrt{g^2(r) - g^2(r_{max})}} - 2 \int_0^{r_{max}} dr \, g(r)$$

$$= \frac{\sqrt{2v_\xi} L^2}{r_{max}} \int_0^1 dw \left[\frac{(2w^{-2} - fv_\xi r_{max}^2)}{\sqrt{\left(1 - \frac{(fv_\xi r_{max}^2 - 2)}{2} w^2\right)(1 - w^2)}} \right.$$

$$\left. -2\frac{1}{w^2}\sqrt{\left(1 - \frac{fv_\xi r_{max}^2}{2} w^2\right)} \right]$$

$$= \frac{\sqrt{2v_\xi} L^2}{r_{max}} \left(\mathcal{I}_{-2}(k, w^{-2}) + \mathcal{I}_0(k, w^0) - \mathcal{I}_g(k', w^{-2}) \right) , \quad (5.141)$$

where $k' = fv_\xi r_{max}^2/2$.

We can easily see that

$$\mathcal{I}_0(k, w^0) = \int_0^1 dw \frac{-fv_\xi r_{max}^2}{\sqrt{\left(1 - \frac{(fv_\xi r_{max}^2 - 2)}{2} w^2\right)(1 - w^2)}} = -fv_\xi r_{max}^2 \mathbf{K}(k) . \quad (5.142a)$$

Consider the substitution $w = \sin u$, so that the second integral is

$$\mathcal{I}_{-2}(k, w^{-2}) = 2 \int_0^{\pi/2} du \frac{1}{\sin^2 u \sqrt{\left(1 - \frac{(fv_\xi r_{max}^2 - 2)}{2} \sin^2 u\right)}}$$

$$= 2 \left[\mathbf{K}(k) - \mathbf{E}(k)\right] - 2 \sqrt{\left(1 - \frac{(fv_\xi r_{max}^2 - 2)}{2} \sin^2 u\right)} \cot u \Bigg|_0^{\pi/2} , \quad (5.142b)$$

and the third integral reads (considering the *arcsin* modulo $2\pi n$)

$$\mathcal{I}_g(k, w^{-2}) = 2 \int_0^{\pi/2} du \frac{\cos u}{\sin^2 u} \sqrt{\left(1 - \frac{fv_\xi r_{max}^2}{2} \sin^2 u\right)}$$

$$= - \left[2\sqrt{\left(1 - \frac{fv_\xi r_{max}^2}{2} \sin^2 u\right)} \csc u \right.$$

$$\left. -\sqrt{2fv_\xi r_{max}^2} \arcsin\left(\sqrt{\frac{fv_\xi r_{max}^2}{2}} \sin u\right) \right] \Bigg|_0^{\pi/2}$$

$$+ 2\pi n \sqrt{2fv_\xi r_{max}^2} . \quad (5.142c)$$

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The terms with $\arcsin(\cdots)$ and the terms in the upper limit $u \to \frac{\pi}{2}$ are constants, but we observe that we have two divergent terms for $u \to 0$, namely

$$I = -2\sqrt{\left(1 - \frac{fv_\xi r_{max}^2}{2}\sin^2 u\right)}\csc u$$

$$II = -2\sqrt{\left(1 - \frac{(fv_\xi r_{max}^2 - 2)}{2}\sin^2 u\right)}\cot u \, , \tag{5.143}$$

and the difference in the equation (5.141) gives

$$\lim_{u \to 0}\left(-\sqrt{\left(1 - \frac{fv_\xi r_{max}^2}{2}\sin^2 u\right)}\csc u + \sqrt{\left(1 - \frac{(fv_\xi r_{max}^2 - 2)}{2}\sin^2 u\right)}\cot u\right) = 0 \, . \tag{5.144}$$

All in all, if $n \in \mathbb{Z}$, the potential energy $V_{q\bar{q}}$ is

$$V_{q\bar{q}}(r_{max}) = \frac{\sqrt{2v_\xi}L^2}{r_{max}}\left[(2 - fv_\xi r_{max}^2)\mathbf{K}(k) - 2\mathbf{E}(k) - 2\pi n\sqrt{2fv_\xi r_{max}^2}\right.$$

$$\left. + \sqrt{2fv_\xi r_{max}^2}\arcsin\left(\sqrt{\frac{fv_\xi r_{max}^2}{2}}\right) + 2\sqrt{1 - \frac{fv_\xi r_{max}^2}{2}}\right] \, . \tag{5.145a}$$

Since $v_\xi \in \left(0, \frac{2}{fr_{max}^2}\right)$, we write $v_\xi = \frac{a}{fr_{max}^2}$ with $a \in (0, 2)$, such that

$$\widehat{V}_{q\bar{q}}(r_{max}) = \frac{1}{L^2 v_\xi \sqrt{f}}V_{q\bar{q}}(r_{max}) = \frac{1}{\sqrt{k^2 + 1}}\left[-2\left(k^2\mathbf{K}(k) + \mathbf{E}(k)\right) - 2\pi n\sqrt{2a}\right.$$

$$\left. + \sqrt{2a}\arcsin\left(\sqrt{\frac{a}{2}}\right) + 2\sqrt{1 - \frac{a}{2}}\right] \, . \tag{5.145b}$$

In the figure 5.2, we plot the graph for three different values of a.

Alternatively, we can write the energy as a function of the distance, $\ell_{q\bar{q}}$, as

$$V_{q\bar{q}}(r_{max}) = \frac{L^2\sqrt{2v_\xi}}{r_{max}}\left[-2\left(\frac{k^2\sqrt{fv_\xi}}{2\sqrt{2}}\sqrt{\frac{-k^2}{1 + k^2}}\ell_{q\bar{q}} + \frac{a}{2}\mathbf{E}(k)\right) - 2\pi n\sqrt{2a}\right.$$

$$\left. + \sqrt{2a}\arcsin\left(\sqrt{\frac{a}{2}}\right) + 2\sqrt{1 - \frac{a}{2}}\right] . \tag{5.145c}$$

and from this last result we see that

$$\frac{dV_{q\bar{q}}}{d\ell_{q\bar{q}}} = L^2\frac{(\sqrt{f}v_\xi)k^4/\sqrt{-k^2}}{r_{max}(1 + k^2)^{1/2}} > 0 \, , \quad \frac{d^2V_{q\bar{q}}}{d\ell_{q\bar{q}}^2} = 0 \, . \tag{5.146}$$

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(a) $a = 0.1$ (b) $a = 1.0$ (c) $a = 1.9$
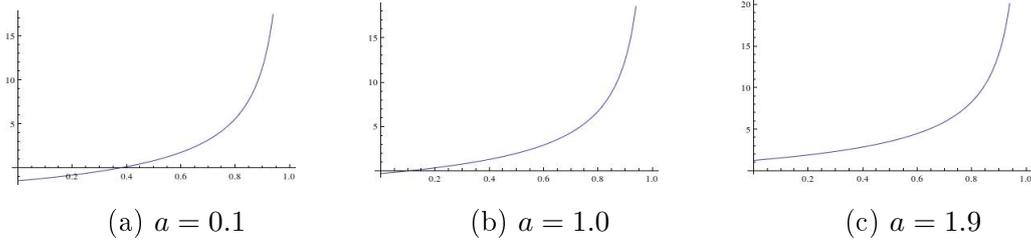
Figure 5.2: Graph of $\widehat{V}_{q\bar{q}}$ against $-k^2$, for three different values of a and $n = 0$.

It is important to notice that, although the potential energy $V_{q\bar{q}}$ exhibits a linear behaviour in relation to the distance $\ell_{q\bar{q}}$, similar to confining theories, we can not say that this theory is confining, since we have a maximum value for the distance $\ell_{q\bar{q}}$ in relation to the maximum distance r_{max}. Therefore, if we suppose that $\ell_{q\bar{q}} < \ell_{max}$, the potential $V_{q\bar{q}}$ is a bounded function of $\ell_{q\bar{q}}$. A similar phenomenon also happens in the calculations of Wilson loops at finite temperature [45]. Moreover, as we said, it is not clear if the interpretation imported from the relativistic gauge theories still holds in this case.

Chapter 6

More on Wilson loops for nonrelativistic backgrounds

D UE to their importance, we want to continue the analysis of the Wilson loops in this chapter. We start with a short review of the usual prescription of Wilson loops defined in [31], reviewed in [45]. Also, we highlight the relevant details for the calculation of the drag forces considered in [51,54].

In this chapter, we examine some string configurations on backgrounds with Schrödinger and Lifshitz symmetries and we see that these systems are tricky. We exclude some configurations and we also find systems that can hardly be solved analytically.

Even though the nonrelativistic systems considered here are at zero temperature, we found a nonzero drag force for them, as in [193]. Finally, reconsidering the systems of [186,188], we perform further analysis and present some speculative ideas on the nature of the nonrelativistic field theory dual to the background.

6.1 Short review

We have presented a review of the prescription for the calculation of Wilson loops in the section (3.4.1), but by completeness, we show again the main results. In summary, we present a short review of the ideas examined in [31,194] and reviewed in [45, 46] for the calculation of the quark-antiquark distance and potential. In addition, we consider some fundamental ideas related to the drag force on a classical

string configuration as in [51, 52, 54].

6.1.1 Quark-antiquark system

As we have seen in the section (3.4.1), we start with a background of the generic form

$$ds^2 = -g_{tt}dt^2 + g_{xx}d\vec{x}^2 + g_{rr}dr^2 + ds_{\mathcal{M}}^2 \tag{6.1}$$

where g_{tt}, g_{xx} and g_{rr} are functions of the radial coordinate r, and the term $ds_{\mathcal{M}}^2$ is a metric of an internal manifold. We can neglect the internal space $ds_{\mathcal{M}}^2$ because we consider a probe string that is not excited along those directions.

We take an ansatz for the string as

$$t = \tau \,, \quad x = x(\sigma) \,, \quad r = r(\sigma) \,, \tag{6.2}$$

and when we calculate the Nambu-Goto action and its equations of motion, we find that this configuration implies

$$\frac{dr}{d\sigma} = \pm \frac{dx}{d\sigma} \frac{f(r)}{C_0 g(r)} \sqrt{f(r)^2 - C_0^2} \,, \tag{6.3}$$

where $f(r)^2 = g_{tt}g_{xx}$, $g(r)^2 = g_{tt}g_{rr}$ and C_0 is an integration constant. The shape of the solution in this background can be pictured as a string whose ends are fixed at $x = 0$ and $x = \ell_{q\bar{q}}$ at the boundary of space, $r \to 0$. In addition, it can extend in the bulk, so that the radial coordinate of the string assumes its maximum value at r_0, that occurs at $x = \ell_{q\bar{q}}/2$. Furthermore, one can show that the integration constant is equal to $C_0 = f(r_0)$, see [46].

Considering the string solution above, we can compute gauge invariant quantities such as the separation and the energy between the endpoints of the string, which can be interpreted as the separation between a quark and an antiquark living on the brane, see [45, 46] for further details. These results are given by

$$\ell_{q\bar{q}}(r_0) = 2f(r_0) \int_0^{r_0} dr \frac{g(r)}{f(r)} \frac{1}{\sqrt{f(r)^2 - f(r_0)^2}} \,, \tag{6.4}$$

$$E_{q\bar{q}}(r_0) = f(r_0)\ell_{q\bar{q}}(r_0) - 2 \int_0^{r_0} dr g(r) + 2 \int_0^{r_0} dr \frac{g(r)}{f(r)} \sqrt{f(r)^2 - f(r_0)^2} \,. \tag{6.5}$$

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6.1.2 Drag force

In [51], the author considered a probe string moving through the AdS$_5$-Schwarzschild background, whose radius of the horizon is related to the temperature of the dual gauge theory. In summary, Gubser considered the metric of the near-extremal D3-brane

$$ds_{10}^2 = \frac{(-h dt^2 + d\vec{x}^2)}{\sqrt{H}} + \sqrt{H}(dr^2/h + d\Omega_5^2) \tag{6.6}$$

where

$$H = 1 + \frac{L^4}{r^4} \ , \quad h = 1 - \frac{r_H^4}{r^4}. \tag{6.7}$$

The near horizon limit is simply

$$ds^2 = \frac{r^2}{L^2}(-h dt^2 + d\vec{x}^2) + \frac{L^2}{r^2}\frac{dr^2}{h} \ , \tag{6.8}$$

where we drop the five dimensional part of the metric, since it plays no role in the present case.

Besides, he considered the following configuration

$$t = \tau \ , \quad x(\tau, \sigma) = v_x \tau + \eta(\sigma) \ , \quad r = \sigma \ , \tag{6.9}$$

with action

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \mathcal{L} \ , \tag{6.10}$$

and density

$$\mathcal{L} = \sqrt{1 - \frac{v_x^2}{h} + \frac{h}{H}\eta'^2} \ . \tag{6.11}$$

From the equation of motion we find that the momentum $\Pi_\eta = \frac{\partial \mathcal{L}}{\partial \eta'}$ is a constant equals to

$$\Pi_\eta = \frac{v_x}{\sqrt{1 - v_x^2}}\frac{r_H^2}{L^2} \ . \tag{6.12}$$

Using this last expression, the authors of [54] showed that the drag force, opposite to the motion of the string, is given by

$$F_\eta = -\frac{1}{2\pi\alpha'}\Pi_\eta, \tag{6.13}$$

and using the relation $\pi L^2 T = r_H$, we see that the drag force depends on the temperature of the system.

In this section we have defined the calculation of the drag force using the holographic principle, but we can reconsider this same calculation for backgrounds without horizon. This is what we intend to do below.

6.2 Schrödinger backgrounds

We begin by reconsidering some of the calculations that have been performed in [186, 188, 193, 195] concerning the calculation of Wilson loops on backgrounds with Schrödinger symmetries. Moreover, we perform further analysis in these solutions, and we study additional string configurations.

