Generalized Geometry Approaches to Gravity

by

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Statutory Declaration

(Declaration on Authorship of a Dissertation)

I, Fech Scen Khoo hereby declare, under penalty of perjury, that I am aware of the consequences of a deliberately or negligently wrongly submitted affidavit, in particular the punitive provisions of § 156 and § 161 of the Criminal Code (up to 1 year imprisonment or a fine at delivering a negligent or 3 years or a fine at a knowingly false affidavit).

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Abstract

We apply techniques in generalized geometry and graded geometry to aspects of gravity theories, reproducing the physical theories with novel mathematical structures underlying the framework. We propose a non-symmetric metric gravity theory from a deformation of a Courant algebroid in generalized geometry where the algebroid axioms remain preserved. This theory extends Einstein's general relativity, by including the kinetic term for a Kalb-Ramond field. In the context of graded geometry, from a particular type of deformation of the graded Poisson brackets, we obtain a teleparallel gravity action, while another type of non-trivial deformation features curvature besides the metric and a connection manifested in the graded Poisson structure.

Using the graded variables from graded geometry, we formulate Galileon theories elegantly in a more compact notation, in which we profit from the formalism in generalizing the theory to arbitrary spacetime dimensions, and arbitrary number and type of fields. The connection to linearised gravity is made by considering mixed-symmetry fields. The main goal is to ensure second order field equations for our Galileon theory.

Finally we end the discussion of the various actions in different theories with investigations on the solutions from Einstein-Gauss-Bonnet-dilaton theory. By studying the black hole solutions of this theory using a quasi-normal mode decomposition, we learn about the mode stability of the black holes, which turn out to be stable under linear pertubations. With data from the future higher sensitive gravitational wave detections, the current large class of modified gravity theory proposals can be strongly constrained, and either favoured or ruled out.

Keywords: Non-symmetric metric; Courant algebroid; Contortion; Graded geometry; Galileon; Mixed-symmetry tensor; Quasi-normal mode; Black hole (mode) stability.

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“. . . I always worked until I had something done and I always stopped when I knew what was going to happen next. That way I could be sure of going on the next day . . . ”

(E. Hemingway, *A Moveable Feast*)

December 15, 2016

Contents

1 Overview and Motivation

Gravitation has long been known to be closely connected to geometry. Einstein's general relativity theory (GR) has since successfully prescribed the geometric nature of gravity. We will begin the thesis by presenting a historical review of attempts in generalizing GR. One of the motivations of the generalization is the quest for the ultimate theory of nature. We will focus on a number of the proposals.

One straightforward generalization of Einstein's theory of gravity takes an anti-symmetric part in addition to the symmetric Riemannian metric into account, as any rank-2 tensor can be expressed generically in symmetric and anti-symmetric parts. This became later known as non-symmetric gravity theory (NGT). A good review of the various models can be found in [1]. It all started with the motivation of Einstein (and Straus) [2] to geometrically unify electromagnetism and gravity. While gravity is described by the symmetric part of the metric, the anti-symmetric part was hoped to represent the anti-symmetric field strength in electromagnetism. The effort was in vain when the theory failed to describe the Lorentz force between charged particles. The history of the conceptual development of the field can be found in [3].

Schrödinger [4] proposed another approach to generalizing Einstein's gravity with a Born-Infeld-like, real Lagrangian, using a non-symmetric Ricci tensor $R_{\mu\nu}$. The Lagrangian density is in a square root of the determinant of the Ricci tensor,

$$\mathcal{L} = \frac{2}{\lambda} \sqrt{-\det R_{\mu\nu}} \ , \tag{1.1}$$

where λ is a real constant. The non-symmetric metric in Schrödinger's proposal is defined as the derivative of the Lagrangian with respect to the Ricci tensor,

$$g^{kl} = \frac{\partial \mathcal{L}}{\partial R_{kl}} \ . \tag{1.2}$$

The connection $\Gamma^i_{k\lambda}$ is treated as a fundamental variable. The Ricci tensor and the non-symmetric metric are connected by a constant λ,

$$R_{kl} = \lambda g_{kl} \ . \tag{1.3}$$

The value of the constant λ, unlike the cosmological constant in Einstein's theory, is irrelevant, though its existence is necessary. The metric g here is a sum of Einstein's symmetric metric and an anti-symmetric tensor. The anti-symmetric tensor is defined like the electromagnetic field strength,

$$F_{ik} = \frac{2}{3} \left(\frac{\partial \Gamma_k}{\partial x_i} - \frac{\partial \Gamma_i}{\partial x_k} \right) \ , \tag{1.4}$$

where

$$\Gamma_\lambda = \frac{1}{2}(\Gamma^\sigma_{\lambda\sigma} - \Gamma^\sigma_{\sigma\lambda}) \ . \tag{1.5}$$

The following relation describes mechano-magnetic phenomena,

$$\partial_\lambda({}^*R_{ik} + F_{ik}) - ({}^*R_{\sigma k} + F_{\sigma k}){}^*\Gamma^\sigma_{i\lambda} - ({}^*R_{i\sigma} + F_{i\sigma}){}^*\Gamma^\sigma_{\lambda k} = 0 \ , \tag{1.6}$$

where

$$^*\Gamma^i_{k\lambda} = \Gamma^i_{k\lambda} + \frac{2}{3}\,\delta^i_k\Gamma_\lambda \tag{1.7}$$

and

$${}^*R_{ik} + F_{ik} = \lambda g_{ik} \ . \tag{1.8}$$

A review of non-symmetric purely affine gravity can be found in [5] . Proposed by Ferraris and Kijowski, the Lagrangian density for the unified electromagnetic and gravitational field is

$$\mathcal{L} = -\frac{e^2}{4}\sqrt{-\det R_{(\mu\nu)}}\; Q_{\alpha\beta}Q_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma} \ , \tag{1.9}$$

where $P^{\mu\nu}$ is reciprocal to $R_{(\mu\nu)}$, $Q_{\mu\nu} = R^\rho{}_{\rho\mu\nu}$ is called the "second Ricci tensor" which represents the electromagnetic field and e has the dimension of an electric charge. The usual Ricci tensor is defined as $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$. In this Lagrangian, the determinant of the symmetric part of Ricci tensor is considered. The expression is further multiplied by quadratic terms of the "second Ricci tensor". This construction is equivalent to the sourceless Einstein-Maxwell equations [6], [7].

