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The D-CTC Condition is Generically Fulfilled in Classical (Non-quantum) Statistical Systems

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Abstract
The D-CTC condition, introduced by David Deutsch as a condition to be fulfilled by analogues for processes of quantum systems in the presence of closed timelike curves, is investigated for classical statistical (non-quantum) bi-partite systems. It is shown that the D-CTC condition can generically be fulfilled in classical statistical systems, under very general, model-independent conditions. The central property used is the convexity and completeness of the state space that allows it to generalize Deutsch's original proof for q-bit systems to more general classes of statistically described systems. The results demonstrate that the D-CTC condition, or the conditions under which it can be fulfilled, is not characteristic of, or dependent on, the quantum nature of a bi-partite system.

Keywords Closed timelike curves · Classical statistical systems · Ergodicity

1 Introduction

In a seminal paper [1], David Deutsch introduced a condition (henceforth referred to as *D-CTC condition*) that is supposed to capture the meaning of processes "running back in time" in bi-partite quantum systems (and more generally, in multi-partite quantum systems, or quantum circuits). In its simplest form, it can be described as follows: Assume a bi-partite quantum mechanical system given, consisting of a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ composed of the Hilbert spaces of two subsystems. Moreover, suppose that U is a unitary operator on $\mathcal{H}$, which is viewed as describing (the result of) a dynamical interaction between the two systems, akin to a time evolution

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operator, or a scattering matrix. Furthermore, let ϱ_A be a density matrix on $\mathcal{H}_A$, or synonymously, a state on the "A"-part of the system with expectation values

$$\langle \mathbf{a} \rangle_A = \mathrm{Tr}_A(\varrho_A \mathbf{a})$$

for all bounded linear operators $\mathbf{a}$ on $\mathcal{H}_A$ where we have written Tr_A to emphasize that the trace is to be understood with respect to the Hilbert space $\mathcal{H}_A$. Relative to these data, a density matrix ϱ on $\mathcal{H}$ is said to *fulfill the D-CTC condition* if the following two conditions are fulfilled (see Fig. 1):

(1) The partial state induced by ϱ on the "A"-part of the system (prior to the interaction U taking effect) equals $\langle . \rangle_A$, i.e.

$$\mathrm{Tr}(\varrho(\mathbf{a} \otimes \mathbf{1}_B)) = \mathrm{Tr}_A(\varrho_A \mathbf{a})$$

holds for all bounded linear operators $\mathbf{a}$ on $\mathcal{H}_A$ (with $\mathbf{1}_B$ denoting the identity operator on $\mathcal{H}_B$). Equivalently, ϱ_A equals $\mathrm{tr}^{\mathcal{H}_B}(\varrho)$, the partial trace of ϱ taken with respect to $\mathcal{H}_B$. Note that in the previous equation, the trace Tr appearing on the left hand side is taken on the full Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

(2) The partial state induced by ϱ on the "B"-part of the system (prior to the interaction U) returns to itself after the interaction U has taken effect, i.e.

$$\mathrm{Tr}(\varrho U^*(\mathbf{1}_A \otimes \mathbf{b})U) = \mathrm{Tr}(\varrho(\mathbf{1}_A \otimes \mathbf{b}))$$

holds for all bounded linear operators $\mathbf{b}$ on $\mathcal{H}_B$ (with $\mathbf{1}_A$ now denoting the identity operator on $\mathcal{H}_A$).

In his proof that the D-CTC condition can always be fulfilled when the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are both finite-dimensional, Deutsch uses that the map

$$\mathcal{S} : \varrho_B \mapsto \mathrm{tr}^{\mathcal{H}_A}(U(\varrho_A \otimes \varrho_B)U^*)$$

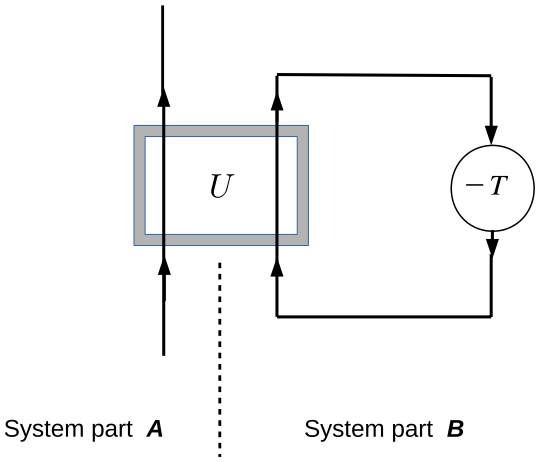

Fig. 1 A process in a quantum circuit is represented a unitary operator U describing the dynamical coupling of two system parts (denoted by A and B). U takes initial states (prior to interaction) to final states (after the interaction has taken effect); that process is supposed to take a time duration T. A "step backward in time" is symbolized by $-T$; the B-part of the result of the process (i.e. the partial state on the B system after the interaction) is again fed into the process as initial state of the B-part

on the set of density matrices ϱ_B on $\mathcal{H}_B$ has a fixed point. However, what is actually being used (and allows the fixed point argument to be applied) is that for quantum mechanical systems, the state space is always convex and complete: It allows for classical statistical (or probabilistic) mixtures of states, and limits (in a suitable sense) thereof. In other words, the D-CTC condition, and the question to which extent it can be fulfilled, is not primarily sensitive to, or dependent on, genuinely quantum mechanical properties of a bi-partite system, such as quantum mechanical superpositions (interference effects), uncertainty relations or entanglement, but really on the convexity and completeness of the state space of the systems in question. Therefore, *the D-CTC condition can also be fulfilled in classical (i.e. non-quantum) statistical physical theories*, such as classical statistical mechanics, under very general, physically realistic conditions; it is the purpose of this article to demonstrate that fact at an appreciable level of mathematical generality and rigour. The authors of [2] reach at a related conclusion, however based on a different reasoning than presented in this article; they argue that in the limit of large Hilbert space dimension of the "B" system part, the D-CTC condition becomes classical. In [3], the authors indicate that the D-CTC condition can be staged in a far more general formal framework than that of quantum mechanics. The feature of the D-CTC condition to be primarily dependent on the ability to form classical statistical mixtures of states has also been observed in [4]. That same article also discusses related investigations of classical "billiard ball" collisions wherein one of the balls enters a "wormhole"-type time machine and re-emerges "prior to entering" exactly such as to be kicked by the other ball into the time machine [5–7]. Such scenarios may be viewed as particular classical counterparts of the D-CTC set-up (or rather—historically more correctly—the D-CTC approach ought to be seen as an attempt at providing a quantum analogue for such "billiard-ball-collisions-with-wormhole-time-machines" set-ups) but we will not follow this line of analogy in the present article. See, however, Sect. 5 for further remarks.

The D-CTC condition is always presented in the context of quantum physics[1] or of quantum computational considerations (as a sample, see the publications [1, 2, 10–15], see also references therein). Therefore, it seems well worth pointing out, and demonstrating, that it is basically of a statistical, but not necessarily quantum physical nature.

In a recent paper [16] (see also [17] for a summary), we have investigated the D-CTC condition in the setting of operator-algebraic quantum field theory [18]. It is useful to briefly outline some of the basic elements of that approach as it helps to make parallels between the D-CTC condition as formulated above for quantum

[1] The article [8] portrays quantum mechanics as a natural mechanism for avoiding paradoxes that would occur in the presence of CTCs in the framework of classical physics; however in [4] it is pointed out that forming statistical mixtures would in a similar way allow it to avoid those paradoxes. Nevertheless, linking the D-CTC condition with quantum physics has become a commonplace because of its origins and potential consequences in quantum computing (see references cited above), and seems to have also gained traction in popular culture: In [9], the lead character Tony Stark verbally mentions "the Deutsch proposition" in an attempt of the protagonists to travel back to the past using a fictional time machine based on quantum physics.

mechanical systems, and the classical statistical physics case to be considered in this article, more easily visible.