The probe string moves on a manifold of the form

$$ds^2 = \frac{R^2}{r^2} \left(\frac{-dt^2}{r^{2(z-1)}} + 2d\xi dt + (dx^i)^2 + dr^2 \right) \tag{6.14}$$

where ξ is a compact timelike coordinate, and the natural number z is the dynamical exponent. It can be shown , see [30] and references therein, that for $i = 1, \cdots, D - 1$, the space (6.14) is the geometric realization of the Schrödinger algebra in D dimensions.

6.2.1 Constant compact direction

First, we consider the following configuration for the probe string [186, 188, 193]

$$t = \tau \ , r = r(\sigma) \ , x = x(\sigma) \ , \xi = constant \ . \tag{6.15}$$

The Nambu-Goto action for this configuration is

$$\mathcal{S} = \frac{T}{2\pi\alpha'} \int d\sigma \sqrt{\frac{R^4}{r^{2(z+1)}} \left((x')^2 + (r')^2 \right)} \ , \tag{6.16}$$

and if we define $f(\sigma) = R^2/r^{(z+1)}$, the equations of motion for x and r are

$$\frac{f^2 x'}{\sqrt{f^2(x'^2 + r'^2)}} = C_0 \tag{6.17}$$

$$\partial_\sigma \left(\frac{f^2 r'}{\sqrt{f^2(x'^2 + r'^2)}} \right) = \frac{(x'^2 + r'^2)}{\sqrt{f^2(x'^2 + r'^2)}} f \frac{df}{dr}. \tag{6.18}$$

Equation (6.17) implies that

$$\frac{dr}{d\sigma} = \pm \frac{dx}{d\sigma} \frac{\sqrt{f^2 - C_0^2}}{C_0} = \pm \frac{dx}{d\sigma} \ V_{eff}(r) \ , \tag{6.19}$$

and the equation (6.18) is solved when this last equation is satisfied. Also, for a ∩-shaped string, the turning point is defined as the point r_0 where $\frac{dr}{dx}\big|_{r_0} = 0$. Using this condition we determine the constant $C_0 = f(r_0)$.

Since we consider a string moving in the bulk with its endpoints lying on the boundary $r \to 0$, the Dirichlet boundary condition $\lim_{r \to 0} \frac{dx}{dr} \to 0$ must be satisfied. We can see that this condition is readily satisfied, since $\lim_{r \to 0} V_{eff} \to \infty$.

The quark-antiquark distance is given by

$$\ell_{q\bar{q}}(r_0, z) = 2r_0 \sqrt{\pi} \frac{\Gamma\left(\frac{z+2}{2(z+1)}\right)}{\Gamma\left(\frac{1}{2(z+1)}\right)}, \tag{6.20}$$

and in $[186, 188]$ it was found that the quark-antiquark potential [1] is

$$\begin{aligned}
V_{q\bar{q}}(r_0, z) &= -2 \frac{R^2 \sqrt{\pi}}{z r_0^z} \frac{\Gamma\left(\frac{z+2}{2(z+1)}\right)}{\Gamma\left(\frac{1}{2(z+1)}\right)} \\
&= -\frac{(2\sqrt{\pi})^{1+z} R^2}{z} \left(\frac{\Gamma\left(\frac{z+2}{2(z+1)}\right)}{\Gamma\left(\frac{1}{2(z+1)}\right)}\right)^{z+1} \frac{1}{\ell_{q\bar{q}}^z} .
\end{aligned} \tag{6.21}$$

From this equation we see that for the special case $z = 1$ we have the behaviour $V_{q\bar{q}} \sim -\frac{1}{\ell_{q\bar{q}}}$ in the potential quark-antiquark, which is consistent with the conformal scaling.

In $[188]$, the author also showed that the convexity conditions $[139, 190]$ of such a configuration are satisfied, that is

$$\frac{dV_{q\bar{q}}}{d\ell_{q\bar{q}}} > 0 , \quad \frac{d^2 V_{q\bar{q}}}{d\ell_{q\bar{q}}^2} \leq 0, \tag{6.22}$$

where the first condition means that the quark-antiquark interaction is always attractive and the second equation means that the potential is a monotone nonincreasing function of ℓ. Therefore, this configuration is physically admissible.

A second configuration with constant compact direction that we would like to explore is given by

$$t = \tau , r = \sigma , x = v_x \tau + \eta(\sigma) , \xi = const . \tag{6.23}$$

As we said in the previous section, the drag force has been studied in $[5, 53]$ in the context of a quark moving in a thermal plasma of $\mathcal{N} = 4$ SYM, and we have seen that the horizon is related to the temperature of the field theory. Despite the fact

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[1]Using the definition of the Gamma function to extend the domain of the Beta function.

that in the present case we do not have a horizon in our geometry, we may apply the very same ideas.

The action is

$$\mathcal{S} = \frac{T}{2\pi\alpha'} \int dr \sqrt{\frac{R^4}{r^2}\left[\frac{1}{r^{2z}} + \frac{\eta'^2}{r^{2z}} - \frac{v^2}{r^2}\right]} \,, \tag{6.24}$$

and the equation of motion implies that Π_η, given by

$$\Pi_\eta = \frac{R^4\eta'}{r^{2z+2}\sqrt{\frac{R^4}{r^2}\left[\frac{1}{r^{2z}} + \frac{\eta'^2}{r^{2z}} - \frac{v^2}{r^2}\right]}} \,, \tag{6.25}$$

is a constant. Therefore, we find

$$\eta' = \Pi_\eta \sqrt{\frac{1 - v^2 r^{2z-2}}{\frac{R^4}{r^{2z+2}} - \Pi_\eta^2}} \,. \tag{6.26}$$

Now observe that if we take $z = 1$, the numerator in the square root is positive for all values $v < 1$, and this is consistent with a relativistic theory.

For the denominator we find that for some large r_* the constant Π_η^2 could be greater than R^4/r_*^4, and in this case, the denominator would be negative. Since there is no upper bound for r, we see that the reality condition of the integral implies that $\Pi_\eta = 0$, which implies that the drag force is zero, as we shall see below.

This result is expected, since for $z = 1$ we have the anti-de Sitter space, which is at zero temperature, and in the relativistic case, the drag force for a system at zero temperature vanishes. Also, we can see that the equation of motion for r is trivially satisfied since $\frac{\partial\mathcal{L}}{\partial r'}\big|_{r=\sigma} = \mathcal{L}$.

For $z = 2$, the values $\Pi_\eta = \pm R^2 v^3$ avoid an imaginary value in (6.26). Essentially these two examples were studied in [193].

In addition, for $z > 1$ we have the general formula

$$\Pi_\eta = \pm R^2 v^{(z+1)/(z-1)} \,. \tag{6.27}$$

Using that the drag force, formally defined as

$$F_{drag} = -\sqrt{-g}G_{xx}g^{\sigma\sigma}\eta' \,, \tag{6.28}$$

and that $\Pi_\eta = \frac{\partial\mathcal{L}}{\partial\eta'}$, we can easily show that $F_{drag} = \Pi_\eta$. The drag force is defined to be contrary to the velocity of the string, hence

$$F_{drag} = -R^2 v^{(z+1)/(z-1)} \,. \tag{6.29}$$

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For the special case $z = 2$, the drag force is $F_{drag} = -R^2 v^3$, which is consistent with the results found in [193].

For a general z, the results above are equal to the case studied in [187] for the Lifshitz spacetime at zero temperature. This happens because any configuration in the Schrödinger and Lifshitz spacetimes have the same Nambu-Goto action when $\xi = constant$.

In the section 6.1.2 we have seen that the radius of the horizon is related to the drag force of the system. On the other hand, the nonrelativistic spaces we consider do not have horizons, but have nontrivial drag forces. As the authors argued in [193], these systems may have a hidden chemical potential that allows such a phenomenon. In fact, making the transformations $t \to \mu t$ and $\xi \to \mu^{-1}\xi$ in (6.14), we can repeat our calculations and see that $F_{drag} \propto 1/\mu^2$, and in the dual field theory, the parameter μ can be interpreted as the chemical potential [30, 196].

In other words, the chemical potential is the conjugate variable to the particle number, and the compact coordinate ξ is directly related to the particle number (see for instance [30, 196]); then it is somewhat expected the presence of this 'hidden' chemical potential. On the other hand, the nature of the coordinate ξ is still a mystery [36–39], and the mechanism (considering that it exists) which allows us to relate the spectrum of the masses (particle number) to the chemical potential is unknown.

6.2.2   Nonconstant compact direction

We now consider that the string also moves on the compact direction ξ. We start with an example studied by [186], where the author concluded that the configuration is not physical. Here we point out some reasons that suggest a richer physical scenario. Furthermore, we study a new configuration in which the compact direction ξ depends on the coordinate σ that parametrizes the string. This configuration is described by a system of nonlinear differential equations and we could not find an explicit solution.

The reader must remember that we do not have a correct interpretation of this coordinate [30], consequently, the physical meaning of the string with its endpoints moving along this direction is uncertain; and maybe it is not even physically admis-

sible. Even so, let us insist on this direction and examine the ansatz

$$t = \tau \ , r = r(\sigma) \ , x = x(\sigma) \ , \xi = \xi(\tau) \ , \tag{6.30}$$

where the Nambu-Goto action reads

$$\mathcal{S} = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{g(\tau,\sigma)^2 \left((x')^2 + (r')^2\right)} \ , \tag{6.31}$$

for $g(r,\xi)^2 = \frac{R^4}{r^2}\left(\frac{1}{r^{2z}} - \frac{2}{r^2}\partial_\tau\xi\right)$. The equations of motion are

$$\partial_\tau g(r,\xi) = 0 \ , \tag{6.32}$$

$$\frac{g^2 x'}{\sqrt{g^2(x'^2 + r'^2)}} = C_1 \ , \tag{6.33}$$

$$\partial_\sigma\left(\frac{g^2 r'}{\sqrt{g^2(x'^2 + r'^2)}}\right) = \frac{(x'^2 + r'^2)}{\sqrt{g^2(x'^2 + r'^2)}}g\frac{dg}{dr}. \tag{6.34}$$

We can see that $\xi(\tau) = v_\xi\tau$, and the third equation is solved by imposing the second one. Therefore, the quark-antiquark distance now reads

$$\ell_{q\bar{q}}^\xi = \frac{2g(r_0)}{R^2} \int_0^{r_0} \frac{dr \ r^{z+1}}{\sqrt{1 - 2v_\xi r^{2z-2} - \frac{g^2(r_0)}{R^4}r^{2z+2}}}, \tag{6.35}$$

and the potential is

$$V_{q\bar{q}} = 2 \int_0^{r_0} \frac{dr \ g(r)}{\sqrt{g^2(r) - g^2(r_0)}} - 2 \int_0^{\hat{r}_0} dr g(r), \tag{6.36}$$

where $\hat{r}_0$ is the end of the space. The last term is necessary to remove the infinity part of the potential [31, 45, 46, 197]. This term is the mass of a W-boson which corresponds to strings stretching from zero to the end of the space $\hat{r}_0$. Additionally, the IR limit is defined such that the maximum value r_0 approaches the end of the space, that is $r_0 \to \hat{r}_0$ [46].

In [186], the author argued that since this integral is imaginary for values of r such that $r^{2(z-1)} > r_*^{2(z-1)} = 1/2v_\xi$, the configuration (6.30) with $\xi = v_\xi\tau$ is unphysical. Even though his arguments seem accurate, we point out some reasons which suggest that, perhaps, it is too early to rule out this configuration, inasmuch as we must be careful in using nonrelativistic spaces in our calculations.

First we need to remember that, except when $z = 1$, which is the *AdS* space, we can have several undesirable features on the background such as curvature singularities at the end of the spacetime $\hat{r}_0 \rightarrow \infty$, see [30, 198, 199]. On the other hand, it is important to notice that not all curvature singularities affect physical quantities [200–202], therefore, these spaces are not severely ill-defined.

In fact, considering spaces with Lifshitz symmetry the authors of [203] considered a configuration that can be interpreted as scattering amplitudes and studied observable consequences of the singularity in the IR structure of the dual field theory. Moreover, we are considering zero temperature systems and these singularities can be removed with finite temperature effects [30, 40].

Therefore, we expect to integrate the counterterm in (6.36) up to some point $\hat{r}_0 < \infty$. In this case, the problem can be fixed if we consider that the "cutoff" is defined at some point $\hat{r}_0 < r_*$, where the integral is well defined.

Evidently, after fixing the end of the space $\hat{r}_0$, we have a maximum (allowed) value for the velocity v_ξ. Then, we can take a velocity v_ξ to be small enough, such that $\hat{r}_0^{2(z-1)} < 1/2v_\xi$. Therefore, we notice that for the case $v_\xi \neq 0$, we have a reasonable configuration under certain conditions, and also that the velocity along the compact direction ξ may have an upper bound.

Furthermore, under the time reversal transformation $t \rightarrow -t$ or the parity $\xi \rightarrow -\xi$, the space (6.14) is not invariant and we have $g^2 = \frac{R^4}{r^2}\left(\frac{1}{r^{2z}} + \frac{2v_\xi}{r^2}\right)$. In this case, the integral for the W-boson mass is always real. This is a hint that field theory dual to supergravity solutions with Galilean symmetries may "perceive" the time direction. This is a point that deserves further investigation.

