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H. Benzair · M. Merad · T. Boudjedaa

A Two-Body Problem for a Class of Coupling Potentials with Harmonic Oscillator Interaction in Noncommutative Phase Space: Path Integral Approach

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Abstract Recently Gnatenko et al. (J Phys Stud 21:4002, 2017) have treated the two-body problem with harmonic oscillator interaction in noncommutative phase space. According to the path integral formulation, we generalize this process by adding the coupling potentials ($\kappa \mathbf{P}^{(a)} \cdot \mathbf{P}^{(b)}$ and $\tilde{\kappa}\mathbf{X}^{(a)} \cdot \mathbf{X}^{(b)}$) together with a Harmonic oscillator interaction in noncommutative phase space. Moreover, in order to convert this problem into a one-particle system, we have added a third condition along with what this paper provided. As well as, when κ and $\tilde{\kappa}$ equal to zero exact expressions for the energy levels of a two-particle system with harmonic oscillator interaction in noncommutative phase space have been found.

1 Introduction

In recent years, systems with noncommutative phase space (NC) have attracted great interest through the multiplicity of its applications in the scientific arenas. This was due to the emergence of some modern theories such as; M-theory compactification, string theory, and quantum Hall effect [1–3], etc. Its idea was old since the mathematical instrument of quantum mechanics was introduced by the physicist Heisenberg in 1930, and was then formulated by Snyder in his famous paper [4]. While the spread of this paper [4] suffered some stagnation for a period of time until the emergence of recent theories. Furthermore, many researches have been published on the scientific scene in the last few years; we find the rotational symmetry can be broken in [5], and consequently, the energy levels of free particle and harmonic oscillator in rotationally invariant noncommutative phase space [6], and constraints of Charge-to-Mass Ratios on the parameters of noncommutative phase space [7], and quantum speed for a relativistic electron [8]. Without forgetting the great important papers devoted to the study of various aspects of quantum physics on NC space by several methods of the analytical and approximate resolution were applied [9–21].

Notably, the two-body system has been studied in a noncommutative phase space i.e., a canonical noncommutative algebra of coordinates and momenta [22–24]. Their work was very lucky when they proposed conditions on the parameters of noncommutativity to be able to reduce the two-particle problem to the one-particle problem. Our purpose is to generalize their work by adding terms which is not used in conventional

H. Benzair (✉)
Laboratoire LRPPS, Université de Ouargla, 30000 Ouargla, Algeria
E-mail: benzair.hadjira@gmail.com

M. Merad (✉)
Laboratoire L.S.D.C, Faculté des Sciences Exactes, Université de Oum El Bouaghi, 04000 Oum El Bouaghi, Algeria
E-mail: meradm@gmail.com

T. Boudjedaa
Laboratoire de Physique Théorique, Université de Jijel, BP 98, 18000 Ouled Aissa, Jijel, Algeria

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physics., called the coupling potentials, defined by $\kappa \mathbf{P}^{(a)} \cdot \mathbf{P}^{(b)}$, $\tilde{\kappa} \mathbf{X}^{(a)} \cdot \mathbf{X}^{(b)}$ in the same context of noncommutative phase space coordinates. Furthermore, the approximate theories require us to have this kind of coupling potentials to describe the motion of molecules, and it appears naturally in the dynamic Hamiltonian vibration describing molecules [25–28]. This type of interaction can be easily treated by the functional method, called path integral and presented in 1972 by Khandekar [29], and then by Christian Grosche [30].

A noncommutative phase space representation can be realized by the simultaneous coordinate operators space-space and momentum-momentum noncommutative relations satisfying

$$\left[\hat{X}_i, \hat{X}_j\right] = \imath\hbar\theta_{ij}, \left[\hat{X}_i, \hat{P}_j\right] = \imath\hbar\delta_{ij}, \left[\hat{P}_i, \hat{P}_j\right] = \imath\hbar\eta_{ij} \text{ with } i, j = 1, 2. \tag{1}$$

which represent the fundamental quantum canonical relation for the four-dimensional noncommutative phase space; i.e., 2D configuration space and 2D momentum space. Here θ_{ij} and η_{ij} are a constant antisymmetric tensor parameter with dimensions of θ_{ij} and η_{ij} as (length)^2 and (momentum)^2, and representing the non-commutativity properties of the phase-space. In fact, in the context of this deformation, the product of any two functions can be realized by substitution of the usual function product by the Weyl–Moyal star product

$$\begin{aligned}(f \star g)(\hat{X}, \hat{P}) &= \exp\left[\frac{i}{2}\left(\theta_{ij}\partial_i^x\partial_j^x + \eta_{ij}\partial_i^p\partial_j^p\right)\right] f(x,p)g(x,p)\\ &= f\left(x_i - \frac{1}{2}\theta_{ij}p_j;\, p_i + \frac{1}{2}\eta_{ij}x_j\right) g(x,p),\end{aligned} \tag{2}$$

where $f(x, p)$ and $g(x, p)$ are two arbitrary infinitely differentiable functions of the commutative variables x_i and p_i, i.e., they generate the Heisenberg algebra of quantum mechanics. However, in the case of two body problem, this Eq. (2) will be applied to the appropriate conditions making then this system to be reduced and solved by one particle case as explained in these works [22,24]. In the next section, we will mention this technique as a preparation for its generalization.

In this paper we study the non-relativistic propagator of the two-particle system with harmonic oscillator interaction, and with presence of the coupling potentials ($\kappa \mathbf{P}^{(a)} \cdot \mathbf{P}^{(b)}$, $\tilde{\kappa} \mathbf{X}^{(a)} \cdot \mathbf{X}^{(b)}$) in the context of a noncommutative phase space coordinates. In the absence of the parameters of non-commutativity, this generalization with only the term $\kappa \mathbf{P}^{(a)} \cdot \mathbf{P}^{(b)}$ has already been studied, for example, the propagator solutions with kinetic coupling potentials [30] and the three-body problem [29]. Furthermore, we find within the framework of non-commutative geometry such as the two-particle system in Coulomb potential for twist-deformed space-time [31], the two-body problems in deformed space with minimal length [32] and in a space with noncommutative coordinates and momenta [22].

