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Aspects of cosmology in supergravity

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Proefschrift ingediend tot het
behalen van de graad van
Doctor in de Wetenschappen

augustus 2007

ISBN 978-90-8649-120-9. Wettelijk depot D/2007/10.705/42

Dankwoord

Bij het schrijven van een thesis zijn heel veel mensen betrokken. Vandaar dat ik hier even plaats wil maken voor een woord van dank.

In de eerste plaats wil ik mijn promotor, Antoine Van Proeyen, bedanken voor al zijn hulp gedurende de vier jaar die dit doctoraat in beslag heeft genomen. In de eerste plaats denk ik hierbij natuurlijk aan de projecten en samenwerkingsverbanden die dankzij hem tot stand zijn gekomen en waarvan deze thesis mag getuigen. Maar uiteraard wens ik hem ook te bedanken voor alles wat ik van hem heb mogen leren, en dat niet alleen op fysica-vlak.

Ook de andere leden van de jury, met name Désiré Bollé, Frederik Denef, Christian Maes, Mario Trigiante en Walter Troost, zou ik willen bedanken voor het zorgvuldig nalezen van het manuscript.

I would also like to thank Luca Martucci, Dieter Van den Bleeken, Pietro Fré, Floriana Gargiulo, Ksenya Rulik, Mario Trigiante, Thomas Van Riet, Dennis Westra, Eric Bergshoeff, Jelle Hartong, Andre Ploegh, Jan De Rydt, Torsten Schmidt and Marco Zagermann for very enjoyable and fruitful collaborations during the past four years and for the offered hospitality in Torino, Groningen and München.

Alle leden van de hoge energie groep verdienen een woord van dank voor alles wat ik heb mogen leren via discussies allerhande. Ik denk hierbij bijvoorbeeld aan de frequente studiegroepjes die we als doctoraatsstudenten onder elkaar belegden. Jan, Bert, Thomas, Walter en Dieter: het was een plezier om met en van jullie te mogen leren in een informele en ongedwongen stijl. Verder wil ik ook alle leden van het ITF bedanken voor het bijdragen tot een leuke werksfeer, af en toe onderbroken door koffie, overheerlijke Alma-maaltijden, eersteloonsdrinken en eervol verlies in sportmanifestaties allerhande. Christine Detroije en Anita Raets wil ik bedanken voor hun hulp bij het doorworstelen van alle administratie.

Mijn ouders en familie bedank ik voor alle kansen en vrijheid die ze mij hebben geboden. Verder bedank ik allen die er voor hebben gezorgd dat de voorbije Leuvense jaren ook op niet-fysica-vlak bijzonder interessant en leuk waren. In het bijzonder denk ik hierbij dan aan Stijn, Katrien, Wouter, Katleen, Tom, Harlinde en Noël, die van de Ziekelingenstraat 43 een aangename plaats maakten om te leven, met een van vloeibare lekkernijen gevulde kelder. En uiteraard zijn er nog de vele anderen, die voorgenoemde kelder geregeld kwamen ledigen.

Tot slot dank ik ook het Fonds voor Wetenschappelijk Onderzoek-Vlaanderen voor de financiële steun.

Everything stated or expressed by man is a note in the margin of a completely erased text. From what's in the note we can extract the gist of what must have been in the text, but there's always a doubt, and the possible meanings are many.

Fernando Pessoa

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Chapter 1

Introduction

1.1 Particles, forces and strings

String theory is an ambitious attempt to reconcile the two cornerstones of twentieth century physics: quantum mechanics and general relativity. Our current understanding of nature is for a large part based on quantum field theory, which is the combination of quantum mechanics and special relativity. Quantum field theory has allowed us to describe the world of elementary particles to great accuracy. Indeed, everything in nature around us is made up of a few elementary particles that interact with each other via four fundamental forces. Two of these forces we are familiar with in everyday life, namely gravity and electromagnetism. At subatomic scales, two other forces, namely the strong and weak force, become important. In the late 1940's, it became clear that quantum field theory was a very successful paradigm to combine quantum mechanics and electromagnetism. In the 1970's, it was understood that also the weak and strong nuclear forces are successfully described by quantum field theory. This led to a theory called the Standard Model, which is a quantum field theory that incorporates electromagnetism and the strong and weak forces. It is a gauge theory, which means that it is invariant under a local symmetry group. For the Standard Model, this gauge group is $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. The Standard Model contains fields associated to spin-1/2 particles, such as quarks and leptons, a spin-0 Higgs boson, as well as spin-1 gauge bosons, that are responsible for the mediation of forces. Although gravity is the dominating force at very large length scales, as far as current particle physics experiments are concerned it is not very important. Indeed, at the energy scales that are nowadays accessible in particle accelerators, gravity is so weak compared to the other forces that its effects can be ignored. Combined with general relativity at long distance scales, the Standard Model thus describes virtually all physics down to the scales that are currently probed by particle experiments. At higher and higher energies, probing smaller and smaller distances, gravity is expected to become more important. So, although a consistent quantum theory of gravity is not needed to describe current particle experiments, it is expected to be important at very high energies. Furthermore, one can easily think of other situations, in which quantum gravity effects become very important, such as at the very early stages of the Universe or in black holes.

One could try to quantize Einstein's theory of gravity in a way similar to other quantum field theories. However, such a quantum field theory of gravity is plagued with infinities that severely limit its predictive power. The appearance of infinities in calculations of physical quantities is a problem of most quantum field theories. For so-called renormalizable theories, there exists a procedure, called renormalization, which removes these infinities and makes it possible to predict quantities in a meaningful way. The Standard Model is an example of such a renormalizable quantum field theory. The renormalization procedure, however, fails in a quantum field theory of gravity. A breakdown of renormalizability often signals the appearance of new physics at some energy scale, which for quantum gravity is given by the Planck mass (in units in which $\hbar = c = 1$):

$$m_{\mathrm{Pl}} = G_N^{-1/2} \,, \tag{1.1}$$

where G_N denotes Newton's constant. Einstein gravity is then considered to be the low energy limit of a more general theory that is also valid at and beyond the Planck scale.

The most promising candidate for such a consistent theory of quantum gravity is superstring theory (see refs. [1, 2, 3]). In string theory, elementary particles correspond to tiny vibrating strings. Strings can be open or closed and different vibration modes of a string lead to different particles. The particle spectrum of a quantized closed superstring contains a massless spin-2 particle. This particle is then identified as the graviton, the particle that mediates the gravitational force. Furthermore, in the spectrum of open and closed strings one can also find massless spin-1 particles, leading to Standard Model-like forces. Apart from massless particles, string theory contains a tower of massive states. Their masses M are of the order of the tension T of the string:

$$M^2 \sim T \,, \qquad T = \frac{1}{2\pi\alpha'} \,, \tag{1.2}$$

where we have introduced the so-called Regge slope parameter α'. The characteristic length scale l_{s} of a string is of order $\sqrt{\alpha'}$. The massive modes of a vibrating string thus have masses of order $1/l_{\mathrm{s}}$. These masses are often taken to be of order 10^{18} GeV[1]. Strings interact with each other by joining and splitting. The strength of these string interactions is controlled by the string coupling constant g_{s}. The endpoints of an open string can always join to form a closed string. Open string theories can thus not exist without closed strings. Hence, any consistent theory contains closed strings, and thus has a graviton in its particle spectrum.

String theory is finite order by order in perturbation theory and thus does not suffer from the infinity problems of quantum field theories. A very special property of string theory is also that it contains no adjustable dimensionless parameters. In particular, the string coupling constant turns out to be determined by the theory itself and cannot be chosen at will. This is in contrast with the Standard Model, which contains 29 parameters such as particle masses, which have to be determined by experiment and put in by hand. Superstring theories also enjoy a symmetry, called supersymmetry, which exchanges bosons and fermions.

Actually, there are five consistent superstring theories, defined perturbatively (i.e., as a series expansion in g_{s}). They are called type IIA, type IIB, type I, heterotic SO(32) and heterotic $E_8 \times E_8$. The type I theory has SO(32) as a gauge group, while the 2 heterotic theories have respectively SO(32) and $E_8 \times E_8$ as gauge group. The type II theories do not

[1] It has been pointed out that it is possible to have a string scale of order 1 TeV in [4].

have gauge groups. All superstring theories can be most easily formulated in 10-dimensional flat Minkowski space. The type II theories are invariant under the maximal amount of supersymmetry, namely under 32 supercharges, while the other theories are invariant under 16 supercharges. As the massive string modes have very high masses, one is usually interested in describing only the light (massless) modes. Their dynamics is captured in a low energy effective action, that correctly reproduces scattering amplitudes for these light modes at low energies. The high degree of supersymmetry determines the form of these effective actions up to two derivatives. It turns out that they are given by 10-dimensional supergravity theories, that are invariant under 16 or 32 local supercharges. Supergravity theories form a supersymmetric extension of Einstein gravity; so, at low energies, string theory indeed recovers (an extension of) general relativity.

Originally, it was thought that only type I theory contains both open and closed strings, while the other theories were believed to describe only closed strings. In the 90's however, it was found that type II theories can also contain open strings. Unlike the closed strings, these cannot move freely through space-time, but their ends are confined on $(p+1)$-dimensional hypersurfaces that were called Dp-branes. These Dp-branes turn out to be dynamical objects that are free to move and change their shapes. The tension $T_{\mathrm{D}p}$ of D-branes is inversely proportional to the string coupling constant : $T_{\mathrm{D}p} \sim 1/g_{\mathrm{s}}$. At weak coupling, D-branes are thus very heavy. They correspond to nonperturbative physical objects in string theory.

Although the five string theories all seem to be completely different, it was found in the mid 90's that they are related through various dualities, which means that two (at first sight) completely different theories can be equivalent. These different theories are often useful in a specific region of their parameter space, where calculations are more easily done. It can happen that calculations that are hard in one theory, are a lot easier in the dual theory. For instance, it can happen that strong coupling behavior in one theory is mapped to weak coupling behavior in the dual theory. Calculations at strong coupling can then be performed by mapping them to computations at weak coupling in the dual theory. An example of such a string duality is given by T-duality. It turns out that a theory where strings are moving on a background of flat space where one direction is compactified on a circle of radius R, can be equivalent to a (possibly different) theory, that is now compactified on a circle with radius α'/R. In this way, one can relate the type IIA and type IIB superstring, as well as the heterotic SO(32) and heterotic $E_8 \times E_8$ string theories. Another duality is S-duality, that relates the strong coupling limit of a string theory to the weak coupling limit of another one. It was for instance shown that the SO(32) heterotic string and the type I string are related in this way, while the type IIB theory is self-dual under S-duality. It also turns out that the type IIA and heterotic $E_8 \times E_8$ string theories at strong coupling lead to an 11-dimensional theory of which little is known. D-branes play their role in the dualities as well. It may happen that D-branes in one perturbative description of string theory correspond to fundamental strings in another theory. An example of this is given by the fundamental string of heterotic SO(32), which is under S-duality mapped to the D1-string of the type I theory. Similarly, the fundamental string of type II theory is at weak coupling much lighter than the D1-brane, while at strong coupling it is the other way around. D-branes are in a sense thus as fundamental as strings. The existence of the various dualities has led to the idea that the five different string theories are just different descriptions of one fundamental theory that has been called M-theory. However, what M-theory precisely is, still remains a mystery.

1.2 Connecting strings to reality

Although superstrings lead to a very compelling consistent theory of quantum gravity, valid at very high energy scales, much work still needs to be done, both from a conceptual and from a more practical point of view. From a conceptual point of view, we still do not have a good idea about how M-theory could be formulated. From a more practical point of view, a very important question is how we can connect string theory, that is most naturally formulated in 10 dimensions, to our four-dimensional reality and whether we can make predictions for physics beyond the Standard Model that could possibly show up in future particle experiments.

In order to make contact with four-dimensional physics, one usually assumes that 6 dimensions are curled up in the shape of a compact internal manifold. The sizes of these compact manifolds are then assumed to be rather small, such that at low energies space-time effectively looks four-dimensional. One often assumes that these internal manifolds are very specific, such that a certain fraction of the original supersymmetries is broken. For instance, by compactifying superstring theories on Calabi-Yau manifolds, only one quarter of the original 32 or 16 supersymmetries are preserved. Compactification of string theories generically leads to a set of massless states as well as states that are highly massive. The dynamics of the massless states at low energies is then governed by a four-dimensional effective supergravity action. In the case of Calabi-Yau compactifications for instance, one obtains supergravities in four dimensions that are invariant under either four or eight supersymmetry transformations. A lot of work was subsequently done on Calabi-Yau compactifications of the $E_8 \times E_8$ heterotic string, as it can lead to four-dimensional effective theories with interesting gauge groups that can be viewed as extensions of the Standard Model. With the advent of D-branes, it was realized that also type II theories could give rise to interesting four-dimensional physics upon compactification. It turns out that the dynamics of open strings that end on a Dp-brane is described by a U(1) gauge theory that lives on the $(p+1)$-dimensional world-volume of the brane. In case N D-branes lie on top of each other, this gauge group is enlarged to U(N). Moreover, stacks of D-branes can intersect each other at angles. This can then lead to interesting spectra of particles that are localized on possibly four-dimensional intersections of the branes. In this way, compactification including D-branes might give a very geometrical way of realizing the Standard Model in a fundamental theory like string theory.

One major drawback of these lower-dimensional effective theories is that they usually contain a lot of scalar fields, called moduli, with undetermined vacuum expectation values. They correspond to continuous deformations of the shape and size of the internal manifold. These scalar fields are unobserved in nature and their vacuum expectation values also determine various other quantities in the low energy effective action, such as coupling constants. String models are thus not predictive, as long as vacuum expectation values for these moduli are not determined. It is thus desirable to devise a mechanism such that these scalars acquire a fixed value. Recently, it was found that one can fix these scalars by allowing non-trivial background fluxes along the internal manifold. Indeed, the massless string spectrum contains various anti-symmetric tensor fields. Allowing non-trivial expectation values for their field strengths along the internal space leads to potentials for the moduli in the lower-dimensional effective theories. If one includes other, nonperturbative effects, coming for instance from Euclidean D-branes that wrap cycles of the internal space, one can in fact engineer models where all moduli are fixed at a minimum of a potential.

Combining the ideas of compactifications with fluxes and D-branes provides a first step towards getting realistic models of four-dimensional low energy physics, which might be useful in obtaining predictions for physics beyond the Standard Model. The current belief is that a very large number of consistent string theory compactifications can be obtained, due to different choices of compactification manifolds, fluxes and minima of the moduli potential. Each of these scenarios leads to different low energy physics (different gauge groups, values of coupling constants, etc.). Although it is by now clear that Standard Model-like constructions (or extensions thereof) can be embedded in string theory in various ways, much work still needs to be done. Compactifications including fluxes, which are crucial in these constructions, are less well understood than ordinary Calabi-Yau compactifications for instance.

1.3 Cosmology as a window into fundamental physics

One of the main problems string theory has to cope with is the lack of experimental input. Cosmology might well provide us with an arena in which string theory scenarios can be confronted with specific data. Indeed, in recent decades, cosmology has turned into a real precision science, where very specific and accurate data can be obtained. Various unexpected facts about our universe have arisen from such data, as we shall now explain.

Cosmological observations have made clear that current physical theories of elementary particles only deal with approximately 5% of the content of the universe. Approximately 25% of the universe seems to consist of what is called dark matter. This corresponds to some form of matter that does not emit electromagnetic radiation, but of which the presence can be inferred thanks to its gravitational effect on ordinary visible matter, like stars and galaxies. Not very much is known about the true nature of dark matter. However, supersymmetric extensions of the Standard Model, which might be obtainable from string theory, often contain weakly interacting particles that are good candidates for this dark matter.

In the late nineties, several research groups obtained data that showed that our universe is currently undergoing a phase in which its expansion is accelerating. Until then, it was generally believed that gravity would slow down the expansion of the universe. The discovery of accelerated expansion thus came as a complete surprise. This acceleration can be explained by assuming that the universe is filled with a strange form of energy, called dark energy, that opposes gravity at large distances. In its simplest form, that seems to fit the observations rather well, dark energy corresponds to a positive cosmological constant. It turns out that dark energy constitutes approximately 70% of the matter/energy content of the universe. Not only is the expansion of the universe currently accelerating, but a number of theoretical problems as well as observations also suggest that the universe underwent a period of exponential expansion soon after the Big Bang. This period is known as the inflationary era.

Various proposals have been made for inflation or for the nature of dark energy. Ultimately however, one might wish to embed these proposals in a more fundamental theory.

As string theory seems to give us a fundamental theory, it is an interesting endeavor to see what its implications are for cosmology and to see whether these can be matched with the large number of cosmological data that are currently obtained or that will be obtained in the near future.

1.4 Topics studied in this thesis

Connecting string theory to known low energy physics and to cosmology can be very useful, as it might shed more light on the structure of the theory and open up the possibility for definite predictions from string theory that can be falsified in the future. The topics that are studied in this thesis should be seen in this light.

Chapters 3 and 4 of the thesis are centered around the issue of finding time-dependent solutions in supergravity theories. In chapter 3, we will study a method that is useful in finding cosmological solutions of supergravity theories. This method is known as the Tits-Satake projection. In chapter 4, we will search for a specific kind of cosmological solutions, the so-called scaling cosmologies, in gauged four-dimensional $\mathcal{N} = 8$ supergravity. The importance of these scaling cosmologies lies in the fact that they can correspond to the early- or late-time behavior of more general cosmological solutions of gauged supergravity. Some of the solutions we find moreover describe accelerating cosmologies. Chapter 5 of this thesis then studies D-branes in backgrounds where fluxes have been turned on. As mentioned, such backgrounds are very useful in constructing realistic models in string theory, as well as in connecting string theory to cosmology. More specifically, we will focus on the structure of the fermionic part of the effective action, describing D-brane dynamics in general backgrounds.

The outline of the thesis is as follows. In chapter 2 we start by giving some background in cosmology, that is meant to put chapters 3 and 4 in a broader context. We start by introducing some basic concepts, such as the assumptions of homogeneity and isotropy, the Hubble law and the Friedmann equations. Next, we review some of the more recent observations that have been obtained in cosmology. We end this part by explaining how scalar fields can account for accelerated expansion. In the context of cosmological solutions in the presence of scalar fields, we introduce the scaling cosmologies that will play an important role in chapter 4. In section 2.2 we give some mathematical background on special geometry that will be important for chapter 3. After a short review on complex and Kähler geometry, we discuss the different special geometries that appear in theories with 8 supercharges, namely special Kähler, very special real and quaternionic-Kähler geometry. We end chapter 2 with a discussion on supergravity theories. We first give a general account on supersymmetry and supergravity, indicating for instance the relation between supersymmetry and geometry. Indeed, supersymmetric theories often contain some scalar fields that can be seen as coordinates on a manifold, called the target space. It turns out that supersymmetry severely restricts the possibilities for these target spaces. We will illustrate this via an example. After this, we briefly discuss the supergravity theories that will play a role in this thesis. We start with the theories with 8 supersymmetries that will enter in chapter 3, putting special emphasis on how the geometrical structures introduced in section 2.2 enter. Next, we introduce some maximal supergravity theories, namely type II supergravities in 10 dimensions and gauged maximal supergravity in four dimensions. In the latter case, a global symmetry group of the Lagrangian is promoted to a local symmetry, using some of the vector fields that are present in the theory. This gauging procedure introduces a potential for the scalars in the theory, which makes gauged supergravities interesting from a cosmological point of view. Ten-dimensional type II supergravities and gauged, maximal four-dimensional supergravity will enter the thesis in chapters 5 and 4 respectively.

Chapter 3 deals with the Tits-Satake projection in the context of supergravity theories with 8 supercharges. Essentially, the Tits-Satake projection is a way of truncating a complicated theory in such a way that after truncation, one obtains a model in which cosmological solutions can often be found in an algorithmic way. More specifically, the Tits-Satake projection is a projection on the scalar target spaces that appear in supergravity theories. The projection is such that solutions of the truncated model also constitute a set of solutions of the original, more complicated theory. Moreover, the Tits-Satake projection also leads to a way of obtaining more general solutions of the original theory, starting from the solutions of the projected theory. The Tits-Satake projection was known in a specific class of supergravity theories, namely the ones with symmetric target spaces. It was used in ref. [5] to construct cosmological solutions of $\mathcal{N}=6$, four-dimensional supergravity. In this thesis, this method is extended to a larger class of supergravities, namely the ones exhibiting so-called homogeneous special geometry. This extension was worked out in collaboration with Pietro Fré, Floriana Gargiulo, Ksenya Rulik, Mario Trigiante and Antoine Van Proeyen in ref. [6]. Homogeneous special geometry is discussed in section 3.2. We first review the classification of homogeneous quaternionic-Kähler geometries, comment on their relation with homogeneous special Kähler and very special real geometry and discuss the structure of their isometry algebras. In section 3.3, we explain how Tits-Satake projections can be obtained for symmetric spaces. The aim of this section is to infer some concepts that can be generalized for the more general class of homogeneous special geometries. This section contains a more theoretical discussion, as well as a specific example. The generalization of the Tits-Satake projection to all homogeneous special geometries is done in section 3.4. We again first give a more theoretical discussion, whose fine points are explained in more detail in subsequent subsections. We end this chapter with a discussion of the results obtained for the Tits-Satake projection of homogeneous special geometries. As an application, we argue that the Tits-Satake projection gives us a tool for grouping $\mathcal{N}=2$ supergravity theories with homogeneous target spaces in a small number of universality classes. We mention the cosmic billiard phenomenon as a physical reason as to why such a grouping in universality classes might be relevant. Finally, we also give an example in which the Tits-Satake projection can be given a microscopic meaning in string compactifications including stacks of D-branes.

In chapter 4, we study scaling cosmologies in $\mathcal{N}=8$ gauged supergravity in four dimensions, in which case there is a potential for the scalars. We perform a truncation of the theory, in which we only keep gravity as well as a subset of the scalar fields in the theory. This truncation, as well as the resulting theory is explained in section 4.2. It turns out that the potential is of the so-called 'multiple exponential' type, meaning that it is a sum of exponential terms. In section 4.3, we review some results concerning scaling cosmologies in theories with multiple exponential potentials and flat target spaces. We indicate that, after truncation, some $\mathcal{N}=8$ gauged supergravity theories allow for scaling cosmologies. We give the different possibilities that can occur and comment on various properties of the corresponding cosmologies. The higher-dimensional origin of these four-dimensional solutions is commented upon in section 4.4. We end this chapter with some comments on the relation between scaling cosmologies and de Sitter vacua in supergravity theories. This chapter is based on work done in collaboration with Thomas Van Riet and Dennis Westra [7].

Chapter 5 then studies fermionic actions for D-branes in general backgrounds with fluxes. We start with some elementary facts about D-branes and comment on the relevance of knowing the structure of the fermionic terms in D-brane effective actions in the presence of fluxes. Building on earlier results of refs. [8, 9], a concise form of this action is given to quadratic order in the fermions in section 5.2. We also indicate how kappa-symmetry can be fixed and how the supersymmetry transformations of the physical D-brane degrees of freedom are found. We moreover show how our action is consistent with T-duality, thus providing an important check on this result. The fermionic action found in section 5.2 is not in canonical form. In section 5.5, we show how, by performing appropriate field redefinitions, one can obtain an action in canonical form. The results discussed in this chapter were obtained in collaboration with Luca Martucci, Dieter Van den Bleeken and Antoine Van Proeyen in ref. [10].

Finally, we end this thesis with an appendix on simple Lie algebras, on real Clifford algebras and an appendix that summarizes some conventions used throughout this thesis.

Chapter 2

Cosmology and supergravity

In this chapter, we will introduce some necessary background material. The first part focuses on some aspects of modern cosmology. We will give a short review of the current status in cosmology, focusing on the recent observation of the accelerated expansion of the universe and the role of scalar fields in finding explanations for this phenomenon. In a second part, we give some mathematical background concerning some geometrical notions that will appear in this work. More specifically, we will define the different types of special geometry that occur in supergravity theories with 8 supercharges. The last section deals with supergravity. After a short general introduction to supersymmetry and supergravity, we will briefly introduce the different supergravity theories that appear in this thesis. More specifically, we will discuss theories with 8 supercharges, maximal supergravities in 10 and 11 dimensions and maximal gauged supergravity in four dimensions. These theories will appear in this thesis in chapters 3, 5 and 4 respectively. The review offered here is very short and by no means complete. We will only highlight some specific aspects of these theories that will be relevant later on.

2.1 An introduction to cosmology

In this section, we will review some basic facts concerning modern cosmology. Some good references for this part are given in [11, 12, 13]. The first part focuses on elementary cosmology, reviewing how the dynamics of an expanding universe is analyzed. The second part gives a short resume of the current observational status in cosmology, focusing on several observational constraints imposed on the value of certain cosmological parameters. These measurements also indicate that the expansion of the universe is currently accelerating. In the third part, we then indicate how coupling scalar fields to gravity can take account of this acceleration. We introduce a specific type of cosmological solutions in models of gravity coupled to scalar fields, that will play a role in chapter 4, namely the scaling cosmologies.

2.1.1 The FLRW metric : the expanding universe

One of the main simplifying assumptions that allows us to use general relativity to the entire universe, is the fact that the universe is spatially homogeneous and isotropic on large scales. Isotropy is the assumption that the universe looks the same in all directions, while homogeneity means that the universe looks the same at every point. Evidence for isotropy mainly comes from observations of the cosmic microwave background. Invoking the Copernican principle, namely that we do not occupy a special point in the universe, then implies isotropy around every point in the universe and hence homogeneity. The upshot of these two assumptions is that space-time on cosmological scales can be described by the so-called Friedmann-Lemaître-Robertson-Walker metric (FLRW metric):

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\Big[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\Big]\,, \tag{2.1}$$

where $a(t)$ is called the scale factor. It describes the relative size of the spatial sections, which are the slices of constant cosmic time t, at different times. The parameter k describes the curvature of the spatial sections and is normalized to either $+1, 0$ or -1. For $k = 1$ one has positively curved spatial sections, that are thus locally isomorphic to three-spheres. For $k = 0$ the hypersurfaces of constant t are locally flat, while for $k = -1$ they are negatively curved (locally hyperbolic).

A very useful quantity in cosmology that can be defined from the scale factor is the Hubble parameter H:

$$H(t) = \frac{\dot{a}(t)}{a(t)}\,, \tag{2.2}$$

where $\dot{}$ means a derivative with respect to cosmic time t. One can show that, when a photon is emitted at time t_e with wavelength λ_e, it will be observed today, at time t_0, with a wavelength λ_0, obeying:

$$\frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} \equiv 1 + z\,. \tag{2.3}$$

When $z > 0$ photons are redshifted, while when $z < 0$ they undergo a blueshift. Around 1920, Slipher, Hubble and Humason measured the shifts in spectral lines of a number of objects that were later identified as galaxies. Almost all spectral lines were redshifted. Since the redshifts $z \ll 1$ in their observations, the Newtonian Doppler-shift formula leads to the conclusion that these galaxies move away from us with radial velocity $v = cz$. Furthermore, their observations also indicated that the redshifts increase with increasing distance from the galaxy to the earth. In fact, Hubble obtained the following linear relation (we have chosen units in which $c = 1$):

$$z = H_0 d\,, \tag{2.4}$$

where d is the distance earth-galaxy, and H_0 is a constant known as Hubble's constant.

These observations immediately lead to the conclusion that our universe is expanding: $\dot{a} > 0$. Hubble's law for nearby sources ($z \ll 1$) can be easily derived. Indeed, small redshifts correspond to small values of $(t_0 - t_e)$, such that we can expand

$$a(t_e) = a(t_0) - \dot{a}(t_0)(t_0 - t_e)\,. \tag{2.5}$$

Since (to leading order in z) the difference $(t_0 - t_e)$ is equal to the distance d to the source, we can rewrite this equation as

$$a(t_e) = a(t_0)\Big[1 - H(t_0)d\Big]\,, \tag{2.6}$$

Again to leading order in z, this immediately leads to

$$1 + z = \frac{1}{1 - H(t_0)d} \approx 1 + H(t_0)d\,, \tag{2.7}$$

leading to Hubble's law. The Hubble constant H_0 is thus given by the value of the Hubble parameter (2.2) at the present time t_0.

Note that for larger redshifts this simple linear behavior is no longer valid. Studying the relation between redshift and distance is however very useful, since it provides us with information regarding the expansion rate of the universe. In fact, as we are going to discuss later, precise measurements of this relation have led to the discovery that the expansion of the universe is accelerating.

2.1.2 Dynamics of the scale factor

In the previous section, we used homogeneity and isotropy to introduce the FLRW metric and the scale factor $a(t)$. The dynamics of the scale factor can be analyzed by plugging the metric (2.1) in the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu}\,, \tag{2.8}$$

where G_N is Newton's constant. These equations relate the time evolution of the scale factor to the matter content of the universe via the energy-momentum tensor $T_{\mu\nu}$.

For simplicity, one often assumes that this energy-momentum tensor assumes the form it has for a perfect fluid[1]:

$$T_{00} = \rho\,, \qquad T_{ij} = p g_{ij} \quad (i,j = 1,2,3)\,, \tag{2.9}$$

where ρ is the energy density of the fluid, p the pressure of the fluid, while g_{ij} represents the spatial part of the metric (2.1). Note that this form of the energy-momentum tensor is consistent with the assumptions of homogeneity and isotropy.

Using the FLRW metric (2.1) in the Einstein equations (2.8), one finds two equations:

$$H^2 = \frac{8\pi G_N}{3}\sum_i \rho_i - \frac{k}{a^2}\,, \tag{2.10}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\sum_i (\rho_i + 3p_i)\,. \tag{2.11}$$

We have denoted the different species of energy/matter present in the universe with the index i. The first of the above equations is known as the Friedmann equation, while the second equation is more commonly known as the acceleration equation. Useful quantities

[1] $T_{\mu\nu}$ is written here in the rest frame of the fluid.

that are often introduced are the density parameter Ω_i of a species i and the total density parameter Ω:

$$\Omega_i = \frac{8\pi G_N}{3H^2}\rho_i = \frac{\rho_i}{\rho_c}\,, \qquad \Omega = \sum_i \Omega_i = \frac{\rho}{\rho_c} \quad (\rho = \sum_i \rho_i)\,, \tag{2.12}$$

where ρ_c is the critical density, that corresponds to the energy density of a flat universe:

$$\rho_c = \frac{3H^2}{8\pi G_N}\,. \tag{2.13}$$

The Friedmann equation can then be rewritten in the following way:

$$\Omega - 1 = \frac{k}{H^2 a^2}\,. \tag{2.14}$$

An immediate consequence of this equation is that the local curvature of the universe is related to the total density as follows:

$$\begin{aligned} \Omega > 1 \quad &\leftrightarrow \quad k = +1\,, \\ \Omega = 1 \quad &\leftrightarrow \quad k = 0\,, \\ \Omega < 1 \quad &\leftrightarrow \quad k = -1\,. \end{aligned} \tag{2.15}$$

Note that the Friedmann equation and the acceleration equation are mutually consistent, provided the following equation holds:

$$\dot\rho + 3H(\rho + p) = 0\,. \tag{2.16}$$

This equation simply expresses energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$.

In order to solve for the scale factor, an equation of state is needed, specifying a relation between ρ and p. For now, we will assume a simple equation of state:

$$p = w\rho\,, \tag{2.17}$$

where w is a constant. Conservation of energy-momentum (2.16) then implies that

$$\rho \propto a^{-3(1+w)}\,. \tag{2.18}$$

Although the equation of state (2.17) is very simple, it turns out that it is obeyed by a lot of interesting fluids. One can for instance consider non-relativistic, non-interacting particles, often denoted as dust. This obeys $p = 0$ and $\rho \propto a^{-3}$. Radiation[2] on the other hand obeys $p = \frac{1}{3}\rho$ and $\rho \propto a^{-4}$. A cosmological constant in Einstein's equations is introduced by taking an energy-momentum tensor of the form:

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G_N} g_{\mu\nu}\,. \tag{2.19}$$

One can see that a cosmological constant also behaves as a perfect fluid, obeying the equation of state (2.17) with $w = -1$.

[2] Usually one not only includes photons here, but also other highly relativistic particles.

To study the behavior of some exact solutions of the Friedmann equation, we will look at the case where the spatial sections are flat ($k = 0$). We will see that recent observations indicate that this choice is not only mathematically but also physically appealing. Assuming a constant equation of state parameter $w > -1$, the following solution for the scale factor is obtained:

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{(2/3(1+w))} , \tag{2.20}$$

where a_0 denotes the present value of the scale factor. Note that for matter domination ($w = 0$) or radiation domination ($w = 1/3$), the universe starts at an initial singularity, where $a = 0$. This singularity is known as the Big Bang. The age of such a universe is given by

$$t_0 = \frac{2}{3(1+w)H_0} . \tag{2.21}$$

When w is not close to -1, a good approximation of the age of such a universe is thus given by the inverse of the Hubble constant. One often calls H_0^{-1} the Hubble time.

One can also study solutions of (2.10) when $k = \pm 1$. The same qualitative behavior is found, irrespective whether one assumes matter domination or radiation domination. For positive spatial curvature, one finds that the universe expands from an initial singularity with $a = 0$ (Big Bang) and then collapses again (Big Crunch). For negative curvature, one finds that the universe expands forever after the Big Bang.

Finally, we remark that all ordinary matter has positive energy density and non-negative pressure. From the acceleration equation (2.11), one can then easily see that the expansion of the universe will always decelerate. It seems however that an accelerated expansion of the universe has occurred at least twice. It is generally believed that a phase of acceleration, called inflation, took place in the very early universe. As we will indicate in the next section, recent observations have also shown that the expansion of our universe is presently accelerating. These phases of acceleration cannot be explained by assuming ordinary forms of matter, but some different forms of matter will have to be introduced. One way of dealing with this is given by the introduction of a cosmological constant. Indeed, it follows from the acceleration equation (2.11) and (2.19) that for a positive cosmological constant Λ, the expansion of the universe will be accelerating. A different way of getting acceleration consists of adding scalar fields; this will be explained in section 2.1.4.

2.1.3 Λ and Cold Dark Matter : the standard model of cosmology

Traditionally, one of the most important problems in observational cosmology was the determination of the Hubble constant H_0. One often writes it in terms of the dimensionless number h as follows:

$$H_0 = 100\, h \, \text{km/sec/Mpc} . \tag{2.22}$$

The value for h found by the Hubble Space Telescope Key Project is [14]:

$$h = 0.71 \pm 0.06 . \tag{2.23}$$

This value is largely agreed upon by other methods [15]. Other cosmological parameters can also be measured. In the following, we describe two recent observations that allowed to severely constrain the values of cosmological parameters such as the density parameters that determine the energy/matter content of our universe.

Measurements of the CMB

The Cosmic Microwave Background (CMB) consists of radiation that exhibits an almost perfect black body spectrum with temperature $T_{\mathrm{CMB}} = 2.725 \pm 0.001$ K. The Hot Big Bang model gives a good explanation for this CMB. Indeed, since the ratio of the energy density of radiation to the energy density of dust scales as $1/a(t)$, and since particles that now contribute to matter used to be hotter and were relativistic at early times, one can conclude that the early universe was dominated by radiation. At early times, photons were energetic enough to ionize hydrogen and hence the universe was filled with a charged and opaque plasma. This phase lasted long enough until the photons redshifted enough to allow for the formation of neutral hydrogen atoms, an era denoted as recombination. After recombination, photons decoupled and went free through the universe. Note that at early times, densities were high enough for matter to be in thermal equilibrium, yielding a black body spectrum. The effect of the expansion of the universe is then to retain this initial black body spectrum, but at lower and lower temperatures : $T \propto 1/a$.

Although nearly isotropic, there are small angular anisotropies $\Delta T/T$ in the CMB temperature, of order 10^{-5}. Studying these anisotropies gives a wealth of information regarding various cosmological parameters. A careful analysis of these anisotropies leads to constraints on essentially all cosmological parameters. Considering for instance the results of the WMAP mission [16, 17, 18], the total density of the universe is constrained as follows:

$$0.98 \leq \Omega_{\mathrm{total}} \leq 1.08\,, \tag{2.24}$$

at 95% confidence level. Note that this gives strong evidence for a flat universe. Much tighter constraints on the parameters can be obtained by assuming either a flat universe or a reasonable value for the Hubble constant. If one assumes a flat universe, one obtains the following set of values for the Hubble constant, the matter density Ω_M, the so-called vacuum energy density Ω_Λ ($= 1 - \Omega_M$ under the assumption $k = 0$) and the baryon density Ω_B:

$$h = 0.72 \pm 0.05\,, \tag{2.25}$$

$$\Omega_M = 1 - \Omega_\Lambda = 0.29 \pm 0.07\,, \tag{2.26}$$

$$\Omega_B = 0.047 \pm 0.006\,. \tag{2.27}$$

A few comments are in order here. First of all, note that the total amount of luminous matter in stars and galaxies is of the order $\Omega_{\mathrm{lum}} \sim 0.001$, so most of the baryonic matter is not in the form of stars, but in the form of ionized gas. Even then, the baryonic matter only constitutes a small fraction of the total matter density. So, we must conclude that most of the matter is not made of particles we know today, but constitutes some other form of matter that is denoted as dark matter. This dark matter is cold (non-relativistic) and has been cold for a long period of time. The remaining energy Ω_Λ is called dark energy. It represents a constituent that has an equation of state parameter w close to -1. It can for instance be modelled by a positive cosmological constant and thus leads to an accelerated expansion of the universe. These results are confirmed by type Ia supernovae measurements, which we now describe.

Type Ia supernovae measurements

The first evidence for the fact that matter does not dominate the universe, but that dark energy is needed to explain observations, came from the study of so-called type Ia supernovae. These events are widely believed to be explosions occurring when the mass of a white dwarf, onto which material from a companion star is accreting, becomes larger than the Chandrasekhar limit. These supernovae are very bright, so they can be detected at high redshifts ($z \sim 1$). Since the Chandrasekhar limit is a universal quantity, these explosions are also of a nearly uniform intrinsic luminosity, a property which makes them very useful in determining distances in the universe. Type Ia supernovae thus allow for more complete measurements of the redshift-distance relation. There are some differences in the observed peak brightness of nearby supernovae, but these differences are fortunately closely related to the shapes of the light curves. So, measuring the apparent luminosity along with the behavior of the light curve allows to perform measurements that are precise enough to distinguish between various cosmological models.

Two groups have independently searched for distant supernovae in order to measure cosmological parameters : the High-Z Supernova Team [19] and the Supernova Cosmology Project [20]. It turns out that their data are much better fitted by a universe dominated by a cosmological constant than by a flat matter-dominated model. The supernova results alone allow for a large range of possible values of Ω_M and Ω_Λ, but if one for instance takes $\Omega_M \sim 0.3$, it turns out that Ω_Λ is highly constrained:

$$\Omega_\Lambda \sim 0.7\,, \tag{2.28}$$

corresponding to a vacuum energy density

$$\rho_\Lambda \sim (10^{-3}\ \mathrm{eV})^4\,. \tag{2.29}$$

These observations thus confirm the existence of dark energy and the accelerated expansion of the universe.

2.1.4 Cosmology with scalar fields : acceleration and scaling

The recent observations described above strongly suggest that the expansion of our universe is currently accelerating due to the presence of dark energy. Furthermore, a number of theoretical problems suggest that the very early universe went through a period of exponential expansion, namely the inflationary age. As already emphasized, accelerated expansion cannot occur taking into account ordinary types of matter and/or radiation. We have already mentioned the cosmological constant as a mean of getting acceleration. The purpose of this section is to show that the inclusion of scalar fields can also lead to accelerated expansion. Although current observations constrain the equation of state parameter w to be close to -1 today (so mimicking a cosmological constant), these measurements generally have little to say about a possible time evolution of w. So, one can actually look a little further than a pure cosmological constant and look at situations where the equation of state changes with time, such as when scalar fields are present in the theory. Scalar fields also appear in a variety of particle physics models such as string theory and supergravity and it's interesting to see whether some of these can be used to construct viable cosmological models. We refer to [21] for a review on different possibilities for modelling dark energy.

To see how scalar fields can give rise to accelerated expansion, consider the simple model of a single scalar field ϕ minimally coupled to gravity. We thus consider the ordinary Einstein-Hilbert action, to which we add the following action for the scalar field:

$$S_{\rm scalar} = \int {\rm d}^4x \sqrt{-g}\Big[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big]\,, \tag{2.30}$$

where $V(\phi)$ denotes a potential term for the scalar field. In a flat FLRW space-time, the equation of motion for the scalar field is given by:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{{\rm d}V}{{\rm d}\phi} = 0\,. \tag{2.31}$$

The $H\dot{\phi}$ term plays the role of a friction term and is called the Hubble friction. The energy-momentum tensor for such a scalar field takes the following form:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\Big[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\Big]\,. \tag{2.32}$$

Restricting to our flat FLRW background and scalars that only depend on time, we obtain from this energy-momentum tensor the following expressions for the energy and pressure densities of the scalar field:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{2.33}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{2.34}$$

The Friedmann equation and acceleration equation then take the form:

$$\begin{aligned} H^2 &= \frac{\kappa^2}{3}\Big[\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big]\,, \\ \frac{\ddot{a}}{a} &= -\frac{\kappa^2}{3}\Big[\dot{\phi}^2 - V(\phi)\Big]\,, \end{aligned} \tag{2.35}$$

where $\kappa^2 = 8\pi G_N$. Acceleration will thus occur when $\dot{\phi}^2 < V(\phi)$. From (2.33), one can also infer that when $\dot{\phi}^2 \ll V(\phi)$, the scalar field will obey an equation of state with parameter $w \simeq -1$, thus mimicking a cosmological constant. This is often summarized by stating that flat potentials give rise to accelerated expansion. In the context of inflation, the requirements on the potential are often expressed in terms of the so-called slow-roll parameters:

$$\epsilon = \frac{m_{\rm Pl}^2}{16\pi}\Big(\frac{1}{V}\frac{{\rm d}V}{{\rm d}\phi}\Big)\,, \qquad \eta = \frac{m_{\rm Pl}^2}{8\pi}\frac{1}{V}\frac{{\rm d}^2V}{{\rm d}\phi^2}\,, \tag{2.36}$$

where $m_{\rm Pl} = G_N^{-1/2} \sim 10^{19}$ GeV denotes the Planck mass. Inflation then occurs when the slow-roll conditions, namely $\epsilon \ll 1,\ \eta \ll 1$ are satisfied. These slow-roll conditions are useful in the context of the inflationary era that occurred in the very early universe. In the context of late-time acceleration, they are not so useful, since apart from dark energy, one should also take (dark) matter into account and the contribution of the latter is not included in the slow-roll parameters. One can however define other parameters, such as $\varepsilon = -\dot{H}/H^2$, to check whether accelerated expansion takes place.

The simplest case in which acceleration can occur, is the case in which the scalar field ϕ is constant: $\phi = \phi_0$. From (2.31), one sees that in this case ϕ_0 must correspond to a stationary point of the potential. The energy-momentum tensor (2.32) then simplifies to

$$T_{\mu\nu} = -g_{\mu\nu} V(\phi_0)\,. \tag{2.37}$$

Comparison with (2.19) shows that the value of the potential at the stationary point plays the role of a cosmological constant. When this value is positive (corresponding to positive cosmological constant), the second equation of (2.35) shows that acceleration occurs. Such solutions correspond to so-called de Sitter universes, namely vacuum solutions of the Einstein equations with positive cosmological constant.

One does not need to restrict oneself to solutions in which the scalar fields are constant. Let us for instance study a model with one scalar field and a simple exponential potential $V(\phi) = \Lambda e^{\alpha\phi}$ in a more systematic way, where we assume that the potential is positive: $\Lambda > 0$. Define the following variables (putting $\kappa^2 = 1/2$):

$$X = \frac{\dot{\phi}}{\sqrt{12}H}\,, \qquad Y = \frac{\Lambda e^{\alpha\phi}}{6H^2}\,. \tag{2.38}$$

In terms of these two variables, the equations of motion (2.10,2.11,2.31) can be rewritten as

$$\begin{aligned} &X^2 + Y = 1\,,\\ &X' = -3XY - \sqrt{3}\alpha Y\,,\\ &Y' = Y(\sqrt{12}\alpha X + 6X^2)\,, \end{aligned} \tag{2.39}$$

where $'$ denotes taking a derivative with respect to ln $a(t)$:

$$f' = \frac{\mathrm{d}f}{\mathrm{d}\ \ln a} = \frac{\dot{f}}{H}\,. \tag{2.40}$$

Note that in terms of the variables (2.38), the dynamical equations are written in a simple first order form. This allows us to identify some simple solutions. Indeed, consider a dynamical system of the following form:

$$\dot{x}^i(t) = f^i(x)\,, \tag{2.41}$$

where the functions f^i only depend on the x^i variables and no longer on their derivatives. Trivial solutions of such a system are given by the zeros of the functions f^i:

$$x^i(t) = x_0^i\,, \qquad \text{where} \quad f^i(x_0) = 0\,. \tag{2.42}$$

These simple solutions are called critical points. Their importance lies in the fact that they can correspond to repellers and attractors and thus capture important information about the dynamics. A general solution of these systems will generically interpolate between two critical points.

Apart from the first equation, the system of equations (2.39) takes the form of (2.41). Note that the first equation in (2.39) can be forgotten if this equation is obeyed at some initial time, since one can show that $X^2 + Y$ is constant along solutions. One can then search for critical points of the last two equations in (2.39). The following 3 critical points are found:

$$(X, Y) = \{(1, 0), (-1, 0), (-\frac{\alpha}{\sqrt{3}}, 1 - \frac{\alpha^2}{3})\} . \tag{2.43}$$

The first two critical points are called non-proper, since $Y = 0$ means that the scalar field assumes an infinite value in this solution. There's no potential energy then, so these solutions are also denoted as kinetic dominated solutions. The scale factor for these solutions is a power-law : $a(t) \sim t^{1/3}$. The third critical point corresponds to a true solution of Einstein's equations and is given by:

$$\phi = -\frac{2}{\alpha}\ln\, t + \frac{1}{\alpha}\ln\Big(\frac{6 - 2\alpha^2}{\alpha^4 \Lambda}\Big) , \qquad a(t) \sim t^{1/\alpha^2} . \tag{2.44}$$

This solution only exists when the potential is not too steep ($\alpha^2 < 3$) and when $\alpha^2 < 1$, the solution describes an accelerating universe.

A specific property of the last solution is that the kinetic energy scales as the potential energy:

$$\Lambda e^{\alpha\phi} \sim \dot{\phi}^2 \sim \frac{1}{t^2} . \tag{2.45}$$

This property is usually referred to as a scaling property and the solutions that obey such a scaling relation are denoted as scaling solutions. Scaling solutions can be divided in three classes, namely matter scaling solutions, curvature scaling solutions and scalar dominated scaling solutions. Curvature scaling solutions have $k \neq 0$ and $\rho_b = 0$, where ρ_b denotes the energy density of a possible extra barotropic fluid that is present in the model. Matter scaling solutions are characterized by $k = 0$ and $\rho_b \neq 0$. For scalar dominated solutions, both k and ρ_b are zero.

In the literature, one can find many definitions for the scaling property. The definition that we will adopt in this thesis is the most restrictive one and entails that the ratio of the energy densities of different constituents remains constant during evolution. For a matter-scaling solution, the energy density of the background barotropic fluid evolves in a constant ratio with respect to the scalar field energy density. Scaling cosmologies have a scale factor that is power-law: $a(t) \sim t^P$. We refer to [21, 22, 23] for more phenomenological issues concerning scaling solutions.

Although we encountered scaling cosmologies here in the context of a model with one scalar field, they also appear in models with multiple scalar fields. In chapter 4 of this thesis, we will consider scaling solutions in models with more than one scalar field, where the potential takes the form of a sum of exponentials. Such systems can generically be obtained by performing a truncation of supergravity theories.

2.2 Geometry

Geometry plays a crucial role in supergravity and string theory. In this section, we will introduce the so-called special geometries, that will enter this thesis in chapter 2. This section merely offers some necessary mathematical background; the precise way in which these structures enter supergravity theories will be explained in the next section. We will first start by recalling some notions of complex and Kähler geometry. Next, we will give a small review of special Kähler, very special real and quaternionic-Kähler geometry. We assume some familiarity with notions of differential geometry, such as differentiable manifolds, metrics, connections and curvatures. We refer to [24, 25] for thorough treatments of differential geometry.

2.2.1 Complex and Kähler geometry

Let us start by recalling the definition of a complex manifold.

2.2.1 Definition. *$\mathcal{M}$ is a complex manifold if the following axioms hold:*

1. *$\mathcal{M}$ is a topological space.*
2. *$\mathcal{M}$ is equipped with a family of pairs $\{(U_\alpha, \phi_\alpha)\}$, where $\{U_\alpha\}$ is a family of open sets that cover $\mathcal{M}$ and ϕ_α is a homeomorphism from U_α to an open subset U of $\mathbb{C}^n$.*
3. *Given U_α and U_β such that $U_\alpha \bigcap U_\beta \neq \emptyset$, the map $\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1}$ from $\phi_\alpha(U_\alpha \bigcap U_\beta)$ to $\phi_\beta(U_\alpha \bigcap U_\beta)$ is holomorphic.*

A complex manifold is thus a space that locally looks like $\mathbb{C}^n$. The number n is called the complex dimension of $\mathcal{M}$ and is also denoted by $n = \dim_{\mathbb{C}}\mathcal{M}$. Note that when one views complex manifolds as real manifolds, they have real dimension $2n$.

The notion of complex manifold can also be stated differently in terms of extra structure defined on the manifold. A good introduction to this can be found in [24, 25, 26]. Suppose that a $2n$-dimensional manifold admits a globally defined $(1,1)$-tensor J with local expression $J_\mu{}^\nu \mathrm{d}x^\mu \otimes \partial_\nu$ with the following property:

$$J_\mu{}^\nu J_\nu{}^\rho = -\delta_\mu^\rho \,. \tag{2.46}$$

The manifold is then called an almost complex manifold and J is called an almost complex structure.

It turns out that complex manifolds are always almost complex. The reverse is not necessarily true : not every almost complex manifold is also a complex manifold. In order to determine whether an almost complex manifold is also complex, one defines the so-called Nijenhuis tensor $N_{\mu\nu}{}^\rho$:

$$N_{\mu\nu}{}^\rho = \frac{1}{6} J_\mu{}^\sigma \partial_{[\sigma} J_{\nu]}{}^\rho - (\mu \leftrightarrow \nu) \,. \tag{2.47}$$

The following theorem holds:

2.2.1 Theorem. *An almost complex manifold is a complex manifold if and only if the Nijenhuis tensor of the associated almost complex structure vanishes.*

So, one can also say that a complex manifold is a $2n$-dimensional real manifold that is equipped with an almost complex structure whose Nijenhuis tensor vanishes. On an arbitrary almost complex manifold, it is always possible to find complex coordinates $\{z^i, \bar{z}^{\bar{i}}\}$ $(i = 1, \cdots, n)$ in a point p such that J assumes the following canonical form:

$$J_i{}^j = \mathrm{i}\delta_i^j\,, \quad J_{\bar{i}}{}^{\bar{j}} = -\mathrm{i}\delta_{\bar{i}}^{\bar{j}}\,, \quad J_{\bar{i}}{}^j = J_i{}^{\bar{j}} = 0\,. \tag{2.48}$$

When the Nijenhuis tensor vanishes, it is possible to find such holomorphic coordinates in an entire neighborhood around the point p. The transition functions relating coordinates in overlapping patches are moreover holomorphic.

When the (almost) complex manifold is endowed with a metric $h_{\mu\nu}$, one can construct a new metric $g_{\mu\nu}$ on the manifold in the following fashion:

$$g_{\mu\nu} = \frac{1}{2}(h_{\mu\nu} + J_\mu{}^\rho J_\nu{}^\sigma h_{\rho\sigma})\,. \tag{2.49}$$

This metric is positive definite if h is and it moreover satisfies the property

$$g_{\mu\nu} = J_\mu{}^\rho J_\nu{}^\sigma g_{\rho\sigma}\,. \tag{2.50}$$

Depending on whether the manifold is almost complex or complex, a metric obeying the property (2.50) is called an almost hermitian metric or a hermitian metric and the corresponding manifold is called almost hermitian or hermitian. In terms of the holomorphic coordinates, a hermitian metric takes a form in which the components that are pure in their indices are zero:

$$\mathrm{d}s^2 = g_{i\bar{j}}\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{j}}\,, \quad g_{ij} = g_{\bar{i}\bar{j}} = 0\,. \tag{2.51}$$

Note that if one defines

$$\mathcal{K}_{\mu\nu} \equiv J_\mu{}^\rho g_{\rho\nu}\,, \tag{2.52}$$

the property (2.50) is equivalent to the antisymmetry of $\mathcal{K}_{\mu\nu}$:

$$\mathcal{K}_{\mu\nu} = -\mathcal{K}_{\nu\mu}\,. \tag{2.53}$$

One thus sees that on an (almost) hermitian manifold, a natural two-form can be defined using the (almost) complex structure. This two-form $\mathcal{K} = \frac{1}{2}\mathcal{K}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is called the fundamental two-form.

For Riemannian manifolds, one can introduce a natural connection that is torsionless and preserves the metric, namely the Levi-Civita connection. In a similar manner, one can introduce a natural connection on hermitian manifolds, defined by imposing that it preserves the metric and the complex structure. This does not uniquely determine the connection yet. It leads to the conditions:

$$\Gamma^k_{i\bar{j}} = \Gamma^{\bar{k}}_{ij} = 0\,, \tag{2.54}$$

together with the complex conjugates of these constraints. If one furthermore imposes that the torsion is pure in its lower indices, the connection is uniquely determined and it is pure in all its indices. Explicitly it is given by:

$$\Gamma^k_{ij} = g^{k\bar{l}}\partial_i g_{j\bar{l}}\,. \tag{2.55}$$

A Kähler manifold is a hermitian manifold that obeys an additional restriction:

2.2.2 Definition. *A hermitian manifold is said to be Kähler if the fundamental two-form $\mathcal{K}$ is closed:*

$$\mathrm{d}\mathcal{K} = 0\,. \tag{2.56}$$

When dealing with Kähler manifolds, one often refers to the fundamental two-form as the Kähler form. In the rest of this chapter, we will adopt a normalization for the Kähler form such that in holomorphic coordinates it is given by:

$$\mathcal{K} = \frac{\mathrm{i}}{2\pi} g_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}\,. \tag{2.57}$$

The closure of $\mathcal{K}$ has some interesting consequences. Indeed, if we write out the condition (2.56) explicitly in terms of holomorphic indices, we get:

$$\mathrm{d}\mathcal{K} \sim \mathrm{i}\partial_i g_{j\bar{k}} \mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^{\bar{k}} + \mathrm{i}\partial_{\bar{i}} g_{j\bar{k}} \mathrm{d}\bar{z}^{\bar{i}} \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^{\bar{k}}\,. \tag{2.58}$$

Both terms should be separately zero, implying that

$$\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}\,, \qquad \partial_{\bar{i}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{i}}\,. \tag{2.59}$$

From this, it follows that in local coordinate patches the hermitian metric can be expressed in terms of a real function $K = K(z,\bar{z})$, called the Kähler potential:

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K\,. \tag{2.60}$$

On the overlap of two coordinate patches U_α and U_β, the respective Kähler potentials $K_{(\alpha)}$ and $K_{(\beta)}$ are related by a Kähler transformation:

$$K_{(\alpha)} = K_{(\beta)} + f_{\alpha\beta}(z) + \bar{f}_{\alpha\beta}(\bar{z})\,. \tag{2.61}$$

A second important property of Kähler manifolds is that due to (2.59), the hermitian connection (2.55) is symmetric in its lower indices and hence coincides with the Christoffel connection. For Kähler manifolds, one thus finds that the Levi-Civita connection also preserves the complex structure.

This fact has implications for the holonomy group of Kähler manifolds. The holonomy group of a manifold is defined by using the notion of parallel transport of tangent vectors. Suppose that the manifold $\mathcal{M}$ is endowed with an affine connection. Consider a point $p \in \mathcal{M}$. Denoting the tangent space at p by $T_p\mathcal{M}$, we can parallel transport a vector $X \in T_p\mathcal{M}$ along a closed loop c going through p. The resulting vector $X' \in T_p\mathcal{M}$ can be different from X, hence parallel transport along closed loops generates an action on $T_p\mathcal{M}$. Upon parallel transport of X along every possible loop, this action on $T_p\mathcal{M}$ defines a group, called the holonomy group of the connection. It turns out that this holonomy group is generated by the curvature tensor of the connection $R_{\mu\nu\rho}{}^\sigma(p)$, seen as a two-form. We will always consider holonomy groups of the Levi-Civita connection. In that case, as the Levi-Civita connection preserves the metric, the length of a tangent vector is not changed upon parallel transport along a closed loop. For an n-dimensional Riemannian manifold, one thus sees that the holonomy group should be contained in SO(n). For a Kähler manifold, the fact that the Levi-Civita connection also preserves the complex structure, implies that the holonomy group of a (complex) n-dimensional Kähler manifold is contained in U(n).

Kähler geometry occurs naturally in supergravity theories with 4 supercharges, as we will see in an explicit example in the next section. The Kähler manifolds that appear in supergravity theories generically obey an extra condition, namely the Kähler form should be of even integer cohomology. By this, we mean that the integral over an arbitrary 2-cycle[3] γ gives an even integer:

$$\int_\gamma \mathcal{K}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = 2n\,, \qquad n \in \mathbb{Z}\,. \tag{2.62}$$

Kähler manifolds obeying (2.62) are then called Hodge-Kähler manifolds or Kähler manifolds of the restricted type.

2.2.2 Special Kähler geometry

There are many equivalent ways to define special Kähler geometry. Moreover, one should also make the distinction between rigid special Kähler geometry and local special Kähler geometry, the former relevant for theories invariant under rigid supersymmetry, the latter appearing in supergravity. As we will be mainly concerned with supergravity theories, we will only discuss local special geometry here. An excellent review on special Kähler geometry can be found in [27].

We will adopt the following definition for special Kähler manifolds:

2.2.3 Definition. *A special Kähler manifold is an n-dimensional Hodge-Kähler manifold $\mathcal{M}$, with the following properties:*

1. *On every coordinate chart there exist complex projective coordinate functions $Z^I(z)$, $I = 0, \cdots, n$ and a holomorphic function $F(Z^I)$ that is homogeneous of second degree, such that the Kähler potential is*

$$K(z,\bar{z}) = -\log\left[\mathrm{i}\bar{Z}^I \frac{\partial}{\partial Z^I} F(Z) - \mathrm{i} Z^I \frac{\partial}{\partial \bar{Z}^I} \bar{F}(\bar{Z})\right]. \tag{2.63}$$

2. *On overlaps of charts U_α and U_β, the corresponding functions of property 1 are connected by transition functions of the form:*

$$\begin{pmatrix} Z \\ \partial F \end{pmatrix}_{(\alpha)} = e^{f_{\alpha\beta}(z)} M_{\alpha\beta} \begin{pmatrix} Z \\ \partial F \end{pmatrix}_{(\beta)}, \tag{2.64}$$

where $f_{\alpha\beta}$ is holomorphic and $M_{\alpha\beta} \in \mathrm{Sp}(2n+2,\mathbb{R})$.

3. *On overlaps of three charts, the transition functions of property 2 satisfy the cocycle conditions:*

$$\begin{aligned} e^{f_{\alpha\beta}} e^{f_{\beta\gamma}} e^{f_{\gamma\alpha}} &= 1\,, \\ M_{\alpha\beta} M_{\beta\gamma} M_{\gamma\alpha} &= 1\,. \end{aligned} \tag{2.65}$$

[3] An m-cycle is an m-dimensional submanifold that has no boundary and is itself not the boundary of an $m+1$-dimensional submanifold.

Note that, using (2.64), the Kähler potentials in overlaps of charts are indeed related by a Kähler transformation:

$$K_{(\alpha)} = K_{(\beta)} - f_{\alpha\beta}(z) - \bar{f}_{\alpha\beta}(\bar{z})\,. \tag{2.66}$$

The holomorphic function F is called the prepotential. The above definition clearly depends on the existence of such a prepotential. There exists a different definition of special Kähler geometry, in which the prepotential is no longer explicitly present.