In the operator-algebraic approach to relativistic quantum field theory [18], there is for any system (quantum field) a C^*-algebra $\mathcal{A}$ whose self-adjoint elements correspond to observables of the system. In most cases, it is no major restriction to suppose that $\mathcal{A}$ is a subalgebra of some $\mathsf{B}(\mathcal{H})$, the algebra of all bounded operators on an (infinite dimensional) Hilbert space $\mathcal{H}$. It is also usually assumed that $\mathcal{A}$ contains an algebraic unit element denoted by $\mathbf{1}$; if $\mathcal{A} \subset \mathsf{B}(\mathcal{H})$, that would be the unit operator on $\mathcal{H}$. Supposing that the spacetime on which the quantum field propagates is Minkowski spacetime (however, the general setting allows for choosing more general, curved spacetimes instead), it is assumed that for any finite (that is, relatively compact) open region O in spacetime there is a C^* subalgebra $\mathcal{A}(O)$ of $\mathcal{A}$ containing the observables that can be measured at times and locations within O, including $\mathbf{1}$. In keeping with this set-up, it is further assumed that $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$ whenever $O_1 \subset O_2$. This property is called *isotony*. Another assumption is *locality*, meaning that $\mathbf{ab} = \mathbf{ba}$ for all $\mathbf{a} \in \mathcal{A}(O_A)$ and $\mathbf{b} \in \mathcal{A}(O_B)$ provided that the spacetime regions O_A and O_B are causally separated, i.e. there is no causal curve beginning in O_A and ending O_B. Particularly in this situation where O_A and O_B are causally separated, one may take the pair of algebras $\mathcal{A}(O_A)$ and $\mathcal{A}(O_B)$ as the mathematical model of a causally separated bi-partite system, with $\mathcal{A}(O_A)$ and $\mathcal{A}(O_B)$ playing roles analogous to $\mathsf{B}(\mathcal{H}_A)$ and $\mathsf{B}(\mathcal{H}_B)$ in the quantum mechanical setting outlined at the beginning.

Another important ingredient of the operator-algebraic approach are states. A state is any expectation value functional $\mathbf{a} \mapsto \langle \mathbf{a} \rangle \quad (\mathbf{a} \in \mathcal{A})$ on the algebra of observables $\mathcal{A}$, and therefore, by definition, $\mathbf{a} \mapsto \langle \mathbf{a} \rangle$ is linear, and fulfills $\langle \mathbf{a}^* \mathbf{a} \rangle \geq 0$ for all $\mathbf{a} \in \mathcal{A}$, as well as $\langle \mathbf{1} \rangle = 1$. Usually, if $\mathcal{A}$ is contained in some $\mathsf{B}(\mathcal{H})$, one considers only *normal* states which arise from density matrices; in other words, a state is normal if it is of the form

$$\langle \mathbf{a} \rangle = \langle \mathbf{a} \rangle_\varrho = \mathrm{Tr}(\varrho \mathbf{a}) \quad (\mathbf{a} \in \mathcal{A})$$

for some density matrix ϱ on the Hilbert space $\mathcal{H}$.

One may now reformulate the D-CTC condition in the operator-algebraic setting as follows. As mentioned, one starts from an observable algebra $\mathcal{A} \subset \mathsf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, together with observable algebras $\mathcal{A}(O_A)$ and $\mathcal{A}(O_B)$ for two causally separated spacetime regions O_A and O_B, representing the observables of a causally separated bi-partite system. Further data assumed given are a normal state $\langle \mathbf{a} \rangle_A = \mathrm{Tr}(\varrho_A \mathbf{a}) \quad (\mathbf{a} \in \mathcal{A}(O_A))$ on $\mathcal{A}(O_A)$ (on the "A"-part of the full system) induced by a density matrix ϱ_A on $\mathcal{H}$, and a unitary operator U on $\mathcal{H}$. Given these data, a state $\langle \mathbf{c} \rangle = \mathrm{Tr}(\varrho \mathbf{c}) \quad (\mathbf{c} \in \mathcal{A})$ is said to fulfill the D-CTC condition if the following two conditions are fulfilled:

(I) The partial state of $\langle . \rangle$ on $\mathcal{A}(O_A)$ coincides with $\langle . \rangle_A$, i.e.

$$\langle \mathbf{a} \rangle = \langle \mathbf{a} \rangle_A \quad (\mathbf{a} \in \mathcal{A}(O_A))$$

(II) The partial state of $\langle . \rangle$ on $\mathcal{A}(O_B)$ returns to itself after the action of the unitary U has taken effect, i.e.

$$\langle U^* \mathbf{b} U \rangle = \langle \mathbf{b} \rangle \quad (\mathbf{b} \in \mathcal{A}(O_B))$$

The analogy with the D-CTC condition with the quantum mechanical case described above should be clear on noting that, since both $\mathcal{A}(O_A)$ and $\mathcal{A}(O_B)$ are in a defined way subalgebras of the larger C^*-algebra $\mathcal{A}$ (or of $\mathsf{B}(\mathcal{H})$), and all the algebras share the common algebraic unit element $\mathbf{1}$, the $\mathbf{a} \in \mathcal{A}(O_A)$ here is analogous to the $\mathbf{a} \otimes \mathbf{1}$ above, and similarly the $\mathbf{b} \in \mathcal{A}(O_B)$ here is analogous to the $\mathbf{1} \otimes \mathbf{b}$ above. We mention however that in general, in quantum field theory the operator algebra generated by $\mathcal{A}(O_A)$ and $\mathcal{A}(O_B)$ in $\mathsf{B}(\mathcal{H})$ need not equal (up to identification) the tensor product $\mathcal{A}(O_A) \otimes \mathcal{A}(O_B)$. A precise statement would require introducing von Neumann algebras at this point which we shall not embark on. Nevertheless, there are criteria as to when such an equality actually does hold, known as *split property* or *statistical independence of states*. We will not further discuss these matters here but refer to [18–20] and references cited there for full details.

The results obtained in [16, 17] are, roughly, as follows (we give here a mainly qualitative description and refer to the cited references for full details). States fulfilling the D-CTC condition cannot be found if the states are also required to fulfill a Reeh-Schlieder like property [18] which implies a strong form of entanglement [21]. On the other hand, if the local algebras of observables fulfill the split property just mentioned, then one can always find states fulfilling the D-CTC condition approximately to any prescribed precision. Since the assumptions are met for a wide range of quantum field theories on globally hyperbolic spacetimes which do not admit closed timelike curves, the latter result makes it doubtful if the D-CTC condition actually relates to quantum processes based on the presence of closed timelike curves in the sense of general relativity. The present work casts doubts on whether the D-CTC condition has quantum physics at its core. We will address these points in a discussion towards the end in Sect. 5.

We now turn to describing the content of the present work. In Sect. 2, we will summarize some basics of commutative C^*-algebras which, in an operator algebraic approach, are used as algebras of observables of classical (non-quantum) statistical systems. The relation to functions (random variables) on locally compact or compact topological Hausdorff spaces and probability measures (states)—through the Riesz Representation Theorem and the Gelfand–Naimark Theorem—is also discussed. We have relied on the references [22–25] for our presentation which, on one hand, is included to make this work self-contained and to introduce the concepts and notation needed, and on the other hand, to explain some points that need to be taken care of when considering limits of states on certain commutative C^*-algebras and the question if they still arise from probability measures. We take up on this topic again in Sect. 3 where the concept of classical statistical bi-partite systems is introduced. A criterion ensuring that limits of sequences of probability measures exist on $C_b(X)$, the C^*-algebra of bounded continuous functions on a locally compact metric space X, and are again probability measures, is provided by Prohorov's Theorem [26, 27] and we use it in Theorem 3.2. In our Theorem 3.1 presented before in Sect. 3, we

prove a very general statement to the effect that the D-CTC condition for classical statistical bi-partite systems is fulfilled but with states in an abstract C^*-algebraic sense which need not be given by probability measures. As indicated, Theorem 3.2 is more specific in that it establishes that the states fulfilling the D-CTC condition are given by probability measures under certain assumptions. A simple example in form of a two-body problem interacting by a binding central potential is discussed in Sect. 4 to illustrate properties of the states fulfilling the D-CTC condition constructed in Theorems 3.1 and 3.2. The example will also serve to point out a relation to ergodicity. In the last section, we collect discussion and conclusion, relating our results also to other literature.

2 Commutative C^*-Algebras and Classical Statistical Systems

2.1 Generalities

Physical systems that are subject to a statistical description of their measurement values, but are *classical* in the sense of not being quantum systems, have observable algebras which are commutative (or Abelian). Let us denote a generic commutative C^*-algebra by A. Commutativity means that $\mathsf{fg} = \mathsf{gf}$ for all $\mathsf{f}, \mathsf{g} \in \mathsf{A}$ and consequently, there are no uncertainty relations among the elements of A which would be indicative of a quantum theory. Likewise, there is no entanglement. Assuming that there is a unit element $\mathsf{1}$ contained in A, a *state* on A is defined as a linear functional $\langle . \rangle : \mathsf{A} \to \mathbb{C}$, $\mathsf{f} \mapsto \langle \mathsf{f} \rangle$ fulfilling $\langle \mathsf{f}^*\mathsf{f} \rangle \geq 0$ (positivity) and $\langle \mathsf{1} \rangle = 1$ (normalization). We recall the the well-known fact that any state w on a C^*-algebra (commutative or not) is norm-continuous: $|w(\mathsf{f})| \leq ||\mathsf{f}||$ for all f where $||\mathsf{f}||$ is the C^*-algebra norm of f [24]. It is worth mentioning that C^*-algebras are algebras over $\mathbb{C}$ (as field of numbers) but that, as in quantum mechanics, only their hermitean elements, fulfilling $\mathsf{f}^* = \mathsf{f}$, are considered as observables yielding real-valued expectation values $w(\mathsf{f}) = \langle \mathsf{f} \rangle$ upon evaluation on states.