We will discuss in the following section, a review of the most basic principles of the two-body system in noncommutative phase space coordinates with harmonic oscillator interaction and then in next section, we will present the two-body problem under harmonic oscillator interaction with kinetic coupling potential $\left(\kappa \mathbf{P}^{(a)} \cdot \mathbf{P}^{(b)}\right)$. In order to make this problem solvable, we add a third condition on the mass of particles, identical masses (i.e., $m_a = m_b$). To be free from this restrictive condition, in the forth section, we will provide exact solutions of normalized wave functions and their corresponding spectra energies by choosing the two-body problem for a class of coupling potentials ($\kappa \mathbf{P}^{(a)} \cdot \mathbf{P}^{(b)}$, $\tilde{\kappa} \mathbf{X}^{(a)} \cdot \mathbf{X}^{(b)}$) with Harmonic oscillator interactions in noncommutative phase space by using the path integral technique. As we will show, this added term $\tilde{\kappa} \mathbf{X}^{(a)} \cdot \mathbf{X}^{(b)}$ delivers us from the condition on the masses (i.e., $m_a = m_b$) but imposes a proportionality between the coefficients κ and $\tilde{\kappa}$. Finally a conclusion is given in Sect. 5.

2 Basic Principles of the Two-Body System in NC Phase Space

In the non-relativistic quantum mechanic for a two-body system with the presence of the noncommutative phase space, the coordinates $\mathbf{X}^{(a,b)}$ and the momenta $\mathbf{P}^{(a,b)}$ of a particle satisfy the following relations,

$$\left[X_1^{(a)}, X_2^{(b)}\right] = i\hbar\delta^{ab}\theta_a, \left[X_i^{(a)}, P_j^{(b)}\right] = i\hbar\delta^{ab}\delta_{ij}, \left[P_1^{(a)}, P_2^{(b)}\right] = i\hbar\delta^{ab}\eta_a, \tag{3}$$

where the symbols (a, b) indicate a number of particle (i.e., $a \equiv$ particle one and $b \equiv$ particle two). As well as the number-case ($i = 1, 2$) denotes the coordinates and the momenta components (i.e., $X_i \equiv (X, Y)$, $P_i \equiv (P_x, P_y)$). While $\theta_{a,b}$ and $\eta_{a,b}$ are antisymmetric non-commutativity parameters of a particle (a) or

(b), and their dimensions are (m^2) and ($kg^2 \times m^2$) respectively. Thanks to the Ref. [6] who have successfully addressed a two-particle system with harmonic oscillator interaction in noncommutative phase space, for this, we will present a brief overview of the main tools to enable separation of the motion of the center-of-mass $\hat{\mathbf{x}}$ and the relative motion $\hat{\mathbf{x}}^{(r)}$, given as:

$$\begin{cases} \hat{\mathbf{x}} = \mu_1 \hat{\mathbf{X}}^{(a)} + \mu_2 \hat{\mathbf{X}}^{(b)}, & \hat{\mathbf{x}}^{(r)} = \hat{\mathbf{X}}^{(a)} - \hat{\mathbf{X}}^{(b)}, \\ \hat{\mathbf{p}} = \hat{\mathbf{P}}^{(a)} + \hat{\mathbf{P}}^{(b)}, & \hat{\mathbf{p}}^{(r)} = \mu_2 \hat{\mathbf{P}}^{(a)} - \mu_1 \hat{\mathbf{P}}^{(b)} \end{cases} \tag{4}$$

where $\mu_i = m_i/M$, $i = a, b$, $(\mu_a + \mu_b = 1)$, $M = m_a + m_b$ is the total mass, and $\mu = m_a m_b/M$ is the reduced mass. The inverse transformations are

$$\begin{cases} \hat{\mathbf{X}}^{(a)} = \hat{\mathbf{x}} + \frac{m_b}{M} \hat{\mathbf{x}}^{(r)}, & \hat{\mathbf{X}}^{(b)} = \hat{\mathbf{x}} - \frac{m_a}{M} \hat{\mathbf{x}}^{(r)}, \\ \hat{\mathbf{P}}^{(a)} = \hat{\mathbf{p}}^{(r)} + \mu_a \hat{\mathbf{p}}, & \hat{\mathbf{P}}^{(b)} = \mu_b \hat{\mathbf{p}} - \hat{\mathbf{p}}^{(r)}. \end{cases} \tag{5}$$

Now using these transformations (5) along with the non-relativistic standard form of the two-particle Hamiltonian

$$\hat{H}_0 = \frac{(\mathbf{P}^{(a)})^2}{2m_a} + \frac{(\mathbf{P}^{(b)})^2}{2m_b} + V\left(\left|\mathbf{X}^{(a)} - \mathbf{X}^{(b)}\right|\right), \tag{6}$$

with $V\left(\left|\mathbf{X}^{(a)} - \mathbf{X}^{(b)}\right|\right)$ being the interaction potential energy between the two particles, the Hamiltonian (6) becomes as

$$\hat{H}_0 = \frac{(\hat{\mathbf{p}}^{(r)})^2}{2\mu} + \frac{\hat{\mathbf{p}}^2}{2M} + V\left(\hat{\mathbf{x}}^{(r)}\right). \tag{7}$$

From Ref. [22], they found that the commutators of the coordinates $\hat{\mathbf{x}}$ and the momenta $\hat{\mathbf{p}}$ of the center-of-mass satisfy the following relations after they substituted Eq. (5) in the commutations relations (3):

$$\left[\hat{x}_1, \hat{x}_2\right] = \imath\hbar\tilde{\theta}, \quad \left[\hat{p}_1, \hat{p}_2\right] = \imath\hbar\tilde{\eta}, \quad \left[\hat{x}_i, \hat{p}_j\right] = \imath\hbar\delta_{ij}. \tag{8}$$

As well as the commutation relations for the coordinates $\hat{\mathbf{x}}^{(r)}$ and momenta $\hat{\mathbf{p}}^{(r)}$ of the relative motion become as follow

$$\left[\hat{x}_1^{(r)}, \hat{x}_2^{(r)}\right] = \imath\hbar\tilde{\theta}^{(r)}, \quad \left[\hat{p}_1^{(r)}, \hat{p}_2^{(r)}\right] = \imath\hbar\tilde{\eta}^{(r)}, \quad \left[\hat{x}_i^{(r)}, \hat{p}_j^{(r)}\right] = \imath\hbar\delta_{ij}, \tag{9}$$

where

$$\tilde{\theta} = \mu_a^2\theta_a + \mu_b^2\theta_b, \quad \tilde{\eta} = \eta_a + \eta_b, \tag{10}$$

$$\tilde{\theta}^{(r)} = \theta_a + \theta_b, \quad \tilde{\eta}^{(r)} = \mu_a^2\eta_b + \mu_b^2\eta_a. \tag{11}$$

In addition, due to the application of non-commutative phase space on the systems that contain interaction between two particles [22,24], we can not return our problem to a single particle system because there are other commutation relations that are not equal zero and are defined as

$$\left[\hat{p}_1, \hat{p}_2^{(r)}\right] = -\left[\hat{p}_2, \hat{p}_1^{(r)}\right] = \imath\hbar\left(\mu_a\eta_b - \mu_b\eta_a\right), \quad \left[\hat{x}_i, \hat{x}_i^{(r)}\right] = \imath\hbar\left(\mu_a\theta_a - \mu_b\theta_b\right). \tag{12}$$

From these above commutation relations which explain the influence between the motion of the center-of-mass and the relative motion in noncommutative space, we can avoid this non-commutativity (12) in the following cases [22,24]:

$$\frac{\eta_{(a,b)}}{m_{(a,b)}} = \alpha, \quad \theta_{(a,b)} m_{(a,b)} = \gamma \tag{13}$$

with α, γ being constants which are the same for particles.