The string theory sigma model incorporates both the symmetric metric and an anti-symmetric tensor field into the theory. It became an inspiration to [8] to attempt a similar combination for gravitation. The idea is to construct a geometric theory, with objects built out of a metric. In this work, the generalized non-symmetric metric consists not only of the symmetric $G_{\mu\nu}$ and anti-symmetric $B_{\mu\nu}$ parts, but also terms expanded in the anti-symmetric tensor fields which carry non-trivial index distribution,

$$g_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu} + \alpha B_{\mu\sigma}B^\sigma{}_\nu + \beta B^{\alpha\beta}B_{\alpha\beta}G_{\mu\nu} + \dots \ , \tag{1.10}$$

where its inverse is

$$g^{\mu\nu} = G^{\mu\nu} + B^{\mu\nu} + (1-\alpha)B^{\mu\alpha}B_\alpha{}^\nu - \beta B^{\alpha\beta}B_{\alpha\beta}G^{\mu\nu} + \dots \ , \tag{1.11}$$

while

$$\sqrt{-g} = \sqrt{-G}\left[1 + \frac{1}{2}\left(\frac{1}{2} - \alpha + \beta D\right)B^{\alpha\beta}B_{\alpha\beta}\right] + \dots \tag{1.12}$$

with constants α, β and spacetime dimension D. The non-symmetric connection coefficient is given by

$$\Gamma^\lambda_{\mu\nu}(g) = {}^c\Gamma^\lambda_{\mu\nu}(G) + \frac{1}{2}\left(\nabla^\lambda B_{\mu\nu} - \nabla_\mu B_\nu{}^\lambda - \nabla_\nu B^\lambda{}_\mu\right) \ , \tag{1.13}$$

up to first order in $B_{\mu\nu}$, with the connection ∇ defined with respect to the metric G. ${}^c\Gamma^\lambda_{\mu\nu}(G)$ is the Christoffel symbol. Higher order terms are defined in equation (A.6) in [8]. The authors found that the massless geometric theory contained propagating ghosts, plus gauge invariance of the theory could not be restored. Hence, a model with various parameters was proposed. The Lagrangian is a sum of the Einstein Lagrangian, a massive ghost-free Lagangian for the anti-symmetric field, and matter coupling terms:

$$L = \frac{\sqrt{-G}}{4\kappa^2} R(G) - \frac{\sqrt{-G}}{\kappa^2}\left[\frac{1}{12}H^2 + \frac{1}{4}\mu^2 B^2\right] - \frac{1}{6} f H_{\lambda\mu\nu} J^{*\lambda\mu\nu} + \dots \ , \tag{1.14}$$

where $\kappa^2 = 4\pi G_{\rm N}$ with Newton's gravitational constant $G_{\rm N}$, μ is the mass of B field, the coupling constant f is dimensionless if $B_{\mu\nu}$ is dimensionless, and J^μ is the fermionic current defined by $J^{*\lambda\mu\nu} = \epsilon^{\lambda\mu\nu\alpha} J_\alpha$ with an anti-symmetric constant $\epsilon_{\lambda\mu\nu\alpha} := \epsilon^{\lambda\mu\nu\alpha} = \pm 1, 0$. The part "$\dots$" in the Lagrangian consists of terms such as $L_{\rm BF} = \alpha(B_{\mu\nu}F^{\mu\nu})^2$ for coupling between B and electromagnetic field and for coupling between matter field and metric as in $L_{\rm Dirac} = \sqrt{-G}\ \bar\psi\gamma^a\left(e^\mu_a\partial_\mu + \frac{1}{2}\hat\Gamma^b_{ca}\sigma^c_b\right)\psi$, where e^μ_a is the vierbein and σ^c_b is a polarization tensor. $\hat\Gamma^b_{ca}$ is a general connection coefficient. It is a sum of the Christoffel symbol and other terms in B and H,

$$\hat\Gamma^b_{ca} = {}^c\Gamma^b_{ca}(G) + c\tilde\nabla^b B_{ca} + d\tilde g^{be} H_{eca} + \dots \ , \tag{1.15}$$

where c, d are additional parameters. The connection $\tilde\nabla$ is defined with respect to the $\tilde g$ which is (1.10) but without the odd number of B.

In the non-symmetric gravitational theory of [9], the non-symmetric metric is defined as

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]} \ , \tag{1.16}$$

where

$$g_{(\mu\nu)} = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu}) \ , \qquad g_{[\mu\nu]} = \frac{1}{2}(g_{\mu\nu} - g_{\nu\mu}) \ . \tag{1.17}$$

The contracted curvature tensor defined in terms of the non-symmetric connection[1] W is

$$R_{\mu\nu}(W) = W^\beta_{\mu\nu,\beta} - \frac{1}{2}(W^\beta_{\mu\beta,\nu} + W^\beta_{\nu\beta,\mu}) - W^\beta_{\alpha\nu}W^\alpha_{\mu\beta} + W^\beta_{\alpha\beta}W^\alpha_{\mu\nu} \tag{1.18}$$

$$= R_{\mu\nu}(\Gamma) + \frac{2}{3}W_{[\mu,\nu]} \ , \tag{1.19}$$

where

$$R_{\mu\nu}(\Gamma) = \Gamma^\beta_{\mu\nu,\beta} - \frac{1}{2}\left(\Gamma^\beta_{(\mu\beta),\nu} + \Gamma^\beta_{(\nu\beta),\mu}\right) - \Gamma^\beta_{\alpha\nu}\Gamma^\alpha_{\mu\beta} + \Gamma^\beta_{(\alpha\beta)}\Gamma^\alpha_{\mu\nu} \ , \tag{1.20}$$

$$W_\mu = \frac{1}{2}(W^\lambda_{\mu\lambda} - W^\lambda_{\lambda\mu}) \ , \tag{1.21}$$

$$W^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{2}{3}\delta^\lambda_\mu W_\nu \ . \tag{1.22}$$

(1.22) leads to

$$\Gamma_\mu = \Gamma^\lambda_{[\mu\lambda]} = 0 \tag{1.23}$$

[1] As usual, $W^\beta_{\mu\nu,\beta}$ denotes $\partial_\beta W^\beta_{\mu\nu}$.

in 4 spacetime dimensions[2]. By

$$R_{\mu\nu}(W) = R^{\lambda}{}_{\mu\lambda\nu}(W) \ , \tag{1.24}$$

we can work out the curvature in W

$$R^{\mu}{}_{\nu\alpha\beta}(W) = \partial_{\alpha} W^{\mu}_{\nu\beta} - W^{\mu}_{\sigma\beta} W^{\sigma}_{\nu\alpha} - \frac{1}{2}(\partial_{\beta} W^{\mu}_{\nu\alpha} + \partial_{\nu} W^{\mu}_{\beta\alpha}) + W^{\mu}_{\sigma\alpha} W^{\sigma}_{\nu\beta} \tag{1.25}$$

and thus in Γ

$$R^{\mu}{}_{\nu\alpha\beta}(\Gamma) = \partial_{\alpha} \Gamma^{\mu}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta} \Gamma^{\sigma}_{\nu\alpha} - \frac{1}{2}(\partial_{\beta} \Gamma^{\mu}_{(\nu\alpha)} + \partial_{\nu} \Gamma^{\mu}_{(\beta\alpha)}) + \Gamma^{\mu}_{(\sigma\alpha)} \Gamma^{\sigma}_{\nu\beta} \ . \tag{1.26}$$

