Phokhara at the frontier of NNLO

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Electron–positron annihilation into hadrons accompanied by an energetic photon provides a powerful tool to measure the hadronic cross-section across a broad energy range at high-luminosity flavour factories such as DAPHNE, CESR, PEP-II, KEK-B, SuperKEKB, and BESIII. The Monte Carlo event generator Phokhara has been widely used and simulates this process with next-to-leading order (NLO) accuracy, incorporating both virtual and soft photon corrections for single-photon emission, as well as events with two hard real photons. In these proceedings, we present recent progress in extending Phokhara beyond NLO precision. The discussion is divided into two parts: (i) we present the plans for adding resummation and refining the hadronic current by incorporating the Generalised Vector Dominance Model as an improvement over the standard scalar QED approximation; and (ii) we outline the preliminary evaluation of two-loop Feynman integrals that form the building blocks of full NNLO predictions.

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1. Introduction and Outlook

With the recent release of the most precise experimental determination of the anomalous magnetic moment of the muon [1], together with the publication of a new White Paper by the $g-2$ Theory Initiative [2], and in light of forthcoming analyses and measurements from the BaBar, BESIII, Belle II, CMD-3, KLOE, and SND experiments, achieving higher theoretical precision in the predictions of both pair-production (also known as energy-scan) and radiative-return processes is of critical importance. The current state-of-the-art Monte Carlo (MC) event generators for the radiative return method, which relies on an energetic photon,

$$e^- e^+ \to X^- X^+ \gamma, \tag{1}$$

with $X \in \{\mu, \pi, e\}$, include NLO corrections with complete dependence on the mass of the initial and final states. A comparison of all available generators on the market and a validation of the different techniques to calculate such processes was recently done in [3].

The MC generator `Phokhara` [4] has been developed and refined for more than two decades; it is widely used by the experimental community to simulate radiative-return processes of interest. It incorporates several hadronic channels that cover a broad spectrum of baryonic and mesonic final states, including kaons, protons, neutrons, and η mesons, among others. However, the primary emphasis has been on the accurate simulation of muon- and pion-pair production accompanied by an energetic photon. Its predecessor, `EVA`, already implemented these reactions at leading order, including the dominant contributions from additional collinear radiation. This framework was subsequently extended in `Phokhara 3.0` to include next-to-leading order corrections for initial-state radiation (ISR), and later further improved to incorporate the complete set of NLO corrections in the final version released, `Phokhara 10`.

The renewed demand for more accurate electron–positron Monte Carlo simulations has revitalized the development of the `Phokhara` generator, with the first set of minor optimizations already implemented. A nonexhaustive list of these improvements, along with a validation of the generator through detailed comparisons with other available MC tools, was presented in [3]. The current development aims to enhance the theoretical precision of the generator along three main fronts:

- First, the fixed-order results already available are being improved through the resummation of soft-photon logarithms. Similar resummation techniques, such as parton showers and the use of collinear structures, have been successfully applied in other MC generators [5, 6].
- Second, the modeling of pions is being refined. Currently, `Phokhara` treats pions as point-like particles described by scalar QED (sQED) rules, weighted by the pion form factor, F_π, obtained from experimental fits. However, this approximation fails in certain regions of phase space where virtual photons can probe the internal composite structure of pions [7, 8].
- Finally, significant progress has been made towards implementing next-to-next-to-leading order (NNLO) predictions within the generator, marking a major step toward improving its overall theoretical robustness. First, the necessary technology to reconstruct QED scattering

amplitudes has been developed [9], and second a new integrator to evaluate the necessary topologies has been built [10].

In the first part, we describe the planned improvements to the Phokhara MC generator, addressing the three main development fronts. We also outline the principal challenges that must be overcome to ensure the successful implementation of these upgrades. In the second part, we focus on the NNLO challenge, and describe how it has been addressed through the development of a new integrator capable of delivering precise and efficient results for two-loop five-point massive Feynman integrals. In particular, we discuss its application to the computation of the pion radiative-return amplitude at NLO to higher orders in the dimensional regulator. Furthermore, we explore the performance of the integrator in the two-loop $pp \to t\bar{t} + \text{jet}$ process, which resembles the low-energy radiative return process, and comment on the possible use of the integrator in other high-energy processes.