2.2.4 Definition. *A special Kähler manifold is an n-dimensional Hodge-Kähler manifold $\mathcal{M}$, that is the base manifold of a* $\mathrm{Sp}(2n+2,\mathbb{R}) \times \mathrm{U}(1)$ *bundle. There should exist a holomorphic section $v(z)$ such that the Kähler potential is given by*

$$K = -\log[-\mathrm{i} < \bar{v}, v >]\,. \tag{2.67}$$

We have denoted by $< v, w >$ the symplectic inner product

$$< v, w > \equiv v^T \Omega w \qquad \text{with} \quad \Omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{2.68}$$

On the overlap of two coordinate charts U_α and U_β, the sections $v_{(\alpha)}$ and $v_{(\beta)}$ are related by transition functions of the form:

$$v_{(\alpha)} = e^{f_{\alpha\beta}(z)} M_{\alpha\beta}\; v_{(\beta)}\,, \tag{2.69}$$

where $f_{\alpha\beta}(z)$ is holomorphic and $M_{\alpha\beta} \in$ $\mathrm{Sp}(2n+2,\mathbb{R})$*. These transition functions should satisfy the cocycle conditions. The section v should moreover satisfy:*

$$< v, \partial_i v > \;=\; 0\,, \tag{2.70}$$

$$< \mathcal{D}_i v, \mathcal{D}_j v > \;=\; 0\,, \tag{2.71}$$

where the Kähler covariant derivative is defined as $\mathcal{D}_i v = \partial_i v + (\partial_i K) v$.

Instead of using the section v, in supergravity one often expresses everything in terms of the section $V \equiv e^{K/2} v$. Introducing derivatives that are covariant with respect to (2.66) and (2.69):

$$\begin{aligned} U_i &\equiv \mathcal{D}_i V \equiv \partial_i V + \frac{1}{2}(\partial_i K) V\,, \qquad && \mathcal{D}_{\bar{i}} V \equiv \partial_{\bar{i}} V - \frac{1}{2}(\partial_{\bar{i}} K) V\,, \\ \bar{U}_{\bar{i}} &\equiv \mathcal{D}_{\bar{i}} \bar{V} \equiv \partial_{\bar{i}} \bar{V} + \frac{1}{2}(\partial_{\bar{i}} K) \bar{V}\,, \qquad && \mathcal{D}_i \bar{V} \equiv \partial_i \bar{V} - \frac{1}{2}(\partial_i K) \bar{V}\,, \end{aligned}$$

one can define:

$$V = \begin{pmatrix} X^I(z,\bar{z}) \\ F_I(z,\bar{z}) \end{pmatrix}, \qquad U_i = \begin{pmatrix} f_i^I \equiv \mathcal{D}_i X^I \\ h_{iI} \equiv \mathcal{D}_i F_I \end{pmatrix}. \tag{2.72}$$

This allows one to introduce the following symmetric matrix:

$$\mathcal{N}_{IJ} \equiv \begin{pmatrix} \bar{h}_{\bar{i}I} & F_I \end{pmatrix} \begin{pmatrix} \bar{f}_{\bar{i}}^J & X^J \end{pmatrix}^{-1}. \tag{2.73}$$

This so-called period matrix will play a special role in $\mathcal{N}=2$ supergravity, as we will see later on. Note that under symplectic transformations, the section V and the period matrix $\mathcal{N}_{IJ}$ transform in the following fashion:

$$\begin{aligned} V &\rightarrow \tilde{V} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} V\,, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n+2,\mathbb{R})\,, \\ \mathcal{N} &\rightarrow \tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\,. \end{aligned} \tag{2.74}$$

We have given two definitions for special Kähler geometry. They are however equivalent, in the sense that, given a section $V = \begin{pmatrix} X^I \\ F_I \end{pmatrix}$, there always exists an $\mathrm{Sp}(2n+2,\mathbb{R})$ transformation leading to a new section $\tilde{V} = \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix}$ such that $\tilde{F}_I$ is the derivative of a holomorphic function $\tilde{F}(\tilde{X}^I)$, that is homogeneous of degree two:

$$\tilde{F}_I = \frac{\partial}{\partial \tilde{X}^I} \tilde{F}(\tilde{X})\,. \tag{2.75}$$

We will come back to the importance of having two definitions when we discuss $\mathcal{N}=2$ supergravity in four dimensions in the next section.

2.2.3 Very special real geometry

Very special real manifolds are determined by a constant symmetric tensor C_{IJK}. Consider the $n+1$-dimensional subspace of $\mathbb{R}^{n+1}$ defined as

$$M = \{ y^I \in \mathbb{R}^{n+1} | C(y) = C_{IJK} y^I y^J y^K > 0 \}\,, \tag{2.76}$$

endowed with the following metric:

$$a_{IJ} = -\frac{1}{3} \partial_I \partial_J \mathrm{ln} C(y)\,. \tag{2.77}$$

The very special real manifold is then defined as the n-dimensional hypersurface $C(y) = 1$, equipped with the metric induced from the embedding space M:

$$g_{wv} = -3 C_{IJK} y^I y^J_{,w} y^K_{,v}\,. \tag{2.78}$$

One can find coordinates ϕ^w on the manifold by looking for a parametric solution of $C_{IJK} h^I(\phi) h^J(\phi) h^K(\phi) = 1$. We have used $y^I_{,w}$ to denote the ordinary derivative of y^I with respect to ϕ^w. For later purposes, we define the object

$$\mathcal{N}_{IJ} = a_{IJ}|_{C(y)=1} = -2 C_{IJK} y^K + 3 y_I y_J\,, \qquad y_I = C_{IJK} y^J y^K\,. \tag{2.79}$$

We will come back later to the role played by very special real geometry and the definition (2.79) in supergravity.

2.2.4 Quaternionic-Kähler geometry

The last type of geometry that will play a prominent role in this thesis is quaternionic-Kähler geometry. We will now review the definition of quaternionic-Kähler manifolds, as well as some important properties.

Let us start by giving the definition of a quaternionic-Kähler manifold.

2.2.5 Definition. *A quaternionic-Kähler manifold is a real $4n$-dimensional manifold $(n > 1)$[4], with coordinates q^X, $X = 1, \cdots, 4n$ and endowed with a metric g_{XY} such that*

1. *there exists an almost quaternionic structure on it. This means that the manifold is endowed with a triplet of $(1,1)$-tensors ${J^\alpha}_X{}^Y$, $\alpha = 1,2,3$ that satisfy the following relation:*

$$J^\alpha J^\beta = -\delta^{\alpha\beta}\mathbb{1} + \varepsilon^{\alpha\beta\gamma}J^\gamma\,. \tag{2.80}$$

A manifold endowed with an almost quaternionic structure is also called an almost quaternionic manifold.

2. *the almost quaternionic structure is integrable, i.e., it is covariantly constant with respect to the Levi-Civita connection ${\Gamma_{XY}}^Z$ and a non-trivial* SU(2)*-connection ${\omega_X}^\alpha$:*

$$\partial_X {J^\alpha}_Y{}^Z - {\Gamma_{XY}}^U {J^\alpha}_U{}^Z + {\Gamma_{XU}}^Z {J^\alpha}_Y{}^U + 2\varepsilon^{\alpha\beta\gamma}{\omega_X}^\beta {J^\gamma}_Y{}^Z = 0\,. \tag{2.81}$$

3. *the metric g_{XY} is hermitian with respect to the three almost complex structures J^α:*

$$g_{XY} = {J^\alpha}_X{}^Z {J^\alpha}_Y{}^U g_{ZU}\,. \tag{2.82}$$

Note that one does not sum over α in the above equation.

The fact that the SU(2)-connection in the second statement of the above definition is non-trivial means that the SU(2)-curvature tensor

$$\mathcal{R}_{XY}{}^\alpha = 2\partial_{[X}\omega_{Y]}{}^\alpha + 2\varepsilon^{\alpha\beta\gamma}{\omega_X}^\beta{\omega_Y}^\gamma\,, \tag{2.83}$$

is non-vanishing. When this curvature is vanishing, the manifold is called hyperkähler. Hyperkähler geometry is relevant to rigid supersymmetry, while quaternionic-Kähler geometry appears in supergravity theories.

In analogy with the construction of the Kähler 2-form in Kähler geometry, one can define three so-called hyperkähler 2-forms:

$$\mathcal{K}^\alpha = \frac{1}{2}\mathcal{K}^\alpha_{XY}\mathrm{d}q^X \wedge \mathrm{d}q^Y\,, \qquad \mathcal{K}^\alpha_{XY} = {J^\alpha}_X{}^Z g_{YZ}\,. \tag{2.84}$$

In contrast to Kähler geometry, they are not closed but only covariantly closed with respect to the SU(2)-connection one-form $\omega^\alpha = {\omega_X}^\alpha \mathrm{d}q^X$:

$$\mathrm{d}\mathcal{K}^\alpha + 2\varepsilon^{\alpha\beta\gamma}\omega^\beta \wedge \mathcal{K}^\gamma = 0\,. \tag{2.85}$$

[4] For $n = 1$ there are some problems with this definition. In this case, one also has to impose that the Riemann tensor is annihilated by the J^α, meaning that ${J^\alpha}_X{}^V R_{VYWZ} + {J^\alpha}_Y{}^V R_{XVWZ} + {J^\alpha}_W{}^V R_{XYVZ} + {J^\alpha}_Z{}^V R_{XYWV} = 0$. This condition is automatically satisfied for $n > 1$.

For Kähler geometry, the existence of a closed Kähler 2-form implies that their holonomy group is contained in $\mathrm{U}(n)$ instead of in $\mathrm{SO}(2n)$. Similarly, also quaternionic-Kähler manifolds have a restricted holonomy group; in this case the holonomy group should be contained in $\mathrm{SU}(2)\times\mathrm{USp}(2n)$.

It is then convenient to introduce a vielbein field f_X^{iA} on the manifold having as 'flat' indices a pair (i,A) consisting of one SU(2)-index $i=1,2$ and one USp($2n$)-index $A=1,\cdots,2n$. This vielbein is related to the metric by:

$$g_{XY}=f_X^{iA}f_Y^{jB}\varepsilon_{ij}\mathbb{C}_{AB}\,, \tag{2.86}$$

where $\varepsilon_{ij}=-\varepsilon_{ji}$ and $\mathbb{C}_{AB}=-\mathbb{C}_{BA}$ are respectively the flat SU(2)- and USp($2n$)-metrics. This vielbein also satisfies a vielbein postulate, meaning that it is covariantly constant with respect to the Levi-Civita connection, the SU(2)-connection ${\omega_X}^\alpha$ and a USp($2n$)-connection ${\Delta_{XA}}^B$:

$$\mathcal{D}_Xf_Y^{iA}=\partial_Xf_Y^{iA}-{\Gamma_{XY}}^Zf_Z^{iA}+{\omega_{Xj}}^if_Y^{jA}+{\Delta_{XB}}^Af_Y^{iB}=0\,, \tag{2.87}$$

where ${\omega_{Xj}}^i$ is defined in terms of ${\omega_X}^\alpha$ and the Pauli-matrices σ^α by

$${\omega_{Xj}}^i=\mathrm{i}{(\sigma^\alpha)_j}^i{\omega_X}^\alpha\,. \tag{2.88}$$

Defining the USp($2n$)-curvature tensor as follows:

$${R_{XYB}}^A=2\partial_{[X}{\Delta_{Y]B}}^A+2{\Delta_{[X|C|}}^A{\Delta_{Y]B}}^C\,, \tag{2.89}$$

the vielbein postulate relates the Riemann tensor[5] to this USp($2n$)-curvature and the SU(2)-curvature:

$${R_{XY}}^{UV}f_U^{iA}f_V^{jB}=\mathrm{i}\varepsilon^{ik}{\mathcal{R}_{XY}}^\alpha{(\sigma^\alpha)_k}^j\mathbb{C}^{AB}+\varepsilon^{ij}{R_{XYC}}^B\mathbb{C}^{AC}\,. \tag{2.90}$$

As the Riemann tensor generates the holonomy group, this equation explicitly tells us that the holonomy group of the manifold is contained in $\mathrm{SU}(2)\times\mathrm{USp}(2n)$.

Finally, let us mention that for quaternionic-Kähler manifolds the SU(2)-curvature is proportional to the hyperkähler 2-form:

$${\mathcal{R}_{XY}}^\alpha=\frac{1}{2}\nu\mathcal{K}^\alpha_{XY}\,. \tag{2.91}$$

In supergravity theories, the number ν is determined by the gravitational coupling constant κ:

$$\nu=-\kappa^2\,. \tag{2.92}$$

The Ricci tensor is moreover proportional to the metric; quaternionic-Kähler manifolds are thus Einstein spaces:

$$R_{XY}=\frac{1}{4n}g_{XY}R\,,\qquad R=4n(n+2)\nu\,. \tag{2.93}$$

As ν is negative in supergravity, the above equation implies that the quaternionic-Kähler manifolds relevant for supergravity have negative scalar curvature.

[5] We have adopted the following definition for the Riemann tensor here : ${R_{XYZ}}^W=2\partial_{[X}{\Gamma_{Y]Z}}^W+2{\Gamma_{V[X}}^W{\Gamma_{Y]Z}}^V$. The Ricci tensor is then defined as $R_{XY}={R_{ZXY}}^Z$.

2.3 Supersymmetry, supergravity and target space geometry

A deep theorem due to Coleman and Mandula [28] investigates all symmetries of a relativistic field theory that are compatible with having non-trivial scattering amplitudes. Their result is that, when there are massive particles present in the theory, the symmetry algebra consists of a direct product of the Poincaré algebra (containing translations P_a and Lorentz transformations M_{ab}) with an algebra G consisting of internal symmetries[6]. They, however, only looked at symmetries that form a closed algebra under commutation relations. When one also includes symmetry algebras that involve anti-commutation relations as well, more general possibilities occur as symmetries of the S-matrix, as was shown by Haag, Łopuszański and Sohnius [29]. Such symmetry algebras are called superalgebras and its generators can be either bosonic or fermionic. The bosonic part of the algebra still consists of a direct product of the Poincaré algebra and internal symmetries[7]. The fermionic generators are called supercharges; they commute with the translations and they form spinor representations of the Lorentz group. The symmetry transformations generated by these supercharges change the spin or helicity of the state on which they are acting, as opposed to the bosonic symmetries. In a quantum theory, one can consequently split the Hilbert space in a part consisting of bosonic states and a part consisting of fermionic states, where the supersymmetry generators Q act as:

$$Q|\text{boson} >= |\text{fermion} > , \qquad Q|\text{fermion} >= |\text{boson} > \ . \tag{2.94}$$

Note that due to the fact that translations commute with the supercharges, bosonic and fermionic states that are related by supersymmetry have the same mass. In general, the supersymmetry generators do not have to form an irreducible representation under the Lorentz group. When they are reducible and decompose into $\mathcal{N}$ irreducible representations, one has so-called $\mathcal{N}$-extended supersymmetry. Working in four dimensions, where one can split spinors in left and right chirality, the fermionic generators can be written as:

$$Q^i_\alpha = \frac{1}{2}(1-\gamma_5)_\alpha{}^\beta Q^i_\beta \, , \qquad Q_{i\alpha} = \frac{1}{2}(1+\gamma_5)_\alpha{}^\beta Q_{i\beta} \, , \tag{2.95}$$

where α is a spinor index and i is an index running from 1 to $\mathcal{N}$, denoting the different irreducible representations under the Lorentz group. Note that we have denoted the chirality of the fermionic generators by the position of the i-index. Let us for definiteness list the non-

[6] When the theory only contains massless particles, the possible algebras consist of a direct product of the conformal algebra and an algebra G of internal symmetries.

[7] As in the Coleman-Mandula theorem, when only massless fields are present, the Poincaré algebra can be generalized to the algebra of conformal transformations.

zero commutators of the superalgebra of $\mathcal{N}$-extended supersymmetry in four dimensions:

$$
\begin{aligned}
&[M_{ab}, M_{cd}] = \eta_{a[c}M_{d]b} - \eta_{b[c}M_{d]a}\,, \qquad [P_a, M_{bc}] = \eta_{a[b}P_{c]}\,,\\
&[M_{ab}, Q^i_\alpha] = -\frac{1}{4}(\gamma_{ab}Q^i)_\alpha\,,\\
&[U_i{}^j, Q^k_\alpha] = \delta^k_i Q^j_\alpha - \frac{1}{2}\delta^j_i Q^k_\alpha\,, \qquad [U(1), Q^i_\alpha] = -\frac{\mathrm{i}}{2}Q^i_\alpha\,,\\
&[U_i{}^j, U_k{}^l] = \delta^l_i U_k{}^j - \delta^j_k U_i{}^l\,,\\
&\{Q^i_\alpha, Q_{\beta j}\} = (\gamma^\mu \mathcal{C}^{-1})_{\alpha\beta}P_\mu \delta^i_j\,.
\end{aligned}
\tag{2.96}
$$

The first line gives the commutation relations of the Poincaré algebra, consisting of translations P_a and Lorentz generators M_{ab}. The second line indicates how the supersymmetry generators Q^i_α transform in a spinor representation of the Lorentz group. It is furthermore also possible that they transform non-trivially under some part of the internal symmetries. These internal symmetries that rotate the supersymmetry generators constitute the R-symmetry group. It can be shown that for $\mathcal{N}$-extended supersymmetry, the R-symmetry is given by $\mathrm{U}(\mathcal{N})$ [8], which we have written above as $\mathrm{SU}(\mathcal{N})\times\mathrm{U}(1)$, generated by respectively $U_i{}^j$ and $U(1)$. The third line then indicates how the R-symmetry acts on the supersymmetry generators, while the fourth line shows how the $U_i{}^j$ close an $\mathrm{SU}(\mathcal{N})$-algebra. The last anticommutator can be seen as the defining relation of supersymmetry, namely the fact that the anticommutator of 2 supersymmetries gives a translation. From this, it follows for instance that, when translations constitute an invertible operation, supersymmetric theories exhibit an equal number of bosonic and fermionic degrees of freedom. This relation also makes clear that theories invariant under local supersymmetry transformations, should be invariant under local translations (diffeomorphisms) as well and hence incorporate gravity. Theories that are invariant under local supersymmetry are correspondingly called supergravity theories.

Note that there is a bound on the number of supercharges in theories invariant under rigid or local supersymmetry. For theories with rigid supersymmetry, this bound is 16 supercharges, while for supergravity, the bound is 32 supercharges. When there are more than 32 supercharges, the representations of the superalgebra generically contain states with helicity higher than two, leading to inconsistent interactions. The possibilities for supersymmetric theories can then be scanned for all space-time dimensions, by combining this bound on the number of supercharges with the dimension of the minimal spinor in the case under consideration. One finds for instance that in space-time dimensions twelve[9] or higher there are no supergravity theories, since the dimension of a minimal spinor is 64. In 4 dimensions, the minimal spinor has four components. One can thus have supergravity theories for values of $\mathcal{N}$ up to 8. Later on, we will be specifically interested in theories with 8 and 32 supercharges, i.e., $\mathcal{N}=2$ and $\mathcal{N}=8$ supergravity.

In field theories, one is interested in a realization of the supersymmetry algebra (2.96) on fields. A set of fields that transform into each other according to an irreducible representation of the superalgebra is then called a (super)multiplet. When one considers supergravity, there is always a gravity multiplet present in the theory. This gravity multiplet contains a spin-2

[8] This conclusion holds in four dimensions. In other dimensions, different R-symmetry groups can occur, depending on the reality conditions that are obeyed by the fermions.

[9] We only consider 1 time-like direction here. When one has two time-like directions, the dimension of a minimal spinor is 32.

graviton, as well as $\mathcal{N}$ spin-3/2 superpartners, that are called gravitini. The gravity multiplet can furthermore also contain additional bosonic and fermionic fields, depending on the space-time dimension and number $\mathcal{N}$ one is considering. For theories with 32 supercharges, this is the only multiplet that occurs. When the number of supercharges is 16 or lower, other (possibly different) supermultiplets can be coupled to the gravity multiplet. These are then denoted as matter multiplets.

Supersymmetry and supergravity have an intimate connection with geometry. Supermultiplets often contain scalar fields. The kinetic terms of these scalar fields generically assume the form of a non-linear sigma model:

$$S_{\rm kin} = -\frac{1}{2}\int {\rm d}^4x \; g_{XY}(q)\partial_a q^X \partial^a q^Y \,. \tag{2.97}$$

The scalars q^X are then interpreted as maps from space-time to a manifold $\mathcal{M}$, called the target space, that is equipped with a metric $g_{XY}(q)$. The scalars can thus be seen as coordinates on $\mathcal{M}$. Supersymmetry leads to restrictions on these target spaces. In order to see how this works, we give the following result of [30]. Consider the following action in 2 dimensions, for scalar fields q^X and fermion fields ψ^X, $X = 1, \cdots, n$:

$$S = -\frac{1}{2}\int {\rm d}^2x \left(g_{XY}(q)\partial_a q^X \partial^a q^Y + g_{XY}(q)\bar\psi^X\gamma^a\mathcal{D}_a\psi^Y + \frac{1}{6}R_{XVYW}(\bar\psi^X\psi^Y)(\bar\psi^V\psi^W)\right) . \tag{2.98}$$

The covariant derivative $\mathcal{D}_a$ on the fermions is defined using the Levi-Civita connection Γ^X_{YZ} in the following way:

$$\mathcal{D}_a\psi^X = \partial_a\psi^X + \Gamma^X_{YZ}\partial_a q^Y\psi^Z \,, \tag{2.99}$$

while R_{XVYW} denotes the Riemann curvature tensor. Note that the action for the scalars takes the form (2.97). The action (2.98) is invariant under the following supersymmetry transformations:

$$\begin{aligned} \delta q^X &= \bar\epsilon\psi^X \,, \\ \delta\psi^X &= -\partial\!\!\!/ q^X\epsilon - \Gamma^X_{YZ}(\bar\epsilon\psi^Y)\psi^Z \,. \end{aligned} \tag{2.100}$$

The action is also invariant under coordinate reparametrizations $q^X \to q'^X(q)$. Under these reparametrizations, the fermions transform as vectors : $\psi'^X = \frac{\partial q'^X}{\partial q^Y}\psi^Y$. These diffeomorphisms moreover also commute with the supersymmetry transformations, due to the presence of the second term in the above transformation of ψ^X.

At this point, the only restriction imposed on the manifold $\mathcal{M}$ is that it is Riemannian, namely that it is equipped with a positive definite metric g_{XY}. The question asked in [30] is whether the action (2.98) admits additional supersymmetry invariances than the ones stated in (2.100). Possible extra supersymmetries can be parametrized using a general ansatz:

$$\begin{aligned} \delta q^X &= J_Y{}^X\bar\epsilon\psi^Y \,, \\ \delta\psi^X &= -I_Y{}^X\partial\!\!\!/\phi^Y\epsilon - S^X_{YZ}(\bar\epsilon\psi^Y)\psi^Z - V^X_{YZ}(\bar\epsilon\gamma^a\psi^Y)\gamma_a\psi^Z - P^X_{YZ}(\bar\epsilon\gamma_5\psi^Y)\gamma_5\psi^Z \,, \end{aligned} \tag{2.101}$$

where I, J, S, V and P denote tensors on the target space $\mathcal{M}$. Invariance of the action (2.98) then imposes the following constraints:

$$\begin{aligned} & g_{XZ}J_Y{}^Z = g_{YZ}I_X{}^Z\,, \quad \nabla_Z J_X{}^Y = 0\,, \quad V^X_{YZ} = P^X_{YZ} = 0\,, \\ & S^X_{YZ} = \Gamma^X_{ZV}J_Y{}^V\,, \quad J_X{}^Z I_Z{}^Y = \delta^Y_X\,, \end{aligned} \tag{2.102}$$

where ∇_Z denotes the covariant derivative with respect to the Levi-Civita connection. The two fermionic invariances (2.100) and (2.101) should also satisfy an $\mathcal{N}=2$ superalgebra. Introducing

$$J^{(0)Y}_X = \delta^Y_X\,, \quad J^{(1)Y}_X = J_X{}^Y\,, \tag{2.103}$$

this leads to:

$$J^{(a)}J^{(b)-1} + J^{(b)}J^{(a)-1} = 2\delta^{ab}\,, \quad a,b = 0,1\,. \tag{2.104}$$

Taking the values $a = 0, b = 1$ for the indices, we learn from this equation that $J^{(1)} = -J^{(1)-1}$, implying:

$$J_X{}^Z J_Z{}^Y = -\delta^Y_X\,. \tag{2.105}$$

The action (2.98) thus allows for an extra supersymmetry if and only if the manifold $\mathcal{M}$ is equipped with a tensor $J_X{}^Y$ such that (2.105) holds. In other words, the manifold is equipped with an almost complex structure. The metric on the manifold is furthermore hermitian with respect to this almost complex structure:

$$g_{XY}J_U{}^X J_V{}^Y = g_{UV}\,. \tag{2.106}$$

One thus finds that $\mathcal{M}$ is an almost hermitian manifold. As the almost complex structure $J_X{}^Y$ is covariantly constant with respect to the Levi-Civita connection, one also finds that the Nijenhuis tensor of $J_X{}^Y$ vanishes and that the Kähler form on $\mathcal{M}$ is closed. One eventually concludes that requiring that (2.98) admits 4 supercharges implies that the target space $\mathcal{M}$ is a complex Kähler manifold.

Although the above analysis was performed in 2 dimensions, one can obtain similar conclusions in 4 dimensions. Indeed, four-dimensional $\mathcal{N}=1$ supersymmetric models are invariant under 4 supercharges and the scalars in these models generically span a Kähler manifold. From this example, we can thus conclude that supersymmetry often puts restrictions on the geometries spanned by scalars in a theory.

2.4 An overview of supergravity theories

In this section, we will give a short overview of the different supergravity theories that will be used in this thesis. We will first look at theories in 5 and 4 dimensions that are invariant under eight supercharges, putting emphasis on the special geometries that appear as target spaces in these theories. We will also indicate how dimensional reduction leads to additional relations between these special geometries. Finally, we will discuss some supergravity theories that are maximally supersymmetric, namely type II theories in 10 dimensions and gauged $\mathcal{N}=8$ supergravity in four dimensions.

2.4.1 Supergravity with eight supercharges

$\mathcal{N}=2$ supergravities in five dimensions

In five-dimensional supergravity with 8 supercharges, the gravity multiplet consists of the fünfbein e_μ^a, 2 gravitini ψ_μ^i and one vector field A_μ. There are furthermore 2 kinds of matter multiplets possible[10], namely vector multiplets and hypermultiplets. A vector multiplet consists of one vector, two spin 1/2-fermions that obey symplectic Majorana reality conditions and one real scalar. The field content of a hypermultiplet is given by four real scalars and two symplectic Majorana fermions. The bosonic part of the Lagrangian, describing the coupling of the gravity multiplet to n vector multiplets and m hypermultiplets, is given by [31]:

$$\begin{aligned}\mathcal{L}_{\mathrm{bos},5d} &= \sqrt{-g}\Big(\frac{R}{2}-\frac{1}{4}\mathcal{N}_{IJ}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J|\mu\nu}-\frac{1}{2}g_{vw}(\phi)\partial_\mu\phi^v\partial^\mu\phi^w-\frac{1}{2}h_{XY}(q)\partial_\mu q^X\partial^\mu q^Y\Big)\\ &\quad+\frac{1}{8}C_{IJK}\varepsilon^{\mu\nu\rho\sigma\tau}\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\sigma}A^K_\tau\,,\end{aligned}\tag{2.107}$$

where the field strengths of the vectors are given by $\mathcal{F}^I_{\mu\nu}=2\partial_{[\mu}A^I_{\nu]}$. The scalars of the vector multiplets are denoted by ϕ^v $(v=1,\cdots,n)$, while q^X $(X=1,\cdots,4m)$ denote the scalars of the hypermultiplets. The index I runs from 0 to n. Indeed, the total number of vectors in the theory is $n+1$, with n vectors residing in the vector multiplets and 1 vector coming from the gravity multiplet. The last term in (2.107) is called the Chern-Simons term. It is determined by a constant, symmetric 3-tensor C_{IJK}.

The kinetic terms of the scalars assume the form of a non-linear sigma model (2.97). In the present case, it turns out that supersymmetry restricts this target space $\mathcal{M}$ to consist of a direct product of 2 factors:

$$\mathcal{M}=\mathcal{VSR}\otimes\mathcal{QK}\,.\tag{2.108}$$

The first factor $\mathcal{VSR}$, describing the geometry of the vector multiplet scalars ϕ^v, is a very special real manifold, determined by the tensor C_{IJK}. The second factor encodes the manifold spanned by the hypermultiplet scalars q^X and corresponds to a quaternionic-Kähler space. Note that the choice of a target space of the form (2.108) determines the bosonic Lagrangian (2.97). Once such a choice is made, the kinetic terms of the scalars can be written in terms of the metric on the product manifold (2.108). The kinetic terms of the vectors are determined by the object $\mathcal{N}_{IJ}$, which for very special real geometry was defined in (2.79). The choice of a very special real manifold also determines the Chern-Simons terms via the tensor C_{IJK}. This conclusion not only holds for the bosonic part of the Lagrangian, but also for the fermionic part as well as for the supersymmetry transformation rules. They are equally well described in terms of geometrical objects, such as curvature tensors, defined on a very special real and a quaternionic-Kähler manifold.

$\mathcal{N}=2$ supergravity in four dimensions

In $\mathcal{N}=2$ supergravity in four dimensions, the gravity multiplet again consists of a vierbein e_μ^a, 2 gravitini ψ_μ^1,ψ_μ^2, that now obey Majorana conditions and one vector A_μ. Again, one

[10] One can also introduce tensor multiplets. However for the theories discussed here, these are equivalent to vector multiplets.

can add vector or hypermultiplets as possible matter multiplets. The field content of the hypermultiplets is the same as in 5 dimensions, with the fermions now obeying Majorana conditions. We will again denote the scalars in the hypermultiplets by q^X. A vector multiplet now consists of one vector, 2 Majorana spinors and one complex scalar. The bosonic action in this case is given by [32]:

$$\begin{aligned}\mathcal{L}_{\mathrm{bos},4d} &= \sqrt{-g}\Big(\frac{R}{2} - g_{i\bar{j}}\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} - \frac{1}{2}h_{XY}(q)\partial_\mu q^X \partial^\mu q^Y + \frac{1}{4}(\mathrm{Im}\mathcal{N}_{IJ})\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J\mu\nu}\Big) \\ &\quad -\frac{1}{8}(\mathrm{Re}\mathcal{N}_{IJ})\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\sigma}\,, \\ &= \sqrt{-g}\Big(\frac{R}{2} - g_{i\bar{j}}\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} - \frac{1}{2}h_{XY}(q)\partial_\mu q^X \partial^\mu q^Y + \frac{1}{2}\mathrm{Im}\Big(\mathcal{N}_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J\mu\nu}\Big)\Big)\,.\end{aligned} \tag{2.109}$$

We have denoted the field strengths of the vectors by $\mathcal{F}^I_{\mu\nu}$. In the last line, we have introduced the (anti-)self-dual field strengths:

$$\mathcal{F}^{\pm I}_{\mu\nu} = \frac{1}{2}(\mathcal{F}^I_{\mu\nu} \pm \tilde{\mathcal{F}}^I_{\mu\nu})\,, \quad \tilde{\mathcal{F}}^I_{\mu\nu} = -\frac{\mathrm{i}}{2}e\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{I\rho\sigma} \quad (e = \sqrt{-g})\,. \tag{2.110}$$

As in five dimensions, the index I on the vectors runs from 0 to n. The complex scalars of the vector multiplets have been denoted by z^i, $i = 1, \cdots, n$.

As in the five-dimensional case, supersymmetry tells us that the target space of the scalars has a direct product form:

$$\mathcal{M} = \mathcal{SK} \otimes \mathcal{QK}\,. \tag{2.111}$$

The manifold $\mathcal{SK}$ describing the geometry of the scalars of the vector multiplets, is now a special Kähler manifold, while the manifold $\mathcal{QK}$ spanned by the hypermultiplet scalars is again a quaternionic-Kähler manifold.

Similar to the situation in five dimensions, the full Lagrangian and supersymmetry transformation rules are determined by geometrical quantities that can be defined on a special Kähler manifold and a quaternionic-Kähler manifold. For instance, given a prepotential F, one can reconstruct the special Kähler metric and period matrix $\mathcal{N}_{IJ}$ appearing in (2.109), as well as other geometrical objects that determine couplings of fermion fields to other fields in the theory. $\mathcal{N} = 2$ supergravity was originally constructed in a formulation in which a prepotential F for the vector multiplet sector is present [32], however, in [33] theories were found that could not be formulated using a prepotential. These models are constructed by applying a duality transformation to a model with prepotential. In order to explain this point, we note that in four dimensional supergravity theories a generalization of Maxwell electric-magnetic dualities is possible. Consider the kinetic terms of the vector fields in (2.109). Defining

$$G^{\mu\nu}_{+I} \equiv 2\mathrm{i}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^{+I}_{\mu\nu}} = \mathcal{N}_{IJ}\mathcal{F}^{+J\mu\nu}\,, \tag{2.112}$$

the set of Bianchi identities and field equations (neglecting possible fermionic terms) can be written as follows

$$\begin{aligned}\partial^\mu \mathrm{Im}\mathcal{F}^{+I}_{\mu\nu} &= 0\,, \\ \partial_\mu \mathrm{Im}G^{\mu\nu}_{+I} &= 0\,.\end{aligned} \tag{2.113}$$

This set of equations is invariant under $\mathrm{G}\,\ell(2n+2,\mathbb{R})$ transformations:

$$\begin{pmatrix} \tilde{\mathcal{F}}^{+} \\ \tilde{G}_{+} \end{pmatrix} = \mathcal{S} \begin{pmatrix} \mathcal{F}^{+} \\ G_{+} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^{+} \\ G_{+} \end{pmatrix} . \tag{2.114}$$

Note that the $G_{\mu\nu}$ are related to $\mathcal{F}_{\mu\nu}$ as in (2.112). We will thus require that under a duality transformation, the matrix $\mathcal{N}_{IJ}$ transforms to $\tilde{\mathcal{N}}_{IJ}$ such that

$$\tilde{G}^{\mu\nu}_{+I} = \tilde{\mathcal{N}}_{IJ} \tilde{\mathcal{F}}^{+J\mu\nu} . \tag{2.115}$$

The matrix $\tilde{\mathcal{N}}$ is then related to the original one as in (2.74). Requiring that $\tilde{\mathcal{N}}$ is again symmetric, one then finds that

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n+2,\mathbb{R}) . \tag{2.116}$$

Note that these symplectic transformations in general do not leave the action invariant, they merely constitute an invariance of the combined set of field equations and Bianchi identities.

Suppose now that we start from a formulation in terms of a prepotential F. One can then perform a duality transformation. Supersymmetry implies that this transformation also acts on the section V, such as to reproduce the transformation law (2.74). So under a duality transformation:

$$V = \begin{pmatrix} X^I \\ F_I \end{pmatrix} \longrightarrow \tilde{V} = \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix} . \tag{2.117}$$

Although the F_I one starts from are derivatives of F with respect to X^I, this is not necessarily true anymore for the $\tilde{F}_I$. In general, it is not possible to find a function $\tilde{F}$, such that $\tilde{F}_I = \frac{\partial \tilde{F}}{\partial \tilde{X}^I}$. The two theories one obtains in this way have different actions and correspond to dual formulations of the same theory.

This is the reason why we adopted two definitions for special Kähler geometry. One definition is very suitable in the case when one has a prepotential at hand, while the other can also be used in more general setups.

Dimensional reduction and special geometry inclusions

In previous sections, we have described the different theories with 8 supercharges in 5 and 4 dimensions as well as the corresponding target space geometries. In this section, we will explain how dimensional reduction gives a recipe to construct mappings from very special real to special Kähler geometry and from special Kähler to quaternionic-Kähler geometry. Let us first briefly explain how dimensional reduction is performed and then apply it to the supergravity theories with 8 supercharges previously introduced.

Consider a massless complex scalar field ϕ in a space-time which is a direct product of four-dimensional Minkowski space-time and a circle. Parametrizing Minkowski space-time by coordinates x^μ and the circle by a coordinate z (where z is periodic with period $2\pi R$), one can expand ϕ as a Fourier series:

$$\phi(x^\mu, z) = \sum_n e^{inz/R} \phi_n(x^\mu) . \tag{2.118}$$

Using the fact that the five-dimensional d'Alembertian $\Box_5 = \Box_4 + \partial_z^2$, one sees that the five-dimensional Klein-Gordon equation $\Box_5 \phi = 0$ leads to separate equations for the different Fourier modes ϕ_n:

$$\Box_4 \phi_n - \frac{n^2}{R^2} \phi_n = 0 \,. \tag{2.119}$$

Each of the Fourier modes can thus be interpreted as a four-dimensional field with a mass given by $(n/R)^2$. Choosing the radius R of the circle to be small, one sees that the modes with $n \neq 0$ become highly massive and we can neglect them when we consider low energies. At energy scales of order $1/R$, the tower of massive modes becomes visible and space-time appears five-dimensional again.

When other fields are present, one can play a similar game. Denoting higher-dimensional coordinates as x^M, $M = 1, \cdots, D+1$, one can split them into a compact (space-like) coordinate z and D non-compact coordinates x^μ. The example given above shows that at sufficiently low energies, one can take all fields to be independent of z, i.e., one only keeps the zero-modes in their Fourier expansion. Consider now for instance the $D+1$-dimensional metric $\hat{g}_{MN}$, where the dependence of z is now suppressed. The index M can now take the values μ or z, leading to the following components : $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$ and $\hat{g}_{zz}$. From a D-dimensional viewpoint, these look like a symmetric tensor, a vector and a scalar respectively. This shows that the higher-dimensional metric decomposes in terms of lower-dimensional fields into a metric $g_{\mu\nu}$, a vector $\mathcal{A}_\mu^{(0)}$ and a scalar ϕ respectively. Similarly, a $D+1$-dimensional vector field $\hat{A}_M$ gives rise to a D-dimensional vector field A_μ and a scalar χ.

Although we have shown this procedure for one compact coordinate, one can also consider more general cases in which there are several compact directions, that span a manifold that is often denoted as the internal manifold. In the simplest case, these internal manifolds are tori, but more general possibilities can occur as well. Note that the procedure can in general be more complicated than described above. One generic feature of compactifications is the appearance of scalar fields in the lower-dimensional effective theory. These scalar fields are often called moduli; they parametrize the size and shape of the internal manifold. The simplest example of such a modulus is given by the scalar field that comes from the reduction of the metric in a circle compactification as described above.

Let us now perform a simple toroidal reduction of the $\mathcal{N} = 2$ supergravity theories of the previous section. It can be shown that for such reductions on tori, the number of supersymmetries is not reduced in the process. This implies that one is led to a lower dimensional theory that is again invariant under eight supersymmetries. Suppose we start from $\mathcal{N} = 2$ supergravity in five dimensions, coupled to n_v vector multiplets. From the reduction of the metric, one gets a four-dimensional metric, a vector field and a scalar. The $n_v + 1$ vectors in the theory reduce to $n_v + 1$ vectors and $n_v + 1$ scalars, while the n_v scalars that were already present in the vector multiplets give rise to n_v four-dimensional scalars. In total, the four-dimensional field content is given by a metric, $n_v + 2$ vectors and $2n_v + 2$ scalars. This is summarized in table 2.1.

So, one sees that the bosonic field content in four dimensions consists of the content of the gravity multiplet and that of $n_v + 1$ vector multiplets. Also at the level of the fermions, this matching occurs. Indeed, the 2 spinors in one vector multiplet reduce to two spinors in four dimensions. The two gravitini $\hat{\psi}_M^i$ give rise to two gravitini ψ_μ^i in four dimensions and two extra fermions ψ_z^i, that sit in a four-dimensional vector multiplet.

5 d	4 d			
	spin-2	spin-1	spin-0	
$\hat{g}_{MN}$	$g_{\mu\nu}$	$\mathcal{A}_\mu^{(0)}$	ϕ	
$\hat{A}_M^I$		A_μ^I	χ^I	$I = 0, \cdots, n_v$
$\hat{\phi}^v$			ϕ^v	$v = 1, \cdots, n_v$

Table 2.1 *In this table, we summarize the dimensional reduction of the bosonic fields of $\mathcal{N} = 2$, 5-dimensional supergravity to four dimensions. The first column shows the 5-dimensional fields. The second, third and fourth columns show the fields that appear under dimensional reduction, classified according to spin. The last column contains remarks concerning index ranges.*

The resulting theory in 4 dimensions is thus $\mathcal{N} = 2$ supergravity coupled to $n_v + 1$ vector multiplets, implying that the four-dimensional action can be described in terms of special Kähler geometry. Indeed, it turns out that the dimensionally reduced action assumes the form of (2.109) (without the kinetic term for the hypermultiplets), where the objects pertaining to special geometry can be derived from the prepotential

$$F(X) = \frac{C_{IJK} X^I X^J X^K}{X^0} . \tag{2.120}$$

Similarly, one can reduce $\mathcal{N} = 2$ supergravity coupled to $n_v + 1$ vector multiplets in four dimensions to three dimensions. The four-dimensional metric again reduces to a metric in three dimensions, a vector field and a scalar. The vector fields reduce to three-dimensional vector fields and scalars. However, in three dimensions one can take into account that a vector field A_k is dual to a scalar φ via the usual Hodge duality:

$$\partial_i \varphi = \epsilon_{ijk} \partial^j A^k , \tag{2.121}$$

where we have used indices i, j, k to denote coordinates in three dimensions. The dimensional reduction from four to three dimensions is summarized in the table 2.2.

4 d	3 d			
$\tilde{g}_{\mu\nu}$	g_{ij}	$\varphi^{(0)}$	σ	
$\tilde{A}_\mu^A$		φ^A	ζ^A	$A = \{0, I\}$
$\tilde{z}^I$			z^I	

Table 2.2 *This table summarizes how the bosonic fields in four dimensions reduce to three dimensions. The first column shows the four-dimensional fields. The second column contains the spin-2 three-dimensional fields. The third column contains three-dimensional spin-0 fields, coming from the dualization of a vector. The fourth column contains the other spin-0 fields. We have denoted four-dimensional fields using a tilde. Indices μ are four-dimensional, while indices i, j are three-dimensional.*

One sees that in total one now has $4(n_v + 2)$ scalar fields, corresponding to the bosonic field content of n_v+2 hypermultiplets. So, in three dimensions one ends up with supergravity with 8 supercharges coupled to n_v+2 hypermultiplets. As expected, the bosonic Lagrangian consists of an Einstein-Hilbert term and a non-linear sigma model exhibiting quaternionic-Kähler geometry.

In this way, dimensional reduction provides a mapping between very special real geometry, special Kähler geometry and quaternionic-Kähler geometry. Performing dimensional reduction from 5 to 4 dimensions leads to a mapping of the set of very special real geometries to a subset of all special Kähler geometries. This map is called the **r**-map and the special Kähler geometries in its image are denoted as very special Kähler geometries. The mapping between special Kähler geometry and quaternionic-Kähler geometry that is induced by reduction from 4 to 3 dimensions is denoted as the **c**-map and its image consists of the special quaternionic-Kähler geometries. Quaternionic-Kähler geometries that are in the image of the $\mathbf{c}\circ\mathbf{r}$-map are often called very special quaternionic-Kähler manifolds. This is summarized in fig. 2.1.

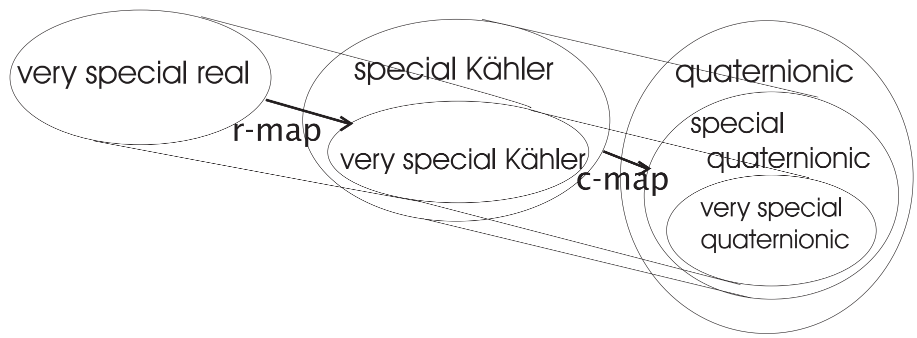


Figure 2.1 *Summary of the actions of the* **r**- *and* **c**-*map, relating the 3 different sorts of geometries appearing in theories with 8 supercharges.*

2.4.2 Eleven-dimensional supergravity and type II supergravities in 10 dimensions

In eleven dimensions, the minimal spinor has 32 components. There is thus a unique supergravity theory, that contains one massless supermultiplet, the gravity multiplet which consists of the following on-shell degrees of freedom[11]

g_{MN}	C_{MNP}	ψ_M
44	84	128

. (2.122)

The corresponding supergravity theory was constructed in [34]. The bosonic part of the Lagrangian is given by

$$S_{11d} = \frac{1}{2\kappa_{11}^2}\int \mathrm{d}^{11}x\sqrt{-g}\Big[R - \frac{1}{2\cdot 4!}G^2\Big] - \frac{1}{12\kappa_{11}^2}\int G\wedge G\wedge C\,, \tag{2.123}$$

[11] We will use latin indices $m, n, p, \cdots$ to denote curved coordinates in 10 dimensions, while in 11 dimensions, we will use capital indices $M, N, P, \cdots$. These theories are used in chapter 4, where this convention will be appropriate. Flat indices in 10 dimensions will be denoted with underlined latin letters.

where $G = \mathrm{d}C$ and κ_{11} denotes the 11-dimensional gravitational coupling constant.

In 10 dimensions, one has more possibilities to construct supergravity theories. Indeed, the minimal spinor is Majorana-Weyl and has 16 real components. This means that minimal $\mathcal{N} = 1$ supergravity in 10 dimensions has 16 supercharges. One can also construct maximal $\mathcal{N} = 2$ theories. For these so-called type II theories, one has two possibilities : either the two independent supersymmetries correspond to two Majorana-Weyl spinors with different chirality (type IIA) or they constitute two Majorana-Weyl spinors with the same chirality (type IIB). Again, there is only one massless multiplet in each case, namely the gravity multiplet. In both cases, this multiplet contains the metric G_{mn}, a two-form $B_{(2)}$ and a scalar Φ (called the dilaton). These fields constitute what is often called the common sector or the NS-NS subsector. In type IIA, the bosonic sector of the gravity multiplet further contains a one-form $C_{(1)}$ and a three-form gauge potential $C_{(3)}$. In type IIB on the other hand, the bosonic sector is completed with an extra scalar $C_{(0)}$, a two-form $C_{(2)}$ and a four-form gauge potential $C_{(4)}$ satisfying a self-duality constraint. These extra form fields $C_{(n)}$ are called R-R forms. In both cases, the fermionic sector consists of the gravitino ψ_m and a dilatino λ. In the type IIA case, the spinors have different chirality and can be combined in one Majorana spinor with 32 components. In type IIB, the spinors are doublets of two spinors with the same chirality. Let us summarize the field content of type II supergravities by listing their fields and their on-shell degrees of freedom for type IIA:

G_{mn}	$B_{(2)}$	Φ	$C_{(1)}$	$C_{(3)}$	ψ_m	λ
35	28	1	8	56	$56+56$	$8+8$

, (2.124)

while for type IIB:

G_{mn}	$B_{(2)}$	Φ	$C_{(0)}$	$C_{(2)}$	$C_{(4)}$	ψ_m	λ
35	28	1	1	28	35	$56+56$	$8+8$

. (2.125)

To write the supergravity actions, let us start by introducing the generalized R-R field-strengths[12]

$$\begin{array}{l} F_{(1)} = \mathrm{d}C_{(0)} \quad , \quad F_{(2)} = \mathrm{d}C_{(1)} \quad , \quad F_{(3)} = \mathrm{d}C_{(2)} + C_{(0)}H_{(3)} \ , \\ F_{(4)} = \mathrm{d}C_{(3)} + H_{(3)} \wedge C_{(1)} \quad , \quad F_{(5)} = \mathrm{d}C_{(4)} + H_{(3)} \wedge C_{(2)} \ , \end{array} \tag{2.126}$$

where $H_{(3)} = \mathrm{d}B_{(2)}$.

The type IIA supergravity action is the following:

$$\begin{aligned} S_{IIA} \;=\; & \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G}\Big\{e^{-2\Phi}\big[R + 4(\partial\Phi)^2 - \frac{1}{2\cdot 3!}(H_{(3)})^2\big] \\ & -\frac{1}{2\cdot 2!}(F_{(2)})^2 - \frac{1}{2\cdot 4!}(F_{(4)})^2\Big\} - \frac{1}{4\kappa_{10}^2}\int B_{(2)}\wedge \mathrm{d}C_{(3)}\wedge \mathrm{d}C_{(3)} \ , \end{aligned} \tag{2.127}$$

whereas the type IIB supergravity action is given by:

$$\begin{aligned} S_{IIB} \;=\; & \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G}\Big\{e^{-2\Phi}\big[R + 4(\partial\Phi)^2 - \frac{1}{2\cdot 3!}(H_{(3)})^2\big] \\ & -\frac{1}{2}(F_{(1)})^2 - \frac{1}{2\cdot 3!}(F_{(3)})^2 - \frac{1}{4\cdot 5!}(F_{(5)})^2\Big\} \end{aligned}$$

[12] We are using essentially the same conventions as in [35].

$$+\frac{1}{4\kappa_{10}^2}\int \mathrm{d}C_{(2)} \wedge H_{(3)} \wedge (C_{(4)} + \frac{1}{2}B_{(2)} \wedge C_{(2)}) \ . \tag{2.128}$$

In both actions, κ_{10} denotes the gravitational coupling constant. In type IIB, one has to supplement the action with a self-duality condition on $F_{(5)}$ that has to be imposed by hand at the level of the equations of motion.

The supersymmetry transformations for both IIA and type IIB can be written in the form

$$\delta_\varepsilon \psi_m = D_m \varepsilon \quad , \quad \delta_\varepsilon \lambda = \Delta\varepsilon \ , \tag{2.129}$$

where the supersymmetry parameter ε is a Majorana spinor for type IIA and a doublet of Majorana-Weyl spinors of positive chirality for type IIB. It is useful to split the operators D_m and Δ as follows:

$$D_m = D_m^{(0)} + W_m \quad , \quad \Delta = \Delta^{(1)} + \Delta^{(2)} \ . \tag{2.130}$$

In type IIA, we have

$$\begin{aligned}
D^{(0)} &= \nabla_m + \frac{1}{4\cdot 2!}H_{mnp}\Gamma^{np}\Gamma_{(10)} \ , \\
W_m &= -\frac{1}{8}e^{\Phi}\Big(\frac{1}{2}F_{np}\Gamma^{np}\Gamma_{(10)} + \frac{1}{4!}F_{npqr}\Gamma^{npqr}\Big)\Gamma_m \ , \\
\Delta^{(1)} &= \frac{1}{2}\Big(\Gamma^m\partial_m\Phi + \frac{1}{2\cdot 3!}H_{mnp}\Gamma^{mnp}\Gamma_{(10)}\Big) \ , \\
\Delta^{(2)} &= \frac{1}{8}e^{\Phi}\Big(\frac{3}{2!}F_{mn}\Gamma^{mn}\Gamma_{(10)} - \frac{1}{4!}F_{mnpq}\Gamma^{mnpq}\Big) \ ,
\end{aligned} \tag{2.131}$$

while in type IIB

$$\begin{aligned}
D^{(0)} &= \nabla_m + \frac{1}{4\cdot 2!}H_{mnp}\Gamma^{np}\sigma_3 \ , \\
W_m &= \frac{1}{8}e^{\Phi}\Big[F_n\Gamma^n(\mathrm{i}\sigma_2) + \frac{1}{3!}F_{npq}\Gamma^{npq}\sigma_1 + \frac{1}{2\cdot 5!}F_{npqrt}\Gamma^{npqrt}(\mathrm{i}\sigma_2)\Big]\Gamma_m \ , \\
\Delta^{(1)} &= \frac{1}{2}\Big(\Gamma^m\partial_m\Phi + \frac{1}{2\cdot 3!}H_{mnp}\Gamma^{mnp}\sigma_3\Big) \ , \\
\Delta^{(2)} &= -\frac{1}{2}e^{\Phi}\Big[F_m\Gamma^m(\mathrm{i}\sigma_2) + \frac{1}{2\cdot 3!}F_{mnp}\Gamma^{mnp}\sigma_1\Big] \ ,
\end{aligned} \tag{2.132}$$

where $\nabla_m = \partial_m + \frac{1}{4}\Omega_m{}^{\underline{np}}\Gamma_{\underline{np}}$ is the covariant derivative.

Note that supergravity in 11 dimensions and 10 dimensions are not unrelated. Upon dimensional reduction, 11-dimensional supergravity leads to a non-chiral supergravity in 10 dimensions with 32 supercharges and hence gives the type IIA theory. Indeed, the 11-dimensional metric gives upon reduction rise to the 10-dimensional metric, a vector field that can be identified with $C_{(1)}$ and a scalar, namely the dilaton Φ. The three-form in 11 dimensions leads to a three-form (when taking all indices in the non-compact directions) and a two-form (when taking one index along the compact direction) in 10 dimensions. These can be identified with $C_{(3)}$ and $B_{(2)}$ respectively.

It turns out that also the type IIA and type IIB supergravity theories are not unrelated. To see this, let us verify how their bosonic field content looks like upon dimensional reduction. The reduction of type IIA gives rise to a metric, one three-form, two two-forms, 3 vector fields and three scalar fields. In the type IIB case, one gets the 9-dimensional metric, two

two-forms, 3 vector fields and 3 scalar fields from the reduction of the metric, $B_{(2)}$, Φ, $C_{(0)}$ and $C_{(2)}$. The reduction of the four-form requires some extra care because of its self-duality. In principle, it gives rise to a four-form and a three-form. However, these two forms are related by the self-duality constraint and one can eliminate one of the two. Usually, one then eliminates the four-form. So, one can conclude that upon dimensional reduction both type IIA and type IIB theories lead to the same spectrum in 9 dimensions. It is also possible to show explicitly that the actions of the type IIA and IIB theories reduce to the same action, describing maximal supergravity in 9 dimensions. As a consequence, it is possible to relate the type IIA and IIB theories using the following rules:

$$\begin{aligned}
&\bar{\phi} = \phi - \frac{1}{2}\ln G_{99}\,, \qquad \bar{G}_{99} = \frac{1}{G_{99}}\,,\\
&\bar{G}_{\bar{m}\bar{n}} = G_{\bar{m}\bar{n}} - \frac{G_{\bar{m}9}G_{\bar{n}9} - B_{\bar{m}9}B_{\bar{n}9}}{G_{99}}\,, \qquad \bar{G}_{\bar{m}9} = \frac{1}{G_{99}}\,,\\
&\bar{B}_{\bar{m}\bar{n}} = B_{\bar{m}\bar{n}} - \frac{B_{\bar{m}9}G_{\bar{n}9} - G_{\bar{m}9}B_{\bar{n}9}}{G_{99}}\,, \qquad \bar{B}_{\bar{m}9} = \frac{G_{\bar{m}9}}{G_{99}}\\
&\bar{C}^{(n)}_{9\bar{m}_2\ldots\bar{m}_n} = C^{(n-1)}_{\bar{m}_2\ldots\bar{m}_n} - (n-1)G_{99}^{-1}G_{9[\bar{m}_2}C^{(n-1)}_{|9|\bar{m}_3\ldots\bar{m}_n]}\,,\\
&\bar{C}^{(n)}_{\bar{m}_1\ldots\bar{m}_n} = C^{(n+1)}_{9\bar{m}_1\ldots\bar{m}_n} - nB_{9[\bar{m}_1}\bar{C}^{(n)}_{|9|\bar{m}_2\ldots\bar{m}_n]}\,,
\end{aligned} \tag{2.133}$$

where the barred fields correspond to the transformed fields, the 9-direction denotes the compact direction and $\bar{m}, \bar{n}, \cdots \neq 9$. As these rules form the supergravity manifestation of a stringy duality, that was called T-duality, they are often called T-duality rules.

2.4.3 Gauged maximal supergravity in four dimensions

Maximal ($\mathcal{N} = 8$) supergravity in four dimensions can be obtained from 11-dimensional supergravity by performing a dimensional reduction in which the compact coordinates are coordinates on a seven-torus. This theory was originally constructed in [36]. As the theory is maximally supersymmetric, only the gravity multiplet is present. Its field content consists of the metric $g_{\mu\nu}$, 28 vector fields A^I_μ with field strengths $F^I_{\mu\nu}$, 70 scalars ϕ^i, 8 gravitini and 56 Majorana fermions, constituting 128 bosonic and 128 fermionic on-shell degrees of freedom. The bosonic part of the Lagrangian is given by:

$$\begin{aligned}
\mathcal{L}_{\text{bos},\mathcal{N}=8} \;=\;& \sqrt{-g}\Big(\frac{R}{2} - \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + \frac{1}{4}(\text{Im}\mathcal{N}_{IJ})F^I_{\mu\nu}F^{J|\mu\nu}\Big)\\
&-\frac{1}{8}(\text{Re}\mathcal{N}_{IJ})\varepsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}\,.
\end{aligned} \tag{2.134}$$

The 70 scalars form a non-linear sigma model with metric g_{ij}. This target space corresponds to the non-compact symmetric coset space

$$\mathcal{M}_{\text{scalar}} = \frac{E_{7(+7)}}{\text{SU}(8)}\,. \tag{2.135}$$

As before, $\mathcal{N}_{IJ}$ represents a scalar-dependent matrix, that can be calculated according to a standard construction [37] that is valid for supergravity theories with symmetric target spaces.

The symmetry group of the kinetic term of the scalars in (2.134) corresponds to the isometry group of (2.135) and is in this case given by $E_{7(+7)}$. This $E_{7(+7)}$ however does not lead to a symmetry of the full Lagrangian, but is only a symmetry of the combined set of field equations and Bianchi identities. The action is only invariant under a subgroup $G \subset E_{7(+7)}$. In fact, one can construct different actions of ungauged $\mathcal{N} = 8$ supergravity, each having a different global invariance group G. These actions are however related via electric-magnetic duality as discussed in section 2.4.1 and hence lead to the same set of equations of motion and Bianchi identities.

In the most known case, described in [38], the invariance group of the action is given by $G = \mathrm{S}\ell(8,\mathbb{R})$. The 28 vector fields in the theory transform in the antisymmetric **28** representation of $\mathrm{S}\ell(8,\mathbb{R})$ and one can use them to gauge a subgroup $K \subset \mathrm{S}\ell(8,\mathbb{R})$, i.e., to promote K from a global to a local invariance of the Lagrangian. This procedure basically consists of replacing ordinary derivatives by derivatives that are gauge-covariant with respect to K and by replacing ordinary abelian field strengths by non-abelian ones. The minimal couplings one thus introduces however explicitly break supersymmetry. It turns out that supersymmetry can be restored by adding parts to the supersymmetry transformations of the fermions as well as extra terms to the Lagrangian. In this way, the gauging procedure also introduces a potential for the scalars that is proportional to the square of the gauge coupling constant. The first example of such a gauged $\mathcal{N} = 8$ supergravity was given in [39], where de Wit and Nicolai gauged an SO(8) subgroup of $\mathrm{S}\ell(8,\mathbb{R})$. Later on, starting from this prime example more general gaugings were considered. It turns out that, when $\mathrm{S}\ell(8,\mathbb{R})$ is the invariance group of the Lagrangian, the most general gaugings consist of the so-called $\mathrm{CSO}(p,q,r)$-gaugings, where $p+q+r = 8$ [40, 41, 42][13]. Denoting the 28 generators of these CSO groups by $\Lambda_{ab} = -\Lambda_{ba}$, with $a, b = 1, \cdots, 8$, they obey the following algebra:

$$[\Lambda_{ab}, \Lambda_{cd}] = \Lambda_{ad}\eta_{bc} - \Lambda_{ac}\eta_{bd} - \Lambda_{bd}\eta_{ac} + \Lambda_{bc}\eta_{ad}\,, \tag{2.136}$$

where

$$\eta_{ab} = \begin{pmatrix} \mathbb{1}_{p\times p} & 0 & 0 \\ 0 & -\mathbb{1}_{q\times q} & 0 \\ 0 & 0 & 0_{r\times r} \end{pmatrix}. \tag{2.137}$$

Note that when $q = r = 0$ the gauge algebra is the one of SO(8). When $r = 0$, the corresponding gauge groups are the non-compact groups $\mathrm{SO}(p,q)$ with $p + q = 8$. When $r \neq 0$, the $\mathrm{CSO}(p,q,r)$ groups are not semi-simple and they can be obtained as group contractions of $\mathrm{SO}(p + r, q)$.