The NGT Lagrangian density proposed in [9] is

$$\mathcal{L}_{\mathrm{NGT}} = \mathcal{L}_R + \mathcal{L}_M \ , \tag{1.27}$$

where

$$\mathcal{L}_R = \mathbf{g}^{\mu\nu} R_{\mu\nu}(W) - 2\lambda\sqrt{-g} - \frac{1}{4}\mu^2 \mathbf{g}^{\mu\nu} g_{[\nu\mu]} - \frac{1}{6} g^{\mu\nu} W_{\mu} W_{\nu} \tag{1.28}$$

with a non-symmetric metric density $\mathbf{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$, where λ is the cosmological constant and μ^2 is here an additional cosmological constant (with dimension of an inverse mass), and the matter Lagrangian density is

$$\mathcal{L}_M = -8\pi g^{\mu\nu} \mathbf{T}_{\mu\nu} \ , \tag{1.29}$$

where $\mathbf{T}_{\mu\nu}$ is an energy-momentum tensor density. Note that

$$g^{\mu\nu} g_{\sigma\nu} = g^{\nu\mu} g_{\nu\sigma} = \delta^{\mu}_{\sigma} \ . \tag{1.30}$$

In first-order formalism, varying the Lagrangian density with respect to the non-symmetric connection $W^{\sigma}_{\mu\nu}$ gives[3]

$$\mathbf{g}^{\mu\nu}{}_{,\sigma} + \mathbf{g}^{\rho\nu} W^{\mu}_{\rho\sigma} + \mathbf{g}^{\mu\rho} W^{\nu}_{\sigma\rho} - \mathbf{g}^{\mu\nu} W^{\rho}_{\sigma\rho} + \frac{2}{3}\delta^{\nu}_{\sigma} \mathbf{g}^{\mu\rho} W^{\beta}_{[\rho\beta]} + \frac{1}{6}(\mathbf{g}^{(\mu\rho)} W_{\beta} \delta^{\nu}_{\sigma} - \mathbf{g}^{(\nu\rho)} W_{\beta} \delta^{\mu}_{\sigma}) = 0 \ . \tag{1.32}$$

Assuming $\lambda = 0$, in linear approximation, the non-symmetric metric is expanded about the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$,

$$g_{\mu\nu} = \eta_{\mu\nu} + {}^{(1)}h_{\mu\nu} + \ldots , \tag{1.33}$$

where ${}^{(1)}h_{[\mu\nu]}$ is let to be equal to $\psi_{\mu\nu}$. The anti-symmetric part of the sourceless field equations can be obtained from the Proca-type Lagrangian,

$$\mathcal{L}_{\psi} = \frac{1}{4}\psi^2_{\mu\nu,\lambda} - \frac{1}{2}\psi^2_{\mu} - \frac{1}{4}\mu^2 \psi^2_{\mu\nu} \ , \tag{1.34}$$

[2]Comment: To consider general spacetime dimension D, we may replace the 3 in the denominator in (1.22) with $D-1$.

[3] Comment: It shows a resemblance to the standard metricity condition from GR,

$$\nabla_i g_{jk} = \partial_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} = 0 \ . \tag{1.31}$$

where $\psi_\mu = \frac{16\pi}{\mu^2} T_{[\mu\nu]}{}^{,\nu}$. It was proven in [10] that the massive Proca-type theory is free of ghosts.

In [11], solutions of the non-symmetric gravity theory [9] containing electromagnetic fields are studied. It was found in [9] that the non-symmetric metric carries only gravitational effect. Therefore, to consider additionally the electromagnetic effect, the following NGT Lagrangian density which includes electromagnetism and sources was proposed [12],

$$\mathcal{L} = \mathbf{g}^{\mu\nu} R_{\mu\nu}(W) + \sqrt{-g}\,(\kappa(g^{[\mu\nu]}F_{\mu\nu})^2 - H^{\mu\nu}F_{\mu\nu}) + \mathcal{L}_M \ . \tag{1.35}$$

$R_{\mu\nu}$ is the NGT contracted curvature tensor (1.18), $F_{\mu\nu}$ is the electromagnetic field, $H_{\mu\nu}$ is a skew tensor defined as $g_{\sigma\beta}g^{\gamma\sigma}H_{\gamma\alpha} + g_{\alpha\sigma}g^{\sigma\gamma}H_{\beta\gamma} = 2g_{\alpha\sigma}g^{\sigma\gamma}F_{\beta\gamma}$, and κ is a coupling constant. The contravariant tensor $g^{\mu\nu}$ satisfies (1.30). When $F_{\mu\nu}$ vanishes, the theory reduces to the NGT of [9].
By introducing an NGT charge ℓ^2 which has been identified with an integration constant resulted from the solution of one of the field equations, and a dimensionless constant s which is also related to the solution of another field equation for the static spherically symmetric case, together with the ordinary mass M and electric charge Q which usually contain in Einstein's gravity theory solutions, the resulting field equations in the theory do not possess singularities. The authors argued to replace the black hole with a new stable, superdense object (SDO). For sufficiently small ℓ^2 and s, the theory agrees well with the experimental observations. Using the $g_{\mu\nu}$ form derived by Papapetrou [13] for a static spherically symmetric field (cf. equation (3.1) in [11]), for a body in this non-singular NGT solution, the proper volume when $Q = \ell^2 = 0$ near $r = 0$ is given by $V = \frac{16\pi r^2 M}{s}\exp(-\pi/s - 2)$, where the surface area is $4\pi r^2$ and the circumference is $2\pi r$. However, it is observationally difficult to distinguish an SDO from a black hole [11].

Another recent motivation to look into modifications of the theory of gravity is the hope that the dark matter will be naturally incorporated into a modified theory. In [14], the authors are interested in the relevant cosmological implications. In the limit of small anti-symmetric field (denoted as B-field), to avoid alteration of the original GR theory, they found that an ad hoc mass m for the B-field is needed. A concern is that an instability of the B-field could occur in the different eras of the universe. The parameters in the Lagrangian thus are needed to be treated or even fine-tuned in the favor of B-field stabilities. A guiding principle to appropriately linearise NGT is still missing. The full linearised B-field Lagrangian was proposed and argued to be of the necessary form [15],

$$\begin{aligned}\mathcal{L} &= \sqrt{-G}\left(R + 2\Lambda - \frac{1}{12}H^2 + (\frac{1}{4}m^2 + \beta R)B^2\right) \\ &\quad + \sqrt{-G}\left(-\alpha R_{\mu\nu}B^{\mu\alpha}B_\alpha{}^\nu - \gamma R_{\mu\alpha\nu\beta}B^{\mu\nu}B^{\alpha\beta}\right) + \mathcal{O}(B^3) \ ,\end{aligned} \tag{1.36}$$

defined with respect to the GR background. The natural choice for the parameters according to the authors is $\alpha = \gamma$. It is remarked that the anti-symmetric B-field as a dark matter candidate indicates a modification to gravity at a certain length scale.