We shall now adopt the mathematical notation and denote a state as $w : \mathsf{A} \to \mathbb{C}$ so that $w(\mathsf{f}) = \langle \mathsf{f} \rangle$ ($\mathsf{f} \in \mathsf{A}$), since this notation has some advantages. It is easy to notice that the set of states on A, henceforth denoted as $\mathscr{S} = \mathsf{A}^*_+$, is closed under finite convex combinations, i.e. if $w_1, \ldots, w_n$ are finitely many states on A and $\lambda_1, \ldots, \lambda_n$ are non-negative numbers such that $\sum_{k=1}^n \lambda_k = 1$, then the convex sum $\sum_{k=1}^n \lambda_k w_k$ that can be formed from the given states is again a state on A. A state is called *pure* if it can be represented in this convex sum form if, and only if, all the w_k coincide; or equivalently, iff all $\lambda_k = 0$ except for exactly one $\lambda_{k'}$ which therefore must be equal to 1. A state which is not pure is called *mixed*. Furthermore, the set of states is closed with respect to taking weak limits: Suppose that $\{w_\kappa\}_{\kappa \in \mathcal{K}}$ is a generalized sequence of states $w_\kappa \in \mathscr{S}$, where $\mathcal{K}$ is an arbitrary directed set. The generalized sequence $\{w_\kappa\}_{\kappa \in \mathcal{K}}$ is called *weakly convergent* (strictly speaking, *weak-*-convergent*) if $\lim_\kappa w_\kappa(\mathsf{f})$ exists for every $\mathsf{f} \in \mathsf{A}$. Then

$$w(\mathsf{f}) = \lim_\kappa w_\kappa(\mathsf{f}) \quad (\mathsf{f} \in \mathsf{A})$$

is again a state on A. We mention also that $\mathscr{S}$ is weakly compact by the Banach–Alaoglu–Theorem [28], which entails that, whenever $\{w_\kappa\}_{\kappa\in\mathcal{K}}$ is a generalized sequence in $\mathscr{S}$, then it admits a weakly convergent generalized subsequence $\{w_{\kappa(\zeta)}\}_{\zeta\in\mathcal{Z}}$ (with suitable directed index set $\mathcal{Z}$).

An *operation* is any map $\tau : \mathscr{S} \to \mathscr{S}$ which preserves convexity, meaning that

$$\tau\left(\sum_{k=1}^{n}\lambda_k w_k\right) = \sum_{k=1}^{n}\lambda_k \tau(w_k)$$

for all finite convex sums of states. Moreover, we will assume operations to be weakly continuous[2] which is defined as follows: τ is *weakly continuous* if, for all weakly converging generalized sequences $\{w_\kappa\}_{\kappa\in\mathcal{K}}$ of states in $\mathscr{S}$, also $\{\tau(w_\kappa)\}_{\kappa\in\mathcal{K}}$ is a weakly converging generalized sequence of states, with $\lim_\kappa \tau(w_\kappa)(\mathsf{f}) = \tau(\lim_\kappa w_\kappa)(\mathsf{f})$ for all $\mathsf{f}\in\mathsf{A}$. We will see some examples soon; obviously, if $\alpha : \mathsf{A}\to\mathsf{A}$ is a C^*-algebra morphism which preserves the unit element, then its dual map $\alpha^*(w)(\mathsf{f}) = w(\alpha(\mathsf{f}))$ is an operation.

2.2 The Gelfand–Naimark Theorem

The next step is to summarize the content of the Gelfand–Naimark theorem (see Lemma 2 in [29]) which characterizes commutative C^*-algebras as sets of number-valued functions and the states as probability measures. To this end, we largely follow the presentations of [22–24] which we recommend for further reading.

Let A denote a commutative C^*-algebra with unit element $\mathbb{1}$. Then the Gelfand–Naimark theorem asserts that there is a compact topological Hausdorff space X and a C^*-algebra isomorphism $\phi : \mathsf{A}\to C^0(X)$ which preserves the unit. Here, $C^0(X)$ is the vector space of all continuous functions on X taking values in $\mathbb{C}$; endowing it with the pointwise product $(fg)(x) = f(x)g(x)$ $(x\in X)$ as an algebra product and complex conjugation as the *-operation, and taking as C^*-norm $||f||_\infty = \sup_{x\in X}|f(x)|$, $C^0(X)$ is a commutative C^*-algebra. Its unit element clearly is the function $\mathbb{1}(x) = 1$ $(x\in X)$ taking identically the value 1. Moreover, for any state w on A, the induced state $w^\phi(f) = w(\phi^{-1}(f))$ on $C^0(X)$ is given by a probability measure μ_w defined on the Borel sets of X:

$$w^\phi(f) = \int_X f(x)\, d\mu_w(x) \quad (f\in C^0(X)).$$

A probability measure is normalized so that $\int_X \mathbb{1}\, d\mu_w = 1$. Furthermore, a state w on A is pure if and only if the measure μ_w is concentrated at a single point x_0 in X (a "Dirac measure"), that is, $w^\phi(f) = f(x_0)$ for all $f\in C^0(X)$. Therefore, the *Gelfand transform* $\mathsf{f}\mapsto f$, $f(x) = w_x(\mathsf{f})$, where the w_x $(x\in X)$ range over the set of pure states on A, provides the concrete realization of the isomorphism ϕ. Any homeomorphism $F : X\to X$ gives rise to a C^*-isomorphism $A_F : C^0(X)\to C^0(X)$ given

[2] Strictly speaking, the continuity property defined here is *weak-*-continuity*.

by $A_F(f) = f \circ F^{-1}$ and one has $A_F(1) = 1$; pulling A_F back by ϕ renders a C^*-algebra isomorphism α_F of A given by $\alpha_F = \phi^{-1} \circ A_F \circ \phi$ which preserves the unit element 1. Consequently, the dual map $\tau_F = \alpha_F^*$ is an operation on the set of states on A. On the Borel measures of X, this operation is given as $\mu \mapsto A_F^*(\mu) = \mu \circ F$ which can be seen from the measure-transformation equation (cf. [25, Theorem 12.46])

$$\int_X f \circ F^{-1} \, d\mu = \int_X f \, d(\mu \circ F) \quad (f \in C^0(X)).$$

Consequently, for a commutative C^*-algebra, operations on the set of states arise from bijective homeomorphisms in the indicated way. There are also operations typically not arising in this way. A simple example is $\tau : w \mapsto \frac{1}{2}(w_0 + w)$ where w_0 is any fixed but arbitrary state. Another class of examples concerns operations on a particular set of states. Assume that a commutative C^*-algebra with unit element is given as $C^0(X)$ for a compact Hausdorff space X, and select any state w_0, i.e. a probability measure μ_0 on the Borel sets of X. Then the Hilbert space of the GNS representation (see Theorem 1 in [29], or Theorem 3.3.3 in [24] for a more modern version)[3] is given as $L^2(X, \mu_0)$ where at this point, one should bear in mind that the L^2 space is formed by equivalence classes of square-integrable functions on X where functions are defined as equivalent iff they deviate on sets of zero μ_0 measure. With respect to the chosen μ_0, one can introduce *normal states* $w_\varrho(f) = \mathrm{Tr}(\varrho f)\,(f \in C^0(X))$ where ϱ is any density matrix on the Hilbert space $L^2(X, \mu_0)$ and $f \in C^0(X)$ acts as multiplication operator on $L^2(X, \mu_0)$. Then any unitary linear operator U on $L^2(X, \mu_0)$ induces the operation $\tau_U : w_\varrho \mapsto w_{U\varrho U^*}$ on the set of normal states with respect to μ_0 .[4] As a side note, a formulation of classical (statistical) mechanics in a related L^2 -space setting appears in [30]; it also serves as a starting point in the so-called geometric quantization [31].

2.3 The Case of $\mathsf{A} = C_b(X)$ for X Non-compact

The discussion up to now should have clarified the bijective relation between communtative C^*-algebras with unit element and their states, and the algebras $C^0(X)$ on compact Hausdorff spaces X and the probability measures on the corresponding Borel sets. The latter mathematical framework is the starting point of classical statistical theories. Here, X usually contains the (classical) degrees of freedom of a physical system; in fact, most commonly $X = T^*Q$ is the phase space of a system whose degrees of freedom can move in some smooth manifold Q. In this situation, there arises the difficulty that even if Q happens to be compact (which needn't be the case), T^*Q is not. Therefore, we are confronted with the circumstance that in

[3] Given any C^*-algebra $\mathcal{A}$ (not necessarily commutative) containing a unit element, and a state ω on $\mathcal{A}$, the *GNS representation* is a triple $(\mathcal{H}, \pi, \Omega)$ where $\mathcal{H}$ is a Hilbert space, π is a unital *-representation of $\mathcal{A}$ by bounded linear operators on $\mathcal{H}$ and Ω is a unit vector in $\mathcal{H}$ so that $\omega(\mathbf{a}) = \langle \Omega, \pi(\mathbf{a})\Omega \rangle$ holds for all $\mathbf{a} \in \mathcal{A}$, and $\pi(\mathcal{A})\Omega$ is dense in $\mathcal{H}$. For every state on a unital C^*-algebra there is such a GNS representation and it is unique up to unitary equivalence.