Note that in (13) there is no summation, and when $\eta_{(a,b)} = 0$, it represents the composite system in noncommutative space, which has been studied in Ref. [33]. These conditions give the possibility to solve a list of problems in the noncommutative phase space, among them, the dependence problem of the motion of the center-of-mass of a composite system on the relative motion; the violation problem of properties of the kinetic energy. It is clear that commutators (12) are equal to zero when relations (13) are satisfied. Therefore, the two-particle problem can be reduced to the problems of the motion of the center-of-mass and the relative motion, which can be studied separately. After these procedures, it is possible to convert these non-commutative variables of the coordinates $\hat{\mathbf{x}}$ and the momenta $\hat{\mathbf{p}}$ of the center-of-mass into the coordinates $\hat{\mathbf{q}}$ and the momenta $\hat{\mathbf{p}}_q$ satisfy the ordinary commutation relations using the following Boop's shift

$$\hat{p}_1 = \hat{p}_{q_1} + \frac{\tilde{\eta}}{2}\hat{q}_2, \hat{p}_2 = \hat{p}_{q_2} - \frac{\tilde{\eta}}{2}\hat{q}_1, \tag{14}$$

$$\hat{x}_1 = \hat{q}_1 - \frac{\tilde{\theta}}{2}\hat{p}_{q_2}, \hat{x}_2 = \hat{q}_2 + \frac{\tilde{\theta}}{2}\hat{p}_{q_1}, \tag{15}$$

as well as for the coordinates $\hat{\mathbf{x}}^{(r)}$ and momentum $\hat{\mathbf{p}}^{(r)}$ of the relative motion can be represented as

$$\hat{p}_1^{(r)} = \sqrt{\frac{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{2\zeta}}\left(\hat{p}_{q_1}^{(r)} + \frac{\zeta}{\tilde{\theta}^{(r)}}\hat{q}_2^{(r)}\right), \hat{p}_2^{(r)} = \sqrt{\frac{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{2\zeta}}\left(\hat{p}_{q_2}^{(r)} - \frac{\zeta}{\tilde{\theta}^{(r)}}\hat{q}_1^{(r)}\right), \tag{16}$$

$$\hat{x}_1^{(r)} = \sqrt{\frac{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{2\zeta}}\left(\hat{q}_1^{(r)} - \frac{\zeta}{\tilde{\eta}^{(r)}}\hat{p}_{q_2}^{(r)}\right), \hat{x}_2^{(r)} = \sqrt{\frac{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{2\zeta}}\left(\hat{q}_2^{(r)} + \frac{\zeta}{\tilde{\eta}^{(r)}}\hat{p}_{q_1}^{(r)}\right), \tag{17}$$

where the coordinates $\hat{\mathbf{q}}^{(r)}$ and the momenta $\hat{\mathbf{p}}_q^{(r)}$ satisfy the ordinary commutation relations and $\zeta = 1 - (1 - \tilde{\theta}^{(r)}\tilde{\eta}^{(r)})^{1/2}$. As a result, substituting the above coordinates variables (14), (15), (16), (8) into the expression of $\hat{H}_0$ related with Harmonic oscillator potential ($V(r) = \frac{\mu\omega^2}{2}r^2$), and with consideration of the two conditions (13), one can get:

$$\hat{H}_0 = \hat{H}_0^{(c)} + \hat{H}_0^{(r)} \tag{18}$$

where the symbols (c) and (r) indicate to the center-of-mass and relative motions, respectively. Hence, the Hamiltonian of the center-of-mass $\hat{H}_0^{(c)}$ is given as

$$\hat{H}_0^{(c)} = \frac{1}{2M}\hat{\mathbf{p}}_q^2 + \frac{\tilde{\eta}^2}{8M}\hat{\mathbf{q}}^2 - \frac{\tilde{\eta}}{2M}\left(\hat{q}_1\hat{p}_{q_2} - \hat{q}_2\hat{p}_{q_1}\right), \tag{19}$$

while the Hamiltonian of the relative motion $\hat{H}_{rel}$ is defined as

$$\hat{H}_0^{(r)} = \frac{1}{2M^{(r)}}(\hat{\mathbf{p}}_q^{(r)})^2 + \left(\frac{\tilde{\eta}^{(r)}\zeta}{4\mu\tilde{\theta}^{(r)}} + \frac{\mu\omega^2\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{4\zeta}\right)(\hat{\mathbf{q}}^{(r)})^2 + \left(\frac{\tilde{\eta}^{(r)}}{2\mu} + \frac{\mu\omega\tilde{\theta}^{(r)}}{2}\right)\left(\hat{q}_1^{(r)}\hat{p}_{q_2}^{(r)} - \hat{q}_2^{(r)}\hat{p}_{q_1}^{(r)}\right) \tag{20}$$

which correspond to the following eigenvalues:

$$E^{(c)} = \hbar\Omega_0^{(c)}\left(n_1 + \frac{1}{2}\right), \quad n_1 = 0, 1, 2, \ldots, \tag{21}$$

and

$$E^{(r)} = \hbar\Omega_0^{(r)}(2n_2 + |\ell_2| + 1) - \ell_2\omega_0^{(r)}, \quad n_2 = 0, 1, 2, \ldots \tag{22}$$

where

$$\Omega_0^{(c)} = \frac{\tilde{\eta}}{M}, \quad \Omega_0^{(r)} = \sqrt{\frac{2}{M^{(r)}}\left(\frac{\tilde{\eta}^{(r)}\zeta}{4\mu\tilde{\theta}^{(r)}} + \frac{\mu\omega^2\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{4\zeta}\right)}, \tag{23}$$

and

$$M^{(r)} = \left[\frac{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{2\mu\zeta} + \frac{\mu\omega^2\tilde{\theta}^{(r)}\zeta}{2\tilde{\eta}^{(r)}}\right]^{-1}, \quad \omega_0^{(r)} = \frac{\tilde{\eta}^{(r)}}{2\mu} + \frac{\mu\omega\tilde{\theta}^{(r)}}{2}. \tag{24}$$