The search for a quantum gravity theory has invited numerous approaches. String theory is an ambitious proposal as a quantum theory to unify gravity and all other fundamental forces in nature. The low energy limit of superstring theory gives the supergravity action (SUGRA). At high energies (that is, short distances), gravity is deemed to behave radically different from GR. SUGRA is viewed as an extension to GR. In SUGRA, infinities in the S-matrix cancel at first and second oder of quantum corrections due to symmetry between bosons and fermions [16]. However, it remains an open question if the cancellations persist at higher order. From the supergravity action (or low-energy effective string action), the higher order corrections which are also known as α'-corrections can be realized. This is an active quest in itself, to study the stringy effects or even beyond in these α' expansion of the effective action.

Einstein's general relativity has been well tested and confirmed by experiments, validated in the weak field regime. Extension to GR or modifications that build on GR provide a rich and illuminating platform to test strong gravitational interactions in nature. The Dvali-Gabadadze-Porrati (DGP) [17] cosmological-model-inspired Galileon theory is a modification to gravity with a scalar degree of freedom. The DGP model has generated interest for its self-accelerating description of the universe without the need of dark energy. It is a model more celebrated at least for its infra-red modification, although it is plagued by some problems at the quantum level [18]. All our successful physical theories, such as general relativity, electrodynamics, and Yang-Mills theory, have field equations that contain derivatives up to second order. This fact is related to Ostrogradsky's theorem [19], which states that theories with field equations beyond second order in derivatives generically lead to instabilities and contain additional ghost-like degrees of freedom. It is hence reasonable to investigate the most general Lagrangian which leads to second order field equations, for a given degree of freedom in arbitrary spacetime dimensions. For the case of a metric tensor, the answer is in the work of Lovelock who classified the Lagrangians in terms of invariants of the Riemann tensor [20]. Later, Horndeski came up with the solution for a scalar-tensor theory in four dimensions [21]. In the case of scalar fields in flat spacetime, such theories are known as Galileons [22], or generalized Galileons when one permits also first or no derivatives acting on the scalar fields in the equations of motion [23], [24], [25]. An interesting correspondence between scalar Galileon theory and general relativity is established in [26], where notions of the Levi-Civita connection and Riemann tensor have a Galileon analogue.

Another approach to modify gravity is to involve higher curvature terms. In particular, we are interested in the Einstein-Gauss-Bonnet-dilaton theory (EGBd), which as well ensures second order field equations, thus has no ghosts in the theory. Similar to the DGP model, this theory passes solar-system-like tests of gravity. However, differences of EGBd with GR can be noticed close to compact objects like black holes. The dilatonic black holes in this theory have a regular event horizon and are asymptotically flat at infinity [27].

Gravitational waves radiated at the unique frequency of the quasi-normal modes shows an indication of spacetime distortion caused by accelerating massive bodies (black holes, neutron stars). These quasi-normal modes are predicted by the perturbation equations. Results from perturbations and full non-linear evolutions of Einstein equations for collisions of black holes are in agreement [28]. The recent LIGO/Virgo detection of the coalescence of two black holes [29], starting from inspiral stage, to merger phase, and to ringing (where quasi-normal modes are relevant) of the newly created black hole, will help us to discriminate the ultimate theory of gravitation. In this thesis, we are working on black hole quasi-normal modes. For neutron star quasi-normal modes, see [30].

1.1 Thesis Outline

This thesis is mainly gravity motivated and implicitly string inspired. As we are interested in the geometrical description of nature (gravity in particular), our work emphasizes greatly the mathematical applications in physics. The different chapters in the thesis are each written in a self-contained way. We will be at times pragmatic with the mathematical subtleties, where the related mathematical background information is mostly stated without proofs. Nonetheless, it has been attempted to be sufficiently well-defined, specially catered for readers with a physics background. Since this is a study in physics, the thesis is inclined to actions or Lagrangians and in the last chapter the equations of motion of the gravity theories. The discussion in the thesis ranges from classical gravity to modified gravity theories, in which aspects of quantum gravity are left for future work. The thesis contains some still to-be-published material. Results from this thesis which have already been published are cited in the reference list of the thesis.

The structure of the main content of the thesis is as follows:

In the opening chapter 2, we introduce geometrical objects, namely metric, connection and Courant algebroids in generalized geometry to obtain a reformulation of gravity that resembles the effective string action. The connection structure in this specific example of a Courant algebroid naturally incorporates both a metric and a 2-form B-field. The resulting action, without dilaton, is a non-symmetric gravity action, which is a classical gravity theory generalized with a contortion which is a skew 3-form.

In chapter 3, graded geometry, which is a mathematical framework closely related to generalized geometry, is employed to study similar structure deformations as performed in the previous chapter 2. Here, a different connection is discovered after introducing a non-symmetric metric into the graded structures. Hence another gravity theory can be laid out. This work was motivated as a step towards quantization of the respective mathematical structure. Roughly speaking, canonical quantization can be achieved by replacing Poisson brackets with commutators. This will be an outlook for further investigations.

In chapter 4, using the graded mathematical objects from the previous chapter 3, linearised Einstein gravity and its corrections are realized from a Galileon-type action for mixed-symmetry tensor fields. Furthermore, we study couplings of curvature with the scalar and p-form Galileons in this graded formalism. Given the polynomials of curvature terms arising from the generalization of the actions in flat spacetime, the Galileon theories in curved spacetime are known as a type of modified gravity theory. Covariantization of the mixed-symmetry fields is also investigated.

The final chapter 5 presents a transition from the theory to physical observations, in specific, gravitational wave radiation. Solutions of a string-inspired modified gravity theory known as the EGBd, that is, the equations of motion are analyzed. The focus is directed to the mode stability of black holes[4] in the EGBd theory against linear perturbations, by studying the axial and polar quasi-normal modes of the solutions. In this analysis, a computer software is used to work out the non-trivial field equations.

[4] The photon sphere is known to be related as well to the stability of a black hole. In [31], we revisited the photon orbits of extremal GR black holes. However we will not cover that topic here. The connection of photon spheres to quasi-normal modes can be found in [32]. One of the outlooks from our work will be to extend the study to modified gravity theories.

2 Generalized Geometry

2.1 Introduction

Let us start with a brief review of Riemannian geometry and then continue with an introduction to generalized geometry.