[4] Such an operation is in general only weakly *sequentially* continuous.

many physically relevant cases, X isn't in a natural way compact. This issue is of some concern for us because it has some consequences for the convergence of states which we need to consider in order to obtain solutions to the D-CTC problem in the following section.

Therefore, assume now that X is a locally compact Hausdorff topological space, and define $C_0(X)$ as the set of all continuous functions $f : X \to \mathbb{C}$ that *vanish at infinity*, i.e. given $f \in C_0(X)$, there is for every $\varepsilon > 0$ some compact set K such that $|f(x)| < \varepsilon$ for all $x \in X \backslash K$. Using the same definitions for the algebraic operations as for $C^0(X)$, the set $C_0(X)$ is a commutative C^*-algebra with C^*-norm $||f||_\infty = \sup\{|f(x)| : x \in X\}$. If X is compact, then $C_0(X) = C^0(X)$ (cf. [25, Sect. (7.13a)]), but if X is not compact, then $C_0(X)$ is a commutative C^*-algebra *without* an algebraic unit element. One can still define probability measures μ on the Borel sets of X as the Borel measures that have unit weight, $\mu(X) = 1$. This is equivalent to requiring that the positive functional $w(f) = \int_X f \, d\mu$ ($f \in C_0(X)$) induced by μ on $C_0(X)$ has unit norm, that is, $||w|| = 1$ where $||w|| = \sup\{|w(f)| : ||f||_\infty = 1\}$.

The Gelfand–Naimark theorem which has been stated above for the case of a commutative C^*-algebra A with an algebraic unit element has the following extension to the case that A doesn't possess an algebraic unit element: There is a locally compact Hausdorff space X and a C^*-algebraic isomorphism $\phi : \mathsf{A} \to C_0(X)$ which again is given by the Gelfand transform; so any state w on A (where the normalization condition, in absence of the algebraic unit 1, is replaced by the condition that w has unit norm, $||w|| = 1$) induces a state $w^\phi(f) = w(\phi^{-1}(f))$ ($f \in C_0(X)$) on $C_0(X)$ which is given by a probability measure on the Borel sets of X (this is exactly the statement of the Riesz' Representation Theorem, see e.g. [25, Theorem 12.36]), and the pure states on A are exactly those which arise as probability measures concentrated at single points of X.

For any locally compact Hausdorff space X, $C_0(X)$ is naturally a C^*-subalgebra of $C_b(X)$, the set of all bounded continuous functions $f : X \to \mathbb{C}$. Clearly, $C_b(X)$ becomes a C^*-algebra using the analogous algebraic operations as defined previously for $C_0(X)$, and again, $||f||_\infty = \sup\{|f(x)| : x \in X\}$ as C^*-norm. However, $C_b(X)$ contains an algebraic unit element given by the function taking the constant value 1, similarly as for $C^0(X)$ for a compact X. While in the case that X is not compact, $C_0(X)$ is a proper C^*-subalgebra of $C_b(X)$, any state w on $C_0(X)$, by being induced by a probability measure on the Borel sets of X, extends uniquely to a state on $C_b(X)$, complying with the normalization condition $w(1) = 1$. Since $C_b(X)$ is a commutative C^*-algebra with an algebraic unit element, by the Gelfand–Naimark theorem it is isomorphic to $C^0(\hat{X})$ for a particular compact Hausdorff space $\hat{X}$, the *Stone-Čech compactification* of X. In fact, for any locally compact Hausdorff space X, "extending" $C_0(X)$ to $C_b(X)$ can be viewed as the "standard model" of the Stone-Čech compactification. We refer to the references [22–24] for further discussion and references on this point.

We therefore choose the commutative C^*-algebra $C_b(X)$ with a locally compact (but not necessarily compact) Hausdorff space X as the most suitable and versatile version of an observable algebra for a classical statistical system since.

There are the following rationales for that choice:

(i) As already mentioned, we view the phase space T^*Q of a mechanical system as the standard example for X, and T^*Q isn't compact in a natural way.

(ii) $C_b(X)$ contains an algebraic unit element while $C_0(X)$ does not for non-compact X. However, having a unit is important since it allows to approximate unbounded functions, which often represent important observables such as the Hamilton function H (assumed continuous), by elements of $C_b(X)$ at the level of expectation values. Namely, in the presence of a unit 1, one can form the resolvents $(1+\epsilon H^2)^{-1}(\epsilon > 0)$, and then the functions $H_\epsilon = (1+\epsilon H^2)^{-1}H$ are in $C_b(X)$ and for sufficiently regular states one obtains $\lim_{\epsilon\to 0} w(H_\epsilon) = w(H)$. Therefore, we see $C_b(X)$, possessing a unit element, as preferred since it allows the approximation of unbounded observables in a canonical way.

(iii) We wish to explore the D-CTC condition in the setting of classical probability theory where, by definition, the states are given by probability measures. As mentioned, any state on $C_0(X)$ is actually induced by a probability measure according to Riesz' theorem, and it extends to a state, induced by the same probability measure, on $C_b(X)$. Therefore, we are not missing any states by choosing $C_b(X)$ as observable algebra instead of $C_0(X)$.

It should be obvious that the operations on states considered previously for $C^0(X)$ with a compact X, in particular those induced by bijective homeomorphisms of X, have their completely analogous counterparts also in the case of $C_b(X)$ with locally compact X.

However, if X is not compact, then there are states on the C^*-algebra $C_b(X)$ which are not given by probability measures on the Borel sets on X. Consider as a particular example the case $X = \mathbb{R}$, and the sequence of states on $C_b(\mathbb{R})$ given by $w_n(f) = f(n)$ $(n \in \mathbb{N})$, i.e. the point-measures concentrated at the integers. If this sequence of states is restricted to $C_0(\mathbb{R})$, it converges for $n \to \infty$ to the zero-functional, $\lim_{n\to\infty} w_n(f) = 0$ for all $f \in C_0(\mathbb{R})$. Clearly, this functional is not induced by a probability measure and therefore the w_n (or any generalized subsequence) do not converge weakly to a *state* on $C_0(\mathbb{R})$. On the other hand, by the Banach–Alaoglu Theorem mentioned before, there is a generalized subsequence $\{n(\kappa)\}_{\kappa\in\mathcal{K}}$ in $\mathbb{N}$ with $\lim_\kappa n(\kappa) = \infty$ so that the generalized subsequence $\{w_{n(\kappa)}\}_{\kappa\in\mathcal{K}}$ of states on $C_b(\mathbb{R})$ converges weakly to a state $w(f) = \lim_\kappa w_{n(\kappa)}(f)$. That state w isn't induced by a probability measure on the Borel sets of $\mathbb{R}$ since $w(1) = 1$ while $w(f) = 0$ for all $f \in C_0(\mathbb{R})$. One may argue that such states have pathological properties and therefore aren't induced by probability measures and should not be regarded as physically realistic states. In order to make the distinction visible in the notation we will, for the commutative C^*-algebra $\mathsf{A} = C_b(X)$, denote the set of all C^*-algebraic states by $\mathscr{S}$ as before, and denote the set of states induced by probability measures by $\mathscr{S}^{(P)}$. If X isn't compact, then $\mathscr{S}^{(P)}$ is a proper subset of $\mathscr{S}$.

The arguments leading to the results on the generic solvability of the D-CTC problem for classical bi-partite statistical systems that we shall derive in the next section make considerable use of the convergence of (generalized) sequences of states. Having made the point that we consider $C_b(X)$ with a possibly non-compact X as algebra of observables, we would like to specify criteria ensuring that solutions

to the D-CTC problem for classical bi-partite statistical systems are given by states which actually are induced by probability measures. As we will see in the next section, the condition of "tightness" on sequences of probability measures in combination with Prohorov's Theorem [26, 27] provide such criteria.