Consequently, in this review, we have presented the basic principles of the two-body system with harmonic oscillator interaction in NC phase space, which corresponds to two parts of the movement, one a free particle in a uniform magnetic field $B_1 = M\Omega_0^{(c)}$ and the other is two-dimensional harmonic oscillator with magnetic field $B_2 = M^{(r)}\omega_0^{(r)}$ with frequency $\Omega_0^{(r)}$. That depends on the effective parameters of non-commutativity $\tilde{\eta}$, $\tilde{\theta}^{(r)}$ and $\tilde{\eta}^{(r)}$. In limit case when $\theta_{a,b} \to 0$ and $\eta_{a,b} \to 0$, one obtains the well known result for the spectrum of a two-particle system with harmonic oscillator interaction.

Thanks to this necessary review of two particles problem in noncommutative phase space coordinates [22], it will be easy for us to study the problem of two particles with the presence of kinetic coupling potential in NC phase space, which we will deal with in the next section.

3 Two-Body System with Kinetic Coupling Potential

In addition to previous Hamiltonian we add a kinetic coupling potential given by

$$\hat{H}_1 = \hat{H}_0 + \kappa \hat{\mathbf{P}}^{(a)} \cdot \hat{\mathbf{P}}^{(b)}, \tag{25}$$

where κ is a kinetic coupling parameter.

The Hamiltonian (25) can be represented by the centre-of-mass ($\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$) and relative ($\hat{\mathbf{x}}^{(r)}$, $\hat{\mathbf{p}}^{(r)}$) coordinates,

$$\hat{H}_1 = \left(\frac{1}{2M} + \kappa\mu_a\mu_b\right)\hat{\mathbf{p}}^2 + \left(\frac{1}{2\mu} - \kappa\right)\left(\hat{\mathbf{p}}^{(r)}\right)^2 + \kappa\left(\mu_b - \mu_a\right)\hat{\mathbf{p}} \cdot \hat{p}^{(r)} + V\left(\hat{\mathbf{x}}^{(r)}\right). \tag{26}$$

In addition to the previous two conditions represented in Eq. (13), we add a third condition that makes a coupling interaction between the relative and the centre-mass momentum is finish (i.e., $m_a = m_b = m$). This condition is obligatory to get rid the coupling between the relative momentum ($\hat{\mathbf{p}}^{(r)}$) and the centre-mass momentum ($\hat{\mathbf{p}}$). Whereas to avoid the issue of equivalence of the two particles mass, we will add in Sect. 4 another coupling potential $\tilde{\kappa}\mathbf{X}^{(a)} \cdot \mathbf{X}^{(b)}$. Where the third condition of the similarity of particle masses is broken, which will be generalized and replaced by terms on the coefficients κ and $\tilde{\kappa}$. In order to make the process more important, it will be formulated by path integral quantization methods.

Under these considerations and with Harmonic oscillator interaction, the Hamiltonian (25) becomes as

$$\hat{H}_1 = \frac{1}{4}\left(\kappa + \frac{1}{m}\right)\hat{\mathbf{p}}^2 - \left(\kappa - \frac{1}{m}\right)\left(\hat{\mathbf{p}}^{(r)}\right)^2 + \frac{m\omega^2}{4}\left(\hat{\mathbf{x}}^{(r)}\right)^2, \tag{27}$$

Hence, we use the generalized Bopp's shift defined in Eqs. (14)–(17). Thus, Eq. (27) yields this equivalence

$$\hat{H}_1 = \hat{H}_1^{(c)} + \hat{H}_1^{(r)}. \tag{28}$$

The Hamiltonian of the center-of-mass $\hat{H}_1^{(c)}$ is taken as

$$\hat{H}_1^{(c)} = \frac{1}{2M_1^{(c)}}\hat{\mathbf{p}}_q^2 + \frac{1}{4}\left(\kappa + \frac{1}{m}\right)\left[\frac{\tilde{\eta}^2}{4}\hat{\mathbf{q}}^2 - \tilde{\eta}\left(\hat{q}_1\hat{p}_{q_2} - \hat{q}_2\hat{p}_{q_1}\right)\right], \tag{29}$$

while $\hat{H}_1^{(r)}$ is given as

$$\begin{aligned}\hat{H}_1^{(r)} = {} & \frac{1}{2M_1^{(r)}}\left(\hat{\mathbf{p}}_q^{(r)}\right)^2 + \frac{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{2\zeta}\left(\frac{m\omega^2}{4} - \frac{\zeta^2}{\left(\tilde{\theta}^{(r)}\right)^2}\left(\kappa - \frac{1}{m}\right)\right)\left(\hat{\mathbf{q}}^{(r)}\right)^2 \\ & + \frac{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{\zeta}\left(\frac{\zeta}{\tilde{\theta}^{(r)}}\left(\kappa - \frac{1}{m}\right) - \frac{m\omega^2}{4}\frac{\zeta}{\tilde{\eta}^{(r)}}\right)\left(\hat{q}_1^{(r)}\hat{p}_{q_2}^{(r)} - \hat{q}_2^{(r)}\hat{p}_{q_1}^{(r)}\right),\end{aligned} \tag{30}$$

for which correspond the following eigenvalues:

$$E_1^{(c)} = \hbar\Omega_1^{(c)}\left(n_1 + \frac{1}{2}\right), \quad n_1 = 0, 1, 2, \ldots, \tag{31}$$

$$E_1^{(r)} = \hbar\Omega_1^{(r)}\left(2n_2 + |\ell_2| + 1\right) - \ell_2\omega_1^{(r)}, \quad n_2 = 0, 1, 2, \ldots \tag{32}$$

where

$$\Omega_1^{(c)} = \frac{\tilde{\eta}}{2}\left(\kappa + \frac{1}{m}\right),\ \Omega_1^{(r)} = \tilde{\eta}^{(r)}\sqrt{\frac{\zeta}{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}M_1^{(r)}}}\sqrt{\frac{1}{m} - \kappa + \frac{m\omega^2}{4}\frac{\left(\tilde{\theta}^{(r)}\right)^2}{\zeta^2}}, \tag{33}$$

and

$$M_1^{(c)} = \frac{2}{\kappa + \frac{1}{m}},\ M_1^{(r)} = \frac{\zeta}{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}\left[\frac{m\omega^2}{4}\frac{\zeta^2}{(\tilde{\eta}^{(r)})^2} - \left(\kappa - \frac{1}{m}\right)\right]^{-1}, \tag{34}$$

$$\omega_1^{(r)} = \left[\tilde{\eta}^{(r)}\left(\kappa - \frac{1}{m}\right) - \frac{m\omega^2}{4}\tilde{\theta}^{(r)}\right]. \tag{35}$$