Riemannian geometry has been known well as the geometry subjected to underlying physics. From general relativity, we are familiar with tensor objects such as metric tensor and curvature tensor, in addition to the non-tensorial connection. The mathematical definition for a Riemannian metric g on a differentiable manifold M, is a type $(0,2)$ tensor field which satisfies the axioms

$$\begin{aligned} g_p(X,Y) &= g_p(Y,X) \ , \qquad (2.1)\\ g_p(X,X) &\geq 0, \text{ where equality holds for } X=0 \ , \qquad (2.2)\end{aligned}$$

at each point $p \in M$ [33]. The Riemannian metric is symmetric and positive definite. It is a bilinear form which takes vectors $X, Y \in T_pM$, where T_pM is the tangent space of M at p. In coordinate basis, it is

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = g_{\nu\mu} \qquad (2.3)$$

at point p.
An affine connection ∇ is a map $\nabla : TM \times TM \to TM$, for the tangent bundle TM. It sends two smooth vector fields to a new smooth vector field and satisfies

$$\begin{aligned} \nabla_X(Y+Z) &= \nabla_X Y + \nabla_X Z \ , \qquad (2.4)\\ \nabla_{(X+Y)}Z &= \nabla_X Z + \nabla_Y Z \ , \qquad (2.5)\\ \nabla_{(fX)}Y &= f\nabla_X Y \ , \qquad (2.6)\\ \nabla_X(fY) &= X[f]Y + f\nabla_X Y \ , \qquad (2.7)\end{aligned}$$

for smooth function, $f \in C^\infty(M)$. In a coordinate basis $\{e_\mu\} = \{\partial/\partial x^\mu\}$, with vector field $X = X^\mu e_\mu$, the covariant derivative of Y with respect to X is

$$\nabla_X Y = X^\mu \left(\frac{\partial Y^\kappa}{\partial x^\mu} + Y^\nu \Gamma^\kappa_{\mu\nu}\right) e_\kappa \ . \qquad (2.8)$$

$\Gamma^\kappa_{\mu\nu}$ is the connection coefficient.
Curvature is a mapping $R : TM \times TM \times TM \to TM$,

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \ . \qquad (2.9)$$

Since curvature is composed of connection, it inherits properties of a connection.

$$\begin{aligned} R(X+Y,W)Z &= R(X,W)Z + R(Y,W)Z \ , \qquad (2.10)\\ R(X,Y)(Z+W) &= R(X,Y)Z + R(X,Y)W \ , \qquad (2.11)\\ R(fX,gY)hZ &= fghR(X,Y)Z \ . \qquad (2.12)\end{aligned}$$

The curvature tensor in a coordinate basis is

$$R^{\mu}{}_{\nu\lambda\kappa} = \partial_\lambda \Gamma^{\mu}_{\kappa\nu} - \partial_\kappa \Gamma^{\mu}_{\lambda\nu} + \Gamma^{\mu}_{\lambda\epsilon}\Gamma^{\epsilon}_{\kappa\nu} - \Gamma^{\mu}_{\kappa\epsilon}\Gamma^{\epsilon}_{\lambda\nu} \ . \tag{2.13}$$

The symmetries of the Riemann curvature tensor are

$$R_{\mu\nu\lambda\kappa} = R_{\lambda\kappa\mu\nu} \tag{2.14}$$
$$R_{\mu\nu\lambda\kappa} = -R_{\mu\nu\kappa\lambda} \tag{2.15}$$
$$R_{\mu\nu\lambda\kappa} = -R_{\nu\mu\lambda\kappa} \ . \tag{2.16}$$

It satisfies the First and Second Bianchi identities

$$R_{\mu[\nu\lambda\kappa]} = 0, \quad R_{\mu\nu[\lambda\kappa;\epsilon]} = 0 \ , \tag{2.17}$$

where ";" denotes a covariant derivative. However, the Bianchi identities are modified when torsion is taken into account.

A smooth manifold M endowed with a Riemannian metric g is a Riemannian manifold (M, g). Examples are d-dimensional Euclidean space $(\mathbb{R}^d, \delta)$ and Minkowski space $(\mathbb{R}^d, \eta)$. With the metric, we can define distance, which is very useful in physics, by the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ . \tag{2.18}$$

For tensors, in particular, for a vector field V, the equation of parallel transport along a path $x^\kappa(\lambda)$ is

$$\frac{dV^\mu}{d\lambda} + \Gamma^{\mu}_{\kappa\nu} V^\nu \frac{dx^\kappa}{d\lambda} = 0 \ , \tag{2.19}$$

with λ being the parameter that parametrizes the path. For the vector field $V^\mu = \frac{dx^\mu}{d\lambda}$, this gives the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\kappa\nu} \frac{dx^\nu}{d\lambda}\frac{dx^\kappa}{d\lambda} = 0 \ . \tag{2.20}$$

Let us first review the definitions of a Lie algebra and a Lie algebroid, before getting to the definition of a Courant algebroid.

Definition.

A Lie algebra $\mathfrak{g}$ is a vector space equipped with a bracket operation $[\cdot,\cdot]_{\text{Lie}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is a $\mathbb{R}$-bilinear map and is anti-symmetric

$$[V, W]_{\text{Lie}} = -[W, V]_{\text{Lie}} \ . \tag{2.21}$$

The Lie bracket satisfies the Jacobi identity

$$[U, [V, W]_{\text{Lie}}]_{\text{Lie}} + [V, [W, U]_{\text{Lie}}]_{\text{Lie}} + [W, [U, V]_{\text{Lie}}]_{\text{Lie}} = 0 \ . \tag{2.22}$$

When E is a collection of isomorphic vector spaces E_p at each point $p \in M$:

Definition.
Let $A \xrightarrow{\pi} M$ be a vector bundle.
Let $[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ be a $\mathbb{R}$-bilinear map, for $e, e' \in \Gamma(A)$ smooth sections.
Map $a : A \to TM$ is an anchor.
$\therefore$ $(A, [\cdot,\cdot]_A, a)$ is a Lie algebroid [34], [35] *when* $[\cdot,\cdot]_A$ is anti-symmetric (Lie bracket), satisfying Jacobi identity and Leibniz rule

$$[e, fe']_A = f[e,e']_A + (a(e)\cdot f)e' \ . \tag{2.23}$$

Generalized geometry can be viewed as an extension of Riemannian geometry, in the sense of summing tangent and cotangent bundles of the manifold. On this generalized bundle, generalized complex structures were introduced by Nigel Hitchin [36] which unify symplectic and complex geometries. In this thesis, we will not focus on generalized complex structures. Instead, Courant algebroids, being the central and fundamental objects in generalized geometry are what we will be discussing. They also turn out to be relevant for non-symmetric gravity.

Definition.
Let $E \xrightarrow{\pi} M$ be a vector bundle.
$\forall e, e' \in \Gamma(E)$ smooth sections,
let $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$,
let $\langle\cdot,\cdot\rangle_E : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ be a symmetric $C^\infty(M)$-bilinear non-degenerate form.
Map $a : E \to TM$ is an anchor. That is, anchor applied to e gives a vector field.
$\therefore$ $(E, [\cdot,\cdot]_E, \langle\cdot,\cdot\rangle_E, a)$ is a Courant algebroid [37], [38], satisfying certain properties, which will become clear in the next section.