3 Classical Statistical Bi-Partite Systems and the D-CTC Condition

We define a *classical statistical bi-partite system* to consist of a direct product $X = X_A \times X_B$ where X_A and X_B are locally compact, Hausdorff topological spaces (or, for one of our results below, metric spaces). Then X is also a locally compact Hausdorff space (res., metric space). We usually think of X_A and X_B as containing the degrees of freedom of two system parts labelled "A" and "B", e.g. $X_A = T^*Q_A$ and $X_B = T^*Q_B$ might be phase spaces over finite dimensional configuration manifolds of many particle systems. The system parts are independent but can be coupled dynamically. Then we take as observable algebras of the subsystems $\mathsf{A}_A = C_b(X_A)$ and $\mathsf{A}_B = C_b(X_B)$; the observable algebra of the full system will be $\mathsf{A} = C_b(X)$. Here, one can think of functions on phase space as the classical example.

Then $\mathsf{A} = C_b(X)$ contains the C^*-subalgebra $\mathsf{A}_{\hat{\otimes}}$ generated by all elements f of the form

$$f = \sum_{j=1}^{N} f_j \otimes g_j$$

where $N \in \mathbb{N}$ and $f_j \in C_b(X_A)$ and $g_j \in C_b(X_B)$ where the tensor product is defined by $f_j \otimes g_j(x_A, x_B) = f_j(x_A)g_j(x_B)$ for all $x_A \in X_A$, $x_B \in X_B$. We will also write $\mathsf{A}_{\hat{\otimes}} = C_b(X_A)\hat{\otimes}C_b(X_B)$. Note that $\mathsf{A}_{\hat{\otimes}}$ contains the unit element of $\mathsf{A} = C_b(X)$.

If w_A is a state on $C_b(X_A)$ and w_B is a state on $C_b(X_B)$, then one can define the *product state* w on $\mathsf{A}_{\hat{\otimes}}$ by setting

$$w(f \otimes g) = w_A(f)w_B(g) \qquad (f \in C_b(X_A), \quad g \in C_b(X_B))$$

and extension by linearity. In the case that $\mathsf{A}_{\hat{\otimes}}$ is a proper C^*-subalgebra of $C_b(X)$, one can still extend the product state w to a state on $\mathsf{A} = C_b(X)$ ([24, Prop. 3.1.6]) which however need not be unique. We call any such state *a product state extension* of w_A and w_B (to $\mathsf{A} = C_b(X)$).

If w_A and w_B are states induced by probability measures μ_A and μ_B on the Borel sets of X_A and X_B respectively, then there is a unique product state w induced by a unique probability measure $\mu = \mu_A \times \mu_B$, the *product measure* of μ_A and μ_B, on the Borel sets of X. The product measure is determined by

$$(\mu_A \times \mu_B)(O \times P) \ = \mu_A(O)\mu_B(P)$$

for Borel sets O of X_A and P of X_B (see Sect. 21 in [25]). We now turn to our results establishing that the D-CTC condition can be fulfilled in classical statistical bi-partite systems in great generality. In Theorem 3.1 we prove a statement to this end entirely set in the C^*-algebraic framework, where the states aren't necessarily

induced by probability measures. Then we present another version in Theorem 3.2 where the states are induced by probability measures; it is for this result that we make use of Prohorov's Theorem, summarized below in this section. In the remark following the statement of Theorem 3.1, we explain how the formulation of the D-CTC condition given here connects to (and is, in fact, more general than) that of [1, 16].

Theorem 3.1 *Let $X = X_A \times X_B$ define a classical statistical bi-partite system where X_A and X_B are locally compact, Hausdorff topological spaces. Let $\tau : \mathscr{S} \to \mathscr{S}$ be an operation and let $w_A \in \mathscr{S}_A$ be a state (in the C^*-algebraic sense) on $C_b(X_A)$.*

Then there is a state $w \in \mathscr{S}$ on $C_b(X)$ (in the C^*-algebraic sense) with the properties

$$w(f_A \otimes 1) = w_A(f_A) \qquad (f_A \in C_b(X_A)) \quad \text{and} \tag{3.1}$$

$$\tau(w)(1 \otimes f_B) = w(1 \otimes f_B) \qquad (f_B \in C_b(X_B)). \tag{3.2}$$

Remark (3.1.A) In line with the terminology in [16], we say that a state w with the properties (3.1) and (3.2) *fulfills the D-CTC condition*, or *is a solution to the D-CTC problem*, with respect to the given $X = X_A \times X_B$, τ and w_A.

(3.1.B) In Sect. 1—where the D-CTC condition of [1] has been summarized—and in [16], the operations are always induced by unitary operators on some Hilbert space on which the algebra of observables is represented; in other words, they are of the form $\tau(\langle . \rangle) = \langle U^* . U \rangle$ with a unitary operator U. The setting here is more general in that this assumption is not being made. In contrast, another assumption on operations which enters here is that the operations are assumed to be weakly continuous in the sense described above which is a natural assumption in the context of states on C^*-algebras. That is not always a natural assumption when operations are induced by unitary operators where usually weak *sequential* continuity is a more natural requirement. Theorem 3.2 below actually only requires weak sequential continuity of the operation τ.

(3.1.C) If both X_A and X_B are compact, then the state w is induced by a probability measure according to the Gelfand–Naimark theorem. One can deduce this also from Theorem 3.2, since the tightness assumptions entering in Theorem 3.2 are automatically fulfilled if both X_A and X_B are compact.

Proof of Theorem 3.1 The proof is analogous to the proof given in [16] in the operator-algebraic quantum field theory context, which in turn is based on the idea of the proof by Deutsch for the quantum mechanical case in finite-dimensional Hilbert spaces [1].

Choose any state w_B° in $\mathscr{S}_B$ and define the state φ_1 in $\mathscr{S}$ by choosing a product state extension of w_A and w_B°, thence obeying

$$\varphi_1(f_A \otimes f_B) = w_A(f_A) w_B^\circ(f_B), \qquad f_A \in C_b(X_A),\ f_B \in C_b(X_B).$$

Then define a sequence of states φ_n ($n \in \mathbb{N}$) in $\mathscr{S}$ inductively choosing product state extensions of w_A and the partial state $f_B \mapsto \tau(\varphi_n)(1 \otimes f_B)$, so that

$$\varphi_{n+1}(f_A \otimes f_B) = w_A(f_A) \cdot \tau(\varphi_n)(1 \otimes f_B) \quad (n \in \mathbb{N}).$$

Note that, as $\tau(\varphi_n)$ is in $\mathscr{S}$, the partial state $f_B \mapsto \tau(\varphi_n)(1 \otimes f_B)$ is in $\mathscr{S}_B$ which then implies that one may choose a product state extension φ_{n+1} from $\mathsf{A}_{\widehat{\otimes}} = C_b(X_A)\widehat{\otimes}C_b(X_B)$ to $\mathsf{A} = C_b(X)$. Without additional conditions however, the product state extensions might be non-unique. The definition of the φ_n implies for all $n \in \mathbb{N}$ (notwithstanding their potential non-unique extension to $\mathsf{A} = C_b(X)$),

$$\begin{aligned}\varphi_{n+1}(f_A \otimes 1) &= w_A(f_A) \cdot \tau(\varphi_n)(1 \otimes 1) = w_A(f_A),\\ \varphi_1(f_A \otimes 1) &= w_A(f_A)w_B^{\circ}(1) = w_A(f_A),\end{aligned}$$

entailing $\varphi_n(f_A \otimes 1) = w_A(f_A)$ for all $f_A \in C_b(X_A)$. Moreover, we have

$$\varphi_{n+1}(1 \otimes f_B) = \tau(\varphi_n)(1 \otimes f_B) \qquad (f_B \in C_b(X_B)) \tag{3.3}$$

for all $n \in \mathbb{N}$.

Another sequence of states $w_{(N)}$ in $\mathscr{S}$ ($N \in \mathbb{N}$) will then be defined from the φ_n by an averaging procedure:

$$w_{(N)}(f) = \frac{1}{N}\left(\sum_{n=1}^{N} \varphi_n(f)\right) \quad (f \in C_b(X)). \tag{3.4}$$

It then follows immediately from the properties of the φ_n that

$$w_{(N)}(f_A \otimes 1) = w_A(f_A) \qquad (f_A \in C_b(X_A),\ N \in \mathbb{N}), \tag{3.5}$$

and furthermore, using (3.3),

$$\begin{aligned}\left|\tau(w_{(N)})(1 \otimes f_B) - w_{(N)}(1 \otimes f_B)\right| &= \left|\frac{1}{N}\left(\sum_{n=1}^{N} \tau(\varphi_n)(1 \otimes f_B) - \varphi_n(1 \otimes f_B)\right)\right|\\ &= \left|\frac{1}{N}\big(\varphi_{N+1}(1 \otimes f_B) - \varphi_1(1 \otimes f_B)\big)\right|\\ &\leq \frac{2}{N}||f_B||_{\infty} \qquad (f_B \in C_b(X_B), \quad N \in \mathbb{N})\end{aligned} \tag{3.6}$$

Owing to the Banach–Alaoglu theorem [28] that we have already mentioned in the previous section, there is a generalized sequence $\{N_\kappa\}_{\kappa \in \mathcal{K}}$, where $\mathcal{K}$ is some directed index set, such that $\lim_\kappa N_\kappa = \infty$ and such that

$$\lim_{\kappa} w_{N_\kappa}(f) = w(f) \qquad (f \in C_b(X))$$

for some state $w \in \mathscr{S}$.