Like ungauged $\mathcal{N} = 8$ supergravity, that can be obtained from higher dimensions by performing dimensional reductions on tori, the $\mathrm{CSO}(p,q,r)$-gaugings of the theory can also be seen to have a higher-dimensional origin. In this case, the compact directions no longer form a torus. The SO(8)-gauging for instance can be obtained by reducing 11-dimensional supergravity on a seven-sphere S^7 [44, 45, 46]. For general CSO-gaugings, the higher-dimensional origin was found in [47]. In this case, the compactification manifold no longer corresponds to a sphere and in fact no longer needs to be compact any more. We will come back to these issues in chapter 4.

13 Other gaugings have been found later, starting from dual actions with different symmetry groups G, see for instance [43]. For the rest of this thesis we only consider the CSO-gaugings.

Due to the fact that gauged supergravities contain potentials for the various scalar fields in the theory, one can wonder about their relevance for cosmology. Various investigations have been performed that were aimed at finding stationary points of the scalar potential in various gauged supergravities that lead to vacua with a positive cosmological constant (see for instance [48, 49, 50, 51, 52, 53]). For the above described CSO-gaugings of $\mathcal{N} = 8$ supergravity, a detailed study has been performed in [54]. It was found that de Sitter vacua can be found for $\mathrm{SO}(p, q)$-gaugings. The de Sitter vacua that are found in this way are however never stable : they always correspond to saddle points of the potential. In chapter 4 of this thesis, we will perform a search for a different type of cosmological solutions in the CSO-gauged supergravities. Instead of searching for de Sitter vacua, we will search for scaling cosmologies, as described in section 2.1.4.

Chapter 3

Tits-Satake projections of homogeneous special geometry

3.1 Introduction : cosmological solutions in ungauged supergravity

The observation that the expansion of our universe is accelerating, invoked a renewed interest in the study of cosmological solutions in supergravity and string theory. In order to gain insight in the structure of time-dependent solutions of supergravity theories, the authors of [55] first focused on ungauged maximal supergravity. More specifically, they focused on time-dependent solutions of maximal ungauged supergravity in three dimensions. All bosonic fields are then exhausted by the metric and scalars. Once one has found interesting cosmological solutions, one can use the inverse process of dimensional reduction[1] to interpret these three-dimensional cosmologies in terms of higher-dimensional ones.

For maximal supergravity in 3 dimensions, the bosonic action takes the form of a non-linear sigma model:

$$\mathcal{L}_{3d} = \sqrt{-g}\Big[\frac{1}{2}R[g] - \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial_\nu\phi^j g^{\mu\nu}\Big]\,. \tag{3.1}$$

There are 128 scalars, so the index $i, j = 1, \cdots, 128$. The target space with metric $g_{ij}(\phi)$ is given by the symmetric coset manifold:

$$\mathcal{M} = \frac{E_{8(8)}}{\mathrm{SO}(16)}\,. \tag{3.2}$$

In [55], a systematic search for solutions of the equations of motion of this theory is performed, where the scalars depend only on time and the metric assumes the following form:

$$\mathrm{d}s^2_{3D} = -A^2(t)\mathrm{d}t^2 + B^2(t)(\mathrm{d}r^2 + r^2\mathrm{d}\phi^2)\,. \tag{3.3}$$

[1] The inverse process of dimensional reduction is also known as dimensional oxidation.

One can show that, after choosing an appropriate time variable, the equations of motion for the scalars reduce to the geodesic equations for the metric $g_{ij}(\phi)$:

$$\ddot{\phi}^i + \Gamma^i_{jk}\dot{\phi}^j\dot{\phi}^k = 0\,, \tag{3.4}$$

where Γ^i_{jk} denotes the Levi-Civita connection for the metric $g_{ij}(\phi)$. Once one has obtained a solution of (3.4), a solution for the metric can be found via the Einstein equations. The problem of finding time-dependent solutions of maximal supergravity is thus reduced to the problem of solving the geodesic equations in the target space manifold (3.2).

Using the solvable parametrization of the coset (3.2), an algorithm was constructed that allows one to find the solutions of the geodesic equations (3.4). This algorithm not only works for the coset (3.2), but holds more generally for non-linear sigma models coupled to gravity, where the target space is a maximally non-compact coset space. These are coset spaces G/H, where G is the split real form of a semi-simple Lie algebra and H is its maximal compact subgroup. In [56], it was shown that for such non-linear sigma models the resulting dynamical equations are indeed integrable and a complete set of solutions can be found.

The algorithm developed in [55] is rather specific for cosets that are maximally non-compact. Such cosets appear generically in maximal supergravity theories, however for phenomenological purposes one is more interested in non-maximal supergravity theories. For these, it is no longer true that the target spaces appearing after reduction to three dimensions are necessarily maximally non-compact cosets. When the target spaces are still symmetric spaces, there exists a way of obtaining a large class of time-dependent solutions, even when the coset is not maximally non-compact. This procedure is based on the so-called Tits-Satake theory in mathematics [57] and was for instance illustrated in [5]. Essentially, one performs a consistent truncation of the original sigma model such that the truncated model corresponds to a sigma model on a maximally non-compact coset. One can then solve the geodesic equations of the truncated model. Since the truncation is a consistent one, these solutions are automatically solutions of the original theory. As was illustrated in a specific example in [5], in this way one can obtain an interesting class of non-trivial time-dependent solutions of non-maximal supergravity theories with symmetric target spaces.

This truncation procedure is known as the Tits-Satake projection of symmetric spaces and can be seen as a mapping that projects non-maximally non-compact coset spaces to maximally non-compact ones. The term 'projection' is a good one, as it happens that in general several different symmetric spaces project onto the same maximally non-compact coset. This feature has led to the grouping of supergravity theories with symmetric target spaces in universality classes. One universality class then consists of all supergravity theories whose target spaces have the same Tits-Satake projection. This organization of theories in classes is physically important since all theories in one class share a similar dynamical behavior. Indeed, all theories in one class share a common set of solutions, namely the solutions of the Tits-Satake projected model. These solutions do not represent the most general solution of a specific member in one class; however it turns out that in a lot of cases this Tits-Satake projection already captures some important part of the time evolution of all theories in the same universality class.

Symmetric spaces only comprise a small fraction of the possible target spaces appearing in supergravity theories. Indeed, the geometry of scalar manifolds is restricted by supersymmetry and depends on the number of supercharges $\sharp_Q$. For instance, when $\sharp_Q \geq 12$ the manifolds are necessarily symmetric spaces within certain classes ($16 \geq \sharp_Q \geq 12$) or just completely fixed cosets for $32 \geq \sharp_Q > 16$. For $12 > \sharp_Q \geq 8$ however, the scalar manifolds are not necessarily symmetric spaces. In fact, they don't have to be homogeneous anymore. A natural question is then whether the procedure of Tits-Satake projection can be extended to a more general class of supergravity theories than just the ones that exhibit symmetric target spaces. The aim of this chapter is to show that one can indeed extend the Tits-Satake projection to a class of $\mathcal{N} = 2$ special geometries, namely the ones that are homogeneous, but not necessarily symmetric. We will mainly focus on homogeneous quaternionic-Kähler manifolds, since these are the ones appearing in three-dimensional supergravity with 8 supercharges. For a lot of these spaces, the **c**- and **r**-maps can be inverted to obtain a higher-dimensional origin in terms of very special real and special Kähler manifolds. Although special geometries are not necessarily homogeneous, a large and rich class of them is. These homogeneous special geometries have all been classified and studied systematically [58, 59]. They describe a large class of supergravity models associated for instance with orbifold or orientifold compactification of superstrings and also with a variety of brane constructions [60, 61, 62]. They occur for instance in $\mathrm{T}^2 \times \mathrm{K3}$ compactifications and in the large radius limit of other Calabi-Yau compactifications. In this chapter, we will make clear that one can still perform the Tits-Satake projection on these spaces. Hence, techniques that were available in the study of time-dependent solutions of supergravity theories with symmetric target spaces, can now be applied to this more general class of $\mathcal{N} = 2$ supergravities.

This chapter is organized as follows. In the next section, we will give a general discussion on homogeneous special geometry. We will review the classification of these geometries as done by [63, 64, 65]. We will describe in detail the structure of their solvable algebras. Furthermore, we will also discuss the structure of their isometry algebras, as this will be relevant later on in the construction. In section 3.3, we will show how the Tits-Satake projection is done for symmetric spaces and we will illustrate this by means of an explicit example. Using this discussion, we will then show in section 3.4 how one can extend this projection to general homogeneous special geometries. Final results and applications will be given in section 3.5. Among the applications, we will in particular discuss the organization of supergravity theories in universality classes. Part of the material in this chapter heavily relies on the theory of semi-simple Lie algebras and their real forms. A summary of some relevant points on this is given in appendix A.

3.2 Homogeneous special geometry

In this section, we will review the classification of homogeneous quaternionic-Kähler manifolds [63, 64, 65]. We will first review some important properties and theorems concerning homogeneous quaternionic-Kähler geometry, that allow one to classify these manifolds in an algebraic way. This algebraic classification will then be presented in more detail. Furthermore, we will also discuss the **r**- and **c**-maps in the context of homogeneous special geometry. Finally, we will give a brief summary of the structure of the isometry groups of these quaternionic-Kähler spaces. Although this is a rather technical review, later sections will heavily make use of the material presented here.

3.2.1 Homogeneous quaternionic-Kähler geometry and solvable algebras

Homogeneous quaternionic-Kähler manifolds correspond to coset spaces. They are thus of the form G/H, where G is the isometry group of the manifold and H the isotropy group that leaves a point in the space fixed. We will not require G to be a semi-simple group. Furthermore, H is not necessarily a symmetric subgroup of G [2], so the space G/H is not necessarily symmetric. We will furthermore only be interested in non-compact homogeneous quaternionic-Kähler manifolds, as these are the ones that are relevant for supergravity.

Alekseevsky conjectured that all non-compact homogeneous quaternionic-Kähler spaces are exhausted by the so-called normal quaternionic manifolds [63]. This means that these spaces admit a completely solvable[3] Lie group of isometries that acts on the manifold in a simply transitive manner[4]. The following theorem [63] then provides the main tool in classifying normal quaternionic spaces:

3.2.1 Theorem. *If a Riemannian manifold $\mathcal{M}$, with metric g, admits a simply transitive solvable group of isometries* $\exp(\mathrm{Solv}_{\mathcal{M}})$*, then it is metrically equivalent to this solvable group manifold:*

$$\begin{aligned} \mathcal{M} &\simeq \exp\left[\mathrm{Solv}_{\mathcal{M}}\right] , \\ g\,|_{e\in\mathcal{M}} &= <,>\,, \end{aligned} \tag{3.5}$$

where $<,>$ is a Euclidean metric defined on the solvable Lie algebra $\mathrm{Solv}_{\mathcal{M}}$.

This theorem then implies that one can identify normal quaternionic-Kähler spaces with solvable group manifolds. These are generated by normal metric Lie algebras $\mathrm{Solv}_{\mathcal{M}}$, i.e., completely solvable Lie algebras endowed with a Euclidean metric. The problem of classifying homogeneous quaternionic-Kähler spaces then boils down to classifying the normal metric algebras that generate them. In this way, one can translate the differential-geometric problem of classifying homogeneous quaternionic-Kähler manifolds into an algebraic problem. All notions of differential geometry of these spaces can be rephrased in an algebraic language. The metric g is translated to the Euclidean metric $<,>$ on $\mathrm{Solv}_{\mathcal{M}}$. The Levi-Civita connection on the manifold is equally well converted to the Nomizu operator which is defined as follows:

$$\begin{aligned} &\mathbb{L} : \mathrm{Solv}_{\mathcal{M}} \times \mathrm{Solv}_{\mathcal{M}} \to \mathrm{Solv}_{\mathcal{M}} : (X,Y) \to \mathbb{L}_X Y\,, \qquad (3.6)\\ &\forall X,Y,Z \in \mathrm{Solv}_{\mathcal{M}} : 2 < \mathbb{L}_X Y, Z >=< [X,Y], Z > - < X, [Y,Z] > - < Y, [X,Z] > \ . \end{aligned}$$

Using the Nomizu operator, one can define the Riemann curvature operator Riem, that is the translation of the usual Riemann tensor:

$$\mathrm{Riem}(X,Y) = [\mathbb{L}_X, \mathbb{L}_Y] - \mathbb{L}_{[X,Y]}\,, \qquad X, Y \in \mathrm{Solv}_{\mathcal{M}} \ . \tag{3.7}$$

[2] H is a symmetric subgroup of G if in the orthogonal decomposition $\mathbb{G} = \mathbb{H} \oplus \mathbb{K}$, their corresponding Lie algebras obey $[\mathbb{H}, \mathbb{K}] \subset \mathbb{K}$ and $[\mathbb{K}, \mathbb{K}] \subset \mathbb{H}$.

[3] A solvable Lie algebra Solv is completely solvable if the adjoint operation ad_X for all generators $X \in \mathrm{Solv}$ has only real eigenvalues.

[4] A group acts on a manifold in a simply transitive way if every two points in the manifold are connected by one and only one group element.

The holonomy algebra Γ of $\mathrm{Solv}_{\mathcal{M}}$ is defined as the Lie algebra generated by the curvature operator Riem and all of its commutators with Nomizu operators:

$$[\mathbb{L}_{X_1}, \ldots, [\mathbb{L}_{X_k}, \mathrm{Riem}(X_{k+1}, X_{k+2})] \ldots], \quad X_k \in \mathrm{Solv}_{\mathcal{M}} \; . \tag{3.8}$$

This holonomy algebra can be shown to coincide with the Lie algebra of the holonomy group of the manifold.

In order to generate quaternionic-Kähler manifolds, the solvable algebras under consideration should also be equipped with a quaternionic structure. This quaternionic structure corresponds to the linear Lie algebra Q generated by three operators J^α, acting on $\mathrm{Solv}_{\mathcal{M}}$ and obeying (2.80). The solvable algebras that generate normal quaternionic-Kähler manifolds are called quaternionic algebras and are defined as follows:

3.2.1 Definition. *A normal metric Lie algebra is called quaternionic if there exists a quaternionic structure Q on it, such that the holonomy algebra Γ is contained in the algebra of* $\mathrm{SU}(2) \times \mathrm{USp}(2n)$.

This is the algebraic counterpart of the fact that the holonomy group of quaternionic-Kähler manifolds is contained in $\mathrm{SU}(2) \times \mathrm{USp}(2n)$. The Riemann tensor splits up in a part corresponding to an SU(2)-curvature $\mathcal{R}^\alpha$ and a $\mathrm{USp}(2n)$-curvature R:

$$\mathrm{Riem}(X,Y) = \mathcal{R}^\alpha(X,Y)\, J^\alpha + R(X,Y) \, . \tag{3.9}$$

This equation is the algebraic equivalent of (2.90). The statement (2.91) in the definition of a quaternionic-Kähler manifold that the SU(2)-curvature should be proportional to the hyper-Kähler 2–forms induced by the three complex structures $J^{1,2,3}$, is then given in algebraic terms as:

$$\mathcal{R}^\alpha(X,Y) = -\tfrac{1}{2}\nu < J^\alpha X, Y > \; . \tag{3.10}$$

The complex structures J^α should also satisfy integrability conditions, expressed in terms of the Nijenhuis tensor [63].

We will now review the classification of normal quaternionic metric Lie algebras. We will just state the results without proofs, putting emphasis on the precise structure of these solvable algebras as this will be relevant later on.

3.2.2 The classification of homogeneous quaternionic-Kähler manifolds

The first step in the classification consists in showing that due to the complete solvability of $\mathrm{Solv}_{\mathcal{M}}$ and the structure equation (3.9) any normal quaternionic algebra contains a subalgebra of quaternionic dimension 1, called the canonical quaternionic subalgebra E. One can moreover show that E can be only one of two possibilities that we will denote for now as $E = \mathrm{Solv}(\mathrm{SU}(2,1))$ or $E = \mathrm{Solv}(\mathrm{USp}(2,2))$. This already leads to two different possibilities.

1. $\underline{E = \mathrm{Solv}(\mathrm{USp}(2,2))}$: One can show that for any quaternionic dimension n there is a unique normal quaternionic Lie algebra Solv_Q admitting $\mathrm{Solv}(\mathrm{USp}(2,2))$ as canonical

quaternionic subalgebra. The corresponding quaternionic manifolds are the following symmetric spaces:

$$\exp \mathrm{Solv}_Q \simeq \frac{\mathrm{USp}(2,2n)}{\mathrm{USp}(2)\times \mathrm{USp}(2n)}\,. \tag{3.11}$$

They are more commonly known as the hyperbolic spaces.

2. $\underline{E = \mathrm{Solv}(\mathrm{SU}(2,1))}$: In this case, the corresponding normal quaternionic algebras Solv_Q have the following structure:

$$\begin{aligned} &\mathrm{Solv}_Q = U + \tilde{U}\,, \\ &[U,U] \subset U\,, \qquad \left[U,\tilde{U}\right] \subseteq \tilde{U}\,, \qquad \left[\tilde{U},\tilde{U}\right] \subset U\,. \end{aligned} \tag{3.12}$$

The subalgebra $U \subset \mathrm{Solv}_Q$ is stable with respect to the action of the complex structure J^1: $J^1\,U = U$. One can show that U is a solvable algebra, that generates a homogeneous Kähler manifold. Such an algebra is called a Kähler algebra and one often calls U the principal Kähler algebra. The subspace $\tilde{U}$ is related to U by the action of a second complex structure J^2: $\tilde{U} = J^2\,U$. The representation $T_U : \tilde{U} \to \tilde{U}$, induced by the adjoint action of U, is called a Q-representation as it has to satisfy certain non-trivial conditions. The most important condition is that T_U is symplectic with respect to a suitable form $\hat{J}$ expressed in terms of J^1. The structure of U can be represented as follows:

$$U \;=\; \sum_{I=1}^{r} U_I\,, \qquad U_I = F_I + X_I\,, \tag{3.13}$$

where U_I are called elementary Kähler subalgebras while r is equal to the dimension of a Cartan subalgebra of Solv_Q, i.e., the rank of Solv_Q. The two–dimensional subalgebras F_I are so-called key algebras. In general, a key algebra F can be described in terms of an orthonormal basis $\{h, g\}$, where $g = J^1\,h$. The commutation relation between the basis elements is $[h, g] = \mu g$. The number μ is called the root of the key algebra; from the requirement that T_U is a Q-representation, it follows that it can only take the values $(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}})$, defining the type I, type II and type III key algebras, respectively. Any key algebra generates a space $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$. An elementary Kähler subalgebra $F + X$ [5] is then defined by the following commutation relations:

$$[h,g] = \mu\, g\,, \qquad [h,x] = \frac{\mu}{2}x\,, \qquad [g,x] = 0\,, \qquad [x,y] = \mu < J^1\,x, y > g\,, \tag{3.14}$$

x, y being elements of X. The collection of r generators h_I of the F_I generate the Cartan subalgebra of Solv_Q.

The canonical quaternionic subalgebra has the structure $E = F_0 + J^2\,F_0$, where F_0 is a key algebra of type I, which is stable under the action of the complex structure J^1.

[5] An elementary Kähler algebra is of type I, II or III, when its key algebra is of type I, II or III.

The intersection of U with E is given by F_0, so the structure of U can alternatively be specified as

$$U = F_0 + \mathrm{Solv}_{SK}\,, \qquad \mathrm{Solv}_{SK} = \sum_{i=1}^{r-1} F_i + X_i\,. \tag{3.15}$$

The second term, denoted as Solv_{SK}, is called the special Kähler subalgebra. It is a normal Kähler subalgebra of rank $r-1$.

It turns out that the possible ranks of normal quaternionic algebras range from 1 to 4. All spaces with $E = \mathrm{Solv}(\mathrm{USp}(2,2))$ have rank 1, while the spaces with $E = \mathrm{Solv}(\mathrm{SU}(2,1))$ have ranks from 1 to 4. We now give an overview of these algebras according to their rank. For later convenience, we will also indicate the weights[6] of the different generators, with respect to the Cartan subalgebra of Solv_Q. These can for instance be found in [63, 64, 66].

Rank 1. For the rank 1 spaces, two important classes can be distinguished, according to whether the canonical quaternionic subalgebra is $\mathrm{Solv}(\mathrm{USp}(2,2))$ or $\mathrm{Solv}(\mathrm{SU}(2,1))$. The algebras with $\mathrm{Solv}(\mathrm{USp}(2,2))$ as canonical quaternionic subalgebra will be denoted as $\mathrm{Solv}_Q(-3,P)$, where $P \geq 0$. They are the only spaces with $\mathrm{Solv}(\mathrm{USp}(2,2))$ as canonical subalgebra and they are given explicitly by the following symmetric spaces:

$$\exp\left[\mathrm{Solv}_Q\,(-3,P)\right] \simeq \frac{\mathrm{USp}(2P+2,2)}{\mathrm{USp}(2P+2)\times \mathrm{SU}(2)}\,. \tag{3.16}$$

When $P=0$, their solvable algebra consists of four generators : one Cartan generator and three generators of weight 1. This weight then corresponds to a positive root of $\mathrm{SU}(1,1)$. For $P>0$, the weight structure is different; in that case, there is an additional set of $4\,P$ generators with weight $\frac{1}{2}$. In this case, the full set of weights of the solvable algebra does not represent a positive root system of a Lie algebra of simple type.

When the canonical quaternionic subalgebra is $\mathrm{Solv}(\mathrm{SU}(2,1))$, there's one corresponding manifold of rank 1 that can be described according to the scheme of (3.12) and (3.15). The corresponding homogeneous space has $\mathrm{Solv}_{SK} = 0$ and can be obtained as the reduction of pure $\mathcal{N}=2$ supergravity in four dimensions to three dimensions. We will denote this space by SG_4. It is given by the symmetric space $\frac{\mathrm{SU}(1,2)}{\mathrm{SU}(2)\times \mathrm{U}(1)}$. The solvable algebra consists of one Cartan generator h_0, while there are two distinct weights associated to two spaces g_0 and q:

$$\boxed{h_0\,:\,(0)\quad g_0\,:\,(1)\quad q\,:\,(\tfrac{1}{2})} \tag{3.17}$$

The weight space g_0 is one-dimensional, whereas q is two-dimensional.

Rank 2. In this case, there are two distinct possibilities. The first is given by $\mathrm{Solv}_{SK} = F$, where F is a key algebra of type III. This corresponds to the quaternionic manifold $\frac{G_{2(+2)}}{\mathrm{SU}(2)\times \mathrm{SU}(2)}$, whose isometry algebra is maximally split. As this case is obtained by

[6] By weights (or gradings) we mean here the eigenvalues of the different generators under the adjoint action of the Cartan subalgebra. So, the weights can be seen as the roots of the solvable algebra.

dimensional reduction of 5-dimensional pure $\mathcal{N}=2$ supergravity to three dimensions, we will denote it by SG_5. The solvable algebra that generates this space has two Cartan generators (h_0, h_1) and the weights are summarized in the following scheme:

$$\begin{array}{|llll|} \hline h_0 \,:\, (0,0) & g_0 \,:\, (1,0) & q_0 \,:\, (1,-\frac{1}{2\sqrt{3}}) & p_0 \,:\, (1,\frac{1}{2\sqrt{3}}) \\ h_1 \,:\, (0,0) & g_1 \,:\, (0,\frac{1}{\sqrt{3}}) & q_1 \,:\, (1,-\frac{\sqrt{3}}{2}) & p_1 \,:\, (1,\frac{\sqrt{3}}{2}) \\ \hline \end{array} \tag{3.18}$$

Note that all weight spaces in the above diagram are one-dimensional.

The second possibility is represented by the series $\frac{\mathrm{SU}(2,P+2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(P+2))}$. They are characterized by the following special Kähler subalgebra:

$$\mathrm{Solv}_{SK} = F + Y\,, \tag{3.19}$$

where F is a key algebra of type I. The corresponding algebras will be denoted by $\mathrm{Solv}_Q(-2,P)$ and they are more explicitly characterized by the following weights:

$$\begin{array}{|llll|} \hline h_0 \,:\, (0,0) & g_0 \,:\, (1,0) & q \,:\, (\frac{1}{2},-\frac{1}{2}) & p \,:\, (\frac{1}{2},\frac{1}{2}) \\ h_1 \,:\, (0,0) & g_1 \,:\, (0,1) & Y \,:\, (0,\frac{1}{2}) & \tilde{Y} \,:\, (\frac{1}{2},0) \\ \hline \end{array} \tag{3.20}$$

These weights are associated to six different subsets of generators, two of which, namely g_0 and g_1 are one-dimensional. The spaces q and p are two-dimensional while the dimension of the spaces Y and $\tilde{Y}$ depends on the value of the parameter P:

$$\dim Y = \dim \tilde{Y} = 2\,P\,. \tag{3.21}$$

For the case of $\mathrm{Solv}_Q(-2,0)$, the spaces Y and $\tilde{Y}$ are absent and the set of weights gives a positive root system of $\mathrm{SO}(3,2)$, whereas the full set of weights (3.20) does not have a simple Lie algebra description for $P \neq 0$.

Rank 3. The special Kähler subalgebra of the quaternionic algebras of rank 3 is a sum of two elementary Kähler algebras of types I and II, respectively, where the second one has no X-part. It is convenient to rename $X_1 = Y$:

$$\mathrm{Solv}_{SK} = (F_1 + Y) + F_2\,, \tag{3.22}$$

The space Y forms a symplectic representation of the type II key algebra F_2, and under this action it splits into two subspaces $Y = Y^+ + Y^-$, with $Y^- = J^1\,Y^+$. There is a corresponding quaternionic algebra for every integer $P \geq 0$, that is denoted as $\mathrm{Solv}_Q(-1,P)$. The algebra consists of 3 Cartan generators h_0, h_1 and h_+, while the weights of the other generators are summarized in the following table:

$$\begin{array}{|llll|} \hline h_0 \,:\, (0,0,0) & g_0 \,:\, (1,0,0) & q_0 \,:\, (\frac{1}{2},-\frac{1}{2},-\frac{1}{\sqrt{2}}) & p_0 \,:\, (\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}) \\ h_1 \,:\, (0,0,0) & g_1 \,:\, (0,1,0) & q_1 \,:\, (\frac{1}{2},-\frac{1}{2},\frac{1}{\sqrt{2}}) & p_1 \,:\, (\frac{1}{2},\frac{1}{2},-\frac{1}{\sqrt{2}}) \\ h_+ \,:\, (0,0,0) & g_+ \,:\, (0,0,\frac{1}{\sqrt{2}}) & q_+ \,:\, (\frac{1}{2},\frac{1}{2},0) & p_+ \,:\, (\frac{1}{2},-\frac{1}{2},0) \\ Y^+ \,:\, (0,\frac{1}{2},\frac{1}{2\sqrt{2}}) & Y^- \,:\, (0,\frac{1}{2},-\frac{1}{2\sqrt{2}}) & \tilde{Y}^+ \,:\, (\frac{1}{2},0,\frac{1}{2\sqrt{2}}) & \tilde{Y}^- \,:\, (\frac{1}{2},0,-\frac{1}{2\sqrt{2}}) \\ \hline \end{array} \tag{3.23}$$

The dimensions of the weight spaces of type Y are related to the parameter P in the following way:

$$\dim Y^+ = \dim Y^- = \dim \tilde{Y}^+ = \dim \tilde{Y}^- = P\,, \tag{3.24}$$

while the other weight spaces are one-dimensional. We can distinguish two cases:

- $\mathrm{Solv}_Q(-1,0)$. In this case, there are no generators of type Y and the solvable algebra corresponds to the solvable algebra that generates the symmetric coset $\frac{\mathrm{SO}(3,4)}{\mathrm{SO}(3)\times\mathrm{SO}(4)}$.
- $\mathrm{Solv}_Q(-1,P)$ with $P \neq 0$. These algebras are not related to any simple Lie algebra. The corresponding quaternionic spaces are not symmetric.

Rank 4. For the rank 4 quaternionic algebras, the subalgebra U has the following structure:

$$\begin{aligned} U &= F_0 + \mathrm{Solv}_{SK} = F_0 + (F_1 + X_1) + (F_2 + X_2) + F_3\,, \\ [F_I,\,F_J] &= 0\,, \qquad I,J = 0,1,2,3\,, \\ F_I &= \{h_I,\,g_I\}\,; \;\; [h_I,\,g_I] = g_I\,, \quad [h_i,\,X_i] = \frac{1}{2}X_i\,, \quad i,j = 1,2,3. \end{aligned} \tag{3.25}$$

It is convenient to set $X_2 = X$ and $X_1 = Y + Z$, where $[F_2, Y] = 0$ and $[F_2, Z] = Z$. Decomposing the spaces X, Y into eigenspaces with respect to the adjoint action of h_3 and the space Z in eigenspaces with respect to h_2, the corresponding eigenspaces are denoted as X^+ and $X^- = J^1 X^+$, etc.

The gradings of the generators with respect to the Cartan subalgebra (h_0, h_1, h_2, h_3) are summarized in the following table [61, 63]:

$h_0\,:\,(0,0,0,0)$	$g_0\,:\,(1,0,0,0)$	$q_0\,:\,(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$p_0\,:\,(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$
$h_1\,:\,(0,0,0,0)$	$g_1\,:\,(0,1,0,0)$	$q_1\,:\,(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$p_1\,:\,(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
$h_2\,:\,(0,0,0,0)$	$g_2\,:\,(0,0,1,0)$	$q_2\,:\,(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	$p_2\,:\,(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$
$h_3\,:\,(0,0,0,0)$	$g_3\,:\,(0,0,0,1)$	$q_3\,:\,(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$p_3\,:\,(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$
$X^+\,:\,(0,0,\frac{1}{2},\frac{1}{2})$	$X^-\,:\,(0,0,\frac{1}{2},-\frac{1}{2})$	$\tilde{X}^+\,:\,(\frac{1}{2},\frac{1}{2},0,0)$	$\tilde{X}^-\,:\,(\frac{1}{2},-\frac{1}{2},0,0)$
$Y^+\,:\,(0,\frac{1}{2},0,\frac{1}{2})$	$Y^-\,:\,(0,\frac{1}{2},0,-\frac{1}{2})$	$\tilde{Y}^+\,:\,(\frac{1}{2},0,\frac{1}{2},0)$	$\tilde{Y}^-\,:\,(\frac{1}{2},0,-\frac{1}{2},0)$
$Z^+\,:\,(0,\frac{1}{2},\frac{1}{2},0)$	$Z^-\,:\,(0,\frac{1}{2},-\frac{1}{2},0)$	$\tilde{Z}^+\,:\,(\frac{1}{2},0,0,\frac{1}{2})$	$\tilde{Z}^-\,:\,(\frac{1}{2},0,0,-\frac{1}{2})$

(3.26)

The corresponding solvable algebras will be denoted by $\mathrm{Solv}_Q(q,P,\dot{P})$. The numbers q, P and $\dot{P}$ are related to the dimensions of the subspaces of type X, Y and Z. The parameter q gives the dimension of the spaces[7] of type X:

$$q = \dim X\,. \tag{3.27}$$

As will be made more explicit later on, the spaces $Y^+ \bigcup Z^+$ form a real representation of the Clifford algebra in $q{+}1$ dimensions with positive signature. A similar result holds for the other spaces of types Y, Z. This representation can in general be reducible.

[7] Here and below, $\dim X = \dim X^+ = \dim X^- = \dim \tilde{X}^+ = \dim \tilde{X}^-$, and similar for Y and Z.

When $q \neq 0$ mod 4 however, there exists only one irreducible representation of this Clifford algebra. The representation formed by the spaces $Y \bigcup Z$ is thus uniquely specified once the number of irreducible representations that constitute it is given. This number is denoted by P. When $q = 0$ mod 4, there exist 2 inequivalent representations of the Clifford algebra and one needs 2 numbers, P and $\dot{P}$ to indicate the representation content of the representation formed by $Y \bigcup Z$. The numbers P and $\dot{P}$ are thus related to the dimension of the union of spaces of type Y and Z:

$$\begin{aligned} q \neq 0 \quad &: \quad (P+\dot{P}) = \frac{\dim Y + \dim Z}{\mathcal{D}_{q+1}}\,, \qquad \dim Y = \dim Z\,, \\ q = 0 \quad &: \quad P = \dim Y\,, \qquad \dot{P} = \dim Z\,, \end{aligned} \tag{3.28}$$

where $\mathcal{D}_{q+1}$ is the dimension of the irreducible representations of the Clifford algebra in $q+1$ dimensions with positive signature. Information concerning real Clifford algebras and their representations can be found in appendix B. In general, the dimension of these manifolds is related to the parameters $(q, P, \dot{P})$ in the following way:

$$\dim \mathrm{Solv}_Q(q, P, \dot{P}) = 4(n+1)\,, \qquad n = 3 + q + (P + \dot{P})\mathcal{D}_{q+1}\,. \tag{3.29}$$

All quaternionic solvable algebras of rank 4 necessarily have the subset of generators (h_I, g_I, q_I, p_I), where $I = 0, 1, 2, 3$, and can have some or all of the three spaces of types X, Y, Z. We can thus distinguish the following particular cases:

- $\mathrm{Solv}_Q(0,0,0) \equiv \mathrm{Solv}(\mathrm{SO}(4,4))$, where the spaces X, Y, Z are all absent. The weights of the generators (h_I, g_I, q_I, p_I) in (3.26) correspond to the positive root system of SO(4,4). The corresponding space is $\frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4)\times\mathrm{SO}(4)}$.
- $\mathrm{Solv}_Q(P,0,0) = \mathrm{Solv}_Q(0,P,0) = \mathrm{Solv}_Q(0,0,P) \equiv \mathrm{Solv}(\mathrm{SO}(4,4+P))$, where only one of the types of spaces X, Y or Z is present. The set of weights of the generators involved in this case (for example $(h_I, g_I, q_I, p_I, Y^{\pm}, \tilde{Y}^{\pm})$) corresponds to the positive root system of the simple Lie algebra SO(4,5). These solvable algebras generate the symmetric spaces $\frac{\mathrm{SO}(4,4+P)}{\mathrm{SO}(4)\times\mathrm{SO}(4+P)}$.
- $\mathrm{Solv}_Q(0,P,\dot{P})$, $P\dot{P} \neq 0$. Note that the set of weights in this case does not correspond to any positive root system of the simple type. These algebras lead to quaternionic spaces that are non-symmetric.
- $\mathrm{Solv}_Q(q,P,\dot{P})$, $P+\dot{P} > 0$. In these cases all three spaces X, Y, Z are present and the complete set of weights given in (3.26) closes the positive root system of the Lie algebra F_4, whereas the full solvable algebra generically does not give rise to a symmetric space.

After we have displayed the weight system of the quaternionic spaces of rank 4 in (3.26), we can see that the other normal quaternionic spaces are truncations of this one (apart from an exception for the case indicated as SG_5).

In general, the non-generic cases can be obtained by deleting some rows of (3.26), and restricting the weights consequently. The full list of rows is $(0123XYZ)$. For $q = 0$, (3.27) already implies that the row X is absent. If $\dot{P} = 0$, we also do not have the Z row, according to (3.28), and for $P = \dot{P} = 0$ neither the Y row. This exhausts the rank 4 cases.

The table for rank 3, i.e., (3.23) can be obtained by deleting also one of the rows that contain generators in the Cartan subalgebra. Keeping only $h_+ = \frac{1}{\sqrt{2}}(h_2 + h_3)$, rather than h_2 and h_3, we have obtained weight vectors for the rank 3 spaces from the general ones in (3.26). The rows Y and Z become identical when restricted to the weights under (h_0, h_1, h_+) such that we only have to keep one of them. For $P = 0$, this row is also absent.

For $q = -2$, the rows 2 and 3 are absent. As such, the weight vectors have only two components, which implies that some weights in (3.26) become identical, namely those of q_0 and q_1, of p_0 and p_1, of Y^+ and Y^- and of $\tilde{Y}^+$ and $\tilde{Y}^-$. This leads to the reduced table in (3.20). The Y generators are absent for $P = 0$.

The other rank 2 system is SG_5, whose weight vectors are modified with respect to the systems denoted generically as $\mathrm{Solv}_Q(q, P, \dot{P})$, as shown in (3.18). Only rows 0 and 1 occur, but due to the modified weights, there is no degeneracy as was the case for $\mathrm{Solv}_Q(-2, 0)$.

Finally, SG_4 consists of only the 0 row, and as such only the first component of the root vectors is relevant. Then q_0 and p_0 are identical, and this leads to (3.17).

The equations (3.27) and (3.28), which were mentioned for rank 4, also have a general validity, except of course for $q < 0$. However, in that case the negative value indicates the number of rows of (0123) that have to be deleted such that (3.29) is also generally valid.

3.2.3 Higher-dimensional origin of homogeneous special geometry

From the above construction, it is clear that there is a one-to-one correspondence between the quaternionic algebra Solv_Q and its special Kähler subalgebra Solv_{SK}, at least in the case when the canonical quaternionic subalgebra E is given by $\mathrm{Solv}(\mathrm{SU}(2,1))$ [8]. This correspondence is in fact the inverse of the **c**-map, discussed in section 2.4.1, which maps the $2n$-dimensional special Kähler manifold $\exp[\mathrm{Solv}_{SK}]$ to the $4(n+1)$-dimensional quaternionic-Kähler manifold $\exp[\mathrm{Solv}_Q]$. This formalism thus allows one to see how the special Kähler subalgebra is enlarged to a quaternionic algebra, upon dimensional reduction from 4 to 3 dimensions. A similar discussion can be developed for the **r**-map, mapping the solvable algebras that generate very special real manifolds to special Kähler algebras. These relations can more generally be summarized as in the table 3.1, representing first the generators of the quaternionic algebra Solv_Q as in (3.26), but rotated over 90°. In this way, the different rows are related by the action of the complex structures[9]. Furthermore, it indicates how the algebras Solv_{SK} and Solv_R are embedded in Solv_Q. The inverse **c**- and **r**-map can then be defined by deleting generators as indicated.

One can in fact also give a six-dimensional origin of a lot of the homogeneous quaternionic-Kähler spaces. As was shown in [59, 67], all homogeneous quaternionic spaces of rank 3 and 4 can be obtained from a reduction of $(1,0)$ supergravities in 6 dimensions over a three-torus. These supergravities have an obligatory gravitational multiplet, consisting of the metric, an anti-selfdual 2-form and two gravitini $(g_{MN}, B^-_{MN}, \psi_{i|M})$, where $M = 0, \ldots, 5$; $i = 1, 2$. Furthermore, three kinds of matter multiplets can be coupled, namely tensor multiplets, vector multiplets and hypermultiplets. One tensor multiplet contains a self–dual 2-form, 2

[8] The hyperbolic spaces generated by $\mathrm{Solv}(-3, P)$, where $E = \mathrm{Solv}(\mathrm{USp}(2,2))$ are not in the image of the **c**-map.

[9] The action of J^1 is really column by column in these tables, but applying J^2 and hence also J^3 on generators of the lowest row leads to a linear combination of the generators in the row indicated by $J^2 A$, resp. $J^3 A$.

Table 3.1 *Generators of the solvable algebras of special manifolds. The generators in different rows are related by complex structures as indicated in the last column. The three tables indicate the generators of the quaternionic algebra* Solv_Q*, the* **c***–dual Kählerian algebra* Solv_{SK} *and the real special algebra* Solv_R*. The last line indicates the multiplicity of any entry in the column, where the last two columns are merged to get to a common expression. If q is zero or negative, the X column is absent, and the negative number indicates the number of columns to the left of it that are absent too, such that the general formula (3.29) always holds.*

$\mathrm{Solv}_Q =$

p_0	p_1	p_2	p_3	$\dot{X}^-$	$\dot{Y}^-$	$\dot{Z}^-$	$-J^3A = J^2J^1A$
q_0	q_1	q_2	q_3	$\dot{X}^+$	$\dot{Y}^+$	$\dot{Z}^+$	J^2A
g_0	g_1	g_2	g_3	X^-	Y^-	Z^-	J^1A
h_0	h_1	h_2	h_3	X^+	Y^+	Z^+	A

$\mathrm{Solv}_{SK} =$

	g_1	g_2	g_3	X^-	Y^-	Z^-	J^1A
	h_1	h_2	h_3	X^+	Y^+	Z^+	A

$\mathrm{Solv}_R =$

				X^-	Y^-	Z^-	A
		h_2	h_3				A

#

1	1	1	1	q	$(P+\dot{P})\mathcal{D}_{q+1}$

spinor fields and a scalar (B^+_{MN}, χ^i, ϕ). A vector multiplet on the other hand consists of a six-dimensional vector and two gaugini (A_M, λ^i). The hypermultiplets contain 4 scalar fields and 2 spinor fields. Hypermultiplets are not relevant in our construction, since hyperscalars already span a quaternionic space in six dimensions that does not become enlarged when stepping down to $d = 5, 4, 3$, so they cannot give rise to chains of manifolds connected with **r**- and **c**-maps. The full theory is given in [68].

The total bosonic field content of the gravity-matter system of $d = 6$, $(1, 0)$ supergravity, excluding hypermultiplets is:

$$(g_{MN}, B^I_{MN}, A^\Lambda_M, \phi^\alpha)\,, \qquad I = 1, \ldots, n_T+1\,, \quad \Lambda = 1, \ldots, n_V\,, \qquad \alpha = 1, \ldots, n_T\,, \tag{3.30}$$

where we included n_V vector multiplets and n_T tensor multiplets. The n_T scalars parametrize the symmetric space

$$\mathcal{M}_{6d} = \frac{\mathrm{SO}(n_T, 1)}{\mathrm{SO}(n_T)}\,. \tag{3.31}$$

The action of these theories contains a topological term [69, 70, 71] that describes couplings of tensor multiplets to vector multiplets:

$$\mathcal{L}_{\rm CS} = C_{I\Lambda\Sigma} B^I \wedge F^\Lambda \wedge F^\Sigma \,, \tag{3.32}$$

where F^Λ are the field strengths of the vectors. This generic form was already conjectured in [72, 73] and was found in [74]. For specific values of the constant tensor $C_{I\Lambda\Sigma}$, these theories give rise to the homogeneous quaternionic-Kähler manifolds of rank 3 and 4 in three dimensions.

The parameters $(q, P, \dot{P})$ are then related to the number of tensor and vector multiplets of $d = 6$ supergravity as follows:

$$n_T = q + 1, \qquad n_V = (P + \dot{P})\mathcal{D}_{q+1} \,. \tag{3.33}$$

3.2.4 Summary : the full classification of homogeneous special geometries

We can now summarize the two previous sections by giving the full classification of homogeneous quaternionic-Kähler manifolds, as well as their possible 4- and 5-dimensional counterparts. So far, we have used the notation Solv_Q to denote the solvable algebras that generate homogeneous quaternionic-Kähler manifolds. Similarly, we will use $\mathrm{Solv}_{SK,R}$ to denote the solvable algebras that generate homogeneous special Kähler, respectively very special real manifolds and that can be obtained from Solv_Q as indicated in table 3.1. As these solvable algebras are basically characterized by the three numbers $q, P, \dot{P}$, we will denote them as $\mathrm{Solv}_{Q,SK,R}\left(q, P, \dot{P}\right)$. As in [75], the family of spaces (very special real, special Kähler and quaternionic-Kähler) will then be denoted as $L(q, P, \dot{P})$. For the family of spaces that have pure supergravity as higher-dimensional origin in 4, respectively 5 dimensions, we will employ the notation SG_4, respectively SG_5.

The full list of homogeneous special geometries is then given in table 3.2. The horizontal lines in the table separate spaces of different rank. When the space is symmetric, we explicitly mention to which coset space it corresponds. When the space is non-symmetric, we mention the solvable algebra by which it is generated. When there is no entry in the table, the corresponding manifold does not exist. For instance for the spaces of type $L(-3, P)$, there is only a quaternionic version, which is not in the image of the **c**-map. The corresponding supergravity theory in 3 dimensions can thus not be uplifted to 4 dimensions. This is in contrast to the case where the entry is SG, which also denotes an empty scalar manifold. In this case however, one can lift the 3-dimensional supergravity theory to pure supergravity in 4 or 5 dimensions, depending on whether one deals with the SG_4 or SG_5 series.
Furthermore, note that the last four lines are in fact already included in $L(q, P)$. We have however listed them separately as they correspond to symmetric spaces. The list of symmetric spaces then consists of $L(-3, P)$, SG_4, $L(-2, P)$, SG_5, $L(-1, 0)$, $L(0, P)$, $L(1, 1)$, $L(2, 1)$, $L(4, 1)$, $L(8, 1)$. The real member of the $L(-1, P)$ series is also symmetric, it's corresponding Kähler and quaternionic-Kähler versions however are non-symmetric.

3.2.5 Isometry groups of homogeneous special geometries

In the previous parts, we have reviewed the classification of homogeneous quaternionic-Kähler manifolds. The key element in this construction is the fact that one can rephrase

Table 3.2 *Homogeneous very special real, special Kähler and quaternionic spaces.*

$C(h)$	range	real	Kähler	quaternionic
$L(-3,P)$	$P \geq 0$			$\frac{\mathrm{USp}(2P+2,2)}{\mathrm{USp}(2P+2)\times\mathrm{SU}(2)}$
SG_4			SG	$\frac{\mathrm{SU}(1,2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$
$L(-2,P)$	$P \geq 0$		$\frac{\mathrm{U}(P+1,1)}{\mathrm{U}(P+1)\times\mathrm{U}(1)}$	$\frac{\mathrm{U}(P+2,2)}{\mathrm{U}(P+2)\times\mathrm{U}(2)}$
SG_5		SG	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$	$\frac{G_{2(2)}}{\mathrm{SU}(2)\times\mathrm{SU}(2)}$
$L(-1,0)$		$\mathrm{SO}(1,1)$	$\left[\frac{\mathrm{SO}(2,1)}{\mathrm{SO}(2)}\right]^2$	$\frac{\mathrm{SO}(3,4)}{\mathrm{SO}(3)\times\mathrm{SO}(4)}$
$L(-1,P)$	$P \geq 1$	$\frac{\mathrm{SO}(P+1,1)}{\mathrm{SO}(P+1)}$	$\mathrm{Solv}_{SK}(-1,P)$	$\mathrm{Solv}_Q(-1,P)$
$L(0,P)$	$P \geq 0$	$\frac{\mathrm{SO}(P+1,1)}{\mathrm{SO}(P+1)}\times \mathrm{SO}(1,1)$	$\frac{\mathrm{SO}(P+2,2)}{\mathrm{SO}(P+2)\times\mathrm{SO}(2)}\times\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$	$\frac{\mathrm{SO}(P+4,4)}{\mathrm{SO}(P+4)\times\mathrm{SO}(4)}$
$L(0,P,\dot{P})$	$P \geq \dot{P} \geq 1$	$\mathrm{Solv}_R(0,P,\dot{P})$	$\mathrm{Solv}_{SK}(0,P,\dot{P})$	$\mathrm{Solv}_Q(0,P,\dot{P})$
$L(q,P)$	$\begin{cases} q \geq 1 \\ P \geq 1 \end{cases}$	$\mathrm{Solv}_R(q,P)$	$\mathrm{Solv}_{SK}(q,P)$	$\mathrm{Solv}_Q(q,P)$
$L(4m,P,\dot{P})$	$\begin{cases} m \geq 1 \\ P \geq \dot{P} \geq 1 \end{cases}$	$\mathrm{Solv}_R(4m,P,\dot{P})$	$\mathrm{Solv}_{SK}(4m,P,\dot{P})$	$\mathrm{Solv}_Q(4m,P,\dot{P})$
$L(1,1)$		$\frac{\mathrm{S}\ell(3,\mathbb{R})}{\mathrm{SO}(3)}$	$\frac{\mathrm{Sp}(6)}{\mathrm{U}(3)}$	$\frac{F_{4(4)}}{\mathrm{USp}(6)\times\mathrm{SU}(2)}$
$L(2,1)$		$\frac{\mathrm{S}\ell(3,\mathbb{C})}{\mathrm{SU}(3)}$	$\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1)}$	$\frac{E_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)}$
$L(4,1)$		$\frac{\mathrm{SU}^*(6)}{\mathrm{Sp}(6)}$	$\frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6)\times\mathrm{U}(1)}$	$\frac{E_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{SU}(2)}$
$L(8,1)$		$\frac{E_{6(-26)}}{F_{4(-52)}}$	$\frac{E_{7(-25)}}{E_{6(-78)}\times\mathrm{U}(1)}$	$\frac{E_{8(-24)}}{E_{7(-133)}\times\mathrm{SU}(2)}$

this as a classification of solvable Lie algebras. These solvable Lie algebras provide translational symmetries of the spaces they generate. They are however only a part of the full symmetry groups. We will now discuss the structure of the full isometry algebras of homogeneous special geometries [75]. This knowledge will be relevant later on in extending the Tits-Satake projection for symmetric spaces to general homogeneous special geometries. In this section, we will concentrate on the families of manifolds that have a very special real member. From table 3.2, it can be seen that this does not include all homogeneous spaces. The other cases however correspond to symmetric spaces and their Tits-Satake projection can be analyzed in the standard way, as will be explained in the next section. We will first discuss these isometry algebras for the special real manifolds occurring in five dimensions. Next, we will indicate what happens upon applying the **r**- and **c**-map.

Isometry algebras of homogeneous very special real spaces

As mentioned in section 2.2.3, very special real manifolds are defined by an equation of the form $C(y) = 1$, where $C(y)$ is a cubic polynomial of variables y that are functions of the physical scalar fields. We will indicate these y-variables here as $y = (y^1, y^2, y^\alpha, y^\Lambda)$ where α and Λ run over $q+1$ and $(P+\dot{P})\mathcal{D}_{q+1}$ values, respectively. For homogeneous very special real manifolds, the polynomial $C(y)$ assumes the following form:

$$C(y) = 3\Big\{ y^1 (y^2)^2 - y^1 y^\alpha y^\alpha - y^2 y^\Lambda y^\Lambda + \gamma_{\alpha\Lambda\Sigma} y^\alpha y^\Lambda y^\Sigma \Big\} . \tag{3.34}$$

In this equation, $\gamma_{\alpha\Lambda\Sigma}$ are the matrix elements of gamma-matrices γ_α, that form a real representation of the Euclidean Clifford algebra in $q+1$ dimensions. As already mentioned in the previous section, this representation is not necessarily irreducible. The number of irreducible representations contained in this representation is denoted by numbers P and $\dot{P}$, depending on whether there is only one or whether there are two inequivalent irreducible representations of this Clifford algebra. These gamma-matrices are thus in general $(P+\dot{P})\mathcal{D}_{q+1}$-dimensional real matrices. The isometry group of the corresponding spaces is then given by the linear transformations of the y that leave (3.34) invariant.

The structure of the isometry algebra $\mathcal{X}$ can be summarized by decomposing it with respect to the adjoint action of one of the Cartan generators $\underline{\lambda}$. One finds that $\mathcal{X}$ has the following structure:

$$\mathcal{X} = \mathcal{X}_0 + \mathcal{X}_{3/2} , \tag{3.35}$$

where the subscript denotes the grading with respect to $\underline{\lambda}$. The space $\mathcal{X}_{3/2}$ consists of generators $\underline{\xi}^\Lambda$, which are always present. Generically, there are no generators with negative gradings, except when the space is symmetric. In that case, in (3.35) there is an extra space $\mathcal{X}_{-3/2}$ consisting of generators $\underline{\zeta}_\Lambda$ and the algebra is semi-simple. The space $\mathcal{X}_0$ has the following structure:

$$\mathcal{X}_0 = \mathfrak{so}(1,1) \oplus \mathfrak{so}(q+1,1) \oplus \mathcal{S}_q(P,\dot{P}) , \tag{3.36}$$

where the $\mathfrak{so}(1,1)$ factor is generated by $\underline{\lambda}$. The generators of the additional invariances of (3.34), which are denoted by $\mathcal{S}_q(P,\dot{P})$ are given by the antisymmetric matrices S that commute with the matrices γ_α. These groups $\mathcal{S}_q(P,\dot{P})$ can be found in appendix B. For later purposes, we mention that the generators of $\mathcal{X}_{3/2}$ transform as a spinor representation under the adjoint action of $\mathfrak{so}(q+1,1)$, while under the adjoint action of $\mathcal{S}_q(P,\dot{P})$ they transform in a vector representation.

The solvable part of this isometry algebra, that acts transitively on the manifold is given by:

$$\begin{aligned} \mathrm{Solv}_R &= \mathfrak{so}(1,1) \oplus \mathrm{Solv}(\mathfrak{so}(q+1,1)) + \mathcal{X}_{3/2} \\ &= \{\underline{\lambda}, \mathrm{Solv}(\mathfrak{so}(q+1,1)), \underline{\xi}^\Lambda\} . \end{aligned} \tag{3.37}$$

The solvable algebra of $\mathfrak{so}(q+1,1)$ consists of one Cartan generator and q nilpotent generators. In this way, we can make contact with previously used notations. Indeed, we have 2 Cartan generators (namely the generator of $\mathfrak{so}(1,1)$ and the Cartan generator of $\mathrm{Solv}(\mathfrak{so}(q+1,1))$). They agree with h_2, h_3 (or suitable linear combinations thereof) in the last part of table 3.1. The q nilpotent generators of $\mathrm{Solv}(\mathfrak{so}(q+1,1))$ constitute the space X^-, while Y^- and Z^- together form $\mathcal{X}_{3/2}$.

Isometry algebras for homogeneous very special Kähler spaces

Upon dimensional reduction, the isometry algebras of real spaces discussed above, are enlarged to isometry algebras of homogeneous special Kähler spaces. The rank of the space is now increased with one unit. Decomposing the isometry algebra $\mathcal{W}$ into eigenspaces with respect to the adjoint action of one of the Cartan generators $\underline{\lambda}'$, the following structure occurs:

$$\begin{aligned}
&\mathcal{W} = \mathcal{W}'_0 + \mathcal{W}'_1 + \mathcal{W}'_2 \,, \\
&\mathcal{W}'_0 = \underline{\lambda}' \oplus \mathfrak{so}(q+2,2) \oplus \mathcal{S}_q(P,\dot{P}) \,, \\
&\mathcal{W}'_1 = \underline{\xi}^\Lambda + \underline{b}_\Lambda = (1, \text{spinor}, \text{vector}) \,, \\
&\mathcal{W}'_2 = \underline{b}_1 = (2,0,0) \,,
\end{aligned} \tag{3.38}$$

where for $\mathcal{W}'_1$ and $\mathcal{W}'_2$ we mention in brackets in which representations the generators transform. The first number in brackets denotes the grading with respect to $\underline{\lambda}'$, while the other two entries denote the representation under the adjoint action of $\mathfrak{so}(q+2,2)$ and $\mathcal{S}_q(P,\dot{P})$ respectively. Note that in general no generators with negative gradings occur. This is different for the symmetric spaces, where there are generators $(\underline{\zeta}_\Lambda, \underline{a}^\Lambda)$ at grading -1, and where there is a generators $\underline{a}^1$ at grading -2. In these cases the algebra is semi-simple.

The solvable subalgebra of translational isometries is given by the following set of generators:

$$\mathrm{Solv}_{SK} = \{\underline{\lambda}', \mathrm{Solv}(\mathfrak{so}(q+2,2)), \underline{\xi}^\Lambda, \underline{b}_\Lambda, \underline{b}_1\} \,. \tag{3.39}$$

Again, it is possible to make contact with the second part of table 3.1. The solvable algebra of $\mathfrak{so}(q+2,2)$ consists of $2q+4$ generators. Two of these belong to the Cartan subalgebra of Solv_{SK}, $2q$ of them constitute the 2 spaces X^+ and X^-, while the remaining 2 generators, together with $\underline{b}_1$ constitute the g-generators. Furthermore, the generators $\underline{\xi}^\Lambda$ and $\underline{b}_\Lambda$ deliver the $Y^\pm$-, $Z^\pm$-generators.

Isometry algebras for homogeneous very special quaternionic spaces

After dimensional reduction from 4 to 3 dimensions, the very special Kähler spaces of the previous section are enlarged to very special quaternionic manifolds. The corresponding isometry algebras are likewise extended and now have the following form (the index M runs over $q+2$ values):

$$\begin{aligned}
\mathcal{V} &= \mathcal{V}'_0 + \mathcal{V}'_1 + \mathcal{V}'_2 \,, \\
\mathcal{V}'_0 &= \underline{\epsilon}' \oplus \mathfrak{so}(q+3,3) \oplus \mathcal{S}_q(P,\dot{P}) \,, \\
\mathcal{V}'_1 &= (\underline{\xi}^\Lambda, \underline{b}_\Lambda) \oplus (\underline{\alpha}_\Lambda, \underline{\beta}^\Lambda) = (1, \text{spinor}, \text{vector}) \,, \\
\mathcal{V}'_2 &= \underline{\epsilon}_+ \oplus (\underline{\alpha}_1, \underline{\beta}^M, \underline{\beta}^0) \oplus \underline{b}_1 = (2, \text{vector}, 0) \,,
\end{aligned} \tag{3.40}$$

where we also indicated the representation of $\mathcal{V}'_1$ and $\mathcal{V}'_2$ under the adjoint action of the three subalgebras of $\mathcal{V}'_0$. As in the previous cases, we decomposed the isometry algebra in terms of the gradings with respect to the Cartan generator $\underline{\epsilon}'$. Again, for the symmetric spaces, the isometry algebra will be extended with additional generators, with gradings -1 and -2 with respect to $\underline{\epsilon}'$, such that the algebra is semi-simple. The solvable algebra in the quaternionic case is spanned by the following generators:

$$\mathrm{Solv}_Q = \{\underline{\epsilon}', \mathrm{Solv}(\mathfrak{so}(q+3,3)), \underline{\xi}^\Lambda, \underline{b}_\Lambda, \underline{b}_1, \underline{\alpha}_\Lambda, \underline{\beta}^\Lambda, \underline{\epsilon}_+, \underline{\alpha}_1, \underline{\beta}^M, \underline{\beta}^0\}\,. \tag{3.41}$$

3.3 Tits-Satake projection for symmetric spaces

In this section, we will review how the Tits-Satake projection is done for symmetric spaces. The aim is to extract some notions in this construction that can later be generalized for general homogeneous special manifolds. We will first give a more theoretical discussion on how the Tits-Satake projection is effectuated. In this discussion, the notion of paint group will appear; this will turn out to be a very important concept. We will then study these paint groups more specifically for the symmetric quaternionic-Kähler manifolds. Finally, we will illustrate the full construction in a concrete example.

3.3.1 The Tits-Satake projection for non-maximally split symmetric spaces

In this section, we will consider symmetric spaces that are non-compact, as these are the ones that are relevant for supergravity. The spaces we will deal with are thus coset spaces of the form:

$$\mathcal{M} = \frac{G_R}{H}\,, \tag{3.42}$$

where G_R is a non-compact real form of a complex semi-simple Lie group and H is its maximal compact subgroup. The Lie algebra of G_R will be denoted by $\mathbb{G}_R$ and is a real form of a complex semi-simple algebra $\mathbb{G}$. For such a non-compact coset, one can decompose $\mathbb{G}_R$ in the Lie algebra $\mathbb{H}$ of H and the orthogonal complement $\mathbb{K}$ (the Cartan decomposition):

$$\mathbb{G}_R = \mathbb{H} \oplus \mathbb{K}\,. \tag{3.43}$$

A Cartan subalgebra $\mathcal{H}_{\mathbb{G}_R}$ of $\mathbb{G}_R$ can be constructed by first searching for a maximal set of commuting elements in $\mathbb{K}$ and then completing this set with appropriate generators in $\mathbb{H}$. The part of $\mathcal{H}_{\mathbb{G}_R}$ that lies in $\mathbb{K}$ defines the non-compact Cartan subalgebra $\mathcal{H}^{\rm nc}$:

$$\mathcal{H}^{\rm nc} \equiv \mathcal{H}_{\mathbb{G}_R} \bigcap \mathbb{K}\,. \tag{3.44}$$

The dimension of $\mathcal{H}^{\rm nc}$ is often denoted as the non-compact rank $r_{\rm nc}$ or simply the rank of the coset G_R/H. This non-compact rank is thus always smaller or equal then the rank of $\mathbb{G}$. When equality is fulfilled, the manifold is maximally non-compact (also called maximally split); in this case $\mathbb{G}_R$ corresponds to the normal real form of some complex Lie algebra. We will however mainly concentrate on the case where the rank of the coset is strictly smaller than the rank of $\mathbb{G}$. In that case, one can make a non-trivial orthogonal split of the Cartan subalgebra[10] in a non-compact and a compact part:

$$\mathcal{H}_{\mathbb{G}_R} = \mathcal{H}^{\rm comp} \oplus \mathcal{H}^{\rm nc}\,. \tag{3.45}$$

[10] We will often abbreviate 'Cartan subalgebra' by CSA.