As a result, the same results were obtained compared to the previous section, but with different parameters. It also corresponds to two parts of the movement, one a free particle in a uniform magnetic field $B_1 = M_1^{(c)}\Omega_1^{(c)}$, and the other is two-dimensional harmonic oscillator with magnetic field $B_2 = M_1^{(r)}\omega_1^{(r)}$ with frequency $\Omega_1^{(r)}$. In limit case when $\kappa \to 0$, one obtains the well known result for the spectrum of a two-particle system with harmonic oscillator interaction in NC phase space, and with equivalent masses (i.e., $m_a = m_b = m$).

4 Two-Body System with Coupling Potentials

In order to be free from the third condition of the equivalence of two-particle mass, let us consider a generalization of two-body system for a class of coupling potentials with Harmonic oscillator interaction in noncommutative phase space defined by

$$\hat{H}_2 = \hat{H}_0 + \kappa\hat{\mathbf{P}}^{(a)} \cdot \hat{\mathbf{P}}^{(b)} + \tilde{\kappa}\hat{\mathbf{X}}^{(a)} \cdot \hat{\mathbf{X}}^{(b)}, \tag{36}$$

here κ and $\tilde{\kappa}$ in Eq. (36) denote an off-diagonal coupling potentials of the two particles (a) and (b), while the coordinates $\hat{\mathbf{X}}^{(a)}$ and the momenta $\hat{\mathbf{P}}^{(a)}$ of a particle satisfy the relation (3). We can switch to centre-of-mass $\hat{\mathbf{x}}$ and relative $\hat{\mathbf{x}}^{(r)}$ by means of the above section. Putting this all together, the Hamiltonian (36) can be represented by the centre-of-mass ($\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$) and relative ($\hat{\mathbf{x}}^{(r)}$, $\hat{\mathbf{p}}^{(r)}$) coordinates,

$$\begin{aligned}\hat{H}_2 = &\left(\frac{1}{2\mu} - \kappa\right)\left(\hat{\mathbf{p}}^{(r)}\right)^2 + \left(\frac{1}{2M} + \kappa\mu_a\mu_b\right)\hat{\mathbf{p}}^2 + (\mu_b - \mu_a)\left(\kappa\hat{\mathbf{p}}\cdot\hat{p}^{(r)} + \tilde{\kappa}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}^{(r)}\right)\\ &+\tilde{\kappa}\hat{\mathbf{x}}^2 - \frac{\tilde{\kappa}\mu}{M}\left(\hat{\mathbf{x}}^{(r)}\right)^2 + V\left(\hat{\mathbf{x}}^{(r)}\right).\end{aligned} \tag{37}$$

As we early discussed, it is difficult to separate the quantum dynamics of problem involving the interaction of two-particle system in noncommutative phase space without imposing some canonical restrictions. In this case (37), it is clear that the factor $(\mu_b - \mu_a)$ remains as the main difficulty of converting the two-particle problem into the single-particle problem using the transformation of the center of mass $\hat{\mathbf{x}}$ and the relative coordinates $\hat{\mathbf{x}}^{(r)}$. For this we supply the two conditions in the case of a two-body system with harmonic oscillator interaction (13) by the third condition in this case of kinetic coupling terms, general and free from the restrictive condition $m_a = m_b = m$, and we will apply after we use Boop's transformations.

We are then interested to determine analytically the propagation K related with Harmonic oscillator interaction ($V(r) = \frac{\mu\omega^2}{2}r^2$) and coupling potentials in NC phase space coordinates, which obeys the equation follow:

$$\left(\hat{H}_2 - \imath\hbar\frac{\partial}{\partial t}\right)_f \star K = \imath\hbar\delta\left(x_f - x_i\right)\delta\left(y_f - y_i\right)\delta\left(t_f - t_i\right), \tag{38}$$

where $\hat{H}_2$ represent the Hamiltonian of our system (37). It can be simplified under the restrictions (10) and (11), to become the two-particle problem can be reduced to the problems of the motion of the center-of-mass and the relative motion, which can be studied separately.

$$\hat{H}_2 = \left(\tfrac{1}{2\mu} - \kappa\right)\left(\hat{\mathbf{p}}^{(r)}\right)^2 + \left(\tfrac{1}{2M} + \kappa\mu_a\mu_b\right)\hat{\mathbf{p}}^2 + (\mu_b - \mu_a)\left(\kappa\hat{\mathbf{p}}\cdot\hat{p}^{(r)} + \tilde{\kappa}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}^{(r)}\right)$$

$$+\tilde{\kappa}\hat{\mathbf{x}}^2+\left(\frac{\mu\omega^2}{2}-\frac{\tilde{\kappa}\mu}{M}\right)\left(\hat{\mathbf{x}}^{(r)}\right)^2. \tag{39}$$

From (38), we can cross into the normal product by use the Boop's shift given in Eqs. (14)–(17). Therefore, we present the transition amplitude $K(\mathbf{x}_f,\mathbf{x}_i,\mathbf{x}_f^{(r)},\mathbf{x}_i^{(r)},T)$ for two-body quantum systems as a matrix element of evolution operator $\hat{U}\,(t_b,t_a)$ [29,30],

$$K(\mathbf{q}_f,\mathbf{q}_i,\mathbf{q}_f^{(r)},\mathbf{q}_i^{(r)},T)=\left\langle \mathbf{q}_f,\mathbf{q}_f^{(r)}\left|\hat{U}\left(t_f,t_i\right)\right|\mathbf{q}_i,\mathbf{q}_i^{(r)}\right\rangle, \tag{40}$$

here $\left|\mathbf{q},\mathbf{q}^{(r)}\right\rangle$ are eigenvectors of some self-adjoint operators of coordinates $(\hat{\mathbf{q}},\hat{\mathbf{q}}^{(r)})$. The corresponding canonical-conjugated operators of momenta are $(\hat{\mathbf{p}}_q,\hat{\mathbf{p}}_q^{(r)})$, so that