Alternatively, since the coordinate σ which parametrizes the string length is compact, we could consider a configuration with $\xi = \xi(\sigma)$, for $\sigma \in [0, L]$. Then, take the ansatz

$$t = \tau \, , r = r(\sigma) \, , x = v_x\tau + \eta(\sigma) \, , \xi = \xi(\sigma), \tag{6.37}$$

with Nambu-Goto action

$$\mathcal{S} = \frac{T}{2\pi\alpha'} \int d\sigma\mathcal{L} \, , \tag{6.38}$$

where

$$\frac{\mathcal{L}^2}{R^4} = \left(\frac{1}{r^{2z}} - \frac{v_x^2}{r^2}\right)\frac{(\eta'^2 + r'^2)}{r^2} + \frac{(\xi')^2}{r^4} \, , \tag{6.39}$$

and equations of motion

$$\partial_\sigma \left(\frac{R^4}{r^4\mathcal{L}}\xi'\right) = 0 \, , \tag{6.40}$$

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$$\partial_\sigma \left[\frac{R^4}{r^2 \mathcal{L}} \left(\frac{1}{r^{2z}} - \frac{v_x^2}{r^2} \right) \eta' \right] = 0 \ , \tag{6.41}$$

$$\frac{\partial \mathcal{L}}{\partial r} = \partial_\sigma \left[\frac{R^4}{r^2 \mathcal{L}} \left(\frac{1}{r^{2z}} - \frac{v_x^2}{r^2} \right) r' \right] \ . \tag{6.42}$$

The first two equations above give

$$\frac{1}{r^2} \left(\frac{R^4}{r^4} - C_1^2 \right) \xi'^2 = C_1^2 \left(\frac{1}{r^{2z}} - \frac{v_x^2}{r^2} \right) (\eta'^2 + r'^2) \ , \tag{6.43}$$

$$\frac{R^4}{r^4} \left(\frac{1}{r^{2z}} - \frac{v_x^2}{r^2} \right)^2 \eta'^2 = C_2^2 \left[\left(\frac{1}{r^{2z}} - \frac{v_x^2}{r^2} \right) \frac{(\eta'^2 + r'^2)}{r^2} + \frac{\xi'^2}{r^4} \right] \ , \tag{6.44}$$

respectively.

We can simplify this system considering the particular case $v_x = 0$. The action is

$$\mathcal{S} = \frac{T}{2\pi\alpha'} \int d\sigma \sqrt{h^2 \left[\frac{(\eta')^2 + (r')^2}{r^{2z}} + \frac{(\xi')^2}{r^2} \right]} \ , \tag{6.45}$$

with $h(\sigma)^2 = R^4/r^2$. Using the notation $2\pi\alpha' \mathcal{S} = T \int d\sigma \mathcal{L}$, we see that the equations of motion are

$$\partial_\sigma \left(\frac{h^2}{\mathcal{L}} \frac{\xi'}{r^2} \right) = 0 \ , \tag{6.46}$$

$$\partial_\sigma \left(\frac{h^2}{\mathcal{L}} \frac{\eta'}{r^{2z}} \right) = 0 \ , \tag{6.47}$$

and

$$\partial_\sigma \left(\frac{h^2}{\mathcal{L}} \frac{r'}{r^{2z}} \right) = \frac{\partial \mathcal{L}}{\partial r} \ . \tag{6.48}$$

The first of these equations implies that

$$\frac{1}{r^2} \left(\frac{h^2}{r^2} - C_1^2 \right) (\xi')^2 = \frac{C_1^2}{r^{2z}} \left(\eta'^2 + r'^2 \right) \ , \tag{6.49}$$

while the second equation gives

$$\frac{1}{r^{2z}} \left(\frac{h^2}{r^{2z}} - C_2^2 \right) (\eta')^2 = C_2^2 \left(\frac{r'^2}{r^{2z}} + \frac{\xi'^2}{r^2} \right) \ . \tag{6.50}$$

In order to find a restricted class of solutions, we consider initially $\xi = constant$, which is the first case considered in this section. Alternatively, we can set $\eta =$

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constant and $r = \sigma$. In the latter, the equation of motion for η is trivially satisfied, which means that $C_2 = 0$, whereas the equation of motion for ξ gives

$$\frac{dr}{d\xi} = \pm \frac{r^{z-1}}{C_1} \sqrt{\frac{h^2}{r^2} - C_1^2} \ . \tag{6.51}$$

The equation (6.48) now gives us the following differential equation

$$\frac{h(\xi')^2}{r^2} = -C_3 \sqrt{\frac{1}{r^{2z}} + \frac{(\xi')^2}{r^2}} \ , \tag{6.52}$$

and if we insert (6.51) into (6.52) we see that this system is not consistent unless $C_1 = C_3 = 0$, which implies that $\xi = constant$. This means that this restricted class of solutions is trivial, and in order to find solutions one may try to solve numerically the coupled equations (6.43 – 6.44) or (6.49 – 6.50) for $v_x = 0$.

In summary, one sees that the motion of string along the compact coordinate ξ of a Schrödinger background is a tricky issue, and deserves further investigation, but in principle, there is no apparent reason to rule out these configurations.

6.3   Lifshitz

Now we would like to study the motion of a string in a space of the form

$$ds^2 = \frac{R^2}{r^2} \left(-\frac{dt^2}{r^{2(z-1)}} + (dx^i)^2 \right) + \frac{R^2}{r^2} dr^2. \tag{6.53}$$

Analogously to what we have done in the last section, we could try to consider the probe string with the following profile

$$t = \tau \ , r = r(\sigma) \ , x = x(\sigma) \ , \tag{6.54}$$

and we get the same equations as in the first example of section (6.2.1), see [185]. Moreover, if we take the example

$$t = \tau \ , r = r(\sigma) \ , x = v_x \tau + \eta(\sigma) \ , \tag{6.55}$$

we find the second example of the same section, since in that case we considered $\xi = constant$.

Additionally, the solution presented in [48] is much more interesting. In this paper, the authors used the methodology of [47], which allowed them to embed a

Schrödinger invariant solutions with $z = 2$ into string theory, to find a supergravity solution with Lifshitz symmetry.

The exterior part of the solution [48] is given by

$$ds^2 = \frac{R^2}{r^2} \left(-2dtd\xi + (dx^i)^2 + r^2 f(\xi)d\xi^2 + dr^2 \right) , \tag{6.56}$$

and in order to make explicitly the Lifshitz symmetry, we write

$$\frac{-2dtd\xi}{r^2} + d\xi^2 \equiv -\frac{dt^2}{r^4} + \left(d\xi - \frac{dt}{r^2} \right)^2 , \tag{6.57}$$

where we considered $f = 1$.

Notice that a configuration with $\xi = constant$, $t = t(\tau)$, $r = r(\sigma)$ and $x = x(\sigma)$ is not allowed, since it would give a zero Nambu-Goto action. On the other hand, we may consider that

$$t = \tau \; , r = r(\sigma) \; , x = v_x \tau + \eta(\sigma) \; , \xi = const, \tag{6.58}$$

and we obtain a nontrivial Nambu-Goto action

$$\mathcal{S} = \frac{T}{2\pi\alpha'} \int d\sigma \frac{R^2 v_x}{r^2} \sqrt{-(\eta')^2 - (r')^2} \; . \tag{6.59}$$

From the reality of the action, we may notice that the functions η and r must be purely imaginary or one of them complex, in such a way that the combination $-(\eta')^2 - (r')^2 > 0$. Such conditions for η and r are unacceptable because they are distances. Therefore, this configuration is unphysical.

In the last chapter we have studied one more configuration, namely

$$t = \tau \; , r = r(\sigma) \; , x = x(\sigma) \; , \xi = v_\xi \tau \; , \tag{6.60}$$

and we saw that the the quark-antiquark distance is given by

$$\ell_{q\bar{q}}(r_{max}) = \frac{2\sqrt{2}}{\sqrt{fv_\xi}} \sqrt{-\frac{k^2 + 1}{k^2}} (\mathbf{K}(k) - \mathbf{E}(k)) \; , \tag{6.61}$$

where $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are the complete elliptic integrals of first and second kind respectively; and $k^2 = (fv_\xi r_{max}^2 - 2)/2$, with $-k^2 \in (0,1)$. The quark antiquark potential is

$$\begin{aligned}
V_{q\bar{q}}(r_{max}) = &\frac{L^2\sqrt{2v_\xi}}{r_{max}} \left[-2 \left(\frac{k^2\sqrt{fv_\xi}}{2\sqrt{2}} \sqrt{\frac{-k^2}{1+k^2}} \ell_{q\bar{q}} + \frac{a}{2}\mathbf{E}(k) \right) - 2\pi n\sqrt{2a} \right. \\
&\left. + \sqrt{2a} \arcsin \left(\sqrt{\frac{a}{2}} \right) + 2\sqrt{1 - \frac{a}{2}} \right] .
\end{aligned} \tag{6.62}$$

where $a \in (0, 2)$.

As we said before, the coordinate ξ is compact, so it is reasonable to take the functional relation $\xi = \xi(\sigma)$. Then we consider the ansatz

$$t = \tau \ , r = r(\sigma) \ , x = v_x \tau + \eta(\sigma) \ , \xi = \xi(\sigma) \ , \tag{6.63}$$

such that

$$\mathcal{S} = \frac{T}{2\pi\alpha'} \int d\sigma \mathcal{L} \ , \tag{6.64}$$

where

$$\mathcal{L} = R^2 \frac{\sqrt{(1 - r^2 v_x^2 f)(\xi')^2 - v_x^2(\eta'^2 + r'^2)}}{r^2}. \tag{6.65}$$

The equations of motion for η and ξ give

$$\eta' = \pm \frac{C_0 r^2}{v_x} \sqrt{\frac{(1 - r^2 v_x^2 f)\xi'^2 - v_x^2 r'^2}{R^4 v_x^2 + C_0^2 r^4}}, \tag{6.66a}$$

$$\frac{v_x^2(\xi')^2 R^4}{2r^2 \mathcal{L}} \partial_\xi f + \partial_\sigma \left(\frac{R^4(1 - r^2 v_x^2 f)\xi'}{r^4 \mathcal{L}} \right) = 0, \tag{6.66b}$$

and if we take f to be a constant, we find

$$\xi' = \pm v_x C_1 r^2 \sqrt{\frac{\eta'^2 + r'^2}{(1 - r^2 v_x^2 f)[C_1^2 r^4 - R^4(1 - r^2 v_x^2 f)]}}. \tag{6.66c}$$

The equation for r is

$$-\partial_\sigma \left(\frac{R^4 v_x^2 r'}{r^4 \mathcal{L}} \right) = \frac{\partial \mathcal{L}}{\partial r} \ . \tag{6.66d}$$

In order to simplify this system, one can try to set one further constraint, $\eta = constant$. From the equation of motion for ξ, we find

$$\xi' = \pm \frac{r^2 C_1 v_x r'}{\sqrt{(1 - fr^2 v_x^2)[r^4 C_1^2 - R^4(1 - fr^2 v_x^2)]}}. \tag{6.67a}$$

We see that if we set $r = \sigma$, we find an inconsistent configuration, since from this last equation

$$\xi' = \pm \frac{r^2 C_1 v_x}{\sqrt{(1 - fr^2 v_x^2)[r^4 C_1^2 - R^4(1 - fr^2 v_x^2)]}} \ , \tag{6.68a}$$

while from the equation of motion for r we have

$$\left(\frac{R^4(1 - r^2 v_x^2 f)}{r^4} \xi'^2 - C_3^2 \right) \xi'^2 = \frac{-C_3^2 v_x^2}{(1 - r^2 v_x^2 f)} \ . \tag{6.68b}$$

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By substitution we can see that both equations are not consistent, unless $C_1 = C_3 = 0$, where the constant

$$C_3 = \frac{R^4 v_x^2}{r^4 \mathcal{L}} + \mathcal{L} \tag{6.69}$$

solves the equation (6.66d) for $r = \sigma$ and $\eta' = 0$.

On the other hand, we can take the system with $\eta' \neq 0$, but with $r = \sigma$. The equations of motion are

$$\eta' = \pm \frac{C_0 r^2}{v_x} \sqrt{\frac{(1 - r^2 v_x^2 f) \xi'^2 - v_x^2}{R^4 v_x^2 + C_0^2 r^4}}, \tag{6.70a}$$

and

$$\xi' = \pm v_x C_1 r^2 \sqrt{\frac{\eta'^2 + 1}{(1 - r^2 v_x^2 f)[C_1^2 r^4 - R^4 (1 - r^2 v_x^2 f)]}}. \tag{6.70b}$$

Finally, we see that the integration constant C_0 is equal to the conserved charge $\Pi_\eta = \frac{\partial \mathcal{L}}{\partial \eta'}$, therefore, the drag force of this configuration is

$$F_{drag} = -\frac{R^4 v_x^2 \eta'}{r^4 \mathcal{L}} , \tag{6.71}$$

but in order for this to be well defined, we need to solve the equation of motion for the coordinate ξ – probably numerically – and we also need to consider the additional condition $(1 - v_x^2 f) \xi'^2 - v_x^2 > 0$, that comes from the reality condition of the equation (6.70a). Therefore, we can see that in this case we can find nontrivial drag forces at zero temperature.

Chapter 7

Conclusions

I N this thesis we studied several properties of the abelian and nonabelian T-duality on backgrounds of type IIA and IIB supergravity. Moreover, we have seen how we can use prescriptions of the gauge/gravity methodology to understand the field theory dual to these T-dual backgrounds.

In chapter 03 of this thesis we have considered a T-duality along an $U(1)$ isometry of a deformation of the MNa solution in [7], such that the resulting type IIA solution consists of D4-branes wrapping a two-cycle. We found a solution with nontrivial RR forms, a nonvanishing Kalb-Ramond field and a complicated metric. We analyzed Maxwell and Page charges associated to this solution.

We then studied properties of the field theory dual to the T-dual gravitational background. From a calculation of the Wilson loops, we saw that the dual gauge theory presents confinement. We also computed the QCD string tension and the gauge coupling of the gauge theory.

From a calculation of the entanglement entropy, we found that the field theory does not have a phase transition, despite being a confining theory; this could be due to the nonlocality of the theory, as suggested in [138]. Finally, considering domain walls in the gravitational background, we generate a Chern-Simons term in the gauge theory.

Also, in the chapter 04 we have studied the nonabelian T-duals of some backgrounds with $\mathcal{N} = 1$ supersymmetry and an AdS factor, that can have an AdS/CFT interpretation. We have considered the nonabelian T-dual of a type IIA solution with an AdS_5 factor, giving a type IIB solution with an AdS_5 factor, and the nonabelian

T-dual of a type IIB domain wall solution that interpolates between $AdS_5 \times T^{1,1}$ and $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$.