Let us tabulate the important differences in the geometries,

Conventional geometry	Generalized geometry
$V \in \Gamma(TM)$	$\mathbb{V} \in \Gamma(E)$
Lie derivative, $\mathcal{L}_V Y = [V,Y]_{\text{Lie}}$	Dorfman derivative
Lie bracket	Dorfman bracket, Courant bracket
Riemannian metric, g	Generalized metric, $\mathbf{G}$
$GL(d)$ symmetry	$O(d,d)$ symmetry
Diffeomorphism	Diffeomorphism + Gauge transformation

Let us go through the objects in generalized geometry and give their formulae:

<u>Generalized vector</u>

We are considering in our work the simplest example of a generalized tangent bundle, $E = TM \bigoplus T^*M$, which is a direct sum of tangent and cotangent bundles. Hence the generalized tangent vector, being the element of E, $\mathbb{V} = V + \lambda$, is a formal sum of a vector V and a 1-form λ, respectively, from the $2d$-dimensional fibers.

Inner product

As a fiber-wise metric on E, the inner product,

$$\langle \mathbb{V}, \mathbb{Y} \rangle = \langle V + \lambda, Y + \sigma \rangle = i_V\sigma + i_Y\lambda \tag{2.24}$$

$$= \sigma(V) + \lambda(Y) \tag{2.25}$$

$$= \begin{pmatrix} V & \lambda \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1}^d \\ \mathbb{1}^d & 0 \end{pmatrix} \begin{pmatrix} Y \\ \sigma \end{pmatrix} , \tag{2.26}$$

is a bilinear pairing. It has an indefinite (d, d) signature, hence the inner product has an $O(d, d)$ symmetry. Examples of $O(d, d)$ transformations are [39]:

B-transform,

$$e^B \begin{pmatrix} V \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} V \\ \lambda \end{pmatrix}, \qquad B : TM \to T^*M, B \in \Omega^2(M) \tag{2.27}$$

Diffeomorphism,

$$O_N \begin{pmatrix} V \\ \lambda \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & N^{-T} \end{pmatrix} \begin{pmatrix} V \\ \lambda \end{pmatrix}, \qquad N : TM \to TM, N \in GL(d), \text{ where } N^{-T} = (N^{-1})^T \tag{2.28}$$

β-transform,

$$e^\beta \begin{pmatrix} V \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V \\ \lambda \end{pmatrix}, \qquad \beta : T^*M \to TM, \beta \in \mathfrak{X}^2(M) \ . \tag{2.29}$$

The inner product is invariant under B-transform,

$$\langle e^B(\mathbb{V}), e^B(\mathbb{Y}) \rangle = \langle \mathbb{V} - B(V, \cdot), \mathbb{Y} - B(Y, \cdot) \rangle \tag{2.30}$$

$$= \langle \mathbb{V}, \mathbb{Y} \rangle - B(V, Y) - B(Y, V) \tag{2.31}$$

$$= \langle \mathbb{V}, \mathbb{Y} \rangle \ , \tag{2.32}$$

as B is anti-symmetric, that is, $B(V, Y) = -B(Y, V)$.

Dorfman derivative and Dorfman bracket

It involves a combination of Lie derivative of vector and of form, and interior product of form,

$$L_{\mathbb{V}}(\mathbb{Y}) = \mathcal{L}_V Y + \mathcal{L}_V \sigma - i_Y d\lambda = [V, Y]_{\text{Lie}} + \mathcal{L}_V \sigma - i_Y d\lambda = [\mathbb{V}, \mathbb{Y}]_{\text{D}} \ . \tag{2.33}$$

The Dorfman bracket $[\mathbb{V}, \mathbb{Y}]_{\text{D}}$ [40], [41] satisfies a Jacobi identity (2.48). It is related to a Dirac structure. A Dirac structure on a manifold M [42] is a maximally isotropic subbundle of the generalized tangent bundle $E = TM \bigoplus T^*M$ under the symmetric pairing (2.24). If it is an integrable Dirac structure, the space of sections of a Dirac bundle $\mathfrak{L}$ (Dirac structure), $\Gamma(\mathfrak{L})$ is closed under the Dorfman bracket. The integrability condition is implied by

$$[\Gamma(\mathfrak{L}), \Gamma(\mathfrak{L})]_{\rm D} \subseteq \Gamma(\mathfrak{L}) \ , \qquad (2.34)$$

that is, $\mathfrak{L}$ is involutive with respect to $[\ ,\ ]_{\rm D}$, or equivalently

$$\langle [e_1, e_2]_{\rm D}, e_3 \rangle = 0 \ , \qquad (2.35)$$

for $e_1, e_2, e_3 \in \Gamma(\mathfrak{L})$.

<u>Courant bracket</u>

$$[\mathbb{V}, \mathbb{Y}]_{\rm Cou} = [V, Y]_{\rm Lie} + \mathcal{L}_V \sigma - \mathcal{L}_Y \lambda - \frac{1}{2} d(i_V \sigma - i_Y \lambda) \qquad (2.36)$$

is the anti-symmetrization of Dorfman bracket. However, it does not obey a Jacobi identity. The Courant bracket [43] is invariant under B-transform,

$$[e^B(\mathbb{V}), e^B(\mathbb{Y})]_{\rm Cou} = e^B\left([\mathbb{V}, \mathbb{Y}]_{\rm Cou}\right) + i_Y i_V dB \ , \qquad (2.37)$$

iff B is closed, that is, $dB = 0$. The derivation of (2.37) is as follows [44], for $\mathbb{V} = V + \lambda$ and $\mathbb{Y} = Y + \sigma$,

$$\begin{aligned}
& [e^B(V + \lambda), e^B(Y + \sigma)]_{\rm Cou} \\
= \ & [V + \lambda + i_V B, Y + \sigma + i_Y B]_{\rm Cou} \\
= \ & [V + \lambda, Y + \sigma]_{\rm Cou} + [i_V B, Y]_{\rm Cou} + [V, i_Y B]_{\rm Cou} \\
= \ & [V + \lambda, Y + \sigma]_{\rm Cou} - \mathcal{L}_Y(i_V B) + \frac{1}{2} d i_Y (i_V B) + \mathcal{L}_V(i_Y B) - \frac{1}{2} d(i_V i_Y B) \\
= \ & [V + \lambda, Y + \sigma]_{\rm Cou} + \mathcal{L}_V i_Y B - \mathcal{L}_Y i_V B + \frac{1}{2}\mathcal{L}_Y i_V B - \frac{1}{2} i_Y d(i_V B) \\
& -\frac{1}{2}\mathcal{L}_V(i_Y B) + \frac{1}{2} i_V d(i_Y B) \\
= \ & [V + \lambda, Y + \sigma]_{\rm Cou} + \mathcal{L}_V i_Y B - \mathcal{L}_Y i_V B \\
& +\frac{1}{2}\mathcal{L}_Y i_V B - \frac{1}{2} i_Y \mathcal{L}_V B + \frac{1}{2} i_Y i_V dB \\
& -\frac{1}{2}\mathcal{L}_V i_Y B - \frac{1}{2} i_V \mathcal{L}_Y B + \frac{1}{2} i_V i_Y dB \\
= \ & [V + \lambda, Y + \sigma]_{\rm Cou} + i_{[V,Y]_{\rm Lie}} B + i_Y i_V dB \\
= \ & e^B([V + \lambda, Y + \sigma]_{\rm Cou}) + i_Y i_V dB \ , \qquad (2.38)
\end{aligned}$$

where formulae

$$\mathcal{L}_V i_Y - i_Y \mathcal{L}_V = i_{[V,Y]_{\rm Lie}} \ , \qquad (2.39)$$

$$i_V i_Y = -i_Y i_V \qquad \text{and} \qquad i_{[V,Y]_{\rm Lie}} = -i_{[Y,V]_{\rm Lie}} \qquad (2.40)$$

have been used.