$$\hat{\mathbf{q}}\left|\mathbf{q},\mathbf{q}^{(r)}\right\rangle=\mathbf{q}\left|\mathbf{q},\mathbf{q}^{(r)}\right\rangle,\ \hat{\mathbf{p}}_q\left|\hat{\mathbf{p}}_q,\hat{\mathbf{p}}_q^{(r)}\right\rangle=\mathbf{p}\left|\hat{\mathbf{p}}_q,\hat{\mathbf{p}}_q^{(r)}\right\rangle;$$
$$\langle\mathbf{q}',\mathbf{q}'^{\,(r)}\mid\mathbf{q},\mathbf{q}^{(r)}\rangle=\delta^2(\mathbf{q}'-\mathbf{q})\delta^2(\mathbf{q}'^{\,(r)}-\mathbf{q}^{\,(r)}) \tag{41}$$
$$I=\int\mid\mathbf{q},\mathbf{q}^{(r)}\rangle\langle\mathbf{q},\mathbf{q}^{(r)}\mid=\int\mid\mathbf{p}_q,\mathbf{p}_q^{(r)}\rangle\langle\mathbf{p}_q,\mathbf{p}_q^{(r)}\mid$$
$$\langle\mathbf{q},\mathbf{q}^{(r)}\mid\mathbf{p}_q,\mathbf{p}_q^{(r)}\rangle=\frac{1}{(2\pi\hbar)^2}e^{\frac{i}{\hbar}\left(\mathbf{p}_q\cdot\mathbf{q}+\mathbf{q}^{(r)}\cdot\mathbf{p}_q^{(r)}\right)}, \tag{42}$$

with the evolution operator $U\left(t_f,t_i\right)=\exp(-\frac{\imath T}{\hbar}\hat{H}_2)$ and $T=t_f-t_i$. We follow the standard discretization method as done in [29,30,34]. The Hamiltonian path integral representation for time evolution from $\mathbf{q}\,(t_i)$ to $\mathbf{q}\left(t_f\right)$ in the time interval $T=t_f-t_i$ can thus be written as

$$K^{(\theta,\eta)}(\mathbf{q}_f,\mathbf{q}_i,\mathbf{q}_f^{(r)},\mathbf{q}_i^{(r)};T)=\lim_{N\to\infty}\prod_{j=1}^{N}\int d\mathbf{q}_j\int d\mathbf{q}_j^{(r)}\prod_{j=1}^{N+1}\int\frac{d\mathbf{p}_{q_j}}{(2\pi\hbar)^2}\int\frac{d\mathbf{p}_{q_j}^{(r)}}{(2\pi\hbar)^2}$$
$$\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\mathbf{p}_{q_j}.\Delta\mathbf{q}_j+\mathbf{p}_q^{(r)}.\Delta\mathbf{q}^{(r)}-\varepsilon H_2(\mathbf{p}_{q_j},\mathbf{q}_j,\mathbf{p}_{q_j}^{(r)},\mathbf{q}_j^{(r)})\right]\right\}. \tag{43}$$

In this case, H_2 consists of three Hamiltonians, one represents the movement of the center-of-mass, while the second represents the relative motion and the third indicates a coupling term between the center-of-mass and relative coordinates. To enable separation of the motion of the center-of-mass and the relative motion, which satisfy the ordinary commutation relations, we aim of this procedure to abolish the third Hamiltonian term under this condition:

$$-\frac{\kappa}{\tilde{\kappa}}=\frac{\tilde{\theta}\zeta}{2\tilde{\eta}^{(r)}}=\frac{2\tilde{\theta}^{(r)}}{\tilde{\eta}\zeta}=\frac{2\zeta}{\tilde{\eta}\tilde{\eta}^{(r)}}=\frac{\tilde{\theta}\tilde{\theta}^{(r)}}{2\zeta}=cst. \tag{44}$$

which is given the relation between the coefficients κ and $\tilde{\kappa}$, as a function of noncommutative parameters. Under these considerations (44), the Hamiltonian H_2 becomes as:

$$H_2(\mathbf{p}_{q_j},\mathbf{q}_j,\mathbf{p}_{q_j}^{(r)},\mathbf{q}_j^{(r)})=\frac{1}{2M_2^{(c)}}\mathbf{p}_{q_j}^2+\frac{M_2^{(c)}\tilde{\omega}^2}{2}\mathbf{q}_j^2+\left[\tilde{\theta}\tilde{\kappa}+\tilde{\eta}\left(\frac{1}{2M}+\kappa\mu_a\mu_b\right)\right]L_{z_j}$$
$$+\frac{1}{2M_2^{(r)}}(\mathbf{p}_{q_j}^{(r)})^2+\frac{M_2^{(r)}(\tilde{\omega}^{(r)})^2}{2}(\mathbf{q}_j^{(r)})^2+\left[\tilde{\eta}^{(r)}\left(\frac{1}{2\mu}-\kappa\right)+\tilde{\theta}^{(r)}\left(\frac{\mu\omega^2}{2}-\frac{\tilde{\kappa}\mu}{M}\right)\right]L_{z_j}^{(r)}, \tag{45}$$

where $L_z=p_{q_1}q_2-p_{q_2}q_1$ and $L_z^{(r)}=p_{q_1}^{(r)}q_2^{(r)}-p_{q_2}^{(r)}q_1^{(r)}$ represents the moment angular along the axis z for center-mass and relative coordinates respectively.

$$\frac{1}{2M_2^{(c)}}=\frac{1}{2M}+\kappa\mu_a\mu_b+\tilde{\kappa}\frac{\tilde{\theta}^2}{4}, \tag{46}$$
$$\frac{1}{2M_2^{(r)}}=\frac{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{2\zeta}\left[\frac{1}{2\mu}-\kappa+\frac{\zeta^2}{(\tilde{\eta}^{(r)})^2}\left(\frac{\mu\omega^2}{2}-\frac{\tilde{\kappa}\mu}{M}\right)\right], \tag{47}$$
$$\frac{M_2^{(c)}\tilde{\omega}^2}{2}=\tilde{\kappa}+(\tilde{\eta}/2)^2\left(\frac{1}{2M}+\kappa\mu_a\mu_b\right), \tag{48}$$

$$\frac{M_2^{(r)}(\tilde{\omega}^{(r)})^2}{2} = \frac{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{2\zeta}\left[\frac{\mu\omega^2}{2} - \frac{\tilde{\kappa}\mu}{M} + \frac{\zeta^2}{(\tilde{\eta}^{(r)})^2}\left(\frac{1}{2\mu} - \kappa\right)\right]. \tag{49}$$