Consequently, every vector in the dual of the full Cartan subalgebra (so in particular every root α) can be decomposed in a part $\alpha_\perp$ that lies along the compact directions of the CSA and a part $\alpha_{||}$ that lies along the non-compact CSA:

$$\alpha = \alpha_\perp \oplus \alpha_{||} \,. \tag{3.46}$$

One can then perform a geometrical projection of the root system onto the non-compact CSA. This geometrical projection constitutes the first step in performing the Tits-Satake projection. More specifically, by putting all $\alpha_\perp$ equal to zero for each root α, one has projected the original root system $\Delta_{\mathbb{G}}$ onto a new system of vectors $\overline{\Delta}$ (the so-called restricted root system) living in a Euclidean space of dimension equal to the non-compact rank r_{nc}. Note that this restricted root system $\overline{\Delta}$ is not really a root system in the usual sense. Indeed, generically roots will occur in $\overline{\Delta}$ with multiplicities and it can moreover also happen that $2\alpha_{||}$ is a restricted root if $\alpha_{||}$ is one. If one however makes abstraction of the fact that roots can occur with multiplicities, one obtains a new root system Δ_{TS}, where the roots are now non-degenerate. This root system Δ_{TS} is the Tits-Satake projection of the original root system. So schematically, the projection is done by putting $\alpha_\perp = 0$ and deleting multiplicities:

$$\Pi_{\text{TS}} \quad : \quad \Delta_{\mathbb{G}} \mapsto \Delta_{\text{TS}} \; : \qquad \Delta_{\mathbb{G}} \stackrel{\alpha_\perp=0}{\longmapsto} \overline{\Delta} \underset{\text{multiplicities}}{\stackrel{\text{deleting}}{\longmapsto}} \Delta_{\text{TS}} \,. \tag{3.47}$$

In a lot of cases, it can happen that Δ_{TS} is actually a root system of simple type (this can for instance happen when $\overline{\Delta}$ contains no roots that are doubles of other roots).

Once the projection is defined at the level of the root system, we can promote it to a projection at the level of the solvable algebra $\text{Solv}(G_R/H)$ that generates the coset G_R/H. A useful observation in constructing $\text{Solv}(G_R/H)$ consists in noticing that the original roots can be divided in three sets, based upon their behavior with respect to the geometrical projection on the non-compact CSA, in the following way:

- ⋆ A first set of roots consists of all roots that vanish upon projection. These are the roots that lie fully along the compact directions of the CSA, henceforth we will denote them as compact roots. The set of compact roots will be denoted by Δ_{comp}.

- ⋆ A second set of roots is formed by all roots for which the projection is one-to-one. We will denote the set of these roots by Δ^η. They then project in a one-to-one way onto a subset of the restricted roots that we will denote by Δ^ℓ_{TS}.

- ⋆ Of course, in the projection it can happen that many different roots project onto the same restricted root. These roots then constitute the third class of roots. We will denote this subset of the original root system as Δ^δ and the subset of the restricted root system onto which they project as Δ^s_{TS}. So, there are several different roots in Δ^δ that project onto the same restricted root in Δ^s_{TS}.

Schematically, this division of the original root system in different subsets can be summarized as follows:

$$
\begin{array}{llllll}
\Delta_{\mathbb{G}} & = & \Delta^{\eta} & \bigcup & \Delta^{\delta} & \bigcup \; \Delta_{\text{comp}} \\
\downarrow \Pi_{\text{TS}} & & \downarrow \Pi_{\text{TS}} & & \downarrow \Pi_{\text{TS}} & \\
\Delta_{\text{TS}} & = & \Delta^{\ell}_{\text{TS}} & \bigcup & \Delta^{s}_{\text{TS}} &
\end{array}
$$

$$
\begin{aligned}
&\forall \alpha^{\ell} \in \Delta^{\ell}_{\text{TS}} \;:\; \dim \Pi^{-1}_{\text{TS}}\left[\alpha^{\ell}\right] = 1\,, \\
&\forall \alpha^{s} \in \Delta^{s}_{\text{TS}} \;:\; \dim \Pi^{-1}_{\text{TS}}\left[\alpha^{s}\right] = m[\alpha^{s}] > 1\,.
\end{aligned} \tag{3.48}
$$

Note that all roots of type η are orthogonal to the compact roots. This follows from the fact that for any two root vectors α and β such that there is no root of the form $\beta + m\alpha$ with $m \in \mathbb{Z}\backslash\{0\}$, the inner product of β and α vanishes. One can also see that in the trivial case of maximally split symmetric spaces, where Δ_{comp} is empty, all root vectors are in Δ^{η} or $\Delta^{\ell}_{\text{TS}}$ (the Tits-Satake projection is then trivial). Furthermore, under addition of root vectors the following properties are satisfied:

$\Delta_{\mathbb{G}}$	Δ_{TS}
$\Delta^{\eta} + \Delta^{\eta} \subset \Delta^{\eta}$	$\Delta^{\ell}_{\text{TS}} + \Delta^{\ell}_{\text{TS}} \subset \Delta^{\ell}_{\text{TS}}$
$\Delta^{\eta} + \Delta^{\delta} \subset \Delta^{\delta}$	$\Delta^{\ell}_{\text{TS}} + \Delta^{s}_{\text{TS}} \subset \Delta^{s}_{\text{TS}}$
$\Delta^{\delta} + \Delta^{\delta} \subset \Delta^{\eta} \bigcup \Delta^{\delta}$	$\Delta^{s}_{\text{TS}} + \Delta^{s}_{\text{TS}} \subset \Delta^{\ell}_{\text{TS}} \bigcup \Delta^{s}_{\text{TS}}$
$\Delta_{\text{comp}} + \Delta^{\eta} = \emptyset$	
$\Delta_{\text{comp}} + \Delta^{\delta} \subset \Delta^{\delta}$	

(3.49)

When constructing $\text{Solv}(G_R/H)$ explicitly via the Iwasawa decomposition, this division of roots in three different subsets implies that the generators of $\text{Solv}(G_R/H)$ can also be divided in three different types. The generators of this solvable algebra are schematically given in the following way:

$$
\text{Solv}(G_R/H) \;\; = \;\; \left\{ H_i, \Phi_{\alpha^{\ell}}, \Omega_{\alpha^{s}|I} \right\} \,. \tag{3.50}
$$

The generators that were denoted by H_i correspond to the non-compact Cartan generators of $\mathbb{G}_R$. Then, there are two kinds of nilpotent generators. First, there are the generators that were denoted by Φ. They correspond to the (positive) η-roots. Since for these, the projection is one-to-one, we can denote them via an index α^{ℓ}, indicating the restricted root upon which the η-root projects. Secondly, there are the generators of type Ω, that roughly correspond to the (positive) δ-roots. Since there are several δ-roots that project to the same restricted root α^{s}, one has to use an extra index I taking values from 1 to the multiplicity $m[\alpha^{s}]$ of that restricted root in the projection.

The crucial observation that allows one to perform the Tits-Satake projection of the solvable algebra (3.50), is the existence of a compact subalgebra $\mathbb{G}_{\text{paint}} \subset \mathbb{G}_R$ that acts as an algebra of outer automorphisms of the solvable algebra $\text{Solv}_{\mathbb{G}_R} \equiv \text{Solv}(G_R/H) \subset \mathbb{G}_R$:

$$
[\mathbb{G}_{\text{paint}}\,, \text{Solv}_{\mathbb{G}_R}] \subset \text{Solv}_{\mathbb{G}_R} \;. \tag{3.51}
$$

This paint algebra $\mathbb{G}_{\text{paint}}$ is essentially given by the compact Cartan generators, together with generators that involve step operators E^{α} corresponding to compact roots. One can then study in more detail how the generators of the solvable algebra behave with respect to the adjoint action of the paint group. The Cartan generators H_i and the generators Φ_{α^ℓ} are singlets under the action of $\mathbb{G}_{\text{paint}}$, i.e., each of them commutes with the whole of $\mathbb{G}_{\text{paint}}$:

$$[H_i \, , \, \mathbb{G}_{\text{paint}}] \, = \, [\Phi_{\alpha^\ell} \, , \, \mathbb{G}_{\text{paint}}] \, = \, 0 \, . \tag{3.52}$$

On the other hand, all generators $\Omega_{\alpha^s|I}$ associated to one restricted root α^s, transform among themselves under the adjoint action of the paint algebra $\mathbb{G}_{\text{paint}}$, according to a linear representation $\mathbf{D}^{[\alpha^s]}$ which, for different restricted roots α^s can be different:

$$\forall\, X \, \in \, \mathbb{G}_{\text{paint}} \quad : \quad \left[X \, , \, \Omega_{\alpha^s|I}\right] \, = \, \left(\mathbf{D}^{[\alpha^s]}[X]\right)_I^{\ J} \, \Omega_{\alpha^s|J} \, . \tag{3.53}$$

This makes it rather easy to define the Tits-Satake projection of $\text{Solv}(G_R/H)$, since one simply has to single out a subalgebra $\mathbb{G}^0_{\text{subpaint}} \subset \mathbb{G}_{\text{paint}}$ of the paint algebra such that with respect to $\mathbb{G}^0_{\text{subpaint}}$, each $m[\alpha^s]$–dimensional representation $\mathbf{D}^{[\alpha^s]}$ branches as follows:

$$\mathbf{D}^{[\alpha^s]} \stackrel{\mathbb{G}^0_{\text{subpaint}}}{\Longrightarrow} \underbrace{\mathbf{1}}_{\text{singlet}} \, \oplus \, \underbrace{\mathbf{J}}_{(m[\alpha^s]-1)-\text{dimensional}} \, . \tag{3.54}$$

Accordingly, one can split the range of the multiplicity index I as follows:

$$I = \{0, x\} \, , \qquad x = 1, \ldots, m[\alpha^s] - 1 \, . \tag{3.55}$$

The index 0 corresponds to the singlet, while x ranges over the representation $\mathbf{J}$. The restriction to the singlets then defines a solvable subalgebra $\text{Solv}_{\mathbb{G}_{\text{TS}}}$ of the original solvable algebra $\text{Solv}_{\mathbb{G}_R}$. $\text{Solv}_{\mathbb{G}_{\text{TS}}}$ then generates a coset $G_{\text{TS}}/H_{\text{TS}}$ that is maximally non-compact. Moreover, the root system of G_{TS} is given by the Tits-Satake projected root system Δ_{TS}. We then denote $\text{Solv}_{\mathbb{G}_{\text{TS}}}$ as the Tits-Satake projection of $\text{Solv}_{\mathbb{G}_R}$ and similarly $G_{\text{TS}}/H_{\text{TS}}$ is identified as the Tits-Satake projection of G_R/H.

As a by-product, one can obtain a more precise relation between $\text{Solv}_{\mathbb{G}_{\text{TS}}}$ and $\text{Solv}_{\mathbb{G}_R}$. It turns out that the tensor product $\mathbf{J} \otimes \mathbf{J}$ contains both the identity representation $\mathbf{1}$ and the representation $\mathbf{J}$ itself. Furthermore, there exists, in the representation $\bigwedge^3 \mathbf{J}$ a $\mathbb{G}^0_{\text{subpaint}}$-invariant tensor a^{xyz} such that the two solvable Lie algebras $\text{Solv}_{\mathbb{G}_R}$ and $\text{Solv}_{\mathbb{G}_{\text{TS}}}$ can be written as indicated in table 3.3.

3.3.2 The paint group for symmetric quaternionic-Kähler spaces

In the previous construction, the paint algebra manifested itself as a crucial notion. Indeed, once the paint group is known, one can study how the solvable algebra splits in representations with respect to this paint group. Knowledge of these representations then allows one to choose an appropriate subpaint group, by which one can single out the Tits-Satake projection of the symmetric space.

Essentially, this paint group is the part of the maximal compact subalgebra $\mathbb{H}$, spanned by the compact CSA $\mathcal{H}^{\text{comp}}$ and step operators associated to compact roots. For symmetric spaces, these paint groups can be read off from the so-called Satake diagrams. These diagrams are a useful tool in classifying real forms of complex Lie algebras (see for instance [76]). They consist of ordinary Dynkin diagrams where two kinds of extra decorations can be added:

$\mathrm{Solv}_{\mathbb{G}_R}$	$\mathrm{Solv}_{\mathbb{G}_{\mathrm{TS}}}$
$[H_i\,,\,H_j]=0$	$[H_i\,,\,H_j]=0$
$[H_i\,,\,\Phi_{\alpha^\ell}]=\alpha^{i\ell}\,\Phi_{\alpha^\ell}$	$\left[H_i\,,\,E^{\alpha^\ell}\right]=\alpha^{i\ell}$
$\left[H_i\,,\,\Omega_{\alpha^s\|I}\right]=\alpha^{is}\,\Omega_{\alpha^s\|I}$	$\left[H_i\,,\,E^{\alpha^s}\right]=\alpha^{is}\,E^{\alpha^s}$
$\left[\Phi_{\alpha^\ell}\,,\,\Phi_{\beta^\ell}\right]=N_{\alpha^\ell\beta^\ell}\,\Phi_{\alpha^\ell+\beta^\ell}$	$\left[E^{\alpha^\ell}\,,\,E^{\beta^\ell}\right]=N_{\alpha^\ell\beta^\ell}\,E^{\alpha^\ell+\beta^\ell}$
$\left[\Phi_{\alpha^\ell}\,,\,\Omega_{\beta^s\|I}\right]=N_{\alpha^\ell\beta^s}\,\Omega_{\alpha^\ell+\beta^s\|I}$	$\left[E^{\alpha^\ell}\,,\,E^{\beta^s}\right]=N_{\alpha^\ell\beta^s}\,E^{\alpha^\ell+\beta^s}$
If $\alpha^s+\beta^s\in\Delta^\ell_{\mathrm{TS}}$: $\left[\Omega_{\alpha^s\|I}\,,\,\Omega_{\beta^s\|J}\right]=\delta^{IJ}\,N_{\alpha^s\beta^s}\,\Phi_{\alpha^s+\beta^s}$	$\left[E^{\alpha^s}\,,\,E^{\beta^s}\right]=N_{\alpha^s\beta^s}E^{\alpha^s+\beta^s}$
If $\alpha^s+\beta^s\in\Delta^s_{\mathrm{TS}}$: $\begin{cases}\left[\Omega_{\alpha^s\|0}\,,\,\Omega_{\beta^s\|0}\right]=N_{\alpha^s\beta^s}\Omega_{\alpha^s+\beta^s\|0}\\ \left[\Omega_{\alpha^s\|0}\,,\,\Omega_{\beta^s\|x}\right]=N_{\alpha^s\beta^s}\Omega_{\alpha^s+\beta^s\|x}\\ \left[\Omega_{\alpha^s\|x}\,,\,\Omega_{\beta^s\|y}\right]=N_{\alpha^s\beta^s}\delta^{xy}\Omega_{\alpha^s+\beta^s\|0}\\ \qquad +N_{\alpha^s\beta^s}a^{xyz}\,\Omega_{\alpha^s+\beta^s\|z}\end{cases}$	$\left[E^{\alpha^s}\,,\,E^{\beta^s}\right]=N_{\alpha^s\beta^s}E^{\alpha^s+\beta^s}$

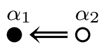
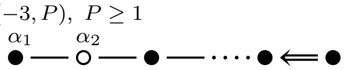
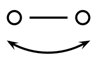
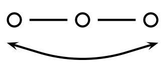
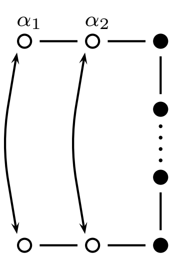
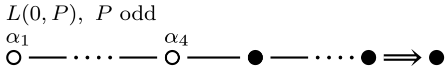
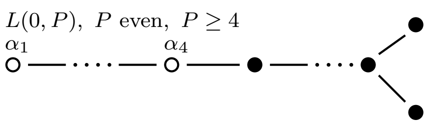
Table 3.3 *This table summarizes the commutation relations of* $\mathrm{Solv}_{\mathbb{G}_R}$ *and* $\mathrm{Solv}_{\mathbb{G}_{\mathrm{TS}}}$. *It is understood that* $N_{\alpha\beta}=0$ *if* $\alpha+\beta\notin\Delta_{\mathrm{TS}}$.

- Some of the dots of the Dynkin diagram can be painted in black. They denote simple roots that lie fully along the compact directions of the CSA (so simple roots in Δ_{comp}).
- Some of the dots can be connected by means of a two-sided arrow. They denote simple roots that result in the same restricted root setting $\alpha_\perp=0$. These necessarily belong to Δ^δ.

Given the Satake diagram, the paint group can then be read from it in the following way. The black dots form a Dynkin diagram of the semi-simple type. The paint group then contains a factor corresponding to this painted subdiagram. This factor contains step operators associated to the roots in Δ_{comp} as well as as many elements of $\mathcal{H}^{\mathrm{comp}}$ as there are roots colored in black in the diagram. Furthermore, for every arrow, there is one additional SO(2)-factor that commutes with the rest of the paint group. Each of these arrows leads to an additional generator in $\mathcal{H}^{\mathrm{comp}}$.

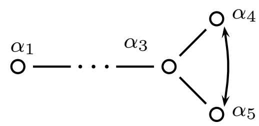
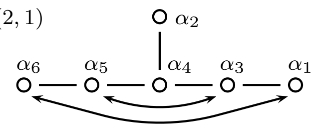
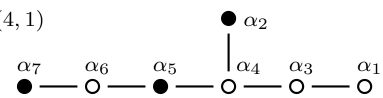
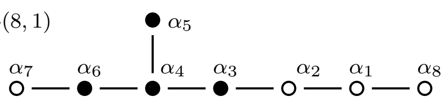
In the following table, we have summarized all Satake diagrams of isometry groups of the symmetric quaternionic-Kähler spaces. The first column contains the Satake diagram as well as a reference to which family $L(q,P)$ the quaternionic-Kähler manifold belongs. The second column contains the total number of dots that form the diagram, while the third column contains the total number of black dots in the diagram. Using this information, one can then infer the paint group, that is given in the last column, along the lines described above.

Table 3.4 *Satake diagrams and paint groups for symmetric quaternionic-Kähler manifolds.*

Satake diagram	dots	black dots	paint group
$L(-3,0)$ 	2	1	SO(3)
$L(-3,P)$, $P \geq 1$ 	$P+2$	$P+1$	$\mathrm{SO}(3) \times \mathrm{USp}(2P)$
SG_4 	2		SO(2)
$L(-2,0)$ 	3		SO(2)
$L(-2,P)$ $P \geq 1$ 	$P+3$	$P-1$	$\mathrm{SO}(2) \times U(P)$
$L(0,P)$, P odd 	$\frac{P+1}{2}+3$	$\frac{P+1}{2}-1$	$\mathrm{SO}(P)$
$L(0,P)$, P even, $P \geq 4$ 	$\frac{P}{2}+4$	$\frac{P}{2}$	$\mathrm{SO}(P)$

Continued on next page

Table 3.4 *continued*

Satake diagram	dots	black dots	paint group
$L(0,2)$ 	5		SO(2)
$L(2,1)$ 	6		SO(2) × SO(2)
$L(4,1)$ 	7	3	$\mathrm{SO}(3)^3$
$L(8,1)$ 	8	4	SO(8)

One could do a similar analysis for the symmetric very special real and special Kähler spaces. One will however find that the paint group is a property of the family $L(q,P)$. In other words, all symmetric spaces that are related by the **c**- and **r**-maps have the same paint groups. The paint group is thus a concept that is invariant under dimensional reduction.

3.3.3 An example

In order to make the previous discussion more clear, we will now illustrate it in a concrete example. The example is chosen such that it is computationally rather simple and yet relevant to $\mathcal{N}=2$ homogeneous special geometry. More precisely, we will discuss the Tits-Satake projection of the quaternionic member of the $L(8,1)$ family, which is given by the following symmetric space:

$$\frac{G_R}{H} = \frac{E_{8(-24)}}{E_{7(-133)} \times \mathrm{SU}(2)} \,. \tag{3.56}$$

We will show more explicitly that performing the Tits-Satake projection along the lines explained above, leads to the following result:

$$\Pi_{\mathrm{TS}} \quad : \quad \frac{E_{8(-24)}}{E_{7(-133)} \times \mathrm{SU}(2)} \longrightarrow \frac{F_{4(4)}}{\mathrm{USp}(6) \times \mathrm{SU}(2)} \,. \tag{3.57}$$

Note that the projected manifold $F_{4(4)}/(\mathrm{USp}(6) \times \mathrm{SU}(2))$ is maximally non-compact and is moreover again a quaternionic manifold. It is the quaternionic member of the $L(1,1)$ family.

The complex E_8 algebra has rank 8 and in the real section $E_{8(-24)}$ there are 4 compact Cartan generators and 4 non-compact ones. Let $\epsilon_i, i = 1, \cdots, 8$ denote an orthonormal basis of 8-component vectors. The E_8 root system is composed of the following 240 roots:

$$\Delta_{E_8} \equiv \left\{ \begin{array}{ccc} \pm\epsilon_i \pm \epsilon_j & (i \neq j) & \mathbf{112} \\ \underbrace{\pm\tfrac{1}{2}\epsilon_1 \pm \tfrac{1}{2}\epsilon_2 \pm \tfrac{1}{2}\epsilon_3 \pm \tfrac{1}{2}\epsilon_4 \pm \tfrac{1}{2}\epsilon_5 \pm \tfrac{1}{2}\epsilon_6 \pm \tfrac{1}{2}\epsilon_7 \pm \tfrac{1}{2}\epsilon_8}_{\text{even number of minus signs}} & & \mathbf{128} \\ \hline & & \mathbf{240} \end{array} \right\} . \tag{3.58}$$

We will make the following choice for the simple roots:

$$\begin{aligned} \alpha_1 &= \{0,1,-1,0,0,0,0,0\}, \\ \alpha_2 &= \{0,0,1,-1,0,0,0,0\}, \\ \alpha_3 &= \{0,0,0,1,-1,0,0,0\}, \\ \alpha_4 &= \{0,0,0,0,1,-1,0,0\}, \\ \alpha_5 &= \{0,0,0,0,0,1,-1,0\}, \\ \alpha_6 &= \{0,0,0,0,0,1,1,0\}, \\ \alpha_7 &= \{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}, \\ \alpha_8 &= \{1,-1,0,0,0,0,0,0\}. \end{aligned} \tag{3.59}$$

The Satake diagram can be found as the last one in table 3.4. Note that the nodes $\alpha_3, \alpha_4, \alpha_5, \alpha_6$ have been painted in black. This means that the corresponding Cartan generators such as e.g., $\alpha_3^i \mathcal{H}_i$ are compact. In this way, these Satake diagrams allow one to construct the splitting (3.45) of the Cartan subalgebra explicitly. Note also that in this case the black roots close a Dynkin diagram of a D_4 algebra. As they correspond to simple compact roots, they correspond to the simple roots of the paint group. From the Satake diagram, one can thus also conclude that the paint group in this case is given by

$$G_{\text{paint}} = \text{SO}(8), \tag{3.60}$$

as was also mentioned in table 3.4. Let us now perform the Tits-Satake projection of the root system explicitly. This case is rather simple since the span of the simple compact roots $\alpha_3, \alpha_4, \alpha_5, \alpha_6$ is just given by the Euclidean space along the orthonormal axes $\epsilon_4, \epsilon_5\, \epsilon_6, \epsilon_7$. These thus span the compact CSA. The Euclidean space along the orthonormal axes $\epsilon_1, \epsilon_2\, \epsilon_3, \epsilon_8$ span the non-compact CSA. Denoting the components of root vectors in the basis ϵ_i by α^i, the splitting (3.46) is very simple. We just have:

$$\alpha_\perp = \left\{\alpha^4 , \alpha^5 , \alpha^6 , \alpha^7\right\} \quad ; \quad \alpha_\parallel = \left\{\alpha^1 , \alpha^2 , \alpha^3 , \alpha^8\right\} , \tag{3.61}$$

and the projection (3.47) immediately yields the following restricted root system:

$$\Delta_{\text{TS}} = \left\{ \begin{array}{ccc} \pm\epsilon_i \pm \epsilon_j & (i \neq j \quad ; \quad i,j = 1,2,3,8) & \mathbf{24} \\ \pm\epsilon_i & (i = 1,2,3,8) & \mathbf{8} \\ \pm\tfrac{1}{2}\epsilon_1 \pm \tfrac{1}{2}\epsilon_2 \pm \tfrac{1}{2}\epsilon_3 \pm \tfrac{1}{2}\epsilon_8 & & \mathbf{16} \\ \hline & & \mathbf{48} \end{array} \right\} , \tag{3.62}$$

which can be seen to coincide with the root system of the simple complex algebra F_4.

With reference to the notations introduced in the previous section, let us now identify the subsets Δ^η and Δ^δ in the positive root subsystem of $\Delta^+_{E_8}$ and their corresponding images in the projection, namely $\Delta^\ell_{\mathrm{TS}}$ and Δ^s_{TS}.

Altogether, performing the projection the following is observed:

- There are 24 roots that have null projection on the non-compact space, namely

$$\alpha_\| = 0 \Leftrightarrow \alpha = \pm\epsilon_i \pm \epsilon_j \quad ; \quad i,j = 4,5,6,7\,. \tag{3.63}$$

  Step operators corresponding to these roots, together with the four compact Cartan generators, form a D_4 algebra, whose dimension is exactly 28. In the chosen real form, such a subalgebra of $E_{8(-24)}$ is the compact algebra SO(8) and its exponential acts as the paint group, as already mentioned in (3.60). All the remaining roots have a non–vanishing projection on the compact space. In particular:

- There are 12 positive roots of E_8 that are exactly projected on the 12 positive long roots of F_4, namely the first line of (3.62), which we therefore identify with $\Delta^\ell_{\mathrm{TS}}$. For these roots, we have $\alpha_\perp = 0$ and they constitute the Δ^η system mentioned above:

$$\Delta^+_{E_8} \supset \Delta^\eta = \{\epsilon_i \pm \epsilon_j\} = \Delta^\ell_{\mathrm{TS}} \quad ; \quad i<j \quad ; \quad i,j = 1,2,3,8\,. \tag{3.64}$$

- There are 8 different positive roots of E_8 that have the same projection on each of the $12 = 4 \oplus 8$ positive short roots of F_4, i.e., the second and third line of (3.62). Namely, all the remaining $12 \times 8 = 96$ roots of E_8 are all projected on short roots of F_4. The set of F_4 positive short roots can be split as follows:

$$\begin{array}{||lcll|c||}
\Delta^s_{\mathrm{TS}} & = & \Delta^s_{\mathrm{vec}} \bigcup \Delta^s_{\mathrm{spin}} \bigcup \Delta^s_{\overline{\mathrm{spin}}} & & \\
\Delta^s_{\mathrm{vec}} & = & \{\epsilon_i\} & i=1,2,3,8 & \mathbf{4} \\
\Delta^s_{\mathrm{spin}} & = & \underbrace{\pm\tfrac{1}{2}\epsilon_1 \pm \tfrac{1}{2}\epsilon_2 \pm \tfrac{1}{2}\epsilon_3 + \tfrac{1}{2}\epsilon_8}_{\text{even number of minus signs}} & & \mathbf{4} \\
\Delta^s_{\overline{\mathrm{spin}}} & = & \underbrace{\pm\tfrac{1}{2}\epsilon_1 \pm \tfrac{1}{2}\epsilon_2 \pm \tfrac{1}{2}\epsilon_3 + \tfrac{1}{2}\epsilon_8}_{\text{odd number of minus signs}} & & \mathbf{4} \\
\hline
 & & & & \mathbf{12}
\end{array} \tag{3.65}$$

  Correspondingly, the subset $\Delta^\delta \subset \Delta_{E_8}$ defined by its projection property

$\Pi_{\rm TS}\left(\Delta^\delta\right) = \Delta^s_{\rm TS}$ is also split in three subsets as follows:

$$
\begin{array}{||lcl|c|c||}
\Delta^\delta_+ & = & \Delta^\delta_{\rm vec} \bigcup \Delta^\delta_{\rm spin} \bigcup \Delta^\delta_{\overline{\rm spin}} & & \\
\Delta^\delta_{\rm vec} & = & \left\{ \underbrace{\epsilon_i}_{\alpha_\parallel} \oplus \underbrace{(\pm\epsilon_j)}_{\alpha_\perp} \right\} \quad , \quad \left(\begin{array}{c} i = 1,2,3,8 \\ j = 4,5,6,7 \end{array} \right) & 4\times \mathbf{8} & 32 \\
\Delta^\delta_{\rm spin} & = & \left\{ \underbrace{\left(\pm\tfrac{1}{2}\epsilon_1 \pm \tfrac{1}{2}\epsilon_2 \pm \tfrac{1}{2}\epsilon_3 + \tfrac{1}{2}\epsilon_8\right)}_{\alpha_\parallel \quad \text{even \# of} - \text{signs}} \oplus \underbrace{\left(\pm\tfrac{1}{2}\epsilon_4 \pm \tfrac{1}{2}\epsilon_5 \pm \tfrac{1}{2}\epsilon_6 \pm \tfrac{1}{2}\epsilon_7\right)}_{\alpha_\perp \quad \text{even \# of} - \text{signs}} \right\} & 4\times \mathbf{8} & 32 \\
\Delta^\delta_{\overline{\rm spin}} & = & \left\{ \underbrace{\left(\pm\tfrac{1}{2}\epsilon_1 \pm \tfrac{1}{2}\epsilon_2 \pm \tfrac{1}{2}\epsilon_3 + \tfrac{1}{2}\epsilon_8\right)}_{\alpha_\parallel \quad \text{odd \# of} - \text{signs}} \oplus \underbrace{\left(\pm\tfrac{1}{2}\epsilon_4 \pm \tfrac{1}{2}\epsilon_5 \pm \tfrac{1}{2}\epsilon_6 \pm \tfrac{1}{2}\epsilon_7\right)}_{\alpha_\perp \quad \text{odd \# of} - } \right\} & 4\times \mathbf{8} & 32 \\
\hline
 & & & & \mathbf{96}
\end{array}
\tag{3.66}
$$

We can now verify the general statements made in the previous sections about the paint group representations to which the various roots are assigned. First of all, we see that the long roots of F_4, namely those 12 given in (3.64) are singlets under the paint group $G_{\rm paint} = \mathrm{SO}(8)$. All other roots fall into multiplets with the same Tits-Satake projection and each of these latter has always the same multiplicity, in our case $m = 8$. So, the short roots of $F_{4(4)}$ fall into 8–dimensional representations of $G_{\rm paint} = \mathrm{SO}(8)$. Let us now determine these representations. SO(8) has three kinds of octets $\mathbf{8_v}$, $\mathbf{8_s}$ and $\mathbf{8_{\overline{s}}}$ and, as we stated, not every root α_s of the Tits-Satake algebra $\mathbb{G}_{\rm TS}$ falls in the same representation $\mathbf{D}$ of the paint group although in this case all $\mathbf{D}^{[\alpha^s]}$ have the same dimension. Looking back at our result, we easily find the answer. The positive roots in the subset $\Delta^\delta_{\rm vec}$ that project on one restricted root, have as compact part $\alpha_\perp$ the weights of the vector representation of SO(8). Hence, the roots of $\Delta^\delta_{\rm vec}$ are assigned to the $\mathbf{8_v}$ of the paint group. The positive roots in $\Delta^\delta_{\rm spin}$ have instead as compact part the weights of the spinor representation of SO(8) and so they are assigned to the $\mathbf{8_s}$ irreducible representation. Finally, with a similar argument, we see that the roots of $\Delta^\delta_{\overline{\rm spin}}$ are in the conjugate spinor representation $\mathbf{8_{\overline{s}}}$. It is now very easy to define the Tits-Satake projection, since one can now easily give the subpaint group $G^0_{\rm subpaint}$[11]. We have to find a subgroup $G^0 \subset \mathrm{SO}(8)$ such that under reduction with respect to it, the three octet representations branch simultaneously as :

$$
\begin{aligned}
\mathbf{8_v} &\stackrel{G^0}{\longrightarrow} \mathbf{1}\oplus\mathbf{7}\,, \\
\mathbf{8_s} &\stackrel{G^0}{\longrightarrow} \mathbf{1}\oplus\mathbf{7}\,, \\
\mathbf{8_{\overline{s}}} &\stackrel{G^0}{\longrightarrow} \mathbf{1}\oplus\mathbf{7}\,.
\end{aligned}
\tag{3.67}
$$

Such group G^0 exists and it is uniquely identified as the 14 dimensional $G_{2(-14)}$. Hence, the subpaint group is $G_{2(-14)}$. Considering now (3.3), we see that the commutation relations of the solvable Lie algebra $\mathrm{Solv}\left(E_{8(-24)}/E_{7(-133)} \times SU(2)\right)$ precisely fall into the general

[11] We will sometimes omit the 'subpaint' indication for convenience.

form displayed in the first column of that table with the index $x = 1, \dots, 7$ spanning the fundamental 7-dimensional representation of $G_{2(-14)}$ and the invariant antisymmetric tensor a^{xyz} being given by the $G_{2(-14)}$-invariant octonionic structure constants. Indeed, the representation $\mathbf{J}$ mentioned in the previous section is the fundamental $\mathbf{7}$ and we have the decomposition:

$$\mathbf{7} \otimes \mathbf{7} = \underbrace{\mathbf{14} \oplus \mathbf{7}}_{\text{antisymmetric}} \oplus \underbrace{\mathbf{27} \oplus \mathbf{1}}_{\text{symmetric}} . \tag{3.68}$$

This shows that the tensor product $\mathbf{J} \otimes \mathbf{J}$ contains both the singlet and $\mathbf{J}$.

Note that this is not the only space that projects onto the quaternionic member of the $L(1,1)$ family. For instance, in the example studied in paper [5], namely

$$\Pi_{\text{TS}} \ : \ \frac{E_{7(-5)}}{\text{SO}(12) \times \text{SU}(2)} \longrightarrow \frac{F_{4(4)}}{\text{USp}(6) \times \text{SU}(2)} , \tag{3.69}$$

the image of the Tits-Satake projection yields the same maximally split coset as in the case presently illustrated, although the original manifold is a different one. The only difference that distinguishes the two cases resides in the paint group. There we have $G_{\text{paint}} = \text{SO}(3) \times \text{SO}(3) \times \text{SO}(3)$ and the subpaint group was identified as $G^0_{\text{subpaint}} = \text{SO}(3)_{\text{diag}}$. Correspondingly, the index $x = 1, 2, 3$ spans the triplet representation of SO(3) which is the $\mathbf{J}$ appropriate to that case and the invariant tensor a^{xyz} is given by the Levi-Civita symbol ε^{xyz}. Similarly, it turns out that also

$$\Pi_{\text{TS}} \ : \ \frac{E_{6(2)}}{\text{SU}(6) \times \text{SU}(2)} \longrightarrow \frac{F_{4(4)}}{\text{USp}(6) \times \text{SU}(2)} , \tag{3.70}$$

where the paint group in this case is given by $\text{SO}(2) \times \text{SO}(2)$.

Let us now consider the group theoretical meaning of the splitting of $F_{4(4)}$ roots into the three subsets Δ^s_{vec}, Δ^s_{spin}, $\Delta^s_{\overline{\text{spin}}}$, which are assigned to different representations of the paint group SO(8). This is easily understood if we recall that there exists a subalgebra $\text{SO}(4,4) \subset F_{4(4)}$ with respect to which we have the following branching rule of the adjoint representation of $F_{4(4)}$:

$$\mathbf{52} \stackrel{\text{SO}(4,4)}{\rightarrow} \mathbf{28}^{\text{nc}} \oplus \mathbf{8}^{\text{nc}}_v \oplus \mathbf{8}^{\text{nc}}_s \oplus \mathbf{8}^{\text{nc}}_{\bar{s}} . \tag{3.71}$$

The superscript *nc* is introduced just in order to recall that these are representations of the non-compact real form SO(4,4) of the D_4 Lie algebra. By $\mathbf{28}$, $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_{\bar{s}}$, we have already denoted and we continue to denote the representations in the compact real form SO(8) of the same Lie algebra. The algebra SO(4,4) is regularly embedded and therefore its Cartan generators are the same as those of $F_{4(4)}$. The 12 positive long roots of $F_{4(4)}$ are the only positive roots of SO(4,4), while the three sets Δ^s_{vec}, Δ^s_{spin}, $\Delta^s_{\overline{\text{spin}}}$ just correspond to the positive weights of the three representations $\mathbf{8}^{\text{nc}}_v$, $\mathbf{8}^{\text{nc}}_s$ and $\mathbf{8}^{\text{nc}}_{\bar{s}}$, respectively. This is in agreement with the branching rule (3.71). So, the conclusion is that the different paint group representation assignments of the various root subspaces correspond to the decomposition of the Tits-Satake algebra $F_{4(4)}$ with respect to what we can call the 'sub Tits-Satake algebra' [12] $\mathbb{G}_{\text{subTS}} = \text{SO}(4,4)$. From this example, we can try to define this sub Tits-Satake algebra

[12] This concept corresponds to the algebra G_s in [77].

in a more intrinsic way. $G_{\rm subTS}$ is the normalizer of the paint group $G_{\rm paint}$ within the original group G_R. Indeed, there is a maximal subgroup:

$$\mathrm{SO}(4,4) \times \mathrm{SO}(8) \,\subset\, E_{8(-24)}\,, \tag{3.72}$$

with respect to which the adjoint of $E_{8(-24)}$ branches as follows:

$$\mathbf{248} \stackrel{\mathrm{SO}(4,4)\times\mathrm{SO}(8)}{\longrightarrow} (1,\mathbf{28}) \oplus (\mathbf{28}^{\rm nc},\mathbf{1}) \oplus (\mathbf{8_v}^{\rm nc},\mathbf{8_v}) \oplus (\mathbf{8_s}^{\rm nc},\mathbf{8_s}) \oplus (\mathbf{8_{\bar{s}}}^{\rm nc},\mathbf{8_{\bar{s}}})\,, \tag{3.73}$$

and the last three terms in this decomposition display the pairing between representations of the paint group and representations of the sub Tits-Satake group. Actually, also the subpaint group $G^0_{\rm subpaint} = G_{2(-14)}$ can be viewed in a similar fashion, namely as the normalizer of the Tits-Satake subgroup $G_{\rm TS} = F_{4(4)}$ within the original group $G_R = E_{8(-24)}$. Indeed, we have a subgroup

$$F_{4(4)} \times G_{2(-14)} \,\subset\, E_{8(-24)}\,, \tag{3.74}$$

such that the adjoint of $E_{8(-24)}$ branches as follows:

$$\mathbf{248} \stackrel{F_{4(4)}\times G_{2(-14)}}{\longrightarrow} (\mathbf{52},\mathbf{1}) \oplus (\mathbf{1},\mathbf{14}) \oplus (\mathbf{26},\mathbf{7})\,. \tag{3.75}$$

The two decompositions (3.73) and (3.75) lead to the same decomposition with respect to the intersection group:

$$\begin{aligned} G_{\rm intsec} &\equiv \left(G_{\rm TS} \times G^0_{\rm subpaint}\right) \bigcap \left(G_{\rm subTS} \times G_{\rm paint}\right) = G_{\rm subTS} \times G^0_{\rm subpaint} \\ &= \left(F_{4(4)} \times G_{2(-14)}\right) \bigcap \left(\mathrm{SO}(4,4) \times \mathrm{SO}(8)\right) = \mathrm{SO}(4,4) \times G_{2(-14)}\,. \end{aligned} \tag{3.76}$$

We find

$$\begin{aligned} \mathbf{248} \;\;\rightarrow\;\; & (\mathbf{1},\mathbf{14}) \oplus (\mathbf{1},\mathbf{7}) \oplus (\mathbf{1},\mathbf{7}) \oplus (\mathbf{8}^{\rm nc}_v,\mathbf{7}) \oplus (\mathbf{8}^{\rm nc}_s,\mathbf{7}) \oplus (\mathbf{8}^{\rm nc}_{\bar{s}},\mathbf{7}) \\ & \oplus (\mathbf{28}^{\rm nc},\mathbf{1}) \oplus (\mathbf{8}^{\rm nc}_v,\mathbf{1}) \oplus (\mathbf{8}^{\rm nc}_s,\mathbf{1}) \oplus (\mathbf{8}^{\rm nc}_{\bar{s}},\mathbf{1})\,. \end{aligned} \tag{3.77}$$

The adjoint of the Tits-Satake subalgebra $G_{\rm TS} = F_{4(4)}$ is reconstructed by collecting together all the singlets with respect to the subpaint group $G^0_{\rm subpaint}$. Alternatively, the adjoint of the paint algebra $G_{\rm paint} = \mathrm{SO}(8)$ is reconstructed by collecting together all the singlets with respect to the sub Tits-Satake algebra $G_{\rm subTS} = \mathrm{SO}(4,4)$.

Finally, we can recognize the sub Tits-Satake algebra as the algebra generated by the CSA and roots Δ^ℓ (and their negatives) in the decomposition (3.48).

3.4 Tits-Satake projection for general homogeneous special geometries

In the previous section, we discussed the Tits-Satake projection in the context of symmetric spaces. Starting from a geometrical projection of the root system, the notions of paint and subpaint groups can be defined. These then constitute the main tools that allow one to single out the Tits-Satake projection of symmetric spaces. As presented so far, the construction was rather specific to symmetric spaces, as it relied on the theory of real forms of semi-simple algebras. In this case, paint groups can be obtained rather easily using some practical tools, such as for instance Satake diagrams. One can wonder whether the construction can be generalized to all homogeneous special geometries, including the non-symmetric ones. In this section, we will explain that this can be done, due to the fact that it is possible to find appropriate paint and subpaint groups. We will first schematically enumerate the several steps that have to be performed in the Tits-Satake projection of homogeneous special geometries. Next, we will focus in more detail on specific steps of the construction. More specifically, we will indicate how the paint and subpaint groups are generalized for homogeneous special geometry.

3.4.1 The general procedure

For a homogeneous special space, generated by the solvable algebra $\mathrm{Solv}_{\mathcal{M}}$, the general procedure relies on the following items:

A] There exists a compact algebra $\mathbb{G}_{\mathrm{paint}}$ which acts as an algebra of outer automorphisms of the solvable algebra $\mathrm{Solv}_{\mathcal{M}}$. The algebra $\mathbb{G}_{\mathrm{paint}}$ can be found as follows. As discussed in section 3.2.5, although $\mathrm{Solv}_{\mathcal{M}}$ consists of translational isometries, the isometry algebra $\mathbb{G}^{\mathrm{iso}}_{\mathcal{M}}$ of the homogeneous special geometries generically contains more symmetries than just the ones in the solvable algebra. Let us define the subalgebra of automorphisms of $\mathrm{Solv}_{\mathcal{M}}$:

$$\mathbb{G}^{\mathrm{iso}}_{\mathcal{M}} \supset \mathrm{Aut}\,[\mathrm{Solv}_{\mathcal{M}}] \quad = \quad \left\{ X \in \mathbb{G}^{\mathrm{iso}}_{\mathcal{M}} \mid \forall \Psi \in \mathrm{Solv}_{\mathcal{M}} \; : \; [X\,,\,\Psi] \in \mathrm{Solv}_{\mathcal{M}} \right\} . \tag{3.78}$$

Since the algebra $\mathrm{Aut}\,[\mathrm{Solv}_{\mathcal{M}}]$ contains $\mathrm{Solv}_{\mathcal{M}}$ as an ideal, we can define the algebra of external automorphisms as the quotient:

$$\mathrm{Aut}_{\mathrm{Ext}}\,[\mathrm{Solv}_{\mathcal{M}}] \equiv \frac{\mathrm{Aut}\,[\mathrm{Solv}_{\mathcal{M}}]}{\mathrm{Solv}_{\mathcal{M}}}\,, \tag{3.79}$$

and we identify $\mathbb{G}_{\mathrm{paint}}$ as the maximal compact subalgebra of $\mathrm{Aut}_{\mathrm{Ext}}\,[\mathrm{Solv}_{\mathcal{M}}]$. Actually, one can see that

$$\mathbb{G}_{\mathrm{paint}} = \mathrm{Aut}_{\mathrm{Ext}}\,[\mathrm{Solv}_{\mathcal{M}}]\,. \tag{3.80}$$

Indeed, the algebra $\mathrm{Aut}_{\mathrm{Ext}}\,[\mathrm{Solv}_{\mathcal{M}}]$ is composed of isometries which belong to the stabilizer subalgebra $\mathbb{H} \subset \mathbb{G}^{\mathrm{iso}}_{\mathcal{M}}$ of any point of the manifold, since $\mathrm{Solv}_{\mathcal{M}}$ acts transitively. In virtue of the Riemannian structure of $\mathcal{M}$, we have $\mathbb{H} \subset \mathfrak{so}(n)$ where $n = \dim(\mathrm{Solv}_{\mathcal{M}})$ and hence also $\mathrm{Aut}_{\mathrm{Ext}}\,[\mathrm{Solv}_{\mathcal{M}}] \subset \mathfrak{so}(n)$ is a compact Lie algebra. Later on, we will give a precise identification of the paint group for all homogeneous special geometries.

We can now reformulate the notion of maximally non-compact or maximally split algebras in such a way that it applies to the case of all considered solvable algebras, independently whether they come from symmetric spaces or not. We will call the algebra $\mathrm{Solv}_{\mathcal{M}}$ maximally split if the paint algebra is trivial, namely:

$$\mathrm{Solv}_{\mathcal{M}} = \text{maximally split} \Leftrightarrow \mathrm{Aut}_{\mathrm{Ext}}\,[\mathrm{Solv}_{\mathcal{M}}] = \emptyset\,. \tag{3.81}$$

For maximally split algebras, the Tits-Satake projection will be trivial, namely $\mathrm{Solv}_{\mathcal{M}}$ will project onto itself.

B] Consider now non-maximally split algebras for which thus $\mathrm{Aut}_{\mathrm{Ext}}\,[\mathrm{Solv}_{\mathcal{M}}] \neq \emptyset$. Let r be the rank of $\mathrm{Solv}_{\mathcal{M}}$ and denote the Cartan generators by H_i. Let there furthermore be n nilpotent generators $\mathcal{W}_\alpha$. The set of singlets under the action of the paint algebra $\mathbb{G}_{\mathrm{paint}}$ closes a solvable subalgebra $\mathrm{Solv}_{\mathrm{subTS}} \subset \mathrm{Solv}_{\mathcal{M}}$. It consists of the whole set of Cartan generators H_i plus a subset of p nilpotent generators $\mathcal{W}_{\alpha^\ell}$ associated with roots α^ℓ:

$$\begin{aligned} \mathrm{Solv}_{\mathrm{subTS}} &= \mathrm{span}\,\{H_i, \mathcal{W}_{\alpha^\ell}\}\,, \\ [\mathrm{Solv}_{\mathrm{subTS}}\,, \mathrm{Solv}_{\mathrm{subTS}}] &\subset \mathrm{Solv}_{\mathrm{subTS}}\,, \\ \forall\, X \in \mathbb{G}_{\mathrm{paint}}\,, \forall \Psi \in \mathrm{Solv}_{\mathrm{subTS}} &: [X, \Psi] = 0\,. \end{aligned} \tag{3.82}$$

We will refer to $\mathrm{Solv}_{\mathrm{subTS}}$ as the 'sub Tits-Satake algebra'. It has the same rank as the original solvable algebra $\mathrm{Solv}_{\mathcal{M}}$. We will see later on that there is a very short list of possible cases for $\mathrm{Solv}_{\mathrm{subTS}}$. In all possible cases, it is the solvable Lie algebra of a symmetric maximally split coset $\mathbb{G}_{\mathrm{subTS}}/\mathbb{H}_{\mathrm{subTS}}$. In this way, eventually, we have the notion of a semi-simple Lie algebra $\mathbb{G}_{\mathrm{subTS}}$. These are given in table 3.5, and correspond to the notion of

Table 3.5 $\mathbb{G}_{\mathrm{subTS}}$. *The solvable algebra of the (maximally split) coset $\mathbb{G}_{\mathrm{subTS}}/\mathbb{H}_{\mathrm{subTS}}$ is the sub Tits-Satake algebra. The lines distinguish spaces of different rank, similar to the scheme in table 3.2. The inverse* **c***–map leads from the last column to the middle one, and the inverse* **r***–map to the first column, each time reducing the rank with 1.*

real	Kähler	quaternionic
		SO(1,1)
		SU(1,1)
	SU(1,1)	SO(2,2)
		$G_{2(2)}$
SO(1,1)	$[\mathrm{SU}(1,1)]^2$	SO(3,4)
$[\mathrm{SO}(1,1)]^2$	$[\mathrm{SU}(1,1)]^3$	SO(4,4)

sub Tits-Satake algebra as it was used for symmetric spaces. However, for homogeneous spaces we start only with the solvable algebras, and as such $\mathrm{Solv}_{\mathrm{subTS}}$ is the algebra that is intrinsically defined as the sub Tits-Satake algebra. This subtlety becomes more relevant for the Tits-Satake algebra itself, where the solvable algebra is not in all cases the solvable algebra of a symmetric space, and thus a corresponding semi-simple group G_{TS} is not always well defined.

C1] Considering the orthogonal decomposition of the original solvable Lie algebra with respect to its sub Tits-Satake algebra:

$$\mathrm{Solv}_{\mathcal{M}} = \mathrm{Solv}_{\mathrm{subTS}} \oplus \mathbb{K}_{\mathrm{short}}\,, \tag{3.83}$$

we find that the orthogonal subspace $\mathbb{K}_{\mathrm{short}}$ decomposes into a sum of q subspaces:

$$\mathbb{K}_{\mathrm{short}} = \bigoplus_{\wp=1}^{q} \mathbb{D}\left[\mathcal{P}^+_\wp, \mathbf{Q}_\wp\right]\,, \tag{3.84}$$

where each $\mathbb{D}\left[\mathcal{P}^+_\wp, \mathbf{Q}_\wp\right]$ is the tensor product

$$\mathbb{D}\left[\mathcal{P}^+_\wp, \mathbf{Q}_\wp\right] \;=\; \mathcal{P}^+_\wp \,\otimes\, \mathbf{Q}_\wp \tag{3.85}$$

of an irreducible representation $\mathbf{Q}_\wp$ of the compact paint algebra $\mathbb{G}_{\text{paint}}$ with an irreducible module $\mathcal{P}^+_\wp$ of the solvable sub Tits-Satake algebra $\text{Solv}_{\text{subTS}}$. An irreducible module $\mathcal{P}^+_\wp$ of $\text{Solv}_{\text{subTS}}$ decomposes in the following way:

$$\mathcal{P}^+_\wp \;=\; \bigoplus_{s=1}^{n_\wp} \mathbb{W}[\alpha^{(\wp,s)}], \qquad n_\wp = \dim \mathcal{P}^+_\wp\,, \tag{3.86}$$

where each $\mathbb{W}[\alpha^{(\wp,s)}]$ is an eigenspace of the CSA of $\mathbb{G}_{\text{subTS}}$, which coincides with that of $\text{Solv}_{\text{subTS}}$ and eventually with the CSA of the original $\text{Solv}_{\mathcal{M}}$. Explicitly, this means:

$$\forall\, H_i \,\in\, \text{CSA}\,(\text{Solv}_{\mathcal{M}})\;,\, \forall\Psi \,\in\, \mathbb{W}[\alpha^{(\wp,s)}] \otimes \mathbf{Q}_\wp \quad : \quad [H_i\,,\,\Psi] \,=\, \alpha^{i\;(\wp,s)}\,\Psi\,. \tag{3.87}$$

The r-component vectors $\alpha^{(\wp,s)}$ are identified as the non-negative weights of some irreducible representation $\mathcal{P}_\wp$ of the simple Lie algebra $\mathbb{G}_{\text{subTS}}$:

$$\mathcal{P}_\wp = \mathcal{P}^+_\wp \,\oplus\, \mathcal{P}^-_\wp\,, \qquad \mathcal{P}^-_\wp = \bigoplus_{s=1}^{n_\wp} \mathbb{W}[-\alpha^{(\wp,s)}]\,. \tag{3.88}$$

C2] The decomposition of $\mathbb{K}_{\text{short}}$ mentioned in (3.84) has actually a general form depending on the rank. We will discuss this here for the quaternionic-Kähler manifolds, as the other ones can be obtained by restriction of the generators using the inverse **c**– and **r**–maps as discussed in section 3.2.3.

r = 4) In this case, there are just three modules of $\mathbb{G}_{\text{subTS}} = \text{SO}(4,4)$ involved in the sum of (3.84) namely $\mathcal{P}_{\mathbf{8_v}}$, $\mathcal{P}_{\mathbf{8_s}}$, $\mathcal{P}_{\mathbf{8_{\bar{s}}}}$, where $\mathbf{8_{v,s,\bar{s}}}$ denotes the vector, spinor and conjugate spinor representation, respectively. All these three modules are 8-dimensional, which means that for all of them there are 4 positive weights and 4 negative ones. Denoting the half-spaces formed by the positive weights by $\mathbf{4^+_{v,s,\bar{s}}}$, we can write:

$$\mathbb{K}_{\text{short}} = \left(\mathbf{4^+_v}, \mathbf{Q_v}\right) \oplus \left(\mathbf{4^+_s}, \mathbf{Q_s}\right) \oplus \left(\mathbf{4^+_{\bar{s}}}, \mathbf{Q_{\bar{s}}}\right)\,, \tag{3.89}$$

where $\mathbf{Q_{v,s,\bar{s}}}$ are three different irreducible representations of $\mathbb{G}_{\text{paint}}$ that will be discussed later in this section. In the generic case, all three representations $\mathbf{Q_{v,s,\bar{s}}}$ are non-vanishing and this corresponds to $L(q,P)$ or $L(4m,P,\dot{P})$ with $q,P,m \geq 1$. Special cases where two of the three representations $\mathbb{G}_{\text{paint}}$ vanish correspond to the classes $L(0,P)$, while for $L(0,P,\dot{P})$ only one of these representations vanishes. The limiting case is that where all three representations are deleted and the full algebra is just $\text{Solv}\left(\frac{\text{SO}(4,4)}{\text{SO}(4)\times\text{SO}(4)}\right)$, which is $L(0,0)$. Note that (3.89) is the generalization of the decomposition (3.73) applying to the case $L(8,1)$. There, we have $\mathbb{G}_{\text{paint}} = \text{SO}(8)$ and the aforementioned irreducible modules are:

$$\mathbf{Q_v} = \mathbf{8_v} \quad ; \quad \mathbf{Q_s} = \mathbf{8_s} \quad ; \quad \mathbf{Q_{\bar{s}}} = \mathbf{8_{\bar{s}}}\,. \tag{3.90}$$

r = 3) In this case, there is only one module of $\mathbb{G}_{\mathrm{subTS}} = \mathrm{SO}(3,4)$ involved in the sum of (3.84) namely $\mathcal{P}_{\mathbf{8_s}}$ where $\mathbf{8_s}$ denotes the 8-dimensional spinor representation of SO(3,4). Denoting by $\mathbf{4}_{\mathbf{s}}^{+}$ the space spanned by the eigenspaces corresponding to positive spinor weights, we can write:

$$\mathbb{K}_{\mathrm{short}} = \left(\mathbf{4}_{\mathbf{s}}^{+}, \mathbf{Q_s}\right) , \tag{3.91}$$

where the representation $\mathbf{Q_s}$ of the paint group will be discussed in later sections. When $\mathbf{Q_s}$ is non-vanishing, we describe the $L(-1,P)$ spaces with $P \geq 1$, which at the quaternionic level are never given by symmetric spaces. When $\mathbf{Q_s}$ vanishes, we degenerate in the case $L(-1,0)$, which is already maximally split.

r = 2) In this case, there is one exceptional case, namely SG_5, where $G_R = G_{\mathrm{subTS}} = G_{2(2)}$. In all other cases, there are two modules of SO(2,2) involved in the sum of (3.84) and these are the spinor module $\mathcal{P}_{\mathbf{4_s}}$ and the vector module $\mathcal{P}_{\mathbf{4_v}}$. Both modules are 4-dimensional and in our adopted notations we can write:

$$\mathbb{K}_{\mathrm{short}} = \left(\mathbf{2}_{\mathbf{s}}^{+}, \mathbf{Q_s}\right) \oplus \left(\mathbf{2}_{\mathbf{v}}^{+}, \mathbf{Q_v}\right) . \tag{3.92}$$

Later on in this section, we will discuss the representations $\mathbf{Q_s}$, $\mathbf{Q_v}$ of the paint group and show how the coset manifolds in the series $L(-2,P)$ can be reconstructed. When $P = 0$, only the representation $\mathbf{Q_v}$ is non-vanishing.

r = 1) In this case, we have to distinguish between $G_{\mathrm{subTS}} = \mathrm{SO}(1,1)$ or $G_{\mathrm{subTS}} = \mathrm{SU}(1,1)$. When $G_{\mathrm{subTS}} = \mathrm{SU}(1,1)$, we have:

$$\mathbb{K}_{\mathrm{short}} = \left(\mathbf{1}_{\mathbf{s}}^{+}, \mathbf{Q_s}\right) , \tag{3.93}$$

where $\mathbf{1}_{\mathbf{s}}^{+}$ denotes the positive weight subspace of the spinor representation of $\mathfrak{so}(1,2)$, i.e., the fundamental of $\mathfrak{su}(1,1)$, which is two-dimensional. The representation $\mathbf{Q_s}$ will be discussed later. When $G_{\mathrm{subTS}} = \mathrm{SO}(1,1)$ on the other hand, we have:

$$\mathbb{K}_{\mathrm{short}} = \left(\mathbf{1}_{\mathbf{s}}^{+}, \mathbf{Q_s}\right) \oplus \left(\mathbf{1}_{\mathbf{v}}^{+}, \mathbf{Q_v}\right) . \tag{3.94}$$

In this case, $\mathbf{1}_{\mathbf{s}}^{+}$ denotes a subspace of weight 1/2 with respect to $\mathbb{G}_{\mathrm{subTS}} = \mathfrak{so}(1,1)$, while the subspace $\mathbf{1}_{\mathbf{v}}^{+}$ has weight 1. When $\mathbf{Q_s}$ is non-vanishing, we describe the spaces $L(-3,P)$, $P \geq 1$. When $\mathbf{Q_s}$ vanishes, we are describing the space $L(-3,0)$. The representations $\mathbf{Q_s}$ and $\mathbf{Q_v}$ of the paint group that appear here will be discussed later.

We can now note a regularity in the decomposition of $\mathbb{K}_{\mathrm{short}}$. For all values of the rank, we generically have the space $(\mathcal{S}^{+}, \mathbf{Q_s})$ that associates a representation of the paint group to the half-spinor representation of the sub Tits-Satake algebra. In addition to this, we can also have the representations $\mathbf{Q_v}$ and $\mathbf{Q_{\bar{s}}}$, which we associate to what we can name the $\mathcal{V}^{+}$ and $\bar{\mathcal{S}}^{+}$ half-modules. These latter can exist in rank 4 and some of them can vanish in lesser rank. Using this notation which covers all the cases, we can now give a general definition of the Tits-Satake projection.

D] The paint algebra $\mathbb{G}_{\text{paint}}$ contains a subalgebra

$$\mathbb{G}^0_{\text{subpaint}} \subset \mathbb{G}_{\text{paint}}\,, \tag{3.95}$$

such that with respect to $\mathbb{G}^0_{\text{subpaint}}$, each of the three irreducible representations $\mathbf{Q}_{\mathbf{v},\mathbf{s},\bar{\mathbf{s}}}$ branches as:

$$\mathbf{Q}_{\mathbf{v},\mathbf{s},\bar{\mathbf{s}}} \stackrel{\mathbb{G}^0_{\text{subpaint}}}{\Longrightarrow} \underbrace{\mathbf{1}}_{\text{singlet}} \oplus \mathbf{J}_{\mathbf{v},\mathbf{s},\bar{\mathbf{s}}}\,, \tag{3.96}$$

where the representation $\mathbf{J}_{\mathbf{v},\mathbf{s},\bar{\mathbf{s}}}$ is in general reducible.