In view of (3.5) and (3.6), and observing the assumed continuity of τ which asserts that $\lim_\kappa \tau(w_{N_\kappa})(f) = \tau(\lim_\kappa w_{N_\kappa})(f)$ for all $f \in C_b(X)$, one now obtains that w has the properties claimed in the statement above,

$$w(f_A \otimes 1) = w_A(f_A) \qquad (f_A \in C_b(X_A)) \quad \text{and} \tag{3.7}$$

$$\tau(w)(1 \otimes f_B) = w(1 \otimes f_B) \qquad (f_B \in C_b(X_B)). \tag{3.8}$$

This proves the Theorem. □

For the remaining part of this section, we introduce some definitions, following [27].

Let μ be a probability measure on the Borel sets of a metric space X. The measure is called *tight* if for any given $\varepsilon > 0$ there is a compact subset K of X such that $\mu(X\backslash K) < \varepsilon$.

Similarly, a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures defined on the Borel sets of a metric space X is called *tight* if for any given $\varepsilon > 0$ there is some compact subset K of X such that

$$\sup_{n\in\mathbb{N}} \mu_n(X\backslash K) < \varepsilon. \tag{3.9}$$

Let X_A and X_B are metric spaces, $X = X_A \times X_B$, and let μ be a probability measure on the Borel sets of X. Then one can define the *marginals* of μ,

$$\mu^{(A)}(O) = \mu(O \times X_B), \qquad \mu^{(B)}(P) = \mu(X_A \times P) \tag{3.10}$$

for Borel sets O of X_A and Borel sets P of X_B. Both $\mu^{(A)}$ and $\mu^{(B)}$ are probability measures.

For later use, we put on record the following statement (see [27], Prob. 5.9).

Lemma 3.1 *Let X_A and X_B be metric spaces and let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on the Borel sets of $X = X_A \times X_B$.*

Then $\{\mu_n\}_{n\in\mathbb{N}}$ is tight if and only if the sequences of marginals $\{\mu_n^{(A)}\}_{n\in\mathbb{N}}$ and $\{\mu_n^{(B)}\}_{n\in\mathbb{N}}$ are both tight.

We shall also make use of the following result.

Prohorov's Theorem [26, 27]

Suppose that X is a locally compact metric space and that $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of probability measures on the Borel sets of X.

If $\{\mu_n\}_{n\in\mathbb{N}}$ is tight, then it is weakly relatively compact: There are a probability measure μ on the Borel sets of X and a subsequence $\{\mu_{n(k)}\}_{k\in\mathbb{N}}$ which converges weakly on $C_b(X)$ to μ, i.e.

$$\lim_{k\to\infty} \int_X f \, d\mu_{n(k)} = \int_X f \, d\mu \quad (f \in C_b(X)). \tag{3.11}$$

Theorem 3.2 *Let $X = X_A \times X_B$ define a classical statistical bi-partite system where X_A and X_B are locally compact metric spaces.*

Let $\tau : \mathscr{S}^{(P)} \to \mathscr{S}^{(P)}$ be an operation on the state space of $C_b(X)$ induced by probability measures on the Borel sets of X, and let $w_A \in \mathscr{S}_A^{(P)}$ be a state on $C_b(X_A)$ which is induced by a probability measure μ_A on the Borel sets of X_A, assumed to be tight.

Suppose also that there is a state $w_B^\circ \in \mathscr{S}_B^{(P)}$ on $C_b(X_B)$ which is induced by a probability measure μ_B° on the Borel sets of X_B, with the property that the sequence of probability measures

$$\mu_B^{(n)}(P) = \tau^n(\mu_A \times \mu_B^\circ)(X_A \times P) \qquad (n \in \mathbb{N})$$

on the Borel sets P of X_B is tight.

Then there is a state $w \in \mathscr{S}^{(P)}$ induced by a probability measure μ on the Borel sets of X such that

$$w(f_A \otimes 1) = w_A(f_A) \qquad (f_A \in C_b(X_A)) \quad \text{and} \tag{3.12}$$

$$\tau(w)(1 \otimes f_B) = w(1 \otimes f_B) \qquad (f_B \in C_b(X_B)). \tag{3.13}$$

Proof Using w_B°, the sequence of states φ_n and whence, the sequence of states $w_{(N)} \in \mathscr{S}^{(P)}$ $(N \in \mathbb{N})$ is constructed as in the proof of Theorem 3.1. It follows easily from the assumptions that the states $w_{(N)}$ on $C_b(X)$ are indeed induced by probability measures, denoted $\mu_{(N)}$. We wish to show that the sequence $\{\mu_{(N)}\}_{N\in\mathbb{N}}$ is tight. According to Lemma 3.1, this follows once it is shown that the sequences of marginals $\{\mu_{(N)}^{(A)}\}_{N\in\mathbb{N}}$ and $\{\mu_{(N)}^{(B)}\}_{N\in\mathbb{N}}$ are both tight. Making use of (3.5), one can see that

$$\mu_{(N)}^{(A)} = \mu_A \qquad (N \in \mathbb{N}) \tag{3.14}$$

and as μ_A has been assumed to be tight, tightness follows for $\{\mu_{(N)}^{(A)}\}_{N\in\mathbb{N}}$. Similarly, (3.3) shows that

$$\mu_{(N)}^{(B)}(P) = \frac{1}{N}\sum_{n=1}^{N} \tau^n(\mu_A \times \mu_B^\circ)(X_A \times P) \tag{3.15}$$

$$= \frac{1}{N}\sum_{n=1}^{N} \mu_B^{(n)}(P) \tag{3.16}$$

holds for all Borel sets P of X_B. Since the sequence $\{\mu_B^{(n)}\}_{n\in\mathbb{N}}$ is by assumption tight, the same can easily be concluded for the averaged sequence $\{\mu_{(N)}^{(B)}\}_{N\in\mathbb{N}}$. Hence, the sequence of probability measures $\{\mu_{(N)}\}_{N\in\mathbb{N}}$ is tight. It can therefore be concluded from Prohorov's Theorem that there is a subsequence $\{\mu_{(N(k))}\}_{k\in\mathbb{N}}$ which converges weakly to a probability measure μ on the Borel sets of X. Then (3.12) follows from

(3.14), and (3.13) is obtained using (3.6) in combination with the weak continuity of τ as in the final part of the proof of Theorem 3.1. □

4 A Simple Example—And Ergodicity

The Cesáro-type limit which enters in the construction of the state w fulfilling the D-CTC condition in Theorems 3.1 and 3.2 is very reminiscent of the discrete time-step evolution averaged limit which is a standard way of obtaining invariant states under a transformation. From this perspective, the construction of w is related to Birkhoff's ergodic theorem [32, Theorem 5.1.1]. In this section, we elaborate a bit on this relation, considering a very simple example: the two-body problem with a spherically symmetric central potential coupling two point masses (particles) in Hamiltonian mechanics. Thus, we have $X_A = X_B = T^*(\mathbb{R}^3) \simeq \mathbb{R}^3 \times \mathbb{R}^3$ as phase spaces for the particles labelled "A" and "B", with Hamiltonian function

$$H_\lambda(q_A, p_A; q_B, p_B) = \frac{1}{2m_A}|p_A|^2 + \frac{1}{2m_B}|p_B|^2 + V(|q_A - q_B|) + \lambda V_{\mathrm{ex}}(q_A, q_B)$$

with particle masses m_A and m_B, and $V : \mathbb{R}_{>0} \to \mathbb{R}$ a smooth function. $V_{\mathrm{ex}}(q_A, q_B)$ is an external potential and $\lambda \geq 0$ is a coupling constant. The coupling constant is introduced mainly to distinguish two cases: $\lambda = 0$, i.e. the center-of-mass moves freely, and $\lambda = 1$, where the center-of-mass moves under the influence of the external potential. We think of Newtonian-type potentials like $V(r) = -\alpha/r$ and $V_{\mathrm{ex}}(q_A, q_B) = -(\beta_A/|q_A| + \beta_B/|q_B|)$ where $\alpha, \beta_A, \beta_B > 0$; however more general (binding, central) potentials are also possible. For the Newtonian-like potentials, we would exclude configurations with $q_A = 0$, $q_B = 0$ and $q_A = -q_B$.

In the case of $\lambda = 0$, the trajectories of bound states (excluding head-on collision) correspond to closed ellipses on which the particles travel in their configuration spaces around the center-of-mass as focal point. For more general binding potentials V, perihelion shifts may occur for bound states, so that the trajectories of the particles in their configuration spaces are rosettae revolving around the center-of-mass in a common orbital plane.