We have probed the interpretation of nonabelian T-duality of these solutions from the point of view of the dual conformal field theory through a calculation of the central charges. We have found that the simple law (4.65) found in [146] for the ratio of central charges before and after the T-duality holds in all cases, with the obvious generalization of N_{D3}^2/N_{D6}^2 to N_{Dp}^2/N_{Dp+3}^2 or to $N_{D3}\bar{N}_{D3}/N_{D6}\bar{N}_{D6}$. In the case of the type IIB dowain wall solution, we obtained the usual $\propto N^2$ behaviour, and on the AdS_3 side we could fix the normalization of the central charge by using a rescaling parameter τ, in order to obtain the same law (4.65) valid on the AdS_5 side of the domain wall. In order to understand better the effect of nonabelian T-duality on gravity duals with AdS factors, one needs to study also other probes of the geometry, but we leave this for future work.

In the fifth chapter, we have studied nonabelian T-duality for nonrelativistic holographic duals. In particular, using a NATD transformation we constructed novel examples of nonrelativistic spaces with the interpretation of holographic duals, one for a conformal Galilean theory in massless type IIA, one for a conformal Galilean theory in massive type IIA, and two for Lifshitz theories in type IIB, coming from NATD of spaces with $T^{1,1}$ and $Y^{p,q}$ internal spaces.

In order to describe the field theories dual to the nonrelativistic gravitational backgrounds, we have calculated the conserved charges of these backgrounds and we compared our results with those obtained in [148].

We have also calculated the Wilson loop observables for the holographic dual spaces, though their true interpretation in the field theory remains to be seen, and it would be very interesting to understand. For the Wilson loops in gravity duals of conformal Galilean theories, we considered that the compact coordinate is constant and we found that the energy potential between quarks is always attractive. For the case of gravity dual of spaces with Lifshitz symmetry, we could not consider a constant compact coordinate, and we do not know the field theoretical interpretation for the string moving in this direction. The Wilson loop that we found for this second class of spaces is proportional to the quark-antiquark distance, but the interpretation of this result is not clear.

It would be useful to characterize further the field theories dual to the nonrelativistic backgrounds considered in this paper, by studying also other properties, like

conductivity or shear viscosity.

Finally, in the chapter 06 we concluded this thesis reconsidering some string configurations that give Wilson loops in the dual nonrelativistic field theory. In summary, we studied strings moving in spacetimes with Schrödinger and Lifshitz symmetries.

We started with Schrödinger spacetimes, and reviewed the string configurations with constant compact dimensions [188]. In this case, we have some physical configurations and we calculated the quark-antiquark distance and potential. By extension, we considered the string moving along the x-direction and we calculated a nonzero drag force for such a configuration.

Taking into account the motion along the compact extra dimension, $\xi = \xi(\tau)$, we reconsidered the configuration of [186]. We pointed that one cannot claim that this configuration is unphysical yet; in fact, there are some issues that must be taking into account: first, the role of the compact coordinate ξ is not clear, and we need to remember that there are genuine singularities at the end of the space. Also, at the present stage of development, we can consider a parity transformation $\xi \rightarrow -\xi$, and, apparently, this transformation makes the system well defined.

Alternatively, we pointed that the coordinate ξ is compact, then the configuration with dependence $\xi = \xi(\sigma)$ may make physical sense. In this case, we found a coupled system of differential equations.

For the Lifshitz case, we saw that there are some cases in which the analysis is the same as in the Schrödinger solution for constant compact dimension [185]. On the other hand, we have considered the Lifshitz solution related to the construction given by [48], and we saw that a rich scenario emerges. For the case with constant compact direction the solution is unphysical.

For a compact dimension ξ with dependence on the dimension τ, we have calculated the quark-antiquark potential in [195]. Finally, for the compact dimension with dependence on the coordinate σ, we calculated the drag force of the string moving through this background.

We recall that we must be careful in using these nonrelativistic spaces. An interesting question is whether the systems of differential equations have solutions or not. It is also very promising to consider the effect of fields of the NS-NS sector on the string, or quantum effects similar to [45].

Appendix A

Type II superstring

N this section we consider a concise review of type IIA and type IIB string theory and their low energy limits. In this appendix I do not try to be self-contained by no means, and it must be seen as a guide to the type II string theory. The topics I consider here can be found in the standard books of string theory and references therein, e.g [10, 11, 55, 58, 93].

A.1   Highlights on the RNS and GS formalisms

There are two equivalent ways we can introduce supersymmetry into superstring theory, namely:

- The Ramond-Neveu-Schwarz (RNS) formalism in which the supersymmetry is realized on the worldsheet

- The Green-Schwarz (GS) formalism in which the supersymmetry is realized on the spacetime.

A.1.1   RNS formalism

Let us consider first the RNS formalism. We consider momentarily that $\alpha' = 1/2$, then the bosonic action in the conformal gauge is simply

$$S_b = -\frac{1}{2\pi} \int \mathrm{d}^2\sigma \partial_\alpha X_\mu \partial^\alpha X^\mu \tag{A.1}$$

and we complement this action with a Dirac action for D massless Majorana fermion, then

$$S = -\frac{1}{2\pi} \int \mathrm{d}^2\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \slashed{\partial} \psi_\mu \right) \tag{A.2}$$

where $\slashed{\partial} = \rho^\alpha \partial_\alpha$ and $\bar{\psi} = i\psi^T \rho^0$. The Dirac matrices ρ^α satisfy the Clifford algebra $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$, such that

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{A.3}$$

If we define

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} \tag{A.4}$$

we can write the fermionic part of (A.2) in the light-cone coordinates as

$$S_f = \frac{i}{\pi} \int \mathrm{d}^2\sigma \left(\psi^\mu_- \partial_+ \psi_{\mu-} + \psi^\mu_+ \partial_- \psi_{\mu+} \right) , \tag{A.5}$$

and the equations of motion are

$$\partial_+ \psi_- = 0 \qquad \partial_- \psi_+ = 0 , \tag{A.6}$$

that describe left-moving and right-moving waves. In addition, these equations implies that these Majorana spinors are also Weyl spinors.

Open strings

We already know that in the open string case, the left-moving modes are related to the right-moving modes by the boundary conditions. In order to be a well defined theory it can be shown that some inequivalent boundary conditions can be satisfied. For instance, if we take

$$\psi^\mu_+|_{\sigma=0} = \psi^\mu_-|_{\sigma=0} , \tag{A.7}$$

we can have two different conditions for the other string endpoint:

1. Ramond (R) boundary condition:

$$\psi^\mu_+|_{\sigma=\pi} = \psi^\mu_-|_{\sigma=\pi} , \tag{A.8}$$

with mode expansion

$$\psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)} \tag{A.9a}$$

$$\psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)} \tag{A.9b}$$

such that $(d_n^\mu)^\dagger = d_{-n}^\mu$.

2. Neveu-Schwarz (NS) boundary condition:

$$\psi_+^\mu|_{\sigma=\pi} = -\psi_-^\mu|_{\sigma=\pi} \; , \tag{A.10}$$

with mode expansion

$$\psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-in(\tau-\sigma)} \tag{A.11a}$$

$$\psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-in(\tau+\sigma)} \; . \tag{A.11b}$$

Closed strings

In the closed string case we have two sets of fermionic modes, hence four different conditions. The periodicities conditions are

$$\psi_\pm^\mu(\tau,\sigma) = \pm\psi_\pm^\mu(\tau,\sigma+\pi) \; , \tag{A.12}$$

that is, we have periodic boundary condition (R), and the antiperiodic boundary condition (NS). Therefore, the right-movers have the expansion

$$\psi_-^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \quad \text{or} \quad \psi_-^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-2in(\tau-\sigma)} \tag{A.13}$$

while the left-movers satisfy

$$\psi_+^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} \quad \text{or} \quad \psi_+^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-2in(\tau+\sigma)} \; . \tag{A.14}$$

All in all, we have four inequivalent possibilities: states in the NS-NS and R-R sectors, describing spacetime bosons, and states in the NS-R and R-NS sector, that describe spacetime fermions.

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From the quantization of the modes, we can be find the spectrum of the open and closed strings. In particular, it can be shown (see [10, 11]) that there are two inequivalent $\mathcal{N} = 2$ superstring theories with only closed strings, the type IIA, in which the left- and right moving R-sector ground states have opposite chirality, and the type IIB string theory, where these modes have same chirality.

A.1.2 GS formalism

In the RNS formalism we can calculate amplitudes in a way that the Lorentz symmetry is preserved, together with the fact that we can find the spectrum of the theory very quickly. Obviously we need to be aware of the GSO projection (see [10,11]) but except this small obstacle, we do not have problems regarding the quantization of string theory in the RNS formalism. On the other, the supersymmetry of the theory in this formalism is not manifest, although we can easily show that the number of degrees of freedom at each mass level match as required by supersymmetry, it is not a direct proof.

On the other hand, we can handle a formalism where the supersymmetry is manifest, the GS formalism. The bosonic string theory is defined as a map from the string worldsheet Σ to a spacetime manifold $\mathcal{M}$, that is

$$X : \Sigma \to \mathcal{M} \, , \tag{A.15}$$

we may think of the GS formalism as a map from string worldsheet Σ to a superspace manifold $\mathcal{M}_\theta$,

$$(X, \Theta) : \Sigma \to \mathcal{M}_\theta \, . \tag{A.16}$$

The GS action is given by

$$
\begin{aligned}
S = &-\frac{1}{2\pi} \int \mathrm{d}^2\sigma \sqrt{-h} h^{\alpha\beta} \Pi_\alpha^\mu \Pi_{\mu\beta} \\
&+ \frac{1}{\pi} \int \mathrm{d}^2\sigma \epsilon^{\alpha\beta} \left[-\partial_\alpha X^\mu \left(\bar{\Theta}^1 \Gamma_\mu \partial_\beta \Theta^1 - \bar{\Theta}^2 \Gamma_\mu \partial_\beta \Theta^2 \right) - \bar{\Theta}^1 \Gamma_\mu \partial_\alpha \Theta^1 \bar{\Theta}^1 \Gamma_\mu \partial_\beta \Theta^2 \right] \, ,
\end{aligned}
\tag{A.17}
$$

where

$$\Pi_\alpha^\mu = \partial_a X^\mu - \bar{\Theta}^A \Gamma_\mu \partial_\alpha \Theta^A \, , \tag{A.18}$$

and Θ^A, with $A = 1, \ldots, \mathcal{N}$, are Majorana-Weyl spinors in $d = 10$. In the type II superstring, we have $\mathcal{N} = 2$. In the type IIA theory, these spinors have opposite

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chirality, while in the type IIB theory, these spinors have same chirality, that is

$$\text{IIA} \; : \; \Gamma_{11}\Theta^A = (-1)^{A+1}\Theta^A \tag{A.19}$$

$$\text{IIB} \; : \; \Gamma_{11}\Theta^A = \Theta^A \; . \tag{A.20}$$

The spectrum of the theory can be analysed through the light-cone quantization, and in this case, only a subgroup of the Lorentz symmetry is manifest, namely, the transverse $SO(8)$ rotational symmetry. Moreover, in ten dimensions a Majorana-Weyl spinor has eight components and each Θ^A fulfills an representation of $SO(8)$, and as we will see below, there will be two inequivalent representations of this symmetry group.

A.2 Triality

In ten dimensions, we can find irreducible spinors that satisfy the Majorana and Weyl conditions simultaneously[1], and in this case our spinors have a minimum of eight complex components. Therefore, we are dealing with representations of the Lie algebra $so(8) = D_4$, whose Dynkin diagram is
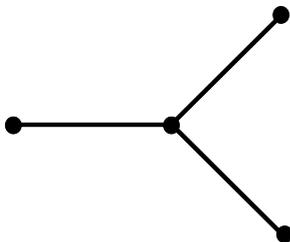
Figure A.1: Dynkin diagram for the Lie algebra $so(8)$.

One conspicuous property of this group is its threefold symmetry, known as *triality*, that permutes inequivalent representations (with the same dimensionality)

---

[1]In fact, given a space with signature (t, s) we can define Majorana-Weyl (MW) spinors if $s - t = 0$ (mod 4). Therefore, in Minkowski spaces, with $d \leq 11$, we can have MW-spinors for $d = 2, 10$ [91, 204]

of this group. In other words, there are three eight dimensional representations of $so(8)$, the fundamental representation, denoted by $\mathbf{8}_v$ and two spinor representations, $\mathbf{8}_s$ and its complex conjugate $\mathbf{8}_c$ [205]. Let us think about it for a moment.

Consider first the familiar rotation group in three dimensions $SO(3)$. The fundamental representation in this case is defined by the action of 3×3 matrices on vectors $\vec{v}$ with components v^i, for $i = 1, 2, 3$, that is $v^i \mapsto R^i_j v^j$, where $R \in SO(3)$. In addition, using the exponential map, we can write the matrix $R \in SO(3)$ as $R = \exp\left(i\hat{\omega} \cdot \vec{L}\right)$ where $\vec{\omega} = \omega \hat{n}$ defines the angle ω we are making the rotation and the components of $\vec{L}$ are the generators of the Lie algebra $so(3)$

$$[L_i, L_j] = i\epsilon_{ijk}L_k \ . \tag{A.21}$$

Moreover, the generators of Lie algebra of the group $SU(2)$ satisfies $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ where σ_i, $i = 1, 2, 3$ are the Pauli matrices. Therefore, the map $\varrho : su(2) \to so(3)$ defined by $L_i \equiv \varrho(\sigma_i) := \sigma_i/2$ is an isomorphism between these two algebras.

On the other hand, the groups $SO(3)$ and $SU(2)$ differ in global topological aspects. In particular, we know that for $n \geq 2$ the orthogonal group $SO(n)$ is not simply connected. At the same time, any matrix $U \in SU(2)$ can be written in the form

$$U = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix} \tag{A.22}$$

with $a, b, c, d \in \mathbb{R}$ and $\det U = a^2 + b^2 + c^2 + d^2 = 1$, which is the equation defining the 3-sphere [2]. The group $SU(2)$ is, as a group manifold, the 3-sphere $\mathbb{S}^3$, therefore, it is simply connected. All in all, one can show that the correspondence between these two groups is two-to-one and in fact $SO(3) \simeq SU(2)/\mathbb{Z}_2$.