E] The restriction to the singlets of $\mathbb{G}^0_{\text{subpaint}}$ defines a Lie subalgebra of $\text{Solv}_{\mathcal{M}}$, namely, if we set:

$$\text{Solv}_{\text{TS}} \equiv \text{Solv}_{\text{subTS}} \oplus \left(\mathcal{V}^+, \mathbf{1}\right) \oplus \left(\mathcal{S}^+, \mathbf{1}\right) \oplus \left(\overline{\mathcal{S}}^+, \mathbf{1}\right)\,, \tag{3.97}$$

we get:

$$[\text{Solv}_{\text{TS}}\,, \text{Solv}_{\text{TS}}] \subset \text{Solv}_{\text{TS}}\,. \tag{3.98}$$

Summarizing, for all homogeneous special geometries, one can define the Tits-Satake projection at the level of solvable algebras by stating:

$$\begin{array}{l} \Pi_{\text{TS}} \quad : \quad \text{Solv}_{\mathcal{M}} \longrightarrow \text{Solv}_{\text{TS}} \subset \text{Solv}_{\mathcal{M}} \\ \Psi \in \text{Solv}_{\text{TS}} \quad \text{if and only if} \quad : \quad \forall X \in \mathbb{G}^0_{\text{subpaint}} \;:\; [X, \Psi] = 0\,. \end{array} \tag{3.99}$$

In other words, we define the Tits-Satake solvable subalgebra Solv_{TS} as spanned by all the singlets under the subpaint group G^0_{subpaint}.

3.4.2 Results for the Tits-Satake projection of homogeneous special manifolds

The discussion of section 3.4.1 outlined the scheme of Tits-Satake projections. We will now demonstrate how the generators of table 3.1 and their weights given in (3.26) [61] fit in this picture, as outlined in point **C]** of the previous section.

Note that we can obtain the following weights

$H_1 : (0,0,0,0)$	$g_0 : (1,1,0,0)$	$q_0 : (0,1,-1,0)$	$p_0 : (1,0,1,0)$
$H_2 : (0,0,0,0)$	$g_1 : (1,-1,0,0)$	$q_1 : (0,1,1,0)$	$p_1 : (1,0,-1,0)$
$H_3 : (0,0,0,0)$	$g_2 : (0,0,1,1)$	$q_2 : (1,0,0,-1)$	$p_2 : (0,1,0,1)$
$H_4 : (0,0,0,0)$	$g_3 : (0,0,1,-1)$	$q_3 : (1,0,0,1)$	$p_3 : (0,1,0,-1)$
$X^+ : (0,0,1,0)$	$X^- : (0,0,0,1)$	$\tilde{X}^+ : (1,0,0,0)$	$\tilde{X}^- : (0,1,0,0)$
$Y^+ : (\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$Y^- : (\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	$\tilde{Y}^+ : (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\tilde{Y}^- : (\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
$Z^+ : (\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$Z^- : (\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$\tilde{Z}^+ : (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$\tilde{Z}^- : (\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$

(3.100)

after changing the basis of the CSA in the following way:

$$H_1 = h_0 + h_1\,, \qquad H_2 = h_0 - h_1\,, \qquad H_3 = h_2 + h_3\,, \qquad H_4 = h_2 - h_3\,. \tag{3.101}$$

The subset (H_i, g_i, q_i, p_i) can be recognized as the CSA and the 12 positive roots of the D_4 simple root system. They generate the solvable sub Tits-Satake algebra:

$$\mathrm{span}\,\{H_i, g_i, q_i, p_i\} = \mathrm{Solv}_{\mathrm{subTS}} \;\equiv\; \mathrm{Solv}\left[\frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4)\times\mathrm{SO}(4)}\right]. \tag{3.102}$$

With reference to the general decomposition of $\mathbb{K}_{\mathrm{short}}$ mentioned in (3.89), the following identification can be made:

$$\begin{aligned}
\mathrm{span}\left\{X^+, X^-, \widetilde{X}^+, \widetilde{X}^-\right\} &= \left(\mathbf{4}^+_{\mathbf{v}}, \mathbf{Q}_{\mathbf{v}}\right)\,,\\
\mathrm{span}\left\{Y^+, Y^-, \widetilde{Y}^+, \widetilde{Y}^-\right\} &= \left(\mathbf{4}^+_{\mathbf{s}}, \mathbf{Q}_{\mathbf{s}}\right)\,,\\
\mathrm{span}\left\{Z^+, Z^-, \widetilde{Z}^+, \widetilde{Z}^-\right\} &= \left(\mathbf{4}^+_{\overline{\mathbf{s}}}, \mathbf{Q}_{\overline{\mathbf{s}}}\right)\,.
\end{aligned} \tag{3.103}$$

Indeed, the weights assigned to the 4 different sets of type X are the 4 positive weights of the 8–dimensional vector representation of $\mathrm{SO}(4,4)$. The weights assigned to the 4 sets of generators of type Y are the 4 positive weights of the 8–dimensional spinor representation $\mathbf{s}$ of $\mathrm{SO}(4,4)$, as there are an even number of minus signs in the eigenvalues $\pm\frac{1}{2}$. The odd number of minus signs for the operators of type Z identifies them with the positive weights of the representation $\overline{\mathbf{s}}$ of $\mathrm{SO}(4,4)$.

Let us examine the cases of lower rank. Note that the case $q = -3$ is not a special quaternionic manifold. Its symmetry structure is not of the form of table 3.1, but its weight structure is summarized under (3.16). Using these weights, one can infer the statements made in (3.94). For the other cases, we have explained at the end of section 3.2.2 how they can be obtained from truncating the general structure of rank 4 spaces. Using these truncations, the statements in point **C2]** of the previous section can be verified. There is an anomaly for the case SG_5 where the weights do not follow the scheme of (3.26), but were given in (3.18). These can be recognized as the CSA and 6 positive roots of G_2. In this case, we have a maximally split algebra and the Tits-Satake projection is trivial.

In general, the Tits-Satake projection of the algebra is a subalgebra consisting of only one generator with the same weight vector. Therefore, it removes the redundancy of the X, Y and Z columns indicated in the last row of table 3.1. In the generic case, the Tits-Satake projected algebra is thus just the algebra with one entry in any entry of table 3.1 that is present. As mentioned in the general outline above, this reduction can be done in a group theoretical way, by restricting to the singlets with respect to the subpaint group. Therefore, one has to identify the paint groups for homogeneous special geometry, as well as the representations of this paint group to which each of the X, Y and Z spaces are assigned. Once this is done, one can choose the appropriate subpaint groups. These steps will be taken in the following subsections.

3.4.3 The paint group in homogeneous special geometries

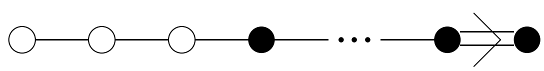
In section 3.3.2, we mentioned how the paint group for non-compact symmetric spaces can be inferred from the corresponding Satake diagrams. In this section, we shall determine the paint groups for general homogeneous special geometries.

From table 3.2, one can see that all homogeneous quaternionic spaces of rank less than 3 are symmetric. For these, it is thus possible to use Satake diagrams to determine the paint groups.

The spaces of rank 3 and 4 all have a five-dimensional origin. We can thus use the structure of their isometry groups, as exhibited in section 3.2 to find the corresponding paint groups. Focusing at the isometry algebra of the quaternionic spaces, one can immediately recognize that the part $\mathcal{S}_q(P,\dot{P})$ (if non-trivial) will always be part of the paint group. Indeed, it acts as a group of external automorphisms on $\mathcal{V}'_1$ and $\mathcal{V}'_2$. Since moreover it commutes with the rest of $\mathcal{V}'_0$, it also acts as a group of external automorphisms on the part of $\mathcal{V}'_0$ that belongs to the solvable algebra (3.41).

Moreover, also the $\mathfrak{so}(q+3,3)$ part of $\mathcal{V}'_0$ acts as a group of automorphisms on $\mathcal{V}'_1$ and $\mathcal{V}'_2$. This implies that the part of $\mathfrak{so}(q+3,3)$ that acts as a group of external automorphisms on its own solvable algebra will also be part of the paint group. But this is nothing but the paint group of $\mathrm{SO}(q+3,3)$ and can be inferred from the corresponding Satake diagram fig. 3.1. It can be easily verified that this contributes an $\mathrm{SO}(q)$-factor to the paint group.

Figure 3.1 *The Satake diagram of* $\mathrm{SO}(q+3,3)$ *for q odd. The paint group is represented by the subdiagram made of filled circles and is seen to be* $\mathrm{SO}(q)$*. For q even, the Satake diagram is different, but a similar conclusion regarding the paint group holds.*



The argument for the Kähler and real spaces is completely analogous. One should just replace $\mathfrak{so}(q+3,3)$ by $\mathfrak{so}(q+2,2)$, $\mathfrak{so}(q+1,1)$ respectively. In each case, one can conclude that the paint group for a general homogeneous special geometry is given by:

$$G_{\text{paint}} = \mathrm{SO}(q) \times \mathcal{S}_q(P,\dot{P})\,. \tag{3.104}$$

Note that for the symmetric special geometries, this formula indeed gives the paint groups that were obtained in table 3.4. The paint group is thus common to all members of a $L(q,P,\dot{P})$ family and is hence invariant under dimensional reduction. Accidentally, this structure is insensitive to the sign of q. Indeed, although the structure of the solvable algebra is very different, for say $q=3$ and for $q=-3$, the paint group is the same in both cases and the same irreducible representations are present. From this, it follows that also the subpaint group and the relevant decompositions will be the same for $\pm q$.

The action of the paint group on the solvable algebra of the corresponding manifolds can also be induced from section 3.2.5. Let us focus on the real spaces for a moment.

- The Cartan generators of the solvable algebra are singlets under the paint group.

- The q nilpotent generators of $\mathrm{Solv}(\mathfrak{so}(q+1,1))$ (corresponding to the generators of type X in previous notations) transform as a vector under the $\mathrm{SO}(q)$ part of the paint group, while they are inert under $\mathcal{S}_q(P,\dot{P})$.

- The generators $\underline{\xi}^\Lambda$ transform in a (in general reducible) spinor representation of $\mathrm{SO}(q+1,1)$. The space $\underline{\xi}^\Lambda$ however splits into two subspaces Y and Z, with different gradings with respect to the non-compact Cartan generator of $\mathrm{SO}(q+1,1)$ (denoted by $\underline{\alpha}$). Since the $\mathrm{SO}(q)$ part of the paint group commutes with this Cartan generator, the $\underline{\xi}^\Lambda$ split into two (in general reducible) spinor representations under the $\mathrm{SO}(q)$ factor of the paint group. These correspond to the spaces Y and Z in previous notations. Since $\mathcal{S}_q(P,\dot{P})$ commutes with $\underline{\alpha}$ as well, the spaces Y and Z will also separately transform in (vector) representations of the $\mathcal{S}_q(P,\dot{P})$ factor of the paint group.

Since for the Kähler and quaternionic-Kähler spaces, the generators of the solvable algebra occur in the same representations as in the case of the real spaces, the story for them is essentially the same as described above. The only difference is that the number of singlets is increased, giving rise to non-trivial $\mathbb{G}_{\mathrm{subTS}}$ algebras.

We can now easily match the above findings with the general discussion of section 3.4.1. There, we worked at the level of the quaternionic member of each family, since this allowed to include all cases, also those that are not in the image of the **c**-map or of the **r**-map. Yet the invariance of the paint group with respect to these maps precisely means that the $\mathbf{Q}_{\mathbf{v},\mathbf{s},\bar{\mathbf{s}}}$ representations remain the same in real, in special Kähler and in quaternionic-Kähler algebras. What changes is just the $\mathbb{G}_{\mathrm{subTS}}$ algebra which, climbing up from quaternionic to real geometry (dimensional oxidation), is progressively reduced in rank. The result was anticipated in item B] of section 3.4.1 and is given in table 3.5.

The information contained in the above discussion is what was needed in order to determine the desired representations $\mathbf{Q}_{\mathbf{v},\mathbf{s},\bar{\mathbf{s}}}$ of the paint group, respectively associated with the vector, spinor and conjugate spinor weights of the sub Tits-Satake algebra. The results for this, which are an immediate consequence of the real Clifford algebra representations discussed in appendix B, are summarized in table 3.6. In writing the spinor representations, one may comment about the way that the spinor representations are denoted. In real components, the representations of $\mathrm{SO}(q)$ are of dimension $\frac{1}{2}\mathcal{D}_{q+1}$. The complex or quaternionic structure acts on the same components. A notation $(\frac{1}{2}\mathcal{D}_{q+1},P)$ as representation of $\mathrm{SO}(q)\times\mathrm{U}(P)$ for the complex case means that it is a complex P-dimensional representation for $\mathrm{U}(P)$ but the complex structure is taken into account for the counting of real components in the first factor. Alternatively, we could have written it as $(\frac{1}{4}\mathcal{D}_{q+1},2P)$ when we take real components for the representation of $\mathrm{U}(P)$ and complex spinor representations. Similarly, in the quaternionic case we can write the representations of $\mathrm{SO}(q)\times\mathrm{USp}(2P)$ as $(\frac{1}{2}\mathcal{D}_{q+1},P)$, as $(\frac{1}{4}\mathcal{D}_{q+1},2P)$ (dividing the complex structures over the two sides) or as $(\frac{1}{8}\mathcal{D}_{q+1},4P)$.

Note that for $q=-2$ the $\mathbf{Q}_{\mathbf{v}}$ representation does not originate from the X-generators as for $q\geq 1$, but from the equality of the roots q_0 and q_1, of p_0 and p_1 as explained at the end of section 3.2.2. On the other hand, for $q=-3$ we do not have the scheme of table 3.1, but the result follows from the known scheme for symmetric spaces.

3.4.4 The subpaint group

The subpaint group whose Lie algebra was denoted as $\mathbb{G}^0_{\mathrm{subpaint}}$ was defined in section 3.4.1 through its property (3.96) relative to the decomposition of the representations $\mathbf{Q}_{\mathbf{v},\mathbf{s},\bar{\mathbf{s}}}$. As noted in the discussion on symmetric spaces, it can alternatively also be seen as the subgroup of the paint group that commutes with the Tits-Satake subalgebra. Searching for subpaint

Table 3.6 *The assignments of paint group representations in homogeneous special geometries.*

Family	**Paint group**	$\mathbf{Q_v}$	$\mathbf{Q_s}$	$\mathbf{Q_{\bar{s}}}$
$L(-3,P)$	SO(3) × USp(2P)	(3,1)	(4,P)	–
SG_4	SO(2)	–	2	–
$L(-2,P)$	SO(2) × U(P)	(2,1)	(2,P)	–
SG_5	**1**	–	–	–
$L(-1,P)$	SO(P)	–	P	–
$L(0,P,\dot{P})$	SO(P) × SO($\dot{\text{P}}$)	–	(P,1)	$(1,\dot{P})$
normal				
$L(q,P)$ $q=1,7 \bmod 8$	$\mathrm{SO}(q)\times\mathrm{SO}(P)$	$(q,1)$	$(\frac{1}{2}\mathcal{D}_{q+1},P)$	$(\frac{1}{2}\mathcal{D}_{q+1},P)$
$L(q,P,\dot{P})$ $q=8 \bmod 8$	$\mathrm{SO}(q)\times\mathrm{SO}(P)$ $\times\mathrm{SO}(\dot{P})$	$(q,1,1)$	$(\frac{1}{2}\mathcal{D}_{q+1},P,1)+$ $(\frac{1}{2}\mathcal{D}_{q+1},1,\dot{P})$	$(\frac{1}{2}\mathcal{D}_{q+1},P,1)+$ $(\frac{1}{2}\mathcal{D}_{q+1},1,\dot{P})$
complex				
$L(q,P)$ $q=2,6 \bmod 8$	$\mathrm{SO}(q)\times\mathrm{U}(P)$	$(q,1)$	$(\frac{1}{2}\mathcal{D}_{q+1},P)$	$(\frac{1}{2}\mathcal{D}_{q+1},P)$
quaternionic				
$L(q,P)$ $q=3,5 \bmod 8$	$\mathrm{SO}(q)\times\mathrm{USp}(2P)$	$(q,1)$	$(\frac{1}{2}\mathcal{D}_{q+1},P)$	$(\frac{1}{2}\mathcal{D}_{q+1},P)$
$L(q,P,\dot{P})$ $q=4 \bmod 8$	$\mathrm{SO}(q)\times\mathrm{USp}(2P)$ $\times\mathrm{USp}(2\dot{P})$	$(q,1,1)$	$(\frac{1}{2}\mathcal{D}_{q+1},P,1)+$ $(\frac{1}{2}\mathcal{D}_{q+1},1,\dot{P})$	$(\frac{1}{2}\mathcal{D}_{q+1},P,1)+$ $(\frac{1}{2}\mathcal{D}_{q+1},1,\dot{P})$

subalgebras is thus the group theoretical formulation of how the Tits-Satake projection is defined.

To single out the subpaint group $G^0_{\text{subpaint}} \subset G_{\text{paint}}$ for a homogeneous space $L(q,P,\dot{P})$, whose paint group is $G_{\text{paint}} = \mathrm{SO}(q)\times\mathcal{S}_q(P,\dot{P})$ the following strategy can be adopted:

1. Since the representation $\mathbf{Q_v}$ corresponding to the X–space generators is always of the form:

$$\mathbf{Q_v} \;=\; (\mathbf{q},\mathbf{1}) \;, \tag{3.105}$$

   where $\mathbf{q}$ denotes the vector representation of $\mathrm{SO}(q)$ and $\mathbf{1}$ the singlet representation of $\mathcal{S}_q(P,\dot{P})$, one first decomposes this representation with respect to the subgroup:

$$\mathrm{SO}(q-1)\subset\mathrm{SO}(q)\,, \tag{3.106}$$

   as then $\mathbf{q}\to\mathbf{1}+(\mathbf{q}-\mathbf{1})$. The singlet, named $X_\bullet$ is the only element of the X–space which survives the Tits-Satake projection.

2. Next, one looks for a subgroup $G^0_{\text{subpaint}} \subset \mathrm{SO}(q-1)\times\mathcal{S}_q(P,\dot{P})$ such that the decompositions of the spinor representations $\mathbf{Q_s}$ and $\mathbf{Q_{\bar{s}}}$, respectively associated with the spaces

Y and Z, contain each at least one singlet. By choosing appropriate singlets that close a subalgebra, one can then complete the Tits-Satake projection. For instance, for very special real spaces, one can infer from the gradings in (3.26), that $[X^-, Z^-] = Y^-$. By choosing appropriate singlets $X^-_\bullet$, $Z^-_\bullet$ in X^- and Z^-, the corresponding singlet $Y^-_\bullet$ in Y^- is defined by $[X^-_\bullet, Z^-_\bullet] = Y^-_\bullet$.

Thus in order to figure out the subpaint group, one should consider the explicit spinor representations of the Clifford algebra in $q+1$ Euclidean dimensions. Here, we explore one by one the cases $-3 \leq q \leq 9$ with arbitrary P. The results depend on the structure of the $\mathcal{S}_q(P, \dot{P})$ part of the paint group, and as such the cases with different q are divided into three groups: normal, almost complex and quaternionic, as explained in appendix B.

There are also two quaternionic spaces that are outside of the $L(q, P, \dot{P})$ families, namely pure $\mathcal{N} = 2$ supergravities in 4 and 5 dimensions. They are both symmetric spaces, out of which only the first one, SG_4, is non–split and has a non–trivial paint group: SO(2). The subpaint group in this case is empty, because in order to get singlets we have to break the paint group completely.

<u>The normal case</u>

$q = \pm 1$. The paint group is $G_{\rm paint} = \mathrm{SO}(P)$. Only P–dimensional vector representations are present in the solvable algebra. These decompose as $\mathbf{P} \rightarrow \mathbf{1} + (\mathbf{P-1})$ under $\mathrm{SO}(P-1)$, which is therefore identified as the subpaint group.

$q = 0$. The paint group is $G_{\rm paint} = \mathrm{SO}(P) \times \mathrm{SO}(\dot{P})$, and there are again only vector representations. Analogously as in the previous case, we thus find that the subpaint group is $G^0_{\rm subpaint} = \mathrm{SO}(P-1) \times \mathrm{SO}(\dot{P}-1)$.

$q = 7$. The paint group is $G_{\rm paint} = \mathrm{SO}(7) \times \mathrm{SO}(P)$. The two representations that are involved are $(\mathbf{7}, \mathbf{1}) = \mathbf{Q_v}$ and $(\mathbf{8}, \mathbf{P}) = \mathbf{Q_s} = \mathbf{Q_{\bar{s}}}$, where $\mathbf{8}$ is the real 8–dimensional spinor representation of SO(7). The subpaint group that allows to find singlets in both is $G^0_{\rm subpaint} = \mathrm{SU}(3) \times \mathrm{SO}(P-1)$, where $\mathrm{SU}(3) \subset G_2 \subset \mathrm{SO}(7)$. Indeed, the representations split as follows:

$$
\begin{aligned}
&(\mathbf{7}, \mathbf{1}) \rightarrow (\mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{1}) + (\overline{\mathbf{3}}, \mathbf{1})\,, \\
&(\mathbf{8}, \mathbf{P}) \rightarrow (\mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{P-1}) + (\mathbf{1}, \mathbf{P-1}) + (\mathbf{6}, \mathbf{1}) \\
&\qquad\qquad + (\mathbf{6}, \mathbf{P-1})\,.
\end{aligned} \tag{3.107}
$$

$q = 8$. The paint group is $G_{\rm paint} = \mathrm{SO}(8) \times \mathrm{SO}(P) \times \mathrm{SO}(\dot{P})$. In this case, there are two inequivalent real spinor representations of SO(8) involved: $\mathbf{8_s}$ and $\mathbf{8_{\bar{s}}}$. The representations of the full paint group are $\mathbf{Q_v} = (\mathbf{8_v}, \mathbf{1}, \mathbf{1})$, $\mathbf{Q_s} = (\mathbf{8_s}, \mathbf{P}, \mathbf{1}) \oplus (\mathbf{8_s}, \mathbf{1}, \dot{\mathbf{P}})$ and $\mathbf{Q_{\bar{s}}} = (\mathbf{8_{\bar{s}}}, \mathbf{P}, \mathbf{1}) \oplus (\mathbf{8_{\bar{s}}}, \mathbf{1}, \dot{\mathbf{P}})$. Following the strategy described above, we select first the subgroup $\mathrm{SO}_0(7) \subset \mathrm{SO}(8)$, which splits the 8–dimensional vector representation (space X) into a singlet plus a 7–dimensional vector. Both spinor representations remain irreducible under the action of this subgroup. Hence, we have to look for a smaller subgroup inside $\mathrm{SO}_0(7)$. This is G_2, which can be defined as the intersection $\mathrm{SO}_0(7) \bigcap \mathrm{SO}_+(7) \bigcap \mathrm{SO}_-(7)$, where $\mathrm{SO}(7)_\pm$ are the stability subgroups of the spinor and conjugate spinor representations, respectively. In this case, in order to obtain singlets in the spinor representation it suffices to split just one of the two vector representations, either $\mathbf{P}$, or $\dot{\mathbf{P}}$. Indeed, e.g., under $G_2 \times \mathrm{SO}(P-1) \times \mathrm{SO}(\dot{P})$ the representations split as follows:

$$\begin{aligned} (\mathbf{8_v},\mathbf{1},\mathbf{1}) &\rightarrow (\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{7},\mathbf{1},\mathbf{1})\,, \\ (\mathbf{8_s},\mathbf{P},\mathbf{1}) + (\mathbf{8_s},\mathbf{1},\dot{\mathbf{P}}) &\rightarrow (\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{7},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{P-1},\mathbf{1}) \\ &\quad +(\mathbf{7},\mathbf{P-1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\dot{\mathbf{P}}) + (\mathbf{7},\mathbf{1},\dot{\mathbf{P}})\,, \\ (\mathbf{8_{\bar{s}}},\mathbf{P},\mathbf{1}) + (\mathbf{8_{\bar{s}}},\mathbf{1},\dot{\mathbf{P}}) &\rightarrow (\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{7},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{P-1},\mathbf{1}) \\ &\quad +(\mathbf{7},\mathbf{P-1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\dot{\mathbf{P}}) + (\mathbf{7},\mathbf{1},\dot{\mathbf{P}})\,. \end{aligned} \tag{3.108}$$

In this way, we obtain singlets in the decomposition of all three involved representations. The subpaint group is thus either $G^0_{\mathrm{subpaint}} = G_2 \times \mathrm{SO}(P-1) \times \mathrm{SO}(\dot{P})$ or $G^0_{\mathrm{subpaint}} = G_2 \times \mathrm{SO}(P) \times \mathrm{SO}(\dot{P}-1)$. In case $\dot{P} = 0$, the subpaint group is $G^0_{\mathrm{subpaint}} = G_2 \times \mathrm{SO}(P-1)$.

$q = 9$**.** The paint group is $G_{\mathrm{paint}} = \mathrm{SO}(9) \times \mathrm{SO}(P)$. Real spinor representations of SO(9) are 16–dimensional. The involved representations of the paint group are $\mathbf{Q_v} = (\mathbf{9},\mathbf{1})$ and $\mathbf{Q_s} = \mathbf{Q_{\bar{s}}} = (\mathbf{16},\mathbf{P})$. The subpaint group is $G^0_{\mathrm{subpaint}} = \mathrm{SO}(7)_+ \times \mathrm{SO}(P-1)$, where $\mathrm{SO}(7)_+ \subset \mathrm{SO}(8) \subset \mathrm{SO}(9)$. This subpaint group induces the following splitting:

$$\begin{aligned} (\mathbf{9},\mathbf{1}) &\rightarrow (\mathbf{1},\mathbf{1}) + (\mathbf{8_s},\mathbf{1})\,, \qquad (3.109)\\ (\mathbf{16},\mathbf{P}) &\rightarrow (\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{P-1}) + (\mathbf{7_v},\mathbf{1}) + (\mathbf{7_v},\mathbf{P-1}) + (\mathbf{8_s},\mathbf{1}) + (\mathbf{8_s},\mathbf{P-1})\,. \end{aligned}$$

In general, we thus conclude that we started from paint groups of the form $\mathrm{SO}(q)\times\mathrm{SO}(P)\times\mathrm{SO}(\dot{P})$. The first factor is broken to the common stability subgroup of the vector and spinor representations. Furthermore, we break $\mathrm{SO}(P)$ to $\mathrm{SO}(P-1)$. In case $P, \dot{P} \geq 1$, we have to break only one of the factors in $\mathrm{SO}(P) \times \mathrm{SO}(\dot{P})$ in this way, except for $q = 0$, which is special due to the fact that in that case the two factors belong to different restricted roots of the solvable algebra.

The almost complex case

The search for the subpaint algebra is analogous to the one for the real case. We only need to be more careful in treating the complex structure. The paint group is $\mathrm{SO}(q) \times U(P)$. We will consider this complex structure as part of the unitary group. In order to find a singlet in the representation $\mathbf{Q_s}$ or $\mathbf{Q_{\bar{s}}}$, we have to consider the stability subgroup of a vector of $\mathrm{U}(P)$, which is $\mathrm{U}(P-1)$. Furthermore, we have to find as in the real case the common stability group of a vector and spinor representation of $\mathrm{SO}(q)$. We will do this explicitly for $q = -2, 2$ and 6. The subpaint group is then the product of the latter with $\mathrm{U}(P-1)$.

$q = \pm 2$**.** SO(2) is already broken by the vector representation. The analysis is thus finished at this point and we obtain $G^0_{\mathrm{subpaint}} = \mathrm{U}(P-1)$. Note that the vector as well as the 2-dimensional spinor representations split in 2 singlets.

$q = 6$**.** The vector representation breaks SO(6) to SO(5). The spinor representation of SO(6), which is a real 8-dimensional representation, is the same as the one of SO(5). To analyze the latter, it is convenient to use the isomorphism $\mathfrak{so}(5) \sim \mathfrak{usp}(4) \sim \mathfrak{su}(2,\mathbb{H})$.

Then the spinor representation becomes a vector representation. A typical vector can be put along one quaternion, such that it is left invariant under the transformations of the other quaternions. Therefore the stability group is $\mathfrak{su}(1,\mathbb{H}) \sim \mathfrak{su}(2)$. Note that we do not have to consider the generator associated with the complex structure on the Clifford algebra here, as this has been taken into account on the side of the U(P) factor. When we consider the decompositions of the vector and spinor representations under SO(6) → SO(5) → SU(2), we can consider, e.g., the **5** as the antisymmetric traceless representation of $\mathfrak{usp}(4)$, and obtain as such:

$$\mathbf{6} \to \mathbf{1}+\mathbf{5} \to \mathbf{1}+\mathbf{1}+\mathbf{2}+\mathbf{2}\,, \qquad \mathbf{8} \to \mathbf{8} \to \mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{4}\,. \tag{3.110}$$

Hence ultimately, in this case $G^0_{\rm subpaint} = {\rm SU}(2) \times {\rm U}(P-1)$, and there are just more singlets in each of the spaces X, Y and Z. In order to define the Tits-Satake projection, we have to select one singlet among the X generators, commute it with one of the Y's and single out the corresponding Z.

The quaternionic case

$q=\pm 3$. In this case, the paint group is $G_{\rm paint} = {\rm SO}(3) \times {\rm USp}(2P)$. The representations present are $\mathbf{Q_v} = (\mathbf{3},\mathbf{1})$ and $\mathbf{Q_s} = \mathbf{Q_{\bar{s}}} = (\mathbf{2},\mathbf{2P})$. Note that the last one was (in real notations) denoted as $(\mathbf{4},\mathbf{P})$ in table 3.6. Splitting the paint group first as SO(3) × USp(2) × USp($2P-2$), and then taking the diagonal of the SO(3) × USp(2) factor, the representation $(2,2P)$ splits as follows under this ${\rm SO}(3)_{\rm diag} \times {\rm USp}(2P-2)$:

$$(\mathbf{2},\mathbf{2P}) \quad \to \quad (\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{1}) + (\mathbf{2},\mathbf{2P-2})\,. \tag{3.111}$$

In order to obtain a singlet in the vector representation, we then take an SO(2) subgroup of ${\rm SO}(3)_{\rm diag}$. The subpaint group is thus:

$$G^0_{\rm subpaint} = {\rm SO}(2)_{\rm diag} \times {\rm USp}(2P-2)\,. \tag{3.112}$$

$q=4$. The story here is similar to the previous case. The paint group is $G_{\rm paint} = {\rm SO}(4) \times {\rm USp}(2P) \times {\rm USp}(2\dot{P})$. One can choose either to break the P or the $\dot{P}$ sector. We will do the former, i.e., break USp($2P$) to USp(2) × USp($2P-2$). It is useful to consider SO(4) as ${\rm SO}(3)_L \times {\rm SO}(3)_R$. The vector representation breaks into one singlet and one triplet under the diagonal subgroup of the two ${\rm SO}(3)_{L/R}$. The subpaint group is the further diagonal with USp(2), and is thus given by:

$$G^0_{\rm subpaint} = {\rm SO}(3)_{\rm diag} \times {\rm USp}(2P-2) \times {\rm USp}(2\dot{P})\,. \tag{3.113}$$

$q=5$. The paint group is $G_{\rm paint} = {\rm SO}(5) \times {\rm USp}(2P)$. The vector representation $\mathbf{Q_v} = (\mathbf{5},\mathbf{1})$ breaks SO(5) to SO(4). We split it as usual into two subgroups ${\rm SO}(3)_{L/R}$. The 8-dimensional spinor representation then splits as $\mathbf{4}+\overline{\mathbf{4}}$, where each one transforms only under one of the factors ${\rm SU}(2)_{L/R}$ mentioned above, such that only one of these factors has to be broken to get the subpaint group. Then, we consider again the subgroup ${\rm USp}(2) \times {\rm USp}(2P-2) \subset {\rm USp}(2P)$ and define the subpaint group as the product:

$$G^0_{\rm subpaint} = {\rm SO}(3)_{\rm diag} \times {\rm SO}(3)_R \times {\rm USp}(2P-2)\,, \tag{3.114}$$

where the diagonal is taken between $\mathrm{SO}(3)_L$ and USp(2). The representations have the following splittings:

$$\begin{aligned}(\mathbf{5},\mathbf{1}) &\rightarrow (\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{2},\mathbf{1})\,,\\ (\mathbf{4},\mathbf{2P}) &\rightarrow (\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{3},\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{2P}-\mathbf{2})+(\mathbf{2},\mathbf{2},\mathbf{1})\\ &\quad +(\mathbf{1},\mathbf{2},\mathbf{2P}-\mathbf{2})\,.\end{aligned} \tag{3.115}$$

We summarize the results in table 3.7.

Table 3.7 *The paint and subpaint algebras of special manifolds for the first* 10 *values of the parameter* q.

q	G_{paint}	G^0_{subpaint}
0	$\mathrm{SO}(P)\times\mathrm{SO}(\dot{P})$	$\mathrm{SO}(P-1)\times\mathrm{SO}(\dot{P}-1)$
1	$\mathrm{SO}(P)$	$\mathrm{SO}(P-1)$
2	$\mathrm{SO}(2)\times\mathrm{U}(P)$	$\mathrm{U}(P-1)$
3	$\mathrm{SO}(3)\times\mathrm{USp}(2P)$	$\mathrm{SO}(2)\times\mathrm{USp}(2P-2))$
4	$\mathrm{SO}(4)\times\mathrm{USp}(2P)\times\mathrm{USp}(2\dot{P})$	$\mathrm{SO}(3)\times\mathrm{USp}(2P-2)\times\mathrm{USp}(2\dot{P})$
5	$\mathrm{SO}(5)\times\mathrm{USp}(2P)$	$\mathrm{SO}(4)\times\mathrm{USp}(2P-2))$
6	$\mathrm{SO}(6)\times\mathrm{U}(P)$	$\mathrm{SU}(2)\times\mathrm{U}(P-1)$
7	$\mathrm{SO}(7)\times\mathrm{SO}(P)$	$\mathrm{SU}(3)\times\mathrm{SO}(P-1)$
8	$\mathrm{SO}(8)\times\mathrm{SO}(P)\times\mathrm{SO}(\dot{P})$	$G_2\times\mathrm{SO}(P-1)\times\mathrm{SO}(\dot{P})$
9	$\mathrm{SO}(9)\times\mathrm{SO}(P)$	$\mathrm{SO}_+(7)\times\mathrm{SO}(P-1)$

3.5 Results and applications

In this section, we will collect the results from previous sections to describe the result of the Tits-Satake projection for all homogeneous special geometries. Next, we will mention some applications of these results. The main application will be the organization of supergravity theories in universality classes. We will give these different universality classes and will also comment on the physical relevance of such a grouping. Finally, we will also give an example in which one can assign a microscopic meaning to the Tits-Satake projection.

3.5.1 Description of the Tits-Satake projections

Let us start by analyzing the very special manifolds that include the $L(q,P,\dot{P})$ families with $q\geq -1$ and pure supergravity in five dimensions: SG_5. In table 3.9, we give the corresponding real, special Kähler and quaternionic versions of the Tits-Satake projected isometry algebras. What is important to note here is that the original infinite set of isometry algebras, extensively discussed in section 3.2, projects onto a finite set of algebras. This means that infinite families all share the same Tits-Satake projections, which reflects the fact that each family has the same system of restricted roots and the only difference between different members of a family comes from multiplicities of these restricted root spaces that are removed in the Tits-Satake projection.

The remaining spaces that are not very special are of lower rank $r \leq 2$, namely $L(-3,0)$, $L(-3,P)$, SG_4, $L(-2,0)$, $L(-2,P)$. Since they are all symmetric spaces, one can determine the corresponding paint groups and Tits-Satake projections, using the Satake diagrams mentioned in table 3.4.

The reason why we call these cases exotic is because the resulting Tits-Satake projected algebras do not correspond to $\mathcal{N}=2$ supergravity models any more, so performing the Tits-Satake projection of, say, a special Kähler space, one arrives at a space that is no longer special Kähler. We analyze the resulting Tits-Satake projected spaces in detail below.

rank = 1

- $L(-3,0)$ corresponds to the symmetric coset $\frac{\mathrm{USp}(2,2)}{\mathrm{USp}(2)\times\mathrm{USp}(2)}$. The Tits–Satake projection of $\mathrm{USp}(2,2)$ leads to $\mathrm{SU}(1,1)$ and the projected manifold

$$\mathcal{M}_{\mathrm{TS}} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \tag{3.116}$$

  is not quaternionic! Indeed, the space $L(-3,0)$ encodes four scalars belonging to just one hypermultiplet, so it cannot be further restricted to a quaternionic submanifold.

- By SG_4, one denotes the quaternionic space $\frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$. The Tits-Satake projection of $\mathrm{Solv}(\mathrm{SU}(2,1))$ gives a so–called bc_1 system[13], given by the solvable algebra

$$\mathrm{Solv}_{\mathrm{TS}} = \mathrm{Span}\{h,\lambda,2\lambda\}\,, \tag{3.117}$$

  where (h,λ) are generators of $\mathrm{Solv}(\mathrm{SU}(1,1))$ and 2λ denotes a generator corresponding to a restricted root that is the double of the restricted root of λ. The resulting space $\mathcal{M}_{\mathrm{TS}}$ is 3–dimensional and hence not quaternionic.

- The $L(-3,P)$ family consists of the symmetric quaternionic spaces $\frac{\mathrm{USp}(2,2P+2)}{\mathrm{USp}(2)\times\mathrm{USp}(2P+2)}$. Their restricted root system can be described in the following way:

$$\mathcal{M}_{\mathrm{TS}}: \quad \bigcirc\ \lambda\,, \quad m_\lambda = 4\,P\,, \quad m_{2\lambda} = 3\,, \tag{3.118}$$

  where $\bigcirc$ means that the Dynkin diagram of this restricted root system is the one of A_1. The simple (restricted) root λ occurs with a multiplicity m_λ. Its double furthermore appears with multiplicity $m_{2\lambda}$. The Tits–Satake projection is not a symmetric space and gives again a bc_1 system, as in (3.117).

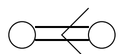
rank = 2

- $L(-2,0)$ is in its quaternionic version given by the symmetric coset $\frac{\mathrm{SU}(2,2)}{\mathrm{SU}(2)\times\mathrm{SU}(2)}$. So, the space is not split. The Tits–Satake projection in this case leads to the six-dimensional manifold:

$$\mathcal{M}_{\mathrm{TS}} = \frac{\mathrm{SO}(2,3)}{\mathrm{SO}(2)\times\mathrm{SO}(3)}\,. \tag{3.119}$$

[13] The names bc_1 and bc_2 that are used to denote some of the Tits-Satake projected algebras denote, respectively, rank one and rank two non-simple Lie algebras, see e.g. [76] for more detailed explanations.

- The quaternionic version of $L(-2,P)$ is the symmetric space $\frac{\mathrm{SU}(2,P+2)}{\mathrm{SU}(2)\times \mathrm{U}(P+2)}$. The restricted root system is again described by giving its Dynkin diagram as well as the multiplicities of the (restricted) simple roots and their possible doubles:



$$\mathcal{M}_{\mathrm{TS}}: \qquad m_{\lambda_1}=2, \quad m_{2\lambda_1}=0, \quad m_{\lambda_2}=2P, \quad m_{2\lambda_2}=1\,. \tag{3.120}$$

  In this case, the Tits-Satake projection gives a bc_2 system.

These results are summarized in the table 3.8, where we organize exotic spaces by just denoting the type of restricted root system that characterizes their Tits-Satake projections.

Table 3.8 *The solvable algebras and Tits-Satake projections of homogeneous special Kähler and quaternionic-Kähler manifolds that are not very special. In this table, P denotes positive definite integers. After the column denoting the series, we mentioned the columns of generators in table 3.1 that are present in this case. The manifolds are always the coset of G_R with its maximal compact subgroup, and the described algebras are the solvable parts of the Iwasawa decomposition of the algebras that are mentioned.*

name	gen.	Special Kähler			Quaternionic-Kähler		
		G_R	$G_{\rm TS}$	$G_{\rm sub-TS}$	G_R	$G_{\rm TS}$	$G_{\rm sub-TS}$
$L(-3,0)$					$\mathfrak{usp}(2,2)$	$\mathfrak{su}(1,1)$	$\mathfrak{so}(1,1)$
$L(-3,P)$					$\mathfrak{usp}(2P+2,2)$	bc_1	$\mathfrak{so}(1,1)$
SG_4	(0)	SG	–	–	$\mathfrak{su}(2,1)$	$bc_1 = \begin{pmatrix} q_0 \\ g_0 \\ h_0 \end{pmatrix}$	$\mathfrak{su}(1,1) = \begin{pmatrix} g_0 \\ h_0 \end{pmatrix}$
$L(-2,0)$	(01)	$\mathfrak{su}(1,1)$	$\mathfrak{su}(1,1) = \begin{pmatrix} g_1 \\ h_1 \end{pmatrix}$	$\mathfrak{su}(1,1) = \begin{pmatrix} g_1 \\ h_1 \end{pmatrix}$	$\mathfrak{su}(2,2)$	$\mathfrak{so}(3,2) = \begin{pmatrix} q_0 & p_1 \\ g_0 & g_1 \\ h_0 & h_1 \end{pmatrix}$	$\mathfrak{so}(2,2) = \begin{pmatrix} g_0 & g_1 \\ h_0 & h_1 \end{pmatrix}$
$L(-2,P)$	$(01Y)$	$\mathfrak{su}(P+1,1)$	$bc_1 = \begin{pmatrix} Y^- \\ g_1 \\ h_1 \end{pmatrix}$	$\mathfrak{su}(1,1) = \begin{pmatrix} g_1 \\ h_1 \end{pmatrix}$	$\mathfrak{su}(P+2,2)$	$bc_2 = \begin{pmatrix} \tilde{Y}^- & Y^- \\ q_0 & p_1 \\ g_0 & g_1 \\ h_0 & h_1 \end{pmatrix}$	$\mathfrak{so}(2,2) = \begin{pmatrix} g_0 & g_1 \\ h_0 & h_1 \end{pmatrix}$

Table 3.9 *The algebras of the very special manifolds. In this table q, m, P and $\dot{P}$ are always integers ≥ 1. After the column denoting the series, we mentioned the columns of generators in table 3.1 that are present in this case (+ denotes the sum of columns 2 and 3). We mention which element of the series gives the Tits-Satake and sub-Tits-Satake algebra, and in case the corresponding manifold is symmetric, we also mention the isometry group G. In case the projected manifold is non-symmetric on the other hand, we mention its solvable algebra.*

name	gen.	Real G_{TS}	Kähler G_{TS}	quaternionic-Kähler G_{TS}	$G_{\mathrm{sub-TS}}$
SG_5	(01)	–	SG_5 : $\mathfrak{su}(1,1)$	SG_5 : $\mathfrak{g}_{2(2)}$	SG_5
$L(-1,0)$	(01+)	$L(-1,0)$: $\mathfrak{so}(1,1)$	$L(-1,0)$: $\mathfrak{su}(1,1)^2$	$L(-1,0)$: $\mathfrak{so}(3,4)$	$L(-1,0)$
$L(-1,P)$	$(01+Y)$	$L(-1,1)$: $\mathfrak{so}(2,1)$	$L(-1,1)$: $\mathrm{Solv}_{SK}(-1,1)$	$L(-1,1)$: $\mathrm{Solv}_Q(-1,1)$	
$L(0,0)$	(0123)	$L(0,0)$: $\mathfrak{so}(1,1)\oplus\mathfrak{so}(1,1)$	$L(0,0)$: $\mathfrak{su}(1,1)^3$	$L(0,0)$: $\mathfrak{so}(4,4)$	$L(0,0)$
$L(0,P)$	$(0123Y)$	$L(0,1)$: $\mathfrak{so}(1,1)\oplus\mathfrak{so}(2,1)$	$L(0,1)$: $\mathfrak{su}(1,1)\oplus\mathfrak{so}(3,2)$	$L(0,1)$: $\mathfrak{so}(5,4)$	
$L(0,P,\dot{P})$	$(0123YZ)$	$L(0,1,1)$: $\mathrm{Solv}_R(0,1,1)$	$L(0,1,1)$: $\mathrm{Solv}_{SK}(0,1,1)$	$L(0,1,1)$: $\mathrm{Solv}_Q(0,1,1)$	
$L(q,P)$ $L(4m,P,\dot{P})$	$(0123XYZ)$	$L(1,1)$: $\mathfrak{s}\ell(3,\mathbb{R})$	$L(1,1)$: $\mathfrak{sp}(6)$	$L(1,1)$: $\mathfrak{f}_{4(4)}$	

3.5.2 The universality classes

We can now group supergravities exhibiting homogeneous special geometry in universality classes, based on the Tits-Satake projection. All manifolds that have the same projection constitute one specific universality class. For the exotic cases discussed above, we conclude that there are 4 such universality classes, exhibited in table 3.11. As far as the other special homogeneous manifolds is concerned, they correspond to infinite families of supergravity theories with 8 supercharges in dimensions $d = 5$, $d = 4$ or $d = 3$, falling into altogether 7 universality classes, displayed in table 3.10. Members within the same class are distinguished by different choices of the paint group and of its representations $\mathbf{Q}_{\mathbf{v},\mathbf{s},\bar{\mathbf{s}}}$. The maximal split representative of the class, namely the Tits-Satake projected manifold is in five out of seven cases a symmetric coset manifold. The only cases where the Tits-Satake manifold is not symmetric are the families that project onto $L(-1,1)$ and $L(0,1,1)$. The most populated universality class, which encompasses most of the homogeneous special geometries is the class $L(1,1)$.

Table 3.10 *The seven universality classes of very special homogeneous geometries.*

Universality Class	Members of the class
$L(1,1)$	$L(q,P)$ for $q,P \geq 1$ $L(4m,P,\dot{P})$ for $m\,P\dot{P} \neq 0$
$L(0,1,1)$	$L(0,P,\dot{P})$ for $P\,\dot{P} \neq 0$
$L(0,1)$	$L(0,P,0)$ for $P \neq 0$ $L(q,0)$ for $q > 0$
$L(0,0)$	$L(0,0)$
$L(-1,0)$	$L(-1,0)$
$L(-1,1)$	$L(-1,P)$ for $P \geq 1$
SG_5	SG_5

Table 3.11 *Exotic universality classes of homogeneous geometries and their Tits Satake root systems.*

Universality Class	Members of the class
bc_2	$L(-2,P)$, $P > 0$
b_2	$L(-2,0)$
bc_1	$\mathrm{SG}_4, L(-3,P)$, $P > 0$
a_1	$L(-3,0)$

That such an organization in universality classes is physically relevant, is shown by the so-called cosmic billiard phenomenon. This effect arises in the study of time-dependent solutions of supergravity theories. Such studies have been mostly performed in the case where the target spaces are symmetric, for instance for pure supergravities or for supergravities with more than eight supercharges. If one assumes configurations where the scalar fields only depend on time and one adopts the solvable parametrization of the scalar manifold, the action (3.1) can be written as follows:

$$S = \int \mathrm{d}t \left[\dot{\vec{h}} \cdot \dot{\vec{h}} + \sum_{\alpha \in \overline{\Delta}^{+}} e^{-2\vec{\alpha}\cdot\vec{h}(t)} N_{\alpha}\left(\phi, \dot{\phi}\right) \right] . \tag{3.121}$$

In this formula, $\vec{h}$ denotes a vector of scalar fields that are associated with the Cartan generators of the solvable algebra, while $N_{\alpha}(\phi, \dot{\phi})$ are a set of polynomial functions depending on the scalar fields ϕ that are associated with the nilpotent generators in the solvable algebra. Note that the sum in the formula runs over all positive restricted roots of the symmetric space. It turns out that the fields h_i evolve within the domain where $\vec{\alpha}\cdot\vec{h} > 0$, for all $\vec{\alpha} \in \overline{\Delta}^{+}$ and they experience bounces when approaching walls defined by $\vec{\alpha}\cdot\vec{h} = 0$ for any $\vec{\alpha}$. This type of scalar field evolution leads to the cosmic billiard phenomenon. Indeed the fields $e^{h_i(t)}$ can be identified with the cosmic scale factors relative to the various compact and non-compact dimensions of the 10- or 11–dimensional space-time manifold and the bouncing phenomena correspond to inversions in the expansion/contraction development of these scale factors. Note that the walls are only determined by different restricted root vectors, irrespective of the multiplicity with which they occur. In other words, different symmetric spaces, that however have the same Tits-Satake projected root system, have the same walls. Hence, in order to define the main features of billiard dynamics, only a subset of the scalar fields ϕ is needed. This subset contains one nilpotent scalar for each positive restricted root. This is sufficient to define the positions of all the walls that cause the inversions in the motion of a billiard ball with coordinates $h_i(t)$.

This clearly shows that a very essential part of the dynamics of all the models in one universality class, is already determined by the Tits-Satake projection that represents the class. Moreover, as was shown in [5] all solutions of the Tits-Satake projected model, which is often completely integrable, are solutions of every member in the class and can be further arbitrarily rotated by means of the paint group to new more general solutions. From the point of view of cosmic billiards, an open problem is that of understanding the relation between the complete integral of the evolutionary equations in the non-projected case with respect to those of the projected one. Indeed, counting of the integration constants shows that the non-projected model contains more solutions than the ones that can be obtained as rotations of solutions of the Tits-Satake projected model. It remains to be seen in future investigations how one can understand the structure of the missing solutions and what their relation with the bulk of solutions produced by the Tits-Satake projection is.

Another example of how the Tits-Satake projection singles out a simple model in one universality class that captures some but not all of the relevant dynamics, is given by the following [66, 78, 79, 80]. One considers the low energy effective theory of type IIB superstring theory, compactified on a $K3 \times T^2/\mathbb{Z}_2$-orientifold, in the presence of n_3 parallel D3-branes and n_7 parallel D7-branes. The branes are all space-time filling and the D7 branes are wrapped on $K3$. One furthermore also allows the field strengths of some form fields to have non-trivial vacuum expectation values along the compact directions. The branes are supposed to be in the Coulomb phase and massive modes are integrated out. It was found that this resulting low energy four-dimensional effective theory is described by an $\mathcal{N} = 2$ supergravity theory. There is a hypermultiplet sector in this model that contains for instance the moduli that describe the shape of $K3$. The vector multiplet sector of the low energy supergravity contains three complex scalars s, t, u originating from the bulk sector and spanning a manifold of the form:

$$\left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)_s \times \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)_t \times \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)_u . \tag{3.122}$$

The s scalar describes the volume of $K3$ and the R–R four–form $C_{(4)}$ on $K3$, t the T^2–complex structure and u the IIB axion–dilaton system:

$$\begin{aligned} s &= C_{(4)} - \mathrm{i}\,\mathrm{Vol}(K_3), \\ t &= \frac{g_{12}}{g_{22}} - \mathrm{i}\,\frac{\sqrt{\det g}}{g_{22}}, \\ u &= C_{(0)} - \mathrm{i}\, e^{\varphi}, \end{aligned} \tag{3.123}$$

where the matrix g denotes the metric on T^2. The corresponding three vector fields A^1_μ, A^2_μ, A^3_μ, together with the vector contained in the graviton multiplet, A^0_μ, originate from the components $B_{\mu a}$, $C_{\mu a}$ of the ten dimensional NS–NS and R–R 2–forms, $a = 1, 2$ labelling the directions of T^2. There are furthermore $n_3 + n_7$ extra vector multiplets, coming from the D-branes. In particular, there are n_3 complex scalars y^r and n_7 complex scalars z^k ($r = 1, \ldots, n_3$; $k = 1, \ldots, n_7$) that describe the positions of the D3- and D7-branes along T^2. Due to the addition of these D-branes, the special Kähler manifold (3.122) is enlarged. It turns out that the scalars of the vector multiplet sector now span a special Kähler space that can be described in terms of the prepotential[14]:

$$F(s, t, u, x^k, y^r) = stu - \frac{1}{2} s z^k z^k - \frac{1}{2} u y^r y^r . \tag{3.124}$$

This special Kähler manifold is the one that belongs to the family $L(0, n_3, n_7)$. It is thus homogeneous, but non-symmetric when both D3- and D7-branes are present.

The Tits-Satake projection of this space is given by the non-symmetric space $L(0, 1, 1)$, corresponding to the situation in which only 1 D3-brane and 1 D7-brane are present. In this case, the Tits-Satake projection has a microscopical meaning, namely it consists of disregarding the number of branes. It is thus clear that one can not describe the full dynamics of a specific model with a generic number of D3- and D7-branes by just looking at the Tits-Satake projection. However, the Tits-Satake projection can be very useful as it can provide us with a tractable model in which some essential parts of the dynamics can be analyzed.

3.6 Conclusion

In this chapter, we have discussed the Tits-Satake projection of homogeneous special geometries that can occur as target spaces in supergravity theories with 8 supercharges. This

[14] Using this prepotential, one can write down the kinetic terms of the scalars. It turns out that the full Lagrangian (for instance the kinetic terms of the vectors) can not be written in terms of a prepotential. It is however related to the theory in terms of the prepotential (3.124) via electric-magnetic duality. We refer to [66, 80] for more details.

projection was known in the context of symmetric spaces. As symmetric spaces only constitute a small fraction of the possible target space geometries that can appear in supergravity, we reconsidered this method in order to give an extension to the more general class of homogeneous special geometries. Due to the fact that crucial notions that can be defined for symmetric spaces (such as paint and subpaint groups), also appear in homogeneous special geometry, we succeeded in giving such an extension.

The Tits-Satake projection offers an efficient tool in analyzing time-dependent solutions of (ungauged) supergravity theories. It allows us to group supergravity theories with homogeneous special target spaces in a small number of universality classes that are such that all geometries in one class have the same projection. The different members of one universality class are distinguished by the fact that they have different paint groups. We commented on the relevance of these universality classes. We mentioned the cosmic billiard phenomenon as an example of how time-dependent solutions of all models in one universality class share some basic features. These basic features are already visible at the level of the Tits-Satake projection. Often, the Tits-Satake projection gives a tractable model, whose dynamics can be analyzed analytically. Solutions that can be found in this way are then common to the whole universality class. As not all solutions of a specific model can be obtained via its Tits-Satake projection, one of the open questions is how the dynamics of specific models in a universality class can be described beyond the Tits-Satake projection. In some cases, such a grouping in universality classes can also be seen from a more microscopical viewpoint. An example of this was given in terms of string compactifications including D-branes, where the Tits-Satake projection corresponds to disregarding the multiplicities of the branes.

Chapter 4

Scaling cosmologies in $\mathcal{N} = 8$ gauged supergravity

4.1 Introduction

In the previous chapter, we studied the Tits-Satake projection in the context of homogeneous special geometry. We mentioned how this procedure is relevant in studying time-dependent solutions of non-maximal supergravity theories. Systematic studies of such solutions have been performed in the context of ungauged supergravity in [5, 55]. In order to understand the possible (late-time) cosmological scenarios in supergravity and string theory, one should however also consider gauged supergravities, in which case there can be a non-trivial potential for the scalars. As was shown in section 2.1.4, this can lead to interesting phenomena, such as acceleration and scaling behavior. In this chapter, we will make a modest step in understanding the behavior of cosmological solutions in the context of $\mathcal{N} = 8$ gauged supergravity.

Extensive research on the vacuum structure of gauged extended supergravities has been carried out; for instance, de Sitter vacua in such theories can generically be found [52, 53, 54, 81, 82, 83, 84, 85]. In most cases, these vacua correspond to saddle points of the scalar potential. Stable de Sitter vacua have only been constructed for $\mathcal{N} \leq 2$ supergravity (see for instance [53, 85] for constructions in $\mathcal{N} = 2$ supergravity). For these examples, it is however unclear whether they have a higher-dimensional origin or a string interpretation.

As mentioned in the introduction, the possibilities for dark energy go beyond a positive cosmological constant. More specifically, we mentioned scaling cosmologies as an interesting class of cosmological solutions. As was shown in an example in section 2.1.4, these correspond to critical points of an autonomous system. As such, they can correspond to repellers and attractors. They can thus correspond to the early-time or late-time behavior of general cosmological solutions of the system under consideration. In contrast to de Sitter vacua, scaling cosmologies have not been given that much attention in supergravity. In reference [86], an unstable accelerating scaling solution was found in $\mathcal{N} = 8$ supergravity as an alternative to acceleration from de Sitter solutions. In [87], an example of a stable scaling solution was found in $\mathcal{N} = 4$ gauged supergravity. Finally, reference [88] considered scaling solutions of 6-dimensional gauged chiral supergravity compactified to 4 dimensions.

This chapter aims at finding scaling cosmologies in $\mathcal{N} = 8$, $d = 4$ supergravity. Although $\mathcal{N} = 8$ supergravity is probably not very realistic from a particle physics point of view, it has some attractive features; it only contains one multiplet, namely the gravity multiplet and its different gaugings are well known. This simplicity makes it easier to oversee all the possibilities for finding interesting solutions. More explicitly, in this chapter we will restrict ourselves to the CSO-gaugings that have been introduced in section 2.4.3. For these gaugings, a higher-dimensional origin is known explicitly [47].

Performing an exhaustive study of cosmological solutions in supergravity theories is notoriously difficult because of the many scalar fields and the corresponding complicated potentials. For instance, in $\mathcal{N} = 8$ supergravity there are 70 scalars that parametrize the coset space $E_{7(+7)}/\,\mathrm{SU}(8)$ and the complexity of the potential depends on the gauge group of the theory. We will therefore truncate the scalar sector and we will try to find solutions in this truncated version of the theory. It turns out that after performing these truncations, the potential is of the so-called 'multiple exponential' type, meaning that it is a sum of exponential terms. This will be explained in section 4.2.

In section 4.3, we will search for scaling solutions of these truncations of $\mathcal{N} = 8$ gauged supergravity. We will first review general results on scaling cosmologies in systems with multiple exponential potentials. We will then show that the potentials in $\mathcal{N} = 8$ CSO-gauged supergravity indeed have the correct form to allow for scaling cosmologies. We will explicitly give these scaling solutions and discuss some of their properties.

The fact that the CSO-gaugings can be given a higher-dimensional interpretation, means that we can view our solutions not only as solutions of $\mathcal{N} = 8,\ d = 4$ gauged supergravity, but that we can also see them as solutions of 11-dimensional supergravity. Some of the scaling solutions that are found in section 4.3 describe accelerating universes. This seems to suggest that we have obtained accelerating cosmologies by compactifying 11-dimensional supergravity. A famous no-go theorem [89, 90, 91] however comments on the impossibility of this. In section 4.4, we will therefore discuss the higher-dimensional origin of our scaling solutions more explicitly and show how some of the assumptions of the no-go theorem are violated.

Finally, in section 4.5, we will mention an interesting interplay between finding de Sitter solutions and scaling cosmologies.

The scaling solutions studied here are of two kinds, namely matter scaling and scalar dominated scaling solutions. In the former type, we will thus allow for the presence of an extra barotropic fluid in the model, while in the latter type the energy density of the barotropic fluid vanishes and the potential energy of the scalar fields scales as the kinetic energy[1]. Note that we make some restrictions in this chapter. We consider only flat FLRW-universes and ignore scaling solutions that exist on the boundary of the scalar manifold[2]. We refer to [93] for a general treatment of scaling solutions in the presence of spatial curvature, that also includes solutions on the boundary of the scalar manifold.

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[1] This is also true for matter scaling solutions so the only difference is that the fluid vanishes.
[2] They are called non-proper solutions in references [92, 93].