Generalized metric

$$\mathbf{G} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \tag{2.41}$$

$$= \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} . \tag{2.42}$$

It unifies Riemannian metric g and 2-form B, and is positive definite [44].
Given a bivector field β on the manifold, investigated in [45], a new generalized metric

$$H' = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} G - bG^{-1}b & bG^{-1} \\ -G^{-1}b & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \tag{2.43}$$

is obtained. Identifying it with a generalized metric with unique Riemannian metric g and 2-form B, one gets the closed-open string backgrounds relation,

$$(g + B)^{-1} = \beta + (G + b)^{-1} : \text{Seiberg-Witten duality [46]} \tag{2.44}$$

for a Poisson β, where metric g and 2-form B in the closed string picture are related to the parameters G and b in the open string sector.

2.2 Courant Algebroid

Given a vector bundle, which we sometimes refer to as the generalized tangent bundle E, endow on it a pairing, a Dorfman bracket and an anchor map, we define a Courant algebroid $(E, \langle\ ,\ \rangle, [\ ,\ ]_{\mathrm{D}}, a)$, which is of our particular interest. A minimal set of Courant algebroid axioms are:
(i) Compatibility with the pairing

$$a(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2]_{\mathrm{D}}, e_3\rangle + \langle e_2, [e_1, e_3]_{\mathrm{D}}\rangle . \tag{2.45}$$

This condition can be interpreted as a Killing equation.
(ii) Non-antisymmetry

$$a^\dagger d\langle e_1, e_2\rangle = [e_1, e_2]_{\mathrm{D}} + [e_2, e_1]_{\mathrm{D}} . \tag{2.46}$$

(iii)

$$\langle fe_1, e_2\rangle = f\langle e_1, e_2\rangle . \tag{2.47}$$

(iv) Jacobi identity

$$[e_1, [e_2, e_3]_{\mathrm{D}}]_{\mathrm{D}} = [[e_1, e_2]_{\mathrm{D}}, e_3]_{\mathrm{D}} + [e_2, [e_1, e_3]_{\mathrm{D}}]_{\mathrm{D}} . \tag{2.48}$$

The Dorfman bracket obeys the Leibniz rule with respect to the bracket itself. We remark that the Dorfman bracket does not satisfy

$$[e_1, [e_2, e_3]_{\mathrm{D}}]_{\mathrm{D}} + [e_2, [e_3, e_1]_{\mathrm{D}}]_{\mathrm{D}} + [e_3, [e_1, e_2]_{\mathrm{D}}]_{\mathrm{D}} = 0 . \tag{2.49}$$

We note that the Leibniz rule,

$$[e_1, fe_2]_{\mathrm{D}} = (a(e_1)f)e_2 + f[e_1, e_2]_{\mathrm{D}} \tag{2.50}$$

follows from (2.45).

Proof:

In (2.45), let $e_2 \to fe_2$,

$$a(e_1)\langle fe_2, e_3\rangle = \langle [e_1, fe_2]_{\mathrm{D}}, e_3\rangle + \langle fe_2, [e_1, e_3]_{\mathrm{D}}\rangle \ . \tag{2.51}$$

On the other hand,

$$\begin{aligned} a(e_1)\langle fe_2, e_3\rangle &= a(e_1)(f\langle e_2, e_3\rangle) && (2.52)\\ &= (a(e_1)f)\langle e_2, e_3\rangle + a(e_1)f(\langle e_2, e_3\rangle) && (2.53)\\ &= (a(e_1)f)\langle e_2, e_3\rangle + f\langle [e_1, e_2]_{\mathrm{D}}, e_3\rangle + f\langle e_2, [e_1, e_3]_{\mathrm{D}}\rangle \ , && (2.54) \end{aligned}$$

where (2.47) has been used.

From (2.51) and (2.54),

$$\begin{aligned} \langle [e_1, fe_2]_{\mathrm{D}}, e_3\rangle &= (a(e_1)f)\langle e_2, e_3\rangle + f\langle [e_1, e_2]_{\mathrm{D}}, e_3\rangle && (2.55)\\ &= \langle (a(e_1)f)e_2 + f[e_1, e_2]_{\mathrm{D}}, e_3\rangle \ , && (2.56) \end{aligned}$$

we conclude

$$[e_1, fe_2]_{\mathrm{D}} = (a(e_1)f)e_2 + f[e_1, e_2]_{\mathrm{D}} \ . \tag{2.57}$$

■

Thus the Dorfman bracket obeys the Leibniz rule with respect to the multiplication of the generalized tangent vector e_2 by a function f, fe_2.

While the homomorphism

$$a([e_1, e_2]_{\mathrm{D}}) = [a(e_1), a(e_2)]_{\mathrm{Lie}} \tag{2.58}$$

follows from the Jacobi identity (2.48).

Proof:

In (2.48), let $e_3 \to fe_3$,

$$[e_1, [e_2, fe_3]_{\mathrm{D}}]_{\mathrm{D}} = [[e_1, e_2]_{\mathrm{D}}, fe_3]_{\mathrm{D}} + [e_2, [e_1, fe_3]_{\mathrm{D}}]_{\mathrm{D}} \ . \tag{2.59}$$

Using Leibniz rule (2.50), on the LHS of (2.59),

$$\begin{aligned} [e_1, [e_2, fe_3]_{\mathrm{D}}]_{\mathrm{D}} &= [e_1, f[e_2, e_3]_{\mathrm{D}} + (a(e_2)f)e_3]_{\mathrm{D}} && (2.60)\\ &= [e_1, f[e_2, e_3]_{\mathrm{D}}]_{\mathrm{D}} + [e_1, (a(e_2)f)e_3]_{\mathrm{D}} && (2.61)\\ &= f[e_1, [e_2, e_3]_{\mathrm{D}}]_{\mathrm{D}} + (a(e_1)f)[e_2, e_3]_{\mathrm{D}} && (2.62)\\ &\quad +(a(e_2)f)[e_1, e_3]_{\mathrm{D}} + (a(e_1)a(e_2)f)e_3 \ , \end{aligned}$$

while on the RHS of (2.59),

$$\begin{aligned}[[e_1,e_2]_{\rm D},fe_3]_{\rm D}+[e_2,[e_1,fe_3]_{\rm D}]_{\rm D} &= f[[e_1,e_2]_{\rm D},e_3]_{\rm D}+(a([e_1,e_2]_{\rm D})f)e_3 \\ &\quad +f[e_2,[e_1,e_3]_{\rm D}]_{\rm D}+(a(e_2)f)[e_1,e_3]_{\rm D} \\ &\quad +(a(e_1)f)[e_2,e_3]_{\rm D}+(a(e_2)a(e_1)f)e_3\ . \end{aligned} \tag{2.63}$$