The important point here is that using this group homomorphism $\phi : SU(2) \to SO(3)$, we can find a representation of $SU(2)$ once we have a representation R of $SO(3)$. Using these facts we define the objects transforming under the fundamental representation of the group $SU(2)$ as the spinor representation $SO(3)$, see [206] for further mathematical details and [69] for a explicit and detailed construction of spinor representations.

One may clearly notice that the fundamental representation of $SO(3)$ is three-dimensional, whilst the spinor representation is two-dimensional, which means that

---

[2]In terms of quaternions $\mathbb{H}$, the Lie group $SU(2)$ is the subset of $\mathbb{H}$ with elements of length equal to one.

the vector and spinor have different dimensions and are obviously intrinsically different. The same occurs for most of the groups we consider, for instance in the Poincaré group $SO(1,3)$ and in the R-symmetry group $SO(6)$ of $\mathcal{N}=4$ SYM, and there is no special reason why it should be different. But the distinguished property of the group $SO(8)$ is that the dimensions of the spinor and vector representation are the same. Surely it is what Bertrand Russell would call *mathematical supreme beauty* [207] and it is astonishing that this amazing symmetry has something deep to say about a mundane theory such as string theory, as we will see now.

A.3 Type II supermultiplet

In order to make contact with string theory, we need to pay attention that in the light-cone quantization of superstring theory we have an $SO(8)$ rotational transverse symmetry, and from the results we have just found, associated to this group we can find three inequivalent representations denoted by $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$, see [10, 11, 55, 90, 208].

Since the theory we supersymmetric, the number of bosonic and fermionic degrees of freedom must match, and one can show that the ground state for open strings (see for instance [11]) is given by $\mathbf{8}_v \oplus \mathbf{8}_s$ (or evidently $\mathbf{8}_v \oplus \mathbf{8}_c$) where $\mathbf{8}_v$ consists of a massless vector and $\mathbf{8}_s$ its spinor partner, together these fields give the *vector multiplet*. Taking tensor products of left and right-movers we obtain the ground state for closed strings. We have two possibilities:

$$
\begin{aligned}
IIA : \ & (\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s) \\
IIB : \ & (\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) \ .
\end{aligned}
\tag{A.23}
$$

Let us now understand what all these products mean. The multiplication table [90, 205, 208] (see also, the appendix B1 of [32] and the D'Hoker's lectures in [58]) of the group $SO(8)$ is

$$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v \quad : \quad [0] + [2] + (2)$$

$$\mathbf{8}_s \otimes \mathbf{8}_s = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_s \quad : \quad [0] + [2] + [4]_s$$

$$\mathbf{8}_c \otimes \mathbf{8}_c = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_c \quad : \quad [0] + [2] + [4]_c$$

$$\mathbf{8}_s \otimes \mathbf{8}_c = \mathbf{8}_v \oplus \mathbf{56}_v \quad : \quad [1]_v + [3]_v$$

$$\mathbf{8}_v \otimes \mathbf{8}_s = \mathbf{8}_c \oplus \mathbf{56}_c$$

$$\mathbf{8}_c \otimes \mathbf{8}_v = \mathbf{8}_s \oplus \mathbf{56}_s$$

Consider first the term $\mathbf{8}_v \otimes \mathbf{8}_v$, which is common to both IIA and IIB string theory and is called NS-NS sector. The decomposition gives a scalar, the *dilaton* ϕ, a anti-symmetric rank 2 tensor, the *Kalb-Ramond field* B_{ij} and the traceless symmetric rank 2 tensor, the *graviton* G_{ij}.

In the type IIA and IIB string theories, we take the product $\mathbf{8}_c \otimes \mathbf{8}_v = \mathbf{8}_s \oplus \mathbf{56}_s$. In this case the decomposition gives a spinor λ_α, the *dilatino* and a vector-spinor ψ_α^i, the *gravitino*, see [208] for a detailed analysis of these decompositions. Together, $\mathbf{8}_v \otimes \mathbf{8}_v$ and $\mathbf{8}_c \otimes \mathbf{8}_v$ give the graviton multiplet.

For the type IIA string theory we have $\mathbf{8}_v \otimes \mathbf{8}_s = \mathbf{8}_c \oplus \mathbf{56}_c$ that is, a dilatino and a gravitino $\lambda_{\dot\alpha}$ and $\psi_{\dot\alpha}^i$. Also, $\mathbf{8}_s \otimes \mathbf{8}_c = \mathbf{8}_v \oplus \mathbf{56}_v$, known as R-R sector, gives a vector $A_i^{(1)}$, the *graviphoton* and an anti-symmetric rank 3-tensor $A_{ijk}^{(3)}$.

Finally, the type IIB string theory has $\mathbf{8}_c \otimes \mathbf{8}_v = \mathbf{8}_s \oplus \mathbf{56}_s$ that gives a dilatino and a gravitino λ_α and ψ_α^i, and the R-R sector is $\mathbf{8}_s \otimes \mathbf{8}_c = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_c$, gives a scalar $A^{(0)}$, an anti-symmetric rank 2-tensor $A_{ij}^{(2)}$ and an anti-symmetric rank 4-tensor $A_{ijkl}^{(4)}$, whose field strength is self-dual, that is $\mathrm{d}A^{(4)} = *\mathrm{d}A^{(4)}$.

sector	type IIA	type IIB
NS-NS (bosons)	ϕ, B_{ij}, G_{ij}	ϕ, B_{ij}, G_{ij}
NS-R (fermions)	$\lambda_{\dot\alpha}, \psi_{\dot\alpha}^i$	$\lambda_{\dot\alpha}, \psi_{\dot\alpha}^i$
R-NS (fermions)	$\lambda_\alpha, \psi_\alpha^i$	$\lambda_{\dot\alpha}, \psi_{\dot\alpha}^i$
R-R (bosons)	$A_i^{(1)}, A_{ijk}^{(3)}$	$A^{(0)}, A_{ij}^{(2)}, A_{ijkl}^{(4)}$

A.4 Type II supergravity

In the previous section we have seen the massless spectrum of the type IIA and IIB string theories. But one can show that for the low energy limit of these theories, $\alpha' \to 0$, the massive spectrum becomes extremely heavy, and we can study the dynamics of the massless modes by a supergravity theory.

A.4.1 D$= 11$ and the type IIA supergravity

We begin by recovering the massless bosonic field content of the type IIA string theory in ten dimensions, that is

> **NS-NS sector:** $G_{\mu\nu}$, $B_{\mu\nu}$, ϕ
> **R-R sector:** $\qquad A_\mu^{(1)}, A_{\mu\nu\rho}^{(3)}$

But the field content of the type IIA supergravity can be obtained from a theory in eleven dimensions, than in principle has nothing to do with string theory, namely, $d = 11$ $\mathcal{N} = 1$ supergravity first studied by [209][3]. Initially, it was realized [213] that we can formulate consistent (with spin ≤ 2) supersymmetric field theories only if the spacetime dimension d is less than or equal to eleven dimensions, than $d \leq 11$. Another important characteristic of eleven dimensional theories is the fact that in principle, if we want to reproduce the standard model gauge group $SU(3) \times SU(2) \times U(1)$ as isometries of compact higher dimensions, then we need at least seven extra dimensions [214], for instance, the manifold $\mathbb{CP}^2 \times \mathbb{S}^2 \times \mathbb{S}^1$. But it became clear that from this procedure we could not obtain a chiral theory.

The field theory of $\mathcal{N} = 1$ $d = 11$ supergravity theory contains the metric G_{MN}, that can be formulated equivalently in terms of *vielbeins* $e_M{}^A$ with 44 degrees of freedom, a Majorana Gravitino Ψ_M with 128 degrees of freedom and an anti-symmetric rank 3-tensor G_{MNP}. Let us now consider just the bosonic sector of the theory, but it important to notice that the fermionic sector could be included in this

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[3]See also [89, 91, 210] for some reviews on this subject and the collections [211, 212] ranging important works from the early days of higher dimensional theories to the Maldacena first work in the AdS/CFT paradigm.

analysis as well [89]. Following the notation of [32], the bosonic action is given by

$$2\kappa_{11}^2 S_{11} = \int \mathrm{d}^{11}x \sqrt{-\hat{G}} \left(R - \frac{1}{4}|\mathrm{d}C|^2 \right) - \frac{1}{6} \int C \wedge \mathrm{d}C \wedge \mathrm{d}C \ , \qquad (A.24)$$

where κ_{11} is related to the Newton's constant G_{11} by $\kappa_{11}^2 = 8\pi G_{11}$. The type IIA supergravity is readily obtained by dimensional reduction of this theory on a 1-sphere $\mathbb{S}^3$.

Consider that the coordinates split as $M = \{\mu, 10\}$, then the metric decomposes as

$$\begin{aligned}
\mathrm{d}s^2 &= \hat{G}_{MN}\mathrm{d}x^M \mathrm{d}x^N \\
&= \hat{G}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{4\phi/3}\left(dx^{10} + A_\nu^{(1)}\mathrm{d}x^\nu\right) \ .
\end{aligned} \qquad (A.25)$$

Therefore, from the eleven dimensional metric G_{MN}, we obtain the ten-dimensional metric $G_{\mu\nu}$, the dilaton ϕ and the graviphoton $A_\mu^{(1)}$. The ten-differential forms $A_{\mu\nu\rho}^{(3)}$, in the R-R sector, and the Kalb-Ramond field $B_{\mu\nu}$, in the NS-NS sector are

$$C_{\mu\nu\rho} \equiv A_{\mu\nu\rho}^{(3)} \ , \quad C_{\mu\nu 10} = B_{\mu\nu} \ . \qquad (A.26)$$

Therefore, if we call $H = \mathrm{d}B$, $F^{(2)}$, $\tilde{F}^{(4)} = \mathrm{d}A^{(3)} + A^{(1)} \wedge H$ the action of the type IIA string theory, obtained by dimensional reduction of the $d = 11$ supergravity is

$$\begin{aligned}
S_{IIA} &= \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G} e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H|^2 \right) \\
&\quad - \frac{1}{4\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G} \left(|F^{(2)}|^2 + |\tilde{F}^{(4)}|^2 \right) - \frac{1}{4\kappa_{10}^2} \int B \wedge \mathrm{d}A^{(3)} \wedge \mathrm{d}A^{(3)} \ ,
\end{aligned} \qquad (A.27)$$

where $\kappa_{10}^2 = \kappa_{11}^2/2\pi R$.

Finally we may observe an important property of the type IIA supergravity. Associated to this theory we have a 2-form and a 4-form field strengths, and using the Hodge duality we can find a 6-form and an 8-form, that is

$$\tilde{F}_6 = *\tilde{F}_4 \ , \qquad \tilde{F}_8 = *F_2 \ . \qquad (A.28)$$

Romans showed in [215] (see for a review [93]) that we can find a deformation of this theory that is not related to the eleven dimensional supergravity. The main observation is that if we consider a 10-form field strength $F_{10} = \mathrm{d}A^{(9)}$, whose equation of motion implies that $\mathrm{d} * F_{10} = 0$ we can introduce a parameter

$$m \equiv *F_{10} \qquad (A.29)$$

in our theory. At the end we find a theory defined by the action

$$S_{IIA}^m = \tilde{S}_{IIA} - \frac{1}{4\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G} m^2 + \frac{1}{2\kappa_{10}^2} \int m F_{10} \, , \qquad (A.30)$$

where $\tilde{S}_{IIA}$ is simply (A.27) but with the replacements

$$F_2 \to F_2 + mB \, , \quad \tilde{F}_4 \to \tilde{F}_4 + \frac{m}{2} B \wedge B \, . \qquad (A.31)$$

A.4.2   Type IIB Supergravity

The massless bosonic field content of the type IIB string theory in ten dimensions is

NS-NS sector: $G_{\mu\nu}$, $B_{\mu\nu}$, ϕ
R-R sector: $A_\mu^{(0)}, A_{\mu\nu}^{(2)}, A_{\mu\nu\rho\sigma}^{(4)}$

and the low energy limit is described by the action

$$S_{IIB} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G} e^{-2\phi} \left(R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}|H|^2 \right)$$
$$- \frac{1}{4\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G} \left(|F^{(1)}|^2 + |\tilde{F}^{(3)}|^2 + \frac{1}{2}|\tilde{F}^{(5)}|^2 \right) - \frac{1}{4\kappa_{10}^2} \int A^{(4)} \wedge H \wedge F^{(3)} \, ,$$
$$(A.32)$$

where

$$\tilde{F}^{(3)} = \mathrm{d}A^{(2)} + A^{(0)}H \, , \quad \tilde{F}^{(5)} = \mathrm{d}A^{(4)} - \frac{1}{2}A^{(2)} \wedge H + \frac{1}{2}B \wedge \mathrm{d}A^{(2)} \, . \qquad (A.33)$$

The equation of motion and the Bianchi identity for $\tilde{F}^{(5)}$ are

$$\mathrm{d} * \tilde{F}^{(5)} = \mathrm{d}\tilde{F}^{(5)} = H \wedge \tilde{F}^{(3)} \, , \qquad (A.34)$$

which is consistent with the constraint

$$\tilde{F}^{(5)} = *\tilde{F}^{(5)} \, . \qquad (A.35)$$

A.4.3 Einstein and String Frame

Finally, the Einstein-Hilbert action in d-dimensions is simply

$$S_{EH} = \frac{1}{2\kappa_d^2} \int \mathrm{d}^d x \sqrt{-G} R \; , \tag{A.36}$$

but in (A.27) and (A.32) we have a factor of $e^{-2\phi}$ in front of the Ricci scalar. This is called, the string frame. We can write the action in the usual Einstein frame (A.36) if we make the transformation

$$G_{\mu\nu} \to G_{\mu\nu}^E = e^{-\phi/2} G_{\mu\nu} \;\Rightarrow\; \sqrt{-G} = e^{5\phi/2} \sqrt{-G^E} \; . \tag{A.37}$$

Evidently, the differential form F_p does not change after this transformation, but the Hodge dual $*F_p$ does, since

$$(*F)_{\mu_{p+1}\cdots\mu_d} = \frac{\sqrt{-G}}{p!} \epsilon_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_d} G^{\mu_1\nu_1} \cdots G^{\mu_p\nu_p} F_{\nu_1\cdots\nu_p} \; , \tag{A.38}$$

so that

$$*F = \frac{1}{(n-p)!} (*F)_{\mu_{p+1}\cdots\mu_d} \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_d} \; , \tag{A.39}$$

and in the Einstein frame we have

$$(*F)_{\mu_{p+1}\cdots\mu_d}^E = e^{(p-5)\phi/2} (*F)_{\mu_{p+1}\cdots\mu_d} \; . \tag{A.40}$$

Bibliography

[1] G. Hardy, *A Mathematician's Apology.* Cambridge paperbacks. Cambridge University Press, 1992.