4.2 Truncating $\mathcal{N} = 8$ gauged supergravity

In this chapter, we will consider an action describing N scalar fields ϕ^i coupled to gravity:

$$S = \int \mathrm{d}^4x \sqrt{-g}\Big[\frac{1}{2\kappa^2}R - \tfrac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\,\partial^\mu\phi^j - V(\phi)\Big] + S_{\text{matter}}\,, \tag{4.1}$$

where $\kappa^2 = 8\pi G_N$ with G_N Newton's constant. In the rest of this chapter, we will often choose units in which $\kappa^2 = \frac{1}{2}$. We have added an action S_{matter}, describing a possible barotropic fluid with energy density ρ and pressure p, subject to a simple equation of state as in (2.17). For later convenience, we rename the parameter w in (2.17) to $\gamma - 1$. In a flat FLRW-background, $\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}\vec{x}^2$, the equations of motion read:

$$\ddot{\phi}^i + \Gamma^i_{jk}\dot{\phi}^j\dot{\phi}^k + 3H\dot{\phi}^i = -g^{ij}\partial_j V\,, \tag{4.2}$$

$$\dot{\rho} + 3\gamma H\rho = 0\,, \tag{4.3}$$

$$p = (\gamma - 1)\rho\,, \tag{4.4}$$

$$H^2 = \frac{\kappa^2}{3}(T + V + \rho)\,, \tag{4.5}$$

$$\dot{H} = -\kappa^2(T + \tfrac{1}{2}\gamma\rho)\,, \tag{4.6}$$

with T the kinetic energy of the scalars, $T = \frac{1}{2}g_{ij}(\phi)\dot{\phi}^i\,\dot{\phi}^j$ and $H = \dot{a}/a$, the Hubble parameter. A scaling cosmology is then a solution of the above equations for which:

$$V(t) \sim T(t) \sim \rho(t)\,. \tag{4.7}$$

From the Friedmann equation (4.5) and the acceleration equation (4.6), we find that the scale factor is power-law[3] : $a(t) \sim t^P$. Note that the solution describes an accelerating cosmology when $P > 1$. Summarizing, a scaling solution is characterized by:

$$V(t) \sim T(t) \sim \rho(t) \sim H^2(t) \sim \dot{H}(t) \sim \frac{1}{t^2}\,. \tag{4.8}$$

We will view the action (4.1) as the bosonic part of the action of an $\mathcal{N} = 8$ gauged supergravity, where the vector fields have been truncated. In that case, the number of scalar fields N is equal to 70 and the metric $g_{ij}(\phi)$ corresponds to the metric on the scalar coset $E_{7(+7)}/\,\mathrm{SU}(8)$. The specific form of the potential depends on which gauging is considered. Obtaining an explicit expression for these potentials is a difficult task. One often makes truncations to get manageable expressions. We will therefore focus on the $\mathrm{S}\ell(8,\mathbb{R})/\,\mathrm{SO}(8)$-submanifold of $E_{7(+7)}/\,\mathrm{SU}(8)$ that contains 35 scalar fields. The other 35 scalars correspond to pseudoscalar fields and can be consistently truncated. After this truncation, the Lagrangian for the metric and the 35 scalars can be written in terms of a coset representative L of the $\mathrm{S}\ell(8,\mathbb{R})/\,\mathrm{SO}(8)$-coset [94]:

$$\mathcal{L} = \sqrt{-g}\Big\{R + \tfrac{1}{4}\mathrm{Tr}[\partial\mathcal{M}\partial\mathcal{M}^{-1}] - V\Big\}\,, \tag{4.9}$$

[3] We will use the capital letter P to denote the power-law of the scale factor. This should not be confused with the capital letter P that was used in denoting families of homogeneous special geometries in the previous chapter.

where $\mathcal{M}=LL^T$. The potential is then given by [94]:

$$V = g^2\Big\{\mathrm{Tr}[(\eta\mathcal{M})^2] - \tfrac{1}{2}(\mathrm{Tr}[\eta\mathcal{M}])^2\Big\}\,, \tag{4.10}$$

where g is the gauge coupling constant. In this expression, η is the diagonal matrix, whose elements appear as structure constants of the CSO(p,q,r)-algebra (see section 2.4.3).

The coset $\mathrm{S}\ell(8,\mathbb{R})/\mathrm{SO}(8)$ is parametrized by 7 dilatons ϕ^i and 28 axions χ^α. In the solvable parametrization of the coset, the dilatons are associated to the Cartan generators H_i of $\mathrm{S}\ell(8,\mathbb{R})$, while the axions are associated to the positive root generators E^α:

$$L = e^{\chi^\alpha E^\alpha} e^{-\frac{1}{2}\phi^i H_i}\,, \tag{4.11}$$

where the exponentials involve a sum over the positive roots α and a sum over the 7 Cartan generators, denoted by the index i.

Note that one can perform an extra truncation; since the axions appear at least squared in the potential it is consistent to put them to zero[4]. Working in the fundamental representation of $\mathrm{S}\ell(8,\mathbb{R})$, the matrix $\mathcal{M}$ then assumes the following diagonal form:

$$\mathcal{M} = \begin{pmatrix} e^{-\vec\beta_1\cdot\vec\phi} & & \\ & \ddots & \\ & & e^{-\vec\beta_8\cdot\vec\phi} \end{pmatrix}\,. \tag{4.12}$$

The 8 vectors $\vec\beta_a$ denote the weights of $\mathrm{S}\ell(8,\mathbb{R})$ in the fundamental representation. They obey the following relations:

$$\sum_a \beta_a^i = 0\,, \quad \sum_a \beta_a^i\beta_a^j = 2\delta^{ij}\,, \quad \vec\beta_a\cdot\vec\beta_b = 2\delta_{ab} - \frac{2}{n}\,. \tag{4.13}$$

An explicit choice is given by the following vectors:

$$\begin{aligned}
\vec\beta_1 &= (1, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{10}}, \tfrac{1}{\sqrt{15}}, \tfrac{1}{\sqrt{21}}, \tfrac{1}{\sqrt{28}})\,,\\
\vec\beta_2 &= (-1, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{10}}, \tfrac{1}{\sqrt{15}}, \tfrac{1}{\sqrt{21}}, \tfrac{1}{\sqrt{28}})\,,\\
\vec\beta_3 &= (0, \tfrac{-2}{\sqrt{3}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{10}}, \tfrac{1}{\sqrt{15}}, \tfrac{1}{\sqrt{21}}, \tfrac{1}{\sqrt{28}})\,,\\
\vec\beta_4 &= (0, 0, \tfrac{-3}{\sqrt{6}}, \tfrac{1}{\sqrt{10}}, \tfrac{1}{\sqrt{15}}, \tfrac{1}{\sqrt{21}}, \tfrac{1}{\sqrt{28}})\,,\\
\vec\beta_5 &= (0, 0, 0, \tfrac{-4}{\sqrt{10}}, \tfrac{1}{\sqrt{15}}, \tfrac{1}{\sqrt{21}}, \tfrac{1}{\sqrt{28}})\,,\\
\vec\beta_6 &= (0, 0, 0, 0, \tfrac{-5}{\sqrt{15}}, \tfrac{1}{\sqrt{21}}, \tfrac{1}{\sqrt{28}})\,,\\
\vec\beta_7 &= (0, 0, 0, 0, 0, \tfrac{-6}{\sqrt{21}}, \tfrac{1}{\sqrt{28}})\,,\\
\vec\beta_8 &= (0, 0, 0, 0, 0, 0, \tfrac{-7}{\sqrt{28}})\,.
\end{aligned} \tag{4.14}$$

After this truncation, one can easily see that the kinetic terms of the scalars become canonical, while the potential is a sum of exponentials:

$$\mathcal{L}_{\text{scalar}} = -\tfrac{1}{2}\delta_{ij}\,\partial\phi^i\partial\phi^j - \tfrac{g^2}{2}\sum_{a=1}^{p+q} e^{-2\vec\beta_a\cdot\vec\phi} + g^2\sum_{a<b}^{p+q}\eta_{aa}\eta_{bb}e^{-(\vec\beta_a+\vec\beta_b)\cdot\vec\phi}\,. \tag{4.15}$$

[4] The kinetic term allows a truncation of all the axions since the dilatons parametrize a totally geodesic submanifold. We refer to [95] for more details concerning this point.

In the next section, we will analyze this model for the existence of scaling solutions.

4.3 Dilatonic scaling cosmologies in $\mathcal{N}=8$ gauged supergravity

The potential in (4.15) belongs to the class of so-called multiple exponential potentials. These are given by a sum of M exponential terms that depend on N scalar fields:

$$V(\phi) = \sum_{a=1}^{M} \Lambda_a \exp[\vec{\alpha}_a \cdot \vec{\phi}\,]\,, \tag{4.16}$$

where $\vec{\alpha}_a \cdot \vec{\phi} = \sum_{i=1}^{N} \alpha_{ai}\phi^i$. Scaling solutions of systems with multiple exponential potentials, where the kinetic terms of the scalars are in canonical form, have been studied in great detail [92, 93, 96, 97, 98, 99, 100]. For convenience, we will first review the main results of these studies. Next, we are going to use this discussion to find scaling solutions of the system (4.15).

Scaling for multiple exponential potentials

It was shown in [92], that it is convenient to separate exponential potentials in two classes I and II. Class I is characterized by the fact that the $\vec{\alpha}_a$-vectors are linearly independent whereas for class II they are linearly dependent. Models that belong to the first class are known in the literature under the name 'Generalized assisted inflation' [100]. Class I generically allows exact scaling solutions, whereas class II can have exact scaling solutions only when the $\vec{\alpha}_a$-vectors are 'affinely related' [92, 93]. This means that there exists a set of R independent $\vec{\alpha}_a$ such that after relabelling $a = 1, \ldots, R$, the remaining $\vec{\alpha}_b$ are expressed as $\vec{\alpha}_b = \sum_{a=1}^{R} c_{ba}\vec{\alpha}_a$, where the coefficients c_{ba} fulfill the constraint:

$$\sum_{a=1}^{R} c_{ba} = 1\,, \quad \text{for all } b = R+1, \ldots, M\,. \tag{4.17}$$

Both types of potentials that allow for scaling solutions have the property that after an orthogonal field redefinition $\vec{\phi} \to \vec{\varphi}$, the potential can always be written as the following product [93, 99]:

$$V(\varphi) = e^{c\varphi_1}\, U(\varphi_2, \ldots, \varphi_N)\,. \tag{4.18}$$

Let us prove (4.18) for class I and then for class II with affinely related $\vec{\alpha}_a$-vectors. We will assume that a field rotation is performed such that the minimal number, R, of scalars appears in the potential and that consequently $N-R$ scalar fields are free. This number R equals the number of linearly independent $\vec{\alpha}_a$-vectors [92]. So, class I has $R=M$ and class II $R<M$.

If the $\vec{\alpha}_a$ are linearly independent, there exists a (unit) vector $\vec{E}$ such that

$$\vec{\alpha}_a \cdot \vec{E} = c\,, \tag{4.19}$$

where c is a number that is independent of the index a. Indeed, if we multiply (4.19) with α_{aj} and sum over a, we obtain:

$$\sum_{a,i} \alpha_{aj}\, \alpha_{ai}\, E^i = c \sum_{a} \alpha_{aj}\,. \tag{4.20}$$

It can be shown that the matrix $B_{ij} = \sum_a \alpha_{aj}\, \alpha_{ai}$ has an inverse (because $R = M$) and the above equation can be solved to find E^i. If we now write the scalar fields in a different basis:

$$\vec{\phi} = \varphi_1\, \vec{E} + \vec{\varphi}_\perp\,, \tag{4.21}$$

then we have in the new basis that $\alpha_{a1} = c$ for all a and consequently the potential takes the form (4.18).

Now assume that the $\vec{\alpha}_a$ are linearly dependent in an affine way. Consider the R independent vectors $\vec{\alpha}_a$ with $a = 1, \ldots, R$. For this subset, we can repeat the same procedure as above to find a unit vector $\vec{E}$ that obeys (4.19). Then we have in the new basis that $\alpha_{a1} = c$ for $a = 1, \ldots, R$. Consider α_{b1} for $b > R$:

$$\alpha_{b1} = \sum_{a=1}^{R} c_{ba} \alpha_{a1} = c \sum_{a=1}^{R} c_{ba} = c\,. \tag{4.22}$$

Again the potential can be factorized as in (4.18).

It is easy to prove the inverse: if the potential can be written as (4.18), then either the $\vec{\alpha}_a$ are linearly independent or they are dependent in an affine way.

As proven in [93, 99], the exact scaling solution is such that the overall scalar φ_1 is non-constant, while all the other scalars are constant. Therefore the exact scaling solutions of multiple exponential potentials are such that the potential is truncated to a single exponential potential[5]. It must then be possible to rewrite the potential like in (4.18). Furthermore, the function U must have stationary points ($\partial U = 0$) in order to have a truncation consistent with the equations of motion. If this is satisfied, the truncated action is given by:

$$S = \int \sqrt{-g}\Big[R - \tfrac{1}{2}(\partial\varphi_1)^2 - \Lambda\, e^{c\varphi_1}\Big] + (S_{\text{matter}})\,, \tag{4.23}$$

where Λ is the value of the function U at the stationary point. If the scaling solution exists, it is given by:

$$a(t) \sim t^P\,, \quad \varphi_1(t) = -\frac{2}{c}\ln t + \frac{\ln(\frac{6P-2}{c^2\,\Lambda})}{c}\,, \quad \rho(t) = 6(1 - \frac{1}{c^2 P})\frac{P^2}{t^2}. \tag{4.24}$$

- Let us first assume that the barotropic fluid vanishes, in which case the scaling solution is the scalar dominated solution with $P = 1/c^2$. The scaling solution exists when $\Lambda > 0$ and $P > 1/3$ or $\Lambda < 0$ and $P < 1/3$. An accelerating solution ($P > 1$) requires $c^2 < 1$. The scaling solution with $\Lambda < 0$ is never stable[6] and the scaling solution with $\Lambda > 0$ is

[5] In [101], the same was proven for purely positive exponential terms and a special class of dilaton couplings.

[6] Strictly speaking, the stability of scaling solutions should be understood in the sense of stability of critical points in autonomous systems. The scaling solutions are critical points of an autonomous system, similar to the simple system discussed in section 2.1.4. The critical point is then called stable, when small fluctuations around the critical point solution don't grow exponentially. Such an analysis was performed in [93].

stable if the extremum of U is a minimum and the fluid perturbations imply an extra stability condition $\frac{1}{c^2} > \frac{2}{3\gamma}$.

- If on the other hand there is a non-zero barotropic fluid, one has a matter scaling solution [96]. The matter scaling solution is such that the energy densities of the barotropic fluid and the scalar fields are non-vanishing and have a fixed ratio. The scale factor of a matter scaling solution is that of a universe containing only the barotropic fluid, that is $a(t) \sim t^P$ with $P = \frac{2}{3\gamma}$. The solution exists when $\Lambda > 0$ and $\frac{1}{c^2} < \frac{2}{3\gamma}$. When U is in a minimum, the solution is stable.

The CSO-dilaton potentials

In order to see whether we can find scaling solutions of the model whose scalar sector is described by (4.15), we will first write the potential in a more suitable form. From the explicit expressions (4.14), it is clear that one can always find a field redefinition such that the scalar matrix $\mathcal{M}$, given in (4.12), assumes the following form:

$$\mathcal{M} = \begin{pmatrix} e^{\frac{c}{2}\varphi}[\tilde{\mathcal{M}}(\phi_1, \cdots, \phi_{7-r})] & 0 \\ 0 & F(\varphi, \phi'_1, \cdots, \phi'_{r-1}) \end{pmatrix} . \tag{4.25}$$

In this expression $\tilde{\mathcal{M}} = \tilde{L}\tilde{L}^T$, where $\tilde{L}$ is now a coset representative of an $S\ell(8-r, \mathbb{R})/SO(8-r)$ coset, where again only the dilatonic scalars are kept. F denotes an $r \times r$-matrix. The scalars $\varphi, \phi'_1, \cdots, \phi'_{r-1}$ on which it depends, are connected to the original dilatons $\phi_{8-r}, \cdots, \phi_7$ via an orthogonal transformation. The number c is determined by

$$c^2 = \frac{8}{p+q} - 1 . \tag{4.26}$$

The potential (4.10) then takes the following form:

$$V = g^2 e^{c\varphi} U(\tilde{\mathcal{M}}) = g^2 e^{c\varphi}\Big[\mathrm{Tr}[(\tilde{\eta}\tilde{\mathcal{M}})^2] - \tfrac{1}{2}(\mathrm{Tr}[\tilde{\eta}\tilde{\mathcal{M}}])^2\Big] , \tag{4.27}$$

where $\tilde{\eta} = \mathrm{diag}(\mathbb{1}_p, -\mathbb{1}_q)$. More explicitly, the following expression for the potential is obtained:

$$V = g^2 e^{c\varphi} U(\phi) = g^2 e^{c\varphi}\Big[\tfrac{1}{2}\sum e^{-2\vec{\beta}'_a\cdot\vec{\phi}} - \sum_{a<b} \tilde{\eta}_{aa}\tilde{\eta}_{bb} e^{-(\vec{\beta}'_a + \vec{\beta}'_b)\cdot\vec{\phi}}\Big] , \tag{4.28}$$

where the vectors $\vec{\beta}'_a$ are now weight vectors of the fundamental representation of $S\ell(p+q, \mathbb{R})$. They can be obtained from the $S\ell(8, \mathbb{R})$-weights, by keeping only the first $p+q$ weights and by deleting the last r columns in these.

For $c \neq 0$, this potential belongs to class II with affinely related $\vec{\alpha}_a$-vectors. Therefore the potential is of the appropriate form for scaling solutions!

To find a scaling solution, it is sufficient to find a stationary point of U that has the correct sign to allow for a scaling solution. The stationary points of U are most easily found using Lagrange multipliers as was shown in [84]. One defines $X_a = e^{-\vec{\beta}'_a\cdot\vec{\phi}}$. We will split the index a in indices $\{i, \bar{i}\}$, where $i = 1, \cdots, p$ and $\bar{i} = p+1, \cdots, p+q$. In this notation, $U(\phi)$ can be rewritten as:

$$U(X) = \sum_i X_i^2 + \sum_{\bar{i}} X_{\bar{i}}^2 - \frac{1}{2}\left(\sum_i X_i - \sum_{\bar{i}} X_{\bar{i}}\right)^2 . \tag{4.29}$$

These $X_i, X_{\bar{i}}$-variables are not unconstrained, since they obey $\prod_a X_a = 1$. One then introduces a Lagrange multiplier to enforce this constraint and demands that the following function

$$S(X, \lambda) = U(X) + \lambda(\prod_i X_i \prod_{\bar{i}} X_{\bar{i}} - 1) , \tag{4.30}$$

has stationary points against variations with respect to X and λ. This leads to the following set of equations:

$$\begin{aligned}
\delta S/\delta X_i = 0 &\Rightarrow X_i^2 - \frac{1}{2} X_i (\sum_j X_j - \sum_{\bar{j}} X_{\bar{j}}) + \frac{\lambda}{2} = 0 , \\
\delta S/\delta X_{\bar{i}} = 0 &\Rightarrow X_{\bar{i}}^2 + \frac{1}{2} X_{\bar{i}} (\sum_j X_j - \sum_{\bar{j}} X_{\bar{j}}) + \frac{\lambda}{2} = 0 , \\
\delta S/\delta \lambda = 0 &\Rightarrow \prod_a X_a = 1 .
\end{aligned} \tag{4.31}$$

Defining $\sigma \equiv (\sum_i X_i - \sum_{\bar{i}} X_{\bar{i}})$, one can view the first two equations of this set as quadratic equations for X_i and $X_{\bar{i}}$ with coefficients determined by σ and λ. The solutions are

$$X_i = \frac{\sigma}{4} \pm \frac{1}{4}\sqrt{\sigma^2 - 8\lambda} , \quad X_{\bar{i}} = -\frac{\sigma}{4} \pm \frac{1}{4}\sqrt{\sigma^2 - 8\lambda} . \tag{4.32}$$

One can show from (4.31) that $\lambda < 0$ and that one therefore has to choose the positive signs in (4.32), since X_a is positive by definition. All X_i therefore have the same value that we will denote by X. Similarly, all $X_{\bar{i}}$ assume an equal value $\bar{X}$. One can also derive that $\lambda = -2X\bar{X}$ from which it follows that:

$$(p-2)X = (q-2)\bar{X} . \tag{4.33}$$

The values of X and $\bar{X}$ can then be found by noting that $X^p \bar{X}^q = 1$.

In order to have the overall exponent $e^{c\varphi}$ in the potential, we require that $p + q < 8$. Going through all possibilities for the extrema of U and taking into account the existency conditions for matter scaling and scalar dominated solutions, one is led to table 4.1.

We remark that there is no scalar dominated solution such that P lies between $1/3$ and 1. This implies that in these models a matter scaling solution can never coexist with a scalar dominated scaling cosmology.

The accelerating scaling solutions ($P > 1$) are found for the CSO$(3, 3, 2)$-gauging and the CSO$(4, 3, 1)$-gauging. The first was found by Townsend in [102] where it was constructed by a reduction of a de Sitter vacuum in 5-dimensional SO$(3, 3)$-gauged supergravity. The second possibility with $P = 7$ is as far as we know not found before. The cosmologies of the CSO$(1, 1, 6)$-gauging were considered before [103] where the solutions were obtained from a reduction of seven-dimensional pure gravity on a group manifold.

Since the matrix $\partial_i\partial_j U$ evaluated at an extremum is not positive definite, the solutions are unstable.

Gauging	matter scaling	scalar dominated
1. CSO(1, 0, 7)	$P = 2/(3\gamma)$	$\nexists$
2. CSO(1, 1, 6)	$P = 2/(3\gamma)$	$\nexists$
3. CSO(2, 2, 4)	$\nexists$	$P = 1$
4. CSO(3, 3, 2)	$\nexists$	$P = 3$
5. CSO(4, 3, 1)	$\nexists$	$P = 7$

Table 4.1 $\mathcal{N} = 8$ *gaugings and their scaling solutions. Whenever a certain type of scaling solution exists, it is indicated in the table via the power of the corresponding scale factor.* $\nexists$ *means that there does not exist a scaling solution of the type under consideration.*

4.4 Higher-dimensional origin

In the previous section, we obtained accelerating scaling cosmologies from CSO-gaugings in $\mathcal{N} = 8$ gauged supergravity. As we are going to review in this section, these solutions can be viewed as solutions of 11-dimensional supergravity. We have thus obtained accelerating cosmologies from a higher-dimensional supergravity. Generally, obtaining accelerating universes from 10- or 11-dimensional supergravity by performing a 'reasonable' compactification, is considered a hard problem due to the existence of a no-go theorem [89, 90, 91]. We will first briefly review the arguments behind this theorem. We will then give the higher-dimensional origin of the scaling solutions we found and mention how they evade the no-go theorem.

A no-go theorem and how to evade it

Consider a warped compactification with the following metric:

$$\mathrm{d}s^2 = \Omega^2(y) g_4(x)_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + g_n(y)_{ab} \mathrm{d}y^a \mathrm{d}y^b \,, \tag{4.34}$$

where g_n denotes the metric on the n-dimensional internal manifold $\mathcal{M}_n$ (with coordinates $y^a, a = 4, \cdots, 3+n$), while g_4 denotes the metric on the 4-dimensional non-compact spacetime (with coordinates $x^\mu, \mu = 0, \cdots, 3$). The function $\Omega(y)$ denotes the warp factor and only depends on the internal coordinates. We will furthermore take it non-vanishing, in order to avoid singularities. Calculating the Ricci-tensor then gives:

$$(R_{4+n})_{00} = (R_4)_{00} + \frac{1}{4} \Omega^{-2}(y) \Box_n \Omega^4(y) \,, \tag{4.35}$$

where $\Box_n$ denotes the Laplacian in the internal manifold and R_{4+n}, R_4 denote the Ricci-tensors of the full metric (4.34) and the metric g_4 respectively. Multiplying (4.35) on both sides with $\Omega^2(y)$ and integrating the resulting equation over the internal manifold, one obtains:

$$\int_{\mathcal{M}_n} \mathrm{d}y^n \, \Omega^2(y) (R_{4+n})_{00} = (R_4)_{00} \int_{\mathcal{M}_n} \mathrm{d}y^n \, \Omega^2(y) \,. \tag{4.36}$$

Note that we dropped the term $\frac{1}{4}\int_{\mathcal{M}_n} \mathrm{d}y^n\, \Box_n \Omega^4(y)$ because we assumed that the internal manifold has no boundary. From (4.36), one immediately notices that

$$(R_{4+n})_{00} > 0 \Rightarrow (R_4)_{00} = -3\frac{\ddot{a}}{a} > 0\,. \tag{4.37}$$

In order to have an accelerating universe, one needs that $(R_{4+n})_{00}$ is negative in some part of space-time. Unfortunately, the energy-momentum tensors of 10- and 11-dimensional supergravity are such that the Einstein equations imply that $(R_{4+n})_{00}$ is positive.

Note, however, that this argument involves a lot of assumptions. One has for instance assumed that the integrals in (4.36) are well-defined. An easy way to evade this consists in giving up the compactness of the internal space. In case of these non-compactifications, one can find de Sitter vacua, see e.g., [47, 83, 102]. The disadvantage of this is that the four-dimensional mass spectrum is continuous and there seems to be no obvious way to circumvent this problem.

Even when considering true compactifications, there exist various ways to evade the no-go theorem. Suppose one has obtained a four-dimensional effective theory from a true compactification, consisting of scalars coupled to gravity with a potential. Consider the case in which there is only one scalar for simplicity. The second equation in (2.35) implies that, when the four-dimensional scalar potential is positive, acceleration is possible whenever $\dot{\phi} = 0$. As was argued in for instance [86], solutions in which $\dot{\phi}$ passes through zero will appear rather generically. Saying that the no-go theorem can not be evaded then almost boils down to saying that these potentials are never positive. This statement is however false. All that can be inferred about these potentials is that they have no positive stationary points [104, 105]. Finding de Sitter vacua (in which the scalar fields are constant) from compactification is thus a very hard problem. On the other hand, if one allows some of the scalars to depend on time, one can find accelerating cosmologies in models that are obtained via compactification of a higher-dimensional theory. Note that the time-dependent scalars correspond to some of the moduli of the internal manifold, implying that the metric on $\mathcal{M}_n$ is time-dependent. In the derivation of the no-go theorem, we however assumed that the internal metric is time-independent. Using time-dependent internal manifolds can thus evade the no-go theorem. An example of this approach is given in [105]. Essentially, they consider compactifications on Einstein spaces of negative curvature. The isometry groups of these spaces are non-compact. So although these spaces are non-compact, one can make them compact by modding out by a discrete subgroup of the isometry group.

Note that also the fact that $(R_{4+n})_{00}$ is positive is an assumption. Although true in 10- or 11-dimensional supergravity, in the context of string or M-theory, this assumption can be easily violated. Indeed, string theory contains orientifolds that can (at least locally) violate this assumption [106]. Also the introduction of branes can lead to various nonperturbative corrections to the potential that can change this picture. It has been shown in [107, 108] that after introduction of such effects one can indeed obtain de Sitter vacua.

Higher-dimensional origin of the scaling solutions

In [47], it was shown that the non-compact gaugings are associated with 11-dimensional supergravity solutions that have a non-compact internal space. For the CSO(p, q, r)-gaugings, the internal space $\mathcal{H}^{p,q,r}$ is a hypersurface in $\mathbb{R}^8$, defined by the following equation:

$$T_{AB}z^A z^B = R^2 \quad \text{and} \quad T = L^T \eta L\,, \tag{4.38}$$

where z^A are Cartesian coordinates of $\mathbb{R}^8$ and R is determined by the flux of the 4-form field strength in 11 dimensions[7].

The metric on the hypersurface is then induced from the Euclidean metric on $\mathbb{R}^8$. Given a solution in 4 dimensions with a metric g_4 and some scalars as only non-vanishing fields, the 11-dimensional metric g_{11} is then determined as:

$$g_{11} = \Delta^{\frac{2}{3}} g_4(x) + \Delta^{-\frac{1}{3}} g_{\mathcal{H}}(x,y)\,, \tag{4.39}$$

where $g_{\mathcal{H}}(x,y)$ is the metric on $\mathcal{H}^{p,q,r}$ and $\Delta(x,y)$ is a warp factor:

$$\Delta = \frac{T^2_{AB} z^A z^B}{R^2}\,. \tag{4.40}$$

From the explicit solutions for the scalar matrix $\mathcal{M}$, we notice that our scaling solutions correspond to SO$(p)\times$SO(q)-invariant directions in the scalar coset[8]:

$$T = e^{\frac{c\varphi}{2}} \begin{pmatrix} X\mathbb{1}_{p\times p} & 0 & 0 \\ 0 & -\bar{X}\mathbb{1}_{q\times q} & 0 \\ 0 & 0 & 0_{r\times r} \end{pmatrix}. \tag{4.41}$$

The constants X and $\bar{X}$ are the constant diagonal components of the $\mathrm{S}\ell(p+q,\mathbb{R})/\mathrm{SO}(p+q)$-scalar matrix $\tilde{\mathcal{M}} = \mathrm{diag}(X\mathbb{1}_p, \bar{X}\mathbb{1}_q)$.

We follow the same spirit of [83] and choose coordinates in which the Euclidean metric on $\mathbb{R}^8$ is given by:

$$\mathrm{d}s^2 = \mathrm{d}\sigma^2 + \sigma^2 \mathrm{d}\Omega^2_{p-1} + \mathrm{d}\tilde{\sigma}^2 + \tilde{\sigma}^2 \mathrm{d}\Omega^2_{q-1} + \sum_{A=p+q+1}^{8} \mathrm{d}z^A \mathrm{d}z^A\,, \tag{4.42}$$

with $\mathrm{d}\Omega^2_n$ the round metric on the unit n-sphere. In terms of these non-Cartesian coordinates, the hypersurface (4.38) is explicitly given by:

$$\sigma^2 - (\frac{\bar{X}}{X})\tilde{\sigma}^2 = \frac{R^2}{X} e^{-\frac{c}{2}\varphi} \sim t\,. \tag{4.43}$$

Because the ratio $\bar{X}/X$ appears often we call it λ. If we introduce new coordinates r, ρ in the following way:

$$\tilde{\sigma} = \rho r, \qquad \sigma = \rho(1+\lambda r^2)^{1/2}\,, \tag{4.44}$$

then the hypersurface (4.43) is defined by $\rho^2 = \frac{R^2}{X} e^{-\frac{c}{2}\varphi}$ and the metric on $\mathcal{H}^{p,q,r}$ is found to be:

$$\mathrm{d}s^2_{\mathcal{H}} = \tfrac{R^2}{X} e^{-\frac{c}{2}\varphi} \Big[\frac{1+(\lambda+\lambda^2)r^2}{1+\lambda r^2} \mathrm{d}r^2 + (1+\lambda r^2)\mathrm{d}\Omega^2_{p-1} + r^2 \mathrm{d}\Omega^2_{q-1} \Big] + \sum_{u=p+q+1}^{8} (\mathrm{d}z^A)^2\,. \tag{4.45}$$

[7] The flux parameter R also corresponds to the inverse of the gauge coupling constant.

[8] In terms of the SO$(p)\times$SO(q) invariant scalars s and t defined in [54, 109, 83], our solutions have constant s and running t.

The warp factor is:

$$\Delta = X(1+(\lambda+\lambda^2)r^2)\, e^{\frac{c}{2}\varphi} \tag{4.46}$$

and the 11-dimensional metric is then given by (4.39).

Note that the uplifted solutions violate the aforementioned no-go theorem in two respects. First of all, they correspond to non-compactifications. One can also not compactify these spaces by modding out by an appropriate discrete set of isometries as was for instance done in [105]. Secondly, the internal metric (4.45) and the warp factor (4.46) both depend explicitly on the running scalar φ. The internal spaces are thus time-dependent.

4.5 Scaling solutions and de Sitter vacua

In section 4.3, we have found accelerating scaling solutions, that are however unstable. A natural question is then whether stable scaling solutions with eternal acceleration are possible at all in supergravity? If one lowers the amount of supersymmetry, then stable solutions are possible. In $\mathcal{N}=4$ gauged supergravity, a (non-accelerating) stable scaling solution was found in reference [87] and, as we shortly outline below, stable eternal accelerating scaling solutions can be present in $\mathcal{N}=2$ theories. These stability properties are similar for de Sitter vacua in supergravity, where stable vacua are only found for $\mathcal{N}\leq 2$.

The existence of stable scaling solutions in $\mathcal{N}=2$ gauged supergravity follows from the fact that there exist stable de Sitter vacua in $d=5$ [53]. Suppose one has a de Sitter critical point in 5-dimensional supergravity. One can then truncate the system to pure gravity and a positive cosmological constant, Λ. If we reduce this theory on a circle and truncate the Kaluza-Klein vector, using the following usual metric ansatz:

$$\mathrm{d}s^2 = e^{\sqrt{\frac{1}{3}}\varphi}\mathrm{d}s_4^2 + e^{-2\sqrt{\frac{1}{3}}\varphi}\,\mathrm{d}z^2\,, \tag{4.47}$$

we find:

$$S_4 = \int \mathrm{d}x^4\sqrt{-g_4}[R - \tfrac{1}{2}(\partial\varphi)^2 - \Lambda\, e^{\sqrt{\frac{1}{3}}\,\varphi}]\,. \tag{4.48}$$

This theory has an accelerating scaling solution:

$$\mathrm{d}s_4^2 = -\mathrm{d}t^2 + t^6\mathrm{d}x_3^2\,, \qquad \sqrt{\frac{1}{3}}\,\varphi = -2\ln t + C\,, \tag{4.49}$$

where C is a constant. Plugging this in the metric ansatz (4.47) and redefining time via $\tau \sim \ln t$, we find that the uplift of the 4-dimensional scaling solution is:

$$\mathrm{d}s_5^2 = -\mathrm{d}\tau^2 + e^{C'\,\tau}\,\mathrm{d}x_4^2\,, \tag{4.50}$$

where C' is also constant. This is indeed a 5-dimensional de Sitter universe in flat FLRW-coordinates. When the de Sitter solution is stable, so is the scaling solution obtained via reduction.

4.6 Conclusion

In this chapter, we investigated scaling solutions in $\mathcal{N} = 8$ gauged supergravity. When restricted to the dilatons, the potential becomes a sum of exponentials. We showed that for the CSO(p, q, r)-gaugings, the exponentials exhibit a special form, namely they have so-called affine couplings. This special form is necessary for the existence of scaling solutions. We find eternal accelerating solutions for which the barotropic fluid vanishes. From the point of view of 11-dimensional supergravity, the solutions correspond to geometries with non-compact and time-dependent internal spaces. If we assume the presence of a barotropic fluid, we also find matter scaling solutions. The solutions we obtained have one running scalar and all other scalars are trapped in a saddle point or maximum of the potential, and are therefore unstable.

It would be interesting to study scaling solutions in non-maximal supergravities. This would entail a detailed study of the possible gaugings and the structure of the potentials that appear in these theories. In this context, we remarked that knowledge of de Sitter vacua in 5 dimensions, leads to knowledge of scaling cosmologies in 4 dimensions.

Scaling solutions correspond to critical points of the cosmological dynamical system and therefore describe the early- or late-time behavior of general cosmological solutions. From this point of view, one can note a similarity with the generic behavior of cosmological solutions in ungauged supergravity. Apart from the intermediate billiard behavior that was mentioned in section 3.5.2, it was found in [55, 56] that the asymptotic behavior of cosmological solutions in maximal ungauged supergravity corresponds to metrics with power-law scale factors and constant axionic fields. Our scaling solutions are thus very similar to the asymptotic behavior of cosmological billiard dynamics in ungauged supergravity. A more complete analysis would involve solutions that interpolate between scaling vacua in order to understand how the cosmic billiard behavior is realized in gauged extended supergravity.

Chapter 5

Dirac actions for D-branes in flux backgrounds

5.1 Introduction

5.1.1 D-branes

One of the great discoveries in string theory concerns the role played by Dp-branes in the theory [110]. These can be introduced by considering so-called Dirichlet boundary conditions for open strings. This means that the endpoints of the string are confined to a $(p+1)$-dimensional hypersurface. The string can move freely along the hypersurface, but is not allowed to move in the transverse directions. Dp-branes are thus introduced as $(p+1)$-dimensional hypersurfaces on which open strings end.

Quantization of open strings with Dirichlet boundary conditions leads to states that correspond to fields that only depend on the coordinates along the brane, i.e., to fields that live on the world-volume of the brane. The open string spectrum then usually contains massless states and a tower of highly massive states. Often, one also finds a mode with negative mass squared, i.e., a tachyon. This tachyonic mode represents an instability of the D-brane. For certain values of p, stable D-branes can occur in superstring theories. In these cases, the spectrum of open strings that start and end on the brane is tachyon-free. It turns out that in type IIA string theory, stable D-branes occur for even values of p, while in type IIB string theory stable branes have odd values of p. In these cases, the D-branes carry a conserved electric or magnetic Ramond-Ramond charge that ensures their stability. These stable D-branes then also preserve half of the 32 supersymmetries of the type II string theory.

The massless states in the open string spectrum correspond to $9-p$ real scalars ϕ, one vector A_α, $\alpha = 0, \cdots, p$, as well as some fermions. The scalars can be interpreted as describing transverse fluctuations of the brane. They thus describe the shape of the brane. Although D-branes were introduced as rigid hyperplanes, the excitations of open strings ending on them imply that they can be seen as dynamical objects in string theory. Together, these massless states comprise the vector multiplet of a $(p+1)$-dimensional U(1) gauge theory with 16 supercharges, indicating that the brane indeed preserves 16 supersymmetries. The low energy dynamics of the massless open string states on a Dp-brane describes a supersymmetric gauge theory in a $(p+1)$-dimensional space-time. This low energy dynamics is described by an effective action. The bosonic parts of the effective actions describing the world-volume dynamics of D-branes are well known. The fermionic part of these actions is however less well known. It is on this fermionic part that this chapter will mainly focus.

Finally, let us mention what happens to D-branes when applying T-duality. We have already mentioned in section 2.4.2 that T-duality exchanges the type IIA and type IIB supergravity theories. Under T-duality, a Dp-brane is mapped to either a D$(p+1)$- or a D$(p-1)$-brane. In the presence of a space-like compact Killing direction x^9, the background fields transform under T-duality as in (2.133). If the 9-direction in which T-duality is performed lies along the world-volume of the Dp-brane, one ends up with a D$(p-1)$-brane. The 9-component of the world-volume gauge field A then becomes a new physical scalar field that describes the fluctuation of the D$(p-1)$-brane in the new transverse direction. If one performs a T-duality along a direction transverse to the Dp-brane, one ends up with a D$(p+1)$-brane. In this case, the world-volume scalar that describes fluctuations in the T-duality direction, becomes a component of the gauge vector. The action of T-duality on the fermionic world-volume fields has been worked out in [111].

5.1.2 Fluxes

As superstring theories naturally live in 10 space-time dimensions, one usually assumes that 6 spatial dimensions are compact, in order to make contact with our four-dimensional reality. Moreover, if string theory is really a unified theory of nature, it should definitely contain the Standard Model of particle physics, which has been tested to great accuracy. Originally, the string theory that was found to be most promising for phenomenological purposes was the heterotic theory with $E_8 \times E_8$ gauge group, as this is large enough to contain the Standard Model gauge group. As was shown in [112], this theory allows for a space-time solution of the form $\mathcal{M}_4 \times \mathrm{CY}_3$, where $\mathcal{M}_4$ is four-dimensional Minkowski space-time and CY_3 is a Calabi-Yau threefold. These vacua moreover preserve only 1/4 of the original 16 supersymmetries. Compactification of heterotic string theory on Calabi-Yau manifolds therefore leads to $\mathcal{N} = 1$ theories in 4 dimensions that might be phenomenologically viable. One of the largest drawbacks of these compactifications is the fact that they lead to a large number of moduli in the lower-dimensional effective theory. These moduli describe changes in size (so-called Kähler moduli) or shape (complex structure moduli) of the internal Calabi-Yau manifold and correspond to massless scalar fields that interact at least gravitationally. As such, they usually lead to disagreement with observations. Generically, coupling constants in the four-dimensional effective theory depend on these moduli. As their vacuum expectation values are undetermined in conventional Calabi-Yau compactifications, this also leads to a problem of predictivity.

This so-called moduli problem can be solved by considering more general compactifications, including fluxes (see refs. [113, 114, 115] for some reviews). This means that one allows for non-trivial fluxes associated to certain n-form fields in the theory, through some non-trivial cycles of the compactification manifold. In other words, if A denotes an n-form potential, with $(n+1)$-form field strength $F = dA$, and γ_{n+1} denotes a non-trivial cycle in the internal manifold, one allows for non-zero values of

$$\int_{\gamma_{n+1}} F\,. \tag{5.1}$$

In the four-dimensional effective theory, these fluxes induce a potential for some of the moduli, thereby fixing some of them at minima of the potential. By taking also various nonperturbative corrections into account, one can generically fix all moduli. Fluxes thus play an important role in searching for realistic vacua of string theory. Moreover, they are also very important in the construction of de Sitter vacua in string theory as was shown in [107].

5.1.3 D-branes in flux backgrounds

In the previous section, we emphasized that compactifications with fluxes play a crucial role in obtaining phenomenologically interesting lower-dimensional effective theories from string theory. Due to the fact that interesting gauge theories live on their world-volume, also D-branes are important in the construction of phenomenological string models. There is thus a natural need to study D-branes that are put in general supergravity backgrounds where fluxes are switched on. The aim of this chapter is to study the world-volume actions that describe D-brane dynamics in such generic supergravity backgrounds.

The bosonic part of the D-brane action in a generic background is well understood (at least in the case of a single D-brane). The fermionic part is however less well understood. In principle, a complete D-brane action is known in a superspace formalism [116, 117]. Although such a superspace action is complete and elegant, the precise manner in which the background fields enter the fermionic terms of the action is hidden. Any explicit calculation or consideration involving the world-volume fermions cannot be done only using such an implicit superspace formalism. The explicit form of these terms is indeed necessary in several interesting situations where the contribution of the world-volume fermionic dynamics becomes relevant. For example, they are necessary to write down the effect of background fluxes in the effective action governing some phenomenologically interesting brane configurations [118, 119, 120]. Also, Euclidean brane configurations are sources of nonperturbative corrections in lower-dimensional effective theories obtained by compactification [121, 122, 123]. Finally, the knowledge of the explicit form of the fermionic terms is indeed necessary for any kind of quantum world-volume computation (see for example [124]).

An important step towards the understanding of the fermionic terms in the Dp-brane actions in backgrounds with fluxes was obtained in [8, 9]. In these papers, the fermionic action was given for any Dp-brane on any (bosonic) supergravity background to quadratic order in the fermions. However, the final results of [8, 9] are rather complicated. For instance, new kinetic terms appear that apparently destroy the Dirac-like form of the actions. It is the aim of this chapter to study these results further. In this chapter, we will show how the seemingly complicated actions of [8, 9] can be rewritten into a more geometrical form that naturally generalizes the Dirac-like operator. Our final result contains a kinetic term of the schematic form

$$(M^{-1})^{\alpha\beta}\Gamma_\beta\nabla_\alpha\ , \tag{5.2}$$

where the precise form of $M_{\alpha\beta}$ will be given later on. This kinetic term (5.2) is still not in canonical form. We will, however, see how a redefinition of the world-volume metric leads to a fermionic action in canonical form.

The organization of this chapter is as follows. In section 5.2.1, we recall some results concerning the world-volume actions describing D-brane dynamics. We recall the structure of the bosonic part of the action and focus on the results of the papers [8, 9] giving the explicit form for the fermionic bilinear action for any Dp-brane on any (bosonic) supergravity background. We then show how it is possible to rewrite this fermionic action in a new, more geometrical form. In section 5.3, we comment on some important symmetries of the D-brane action and how to fix some of them in order to obtain a description in terms of physical degrees of freedom. As an explicit check on our result for the fermionic action, we consider its invariance under T-duality in section 5.4. In section 5.5, we then explain how one can obtain a canonical action. For some notational aspects concerning this chapter, we refer to appendix C.

5.2 The quadratic fermionic action on a general background

D-brane actions on general backgrounds have been formulated using the superspace formalism in [116, 117]. Unfortunately, although superspace actions are in principle complete, the explicit couplings between the physical fields as well as the fermionic sector of the theory are quite obscure in this formalism. In order to be able to make any calculations involving the world-volume fermions, one has to expand these superactions in components. If one starts from the superactions of [116, 117], such a calculation can be really cumbersome and requires a case by case study (see for example [125] and the recent [126]). The fermionic action for any D-brane on any background was obtained in [8, 9] to quadratic order in the fermions by following a somehow different route. The starting point was the normal coordinate expansion of the M2-brane superaction presented in [127]. Then, by dimensional reduction and T-duality all the Dp-brane actions quadratic in the fermions were derived in a unified and compact form dictated by the consistency with T-duality. We will then start from the results of [8, 9] and show how these results can be recast in a more geometrical form.

5.2.1 The quadratic fermionic action on a general background

The dynamics of the world-volume of the brane is usually described in the following way. The world-volume of the brane is parametrized by the coordinates ξ^α, $\alpha = 0, \ldots, p$. The embedding of the brane in the full 10-dimensional space-time is then given by specifying the 10-dimensional space-time coordinates x^m, $m = 0, \ldots, 9$ as functions of these world-volume coordinates : $x^m = x^m(\xi^\alpha)$. Note that from the point of view of the world-volume, the $x^m(\xi^\alpha)$ correspond to scalar fields living on the world-volume. There is furthermore also a U(1)-vector field A^α living on the world-volume, as well as a pair of Majorana-Weyl fermions θ_1 and θ_2. In the type IIA case, θ_1 and θ_2 have opposite chirality and they can be combined in one 32-component Majorana spinor $\theta = \theta_1 + \theta_2$. In type IIB, both spinors in this pair have the same positive chirality. Our conventions regarding spinors are summarized in appendix C.

The bosonic part of the Dp-brane action is given by:

$$S_{\mathrm{D}p}^{(B)} = -\tau_{\mathrm{D}p} \int \mathrm{d}^{p+1}\xi\, e^{-\Phi} \sqrt{-\det(g+\mathcal{F})} + \tau_{\mathrm{D}p} \int \sum_n P[C_{(n)}]\, e^{\mathcal{F}}, \tag{5.3}$$

where $\tau_{\mathrm{D}p}$ is the brane tension[1]. The first term in this action is usually called the Dirac-Born-Infeld (DBI) action, while the second term is known as the Chern-Simons (CS) action. Denoting the background metric by G_{mn}, $g_{\alpha\beta}$ is its pull-back on the world-volume:

$$g_{\alpha\beta} = \frac{\partial x^m}{\partial \xi^\alpha}\frac{\partial x^n}{\partial \xi^\beta} G_{mn}\,. \tag{5.4}$$

Pull-backs of background fields on the world-volume will also be indicated by the symbol $P[.]$; thus $g_{\alpha\beta}$ could alternatively be written as $P[G]_{\alpha\beta}$. The other field $\mathcal{F}_{\alpha\beta}$ appearing in the DBI action is a combination of the field strength $f = dA$ of the gauge field living on the brane and the pull-back of the background NS-NS two-form B_{mn}:

$$\mathcal{F}_{\alpha\beta} = P[B]_{\alpha\beta} + f_{\alpha\beta}\,. \tag{5.5}$$

The CS action involves the same combination $\mathcal{F}$ as well as the pull-backs of the R-R forms $C_{(n)}$ on the world-volume. The notation of the CS action employed in (5.3) is a formal one. The exponential of $\mathcal{F}$ appearing in there should be interpreted via its series expansion:

$$e^{\mathcal{F}} = \sum_{n=0} \frac{1}{n!}\mathcal{F}^n\,, \tag{5.6}$$

where multiplication of forms should be understood as taking wedge products of forms. The CS action then involves a formal sum of wedge products of this exponential with R-R forms. It is then understood that the CS action consists of the forms in this formal sum that have the correct degree for the integration to make sense, i.e., one only keeps the forms of degree $p+1$.

The fermionic part of the world-volume action was calculated to quadratic order in the fermions in [8, 9] and is given by:

$$S_{\mathrm{D}p}^{(F)} = \frac{\tau_{\mathrm{D}p}}{2} \int \mathrm{d}^{p+1}\xi\, e^{-\Phi} \sqrt{-\det(g+\mathcal{F})}\,\bar{\theta}(1-\Gamma_{\mathrm{D}p})(\Gamma^\alpha D_\alpha - \Delta + L_{\mathrm{D}p})\theta, \tag{5.7}$$

where Γ_α are pull-backs of the gamma matrices Γ_m. This action is determined by operators $\Gamma_{\mathrm{D}p}$ and $L_{\mathrm{D}p}$. For type IIA D-branes, these are given by:

$$\Gamma_{\mathrm{D}(2n)} = \sum_{q+r=n} \frac{(-)^{r+1}(\Gamma_{(10)})^{r+1}\epsilon^{\alpha_1\ldots\alpha_{2q}\beta_1\ldots\beta_{2r+1}}}{q!(2r+1)!2^q\sqrt{-\det(g+\mathcal{F})}} \mathcal{F}_{\alpha_1\alpha_2}\cdots\mathcal{F}_{\alpha_{2q-1}\alpha_{2q}}\Gamma_{\beta_1\ldots\beta_{2r+1}}\,, \tag{5.8}$$

$$L_{\mathrm{D}(2n)} = \sum_{q\geq 1, q+r=n} \frac{(-)^{r+1}(\Gamma_{(10)})^{r+1}\epsilon^{\alpha_1\ldots\alpha_{2q}\beta_1\ldots\beta_{2r+1}}}{q!(2r+1)!2^q\sqrt{-\det(g+\mathcal{F})}} \mathcal{F}_{\alpha_1\alpha_2}\cdots\mathcal{F}_{\alpha_{2q-1}\alpha_{2q}}\Gamma_{\beta_1\ldots\beta_{2r+1}}{}^{\gamma}D_\gamma\,, \tag{5.9}$$

[1] In terms of the string coupling constant g_s and the Regge slope α', $\tau_{\mathrm{D}p}^{-1}$ is given by $= (2\pi)^p(\alpha')^{\frac{p+1}{2}} g_s$.

while for type IIB D-branes:

$$\Gamma_{\mathrm{D}(2n+1)} = \sum_{q+r=n+1} \frac{(-)^{r+1}(\mathrm{i}\sigma_2)(\sigma_3)^r \epsilon^{\alpha_1 \ldots \alpha_{2q}\beta_1\ldots\beta_{2r}}}{q!(2r)!2^q\sqrt{-\det(g+\mathcal{F})}} \mathcal{F}_{\alpha_1\alpha_2}\cdots\mathcal{F}_{\alpha_{2q-1}\alpha_{2q}}\Gamma_{\beta_1\ldots\beta_{2r}} \,, \quad (5.10)$$

$$L_{\mathrm{D}(2n+1)} = \sum_{q\geq 1, q+r=n+1} \frac{(-)^{r+1}(\mathrm{i}\sigma_2)(\sigma_3)^r \epsilon^{\alpha_1 \ldots \alpha_{2q}\beta_1\ldots\beta_{2r}}}{q!(2r)!2^q\sqrt{-\det(g+\mathcal{F})}} \mathcal{F}_{\alpha_1\alpha_2}\cdots\mathcal{F}_{\alpha_{2q-1}\alpha_{2q}}\Gamma_{\beta_1\ldots\beta_{2r}}{}^{\gamma}D_\gamma \,. \quad (5.11)$$

In the above expressions, D_m (which enters through its pull-back D_α) and Δ are the operators appearing in the supersymmetry transformation laws of the background gravitino and dilatino (2.130), (2.131) and (2.132).

Note that a naive counting of degrees of freedom leads to a paradoxical result. There are 10 scalar fields x^m and one gauge vector A^α with $p-1$ physical degrees of freedom. So, it seems that in total there are $9+p$ bosonic degrees of freedom. On the other hand, the fields θ_1 and θ_2 together have 32 real components. The Dirac equation however implies that only half of these survive as independent propagating degrees of freedom. So, it seems that the number of physical fermionic degrees of freedom is 16. The numbers of bosonic and fermionic degrees of freedom therefore do not seem to match, in contradiction to what supersymmetry tells us. The solution to this paradox is the observation that the world-volume action enjoys a number of extra local symmetries. First of all, it is invariant under world-volume diffeomorphisms, i.e., reparametrizations of the world-volume coordinates ξ^α. Secondly, it is also invariant under a fermionic symmetry, called kappa-symmetry. It was shown in [9] that the full action given by (5.3) plus (5.7), is symmetric under the following kappa-symmetry transformations:

$$\begin{aligned} \delta_\kappa\bar\theta &= \bar\kappa(1+\Gamma_{\mathrm{D}p}) \,, \\ \delta_\kappa x^m &= -\frac{1}{2}\delta_\kappa\bar\theta\Gamma^m\theta \,, \\ \delta_\kappa A_\alpha &= \frac{1}{2}\delta_\kappa\bar\theta\tilde\Gamma_{(10)}\Gamma_\alpha\theta - \frac{1}{2}B_{\alpha m}\delta_\kappa\bar\theta\Gamma^m\theta \,, \end{aligned} \quad (5.12)$$

where

$$\text{for type IIA} : \tilde\Gamma_{(10)} = \Gamma_{(10)} \,, \qquad \text{for type IIB} : \tilde\Gamma_{(10)} = \Gamma_{(10)}\otimes\sigma_3 \,, \quad (5.13)$$

and κ is the fermionic parameter of the kappa-symmetry. It turns out that the number of physical fermionic degrees of freedom is reduced by a factor of one half by fixing the kappa-symmetry, thus leaving only 8 physical fermionic degrees of freedom. Reparametrization invariance on the other hand allows one to fix $p+1$ of the x^m, leaving only $9-p$ of the x^m as physical degrees of freedom. Together with the $p-1$ degrees of freedom of the gauge vector this then also gives 8 bosonic degrees of freedom, as required by supersymmetry. Later on in this chapter, we will come back to the problem of how kappa-symmetry and reparametrization invariance can be fixed in a suitable fashion.

In the restricted case when $\mathcal{F}_{\alpha\beta}=0$, the action for the fermions takes an explicit canonical Dirac-like form as can be seen from (5.7). The background fluxes then contribute with mass terms through the operators D_α and Δ. The effect of a non-zero $\mathcal{F}_{\alpha\beta}$ is twofold. First, it is included in the $\Gamma_{\mathrm{D}p}$ operators (5.8) and (5.10). Secondly, we have the new terms $L_{\mathrm{D}p}$, which not only add new non-derivative couplings between background and world-volume fields, but also add new kinetic terms. The rest of this section will then be devoted to showing how these new terms can be reorganized such that the fermionic Lagrangian (5.7) is written in a more geometrical form where the effect of the field $\mathcal{F}_{\alpha\beta}$ can be reabsorbed in a shift of the pulled-back world-volume metric $g_{\alpha\beta}$. This simplifies the final form of the action considerably and makes its structure more transparent.

First of all, let us introduce the operator

$$\Gamma^{(0)}_{\mathrm{D}p} = \frac{\epsilon^{\alpha_1\ldots\alpha_{p+1}}}{(p+1)!\sqrt{-\det g}}\Gamma_{\alpha_1\ldots\alpha_{p+1}} \ . \tag{5.14}$$

Note that $\Gamma^{(0)}_{\mathrm{D}p}$ squares to $(-1)^{(p-1)(p-2)/2}$. By using the formula

$$\epsilon^{\alpha_1\ldots\alpha_r\beta_{r+1}\ldots\beta_{p+1}}\Gamma_{\beta_{r+1}\ldots\beta_{p+1}} = (-)^{r(r-1)/2}(p+1-r)!\sqrt{-\det g}\,\Gamma^{\alpha_1\ldots\alpha_r}\Gamma^{(0)}_{\mathrm{D}p}\,, \tag{5.15}$$

one can write the chiral operators (5.8) and (5.10) in the following form [117]:

$$\Gamma_{\mathrm{D}(2n)} = \frac{\sqrt{-\det g}}{\sqrt{-\det(g+\mathcal{F})}}\Gamma^{(0)}_{\mathrm{D}(2n)}(\Gamma_{(10)})^{n+1}\sum_q \frac{(-)^q(\Gamma_{(10)})^q}{q!2^q}\Gamma^{\alpha_1\ldots\alpha_{2q}}\mathcal{F}_{\alpha_1\alpha_2}\cdots\mathcal{F}_{\alpha_{2q-1}\alpha_{2q}} \ , \tag{5.16}$$

$$\Gamma_{\mathrm{D}(2n+1)} = \frac{\sqrt{-\det g}}{\sqrt{-\det(g+\mathcal{F})}}\Gamma^{(0)}_{\mathrm{D}(2n+1)}(\sigma_3)^{n+1}(-\mathrm{i}\sigma_2)\sum_q \frac{(\sigma_3)^q}{q!2^q}\Gamma^{\alpha_1\ldots\alpha_{2q}}\mathcal{F}_{\alpha_1\alpha_2}\cdots\mathcal{F}_{\alpha_{2q-1}\alpha_{2q}} \ . \tag{5.17}$$

From (5.9), one analogously finds that in the type IIA case

$$\begin{aligned} L_{\mathrm{D}(2n)} &= \frac{-\sqrt{-\det g}}{\sqrt{-\det(g+\mathcal{F})}}\Gamma^{(0)}_{\mathrm{D}(2n)}\left(\Gamma_{(10)}\right)^n \times \\ &\quad \sum_{q\geq 1}\frac{\left(-\Gamma_{(10)}\right)^{q-1}}{(q-1)!2^{q-1}}\Gamma^{\alpha_1\ldots\alpha_{2q-1}}\mathcal{F}_{\alpha_1\alpha_2}\ldots\mathcal{F}_{\alpha_{2q-3}\alpha_{2q-2}}\mathcal{F}_{\alpha_{2q-1}}{}^{\gamma}D_\gamma \ , \end{aligned} \tag{5.18}$$

which can in turn be rewritten as

$$\begin{aligned} L_{\mathrm{D}(2n)} &= -\Gamma_{\mathrm{D}(2n)}\Gamma_{(10)}\Gamma^\alpha\mathcal{F}_\alpha{}^\beta D_\beta - \frac{\sqrt{-\det g}}{\sqrt{-\det(g+\mathcal{F})}}\Gamma^{(0)}_{\mathrm{D}(2n)}\left(\Gamma_{(10)}\right)^{n+1} \times \\ &\quad \sum_{q\geq 2}\frac{\left(-\Gamma_{(10)}\right)^{q-2}}{(q-2)!2^{q-2}}\Gamma^{\alpha_1\ldots\alpha_{2q-3}}\mathcal{F}_{\alpha_1\alpha_2}\ldots\mathcal{F}_{\alpha_{2q-5}\alpha_{2q-4}}\mathcal{F}_{\alpha_{2q-3}}{}^{\gamma_1}\mathcal{F}_{\gamma_1}{}^{\gamma_2}D_{\gamma_2} \ . \end{aligned} \tag{5.19}$$

By iterating this last step (and by doing an analogous calculation for the IIB case), the following formulae can be found:

$$\begin{aligned} L_{\mathrm{D}(2n)} &= -\Gamma_{\mathrm{D}(2n)}\sum_{q\geq 1}(\Gamma_{(10)})^q(\mathcal{F}^q)^{\alpha\beta}\Gamma_\alpha D_\beta \ , \\ L_{\mathrm{D}(2n+1)} &= -\Gamma_{\mathrm{D}(2n+1)}\sum_{q\geq 1}(-\sigma_3)^q(\mathcal{F}^q)^{\alpha\beta}\Gamma_\alpha D_\beta \ , \end{aligned} \tag{5.20}$$

where $(\mathcal{F}^q)^{\alpha\beta} = \mathcal{F}^{\alpha}{}_{\gamma_1}\mathcal{F}^{\gamma_1}{}_{\gamma_2}\cdots\mathcal{F}^{\gamma_{q-2}}{}_{\gamma_{q-1}}\mathcal{F}^{\gamma_{q-1}\beta}$.

By putting these results together, it is easy to see that we can write the fermionic action (5.7) in the compact and elegant form:

$$S^{(F)}_{\mathrm{D}p} = \frac{\tau_{\mathrm{D}p}}{2}\int \mathrm{d}^{p+1}\xi\, e^{-\Phi}\sqrt{-\det(g+\mathcal{F})}\,\bar\theta(1-\Gamma_{\mathrm{D}p})\big[(\tilde M^{-1})^{\alpha\beta}\Gamma_\beta D_\alpha - \Delta\big]\theta\,. \qquad (5.21)$$

where the operator $M_{\alpha\beta}$ is defined as

$$\tilde M_{\alpha\beta} = g_{\alpha\beta} + \tilde\Gamma_{(10)}\mathcal{F}_{\alpha\beta}\,. \qquad (5.22)$$

One can now see that the operators $L_{\mathrm{D}p}$ that in the action (5.7) were disturbing, are the source of the natural geometrical coupling to the matrix

$$M_{\alpha\beta} = g_{\alpha\beta} + \mathcal{F}_{\alpha\beta}\;. \qquad (5.23)$$

This matrix substitutes the metric already in the bosonic action or in the definition of the natural volume element defined on the brane. In this way, we see how the effect of the $\mathcal{F}_{\alpha\beta}$ field, given by the couplings contained in the operators $L_{\mathrm{D}p}$, can be schematically reabsorbed in the following redefinition of the kinetic term:

$$g^{\alpha\beta}\Gamma_\beta\nabla_\alpha \quad\rightarrow\quad (\tilde M^{-1})^{\alpha\beta}\Gamma_\beta D_\alpha\;. \qquad (5.24)$$

We will discuss the effect of non-zero $\mathcal{F}_{\alpha\beta}$ on the world-volume geometry in more detail in section 5.5. First however, we will discuss how one can fix the κ-symmetry (5.12) and reparametrization invariance in order to obtain an action that is expressed in terms of physical fields. We will also establish the supersymmetry transformations of the physical fields under which the complete κ-fixed action is invariant. Finally, we will show how the action we have obtained is consistent with T-duality, giving an important check on our result.

5.3 κ-fixing and supersymmetry

In section 5.2.1, we already alluded to the fact that the action (5.21), once completed with the bosonic action (5.3), is invariant under world-volume diffeomorphisms and κ-symmetry (5.12). In particular, we have mentioned how these extra symmetries account for the correct number of physical bosonic and fermionic degrees of freedom. In this section, we would like to discuss some aspects regarding their gauge-fixing and the consequent effects on the way the possible background supersymmetries are realized on the brane.