2. Towards a new release of Phokhara MC generator

Advancing beyond NLO theoretical predictions represents a major challenge in precision collider phenomenology. To address the challenges that come with it, this section is organized into two subsections, each devoted to a distinct line of development. The first focuses on extending the fixed-order framework through the resummation of logarithmically enhanced soft-photon contributions. The second discusses improvements to the modelling of hadronic final states, with particular emphasis on a more accurate description of pion dynamics beyond the point-like approximation.

2.1 Resummation of soft photon logarithms

The use of resummation techniques has been explored in different approaches in various MC simulations. The generator Babayaga@NLO [5] has done so via parton showers combined with leading fixed order, and MCGPJ [6] and AfkQed have explored the inclusion of dominant radiative corrections from additional collinear radiation through structure function techniques. Furthermore, the generators KKMCee [11] and Sherpa rely on the Yennie–Frautschi–Suura (YFS) [12] exponentiation to resum all QED IR divergences to all orders. In particular, KKMCee uses the Coherent Exclusive Exponentiation (CEEX) [13], which also accounts for QED interference and inclusion of narrow resonances. The ongoing effort is to include the CEEX framework in Phokhara, with two crucial extensions. First, the combination of the resummed effects with fixed order NLO predictions, going beyond the current prediction by KKMCee, for both pion and muon radiative return modes. Second, the resummation of collinear logarithms, consistent with the CEEX framework, will be implemented. These two aspects will provide valuable insights on the impact of radiative corrections to the radiative processes and improve the accuracy of the predictions in key phase-space regions.

2.2 Pions beyond sQED

The conventional treatment of pions as point-like particles, weighted by an empirical form factor, can be improved. The significance of this refinement was extensively discussed in Ref. [7], where a new approach inspired by the Generalized Vector Meson Dominance (GVMD) model was

PoS(EPS-HEP2025)294

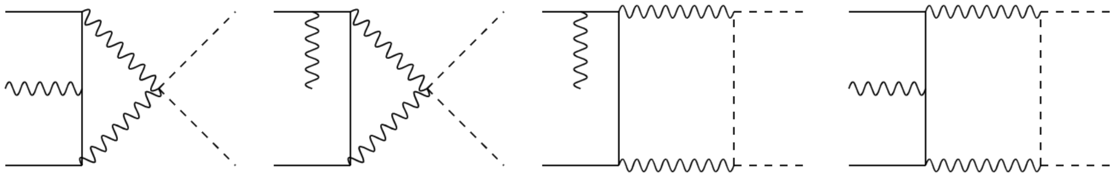

Figure 1: Diagrams for the ISR TVP gauge invariant group, omitting crossing of the external pions.

proposed and implemented for the energy-scan mode in MCGPJ. This model improves the prediction of two-virtual-photon (TVP) diagrams by incorporating the composite internal structure of the pion, particularly affecting configurations where one of the virtual photons is soft. Within the GVMD framework, the pion form factor is represented as a sum over Breit–Wigner resonance functions, which are incorporated into the sQED $\pi^+\pi^-\gamma$ vertex. Using standard loop-integration techniques, the modification of the Feynman rules can effectively be interpreted as assigning masses to the virtual photons, with values determined by the fitted parameters of the pion form factor. This modification preserves both the ultraviolet and the infrared finiteness of the resulting amplitudes. Although additional work is required to achieve an effective description of the vertex with one additional photon, $\pi^+\pi^- \to 3\gamma^{(*)}$, the impact of adopting GVMD—or its dispersive counterpart, FsQED [8]—in the radiative-return channel has not yet been investigated. Such an extension is expected to significantly enhance the accuracy of the predictions for ISR TVP diagrams.

3. Towards NNLO

A natural and necessary step toward improving theoretical precision is the inclusion of full or partial NNLO corrections to the amplitudes. However, the computation of $2 \to 3$ scattering amplitudes involving massive particles in both initial and final states presents substantial theoretical and computational challenges. In particular, the evaluation of the two-loop Feynman integrals arising in such processes constitutes one of the main bottlenecks. These integrals involve intricate analytic structures, a high dimensionality of the kinematic space and contain functions beyond polylogarithms. Developing efficient and reliable methods for their numerical evaluation is therefore of critical importance. In fact, the demand for fast and accurate computation of multi-scale, two-loop topologies has already emerged in related high-energy processes studied by the theoretical community [14], underscoring the relevance and urgency of advancing these techniques.