[2] T. Buscher, "Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models," *Phys.Lett.* **B201** (1988) 466.

[3] X. C. de la Ossa and F. Quevedo, "Duality symmetries from nonAbelian isometries in string theory," *Nucl.Phys.* **B403** (1993) 377–394, `arXiv:hep-th/9210021 [hep-th]`.

[4] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int.J.Theor.Phys.* **38** (1999) 1113–1133, `arXiv:hep-th/9711200 [hep-th]`.

[5] S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys.Lett.* **B428** (1998) 105–114, `arXiv:hep-th/9802109 [hep-th]`.

[6] E. Witten, "Anti-de Sitter space and holography," *Adv.Theor.Math.Phys.* **2** (1998) 253–291, `arXiv:hep-th/9802150 [hep-th]`.

[7] J. M. Maldacena and H. S. Nastase, "The Supergravity dual of a theory with dynamical supersymmetry breaking," *JHEP* **09** (2001) 024, `arXiv:hep-th/0105049 [hep-th]`.

[8] J. M. Maldacena and C. Nunez, "Towards the large N limit of pure N=1 superYang-Mills," *Phys. Rev. Lett.* **86** (2001) 588–591, `arXiv:hep-th/0008001 [hep-th]`.

[9] F. Canoura, P. Merlatti, and A. V. Ramallo, "The Supergravity dual of 3d supersymmetric gauge theories with unquenched flavors," *JHEP* **05** (2008) 011, `arXiv:0803.1475 [hep-th]`.

[10] J. Polchinski, *String Theory: Volume 1, An Introduction to the Bosonic String & Volume 2, Superstring Theory and Beyond.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1998.

[11] K. Becker, M. Becker, and J. Schwarz, *String Theory and M-Theory: A Modern Introduction.* Cambridge University Press, 2006.

[12] R. Blumenhagen, D. Lüst, and S. Theisen, *Basic Concepts of String Theory.* Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 2012.

[13] P. West, *Introduction to Strings and Branes.* Cambridge University Press, 2012.

[14] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys.Rept.* **323** (2000) 183–386, `arXiv:hep-th/9905111 [hep-th]`.

[15] H. Nastase, *Introduction to the AdS/CFT Correspondence.* Cambridge University Press, 2015.

[16] J. H. Schwarz and A. Sen, "Duality symmetries of 4-D heterotic strings," *Phys. Lett.* **B312** (1993) 105–114, `arXiv:hep-th/9305185 [hep-th]`.

[17] A. Sen, "Dyon - monopole bound states, selfdual harmonic forms on the multi - monopole moduli space, and SL(2,Z) invariance in string theory," *Phys. Lett.* **B329** (1994) 217–221, `arXiv:hep-th/9402032 [hep-th]`.

[18] E. Frenkel, "Lectures on the Langlands program and conformal field theory," in *DARPA Workshop on Langlands Program and Physics Princeton, New Jersey, March 8-10, 2005.* 2005. `arXiv:hep-th/0512172 [hep-th]`.

[19] E. Frenkel, "Gauge Theory and Langlands Duality," in *Bourbaki Seminar Paris, France, June 13-14, 2009.* 2009. `arXiv:0906.2747 [math.RT]`.

[20] A. Strominger, S.-T. Yau, and E. Zaslow, "Mirror symmetry is T duality," *Nucl. Phys.* **B479** (1996) 243–259, `arXiv:hep-th/9606040 [hep-th]`.

[21] B. R. Greene, "String theory on Calabi-Yau manifolds," in *Fields, strings and duality. Proceedings, Summer School, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI'96, Boulder, USA, June 2-28, 1996*, pp. 543–726. 1996. `arXiv:hep-th/9702155 [hep-th]`.

[22] K. Sfetsos and D. C. Thompson, "On non-abelian T-dual geometries with Ramond fluxes," *Nucl.Phys.* **B846** (2011) 21–42, `arXiv:1012.1320 [hep-th]`.

[23] G. Itsios, C. Núñez, K. Sfetsos, and D. C. Thompson, "Non-Abelian T-duality and the AdS/CFT correspondence: new N=1 backgrounds," *Nucl.Phys.* **B873** (2013) 1–64, `arXiv:1301.6755 [hep-th]`.

[24] I. R. Klebanov and E. Witten, "Superconformal field theory on three-branes at a Calabi-Yau singularity," *Nucl. Phys.* **B536** (1998) 199–218, `arXiv:hep-th/9807080 [hep-th]`.

[25] G. Itsios, C. Núñez, K. Sfetsos, and D. C. Thompson, "On Non-Abelian T-Duality and new N=1 backgrounds," *Phys.Lett.* **B721** (2013) 342–346, `arXiv:1212.4840`.

[26] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Supersymmetric AdS(5) solutions of M theory," *Class. Quant. Grav.* **21** (2004) 4335–4366, `arXiv:hep-th/0402153 [hep-th]`.

[27] A. Donos and J. P. Gauntlett, "Flowing from AdS_5 to AdS_3 with $T^{1,1}$," *JHEP* **08** (2014) 006, `arXiv:1404.7133 [hep-th]`.

[28] C. R. Hagen, "Scale and conformal transformations in galilean-covariant field theory," *Phys. Rev.* **D5** (1972) 377–388.

[29] U. Niederer, "The maximal kinematical invariance group of the free Schrodinger equation.," *Helv. Phys. Acta* **45** (1972) 802–810.

[30] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," *Class.Quant.Grav.* **26** (2009) 224002, `arXiv:0903.3246 [hep-th]`.

[31] J. M. Maldacena, "Wilson loops in large N field theories," *Phys. Rev. Lett.* **80** (1998) 4859–4862, `arXiv:hep-th/9803002 [hep-th]`.

[32] J. Polchinski, "Introduction to Gauge/Gravity Duality," `arXiv:1010.6134 [hep-th]`.

[33] P. Anderson, *Basic Notions Of Condensed Matter Physics*. Advanced Books Classics Series. Westview Press, 2008.

[34] S. Sachdev, *Quantum Phase Transitions*. Troisième Cycle de la Physique. Cambridge University Press, 2011.

[35] Y. Nishida and D. T. Son, "Nonrelativistic conformal field theories," *Phys.Rev.* **D76** (2007) 086004, `arXiv:0706.3746 [hep-th]`.

[36] D. Son, "Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrodinger symmetry," *Phys.Rev.* **D78** (2008) 046003, `arXiv:0804.3972 [hep-th]`.

[37] K. Balasubramanian and J. McGreevy, "Gravity duals for non-relativistic CFTs," *Phys.Rev.Lett.* **101** (2008) 061601, `arXiv:0804.4053 [hep-th]`.

[38] C. P. Herzog, M. Rangamani, and S. F. Ross, "Heating up Galilean holography," *JHEP* **0811** (2008) 080, `arXiv:0807.1099 [hep-th]`.

[39] J. Maldacena, D. Martelli, and Y. Tachikawa, "Comments on string theory backgrounds with non-relativistic conformal symmetry," *JHEP* **0810** (2008) 072, `arXiv:0807.1100 [hep-th]`.

[40] A. Adams, K. Balasubramanian, and J. McGreevy, "Hot Spacetimes for Cold Atoms," *JHEP* **0811** (2008) 059, `arXiv:0807.1111 [hep-th]`.

[41] S. Kachru, X. Liu, and M. Mulligan, "Gravity duals of Lifshitz-like fixed points," *Phys.Rev.* **D78** (2008) 106005, `arXiv:0808.1725 [hep-th]`.

[42] P. Chaikin and T. Lubensky, *Principles of Condensed Matter Physics*. Cambridge University Press, 2000.

[43] A. V. Smilga, *Lectures on Quantum Chromodynamics*. World Scientific,, 2001.

[44] Y. Makeenko, "A Brief Introduction to Wilson Loops and Large N," *Phys. Atom. Nucl.* **73** (2010) 878–894, `arXiv:0906.4487 [hep-th]`.

[45] J. Sonnenschein, "What does the string / gauge correspondence teach us about Wilson loops?," arXiv:hep-th/0003032 [hep-th].

[46] C. Núñez, M. Piai, and A. Rago, "Wilson Loops in string duals of Walking and Flavored Systems," *Phys.Rev.* **D81** (2010) 086001, arXiv:0909.0748 [hep-th].

[47] A. Donos and J. P. Gauntlett, "Solutions of type IIB and D=11 supergravity with Schrodinger(z) symmetry," *JHEP* **0907** (2009) 042, arXiv:0905.1098 [hep-th].

[48] A. Donos and J. P. Gauntlett, "Lifshitz Solutions of D=10 and D=11 supergravity," *JHEP* **1012** (2010) 002, arXiv:1008.2062 [hep-th].

[49] K. Balasubramanian and K. Narayan, "Lifshitz spacetimes from AdS null and cosmological solutions," *JHEP* **1008** (2010) 014, arXiv:1005.3291 [hep-th].

[50] R. Gregory, S. L. Parameswaran, G. Tasinato, and I. Zavala, "Lifshitz solutions in supergravity and string theory," *JHEP* **12** (2010) 047, arXiv:1009.3445 [hep-th].

[51] S. S. Gubser, "Drag force in AdS/CFT," *Phys.Rev.* **D74** (2006) 126005, arXiv:hep-th/0605182 [hep-th].

[52] M. Chernicoff, J. A. Garcia, and A. Guijosa, "The Energy of a Moving Quark-Antiquark Pair in an N=4 SYM Plasma," *JHEP* **09** (2006) 068, arXiv:hep-th/0607089 [hep-th].

[53] C. P. Herzog, "Energy Loss of Heavy Quarks from Asymptotically AdS Geometries," *JHEP* **0609** (2006) 032, arXiv:hep-th/0605191 [hep-th].

[54] C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz, and L. G. Yaffe, "Energy loss of a heavy quark moving through N=4 supersymmetric Yang-Mills plasma," *JHEP* **07** (2006) 013, arXiv:hep-th/0605158 [hep-th].

[55] M. Green, J. Schwarz, and E. Witten, *Superstring Theory: Volume 1, Introduction & Volume 2, Loop Amplitudes, Anomalies and Phenomenology:*

25th Anniversary Edition. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2012.

[56] G. Veneziano, "Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories," *Nuovo Cim.* **A57** (1968) 190–197.

[57] P. Di Vecchia, "The Birth of string theory," *Lect. Notes Phys.* **737** (2008) 59–118, `arXiv:0704.0101 [hep-th]`.

[58] P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, and E. Witten, eds., *Quantum fields and strings: A course for mathematicians. Vol. 1, 2.* 1999.

[59] C. Vafa, "Geometric physics," in *International Congress of Mathematicians (ICM 1998) Berlin, Germany, August 18-27, 1998.* 1998. `arXiv:hep-th/9810149 [hep-th]`.

[60] J. Teschner, "Exact results on N=2 supersymmetric gauge theories," `arXiv:1412.7145 [hep-th]`.

[61] J. Polchinski, "Dualities of Fields and Strings," `arXiv:1412.5704 [hep-th]`.

[62] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory.* Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[63] S. R. Coleman, "The Quantum Sine-Gordon Equation as the Massive Thirring Model," *Phys. Rev.* **D11** (1975) 2088.

[64] J. von Delft and H. Schoeller, "Bosonization for beginners: Refermionization for experts," *Annalen Phys.* **7** (1998) 225–305, `arXiv:cond-mat/9805275 [cond-mat]`.

[65] D. Senechal, "An Introduction to bosonization," in *CRM Workshop on Theoretical Methods for Strongly Correlated Fermions Montreal, Quebec, Canada, May 26-30, 1999.* 1999. `arXiv:cond-mat/9908262 [cond-mat]`.

[66] P. A. M. Dirac, "Quantized Singularities in the Electromagnetic Field," *Proc. Roy. Soc. Lond.* **A133** (1931) 60–72.

[67] G. 't Hooft, "Magnetic Monopoles in Unified Gauge Theories," *Nucl. Phys.* **B79** (1974) 276–284.

[68] A. M. Polyakov, "Particle Spectrum in the Quantum Field Theory," *JETP Lett.* **20** (1974) 194–195. [Pisma Zh. Eksp. Teor. Fiz.20,430(1974)].

[69] L. Ryder, *Quantum Field Theory.* Cambridge University Press, 1996.

[70] M. Shifman and A. Yung, *Supersymmetric Solitons.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2009.

[71] M. Shifman and A. Yung, "Supersymmetric Solitons and How They Help Us Understand Non-Abelian Gauge Theories," *Rev. Mod. Phys.* **79** (2007) 1139, `arXiv:hep-th/0703267 [hep-th]`.

[72] R. Rajaraman, *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory.* North-Holland personal library. North-Holland Publishing Company, 1982.

[73] N. Manton and P. Sutcliffe, *Topological Solitons.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2004.

[74] D. Tong, "TASI lectures on solitons: Instantons, monopoles, vortices and kinks," in *Theoretical Advanced Study Institute in Elementary Particle Physics: Many Dimensions of String Theory (TASI 2005) Boulder, Colorado, June 5-July 1, 2005.* 2005. `arXiv:hep-th/0509216 [hep-th]`.

[75] P. Goddard, J. Nuyts, and D. I. Olive, "Gauge Theories and Magnetic Charge," *Nucl. Phys.* **B125** (1977) 1.