In order to consider the problem of κ-fixing more clearly, it is convenient to write (5.12) in a double spinor convention for both type IIA and type IIB. Details about this convention are given in appendix C. In this notation, the first two transformation rules of (5.12) can be written in exactly the same form but with a $\Gamma_{\mathrm{D}p}$ given by[2] (for both type IIA and IIB):

$$\Gamma_{\mathrm{D}p} = (-)^p\Gamma^{(0)}_{\mathrm{D}p}(\sigma_3)^{\frac{p(p+1)}{2}}(\mathrm{i}\sigma_2)\frac{\sqrt{-\det g}}{\sqrt{-\det(g+\mathcal{F})}}\sum_q(\sigma_3)^q\frac{\Gamma^{\alpha_1\ldots\alpha_{2q}}}{q!2^q}\mathcal{F}_{\alpha_1\alpha_2}\cdots\mathcal{F}_{\alpha_{2q-1}\alpha_{2q}}\;. \qquad (5.25)$$

[2] Note that, for type IIA, the $\Gamma_{\mathrm{D}p}$ in double spinor notation is not the same as (5.16) but is given by $\Gamma^{\text{double}}_{\mathrm{D}p} = \sigma_1\Gamma_{\mathrm{D}p}\sigma_1$ due to the chosen representation of the charge conjugation matrix. On the other hand, the σ_1 factors in (C.16) have already been extracted from $\Gamma^{(0)}_{\mathrm{D}p}$, which is thus considered here as the diagonal matrix in the extension index.

On the other hand, the last transformation of (5.12) takes the form:

$$\delta_\kappa A_\alpha = -\frac{1}{2}\delta_\kappa\bar\theta\Gamma_\alpha\sigma_3\theta - \frac{1}{2}B_{\alpha m}\delta_\kappa\bar\theta\Gamma^m\theta \ . \tag{5.26}$$

As stressed in [128], it is important to note that the $\Gamma_{\mathrm{D}p}$ operators entering the κ-symmetry transformations (5.12) are off-diagonal:

$$\Gamma_{\mathrm{D}p} = \begin{pmatrix} 0 & \check\Gamma^{-1}_{\mathrm{D}p} \\ \check\Gamma_{\mathrm{D}p} & 0 \end{pmatrix} , \tag{5.27}$$

where

$$\check\Gamma_{\mathrm{D}p} = (-)^{\frac{(p-2)(p-3)}{2}}\Gamma^{(0)}_{\mathrm{D}p}\frac{\sqrt{-\det g}}{\sqrt{-\det(g+\mathcal{F})}}\sum_q \frac{\Gamma^{\alpha_1\ldots\alpha_{2q}}}{q!2^q}\mathcal{F}_{\alpha_1\alpha_2}\cdots\mathcal{F}_{\alpha_{2q-1}\alpha_{2q}} \ , \tag{5.28}$$

and $\check\Gamma^{-1}_{\mathrm{D}p}(\mathcal{F}) = (-)^{\frac{(p-1)(p-2)}{2}}\check\Gamma_{\mathrm{D}p}(-\mathcal{F})$. This property allows one to fix the κ-symmetry in a simple manner. Using an irreducible 16-dimensional spinor κ for which $\tilde\Gamma_{(10)}\kappa = -\kappa$, one can rewrite the transformations (5.12) in the following way:

$$\delta_\kappa\bar\theta_1 = \bar\kappa\check\Gamma_{\mathrm{D}p} \quad , \quad \delta_\kappa\bar\theta_2 = \bar\kappa \ . \tag{5.29}$$

It is then clear that one can fix κ-symmetry by adopting a covariant gauge-fixing $\tilde\Gamma_{(10)}\theta = \theta$ (i.e., $\theta_2 = 0$), as was for instance discussed in [128, 129]. The resulting κ-fixed action can be easily seen to be expressible in terms of only θ_1 in the following way:

$$\begin{aligned} S^{(F)}_{\mathrm{D}p} = \frac{T_{\mathrm{D}p}}{2}\int \mathrm{d}^{p+1}\xi\, e^{-\Phi}\sqrt{-\det(g+\mathcal{F})}\Big\{&\bar\theta_1\big[(M^{-1})^{\alpha\beta}\Gamma_\alpha D^{(0)}_\beta - \Delta^{(1)}\big]\theta_1 \\ &-\bar\theta_1\check\Gamma^{-1}_{\mathrm{D}p}\big[(M^{-1})^{\alpha\beta}\Gamma_\beta W_\alpha - \Delta^{(2)}\big]\theta_1\Big\}, \end{aligned} \tag{5.30}$$

Note that the explicit form of this action involves terms that vanish by means of the symmetry properties of the gamma matrices. Extracting all couplings is however a straightforward task.

Let us now discuss how possible background supersymmetries are realized on the world-volume. When the supergravity background possesses a Killing spinor ε (so $D_m\varepsilon = \Delta\varepsilon = 0$), then the gauge-unfixed D-brane action is symmetric under the following (leading order) induced supersymmetry transformations [9] (in standard notation for IIA):

$$\begin{aligned} \delta_\varepsilon\theta &= \varepsilon \ , \\ \delta_\varepsilon x^m &= -\frac{1}{2}\bar\theta\Gamma^m\varepsilon \ , \\ \delta_\varepsilon A_\alpha &= \frac{1}{2}\bar\theta\tilde\Gamma_{(10)}\Gamma_\alpha\varepsilon - \frac{1}{2}B_{\alpha m}\bar\theta\Gamma^m\varepsilon \ . \end{aligned} \tag{5.31}$$

These transformations have the same form in double spinor notation, up to the substitution $\tilde\Gamma_{(10)} \to -\sigma_3$ in the last line. In order to write these transformations in their gauge-fixed form, we have to be careful with the fact that the supersymmetry transformations do not necessarily respect the chosen gauge for κ-symmetry. Indeed, after a supersymmetry transformation (5.31), $\theta_2 = \epsilon_2$ and hence no longer zero. In order to compensate for this breaking of the κ-fixing condition $\theta = \tilde\Gamma_{(10)}\theta$, one has to add a compensating κ-transformation (5.29) with $\kappa = -\varepsilon_2$. Then, the resulting supersymmetry transformation of the physical fermionic field θ_1 becomes:

$$\delta_\varepsilon \bar{\theta}_1 = \bar{\varepsilon}_1 - \bar{\varepsilon}_2 \check{\Gamma}_{\mathrm{D}p} \ . \tag{5.32}$$

Supersymmetry is thus preserved by a classical configuration, in which the fermions are equal to zero, only if:

$$\bar{\varepsilon}_1 = \bar{\varepsilon}_2 \check{\Gamma}^{(\mathrm{cl})}_{\mathrm{D}p} \ . \tag{5.33}$$

The condition that has to be satisfied in order for the D-brane to preserve the background supersymmetry ε is thus given by $\bar{\varepsilon}\Gamma^{(\mathrm{cl})}_{\mathrm{D}p} = \bar{\varepsilon}$. Then, using the relation (5.33), it is possible to write the preserved supersymmetry transformations for the κ-fixed action in terms of only ε_1 as follows:

$$\begin{aligned}
\delta_\varepsilon \bar{\theta}_1 &= \bar{\varepsilon}_1 (1 - \check{\Gamma}^{(\mathrm{cl})-1}_{\mathrm{D}p} \check{\Gamma}_{\mathrm{D}p}) \ , \\
\delta_\varepsilon x^m &= \frac{1}{2} \bar{\varepsilon}_1 (1 + \check{\Gamma}^{(\mathrm{cl})-1}_{\mathrm{D}p} \check{\Gamma}_{\mathrm{D}p}) \Gamma^m \theta_1 \ , \\
\delta_\varepsilon A_\alpha &= \frac{1}{2} \bar{\varepsilon}_1 (1 + \check{\Gamma}^{(\mathrm{cl})-1}_{\mathrm{D}p} \check{\Gamma}_{\mathrm{D}p}) \Gamma_\alpha \theta_1 + \frac{1}{2} B_{\alpha m} \bar{\varepsilon}_1 (1 + \check{\Gamma}^{(\mathrm{cl})-1}_{\mathrm{D}p} \check{\Gamma}_{\mathrm{D}p}) \Gamma^m \theta_1 \ .
\end{aligned} \tag{5.34}$$

It is important to remark that these expressions only give the supersymmetry transformations in the lowest order of fermions (without fermion fields for the transformations of the fermions, and linear in fermions for the transformations of the bosons). The supersymmetry of the full action needs higher order terms. However, the transformations in (5.34) are sufficient for determining the variation of the action linear in fermions. Therefore, they are exact supersymmetries for the completely truncated action quadratic in both bosons and fermions around some particular classical configuration.

To see how the transformations (5.34) look like in this linearized approximation, it is convenient to fix the residual gauge invariance under world-volume diffeomorphisms in order to identify the physical world-volume scalar fields. Usually, this is done by adopting the so-called static gauge condition. In this case, one uses reparametrization invariance to identify $p+1$ of the space-time coordinates x^m with the world-volume coordinates $\xi^\alpha : x^\alpha = \xi^\alpha$. Only the fluctuations $\delta x^{\widehat{m}}$, where $\widehat{m} = p+1, \ldots, 9$ labels the directions transverse to the brane, are considered as physical, and one has to impose the condition $\delta x^\alpha = 0$. However, since we are working in a general curved space, this kind of gauge-fixing is not the most geometrical one due to the arbitrariness of the coordinate choice. It is then natural to break explicitly the local $SO(1,9)$ Lorentz invariance of the theory into $SO(1,p) \times SO(9-p)$ and select a class of adapted co-vielbeine $e^{\underline{m}} = (e^{\underline{\alpha}}, e^{\underline{\widehat{m}}})$, such that the pull-back on the brane of the $e^{\underline{\widehat{m}}}$ is vanishing and the pulled-back $e^{\underline{\alpha}}$ form a world-volume vielbein. Now one can consider the fluctuations of the brane as described by a section $\phi^{\underline{\widehat{m}}}$ of the normal bundle (i.e., $\phi^{\underline{\widehat{m}}} = e^{\underline{\widehat{m}}}_m \delta x^m$). This means that the natural gauge-fixing condition is:

$$e^{\underline{\alpha}}_m \delta x^m = 0 \, . \tag{5.35}$$

In order to write the supersymmetry transformations for the completely gauge-fixed linearized action, we now have to compensate the transformation (5.34) with a world-volume diffeomorphism $\delta\xi^\alpha(\epsilon)$ defined by the condition:

$$\delta\xi^\alpha(\epsilon) P[e^{\underline{\alpha}}]_\alpha = -e^{\underline{\alpha}}_m \delta_\epsilon x^m \ . \tag{5.36}$$

Taking into account this compensation and using the fact that $\phi^{\underline{\widehat{m}}} = 0$ when evaluated on the classical configuration, the linearized gauge-fixed supersymmetry transformations (5.34) become:

$$\begin{aligned}
\delta_\varepsilon \theta_1 &= (M^{-1})^{\beta\alpha}(\nabla^N_\alpha \phi^{\underline{\widehat{m}}})\Gamma_\beta \Gamma_{\underline{\widehat{m}}}\varepsilon_1 + (M^{-1})^{\beta\alpha}\Gamma_\beta{}^\gamma K_{\gamma\alpha}{}^{\underline{\widehat{m}}}\phi_{\underline{\widehat{m}}} \\
&\quad -\frac{1}{2}(M^{-1})^{\alpha\gamma}(M^{-1})^{\beta\delta}(\phi^{\underline{\widehat{m}}} H_{\underline{\widehat{m}}\gamma\delta} + f_{\gamma\delta})\Gamma_{\alpha\beta}\varepsilon_1 \ , \\
\delta_\varepsilon \phi^{\underline{\widehat{m}}} &= \bar{\varepsilon}_1 \Gamma^{\underline{\widehat{m}}}\theta_1 \ , \\
\delta_\varepsilon A_\alpha &= \bar{\varepsilon}_1 \Gamma_\alpha \theta_1 + f^{(\mathrm{cl})}_{\alpha\beta}\bar{\varepsilon}_1 \Gamma^\beta \theta_1 + B_{\alpha m}\bar{\varepsilon}_1 \Gamma^m \theta_1 \ ,
\end{aligned} \tag{5.37}$$

where $\nabla^N_\alpha = \partial_\alpha + \frac{1}{4}\mathcal{A}_\alpha{}^{\underline{\widehat{m}\widehat{n}}}\Gamma_{\underline{\widehat{m}\widehat{n}}}$ indicates the normal bundle covariant derivative with connection

$$\mathcal{A}_\alpha{}^{\underline{\hat{m}\hat{n}}} = \Omega_\alpha{}^{\underline{\hat{m}\hat{n}}} \ , \tag{5.38}$$

where $\Omega_\alpha{}^{\underline{mn}}$ is the pull-back of the spin connection of the target space vielbein $e^{\underline{m}}_m$ and $K_{\alpha\beta}{}^{\underline{\hat{n}}}$ is the extrinsic curvature of the world-volume of the brane, defined by

$$K_{\alpha\beta}{}^{\underline{\hat{n}}} = K_{\beta\alpha}{}^{\underline{\hat{n}}} = e^{\underline{\delta}}_\beta \Omega_{\alpha\underline{\delta}}{}^{\underline{\hat{n}}} \ . \tag{5.39}$$

The derivation of the first of (5.37) is straightforward but tedious, as it involves several rearrangements using gamma matrix properties along the lines followed to derive (5.20) from (5.9) and (5.11).

5.4 Consistency with T-duality

The original fermionic action (5.7) was constructed in [9] by using T-duality and assembling the different terms in a rather indirect way, using the partial results obtained previously in [8] and completing them by means of consistency conditions. Let us rederive the proof of the T-duality consistency of the above actions given in [8, 9], starting from their new expression given in (5.21). This is an important consistency check and it clarifies also the validity of the arguments, given in [8, 9], to obtain the final form of the action (5.7). In order to do this, let us first of all prove that the term

$$\frac{\tau_{\mathrm{D}p}}{2}\int \mathrm{d}^{p+1}\xi\, e^{-\Phi}\sqrt{-\det(g+\mathcal{F})}\,\bar{\theta}\big[(\tilde{M}^{-1})^{\alpha\beta}\Gamma_\beta D_\alpha - \Delta\big]\theta \ , \tag{5.40}$$

in the fermionic action (5.21) is left invariant in form by T-duality. One can check this property directly by using the usual T-duality rules for the bosonic fields and Hassan's T-duality rules for the fermions [111]. It is, however, easier to derive this property in a less direct way. Let us first introduce the following combination of bosonic and fermionic fields:

$$\begin{aligned}
\mathbf{\Phi} &= \Phi - \frac{1}{2}\bar{\theta}\Delta\theta \ , \\
\mathbf{G}_{mn} &= G_{mn} - \bar{\theta}\Gamma_{(m}D_{n)}\theta \ , \\
\mathbf{B}_{mn} &= B_{mn} - \bar{\theta}\tilde{\Gamma}_{(10)}\Gamma_{[m}D_{n]}\theta \ .
\end{aligned} \tag{5.41}$$

These can be seen as superfields expanded up to second order. One of the basic observations of [9, 130] is that, using Hassan's T-duality rules for fermions [111], these second-order superfields transform (up to second order) in the same way the corresponding bosonic fields do. The next step is to note that the term (5.40) can be seen as the second-order term arising in the expansion of a DBI action for the superfields (5.41):

$$S_{\mathrm{D}p} = -\tau_{\mathrm{D}p} \int \mathrm{d}^{p+1}\xi \, e^{-\mathbf{\Phi}} \sqrt{-\det(P[\mathbf{G}+\mathbf{B}]+f)} \ . \tag{5.42}$$

Once we know that the superfields (5.41) transform as the corresponding bosonic fields under T-duality, we can immediately conclude that the action (5.42) is left invariant in form by T-duality and then also its second-order term (5.40) is.

It is now easy to see that also the second contribution in the second-order action (5.21):

$$-\frac{\tau_{\mathrm{D}p}}{2} \int \mathrm{d}^{p+1}\xi \, e^{-\Phi} \sqrt{-\det(g+\mathcal{F})}\, \bar{\theta}\Gamma_{\mathrm{D}p}\big[(\tilde{M}^{-1})^{\alpha\beta}\Gamma_\beta D_\alpha - \Delta\big]\theta \ , \tag{5.43}$$

is left invariant by T-duality. Since we already know that the term $\big[(\tilde{M}^{-1})^{\alpha\beta}\Gamma_\beta D_\alpha - \Delta\big]$ is invariant in form under T-duality, we only need that the $\Gamma_{\mathrm{D}p}$ are transformed into themselves under T-duality. But this is indeed the case by definition, since in [9] these operators were obtained one from the other by using T-duality. Then the whole action (5.21) is clearly invariant in form under T-duality.

5.5 The world-volume geometry and a canonical action

In section 5.2.1, we have found a simple form for the quadratic fermionic action for a D-brane in which couplings to the background geometry are more transparent. The action (5.21) is however not in a canonical form, as the kinetic term is given by:

$$(\tilde{M}^{-1})^{\alpha\beta}\Gamma_\beta D_\alpha \ . \tag{5.44}$$

This feature is already visible in the bosonic action, as can be seen from the fact that the natural integration measure is given by:

$$\sqrt{-\det(g+\mathcal{F})} \ . \tag{5.45}$$

When one studies the world-volume physics around a particular background brane configuration with a nonzero $\mathcal{F}_{\alpha\beta}$, a general fluctuation $\delta M_{\alpha\beta}$ of (5.23) then has a Lagrangian containing terms of the schematic form

$$\sqrt{-\det M}(M^{-1}\cdots M^{-1}\delta M\cdots\delta M) \ . \tag{5.46}$$

This means that not only the natural volume element is given by $\sqrt{-\det M}$, but also that the lower indices of the different $(\delta M)_{\alpha\beta}$ are raised not with a metric but with $(M^{-1})^{\alpha\beta}$, analogously to what happens in the term (5.44) of the fermionic action. The effect is that the kinetic terms arising from the expansion of the bosonic action are not in canonical form, just like in the fermionic case.

In this section, we will explore the geometry characterizing the theory that lives on the world-volume of the Dp-brane in more detail. We will from now on consider the dynamics of the brane around some classical configuration and use the condition (5.35) to fix the world-volume reparametrization invariance. Furthermore, we will always consider the (bosonic) fields as evaluated at their classical value and write the contribution coming from the dynamical fluctuations explicitly.

In the first subsection 5.5.1, we focus on the bosonic part of the full action. We show how one can obtain an action in canonical form by deforming the world-volume geometry. In the following subsection 5.5.2, we also obtain the fermionic action in canonical form by writing it in terms of this deformed geometry.

5.5.1 The world-volume geometry

We will start our study of the world-volume geometry by first looking at the bosonic part of the action. As we will eventually work with fermions, it is convenient to define the deformed theory in terms of the vielbein instead of the metric. In particular, since we will work around some fixed classical configuration, we can restrict to the class of adapted co-vielbeine $e^{\underline{m}} = (e^{\underline{\alpha}}, e^{\underline{\widehat{m}}})$ such that $P[e^{\underline{\widehat{m}}}] = 0$, introduced previously. The presence of a world-volume field $\mathcal{F}$, naturally selects a subclass of world-volume vielbeine $e^{\underline{\alpha}}$ such that[3]:

$$\mathcal{F} = \tanh\phi_0 \; e^{\underline{0}} \wedge e^{\underline{1}} \; + \sum_{r=1}^{[(p-1)/2]} \tan\phi_r \; e^{\underline{2r}} \wedge e^{\underline{2r+1}} \; . \tag{5.47}$$

This form is no longer invariant under the full world-volume $\mathrm{SO}(1,p)$ symmetry but only preserves a residual $SO(1,1)\times[SO(2)]^{[(p-1)/2]}$ symmetry. This decomposition has been used in [129] to write the world-volume chiral operators (5.16) and (5.17) (up to a sign) in a nice form. We will then show that the effect of the field $\mathcal{F}$ can be reabsorbed in a non-isotropic deformation of the world-volume metric.

Let us first make a preliminary observation. If we define:

$$X_{\underline{\alpha}}{}^{\underline{\beta}} = \mathcal{F}_{\underline{\alpha}}{}^{\underline{\beta}} \; , \tag{5.48}$$

then the action of the matrix $(1+X)$ can be seen as the product of a rotation $\Lambda \in SO(1,1)\times[SO(2)]^{[(p-1)/2]}$ and an operator T defined as:

$$T = \sqrt{1 - X^2} \quad , \quad \Lambda = (1+X)T^{-1} \; . \tag{5.49}$$

These properties can be immediately understood by writing Λ and T in our preferred vielbein satisfying (5.47):

$$T_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} \frac{1}{\cosh\phi_0} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\cosh\phi_0} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\cos\phi_1} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\cos\phi_1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \tag{5.50}$$

[3] Here and in the following, we often do not write explicitly the pull-back symbol $P[.]$ and our notation does not distinguish between the world-volume vielbein and the $e^{\underline{\alpha}}$ belonging to the target space vielbein. The resolution of these ambiguities should be clear from the context.

and

$$\Lambda_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} \cosh\phi_0 & \sinh\phi_0 & 0 & 0 & \dots \\ \sinh\phi_0 & \cosh\phi_0 & 0 & 0 & \dots \\ 0 & 0 & \cos\phi_1 & \sin\phi_1 & \dots \\ 0 & 0 & -\sin\phi_1 & \cos\phi_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{5.51}$$

Using the matrix T, we can define a new non-isotropically deformed vielbein

$$\hat{e}^{\underline{\alpha}} = e^{\underline{\beta}} T_{\underline{\beta}}{}^{\underline{\alpha}} , \tag{5.52}$$

and consequently a deformed metric

$$\hat{g}_{\alpha\beta} = \eta_{\underline{\alpha\beta}} \hat{e}^{\underline{\alpha}}_{\alpha} \hat{e}^{\underline{\beta}}_{\beta} . \tag{5.53}$$

The deformed metric is then related to the original one and $\mathcal{F}_{\alpha\beta}$ by

$$\hat{g}_{\alpha\beta} = g_{\alpha\beta} - \mathcal{F}_{\alpha\gamma} g^{\gamma\delta} \mathcal{F}_{\delta\beta} . \tag{5.54}$$

Writing the natural volume element entering the action in the form:

$$\sqrt{-\det(g+\mathcal{F})} = \sqrt{-\det g \det(1+X)} = \frac{\sqrt{-\det \hat{g}}}{\sqrt{\det(1+X)}} , \tag{5.55}$$

one is led to the interpretation that we now have a world-volume theory defined in a deformed string frame with a standard volume element $\sqrt{-\det \hat{g}}$, and a coupling to a world-volume rescaled dilaton $\widehat{\Phi}$:

$$e^{\widehat{\Phi}} = e^{\Phi} \sqrt{\det(1+X)} . \tag{5.56}$$

Let us now show that one can rewrite the bosonic action in a canonical form. Note that the matrix $(M^{-1})^{\alpha\beta}$ entering the general expansion (5.46) takes the form:

$$(M^{-1})^{\alpha\beta} = \hat{g}^{\alpha\beta} - \hat{\mathcal{F}}^{\alpha\beta} , \tag{5.57}$$

where $\hat{g}^{\alpha\beta}$ is the inverse of $\hat{g}_{\alpha\beta}$ and $\hat{\mathcal{F}}^{\alpha\beta} = \hat{e}^{\alpha}_{\underline{\alpha}} \hat{e}^{\beta}_{\underline{\beta}} X^{\underline{\alpha\beta}}$. In the deformed theory, the inverse metric $\hat{g}^{\alpha\beta}$ thus separates from the contribution given by $\hat{\mathcal{F}}^{\alpha\beta}$ which can be directly identified as a "deformed" version of the background world-volume field strength. Then, when formulated in terms of the new deformed geometry, the kinetic terms come in a canonical form. The effect of a non-zero background $\mathcal{F}_{\alpha\beta}$ can not be completely reabsorbed in a deformation of the metric. Indeed, the $\hat{\mathcal{F}}^{\alpha\beta}$ appearing in (5.57) adds other couplings in the expansion (5.46), involving also derivatives of the bosonic world-volume fields, but since $\hat{\mathcal{F}}^{\alpha\beta}$ is antisymmetric, these are different in nature from kinetic terms and can be interpreted as generalized electromagnetic couplings. In the following section, we will see how the deformation of the world-volume geometry introduced here will allow us to isolate a kinetic term, writing the fermionic action as a standard Dirac action plus mass terms coming from the embedding in a curved background with fluxes and from the world-volume background field strength $\mathcal{F}_{\alpha\beta}$.

5.5.2 A canonical fermionic action

We will now reconsider the fermionic action (5.21) and rewrite it in terms of the deformed vielbein (5.52) defined in the previous section. Following [129], we write the chiral operators (5.8) and (5.10) that enter the fermionic action in the form:

$$\Gamma_{\mathrm{D}p} = e^{R\tilde{\Gamma}_{(10)}}\Gamma^{(0)\prime}_{\mathrm{D}p}e^{-R\tilde{\Gamma}_{(10)}} \ . \tag{5.58}$$

We have defined the operators

$$\Gamma^{(0)\prime}_{\mathrm{D}p} = \begin{cases} \Gamma^{(0)}_{\mathrm{D}p}(\Gamma_{(10)})^{\frac{p+2}{2}} & \text{for type IIA} \ , \\ \Gamma^{(0)}_{\mathrm{D}p}(\mathrm{i}\sigma_2)(\sigma_3)^{\frac{p+1}{2}} & \text{for type IIB} \ , \end{cases} \tag{5.59}$$

and

$$R = \frac{1}{4}Y_{\underline{\alpha\beta}}\Gamma^{\underline{\alpha\beta}} \tag{5.60}$$

is a Lorentz generator expressed easily in our preferred vielbein in the following way:

$$Y_{(2)} = \phi_0 e^{\underline{0}}\wedge e^{\underline{1}} + \sum_{r=1}^{[(p-1)/2]}\phi_r e^{\underline{2r}}\wedge e^{\underline{2r+1}} \ . \tag{5.61}$$

Let us observe that $R\tilde{\Gamma}_{(10)}$ generates on each irreducible component of a type IIA or IIB spinor a Lorentz transformation belonging to the unbroken $SO(1,1)\times[SO(2)]^{[(p-1)/2]}$, but also that it rotates the two irreducible components of a type IIA/IIB spinor in opposite directions. We can then define, for both type IIA and IIB, a new "rotated" fermionic field (recalling that the two irreducible components are rotated in opposite directions)[4]:

$$\Theta = e^{R\tilde{\Gamma}_{(10)}}\theta. \tag{5.62}$$

This sort of generalized chiral rotation is naturally accompanied by the following redefinition of the operators entering the fermionic action:

$$\begin{aligned} \hat{D}^{(0)}_{\alpha} &= e^{R\tilde{\Gamma}_{(10)}}D^{(0)}_{\alpha}e^{-R\tilde{\Gamma}_{(10)}} \ , \\ \hat{W}_{\alpha} &= e^{-R\tilde{\Gamma}_{(10)}}W_{\alpha}e^{-R\tilde{\Gamma}_{(10)}} \ , \\ \hat{\Delta}^{(1)} &= e^{-R\tilde{\Gamma}_{(10)}}\Delta^{(1)}e^{-R\tilde{\Gamma}_{(10)}} \ , \\ \hat{\Delta}^{(2)} &= e^{R\tilde{\Gamma}_{(10)}}\Delta^{(2)}e^{-R\tilde{\Gamma}_{(10)}} \ , \end{aligned} \tag{5.63}$$

where the operators involved are defined in (2.131,2.132).

In the above redefinition, it can be useful to write the operator W_m entering the action, in terms of a new operator W defined by the relation $W_m = W\Gamma_m$ and then

$$\hat{W} = e^{-R\tilde{\Gamma}_{(10)}}We^{R\tilde{\Gamma}_{(10)}} \ . \tag{5.64}$$

Then, it is possible to write the fermionic action (5.21) in a canonical form. Indeed, using the fact that

$$e^{-R}\Gamma_{\underline{\alpha}}e^{R} = \Lambda_{\underline{\alpha}}{}^{\underline{\beta}}\Gamma_{\underline{\beta}} \ , \tag{5.65}$$

[4] A similar rotation of the fermions accompanied by a vielbein redefinition of the kind given in (5.52) was discussed in [131, 132].

and that, for example,

$$\Lambda_{\underline{\alpha}}{}^{\underline{\beta}} e_{\underline{\beta}}^{m} = (1+X)_{\underline{\alpha}}{}^{\underline{\beta}} \hat{e}_{\underline{\beta}}^{m} \ , \tag{5.66}$$

it is possible to show that the action (5.21) can be written in the form:

$$\begin{aligned} S_{\mathrm{D}p}^{(F)} &= \frac{\mathcal{T}_{\mathrm{D}p}}{2}\int \mathrm{d}^{p+1}\xi\, e^{-\widehat{\Phi}}\sqrt{-\det \hat{g}}\Big\{\bar{\Theta}\big(1-\Gamma_{\mathrm{D}p}^{(0)\prime}\big)\big(\hat{\Gamma}^{\alpha}\hat{D}_{\alpha}^{(0)} - \hat{\Delta}^{(1)}\big)\Theta \\ &\quad +\bar{\Theta}\big(1-\Gamma_{\mathrm{D}p}^{(0)\prime}\big)\, e^{-2R\tilde{\Gamma}_{(10)}}\big(\hat{g}^{\alpha\gamma}[(1+\tilde{\Gamma}_{(10)}X)^{-1}]_{\gamma}{}^{\beta}\hat{\Gamma}_{\beta}\hat{W}\hat{\Gamma}_{\alpha} - \hat{\Delta}^{(2)}\big)\Theta\Big\}, \end{aligned} \tag{5.67}$$

where

$$\hat{\Gamma}_{\alpha} = \hat{e}_{\alpha}^{\underline{\beta}}\Gamma_{\underline{\beta}} \ , \qquad \hat{\Gamma}^{\alpha} = \hat{g}^{\alpha\beta}\hat{\Gamma}_{\beta} \ . \tag{5.68}$$

Using the equations (5.65) and (5.66), it is possible to write the explicit form of the new operators defined in (5.63) and (5.64). To do this, let us first of all extend the operator $X_{\underline{\alpha}}{}^{\underline{\beta}}$ to an operator acting on all the indices, by simply putting all the remaining components equal to zero. This redefinition can then be extended in an obvious way to all the other operators constructed from $X_{\underline{\alpha}}{}^{\underline{\beta}}$. Then, for example, we can define the complete deformed vielbein $\hat{e}_{m}^{\underline{m}}$, by simply generalizing the definition (5.52) into the definition $\hat{e}_{m}^{\underline{n}} T_{\underline{n}}{}^{\underline{m}}$. Of course, these extended redefinitions only make sense when restricted to the world-volume of the brane. The operators $\Delta^{(1)}$ and $\Delta^{(2)}$ are defined in terms of operators $\mathcal{T}$ of the form:

$$\mathcal{T} \sim \mathcal{T}_{m_1 m_2 \ldots}\Gamma^{m_1 m_2 \ldots} \ . \tag{5.69}$$

Then, it is possible to see that the corresponding "hatted" operators have the form[5]:

$$\hat{\mathcal{T}} \sim \mathcal{T}_{k_1 k_2 \ldots}\hat{\Gamma}^{m_1 m_2 \cdots}(1+\tilde{\Gamma}_{(10)}X)_{m_1}{}^{k_1}(1+\tilde{\Gamma}_{(10)}X)_{m_2}{}^{k_2}\cdots \ . \tag{5.70}$$

One can do a completely analogous computation for $\hat{W}$, where however in this case $\tilde{\Gamma}_{(10)}$ in (5.70) is replaced by $-\tilde{\Gamma}_{(10)}$. Note further that:

$$e^{-2R\tilde{\Gamma}_{(10)}} = \frac{1}{\sqrt{1+X}}\sum_{q}\frac{(-)^q(\tilde{\Gamma}_{(10)})^q}{q!2^q} X_{\underline{\alpha_1\alpha_2}}\cdots X_{\underline{\alpha_{2q-1}\alpha_{2q}}}\Gamma^{\underline{\alpha_1\ldots\alpha_{2q}}}, \tag{5.71}$$

and that the operator $\Gamma_{\mathrm{D}p}^{(0)}$ entering the definition of $\Gamma_{\mathrm{D}p}^{(0)\prime}$ in (5.59) can be written in terms of the deformed quantities by simply adding "hats" everywhere in the definition (5.14).

It remains to rewrite $\hat{\Gamma}^{\alpha}\hat{D}_{\alpha}^{(0)}$ in terms of the deformed variables. The contribution by the B field deforms analogous to (5.70). The pull-back of the target space covariant derivative can be rewritten as:

$$\begin{aligned} \hat{\Gamma}^{\alpha}\hat{\nabla}_{\alpha} &= \hat{\Gamma}^{\alpha}\hat{\mathcal{D}}_{\alpha} + \frac{1}{4}\hat{\Gamma}^{\alpha}\hat{\mathcal{B}}_{\alpha}{}^{\underline{\alpha\beta}}\Gamma_{\underline{\alpha\beta}} - \frac{1}{2}X_{\eta}{}^{\beta}K_{\alpha\beta}{}^{\underline{\hat{n}}}\,\tilde{\Gamma}_{(10)}\hat{\Gamma}^{\alpha\eta}{}_{\underline{\hat{n}}} \\ &\quad + \frac{1}{2}\hat{g}^{\alpha\beta}K_{\alpha\beta}{}^{\underline{\hat{n}}}\,\Gamma_{\underline{\hat{n}}} + \frac{1}{4}\hat{\Gamma}^{\alpha}\mathcal{A}_{\alpha}{}^{\underline{\hat{n}\hat{m}}}\Gamma_{\underline{\hat{n}\hat{m}}} \, . \end{aligned} \tag{5.72}$$

[5] Since $\hat{X}_{\alpha}{}^{\beta} = \hat{e}_{\alpha}{}^{\underline{\alpha}}\hat{e}_{\underline{\beta}}{}^{\beta}X_{\underline{\alpha}}{}^{\underline{\beta}} = e_{\alpha}{}^{\underline{\alpha}}e_{\underline{\beta}}{}^{\beta}X_{\underline{\alpha}}{}^{\underline{\beta}} = X_{\alpha}{}^{\beta}$, the objects in (5.67) and (5.70) are unambiguously defined.

In this expression, we have made use of the splitting of the pull-backed target space connection into a world-volume connection plus a part related to the extrinsic curvature and the normal bundle connection, introduced in (5.38) and (5.39). The normal bundle connection and extrinsic curvature are given by the embedding in the target space, and as such they do not depend on the world-volume geometry. As we want to find the kinetic term in a canonical form, we needed to introduce in (5.72) a covariant derivative $\mathcal{D}$ with respect to the deformed frame, since this is the frame in which the world-volume geometry is naturally described. This new derivative is defined using the connection $\hat{\omega}$ of the deformed vielbein, which is related to the original world-volume connection ($\omega_\alpha{}^{\underline{\alpha\beta}} = \Omega_\alpha{}^{\underline{\alpha\beta}}$) by:

$$\omega_\alpha{}^{\underline{\alpha\beta}} = \hat{\omega}_\alpha{}^{\underline{\alpha\beta}} + \hat{\mathcal{B}}_\alpha{}^{\underline{\alpha\beta}}\,, \tag{5.73}$$

with,

$$\begin{aligned}\mathcal{B}_\alpha{}^{\underline{\alpha\beta}} &= (1+\tilde{\Gamma}_{(10)}X)^{\underline{[\alpha|\rho}}\Big[(1+\tilde{\Gamma}_{(10)}X)^{-1}\hat{\mathcal{D}}_{\underline{\rho}}X(1+\tilde{\Gamma}_{(10)}X)^{-1} \\ &\quad -(1-X^2)\hat{\mathcal{D}}_{\underline{\rho}}X\Big]_{\underline{\gamma}}{}^{\underline{|\beta]}}\,\hat{e}_{\underline{\alpha}}^{\underline{\gamma}}\,\tilde{\Gamma}_{(10)} - \Big[\hat{\mathcal{D}}_\alpha X(1+\tilde{\Gamma}_{(10)}X)^{-1}\Big]^{\underline{[\alpha\beta]}}\,\tilde{\Gamma}_{(10)}\ ,\end{aligned} \tag{5.74}$$

where we have raised and lowered the flat indices by using $\eta_{\underline{\alpha\beta}}$ and its inverse as usual. This discussion makes explicit that the operator $\hat{\Gamma}^\alpha\hat{D}^{(0)}_\alpha$ contains the covariant derivative with respect to the deformed metric together with some covariant couplings of the world-volume fields to the background. If one takes the world-volume geometry as given by the deformed metric (5.53), it can be seen that (5.67) consists of a canonical Dirac operator together with some additional interactions, given by the embedding and the fluxes.

Let us next consider the effect of gauge fixing kappa-symmetry. We impose the gauge fixing condition:

$$\Theta = \tilde{\Gamma}_{(10)}\Theta\,, \tag{5.75}$$

which simply means that the second component of Θ is equal to zero. It can then easily be seen that the action (5.67) written in terms of the first component of Θ (which we indicate again with Θ) reduces to:

$$\begin{aligned}S'^{(F)}_{\mathrm{D}p} &= \frac{T_{\mathrm{D}p}}{2}\int \mathrm{d}^{p+1}\xi\, e^{-\hat{\Phi}}\sqrt{-\det\hat{g}}\,\Big\{\bar{\Theta}\big(\hat{\Gamma}^\alpha\hat{D}^{(0)}_\alpha - \hat{\Delta}^{(1)}\big)\Theta \\ &\quad -\bar{\Theta}\check{\Gamma}^{-1}_{\mathrm{D}p}\big(\hat{g}^{\alpha\gamma}[(1+X)^{-1}]_\gamma{}^\beta\hat{\Gamma}_\beta\hat{W}\hat{\Gamma}_\alpha - \hat{\Delta}^{(2)}\big)\Theta\Big\}.\end{aligned} \tag{5.76}$$

We conclude this section by writing the linearized supersymmetry transformation rules (5.37) in the new deformed variables:

$$\begin{aligned}\delta_\varepsilon\Theta &= \hat{\Gamma}^\alpha\nabla^N_\alpha\phi^{\underline{\hat{m}}}\Gamma_{\underline{\hat{m}}}\chi - X_\alpha{}^\beta K_{\beta\gamma}{}^{\underline{\hat{m}}}\phi_{\underline{\hat{m}}}\widehat{\Gamma}^{\alpha\gamma}\chi - \frac{1}{2}(\phi^{\underline{\hat{m}}}H_{\underline{\hat{m}}\alpha\beta} + f_{\alpha\beta})\hat{\Gamma}^{\alpha\beta}\chi\ , \\ \delta_\chi\phi^{\underline{\hat{m}}} &= \bar{\chi}\Gamma^{\underline{\hat{m}}}\Theta\ , \\ \delta_\chi A_\alpha &= \bar{\chi}\hat{\Gamma}_\alpha\Theta + B_{\alpha m}e^m_{\underline{\hat{m}}}\bar{\chi}\Gamma^{\underline{\hat{m}}}\Theta\ ,\end{aligned} \tag{5.77}$$

where $\chi = e^R\varepsilon_1$ and the scalar fields $\phi^{\underline{\hat{m}}}$ describe the brane fluctuations in the normal directions and $f_{\alpha\beta}$ is the dynamical world-volume field strength. Note that the world-volume part of these transformations takes the usual form valid when the world-volume field strength $\mathcal{F}_{\alpha\beta}$ is vanishing, while the terms in (5.77) involving explicitly $B_{(2)}$ and its field strength $H_{(3)}$, are non-zero only if some of the off-diagonal components $B_{\alpha m}e^m_{\underline{\hat{m}}}$ of $B_{(2)}$ are non-vanishing.

5.6 Conclusion

The main results of this chapter are the transparent formulae for the quadratic fermionic part of the Dp-brane actions on any supergravity background including possible fluxes. In particular, (5.21) gives this parametrization and κ-symmetric action, and (5.76) gives a convenient κ-fixed form.

We started by re-expressing the results of [8, 9] by re-organizing terms in a more compact notation. We clarified the underlying geometric structure, using the tensor $\tilde{M}_{\alpha\beta}$, see (5.22), in both types IIA and IIB. The formulation makes the invariance under T-duality easy to be verified. Using a similar doublet notation for IIA and IIB, the κ-gauge-fixing can be discussed uniformly. The preserved supersymmetry transformations after gauge-fixing the κ-symmetry and world-volume reparametrizations are obtained in a linearized form. In order to clarify the world-volume geometry, we have identified a new natural world-volume vielbein such that the measure of integration is its determinant and all kinetic terms for the fermions are recollected in the standard Dirac operator. Also the supersymmetry transformations in this new metric are obtained, and the explicit form of the terms entering the κ-fixed action is discussed.

These new results can be useful for any kind of quantum calculation on the brane, in particular for the understanding of nonperturbative effects in string theory. The results can also be useful for constructing effective actions for string configurations where D-branes are involved.

Appendix A

Simple Lie algebras

This appendix collects some facts on simple Lie algebras. We start by giving some basic definitions on Lie algebras. We then discuss the structure of complex simple Lie algebras and their classification. Finally, we discuss some aspects of real forms of Lie algebras, cosets, symmetric spaces and Iwasawa decompositions. The goal of this appendix is to present a short summary of material that is frequently used in this thesis. It is by no means meant to offer a complete treatment. A very good reference concerning the material presented here is given by [76, 133].

A.1 Basic definitions

A Lie algebra $\mathbb{G}$ is a (finite-dimensional) vector space over a field K, equipped with a bilinear product $[,]$ (called commutator):

$$[,] : \mathbb{G} \times \mathbb{G} \to \mathbb{G}\,, \tag{A.1}$$

satisfying

1. antisymmetry : $[a, b] = -[b, a]\,, \quad \forall a, b \in \mathbb{G}\,,$
2. the Jacobi identity : $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0\,, \quad \forall a, b, c \in \mathbb{G}\,.$

We will always consider cases in which $K = \mathbb{R}, \mathbb{C}$, i.e., we will restrict to real, respectively complex Lie algebras. A Lie algebra $\mathbb{G}$ is called abelian when $[\mathbb{G}, \mathbb{G}] = 0$. Introducing a basis $\{X_1, \ldots, X_n\}$, the commutator of two basis elements can be expanded in this basis:

$$[X_i, X_j] = \sum_k f_{ij}{}^k X_k\,. \tag{A.2}$$

The X_i are called generators of the algebra, while the constants $f_{ij}{}^k$ are denoted as structure constants. They satisfy:

1. antisymmetry : $f_{ij}{}^k = -f_{ji}{}^k\,,$
2. Jacobi identity : $f_{[ij}{}^k f_{m]k}{}^l = 0$.

A representation of a Lie algebra is a homomorphism ϕ from $\mathbb{G}$ to $M(V)$, where $M(V)$ denotes the set of linear transformations of a vector space V, such that:

$$\phi([X,Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X)\,, \quad \forall X,Y \in \mathbb{G}\,. \tag{A.3}$$

The adjoint representation ad is defined by taking $V = \mathbb{G}$:

$$\mathrm{ad}\ :\ X \in \mathbb{G} \to \mathrm{ad}_X \in M(\mathbb{G})\ :\ \mathrm{ad}_X Y = [X,Y] \quad X,Y \in \mathbb{G}\,. \tag{A.4}$$

In terms of this adjoint representation, one can define a scalar product $(,)$ on the algebra as follows:

$$(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y) \quad \forall X,Y \in \mathbb{G}\,, \tag{A.5}$$

i.e., (X,Y) is given by the trace of the product of the matrices representing X and Y in the adjoint representation. This inner product is called the Killing form; taking the inner product of the generators leads to a metric g_{ij}, the so-called Cartan metric[1]:

$$g_{ij} = (X_i, X_j) = f_{ik}{}^{l} f_{jl}{}^{k}\,. \tag{A.6}$$

As one can always choose the structure constants to be real, g_{ij} is real and symmetric.

A *(Lie) subalgebra* $\mathbb{H}$ of $\mathbb{G}$ is a vector subspace of $\mathbb{G}$ that is itself a Lie algebra, namely $[\mathbb{H},\mathbb{H}] \subset \mathbb{H}$. . An *ideal* $\mathbb{I}$ of $\mathbb{G}$ is a subalgebra of $\mathbb{G}$ satisfying $[\mathbb{I},\mathbb{G}] \subset \mathbb{I}$.

A Lie algebra $\mathbb{G}$ is called *simple* if it is nonabelian and the only ideals are $\mathbb{G}$ and the trivial subalgebra containing only the null vector. A Lie algebra is called *semi-simple* if it has no abelian ideals. An important theorem due to Cartan states that a Lie algebra is semi-simple if and only if the Cartan metric is non-singular, $\det(g_{ij}) \neq 0$. This is equivalent to saying that the Killing form is non-degenerate, i.e., $(X,Y) = 0\ \forall Y \in \mathbb{G} \Rightarrow X = 0$. A *solvable* algebra is a Lie algebra such that the series:

$$\mathbb{G}^{(0)} = \mathbb{G}\,, \quad \mathbb{G}^{(1)} = [\mathbb{G}^{(0)}, \mathbb{G}^{(0)}]\,, \quad \ldots\,, \quad \mathbb{G}^{(k)} = [\mathbb{G}^{(k-1)}, \mathbb{G}^{(k-1)}]\,, \tag{A.7}$$

stops for some value of k. The importance of these definitions lies in the fact that they provide building blocks for Lie algebras. One can show that every semi-simple Lie algebra can be decomposed into the direct sum of its simple ideals. A theorem due to Levi states that every Lie algebra can be decomposed into the direct sum of simple Lie algebras and solvable algebras.

A.2 Structure of simple Lie algebras

In the previous section, we have mentioned the Levi theorem that implies that simple Lie algebras play a prominent role in Lie algebra theory, as they are building blocks for more general Lie algebras. In this section, we will elaborate more on the structure of simple Lie algebras.

[1] In principle, we are considering non-abelian Lie algebras here.

A.2.1 The Cartan-Weyl basis and roots

In this section, we will only treat simple complex Lie algebras. In the so-called *Cartan-Weyl basis*, the generators of the algebra are constructed as follows. First, one finds a maximal set of commuting generators H_i, $i = 1, \ldots, r$, such that ad_{H_i} is semi-simple for all i [2]. This set of generators is called the *Cartan subalgebra* of $\mathbb{G}$ ($\mathrm{CSA}_{\mathbb{G}}$). The rank of $\mathbb{G}$ is then defined as the dimension r of this Cartan subalgebra.

Suppose we have a representation R of the algebra that associates to every element X an $n \times n$-matrix $R(X)$. Since the matrices $R(H_i)$ commute, we can simultaneously diagonalize them. Denoting the eigenvectors by $|\mu>$, the eigenvalues are given by μ^i:

$$H_i|\mu>=\mu^i|\mu> \ . \tag{A.8}$$

The r-component vector $(\mu^1, \cdots, \mu^r)$ is called a weight vector (or weight) of the representation R.

The weight vectors of the adjoint representation are called the roots (or root vectors) of the algebra $\mathbb{G}$. The eigenvectors that correspond to non-zero roots α are denoted by E^α and are often called step operators (often also denoted as roots). One thus has:

$$\mathrm{ad}_{H_i} E^\alpha = [H_i, E^\alpha] = \alpha^i E^\alpha \,, \tag{A.9}$$

where α^i are the components of the root α. The set of all roots of $\mathbb{G}$ is then called the root system of $\mathbb{G}$. The root system is denoted as $\Delta_{\mathbb{G}}$. One can then make the following root-decomposition of the algebra:

$$\mathbb{G} = \mathrm{CSA}_{\mathbb{G}} \bigoplus_{\alpha \in \Delta_{\mathbb{G}}} \mathbb{G}_\alpha \,, \quad \mathbb{G}_\alpha = \{X \in \mathbb{G} \mid [H_i, X] = \alpha^i X \ \forall H_i \in \mathrm{CSA}_{\mathbb{G}}\} \,, \tag{A.10}$$

where the direct sums denote direct sums of vector spaces. The root system $\Delta_{\mathbb{G}}$ has the following properties:

1. If $\alpha \in \Delta_{\mathbb{G}}$, then $-\alpha \in \Delta_{\mathbb{G}}$.
2. If $\alpha \in \Delta_{\mathbb{G}}$, then the only multiples of α in the root system are 0 and $\pm\alpha$.
3. $\alpha = 0 \Leftrightarrow \mathbb{G}_\alpha = \mathrm{CSA}_{\mathbb{G}}$.
4. If α, β and $\alpha + \beta$ are roots, then $[\mathbb{G}_\alpha, \mathbb{G}_\beta] \subset \mathbb{G}_{\alpha+\beta}$. If α, β are roots but $\alpha + \beta$ is no root, then $[\mathbb{G}_\alpha, \mathbb{G}_\beta] = 0$.
5. The eigenspaces $\mathbb{G}_\alpha$ for $\alpha \neq 0$ are one-dimensional.
6. If $\alpha, \beta \in \Delta_{\mathbb{G}}$ and $\alpha + \beta \neq 0$, then the Killing form vanishes on $\mathbb{G}_\alpha \times \mathbb{G}_\beta$.

The upshot of all this is that the commutators of the generators in the Cartan-Weyl basis can be summarized as:

$$\begin{aligned} [H_i, H_j] &= 0 \,, \\ [H_i, E^\alpha] &= \alpha^i E^\alpha \,, \\ \left[E^\alpha, E^{-\alpha}\right] &= \alpha^i H_i \,, \\ \left[E^\alpha, E^\beta\right] &= N_{\alpha\beta} E^{\alpha+\beta} \,, \qquad \alpha + \beta \neq 0, \ \alpha + \beta \in \Delta_{\mathbb{G}} \,, \end{aligned} \tag{A.11}$$

[2] An endomorphism is called semi-simple if in a suitable basis it can be expressed by means of a diagonal matrix.

while the scalar products of these generators are given by:

$$\begin{aligned} (H_i, H_j) &= \delta_{ij}\,, \\ (H_i, E^\alpha) &= 0\,, \\ (E^\alpha, E^{-\alpha}) &= 1\,, \\ (E^\alpha, E^\beta) &= 0\,, \qquad \alpha+\beta \neq 0\,. \end{aligned} \tag{A.12}$$

A.2.2 Simple roots, the Cartan matrix and Dynkin diagrams

Using a fixed basis $\{H_1, \ldots, H_r\}$ of $\mathrm{CSA}_{\mathbb{G}}$, one can introduce an ordering in $\Delta_{\mathbb{G}}$. A root α is called *positive* if its first non-zero component is greater than zero. The set of positive roots is then denoted as $\Delta_{\mathbb{G}}^+$. A root is then called negative when it is not positive.

A root is called *simple* if it is positive and if it can not be written as the sum of two other positive roots. The number of simple roots is given by the rank of the Lie algebra. We will denote the simple roots by α_i, $i = 1, \ldots, r$ [3]. The simple roots obey the following properties:

1. The simple roots α_i are linearly independent and span the dual of $\mathrm{CSA}_{\mathbb{G}}$.
2. For α_i, α_j simple roots : $\alpha_i \cdot \alpha_j \leq 0$.
3. If α_i and α_j are simple roots, then $\alpha_i - \alpha_j$ is not a root.
4. Every positive root is a sum of simple roots with nonnegative integer coefficients : $\forall \alpha \in \Delta_{\mathbb{G}}^+ : \alpha = \sum_i n_i \alpha_i$ with n_i nonnegative integers.

The scalar product used in property 2 is simply defined as $\alpha_i \cdot \alpha_j = \sum_k \alpha_i^k \cdot \alpha_j^k$. In terms of the simple roots, one can define an $r \times r$-matrix A_{ij}, the so-called *Cartan matrix* as follows:

$$A_{ij} = 2\frac{\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j}\,. \tag{A.13}$$

The following properties of this Cartan matrix are worth noting:

1. $A_{ii} = 2$ and if $i \neq j$ then $A_{ij} \leq 0$.
2. A_{ij} is an integer and if $i \neq j$ then $A_{ij} = 0, -1, -2$ or -3.
3. The determinant of A_{ij} is positive[4].

The Cartan matrix can alternatively be represented in a graphical way. To every Cartan matrix, one can associate a diagram consisting of nodes connected by lines in the following way. To every simple root α_i, one associates a node in the diagram. The number of lines between the nodes corresponding to α_i and α_j then equals $A_{ij}A_{ji}$. From the properties of the roots, it then follows that the number of lines is minimally 0, if $\alpha_i \cdot \alpha_j = 0$, and maximally 3. If $A_{ij}A_{ji} > 1$, the simple roots α_i and α_j have different lengths, in which case one draws an arrow on the lines connecting the nodes of α_i and α_j, pointing in the direction

[3] Note that the simple roots are denoted with a lower index. This should not be confused with the components of the roots, which carry an upper index.
[4] This is true for finite-dimensional algebras.

of the shorter root. A diagram obtained from the Cartan matrix in this way is called a *Dynkin diagram*. Given a Dynkin diagram, it is possible to reconstruct the Cartan matrix.

The importance of the simple roots and the Cartan matrix lies in the fact that from the simple roots the full set of roots can be reconstructed. The simple roots on the other hand can be reconstructed from the Cartan matrix. The Cartan matrix thus specifies the complete commutation relations of the algebra. This point can be made more manifest by going to the so-called *Chevalley basis*. To each simple root α_i, one associates three generators:

$$e^i = E^{\alpha_i}\,, \quad f^i = E^{-\alpha_i}\,, \quad h_i = 2\frac{\alpha_i \cdot H}{\alpha_i \cdot \alpha_i}\,. \tag{A.14}$$

The commutation relations of these generators are then given by[5]:

$$\begin{aligned} [h_i, h_j] &= 0\,, \\ \left[h_i, e^j\right] &= A_{ji} e^j\,, \\ \left[h_i, f^j\right] &= -A_{ji} f^j\,, \\ \left[e^i, f^j\right] &= \delta_{ij} h_j\,. \end{aligned} \tag{A.15}$$

The remaining step operators can then be obtained by repeatedly taking commutators of these basic generators, subject to the so-called Serre relations:

$$\begin{aligned} [\mathrm{ad}_{e^i}]^{1-A_{ji}}\, e^j &= 0\,, \\ \left[\mathrm{ad}_{f^i}\right]^{1-A_{ji}}\, f^j &= 0\,. \end{aligned} \tag{A.16}$$

In the above equations, the power of $(1-A_{ji})$ indicates the number of nestings in the commutator, for instance $[\mathrm{ad}_{e^i}]^2\, e^j = \left[e^i, \left[e^i, e^j\right]\right]$. That the Serre relations and the commutators (A.15) can be written in terms of the Cartan matrix shows that the Cartan matrix indeed contains all information on the structure of $\mathbb{G}$.

Classifying complex simple Lie algebras thus boils down to classifying simple root systems or equivalently classifying Cartan matrices.

A.2.3 Classification

The possible Cartan matrices and corresponding Dynkin diagrams turn out to be highly restricted.

It turns out that there are four infinite series of Lie algebras:

1. the A_n series containing the $\mathrm{S}\,\ell(n+1,\mathbb{C})$ algebras.
2. the B_n series containing the $\mathrm{SO}(2n+1,\mathbb{C})$ algebras.
3. the C_n series containing the $\mathrm{Sp}(n,\mathbb{C})$ algebras.
4. the D_n series containing the $\mathrm{SO}(2n,\mathbb{C})$ algebras.

Besides these so-called *classical* algebras, it turns out that 5 extra Dynkin diagrams are allowed, corresponding to the so-called *exceptional* algebras. These five exceptional algebras are denoted as E_6, E_7, E_8, F_4 and G_2.

[5] We have used a different normalization for the E^{α_i} generators, with respect to (A.11).

A.3 Real forms

In the previous section, we have discussed the classification of complex simple Lie algebras. The procedure of determining the structure of Lie algebras in terms of eigenvalue subspaces of the Cartan subalgebra in fact requires the field of the algebra to be $\mathbb{C}$. A natural question is whether one can also classify real simple algebras. The answer to this is contained in the theory of *real forms*. We will first define the notion of a real form of a complex algebra and illustrate this with some examples. Next, we will introduce some important decompositions of real semi-simple algebras, that are used in this thesis.

A.3.1 Definitions and examples

Let us start by recalling some definitions. Let V be a vector space over $\mathbb{R}$. $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ is called the complexification of V. Note that $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{\mathbb{C}}$. In other words, the complexification of a real vector space is obtained by multiplying vectors with complex numbers instead of real numbers. Let W be a vector space over $\mathbb{C}$. Restricting the definition of scalar multiplication to $\mathbb{R}$ leads to a vector space $W^{\mathbb{R}}$ over $\mathbb{R}$ and $\dim_{\mathbb{C}} W = 1/2 \dim_{\mathbb{R}} W^{\mathbb{R}}$. If $\{w_1, \ldots, w_n\}$ is a basis of W, a basis of $W^{\mathbb{R}}$ is obtained as $\{w_1, \ldots, w_n, \mathrm{i}w_1, \ldots, \mathrm{i}w_n\}$.

Let $\mathbb{G}$ be a complex Lie algebra. A real form of $\mathbb{G}$ is a subalgebra $\mathbb{G}_\circ$ of the real Lie algebra $\mathbb{G}^{\mathbb{R}}$ such that:

$$\mathbb{G}^{\mathbb{R}} = \mathbb{G}_\circ \oplus \mathrm{i}\mathbb{G}_\circ \,, \tag{A.17}$$

where the direct sum should be interpreted as a direct sum of vector spaces. The real forms of a complex algebra are thus such that their complexification is equal to the original complex algebra. A given complex algebra in general admits more then one real form. The importance of the notion of a real form lies in the fact that classifying all real forms of simple complex Lie algebras gives a classification of all simple real Lie algebras.

Two special real forms can always be distinguished, namely the *split real form* and the *compact real form*. The split real form (also called normal real form) of the complex algebra

$$\sum_{i=1}^{r} c_i H_i + \sum_{\alpha \in \Delta_{\mathbb{G}}} c_\alpha E^\alpha \,, \qquad c_i, c_\alpha \in \mathbb{C} \tag{A.18}$$

is obtained by restricting the complex coefficients to real coefficients. As all structure constants are real numbers, this normal real form is closed under commutation. In terms of the basis $\{H_i, E^\alpha\}$, the Cartan metric has the following form:

$$g = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & & 1 & & \\ & & & 1 & & & \\ & & & & & & 1 \\ & & & & & 1 & \end{pmatrix} \begin{matrix} H_1 \\ \vdots \\ H_r \\ E^\alpha \\ E^{-\alpha} \\ E^\beta \\ E^{-\beta} \end{matrix} \,. \tag{A.19}$$

By going to the new basis $\{H_i, \frac{1}{\sqrt{2}}(E^\alpha + E^{-\alpha}), \frac{1}{\sqrt{2}}(E^\alpha - E^{-\alpha})\}$, this metric is diagonalized:

$$g = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & 1 & \\ & & & & & & -1 \end{pmatrix} \begin{matrix} H_1 \\ \vdots \\ H_r \\ (E^\alpha + E^{-\alpha})/\sqrt{2} \\ (E^\alpha - E^{-\alpha})/\sqrt{2} \\ (E^\beta + E^{-\beta})/\sqrt{2} \\ (E^\beta - E^{-\beta})/\sqrt{2} \end{matrix} . \tag{A.20}$$

The compact real form can be obtained from the split real form by performing the so-called *Weyl unitary trick*. This means that one multiplies the generators of the split form with factors of i such that the Cartan metric becomes minus the identity. In other words, the compact real form is given by linear combinations with real coefficients of the following generators:

$$\left\{ \mathrm{i}H_j, \frac{\mathrm{i}}{\sqrt{2}}(E^\alpha + E^{-\alpha}), \frac{1}{\sqrt{2}}(E^\alpha - E^{-\alpha}) \right\} . \tag{A.21}$$

By explicit computation, one can check that these generators close under commutation. The complexification of both the normal and the compact real form is then indeed given by (A.18). Note that for simple Lie groups, it holds that the group is compact if and only if the Cartan metric on its Lie algebra is negative definite.