We can perform the momenta integrations to obtain the Lagrangian path integral for the coupled system, which represent the product of two propagators are identical to the path integral solution for charged particle in two-dimensional harmonic oscillator with magnetic field, and with different parameters. It will be solved easily by the polar coordinates [34]. Consequently, the continuous expression for each path integral can be written as

$$\begin{aligned} &\int_{\mathbf{x}(t_a)=\mathbf{x}_a}^{\mathbf{x}(t_b)=\mathbf{x}_b} D\mathbf{x}(t) \exp\left\{\frac{im}{2\hbar}\int_{t_a}^{t_b}\left[\dot{\mathbf{x}}^2 - \omega_0^2\left(x^2+y^2\right) - 2\omega_1\left(x\dot{y} - y\dot{x}\right)\right]\right\} \\ &= \int_{r(t_a)=r_a}^{r(t_b)=r_b} rDr(t) \int_{\varphi(t_a)=\varphi_a}^{\varphi(t_b)=\varphi_b} D\varphi \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2 - \omega_0^2 r^2\right) + m\omega_1 r^2\dot{\varphi} + \frac{\hbar^2}{8m}\right]dt\right\}. \end{aligned} \tag{50}$$

The solution of this Kernel path integral is given as [34]

$$\sum_n\sum_\ell \sqrt{\frac{m\Omega}{\hbar\pi}\frac{n!}{(n+|\ell|)!}}e^{-\frac{iT}{\hbar}[\Omega(2n+|\ell|+1)-\ell\omega_1]}e^{i\ell\varphi - \frac{m\Omega}{2\hbar}r^2}\left(\frac{m\Omega}{\hbar}r^2\right)^{|\ell|/2}L_n^{|\ell|}\left(\frac{m\Omega}{\hbar}r^2\right), \tag{51}$$

where $L_n^{|\ell|}(z)$ are generalized Laguerre polynomials and $\Omega = \sqrt{\omega_1^2 + \omega_0^2}$. However, the transition amplitude of our system is product of two propagators separated given in Eq. (51), which distinguish two parameters of Ω and m. The first represent the path integral solution for center-of-mass $\mathbf{q}-$coordinates. It is characterized by the following information

$$M_2^{(c)} = \left[\frac{1}{M} + 2\kappa\mu_a\mu_b + \tilde{\kappa}\frac{\tilde{\theta}^2}{2}\right]^{-1}, \tag{52}$$

$$\omega_0^2 = \left[\tilde{\omega}^2 - \left[\tilde{\theta}\tilde{\kappa} + \tilde{\eta}\left(\frac{1}{2M} + \kappa\mu_a\mu_b\right)\right]^2\right], \tag{53}$$

$$\omega_2^{(c)} = \left[\tilde{\theta}\tilde{\kappa} + \tilde{\eta}\left(\frac{1}{2M} + \kappa\mu_a\mu_b\right)\right], \tag{54}$$

which leads to

$$\Omega_2^{(c)} = \sqrt{\left[\tilde{\kappa} + (\tilde{\eta}/2)^2\left(\frac{1}{2M} + \kappa\mu_a\mu_b\right)\right]\left[\frac{2}{M} + 4\kappa\mu_a\mu_b + \tilde{\kappa}\tilde{\theta}^2\right]}. \tag{55}$$

While the second Kernel represents the path integral representation of the transition amplitude with relative $\mathbf{q}^{(r)}$-coordinates, and it is characterized as

$$M^{(r)} = \frac{\zeta}{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}\left[\frac{1}{2\mu} - \kappa + \frac{\zeta^2}{(\tilde{\eta}^{(r)})^2}\left(\frac{\mu\omega^2}{2} - \frac{\tilde{\kappa}\mu}{M}\right)\right]^{-1}, \tag{56}$$

$$\omega_0^{(r)} = \sqrt{(\tilde{\omega}^{(r)})^2 - \left[\tilde{\eta}^{(r)}\left(\frac{1}{2\mu} - \kappa\right) + \tilde{\theta}^{(r)}\left(\frac{\mu\omega^2}{2} - \frac{\tilde{\kappa}\mu}{M}\right)\right]^2}, \tag{57}$$

$$\omega_2^{(r)} = \left[\tilde{\eta}^{(r)}\left(\frac{1}{2\mu} - \kappa\right) + \tilde{\theta}^{(r)}\left(\frac{\mu\omega^2}{2} - \frac{\tilde{\kappa}\mu}{M}\right)\right]. \tag{58}$$

This leads to

$$\begin{aligned} \Omega_2^{(r)} = \frac{1}{\sqrt{2}}\left(\frac{\tilde{\theta}^{(r)}\tilde{\eta}^{(r)}}{\zeta}\right)\left[\frac{\mu\omega^2}{2} - \frac{\tilde{\kappa}\mu}{M} + \frac{\zeta^2}{(\tilde{\eta}^{(r)})^2}\left(\frac{1}{2\mu} - \kappa\right)\right]^{1/2} \\ \times\left[\frac{1}{2\mu} - \kappa + \frac{\zeta^2}{(\tilde{\eta}^{(r)})^2}\left(\frac{\mu\omega^2}{2} - \frac{\tilde{\kappa}\mu}{M}\right)\right]^{1/2}. \end{aligned} \tag{59}$$

Finally, the spectral decomposition of the transition amplitude takes the standard symmetry form:

$$K^{(\theta,\eta)}(\mathbf{q}_f, \mathbf{q}_i, \mathbf{q}_f^{(r)}, \mathbf{q}_i^{(r)}; T) = \prod_{k=1,2}\sum_{n_k=0}^{\infty}\sum_{\ell_k}\Phi_{n_k,\ell_k}(\mathbf{q}_f^{(k)})\Phi^*_{n_k,\ell_k}(\mathbf{q}_i^{(k)})e^{-\frac{i}{\hbar}E_{n_k,\ell_k}T}. \tag{60}$$

The indices $k = 1, 2$ indicates the coordinates code of the center-of-mass and relative motion, respectively. Therefore we can write the eigenfunctions of a two-particle system with harmonic oscillator and coupling potentials in noncommutative phase space

$$\Phi_{n_k,\ell_k}\left(\mathbf{q}^{(k)}\right) = \sqrt{\frac{m_k\Omega_k}{\hbar\pi}\frac{n_k!}{(n_k+|\ell_k|)!}}\left(\frac{m_k\Omega_k}{\hbar}r_k^2\right)^{|\ell_k|/2} \times \exp\left[i\ell_k\varphi_k - \frac{m_k\Omega_k}{2\hbar}r_k^2\right] L_{n_k}^{|\ell_k|}\left(\frac{m_k\Omega_k}{\hbar}r_k^2\right). \tag{61}$$