[76] C. Montonen and D. I. Olive, "Magnetic Monopoles as Gauge Particles?," *Phys. Lett.* **B72** (1977) 117.

[77] E. Witten and D. I. Olive, "Supersymmetry Algebras That Include Topological Charges," *Phys. Lett.* **B78** (1978) 97.

[78] H. Osborn, "Topological Charges for N=4 Supersymmetric Gauge Theories and Monopoles of Spin 1," *Phys. Lett.* **B83** (1979) 321.

[79] N. Seiberg, "Exact results on the space of vacua of four-dimensional SUSY gauge theories," *Phys. Rev.* **D49** (1994) 6857–6863, `arXiv:hep-th/9402044 [hep-th]`.

[80] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," *Nucl. Phys.* **B426** (1994) 19–52, `arXiv:hep-th/9407087 [hep-th]`. [Erratum: Nucl. Phys.B430,485(1994)].

[81] A. Kapustin and E. Witten, "Electric-Magnetic Duality And The Geometric Langlands Program," *Commun. Num. Theor. Phys.* **1** (2007) 1–236, `arXiv:hep-th/0604151 [hep-th]`.

[82] C. Vafa, "Geometric origin of Montonen-Olive duality," *Adv. Theor. Math. Phys.* **1** (1998) 158–166, `arXiv:hep-th/9707131 [hep-th]`.

[83] S. Gukov and E. Witten, "Gauge Theory, Ramification, And The Geometric Langlands Program," `arXiv:hep-th/0612073 [hep-th]`.

[84] J. Figueroa-O'Farrill, "Electromagnetic duality for children." 8 October 1998; Unpublished.

[85] A. Kapustin, "Lectures on electric-magnetic duality and the geometric langlands program." January 1, 2008; Unpublished.

[86] J. A. Harvey, "Magnetic monopoles, duality and supersymmetry," in *Fields, strings and duality. Proceedings, Summer School, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI'96, Boulder, USA, June 2-28, 1996*. 1996. `arXiv:hep-th/9603086 [hep-th]`.

[87] L. Alvarez-Gaume and F. Zamora, "Duality in quantum field theory and string theory," in *Fundamental particles and interactions. Proceedings, 1st Meeting on Frontiers in Contemporary Physics, FCP'97, Nashville, USA, May 11-16, 1997*. 1997. `arXiv:hep-th/9709180 [hep-th]`.

[88] P. Van Nieuwenhuizen, "Supergravity," *Phys. Rept.* **68** (1981) 189–398.

[89] M. J. Duff, B. E. W. Nilsson, and C. N. Pope, "Kaluza-Klein Supergravity," *Phys. Rept.* **130** (1986) 1–142.

[90] B. de Wit, "Supergravity," in *Unity from duality: Gravity, gauge theory and strings. Proceedings, NATO Advanced Study Institute, Euro Summer School, 76th session, Les Houches, France, July 30-August 31, 2001*, pp. 1–135. 2002. `arXiv:hep-th/0212245 [hep-th]`.

[91] D. Freedman and A. Van Proeyen, *Supergravity*. Cambridge University Press, 2012.

[92] E. Álvarez, L. Álvarez-Gaumé, and Y. Lozano, "An Introduction to T duality in string theory," *Nucl.Phys.Proc.Suppl.* **41** (1995) 1–20, `arXiv:hep-th/9410237 [hep-th]`.

[93] T. Ortín, *Gravity and Strings*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2004.

[94] E. Bergshoeff, C. M. Hull, and T. Ortin, "Duality in the type II superstring effective action," *Nucl. Phys.* **B451** (1995) 547–578, `arXiv:hep-th/9504081 [hep-th]`.

[95] C. Johnson, *D-Branes*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2006.

[96] S. F. Hassan, "T duality, space-time spinors and RR fields in curved backgrounds," *Nucl. Phys.* **B568** (2000) 145–161, `arXiv:hep-th/9907152 [hep-th]`.

[97] S. F. Hassan, "SO(d,d) transformations of Ramond-Ramond fields and space-time spinors," *Nucl. Phys.* **B583** (2000) 431–453, `arXiv:hep-th/9912236 [hep-th]`.

[98] M. Cvetic, H. Lu, C. N. Pope, and K. S. Stelle, "T duality in the Green-Schwarz formalism, and the massless / massive IIA duality map," *Nucl. Phys.* **B573** (2000) 149–176, `arXiv:hep-th/9907202 [hep-th]`.

[99] B. Kulik and R. Roiban, "T duality of the Green-Schwarz superstring," *JHEP* **09** (2002) 007, `arXiv:hep-th/0012010 [hep-th]`.

[100] R. Benichou, G. Policastro, and J. Troost, "T-duality in Ramond-Ramond backgrounds," *Phys. Lett.* **B661** (2008) 192–195, `arXiv:0801.1785 [hep-th]`.

[101] K. Sfetsos, K. Siampos, and D. C. Thompson, "Canonical pure spinor (Fermionic) T-duality," *Class. Quant. Grav.* **28** (2011) 055010, `arXiv:1007.5142 [hep-th]`.

[102] K. Hori, *Mirror Symmetry*. Clay mathematics monographs. American Mathematical Society, 2003.

[103] A. Neitzke and C. Vafa, "Topological strings and their physical applications," `arXiv:hep-th/0410178 [hep-th]`.

[104] D. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*. Mathematical surveys and monographs. American Mathematical Society, 1999.

[105] D. Joyce, *Riemannian Holonomy Groups and Calibrated Geometry*. Oxford graduate texts in mathematics. Oxford University Press, 2007.

[106] T. Hübsch, *Calabi-Yau Manifolds: A Bestiary for Physicists*. World Scientific, 1994.

[107] V. Bouchard, "Lectures on complex geometry, Calabi-Yau manifolds and toric geometry," `arXiv:hep-th/0702063 [HEP-TH]`.

[108] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," *Nucl. Phys.* **B72** (1974) 461.

[109] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014.

[110] A. V. Manohar, "Large N QCD," in *Probing the standard model of particle interactions. Proceedings, Summer School in Theoretical Physics, NATO Advanced Study Institute, 68th session, Les Houches, France, July 28-September 5, 1997. Pt. 1, 2*, pp. 1091–1169. 1998. `arXiv:hep-ph/9802419 [hep-ph]`.

[111] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS / CFT correspondence," in *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings*, pp. 3–158. 2002. `arXiv:hep-th/0201253 [hep-th]`.

[112] J. M. Maldacena, "TASI 2003 lectures on AdS / CFT," in *Progress in string theory. Proceedings, Summer School, TASI 2003, Boulder, USA, June 2-27, 2003*, pp. 155–203. 2003. `arXiv:hep-th/0309246 [hep-th]`.

[113] J. D. Edelstein and R. Portugues, "Gauge/string duality in confining theories," *Fortsch. Phys.* **54** (2006) 525–579, `arXiv:hep-th/0602021 [hep-th]`.

[114] H. Nastase, "Introduction to AdS-CFT," `arXiv:0712.0689 [hep-th]`.

[115] A. V. Ramallo, "Introduction to the AdS/CFT correspondence," *Springer Proc. Phys.* **161** (2015) 411–474, `arXiv:1310.4319 [hep-th]`.

[116] H. Nastase, *Introduction to the AdS/CFT Correspondence*. Cambridge University Press, 2015.

[117] E. Álvarez, L. Álvarez-Gaumé, and Y. Lozano, "On nonAbelian duality," *Nucl.Phys.* **B424** (1994) 155–183, `arXiv:hep-th/9403155 [hep-th]`.

[118] Y. Lozano, "NonAbelian duality and canonical transformations," *Phys. Lett.* **B355** (1995) 165–170, `arXiv:hep-th/9503045 [hep-th]`.

[119] K. Sfetsos and D. C. Thompson, "New $\mathcal{N} = 1$ supersymmetric AdS_5 backgrounds in Type IIA supergravity," *JHEP* **1411** (2014) 006, `arXiv:1408.6545 [hep-th]`.

[120] E. Gevorgyan and G. Sarkissian, "Defects, Non-abelian T-duality, and the Fourier-Mukai transform of the Ramond-Ramond fields," *JHEP* **1403** (2014) 035, `arXiv:1310.1264 [hep-th]`.

[121] Ö. Kelekci, Y. Lozano, N. T. Macpherson, and E. Ó Colgáin, "Supersymmetry and non-Abelian T-duality in type II supergravity," *Class.Quant.Grav.* **32** (2015) no. 3, 035014, `arXiv:1409.7406 [hep-th]`.

[122] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, and A. Van Proeyen, "New formulations of D = 10 supersymmetry and D8 - O8 domain walls," *Class. Quant. Grav.* **18** (2001) 3359–3382, `arXiv:hep-th/0103233 [hep-th]`.

[123] A. H. Chamseddine and M. S. Volkov, "NonAbelian vacua in D = 5, N=4 gauged supergravity," *JHEP* **04** (2001) 023, `arXiv:hep-th/0101202 [hep-th]`.

[124] M. Schvellinger and T. A. Tran, "Supergravity duals of gauge field theories from SU(2) x U(1) gauged supergravity in five-dimensions," *JHEP* **06** (2001) 025, `arXiv:hep-th/0105019 [hep-th]`.

[125] M. Bershadsky, C. Vafa, and V. Sadov, "D-branes and topological field theories," *Nucl. Phys.* **B463** (1996) 420–434, `arXiv:hep-th/9511222 [hep-th]`.

[126] J. M. Maldacena and C. Nunez, "Supergravity description of field theories on curved manifolds and a no go theorem," *Int. J. Mod. Phys.* **A16** (2001) 822–855, `arXiv:hep-th/0007018 [hep-th]`. [,182(2000)].

[127] A. Karch and E. Katz, "Adding flavor to AdS / CFT," *JHEP* **06** (2002) 043, `arXiv:hep-th/0205236 [hep-th]`.

[128] C. Nunez, A. Paredes, and A. V. Ramallo, "Flavoring the gravity dual of N=1 Yang-Mills with probes," *JHEP* **12** (2003) 024, `arXiv:hep-th/0311201 [hep-th]`.

[129] R. Casero, C. Nunez, and A. Paredes, "Towards the string dual of N=1 SQCD-like theories," *Phys. Rev.* **D73** (2006) 086005, `arXiv:hep-th/0602027 [hep-th]`.

[130] C. Nunez, A. Paredes, and A. V. Ramallo, "Unquenched Flavor in the Gauge/Gravity Correspondence," *Adv. High Energy Phys.* **2010** (2010) 196714, `arXiv:1002.1088 [hep-th]`.

[131] N. T. Macpherson, "The Holographic Dual of 2+1 Dimensional QFTs with N=1 SUSY and Massive Fundamental Flavours," *JHEP* **06** (2012) 136, `arXiv:1204.4222 [hep-th]`.

[132] N. T. Macpherson, "SuGra on G_2 Structure Backgrounds that Asymptote to AdS_4 and Holographic Duals of Confining $2 + 1d$ Gauge Theories with $N = 1$ SUSY," *JHEP* **04** (2013) 076, `arXiv:1301.5178 [hep-th]`.

[133] N. T. Macpherson, "Non-Abelian T-duality, G_2-structure rotation and holographic duals of $N = 1$ Chern-Simons theories," *JHEP* **11** (2013) 137, `arXiv:1310.1609 [hep-th]`.

[134] J. Gaillard and D. Martelli, "Fivebranes and resolved deformed G_2 manifolds," *JHEP* **05** (2011) 109, `arXiv:1008.0640 [hep-th]`.

[135] D. Marolf, "Chern-Simons terms and the three notions of charge," in *Quantization, gauge theory, and strings. Proceedings, International Conference dedicated to the memory of Professor Efim Fradkin, Moscow, Russia, June 5-10, 2000. Vol. 1+2*, pp. 312–320. 2000. `arXiv:hep-th/0006117 [hep-th]`.

[136] S.-J. Rey and J.-T. Yee, "Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity," *Eur. Phys. J.* **C22** (2001) 379–394, `arXiv:hep-th/9803001 [hep-th]`.

[137] E. Caceres, N. T. Macpherson, and C. Núñez, "New Type IIB Backgrounds and Aspects of Their Field Theory Duals," *JHEP* **1408** (2014) 107, `arXiv:1402.3294 [hep-th]`.

[138] U. Kol, C. Nunez, D. Schofield, J. Sonnenschein, and M. Warschawski, "Confinement, Phase Transitions and non-Locality in the Entanglement Entropy," *JHEP* **06** (2014) 005, `arXiv:1403.2721 [hep-th]`.

[139] R. E. Arias and G. A. Silva, "Wilson loops stability in the gauge/string correspondence," *JHEP* **1001** (2010) 023, `arXiv:0911.0662 [hep-th]`.

[140] S. Ryu and T. Takayanagi, "Aspects of Holographic Entanglement Entropy," *JHEP* **08** (2006) 045, `arXiv:hep-th/0605073 [hep-th]`.

[141] T. Nishioka, S. Ryu, and T. Takayanagi, "Holographic Entanglement Entropy: An Overview," *J. Phys.* **A42** (2009) 504008, `arXiv:0905.0932 [hep-th]`.

[142] I. R. Klebanov, D. Kutasov, and A. Murugan, "Entanglement as a probe of confinement," *Nucl. Phys.* **B796** (2008) 274–293, `arXiv:0709.2140 [hep-th]`.

[143] A. Lewkowycz, "Holographic Entanglement Entropy and Confinement," *JHEP* **05** (2012) 032, `arXiv:1204.0588 [hep-th]`.

[144] B. S. Acharya and C. Vafa, "On domain walls of N=1 supersymmetric Yang-Mills in four-dimensions," `arXiv:hep-th/0103011 [hep-th]`.

[145] Y. Lozano, E. Ó Colgáin, and D. Rodríguez-Gómez, "Hints of 5d Fixed Point Theories from Non-Abelian T-duality," *JHEP* **1405** (2014) 009, `arXiv:1311.4842 [hep-th]`.