A.3.2 Cartan decomposition

The examples of the normal and compact real forms have shown that real forms are related via the Weyl unitary trick, i.e., by multiplying certain generators with the imaginary unit. One can also restate this example as follows. On the compact real form, one can consider a notion of complex conjugation, such that generators of type $(1/\sqrt{2})(E^\alpha - E^{-\alpha})$ are real and have eigenvalue $+1$ under this complex conjugation, while generators of type $\mathrm{i}H_j$ and $(\mathrm{i}/\sqrt{2})(E^\alpha + E^{-\alpha})$ are purely imaginary and have eigenvalue -1 under complex conjugation. The normal real form can then be obtained from the compact form by applying the Weyl unitary trick on the subspace of purely imaginary generators.

The lesson from this is that new real forms can be obtained from the compact real form by defining a suitable notion of complex conjugation. Such a complex conjugation corresponds to an involutive automorphism σ of the Lie algebra, namely an automorphism σ satisfying:

$$\sigma^2 = \mathbb{1} . \tag{A.22}$$

An involutive automorphism then has eigenvalues ± 1. Let $\mathbb{G}$ be a compact simple Lie algebra. Given such an involution σ, one can decompose $\mathbb{G}$ into its eigenspaces of σ:

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K} , \tag{A.23}$$

where $\mathbb{H}$ corresponds to the eigenspace with eigenvalue $+1$, while $\mathbb{K}$ is the eigenspace corresponding to eigenvalue -1. The subspaces $\mathbb{H}$ and $\mathbb{K}$ are then orthogonal:

$$(\mathbb{H}, \mathbb{K}) = 0 . \tag{A.24}$$

One can furthermore also show that $\mathbb{H}$ is closed under commutation and hence forms a subalgebra. In fact, the following commutation relations can be seen to hold:

$$\begin{aligned} [\mathbb{H},\mathbb{H}] &= \mathbb{H}\,, \\ [\mathbb{H},\mathbb{K}] &= \mathbb{K}\,, \\ [\mathbb{K},\mathbb{K}] &= \mathbb{H}\,. \end{aligned} \tag{A.25}$$

Performing the Weyl unitary trick on the subspace $\mathbb{K}$ leads to another algebra $\mathbb{G}^*$:

$$\mathbb{G}^* = \mathbb{H} \oplus \mathrm{i}\mathbb{K}\,. \tag{A.26}$$

This new algebra is closed under commutation with real structure constants. Unless $\mathbb{H} = \mathbb{G}$, this new algebra is not compact.

The decomposition (A.26) is called the Cartan decomposition. The subspace $\mathbb{H}$ is then the maximal compact subalgebra of $\mathbb{G}^*$. For instance, for the normal real form, the maximal compact subalgebra is spanned by the generators of type $(1/\sqrt{2})(E^{\alpha} - E^{-\alpha})$.

By this construction, it is possible to associate a non-compact real form $\mathbb{G}^*$ with the compact real form through the involution σ. One can also show that as σ runs through all possible involutive automorphisms of the compact real form $\mathbb{G}$, the corresponding $\mathbb{G}^*$ runs through all real forms associated with the complex semi-simple algebra of which $\mathbb{G}$ is the compact form. In this way, one can classify all possible real forms of complex semi-simple algebras. The full classification is shown in table (A.1)[6].

A.3.3 Cosets and symmetric spaces

Given a semi-simple Lie group G [7] and its maximal compact subgroup H, one can define an equivalence relation in G. Two elements g and g' in G are equivalent when they can be connected by right multiplication with an element of H:

$$g \sim g' \quad \text{if} \quad g = g'h\,, \quad h \in H\,. \tag{A.27}$$

The corresponding equivalence class is called a left coset. The set of all left cosets then constitutes the coset space G/H.

Each coset can be characterized by a coset representative $L(\phi)$, labelled by as many coordinates ϕ^i as needed. One supposes that each coset contains exactly one of the $L(\phi)$, such that the coset representatives give a decent parametrization of the coset space. Once a representative $L(\phi)$ is chosen, every group element g can be decomposed as:

$$g = L(\phi)h\,, \quad h \in H\,. \tag{A.28}$$

Multiplying a coset representative from the left with an arbitrary group element g of G brings one to another coset:

$$gL(\phi) = L(\phi')h\,, \tag{A.29}$$

6 USp($2N$), SU*($2N$), SO*($2N$) can be defined as groups of matrices over the quaternions $\mathbb{H}$. One has that USp($2N$) = U($N,\mathbb{H}$), SO*($2N$) = O($N,\mathbb{H}$). SU*($2N$) = Sℓ($N,\mathbb{H}$), where Sℓ($N,\mathbb{H}$) consists of linear transformations that have a determinant with modulus 1.

7 Note that we will often apply the same terminology to the Lie algebra $\mathbb{G}$ and the group G that is generated by it via the exponential mapping.

Compact form	Real form	Maximal compact subalgebra
$\mathrm{SU}(n)$	$\mathrm{SU}(p,n-p)$	$\mathrm{SU}(p)\times\mathrm{SU}(n-p)\times\mathrm{U}(1), \quad 1\leq p<n$
	$\mathrm{S}\,\ell(n)$	$\mathrm{SO}(n)$
$\mathrm{SU}(2n)$	$\mathrm{SU}^*(2n)$	$\mathrm{USp}(2n)$
$\mathrm{SO}(n)$	$\mathrm{SO}(p,n-p)$	$\mathrm{SO}(p)\times\mathrm{SO}(n-p)$
$\mathrm{SO}(2n)$	$\mathrm{SO}^*(2n)$	$\mathrm{U}(n)$
$\mathrm{USp}(2n)$	$\mathrm{Sp}(2n)$	$\mathrm{U}(n)$
	$\mathrm{USp}(2p,2n-2p)$	$\mathrm{USp}(2p)\times\mathrm{USp}(2n-2p)$
$G_{2,-14}$	$G_{2,-14}$	$G_{2,-14}$
	$G_{2,2}$	$\mathrm{SU}(2)\times\mathrm{SU}(2)$
$F_{4,-52}$	$F_{4,-52}$	$F_{4,-52}$
	$F_{4,-20}$	$\mathrm{SO}(9)$
	$F_{4,4}$	$\mathrm{USp}(6)\times\mathrm{SU}(2)$
$E_{6,-78}$	$E_{6,-78}$	$E_{6,-78}$
	$E_{6,-26}$	$F_{4,-52}$
	$E_{6,-14}$	$\mathrm{SO}(10)\times\mathrm{SO}(2)$
	$E_{6,2}$	$\mathrm{SU}(6)\times\mathrm{SU}(2)$
	$E_{6,6}$	$\mathrm{USp}(8)$
$E_{7,-133}$	$E_{7,-133}$	$E_{7,-133}$
	$E_{7,-25}$	$E_{6,-78}\times\mathrm{SO}(2)$
	$E_{7,-5}$	$\mathrm{SO}(12)\times\mathrm{SU}(2)$
	$E_{7,7}$	$\mathrm{SU}(8)$
$E_{8,-248}$	$E_{8,-248}$	$E_{8,-248}$
	$E_{8,-24}$	$E_{7,-133}\times\mathrm{SU}(2)$
	$E_{8,8}$	$\mathrm{SO}(16)$

Table A.1 *Classification of real forms of complex simple algebras. The first column indicates the compact real form. The second column denotes the several real forms that can be constructed from this compact form. The third column then contains the maximal compact subalgebra. The second number in subscript in the notation for the real forms of the exceptional algebras denotes the number of non-compact minus the number of compact generators.*

where in general ϕ' and h depend on ϕ and g. The decomposition of the (real) Lie algebra $\mathbb{G}$ of G in the algebra $\mathbb{H}$ of H and the orthogonal complement $\mathbb{K}$:

$$\mathbb{G}=\mathbb{H}\oplus\mathbb{K}\,, \tag{A.30}$$

leads to a useful coset representative:

$$L=\exp(\mathbb{K})\,. \tag{A.31}$$

These coset manifolds are homogeneous spaces, i.e., any two points can be connected via an isometry transformation. The isometry group of the space is then given by G, while H corresponds to the group that leaves a point fixed, the so-called isotropy group.

Given a Lie algebra $\mathbb{G}$, a subalgebra $\mathbb{H}$ such that an orthogonal decomposition (A.23) holds, together with the commutation relations (A.25), is called a *symmetric* subalgebra. The associated Lie group H is similarly called a symmetric subgroup. The coset space G/H that one can associate to this is then also called a *symmetric* space[8]. The cosets found by dividing a real form of a semi-simple algebra by its maximal compact subgroup thus correspond to symmetric spaces.

The metric on a symmetric space can be found as follows. Using the coset representative $L(\phi)$, one can define the one-form $L^{-1}\mathrm{d}L$. This one-form is Lie algebra valued and can according to (A.30) be decomposed as follows:

$$L^{-1}\mathrm{d}L = E + \Omega\,, \tag{A.32}$$

where E is a one-form taking values in $\mathbb{K}$, while Ω takes values in $\mathbb{H}$. Note that $L^{-1}\mathrm{d}L$ is invariant under left multiplication of L with a ϕ-independent element $g \in G$. Under right multiplication of L with local elements $h \in H$, one has:

$$\begin{aligned} E &\rightarrow h^{-1}Eh\,, \\ \Omega &\rightarrow h^{-1}\Omega h + h^{-1}\mathrm{d}h\,. \end{aligned} \tag{A.33}$$

Denoting the generators of $\mathbb{K}$ as K_A, E can be written as:

$$E = E_i^A \mathrm{d}\phi^i K_A\,. \tag{A.34}$$

Working in a specific matrix representation of the algebra, the metric on G/H is then constructed as follows:

$$\mathrm{d}s^2 = \mathrm{Tr}[E^2] = E_i^A \eta_{AB} E_j^B \mathrm{d}\phi^i \mathrm{d}\phi^j\,, \qquad \eta_{AB} \sim \mathrm{Tr}(K_A K_B)\,. \tag{A.35}$$

Note that for the algebras that we consider, η_{AB} is proportional to the Cartan-Killing metric restricted to $\mathbb{K}$. One can thus interpret E as a vielbein one-form on the manifold, where the flat metric is proportional to the Cartan-Killing metric restricted to $\mathbb{K}$.

Note that sometimes one can give an alternative formula for the metric. Let us for instance consider the cosets $\mathrm{S}\,\ell(n,\mathbb{R})/\,\mathrm{SO}(n)$. If we work in the fundamental representation, the Lie algebra of $\mathbb{H} = \mathrm{SO}(n)$ is spanned by the anti-symmetric matrices. We thus have:

$$E = \frac{L^{-1}\mathrm{d}L + (L^{-1}\mathrm{d}L)^T}{2}\,, \quad \Omega = \frac{L^{-1}\mathrm{d}L - (L^{-1}\mathrm{d}L)^T}{2}\,. \tag{A.36}$$

One then has

$$\mathrm{Tr}[E^2] = \frac{1}{2}\mathrm{Tr}\Big[L^{-1}\mathrm{d}L L^{-1}\mathrm{d}L + L^{-1}\mathrm{d}L(L^{-1}\mathrm{d}L)^T\Big]\,. \tag{A.37}$$

Defining $\mathcal{M} = LL^T$, one can see that this expression is also given by

$$\mathrm{Tr}[E^2] = -\frac{1}{4}\Big[\mathrm{d}\mathcal{M}\mathrm{d}\mathcal{M}^{-1}\Big]\,. \tag{A.38}$$

This form of the metric is for instance used in (4.9) to write down the kinetic terms of the scalar fields in $\mathcal{N} = 8$ supergravity in a suitable form.

[8] Strictly speaking, symmetric spaces are spaces for which the covariant derivative of the Riemann curvature tensor is zero. One can however show that this leads to coset spaces obtained by dividing a Lie group by a symmetric subgroup.

A.3.4 The Iwasawa decomposition

In the previous subsection, we saw how the Cartan decomposition leads to a specific parametrization of cosets G/H. There exists a different parametrization of cosets which is often very useful. This is based on the fact that one can alternatively decompose an element of a semi-simple Lie group G in a unique way as the product of an element of the maximal compact subgroup H and an element of a solvable group S:

$$G = HS\,. \tag{A.39}$$

This decomposition is known as the Iwasawa decomposition. At the level of the Lie algebra $\mathbb{G}$ of G, this decomposition is given by

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{S}\,, \tag{A.40}$$

where $\mathbb{H}$ is the maximal compact subalgebra and $\mathbb{S}$ is a solvable Lie algebra. Note that $\mathbb{S}$ is a subalgebra of $\mathbb{G}$, in contrast to the subspace $\mathbb{K}$ appearing in the Cartan decomposition. The solvable algebra $\mathbb{S}$ then determines the coset G/H as follows:

$$\frac{G}{H} = \exp(\mathbb{S})\,. \tag{A.41}$$

Let us now outline how one can construct the solvable algebra $\mathbb{S}$ explicitly. Considering the Cartan decomposition of a non-compact algebra $\mathbb{G}$, one starts by choosing a maximal set $\mathcal{H}_{\mathbb{K}}$ of commuting elements of $\mathbb{K}$. Choosing a basis $\{H_1, \ldots, H_r\}$ for $\mathcal{H}_{\mathbb{K}}$, one can then define the following subspaces of $\mathbb{G}$:

$$\mathbb{G}_\lambda := \left\{X \in \mathbb{G} \mid [H_i, X] = \lambda^i X\,,\ \lambda^i \in \mathbb{R}\,,\ \forall H_i \in \mathcal{H}_{\mathbb{K}}\right\}\,. \tag{A.42}$$

In this way, $\lambda = (\lambda^1, \ldots, \lambda^r)$ form r-component vectors that, when non-zero, are called *restricted roots*. Note that now the dimension of $\mathbb{G}_\lambda$ can be bigger than 1. Denoting the set of restricted roots by Σ, one can split it in a set of positive restricted roots Σ^+ and negative restricted roots Σ^- in the same way as was done for ordinary roots. The solvable algebra $\mathbb{S}$ is then given by:

$$\mathbb{S} = \mathcal{H}_{\mathbb{K}} \bigoplus_{\lambda \in \Sigma^+} \mathbb{G}_\lambda\,. \tag{A.43}$$

The parametrization (A.41) is often called the solvable parametrization of the coset.

Appendix B

Properties of real Clifford algebras

In this appendix, we will recall some properties of real Clifford algebras. Some reviews are in [134, 135]. We will restrict to Clifford algebras with positive signature. The $(q+1)$-dimensional real Clifford algebra $\mathcal{C}(q+1,0)$ is generated by real matrices γ_μ ($\mu = 1, \cdots, q+1$) satisfying:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \mathbb{1}. \tag{B.1}$$

The main properties are given in table B.1, which we will now further explain.

Table B.1 *Real Clifford algebras $\mathcal{C}(q+1,0)$, the dimension $\mathcal{D}_{q+1}$ of their irreducible representations, and the metric preserving group in the centralizer of the Clifford algebra in the $(P+\dot{P})\mathcal{D}_{q+1}$-dimensional representation. Here $\mathbf{F}(n)$ stands for $n \times n$ matrices with entries in the field $\mathbf{F}$.*

q	$q+1$	$\mathcal{C}(q+1,0)$	$\mathcal{D}_{q+1}$	$\mathcal{S}_q(P,\dot{P})$
-1	0	$\mathbf{R}$	1	$\mathrm{SO}(P)$
0	1	$\mathbf{R} \oplus \mathbf{R}$	1	$\mathrm{SO}(P) \times \mathrm{SO}(\dot{P})$
1	2	$\mathbf{R}(2)$	2	$\mathrm{SO}(P)$
2	3	$\mathbf{C}(2)$	4	$\mathrm{U}(P)$
3	4	$\mathbf{H}(2)$	8	$\mathrm{U}(P,\mathbf{H}) \equiv \mathrm{USp}(2P)$
4	5	$\mathbf{H}(2) \oplus \mathbf{H}(2)$	8	$\mathrm{USp}(2P) \times \mathrm{USp}(2\dot{P})$
5	6	$\mathbf{H}(4)$	16	$\mathrm{U}(P,\mathbf{H}) \equiv \mathrm{USp}(2P)$
6	7	$\mathbf{C}(8)$	16	$\mathrm{U}(P)$
7	8	$\mathbf{R}(16)$	16	$\mathrm{SO}(P)$
$n+7$	$n+8$	$\mathbf{R}(16) \times \mathcal{C}(n,0)$	$16\ \mathcal{D}_n$	as for $q+1=n$

When $q+1 = 0, 1, 2 \pmod 8$, the matrices of the complex Clifford algebra can be chosen to be real. So in these cases, the dimension of an irreducible representation is given by the dimension of the corresponding complex representation. If this occurs, the real Clifford algebra is said to be of the normal type. In the other cases, it is possible to obtain a real representation of dimension twice that of the complex representation. Indeed, many representations contain only purely real or purely imaginary matrices. Real matrices of double dimension are then obtained by considering the following matrices:

$$\Gamma^a = \gamma^a \otimes \mathbb{1}_2 \quad \text{if } \gamma^a \text{ is real}, \qquad \Gamma^a = \gamma^a \otimes \sigma_2 \quad \text{if } \gamma^a \text{ is imaginary}\,. \tag{B.2}$$

Consider now a real irreducible representation of the Clifford algebra $\mathcal{C}(q+1,0)$, given by $\mathcal{D}_{q+1} \times \mathcal{D}_{q+1}$-matrices γ_μ, where $\mathcal{D}_{q+1}$ is given in table B.1. Consider a real $\mathcal{D}_{q+1} \times \mathcal{D}_{q+1}$-matrix S satisfying:

$$[S, \gamma_\mu] = 0\,. \tag{B.3}$$

According to Schur's lemma, matrices that commute with an irreducible representation of the Clifford algebra must form a division algebra. This leads to distinction in a normal, almost complex and quaternionic case.

B.0.5 The normal case

As already mentioned, this occurs when

$$q + 1 = 0, 1, 2 \mod 8\,. \tag{B.4}$$

In this case the general form of the matrices S, commuting with all γ-matrices, is

$$S = a\mathbb{1}\,, \tag{B.5}$$

where a is a real constant. The dimension of the irreducible representation is given by $\mathcal{D}_{q+1} = 2^l$, where $q+1 = 2l$ or $2l+1$. For $q+1$ even this irreducible representation is unique (up to similarity transformations), while for $q+1$ odd, the representations γ_μ and $-\gamma_\mu$ are inequivalent and constitute the 2 possible irreducible representations one can have. In this case the product of all γ-matrices is moreover given by plus or minus the identity.

B.0.6 The almost complex case

This occurs when

$$q + 1 = 3, 7 \mod 8\,. \tag{B.6}$$

The irreducible representation is unique and has dimension $\mathcal{D}_{q+1} = 2^{l+1}$. The general form of the matrices S is given by:

$$S = a\mathbb{1} + bJ\,, \tag{B.7}$$

where a, b are real constants and where the real $\mathcal{D}_{q+1} \times \mathcal{D}_{q+1}$-matrix J commutes with all γ-matrices and squares to $-\mathbb{1}$. J is given by:

$$J = \pm\gamma_1 \cdots \gamma_{q+1}. \tag{B.8}$$

B.0.7 The quaternionic case

This occurs when

$$q + 1 = 4, 5, 6 \mod 8\,. \tag{B.9}$$

The dimension of the irreducible representations is given by $\mathcal{D}_{q+1} = 2^{l+1}$. It is unique for $q+1$ even, while there exist two inequivalent irreducible representations when $q+1$ is odd. The two irreducible representations are again related to each other by a minus-sign. The general form of the matrices S is now given by:

$$S = a_0 \mathbb{1} + \sum_{j=1}^{3} a_j E_j\,, \tag{B.10}$$

where the constants a_0, a_i are all real. The three matrices E_i commute with the γ-matrices and they satisfy a quaternion relation:

$$E_j E_k = -\delta_{jk} \mathbb{1} + \sum_{l=1}^{3} \epsilon_{jkl} E_l\,. \tag{B.11}$$

B.0.8 The structure of $\mathcal{S}_q(P, \dot{P})$

The representations of the real Clifford algebras we are working with, need not be irreducible. If one has a reducible representation, one can choose it to be of the form

$$\gamma_\mu = \mathbb{1}_P \otimes \gamma_\mu^{\rm irr} \quad \text{for } q \neq 0 \text{ mod } 4\,, \tag{B.12}$$

$$\gamma_\mu = \eta \otimes \gamma_\mu^{\rm irr} \quad \text{for } q = 0 \text{ mod } 4\,. \tag{B.13}$$

where $\gamma_\mu^{\rm irr}$ is an irreducible representation of the Clifford algebra, and $\eta = \text{diag}(\mathbb{1}_P, -\mathbb{1}_{\dot{P}})$. The group $\mathcal{S}_q(P, \dot{P})$, appearing in the isometry groups of homogeneous very special spaces is generated by all antisymmetric matrices that commute with all γ-matrices. In the normal case and when $q \neq 0$ mod 4, the generators of $\mathcal{S}_q(P, \dot{P})$ are given by:

$$S = A \otimes \mathbb{1}\,. \tag{B.14}$$

where A is an antisymmetric $P \times P$-matrix. When $q = 0$ mod 4, the matrix A has to be replaced by a matrix consisting of 2 blocks : one $P \times P$ and one $\dot{P} \times \dot{P}$ antisymmetric block. In the almost complex case, the generators of $\mathcal{S}_q(P, \dot{P})$ are of the following form:

$$S = A \otimes \mathbb{1}\,, \qquad \text{or} \quad S = B \otimes J\,, \tag{B.15}$$

where A, B are antisymmetric, respectively symmetric $P \times P$-matrices. In the quaternionic case, when $q \neq 0$ mod 4, the generators of $\mathcal{S}_q(P, \dot{P})$ are:

$$S = A \otimes \mathbb{1}\,, \qquad \text{or} \quad S = B \otimes \left(\sum_{j=1}^{3} a_j E_j \right)\,, \tag{B.16}$$

where A, B are antisymmetric, respectively symmetric, $P \times P$-matrices. Again, when $q = 0$ mod 4, one should look upon A and B as (anti)symmetric matrices consisting of (anti)symmetric $P \times P$ and $\dot{P} \times \dot{P}$ blocks.

Appendix C

Conventions

Throughout this work, we use the mostly plus convention for the space-time metric, i.e., the metric has signature $(-,+,\cdots,+)$. The Levi-Civita connection is defined by:

$$\Gamma^\mu_{\nu\rho} \equiv \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho})\,. \tag{C.1}$$

The space-time curvature tensor is then defined by:

$$R^\mu{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\tau}\Gamma^\tau{}_{\nu\sigma} - \Gamma^\mu_{\sigma\tau}\Gamma^\tau{}_{\nu\rho}\,. \tag{C.2}$$

The Ricci tensor and the scalar curvature are given by:

$$R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}\,, \qquad R \equiv g^{\mu\nu} R_{\mu\nu}\,. \tag{C.3}$$

The Einstein equation reads:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}\,, \tag{C.4}$$

where κ is the gravitational coupling constant. In four space-time dimensions, it is given by $\kappa^2 = 8\pi G_N/c^4$, where G_N is Newton's constant. These equations can be derived from the following action:

$$S = \int \mathrm{d}^D x \Big(\sqrt{-g} \frac{R}{2\kappa^2} + \mathcal{L}_m \Big)\,. \tag{C.5}$$

The energy-momentum tensor $T_{\mu\nu}$ is then defined in terms of the Lagrangian of the matter fields $\mathcal{L}_m$ as:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}}\,. \tag{C.6}$$

Unless explicitly stated otherwise, we use units in which $\hbar = c = \kappa^2 = 1$.

We use the following terminology for (irreducible) spinors. In even dimensions, we can project a spinor λ on its left- or right-handed part:

$$\lambda_{L,R} = \frac{1}{2}(1 \pm \gamma_*)\lambda\,, \tag{C.7}$$

where γ_* denotes the product of all gamma matrices. Note that in four dimensions γ_* is also denoted as γ_5. The projection (C.7) is compatible with Lorentz transformations, i.e., Lorentz transformations preserve the handedness of the spinor. A spinor of definite handedness (or chirality) is called a Weyl spinor. It can also be possible to impose a reality condition on the spinor:

$$\lambda^* = \tilde{B}\lambda\,, \tag{C.8}$$

where $*$ denotes complex conjugation. This reality condition must be compatible with Lorentz transformations, $\tilde{B}$ must be unitary and λ^{**} is supposed to be equal to λ. Whenever this is possible, a spinor obeying (C.8) is called a Majorana spinor. When the condition (C.8) respects the chiral projection (C.7), one speaks of Majorana-Weyl spinors. Whenever the Majorana condition is not possible, it might still be possible to impose a twisted reality condition:

$$(\lambda^i)^* = \tilde{B}\Omega_{ij}\lambda^j\,. \tag{C.9}$$

This should satisfy the same consistency conditions as for Majorana spinors and Ω is an antisymmetric matrix that satisfies $\Omega\Omega^* = \mathbb{1}$. Note that this condition involves multiple spinors, as denoted by the index i. The condition (C.9) is called the symplectic Majorana condition. We refer to [136] for more information, such as which conditions can be imposed in specific dimensions and signatures.

We use Latin indices $m, n, \ldots = 0, \ldots, 9$ for 10-dimensional curved coordinates, whereas for Dp-brane world-volume coordinates we use Greek indices $\alpha, \beta, \ldots = 0, \ldots, p$. The corresponding flat indices are underlined, e.g., the vielbein is given by $e^{\underline{m}} = e_n^{\underline{m}}\mathrm{d}x^n$. The ten dimensional (flat) gamma matrices are $\Gamma_{\underline{m}}$; they obey:

$$\{\Gamma_{\underline{m}}, \Gamma_{\underline{n}}\} = 2\eta_{\underline{mn}}\,. \tag{C.10}$$

Note that Clifford algebras in other dimensions obey a similar algebra. The 10-dimensional chiral operator is $\Gamma_{(10)} = \Gamma^{\underline{01\cdots 9}}$. Pulled back gamma matrices are then $\Gamma_\alpha = \Gamma_{\underline{m}} e_m^{\underline{m}} \partial_\alpha x^m$. The Levi-Civita symbol $\epsilon^{\alpha_1\cdots\alpha_{p+1}}$ is a density, i.e., it takes values ± 1.

We use the standard convention for the expansion of the forms in components, i.e., a p-form $\chi^{(p)}$ is expanded as:

$$\chi_{(p)} = \frac{1}{p!}\chi_{m_1\ldots m_p}\mathrm{d}x^{m_1}\wedge\ldots\wedge\mathrm{d}x^{m_p}\;. \tag{C.11}$$

Note that this differs from the conventions adopted in [8, 9], where a convention is adopted in which the $\mathrm{d}x^m$ in the above expression are multiplied in the opposite order. The R-R gauge fields $C^{(n)}$ are related to those used in [8, 9] by the substitution:

$$C_{m_1\ldots m_n} \to (-)^{\frac{n(n-1)}{2}} C_{m_1\ldots m_n}\;, \tag{C.12}$$

in such a way that the associated differential forms in the two conventions are the same. Another difference with these papers is that we have changed the definition of the charge conjugation matrix such that we avoid a factor i for any barred spinor. I.e., we take $\bar{\theta} = \mathrm{i}\theta^T\Gamma^{\underline{0}}$. Complex conjugation reverses the order of spinors. The spinors we use in 10 dimensions are real and chiral.

Finally, we end this appendix with some comments on the double spinor notation used from section 5.3 onwards. From the start, the spinors in type IIB are doublets. This means that θ stands for the 64-component spinor

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} , \tag{C.13}$$

whose 32-component parts are both left-handed, i.e., $\theta_i = \Gamma_{(10)}\theta_i$ with $i = 1, 2$. The Γ matrices do not mix with the extension index, i.e., $\Gamma_m\theta$ stands for:

$$\begin{pmatrix} \Gamma_m\theta_1 \\ \Gamma_m\theta_2 \end{pmatrix} , \tag{C.14}$$

of which both components are now right-handed. In other words, the Clifford matrices in the large space act as $\Gamma_m \otimes \mathbb{1}_2$. The conjugate spinor $\bar{\theta}$ is represented by:

$$\begin{pmatrix} \bar{\theta}_1 & \bar{\theta}_2 \end{pmatrix} , \tag{C.15}$$

and is from the right projected onto itself by $\frac{1}{2}(1 - \Gamma_{(10)}) \otimes \mathbb{1}_2$.

For type IIA, in sections 5.2.1 and 5.4 and in the heavily-used references [8, 9] as in many other papers, the two spinors are combined in a 32-component Majorana spinor $\theta = \theta_1 + \theta_2$, where $\theta_1 = \Gamma_{(10)}\theta_1$ (left-handed) and $\theta_2 = -\Gamma_{(10)}\theta_2$ (right-handed). We now define here also the doublet spinor (C.13), where now both 32-component parts have opposite chiralities. To obtain formulae that are similar to the IIB formulae, we use in the 64-component representation different Clifford representations. The Clifford matrices in the large space are represented by:

$$\Gamma_m^{\text{double}} = \begin{pmatrix} 0 & \Gamma_m \\ \Gamma_m & 0 \end{pmatrix} = \Gamma_m \otimes \sigma_1 \ . \tag{C.16}$$

The charge conjugation matrix in the large space is taken to be $\mathcal{C} \otimes \sigma_1$, where $\mathcal{C}$ is the 32×32 charge conjugation matrix. This implies that the conjugate spinor is:

$$\begin{pmatrix} \bar{\theta}_2 & \bar{\theta}_1 \end{pmatrix} . \tag{C.17}$$

These two choices imply, e.g., that the expression $\delta_\kappa \bar{\theta}\Gamma^m\theta$ maintains its form when we go from the 32-component notation to the 64-component notation. The matrix $\Gamma_{(10)}$ is still represented by $\Gamma_{(10)} \otimes \mathbb{1}_2$, but on the doublet (C.13) it acts as $\mathbb{1}_{32} \otimes \sigma_3$. Therefore $\tilde{\Gamma}_{(10)}$, see (5.13), is represented on θ in both IIA and IIB as $\mathbb{1}_{32} \otimes \sigma_3$. In any case it anticommutes with the representations of the Γ-matrices.

Appendix D

Nederlandse samenvatting

D.1 Snaartheorie

De natuur rondom ons is samengesteld uit een aantal elementaire deeltjes die met elkaar interageren via vier fundamentele natuurkrachten. Twee van deze krachten zijn welbekend uit het dagelijkse leven, met name de elektromagnetische kracht en gravitatie. De sterke en zwakke wisselwerking, die de andere twee krachten uitmaken, zijn voornamelijk werkzaam op subatomaire schaal. Onze beschrijving van de wereld van elementaire deeltjes en fundamentele wisselwerkingen is voornamelijk gebaseerd op relativistische quantumveldentheorie, de combinatie van quantummechanica met speciale relativiteit. Met name de elektromagnetische, sterke en zwakke wisselwerkingen blijken accuraat te kunnen worden beschreven door quantumveldentheorieën. Dit heeft in de jaren '70 geleid tot de ontwikkeling van het Standaard Model van de deeltjesfysica. Het Standaard Model is een voorbeeld van een ijktheorie, wat betekent dat er invariantie is onder een lokale symmetriegroep. In het geval van het Standaard Model is die ijkgroep gegeven door $SU(3) \times SU(2) \times U(1)$. Verder bevat het Standaard Model velden die geassocieerd zijn met spin-$1/2$ deeltjes, zoals quarks en leptonen, een spin-0 Higgs boson en spin-1 ijkbosonen, die verantwoordelijk zijn voor de diverse wisselwerkingen tussen de deeltjes.

De gravitatiekracht wordt niet beschreven in het Standaard Model. Hoewel gravitatie alomtegenwoordig is op grote afstandsschalen, waar Einstein's algemene relativiteit een bijzonder goede beschrijving geeft, hoeft men er niet echt rekening mee te houden bij het beschrijven van deeltjesfysica-experimenten. Voor de energieën die op dit moment in deeltjesversnellers bereikt kunnen worden, is het zo dat de gravitatiekracht bijzonder zwak is in vergelijking met de drie andere krachten. Het effect van gravitatie kan dus verwaarloosd worden en men heeft dus niet echt nood aan een quantumtheorie voor de gravitatiekracht om huidige experimentele resultaten te beschrijven. Bij zeer hoge energieën wordt er wel verwacht dat gravitatie een belangrijke rol zal spelen. Bovendien kan men gemakkelijk situaties bedenken waarin een theorie van quantumgravitatie nodig is, zoals bijvoorbeeld in het begin van het universum of in het inwendige van zwarte gaten.

Einstein's algemene relativiteitstheorie verzoenen met quantummechanica is echter geen eenvoudige taak. Berekeningen in quantumveldentheorieën leveren vaak oneindig als antwoord op. Voor zogenaamde renormalizeerbare veldentheorieën bestaat er evenwel een procedure, renormalizatie genaamd, die toelaat eindige resultaten te bekomen en zinvolle voorspellingen te doen. Het Standaard Model is een voorbeeld van zo'n renormalizeerbare theorie. Indien men echter Einstein's gravitatietheorie als veldentheorie probeert te quantiseren, vindt men dat deze theorie niet renormalizeerbaar is. Niet-renormalizeerbaarheid is vaak het signaal dat op een bepaalde energieschaal nieuwe fysica nodig is om fenomenen correct te beschrijven. Voor quantumgravitatie is die energieschaal gegeven door de Planck massa:

$$m_{\rm Pl} = G_N^{-1/2}\,, \tag{D.1}$$

waarbij G_N de constante van Newton is en we in eenheden hebben gewerkt waarin $\hbar = c = 1$. In dit opzicht dient Einstein's gravitatietheorie beschouwd te worden als een lage-energielimiet van een meer algemene theorie die ook geldig is op en boven de Planckschaal.

De meest beloftevolle kandidaat voor een dergelijke consistente theorie van quantumgravitatie is supersnaartheorie. In snaartheorie veronderstelt men dat de elementaire deeltjes overeenstemmen met kleine, trillende snaren. Snaren kunnen open of gesloten zijn en verschillende trillingswijzen van een snaar geven aanleiding tot verschillende deeltjes. Indien men de gesloten snaar quantiseert vindt men dat het spectrum een massaloos spin-2 deeltje bevat, dat geïdentificeerd wordt als het graviton, het deeltje dat zorgt voor gravitationele wisselwerkingen. Men kan bovendien ook massaloze spin-1 deeltjes vinden in het spectrum van open en gesloten snaren, die verantwoordelijk kunnen zijn voor Standaard-Model-achtige wisselwerkingen. Naast massaloze deeltjes bevat snaartheorie ook massieve deeltjes. Hun massa's M worden bepaald door de spanning van de snaar T:

$$M^2 \sim T\,, \qquad T = \frac{1}{2\pi\alpha'}\,, \tag{D.2}$$

waarbij we de zogenaamde Regge-parameter α' hebben ingevoerd. De karakteristieke lengte $l_{\rm s}$ van een snaar is van de orde $\sqrt{\alpha'}$. De massieve toestanden van een trillende snaar hebben dus massa's van de orde $1/l_{\rm s}$. Vaak neemt men aan dat die massa's van de orde van 10^{18} GeV zijn. Snaren interageren met elkaar door op te splitsen of samen te smelten. De sterkte van deze snaarinteracties wordt bepaald door de snaarkoppelingsconstante $g_{\rm s}$. Open snaren kunnen niet voorkomen zonder gesloten snaren, daar de eindpunten van een open snaar zich altijd kunnen samenvoegen om een gesloten snaar te vormen. Elke consistente snaartheorie bevat dus gesloten snaren en heeft dus een graviton in zijn deeltjesspectrum.

Een groot voordeel van snaartheorie is dat ze eindig is orde per orde in perturbatietheorie. Bovendien bevat snaartheorie geen onbepaalde dimensieloze parameters. De snaarkoppelingsconstante bijvoorbeeld wordt bepaald door de theorie zelf en kan niet zomaar gekozen worden. Dit is een groot verschil met het Standaard Model, dat 29 parameters bevat die experimenteel dienen te worden bepaald. Supersnaartheorie wordt ook gekenmerkt door een symmetrie die bosonen en fermionen uitwisselt en die supersymmetrie wordt genoemd.

Eigenlijk zijn er vijf consistente supersnaartheorieën. Dit zijn de zogenaamde IIA, IIB, type I, heterotische SO(32) en heterotische $E_8 \times E_8$ theorieën. Type I heeft SO(32) als ijkgroep, terwijl de twee heterotische theorieën respectievelijk SO(32) en $E_8 \times E_8$ ijkgroepen hebben. Alle supersnaartheorieën zijn het gemakkelijkst te formuleren in een 10-dimensionale vlakke Minkowskiruimte. De type II theorieën zijn bovendien invariant onder 32 supersymmetrieën, terwijl de andere 16 supersymmetrieën hebben. Aangezien de massieve toestanden zeer grote massa's hebben, is men meestal geïnteresseerd in het beschrijven van de lichte (massaloze) snaarexcitaties. Hun dynamica kan worden beschreven door middel van lage-energie effectieve acties. Deze acties reproduceren verstrooiingsamplitudes voor de massaloze snaardeeltjes bij lage energie. Deze effectieve acties zijn gegeven door 10-dimensionale supergravitatietheorieën, die invariant zijn onder 16 of 32 superladingen. Deze supergravitatietheorieën vormen een supersymmetrische extensie van Einstein gravitatie, zodat men bij lage energieën inderdaad een extensie van algemene relativiteitstheorie bekomt.

Aanvankelijk dacht men dat alleen de type I theorie open en gesloten snaren bevat, terwijl de andere theorieën geacht werden enkel gesloten snaren te beschrijven. In de jaren '90 werd echter gevonden dat type II theorieën ook open snaren kunnen bevatten. Deze kunnen zich echter niet vrij doorheen de ruimte-tijd bewegen, maar zitten met hun uiteinden vast aan $(p+1)$-dimensionale hyperoppervlakken, die Dp-branen worden genoemd. Deze Dp-branen blijken eveneens dynamische objecten te zijn die vrij kunnen bewegen en van vorm kunnen veranderen. De spanning van een dergelijk Dp-braan is omgekeerd evenredig met de snaarkoppelingsconstante : $T_{\mathrm{D}p} \sim 1/g_{\mathrm{s}}$. Bij zwakke koppeling zijn D-branen dus bijzonder zwaar. Snaartheorie is dus niet alleen een theorie van snaren maar bevat ook niet-perturbatieve objecten als D-branen.

Hoewel aanvankelijk werd gedacht dat de vijf supersnaartheorieën allen verschillend zijn, werd in de jaren '90 vastgesteld dat ze in feite allemaal verbonden zijn door dualiteiten. Verschillende theorieën blijken dus equivalent te zijn. Vaak is het zo dat berekeningen die in een theorie moeilijk zijn, gemakkelijk blijken te zijn in de duale theorie, doordat bijvoorbeeld gedrag bij sterke koppeling afgebeeld wordt op gedrag bij zwakke koppeling onder de dualiteit. Een voorbeeld van zo'n snaardualiteit is gegeven door T-dualiteit. Dit betekent dat een theorie waarin snaren bewegen in een vlakke ruimte waarin een richting gecompactificeerd is als een cirkel met straal R, equivalent is aan een (mogelijk andere) theorie die gecompactificeerd is op een cirkel met straal α'/R. Op die manier kan men bijvoorbeeld de type IIA supersnaar linken aan de IIB snaar. Een andere dualiteit is de zogenaamde S-dualiteit, die de sterke koppelingslimiet van een snaartheorie linkt aan de zwakke koppelingslimiet van een andere. De SO(32) heterotische snaar is via S-dualiteit verbonden met de type I snaar, terwijl IIB snaartheorie zelf-duaal is onder S-dualiteit. Men heeft ook gevonden dat de type IIA en heterotische $E_8 \times E_8$ snaartheorieën bij sterke koppeling equivalent worden met een 11-dimensionale theorie waarover nog weinig geweten is. D-branen spelen ook een specifieke rol in de dualiteiten. Zo kan het bijvoorbeeld gebeuren dat D-branen in een perturbatieve beschrijving van snaartheorie overeenstemmen met fundamentele snaren in een andere theorie. De fundamentele snaar van de heterotische SO(32) theorie wordt onder S-dualiteit bijvoorbeeld afgebeeld op de D1-snaar van de type I theorie. Bij zwakke koppeling is de fundamentele snaar van type II veel lichter dan de D1-braan, terwijl het bij sterke koppeling juist andersom is. Dit toont nogmaals aan dat snaartheorie niet enkel snaren bevat, maar dat D-branen even fundamenteel zijn als snaren. Het bestaan van de verschillende dualiteiten heeft geleid tot het beeld dat de diverse snaartheorieën slechts verschillende beschrijvingen zijn van een enkele fundamentele theorie, de zogenaamde M-theorie. Wat M-theorie precies is, is echter nog steeds een groot mysterie.

D.2 Realistische snaren

Hoewel supersnaren leiden tot een consistente quantumgravitatietheorie, die geldig is bij zeer hoge energieën, stellen er zich toch nog veel problemen, die zowel van conceptuele als meer praktische aard zijn. Een belangrijk conceptueel probleem is dat een goeie formulering van M-theorie nog niet gekend is. Vanuit praktisch oogpunt, is het belangrijk om na te gaan hoe de snaartheorie, die het best te formuleren is in 10 dimensies, kan gelinkt worden aan onze vier-dimensionale realiteit. Met de komst van nieuwe deeltjesversnellers is van belang om te zien of en hoe men uit snaartheorie voorspellingen kan bekomen over hoe de natuur zich gedraagt bij hogere energieën dan momenteel bereikbaar zijn.

Om contact te maken met vier-dimensionele fysica veronderstelt men meestal dat 6 van de 10 dimensies gecompactificeerd zijn. Deze vormen dan een compacte interne variëteit, waarvan het volume voldoende klein wordt verondersteld, zodat bij lage energieën de ruimtetijd er effectief vier-dimensionaal uitziet. Vaak neemt men aan dat die interne variëteiten specifiek gekozen zijn, zodat een gedeelte van de supersymmetrie gebroken is. Indien men bijvoorbeeld een Calabi-Yau-variëteit kiest, blijkt dat slechts een vierde van de oorspronkelijke 32 of 16 supersymmetrieën behouden blijft. Compactificatie van snaartheorieën leidt tot massaloze toestanden en toestanden die zeer massief zijn. De dynamica van de massaloze toestanden bij lage energie wordt dan beschreven door een vier-dimensionale supergravitatietheorie. In het geval van Calabi-Yau-compactificaties bijvoorbeeld zijn die invariant onder vier of acht superladingen. Aanvankelijk werd veel gewerkt aan Calabi-Yau-compactificaties van de $E_8 \times E_8$ heterotische snaartheorie, aangezien die leiden tot vier-dimensionale theorieën die het Standaard Model kunnen bevatten. Dankzij de ontdekking van D-branen, realiseerde men zich dat ook type II theorieën kunnen leiden tot interessante vier-dimensionale fysica. De dynamica van open snaren die eindigen op een Dp-braan wordt beschreven door een U(1) ijktheorie, die leeft op het $(p+1)$-dimensionale wereldvolume van de braan. In geval men N opeengepakte D-branen beschouwt, wordt deze ijkgroep zelfs U(N). Bovendien kan men ook verschillende opeenliggende D-branen beschouwen die elkaar onder bepaalde hoeken snijden. Dit kan eventueel leiden tot interessante spectra van deeltjes, die gelocaliseerd zijn op vier-dimensionale intersecties van de branen. Op die manier kan men met behulp van D-branen het Standaard Model op een meetkundige wijze realiseren in snaartheorie.

Een groot nadeel van die vier-dimensionale effectieve theorieën is dat ze vaak een groot aantal scalaire velden bevatten, waarvan de vacuum-verwachtingswaarde onbepaald is. Deze zogenaamde moduli stemmen overeen met continue vervormingen van de vorm en grootte van de interne variëteiten. Dergelijke scalaire velden zijn nog niet waargenomen en hun vacuum-verwachtingswaarde bepaalt allerlei grootheden (zoals diverse koppelingsconstanten) in de lage-energie effectieve acties. Zolang die scalaire velden niet vastliggen, is het moeilijk om voorspellingen te halen uit snaartheorie. Recent werden echter mechanismes ontwikkeld om die moduli vast te leggen. Onder de massaloze velden in het snaarspectrum bevinden zich verschillende anti-symmetrische tensorvelden. Indien men hun veldsterktes een niet-triviale verwachtingswaarde geeft langs de compacte richtingen (men spreekt van het aanzetten van fluxen), bekomt men een potentiaal voor deze moduli in de lager-dimensionale effectieve theorieën. Indien men deze zogenaamde fluxcompactificaties combineert met niet-perturbatieve effecten, afkomstig van bijvoorbeeld Euclidische D-branen die gewikkeld zijn rond bepaalde cykels van de interne variëteit, kan men modellen bekomen waarbij alle moduli vastgelegd kunnen worden in een minimum van de potentiaal.

D-branen en compactificaties met fluxen vormen een eerste stap in het verkrijgen van realistische modellen voor vier-dimensionale fysica uit snaartheorie. Op dit moment denkt men dat een groot aantal consistente compactificaties van snaartheorie kan worden bekomen, gekenmerkt door verschillende keuzes voor de interne variëteit, fluxen en minima van de moduli-potentiaal. Elk van deze scenario's leidt tot andere lage-energie-fysica. Hoewel het stilaan duidelijk wordt dat het mogelijk is om Standaard-Model-achtige theorieën in snaartheorie in te bedden, dient er echter nog heel wat werk verzet te worden. Compactificaties met fluxen zijn bijvoorbeeld veel minder goed begrepen dan standaard Calabi-Yau-compactificaties.

D.3 Kosmologie

Eén van de problemen waar snaartheorie mee te kampen heeft, is het gebrek aan experimentele input. Het zou echter wel eens kunnen dat kosmologie data aanbrengt waaraan kosmologische scenario's, die geïnspireerd zijn door snaartheorie, kunnen worden getoetst. Kosmologie is de laatste jaren immers uitgegroeid tot een preciese wetenschap, waar specifieke en accurate data kunnen worden bekomen. Uit dergelijke data zijn enkele merkwaardige feiten omtrent het heelal naar boven gekomen.

Uit kosmologische waarnemingen is gebleken dat de huidige fysische theorieën slechts 5% van de inhoud van het universum behandelen. Ongeveer 25% van het heelal blijkt te bestaan uit wat men donkere materie noemt. Deze naam doelt op een vorm van materie die geen elektromagnetische straling uitzendt, maar waarvan de aanwezigheid kan worden nagegaan dankzij de gravitationele aantrekkingskracht op gewone, zichtbare materie zoals sterren en melkwegstelsels. Over donkere materie is niet echt veel geweten. Supersymmetrische extensies van het Standaard Model, zoals die eventueel uit snaartheorie kunnen worden bekomen, bevatten echter vaak zwak interagerende deeltjes die goede kandidaten vormen voor donkere materie.

In de late jaren negentig werden ook heel wat data bekomen die erop wijzen dat de uitdijing van ons universum momenteel versnelt. Voordien werd steeds aangenomen dat de gravitatiekracht de uitdijing van het universum zou vertragen. De ontdekking van versnelde expansie was dan ook verrassend. Deze versnelling kan worden verklaard door aan te nemen dat het heelal gevuld is met een vreemde vorm van energie, de zogenaamde donkere energie, die het effect van gravitatie op grote afstanden tegenwerkt. De eenvoudigste vorm van dergelijke donkere energie, die goed in overeenstemming is met de waarnemingen, is gegeven door een positieve kosmologische constante. Het blijkt dat donkere energie ongeveer 70% uitmaakt van de materie/energie inhoud van het heelal. Op basis van allerlei theoretische problemen neemt men overigens ook vaak aan dat het heelal niet alleen nu een periode van versnelde expansie ondergaat, maar dat dit ook kort na de Big Bang gebeurde. Men spreekt dan over de periode van inflatie.

Voor inflatie en donkere energie zijn al verschillende scenario's voorgesteld. Men wil deze uiteindelijk echter inbedden in een meer fundamentele theorie, zoals bijvoorbeeld snaartheorie. Het is dus een interessante uitdaging om na te gaan wat de gevolgen van snaartheorie op kosmologisch vlak zijn en of deze kunnen worden getoetst aan de weelde van kosmologische data die op dit moment worden verkregen of die nog zullen worden verkregen in de nabije toekomst.

D.4 Deze thesis

Snaartheorie in contact brengen met gekende lage-energie-fysica en kosmologie is nuttig, aangezien dit kan toelaten de structuur van de theorie beter te begrijpen en in de toekomst zou kunnen leiden tot voorspellingen die falsifieerbaar zijn. Het werk in deze thesis dient dan ook in deze context gezien te worden.

Hoofdstukken 3 en 4 van deze thesis behandelen diverse aspecten omtrent het vinden van tijdsafhankelijke (kosmologische) oplossingen in supergravitatietheorieën. In hoofdstuk 3 bestuderen we een methode die nuttig is bij het zoeken van kosmologische oplossingen van supergravitatietheorieën. Deze methode staat bekend als de Tits-Satake projectie. In hoofdstuk 4 zullen we een specifieke soort van kosmologische oplossingen, de zogenaamde schaalkosmologieën, zoeken in geijkte vier-dimensionale $\mathcal{N} = 8$ supergravitatie. Deze schaalkosmologieën kunnen overeenstemmen met het gedrag van meer algemene kosmologische oplossingen van geijkte supergravitaties bij vroege of late tijden. In hoofdstuk 5 zullen we branen beschouwen in achtergronden waar fluxen zijn aangezet. Zoals vermeld zijn dergelijke achtergronden belangrijk bij het construeren van realistische deeltjesfysica modellen uit snaartheorie en bij het linken van snaartheorie aan kosmologie. In hoofdstuk 5 zullen we ons meer specifiek concentreren op de structuur van het fermionische deel van de effectieve actie, die de dynamica van D-branen beschrijft in algemene achtergronden met fluxen.

De structuur van deze thesis is als volgt. In hoofdstuk 2 beginnen we met wat achtergrond te geven over kosmologie, die bedoeld is om de hoofdstukken 3 en 4 in een bredere context te plaatsen. We introduceren enkele basisconcepten, zoals de wet van Hubble en de Friedmann-vergelijkingen. Vervolgens bespreken we enkele van de recente kosmologische waarnemingen. We beëindigen dit gedeelte met een discussie over hoe scalaire velden aanleiding kunnen geven tot versnellende kosmologieën. Hier voeren we ook de schaalkosmologieën in die in hoofdstuk 4 een prominente rol spelen. In sectie 2.2 verstrekken we wat wiskundige achtergrond die nodig zal zijn voor hoofdstuk 3. Na een korte introductie in Kähler-meetkunde, bespreken we de verschillende meetkundige structuren die verschijnen in theorieën met 8 superladingen. Het betreft hier de zogenaamde speciale meetkundes, met name speciale Kähler, zeer speciale reële en quaternionische-Kähler meetkunde. We beëindigen hoofdstuk 2 met een bespreking van supergravitatietheorieën. We belichten eerst wat algemene aspecten van supersymmetrie en supergravitatie, zoals het verband tussen supersymmetrie en meetkunde. Supergravitatietheorieën bevatten vaak een groot aantal scalaire velden. Deze kunnen gezien worden als coördinaten op een welbepaalde doelruimte. Supersymmetrie legt restricties op op de doelruimtes beschreven door deze scalairen. Hoe dit gebeurt wordt geïllustreerd aan de hand van een voorbeeld. Vervolgens bespreken we kort de verschillende supergravitatietheorieën die voorkomen in deze thesis. We beginnen met de theorieën in 3, 4 en 5 dimensies die invariant zijn onder 8 superladingen. We besteden hierbij speciale aandacht aan hoe deze theorieën bepaald worden door de geometrieën die beschreven werden in sectie 2.2. Vervolgens bespreken we kort enkele theorieën die invariant zijn onder het maximale aantal supersymmetrieën, namelijk 32. Meer specifiek behandelen we de type II supergravitaties in 10 dimensies en de geijkte maximale supergravitatie in 4 dimensies. Die laatste theorie wordt gekenmerkt door het feit dat een globale symmetriegroep van de Lagrangiaan lokaal wordt gemaakt met behulp van ijkvelden die in de theorie aanwezig zijn. Deze ijkingsprocedure introduceert een potentiaal voor de scalaire velden in de theorie. Geijkte supergravitatietheorieën bevatten dus een aantal scalaire velden met potentiaal, wat ze interessant maakt voor het vinden van eventueel versnellende kosmologische oplossingen. Type II supergravitaties in 10 dimensies en geijkte maximale supergravitatie in 4 dimensies worden in de thesis respectievelijk gebruikt in hoofdstukken 5 en 4.

In hoofdstuk 3 beginnen we dan met de studie van de Tits-Satake projectie in de context van supergravitatietheorieën met 8 superladingen. De Tits-Satake projectie is een nuttig instrument bij de studie van tijdsafhankelijke oplossingen van supergravitatietheorieën, omdat ze toelaat een ingewikkelde theorie te trunceren naar een simpeler model waarvan men oplossingen kan vinden met behulp van algoritmes. Meer specifiek gaat het hier om een projectie van een ingewikkelde doelruimte van scalairen op een eenvoudiger doelruimte. De truncatie is zo dat oplossingen van de getrunceerde theorie ook oplossingen zijn van de oorspronkelijke theorie. De Tits-Satake projectie levert eveneens een manier om meer algemene oplossingen van het oorspronkelijke model te bekomen vanuit de oplossingen van de getrunceerde theorie. De Tits-Satake projectie is welgekend in geval de scalaire doelruimtes symmetrische ruimtes zijn. Deze methode werd bijvoorbeeld in [5] toegepast om kosmologische oplossingen van vier-dimensionale $\mathcal{N} = 6$ supergravitatie te construeren. In hoofdstuk 3 zullen we deze methode uitbreiden tot een grotere klasse van supergravitaties, waarbij de doelruimtes niet noodzakelijk meer symmetrische ruimtes zijn. We zullen aantonen dat de Tits-Satake projectie ook kan worden gedefinieerd voor speciale meetkundes die homogeen zijn. Deze extensie werd uitgewerkt in samenwerking met P. Fré, F. Gargiulo, K. Rulik, M. Trigiante en A. Van Proeyen in [6]. Dit hoofdstuk begint met een overzicht van de classificatie van de homogene quaternionische-Kähler variëteiten en hun relatie met homogene speciale Kähler en zeer speciale reële ruimtes. We besteden hier ook aandacht aan de structuur van de isometrie algebra's van deze ruimtes. In sectie 3.3 bespreken we vervolgens hoe de Tits-Satake projectie van een symmetrische ruimte bekomen wordt. De bedoeling van dit gedeelte is om enkele begrippen in te voeren die belangrijk zijn in de veralgemening van de Tits-Satake projectie naar algemene homogene speciale meetkundes. Deze sectie start met een theoretische discussie, die gevolgd wordt door een concreet voorbeeld. De veralgemening van de projectie voor algemene homogene speciale doelruimtes wordt uitgewerkt in sectie 3.4. We geven hier opnieuw een algemene discussie, waarvan bepaalde punten vervolgens meer expliciet worden uitgewerkt. Dit hoofdstuk wordt beëindigd met een samenvatting van de resultaten. We belichten tevens enkele toepassingen. Als voornaamste toepassing vermelden we dat de Tits-Satake projectie ons toelaat supergravitatietheorieën in een kleine verzameling universaliteitsklassen onder te verdelen. Alle theorieën in zo'n klasse hebben dezelfde Tits-Satake projectie en hun dynamica vertoont een gelijkaardig gedrag. Dit laatste punt wordt geïllustreerd aan de hand van het kosmisch-biljart-fenomeen. Dit fenomeen toont duidelijk aan dat een aantal (maar niet alle) cruciale aspecten van de dynamica van een bepaalde theorie al kunnen afgeleid worden door zich te beperken tot de Tits-Satake projectie. We illustreren dit ook aan de hand van een voorbeeld waarin de Tits-Satake projectie op een microscopische manier kan worden geïnterpreteerd in compactificaties van snaartheorie met opeengepakte D-branen.

In hoofdstuk 4 beschouwen we dan de geijkte maximale supergravitietheorieën in 4 dimensies. Zoals reeds vermeld is er nu een potentiaal voor de scalairen. Dit heeft vaak interessante gevolgen voor kosmologie. Zo bijvoorbeeld is er veel werk verricht naar configuraties waarbij de scalairen constant zijn en in een minimum van de potentiaal vastgelegd zijn. Indien de waarde van de potentiaal in dit minimum positief is, geeft dit aanleiding tot een vacuum met positieve kosmologische constante, een zogenaamd de-Sitter-vacuum. In dit hoofdstuk leggen we ons echter toe op een ander soort kosmologische oplossingen, die minder vaak beschouwd werden in supergravitatie. We onderzoeken namelijk deze geijkte supergravitaties op de aanwezigheid van schaalkosmologieën. Dit zijn kosmologische oplossingen waarbij de verschillende energiecomponenten (kinetische energie van de scalairen, potentiële energie van de scalairen, energiedichtheid van mogelijke andere materie) constante verhoudingen hebben. Het belang van dergelijke oplossingen is dat ze kunnen overeenkomen met het gedrag van meer algemene kosmologische oplossingen bij vroege of late tijden. We werken in dit hoofdstuk meer specifiek in een truncatie van de geijkte theorieën. Deze truncatie wordt uitgelegd in sectie 4.2. De potentialen in deze getrunceerde theorie hebben een specifieke vorm; ze zijn namelijk een som van exponentiële termen. Resultaten omtrent schaalkosmologieën in theorieën met dergelijke zogenaamde multi-exponentiële potentialen zijn vrij goed gekend en worden besproken in sectie 4.3. In deze sectie wordt eveneens duidelijk gemaakt dat de potentialen voor bepaalde ijkingen de goede vorm hebben om schaalkosmologieën toe te laten. We geven de gevallen waarin schaalkosmologieën optreden en bespreken enkele eigenschappen van deze oplossingen. In het bijzonder stemmen enkele van de oplossingen overeen met versnellende kosmologieën. In sectie 4.3 bekijken we de hoger-dimensionale oorsprong van de bekomen oplossingen. We beëindigen dit hoofdstuk met een interessant verband tussen de-Sitter-vacua en schaalkosmologieën in supergravitatietheorieën. Het werk in dit hoofdstuk gebeurde in samenwerking met T. Van Riet en D. B. Westra en verscheen in [7].

In hoofdstuk 5 tenslotte bestuderen we de fermionische acties voor D-branen in willekeurige supergravitatie-achtergronden met fluxen. We beginnen met enkele elementaire feiten omtrent D-branen en motiveren waarom kennis van het fermionische deel van de effectieve D-braan-actie relevant is. Deze fermionische actie werd bekomen op kwadratische orde in de fermionen in [8, 9]. De vorm van deze actie is echter vrij ingewikkeld. In dit hoofdstuk bouwen we verder op deze resultaten en bekomen we een compacte en elegante vorm voor deze fermionische actie. Dit wordt gedaan in sectie 5.2. We bespreken hier tevens enkele specifieke symmetrieën van deze actie, zoals invariantie onder reparametrizaties van de coördinaten langsheen de braan, een fermionische symmetrie, kappa-symmetrie genaamd en supersymmetrie. We tonen ook aan dat deze actie consistent is met T-dualiteit, wat een belangrijke check geeft op dit resultaat. De actie die we bekomen is niet in canonische vorm. In sectie 5.5 tonen we aan hoe, door geschikte veldherdefinities te doen, deze actie toch in canonische vorm kan worden geschreven. De resultaten in dit hoofdstuk werden bekomen in samenwerking met L. Martucci, D. Van den Bleeken en A. Van Proeyen in [10].

Samengevat zijn de belangrijkste resultaten van deze thesis:

- De uitbreiding van de Tits-Satake projectie, die gedefinieerd was voor symmetrische ruimtes, naar willekeurige homogene speciale meetkundes.

- De groepering van supergravitatietheorieën met homogene speciale meetkunde in een klein aantal universaliteitsklassen.

- Het geven van een voorbeeld waarin de Tits-Satake projectie microscopisch kan worden geïnterpreteerd. Het betreft hier een compactificatie met meerdere D-branen opeen. De Tits-Satake projectie correspondeert dan met het geval waarin slechts 1 D-braan in beschouwing wordt gebracht.

- Het geven van schaalkosmologieën in geijkte, vier-dimensionale maximale supergravitatietheorieën. De identificatie van de ijkingen die dergelijke kosmologieën toelaten. De identificatie van de gevallen waarin de schaalkosmologie versnellend is.
- Het geven van de hoger-dimensionale oorsprong van de gevonden schaalkosmologieën.
- Het geven van een verband tussen schaalkosmologieën en de Sitter vacua dat mogelijkerwijze toelaat eenvoudig stabiele schaalkosmologieën te vinden.
- De constructie van een eenvoudige en elegante vorm voor het fermionisch gedeelte van de effectieve actie, die D-braan dynamica beschrijft. Het herschrijven van deze actie in canonische vorm.

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