The energy eigenvalues $E^{\theta,\eta}$ are easily obtained from Eqs. (51) and (60)

$$E^{\theta,\eta} = \hbar\left[\Omega_1\left(2n_1 + |\ell_1| + 1\right) - \ell_1\omega_1 + \Omega_2\left(2n_2 + |\ell_2| + 1\right) - \ell_2\omega_2\right], \tag{62}$$

where n_1, n_2 are the quantum numbers $n_1 = 0, 1, 2, \ldots$, $n_2 = 0, 1, 2, \ldots$, and $\Omega_1 = \Omega_2^{(c)}$, $\Omega_2 = \Omega_2^{(r)}$, $\omega_1 = \omega_2^{(c)}$, $\omega_2 = \omega_2^{(r)}$ are defined by (55) and (59), respectively. As it appears from the terms of energy and wave functions, for $\kappa = \tilde{\kappa} = 0$ we will reach the same results presented by Gnatenko and Tkachuk [24]. Note that the absence of these parameters κ and $\tilde{\kappa}$ did not change the expression of energy $E^{\theta,\eta}$, there is only a change in coefficients $\Omega_0^{(c)}$ and $\Omega_0^{(r)}$.

In the case when $\eta^{(a,b)} = 0$, the expected result here, is very simple and we do not need to impose a condition on the mass of the two particles as we did in this work. Through two searches [24,30], we can be separated them using the same technique that we have done, where the result is a product of two propagators, the first is in free case and the second represent the path integral representation of the transition amplitude with relative coordinates of a point particle moving in constant magnetic field and in two-dimensional anisotropic oscillator.

In addition, when $\theta^{(a,b)} = 0$, we find the same situation, and we can follow the same steps which we carried in this paper, only with simple calculations and different parameters.

5 Conclusion

In this paper, we have successfully studied systems that contain the two-body interaction, for example, a harmonic oscillator with coupling potentials in 2-dimensional noncommutative phase space that show the applicability it using path integral technique. As we know the two-particle problem has been examined in usual quantum mechanics by transforming the coordinates particles to center-of-mass and relative coordinates. Either in the presence of noncommutative phase space, the motion of the center-of-mass ($\mathbf{x}$) and the relative motion ($\mathbf{x}^{(r)}$) are not independent, and therefore can not achieve complete separation of these variables. To overcome this difficulty when kinetic coupling potential is present, we have set two important conditions. One of them is on noncommutative parameters which correspond to a particle are determined by its mass [see Eq. (13)], and the other on the mass of the two particles $m_1 = m_2$. While in the general case, when using coupling potentials, the requirement of equal masses is being generalized by Eq. (44). As well as, when κ and $\tilde{\kappa}$ equal to zero exact expressions for the energy levels of a two-particle system with harmonic oscillator interaction in noncommutative phase space have been found. In the limit $\left(\theta^{(a,b)}, \eta^{(a,b)} \to 0\right)$, we recover the usual case.

References

1. A. Connes, M.R. Douglas, A. Schwarz, JHEP **9802**, 003 (1998)
2. N. Seiberg, E. Witten, JHEP **9909**, 032 (1999)
3. L. Susskind, arXiv:hep-th/0101029
4. H. Snyder, J. Weinberg, Phys. Rev. **57**, 307 (1940)
5. M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001)
6. Kh.P. Gnatenko, O.V. Shyiko, Mod. Phys. Lett. A **33**, 1850091 (2018)
7. K. Ma, Adv. High Energy Phys. (2017). https://doi.org/10.1155/2017/1945156
8. K. Wang, Y.-F. Zhang, Q. Wang, Z.-W. Long, J. Jing, Int. J. Mod. Phys. A **32**, 1750143 (2017)
9. M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001)
10. J. Gamboa, M. Loewe, J.C. Rojas, Phys. Rev. D **64**, 067901 (2001)

11. S. Bellucci, A. Nersessian, C. Sochichiu, Phys. Lett. B **522**, 345 (2001)
12. A. Kijanka, P. Kosinski, Phys. Rev. D **70**, 127702 (2004)
13. J. Zhang, Phys. Rev. Lett. **93**, 043002 (2004)
14. K. Li, J. Wang, Eur. Phys. J. C **50**, 1007 (2007)
15. M. Rosenbaum, J. David Vergara, L. Roman Juarez, Phys. Lett. A **367**, 1 (2007)
16. J. Jing, S.H. Zhao, J.F. Chen, Z.W. Long, Eur. Phys. J. C **54**, 685 (2008)
17. C. Acatrinei, JHEP **0109**, 007 (2001)
18. A. Smailagic, E. Spalucci, J. Phys. A **36**, 467 (2003)
19. S. Gangopadhyay, F.G. Scholtz, Phys. Rev. Lett. **102**, 241602 (2009)
20. H. Benzair, M. Merad, T. Boudjedaa, A. Makhlouf, Z. Naturforsch, **67a**, 77 (2012)
21. H. Benzair, M. Merad, T. Boudjedaa, Mod. Phys. Lett. A **28**, 1350144 (2013)
22. Kh.P. Gnatenko, V.M. Tkachuk, Phys. Lett. A **381**, 2463 (2017)
23. Kh.P. Gnatenko, Mod. Phys. Lett. A **32**, 1750166 (2017)
24. Kh.P. Gnatenko, V.M. Tkachuk, J. Phys. Stud. **21**, 3001 (2017)
25. I.M. Mills, A.G. Robiette, Mol. Phys. **56**, 743 (1985)
26. H.-Y. Fan, B.-Z. Chen, Y. Fan, Phys. Lett. A **213**, 226 (1996)
27. H.-Y. Fan, L.-S. Li, Phys. Lett. A **229**, 264 (1997)
28. H.-Y. Fan, J.R. Klauder, Phys. Rev. A **49**, 704 (1994)
29. D.C. Khandekar, S.V. Lawande, J. Phys. A Gen. Phys. **5**, 812 (1972)
30. C. Grosche, Czechoslov. J. Phys. **49**, 1413 (1999)
31. M. Daszkiewicz, Phys. Scr. **93**, 1 (2018)
32. M.I. Samar, V.M. Tkachuk, J. Math. Phys. **58**, 122108 (2017)
33. Kh.P. Gnatenko, Phys. Lett. A **377**, 3061 (2013)
34. C. Grosche, F. Steiner, *Handbook of Feynman Path Integrals* (Springer, Berlin, 1998)