For $\lambda = 1$, the trajectories of the bound states are approximately similar; however the center-of-mass trajectory is almost an ellipse with the center of the potential at $q_A = 0$ and $q_B = 0$ as focal point. This case corresponds to a planet (the "A"-system) with a moon or satellite (the "B" system) that are bound in the gravitational field of a very heavy central star which under the mutual gravitational interaction remains almost at rest and can therefore be effectively described as an external potential. (For this to be a good approximation, the stellar mass is to be very much larger than m_A and m_B, and also $|q_A|$ and $|q_B|$ are to be very much larger than $|q_A - q_B|$.)

Then let $F_t : X_A \times X_B \to X_A \times X_B$ denote the phase flow map for the two-body system with the Hamiltonian function H_λ, taking phase space points from some "initial" time t_i to some "final" time $t_f = t_i + t$. It induces the C^*-algebra isomorphism

$$A_t f = f \circ F_t^{-1} \qquad (f \in C_b(X_A \times X_B))$$

on the phase space functions, and in turn it induces the operation

$$(\tau_t w)(f) = w(A_t f)$$

on the state space $\mathscr{S}$ of $C_b(X_A \times X_B)$. Note that one also has $\tau_t : \mathscr{S}^{(P)} \to \mathscr{S}^{(P)}$, i.e. it maps the set of probability measures on the Borel sets of $X = X_A \times X_B$ to itself.

We wish to illustrate the significance of the tightness assumption in Theorem 3.2. Let us look at phase space points $x_A = (q_{A,i}, p_{A,i})$ and $x_B = (q_{B,i}, p_{B,i})$ at "initial" time t_i. The points correspond to pure states on $C_b(X_A)$ and $C_b(X_B)$, induced by Dirac-measures δ_{x_A} and δ_{x_B} concentrated at x_A and x_B, respectively. The configuration space points q_A and q_B correspond to the inertial coordinates of the particles.

Assume first that $\lambda = 0$. Then, given any x_A, it is always possible to find an x_B so that the resulting particle trajectories form a bound system, but in general, the center-of-mass will then move with a constant (non-zero) velocity. In this case one cannot expect that the sequence of measures[5] $\tau_t^n(\delta_{x_A} \times \delta_{x_B})$ $(n \in \mathbb{N})$ will be tight because (i) of the validity of Liouville's theorem and (ii) the support of these measures in the q-components remains within a ball of sufficiently large, fixed radius around the center of mass at time $n \cdot t$, so it moves to infinity as $n \to \infty$. (One could compensate that by re-defining F_t so as to subtract the center-of-mass motion, but that re-definition depends on the choice of x_A and x_B.)

Consider now the case $\lambda = 1$. Then there are a non-empty open set Y and a compact set K in $X = X_A \times X_B$ so that $F_t^n(Y) \subset K$ for all $n \in \mathbb{N}$. Therefore, whenever one chooses a point $x_A \in X_A$ that is in the A-component of Y, there is some x_B in X_B with $(x_A, x_B) \in Y$. Consequently, one obtains that the sequence of marginal measures $P \mapsto \tau^n(\delta_{x_A} \times \delta_{x_B})(X_A \times P)$ $(n \in \mathbb{N})$ is tight because all of these measures have their support in the B-component projection of K, which is a compact subset of X_B. Hence, in this setting, we can apply Theorem 3.2. As already explained, τ is identified with τ_t, and we may choose w_A as being induced by a probability measure δ_{x_A} for a phase space point x_A that is part of a bound state in Y as just explained; there is actually a wide range of choice for such x_A. Then we may choose w_B° as any δ_{x_B} so that $(x_A, x_B) \in Y$. As discussed, the assumptions of Theorem 3.2 are fulfilled, and there is a state w given by a probability measure on the Borel sets of X so that the properties (3.12) and (3.13) are fulfilled. On the other hand, if one chooses δ_{x_A} for w_A as before, but selects as w_B° a δ_{x_B} so that (x_A, x_B) does not correspond to a bound state, one cannot expect that the required tightness assumption is fulfilled, by a reasoning similar to the $\lambda = 0$ case before.

In the case $\lambda = 1$ and $(x_A, x_B) \in Y$, let us try to understand the properties of the state w constructed in Theorem 3.2 with the properties (3.12) and (3.13). To this end, to ease the illustration, we consider a very much simplified situation. We assume that m_A is very much larger than m_B so that the motion of the "A" particle

[5] Here we slightly abuse notation and identify states with the measures by which they are induced; instead of $\tau_t^n(\delta_{x_A} \times \delta_{x_B})$ we should write, more correctly, the transformed measures $(\delta_{x_A} \times \delta_{x_B}) \circ F_t^n$.

coincides to very good approximation with the center-of-mass motion. Furthermore, we assume that $|q_A|$ is extremely large so that, even for a very high number of orbits of the "B" particle around the "A" particle, the center-of-mass motion is approximately a free motion. This corresponds to a satellite, or "piece of rock" ("B" particle) orbiting a planet ("A" particle) which is on a very remote orbit around a star. We assume that the orbital planes are coincident, and that the orbit of the satellite around the planet is, to good approximation, circular. Then we consider the measures (3.15) constructed in Theorem 3.2 for the present situation,

$$\mu^{(B)}_{(N)}(P) = \frac{1}{N}\sum_{n=1}^{N} \tau_t^n(\delta_{x_A} \times \delta_{x_B})(X_A \times P). \tag{4.1}$$

and change to inertial coordinates q_A, q_B in which (according to our simplifying assumptions) the planet is approximately at rest. The measures $\mu^{(B)}_{(N)}$ depend, of course, on how t is chosen. For the assumed (approximately) circular orbit, let T denote the orbital period. There are several cases that one can consider:

(i) $t = kT$ for some $k \in \mathbb{N}$. Then $\tau_t^n(\delta_{x_A} \times \delta_{x_B})(X_A \times P)$ is independent of n for all $n \in \mathbb{N}$ since we have (in our approximation) $F_t(x_A, x_B) = (x_A, x_B)$ in this case. Therefore, $\mu^{(B)}_{(N)} = (\delta_{x_A} \times \delta_{x_B})(X_A \times P)$ is also independent of N: Applying τ_t just reproduces the initial phase space points.

(ii) $t = kT/\ell$ for some $k, \ell \in \mathbb{N}$. Then $F_t^\ell(x_A, x_B) = (x_A, x_B)$ and hence $\tau_t(\sum_{j=1}^{\ell}(\delta_{x_A} \times \delta_{x_B})(X_A \times P) = \sum_{j=1}^{\ell} \tau_t^j(\delta_{x_A} \times \delta_{x_B})(X_A \times P)$. Therefore, $\mu^{(B)}_{(m\cdot\ell)}(P) = \mu^{(B)}_{(\ell)}(P)$ for all $m \in \mathbb{N}$.

Thus, in case (i), there is a state w fulfilling (3.12) and (3.13) whose partial state on the "B"-part of the system, at initial time t_i, is given just by δ_{x_B}. In case (ii), there is a state w fulfilling (3.12) and (3.13) with partial state on the "B"-part given by

$$\frac{1}{\ell}\sum_{j=1}^{\ell} \delta_{x_B(jkT/\ell)}$$

where $(x_A, x_B(t')) = F_{t'}(x_A, x_B)$. This "statistical mixture of phase space points" can be represented as ℓ copies of the B-particle, i.e. ℓ identical satellites, each on the same circular orbit, separated in position and momenta by a $1/\ell$ fraction of the orbit, so that this phase space distribution gets mapped to itself under the phase space map $F^\ell_{kT/\ell}$. Quite clearly, the cases (i) and (ii) correspond to periodic orbits of the (effective) motion of the satellite around the planet.