The method of canonical differential equations [15] has been successfully used to obtain analytic expressions for a wide range of multi-loop integrals. Furthermore, in the cases where analytic functions are costly to compute, solving the set of differential equations via the Frobenius method, as implemented in several public packages [16–18], has allowed for the construction of interpolation grids for $2 \to 2$ topologies. Nevertheless, when processes with higher dimensionality are considered, interpolation is not good enough, and one has to rely on on-the-fly evaluations. In such cases, sufficiently high precision and low CPU times are hard to reach. The method of PentagonFunctions [19–21], was developed for this specific purpose and has had great success with massless processes, as well as with an external massive leg. However, the presence of non-polylogarithmic functions in topologies with more scales is a significant challenge to overcome with potentially high impact on evaluation times. For this reason, we explore an alternative method, which is to solve the differential equations fully numerically, and develop a custom integrator.

3.1 A new integrator for Feynman integrals

Numerical solutions of differential equations constitute a vast and well-established field with a wide range of applications. In the context of high-energy physics, this approach has already been applied to the evaluation of Feynman integrals [22–24], particularly for $2 \to 2$ topologies. In Ref. [10], we extend and refine these techniques to address three specific gaps in the existing literature: (i) the automation of pathfinding between boundary conditions and the target evaluation point, (ii) the systematic exploitation of the analytical structure inherent in differential equations expressed in canonical or ϵ-factorized form, and (iii) the implementation of optimizations necessary to achieve minimal computational run-times.

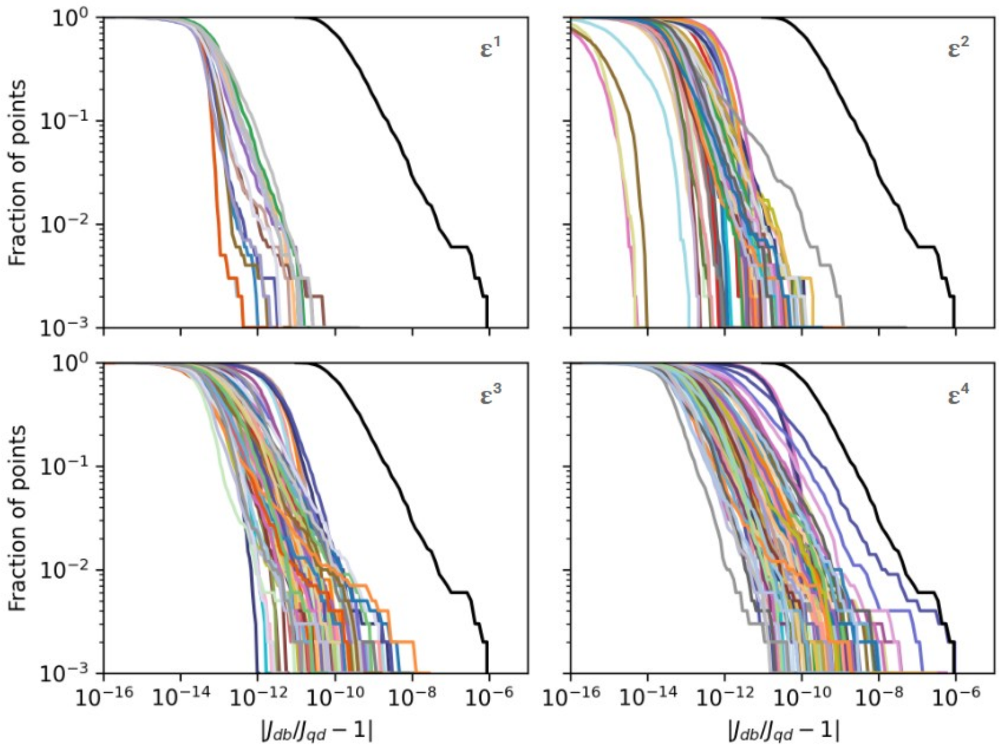

Figure 2: Cumulative sum of the relative difference between ϵ coefficients of the master integrals of PBb family of Ref.[14], as evaluated with double or quadruple precision for a thousand arbitrary points. The black line denotes the cumulative sum of all coefficients.

We demonstrate the impact of these improvements by evaluating all the master integrals of the elliptic PBb family from Ref. [14] up to ϵ^4, for the reaction $pp \to t\bar{t} + \text{jet}$. We show the precision reached by the integrator in Figure 2. The integrator takes, on average, $O(0.1)$ seconds per phase-space point to evaluate all coefficients with double precision. In quadruple precision, the time increases to the order of hundreds of seconds. With these small CPU times, we are confident that evaluating the two-loop integrals will not be the bottleneck to evaluating the NNLO-accurate amplitudes.

Acknowledgments

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