[146] N. T. Macpherson, C. Núñez, L. A. Pando Zayas, V. G. J. Rodgers, and C. A. Whiting, "Type IIB supergravity solutions with AdS_5 from Abelian and non-Abelian T dualities," *JHEP* **1502** (2015) 040, `arXiv:1410.2650 [hep-th]`.

[147] A. Barranco, J. Gaillard, N. T. Macpherson, C. Núñez, and D. C. Thompson, "G-structures and Flavouring non-Abelian T-duality," *JHEP* **08** (2013) 018, `arXiv:1305.7229 [hep-th]`.

[148] Y. Lozano and N. T. Macpherson, "A new AdS_4/CFT_3 dual with extended SUSY and a spectral flow," *JHEP* **1411** (2014) 115, `arXiv:1408.0912 [hep-th]`.

[149] Y. Lozano, E. Ó Colgáin, D. Rodríguez-Gómez, and K. Sfetsos, "Supersymmetric AdS_6 via T Duality," *Phys.Rev.Lett.* **110** (2013) no. 23, 231601, `arXiv:1212.1043 [hep-th]`.

[150] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Sasaki-Einstein metrics on S**2 x S**3," *Adv.Theor.Math.Phys.* **8** (2004) 711–734, `arXiv:hep-th/0403002 [hep-th]`.

[151] J. P. Gauntlett, D. Martelli, J. F. Sparks, and D. Waldram, "A New infinite class of Sasaki-Einstein manifolds," *Adv. Theor. Math. Phys.* **8** (2006) 987–1000, `arXiv:hep-th/0403038 [hep-th]`.

[152] T. R. Araujo and H. Nastase, "$\mathcal{N} = 1$ SUSY backgrounds with an AdS factor from non-Abelian T duality," *Phys. Rev.* **D91** (2015) no. 12, 126015, `arXiv:1503.00553 [hep-th]`.

[153] Y. Bea, J. D. Edelstein, G. Itsios, K. S. Kooner, C. Núñez, *et al.*, "Compactifications of the Klebanov-Witten CFT and new AdS$_3$ backgrounds," *JHEP* **1505** (2015) 062, `arXiv:1503.07527 [hep-th]`.

[154] N. T. Macpherson, C. Nunez, D. C. Thompson, and S. Zacarias, "Holographic Flows in non-Abelian T-dual Geometries," `arXiv:1509.04286 [hep-th]`.

[155] H. Dimov, S. Mladenov, R. C. Rashkov, and T. Vetsov, "Non-abelian T-duality of Pilch-Warner background," `arXiv:1511.00269 [hep-th]`.

[156] J. P. Gauntlett, E. O Colgain, and O. Varela, "Properties of some conformal field theories with M-theory duals," *JHEP* **02** (2007) 049, `arXiv:hep-th/0611219 [hep-th]`.

[157] E. Witten, "Baryons and branes in anti-de Sitter space," *JHEP* **07** (1998) 006, `arXiv:hep-th/9805112 [hep-th]`.

[158] N. Bobev, A. Kundu, and K. Pilch, "Supersymmetric IIB Solutions with Schrodinger Symmetry," *JHEP* **0907** (2009) 107, `arXiv:0905.0673 [hep-th]`.

[159] A. Donos and J. P. Gauntlett, "Supersymmetric solutions for non-relativistic holography," *JHEP* **0903** (2009) 138, `arXiv:0901.0818 [hep-th]`.

[160] H. Ooguri and C.-S. Park, "Supersymmetric non-relativistic geometries in M-theory," *Nucl.Phys.* **B824** (2010) 136–153, `arXiv:0905.1954 [hep-th]`.

[161] A. Donos and J. P. Gauntlett, "Schrodinger invariant solutions of type IIB with enhanced supersymmetry," *JHEP* **0910** (2009) 073, `arXiv:0907.1761 [hep-th]`.

[162] H. Singh, "Galilean type IIA backgrounds and a map," *Mod.Phys.Lett.* **A26** (2011) 1443–1451, `arXiv:1007.0866 [hep-th]`.

[163] J. Jeong, H.-C. Kim, S. Lee, E. Ó Colgáin, and H. Yavartanoo, "Schrödinger invariant solutions of M-theory with Enhanced Supersymmetry," *JHEP* **1003** (2010) 034, `arXiv:0911.5281 [hep-th]`.

[164] S. M. Ko, C. Melby-Thompson, R. Meyer, and J.-H. Park, "Dynamics of Perturbations in Double Field Theory and Non-Relativistic String Theory," arXiv:1508.01121 [hep-th].

[165] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Supersymmetric AdS backgrounds in string and M-theory," arXiv:hep-th/0411194 [hep-th].

[166] U. Gran, G. Papadopoulos, and D. Roest, "Systematics of M-theory spinorial geometry," *Class.Quant.Grav.* **22** (2005) 2701–2744, arXiv:hep-th/0503046 [hep-th].

[167] C. I. Lazaroiu and E.-M. Babalic, "Geometric algebra and M-theory compactifications," *Rom.J.Phys.* **58** (2013) 5–6, arXiv:1301.5094 [hep-th].

[168] M. Rocek and E. P. Verlinde, "Duality, quotients, and currents," *Nucl.Phys.* **B373** (1992) 630–646, arXiv:hep-th/9110053 [hep-th].

[169] A. Giveon and M. Rocek, "Generalized duality in curved string backgrounds," *Nucl.Phys.* **B380** (1992) 128–146, arXiv:hep-th/9112070 [hep-th].

[170] E. Álvarez, L. Álvarez-Gaumé, J. Barbon, and Y. Lozano, "Some global aspects of duality in string theory," *Nucl.Phys.* **B415** (1994) 71–100, arXiv:hep-th/9309039 [hep-th].

[171] A. Giveon and M. Rocek, "On nonAbelian duality," *Nucl.Phys.* **B421** (1994) 173–190, arXiv:hep-th/9308154 [hep-th].

[172] M. Gasperini, R. Ricci, and G. Veneziano, "A Problem with nonAbelian duality?," *Phys.Lett.* **B319** (1993) 438–444, arXiv:hep-th/9308112 [hep-th].

[173] Y. Lozano, E. Ó Colgáin, K. Sfetsos, and D. C. Thompson, "Non-abelian T-duality, Ramond Fields and Coset Geometries," *JHEP* **1106** (2011) 106, arXiv:1104.5196 [hep-th].

[174] J. Gaillard, N. T. Macpherson, C. Núñez, and D. C. Thompson, "Dualising the Baryonic Branch: Dynamic SU(2) and confining backgrounds in IIA," *Nucl. Phys.* **B884** (2014) 696–740, arXiv:1312.4945 [hep-th].

[175] K. S. Kooner and S. Zacarías, "Non-Abelian T-Dualizing the Resolved Conifold with Regular and Fractional D3-Branes," *JHEP* **08** (2015) 143, `arXiv:1411.7433 [hep-th]`.

[176] Y. Lozano, N. T. Macpherson, J. Montero, and E. Ó Colgáin, "New $AdS_3 \times S^2$ T-duals with $\mathcal{N} = (0,4)$ supersymmetry," *JHEP* **08** (2015) 121, `arXiv:1507.02659 [hep-th]`.

[177] Y. Lozano, N. T. Macpherson, and J. Montero, "A $\mathcal{N} = 2$ Supersymmetric AdS_4 Solution in M-theory with Purely Magnetic Flux," `arXiv:1507.02660 [hep-th]`.

[178] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," *JHEP* **0810** (2008) 091, `arXiv:0806.1218 [hep-th]`.

[179] T. Nishioka and T. Takayanagi, "On Type IIA Penrose Limit and N=6 Chern-Simons Theories," *JHEP* **0808** (2008) 001, `arXiv:0806.3391 [hep-th]`.

[180] M. Cvetic, H. Lu, and C. Pope, "Consistent warped space Kaluza-Klein reductions, half maximal gauged supergravities and CP**n constructions," *Nucl.Phys.* **B597** (2001) 172–196, `arXiv:hep-th/0007109 [hep-th]`.

[181] C. M. Hull, "Massive string theories from M theory and F theory," *JHEP* **11** (1998) 027, `arXiv:hep-th/9811021 [hep-th]`.

[182] A. Smilga, *Lectures on Quantum Chromodynamics*. World Scientific, 2001.

[183] J. Erdmenger and M. Ammon, *Gauge/gravity duality. Foundations and applications*. Cambridge University Press, 2015.

[184] T. R. Araujo and H. Nastase, "Comments on the T-dual of the gravity dual of D5-branes on S³," *JHEP* **1504** (2015) 081, `arXiv:1409.7333 [hep-th]`.

[185] U. H. Danielsson and L. Thorlacius, "Black holes in asymptotically Lifshitz spacetime," *JHEP* **0903** (2009) 070, `arXiv:0812.5088 [hep-th]`.

[186] J. Kluson, "Open String in Non-Relativistic Background," *Phys.Rev.* **D81** (2010) 106006, `arXiv:0912.4587 [hep-th]`.

[187] K. B. Fadafan, "Drag force in asymptotically Lifshitz spacetimes,"
`arXiv:0912.4873 [hep-th]`.

[188] H. M. Siahaan, "On non-relativistic $Q\bar{Q}$ potential via Wilson loop in
Galilean space-time," *Mod.Phys.Lett.* **A26** (2011) 1719–1724,
`arXiv:1106.5008 [hep-th]`.

[189] K. B. Fadafan and F. Saiedi, "On Holographic Non-relativistic Schwinger
Effect," `arXiv:1504.02432 [hep-th]`.

[190] C. Bachas, "Convexity of the Quarkonium Potential," *Phys.Rev.* **D33** (1986)
2723.

[191] A. Jeffrey and D. Zwillinger, *Table of Integrals, Series, and Products.*
Elsevier Science, 2000.

[192] P. Byrd and M. Friedman, *Handbook of Elliptic Integrals for Engineers and
Physicists.* Grundlehren der mathematischen Wissenschaften. Springer
Berlin Heidelberg, 2013.

[193] A. Akhavan, M. Alishahiha, A. Davody, and A. Vahedi, "Non-relativistic
CFT and Semi-classical Strings," *JHEP* **0903** (2009) 053, `arXiv:0811.3067
[hep-th]`.

[194] Y. Kinar, E. Schreiber, and J. Sonnenschein, "Q anti-Q potential from
strings in curved space-time: Classical results," *Nucl. Phys.* **B566** (2000)
103–125, `arXiv:hep-th/9811192 [hep-th]`.

[195] T. R. Araujo and H. Nastase, "Non-Abelian T-duality for nonrelativistic
holographic duals," *JHEP* **11** (2015) 203, `arXiv:1508.06568 [hep-th]`.

[196] K. Balasubramanian and J. McGreevy, "The Particle number in Galilean
holography," *JHEP* **1101** (2011) 137, `arXiv:1007.2184 [hep-th]`.

[197] N. Drukker, D. J. Gross, and H. Ooguri, "Wilson loops and minimal
surfaces," *Phys. Rev.* **D60** (1999) 125006, `arXiv:hep-th/9904191
[hep-th]`.

[198] G. T. Horowitz and S. F. Ross, "Naked black holes," *Phys. Rev.* **D56** (1997)
2180–2187, `arXiv:hep-th/9704058 [hep-th]`.

[199] M. Blau, J. Hartong, and B. Rollier, "Geometry of Schrodinger Space-Times, Global Coordinates, and Harmonic Trapping," *JHEP* **07** (2009) 027, `arXiv:0904.3304 [hep-th]`.

[200] S. S. Gubser, "Curvature singularities: The Good, the bad, and the naked," *Adv. Theor. Math. Phys.* **4** (2000) 679–745, `arXiv:hep-th/0002160 [hep-th]`.

[201] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis, and R. Meyer, "Effective Holographic Theories for low-temperature condensed matter systems," *JHEP* **11** (2010) 151, `arXiv:1005.4690 [hep-th]`.

[202] M. Lippert, R. Meyer, and A. Taliotis, "A holographic model for the fractional quantum Hall effect," *JHEP* **01** (2015) 023, `arXiv:1409.1369 [hep-th]`.

[203] T. Andrade, Y. Lei, and S. F. Ross, "Scattering amplitudes in Lifshitz spacetime," *Class. Quant. Grav.* **31** (2014) no. 21, 215002, `arXiv:1406.6389 [hep-th]`.

[204] A. Van Proeyen, "Tools for supersymmetry," in *Proceedings, Spring School on Quantum Field Theory. Supersymmetries and Superstrings*, vol. 9, pp. I.1–I.48. 1999. `arXiv:hep-th/9910030 [hep-th]`.

[205] R. Slansky, "Group Theory for Unified Model Building," *Phys. Rept.* **79** (1981) 1–128.

[206] J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2003.

[207] B. Russell, The Study of Mathematics *in Mysticism and Logic: And Other Essays*. Longmans, Green and Company, 1919.

[208] B. de Wit and J. Louis, "Supersymmetry and dualities in various dimensions," in *Strings, branes and dualities. Proceedings, NATO Advanced Study Institute, Cargese, France, May 26-June 14, 1997*. 1997. `arXiv:hep-th/9801132 [hep-th]`.

[209] E. Cremmer, B. Julia, and J. Scherk, "Supergravity Theory in Eleven-Dimensions," *Phys. Lett.* **B76** (1978) 409–412.

[210] H. Nastase, "Introduction to Supergravity," `arXiv:1112.3502 [hep-th]`.

[211] T. Appelquist, A. Chodos, and P. G. O. Freund, eds., *Modern Kaluza-Klein Theories.* 1987.

[212] M. J. Duff, ed., *The world in eleven-dimensions: Supergravity, supermembranes and M theory.* 1999.

[213] W. Nahm, "Supersymmetries and their Representations," *Nucl. Phys.* **B135** (1978) 149.

[214] E. Witten, "Search for a Realistic Kaluza-Klein Theory," *Nucl. Phys.* **B186** (1981) 412.

[215] L. J. Romans, "Massive N=2a Supergravity in Ten-Dimensions," *Phys. Lett.* **B169** (1986) 374.