(iii) $t = rT$ with r irrational. In this case, there are no "periods" in the sequence $\mu^{(B)}_{(N)}$ $(N \in \mathbb{N})$ and thus any state w constructed in the proof of Theorem 3.2 with the properties (3.12) and (3.13) can be deduced to be induced, on the "B"-part of the system, by a measure $P \mapsto \mu(X_A \times P)$ which is supported on a dense set on the circular orbit of the satellite around the planet. This follows since on the circle, the irrational rotations (1) are known to produce dense orbits under

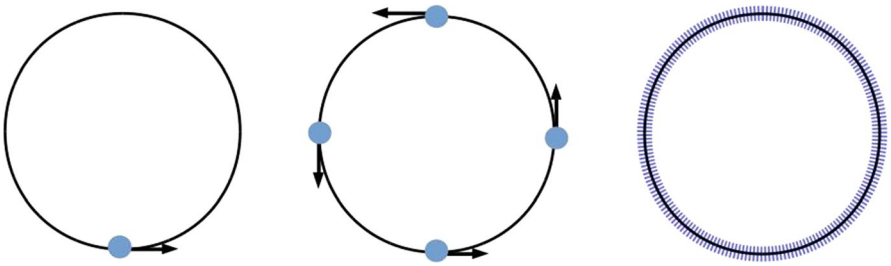

Fig. 2 Illustration of the three cases (i), (ii) and (iii) for the distribution of the state w on the "B"-part of the system at initial time t_i mentioned in the text, from left to right. For case (i), the initial position $q_{B,i}$ of the "satellite" on the circular orbit is depicted by the blue dot and its initial momentum $p_{B,i}$ is represented by the arrow. Case (ii) shows a distribution for $k = 4$, with four copies of the "satellite" separated by moving the phase space points by $T/4$ on the orbit. In case (iii), one obtains a dense distribution over the orbit; the momenta are not indicated (Color figure online)

successive applications by a classic result of Kronecker [32, Theorem 3.2.3] and they (2) are ergodic with respect to the Lebesgue-measure on the the circle [32, Theorem 3.5.7].

The three cases are illustrated in Fig. 2. For an interactive illustration, see [33]

Thus, under the very idealized assumptions made for the simplified situation as described, there is actually a state w fulfilling the D-CTC condition for all F_t ($t \in \mathbb{R}$): The ergodic state, obtained by the construction of w in Theorem 3.1 for the case that $\tau = \tau_{t'}$ induced by $F_{t'}$ for any $t' = rT$ with r irrational. This amounts to taking the Lebesgue measure on the circular trajectory of the satellite. It is worth pointing out that in [3] (Appendix B), solutions to the D-CTC problem in a more general formal framework are also obtained by means of an ergodic averaging.

Going back to the days of the very inception of ideas on ergodicity, when thinking of a satellite orbiting a planet, the rings of the planet Saturn are an example that one might envisage as an approximate realization of the ergodic state. (This "example" appears in publications of Boltzmann, see [34] for references and discussion.) Indeed, if one evolves the ring system by an arbitrary time-step, it appears unchanged, at least at scales larger than about 10 km which is tiny compared to the dimensions of the ring orbits—at scales larger than around 10 km, the rings, which are mostly formed by rocks and pieces of ice of various sizes between the millimeter and kilometer scale, appear almost homogeneous in the angular direction (while there are significant density variations in the radial direction) [35]. (It should be noted that the dynamics of the rings of Saturn is only approximately ergodic, see e.g. [35, 36] and literature cited there for investigations on this problem.) However, our discussion in this section should serve to illustrate that the D-CTC condition is nothing extraordinary in classical statistical mechanics, that it relates to ergodicity and can be viewed as approximately realized in macroscopic physical systems at appropriate scales.

5 Discussion and Conclusion

We have shown that the D-CTC condition is generically fulfilled for classical statistical bi-partite systems, under very general (yet mathematically precise) conditions. The D-CTC condition originated from Deutsch's proposal for giving a description of what it means that parts of quantum systems undergo processes that might be viewed as analogues to "going backwards in time". However, whether or not that condition can be fulfilled rests mainly on convexity and completeness of the space of states of a system, irrespective of its quantum or classical nature. Moreover, in the light of the results of [16], as we have indicated in Sect. 1, one might also doubt if the D-CTC condition actually says very much about closed timelike curves as they are understood in general relativity. This may call into question if the D-CTC condition really is a means of "treating time travel quantum mechanically" (title of the article [11]) or if statements like "quantum mechanics therefore allows for causality violation without paradoxes whilst remaining consistent with relativity" [37] are well-founded.

The starting point of Deutsch's discussion was a classical system where the states consist of finite sequences of "bits", i.e. the state space is a discrete, finite set, not admitting convex combinations. That assumption restricts the choice of states fulfilling the D-CTC considerably, as is shown in an example in [1]. In contrast, taking "q-bits" as the "quantized" version of a classical "bit" system naturally renders a convex and complete state space so that the D-CTC problem generically has many solutions. Nevertheless, it is rather the possibility to take classical statistical mixtures of states than anything specifically quantum mechanical that warrants solutions to the D-CTC condition in a q-bit system. Allowing classical statistical mixtures of "bit" states would have the same effect to this end. (However, in applications, "bit" states are introduced exactly for the purpose of avoiding uncertainties in state discrimination that may occur e.g. in the form of classical mixtures of states, so that from that perspective, forming statistical mixtures doesn't appear natural for "bit" state systems. Yet it is a viable theoretical possibility.)

Therefore, one should be careful not to jump to explanations as to why the D-CTC condition is fulfilled in quantum systems which rely on typical features of quantum systems or their behaviour in spacetime (e.g. interference or uncertainty relations) as this does not relate to what the D-CTC condition—or the fact that it can be generically fulfilled—is based on; such explanations may result in inadequate interpretations and are therefore misleading. We would regard the attempt in [1] to give an explanation for the solvability of the D-CTC problem in quantum mechanics based on a many-worlds interpretation, in this sense, as unconvincing (it has elsewhere been criticized on other grounds [38]).

Yet, the fact that the D-CTC condition can generically be solved in q-bit systems can open interesting aspects for quantum computing and quantum communication [1, 2, 10–15]. In this context, the central point of investigation is—using the notation of the beginning part of Sect. 1—the question what "output" states on the "A" part of the bi-partite system,

$$\langle \mathbf{a} \rangle_{\tilde{A}} = \mathrm{Tr}_A(\tilde{\varrho}_A \mathbf{a}) = \mathrm{Tr}(\varrho U^* (\mathbf{a} \otimes \mathbf{1}) U),$$

can be derived from the density matrix ϱ fulfilling the D-CTC condition for a given unitary U on the full system, and an "input" state $\langle \mathbf{a} \rangle_A = \mathrm{Tr}_A(\varrho_A \mathbf{a})$ on the "A" part of the system. In other words, the investigation is on the map $\phi : \varrho_A \mapsto \tilde{\varrho}_A$ for given U. There are some difficulties here. First, since ϱ is not uniquely determined by U and ϱ_A, ϕ is not naturally defined as a map on the state space of the "A" part of the bi-partite quantum mechanical system. Secondly, given that a map ϕ can be determined by imposing additional selection criteria, if ϱ is constructed as in [1], then ϕ fails to be convex, i.e. it isn't an operation. This is also to be expected in the classical (measure-theoretic) framework which we have considered in the present article in the sense that in general, the dependence of the state w in the proof of Theorems 3.1 and 3.2 and of the partial state $\tilde{w}_A(f_A) = \tau(w)(f_A \otimes 1)$ is not convex in the given state w_A on the "A"-part of the system.

That failure of ϕ to be convex in ϱ_A is in the literature usually referred to by saying that (solutions to the) D-CTC condition are "non-linear" in the input state ϱ_A, and it has been discussed that this may impede the utility of the solvability of the D-CTC problem for the purposes of quantum computing. For considerable further investigation on this issue, see again the articles just cited, and also references given there. A contention expressed in [2] is that due to the failure of ϕ to be convex, the D-CTC condition is incomplete. Basically, that is also our conclusion, however potentially at a more fundamental level, in the sense that the D-CTC condition doesn't depend on typically quantum mechanical features of a bi-partite system. When claiming that quantum mechanics is an important ingredient in avoiding the notorious paradoxes of time travel, but then implementing that formally at the level of the D-CTC condition which is not sensitive to whether a bi-partite system is of quantum mechanical nature or not, and instead just depends on its basic statistical properties, something seems to be missing.

Concerning the possibility that the D-CTC condition isn't sufficiently complete or specific to really allow statements connecting quantum processes and closed timelike curves, we have commented in [16] that a possible approach could be to include spacetime localization into the description, in the spirit of the algebraic framework of quantum field theory as sketched in Sect. 1. Still, one would have to connect locality properties of the unitary operator U, or of the operation τ with the locality properties of the observables. In the present paper, we have not considered locality properties of the observables and that is, in a certain sense, an omission. Therefore, it would be interesting to see if, and how, our results might extend to the analysis of billiard ball collisions in the presence of wormhole-type time machines [5–7].

In [17] we have pointed out that the results in [16] (as well as in [17]) on whether the D-CTC condition can be fulfilled or not depend very much on the assumptions made, and on whether one insists that the D-CTC condition is fulfilled exactly, or just approximately (to arbitrary precision). In fact the question of the adequate mathematical idealization is a common problem when trying to explore unchartered territory in physics by theoretical methods. In the context of the question if one may ascribe physical reality to anything which one might bear the properties of a "time machine", i.e. processes which can be interpreted as brought about by the presence of closed timelike curves, the problem of what constitutes an adequate mathematical idealization acquires considerable importance, and we think that one of the inspiring

aspects about the investigation of the D-CTC condition is to highlight that issue, and potentially gain some insight.

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