Equate (2.62) and (2.63),

$$f[e_1,[e_2,e_3]_{\rm D}]_{\rm D}+(a(e_1)a(e_2)f)e_3 = f[[e_1,e_2]_{\rm D},e_3]_{\rm D}+(a([e_1,e_2]_{\rm D})f)e_3 +f[e_2,[e_1,e_3]_{\rm D}]_{\rm D}+(a(e_2)a(e_1)f)e_3 \tag{2.64}$$

$$(a(e_1)a(e_2)f)e_3 = (a([e_1,e_2]_{\rm D})f)e_3+(a(e_2)a(e_1)f)e_3 \tag{2.65}$$

$$([a(e_1),a(e_2)]_{\rm Lie}f)e_3 = (a([e_1,e_2]_{\rm D})f)e_3 \tag{2.66}$$

hence,

$$a([e_1,e_2]_{\rm D})=[a(e_1),a(e_2)]_{\rm Lie}\ . \tag{2.67}$$

■

Most of the above proofs can be found in [47].

Another viewpoint on axioms (2.45) and (2.46) is the following,

$$a(e)\langle\tilde e,\tilde e\rangle = 2\langle[e,\tilde e]_{\rm D},\tilde e\rangle \tag{2.68}$$

$$= 2\langle[\tilde e,\tilde e]_{\rm D},e\rangle\ , \tag{2.69}$$

where (2.68) corresponds to (2.45) and (2.69) corresponds to (2.46), for $\tilde e=e_1+e_2$.

Proof:

By polarization technique, $\tilde e=e_1+e_2$,

$$a(e)\langle e_1+e_2,e_1+e_2\rangle=a(e)\left(\langle e_1,e_1\rangle+\langle e_2,e_2\rangle+\langle e_1,e_2\rangle+\langle e_2,e_1\rangle\right)\ , \tag{2.70}$$

while RHS of (2.68),

$$2\langle[e,e_1{+}e_2]_{\rm D},e_1{+}e_2\rangle=2\left(\langle[e,e_1]_{\rm D},e_1\rangle+\langle[e,e_1]_{\rm D},e_2\rangle+\langle[e,e_2]_{\rm D},e_1\rangle+\langle[e,e_2]_{\rm D},e_2\rangle\right)\ . \tag{2.71}$$

After identifying

$$a(e)\langle e_1,e_1\rangle=2\langle[e,e_1]_{\rm D},e_1\rangle \tag{2.72}$$

and

$$a(e)\langle e_2,e_2\rangle=2\langle[e,e_2]_{\rm D},e_2\rangle\ , \tag{2.73}$$

we are left with

$$a(e)\langle e_1,e_2\rangle+a(e)\langle e_2,e_1\rangle=2\langle[e,e_1]_{\rm D},e_2\rangle+2\langle[e,e_2]_{\rm D},e_1\rangle\ . \tag{2.74}$$

Due to the symmetric pairing $\langle\ ,\ \rangle$, we arrive at

$$a(e)\langle e_1,e_2\rangle=\langle[e,e_1]_{\rm D},e_2\rangle+\langle[e,e_2]_{\rm D},e_1\rangle\ , \tag{2.75}$$

which is (2.45).

■

In the non-anti-symmetry nature of the Dorfman bracket (2.46), the conjugate transpose map $a^\dagger : T^*M \to E^* \simeq E$ is defined by identifying E^* and E. The RHS of (2.46) (that is, $a^\dagger d\langle e_1, e_2\rangle$) is an element of E^*, while LHS of (2.46) (that is, $[e_1, e_2]_{\mathrm{D}} + [e_2, e_1]_{\mathrm{D}}$) is an element of E. The identification is done by using the symmetric, non-degenerate pairing $\langle\ ,\ \rangle$, as outlined in the following,

$$[e_1, e_2]_{\mathrm{D}} + [e_2, e_1]_{\mathrm{D}} = \kappa^{-1} a^T d\langle e_1, e_2\rangle \tag{2.76}$$

$$\kappa\left([e_1, e_2]_{\mathrm{D}} + [e_2, e_1]_{\mathrm{D}}\right) = a^T d\langle e_1, e_2\rangle\ , \tag{2.77}$$

where we have used $\kappa^{-1} a^T = a^\dagger$,

$$\langle [e_1, e_2]_{\mathrm{D}} + [e_2, e_1]_{\mathrm{D}}, e\rangle = a^T d\langle e_1, e_2\rangle\,(e) = d\langle e_1, e_2\rangle\,(ae) = (ae)\langle e_1, e_2\rangle\ . \tag{2.78}$$

2.3 Deformations of a Courant Algebroid

In this work we are interested in using one example of a Courant algebroid and perform deformations and twists on the objects that characterize this algebroid. To be specific, it is the Courant algebroid $(E, a, [\ ,\ ]_{\mathrm{D}}, \langle\ ,\ \rangle)$, with generalized tangent bundle $E = TM \bigoplus T^*M$ (hence the generalized vector $\mathbb{V} = V + \lambda$), an anchor map $a : E \to TM$, Dorfman bracket and the natural pairing. We will keep the 5 properties (axioms) of the algebroid in check, namely homomorphism (2.58), Leibniz rule (2.50), Jacobi identity (2.48), Dorfman bracket's non-anti-symmetry (2.46) and pairing compatibility condition (2.45).

We propose deformations only of the pairing and Dorfman bracket in the Courant algebroid. The deformed structures are denoted with a prime.
(a) Deformed pairing

$$\langle e_1, e_2\rangle' = \langle e^{\mathcal{G}}(e_1), e^{\mathcal{G}}(e_2)\rangle \tag{2.79}$$

and
(b) deformed Dorfman bracket

$$[e_1, e_2]'_{\mathrm{D}} = e^{-\mathcal{G}}[e^{\mathcal{G}}(e_1), e^{\mathcal{G}}(e_2)]_{\mathrm{D}}\ , \tag{2.80}$$

where $\mathcal{G} = g + B$, $\mathcal{G} : TM \to T^*M$ maps vectors to 1-forms, where g is non-degenerate. We will name $\mathcal{G}$ a non-symmetric metric, as it is a sum of a symmetric metric g and anti-symmetric 2-form B. $\mathcal{G}$ was determined from examination on the 5 properties of the Courant algebroid. The exact expressions for the deformations are
(a)

$$\langle e_1, e_2\rangle' = \langle e_1, e_2\rangle + 2g\left(a(e_1), a(e_2)\right) \tag{2.81}$$

and

(b)

$$[e_1, e_2]'_{\mathrm{D}} = [e_1, e_2]_{\mathrm{D}} + 2g(\nabla a(e_1), a(e_2)) \ , \tag{2.82}$$

with a torsionful connection.

Let us work out the explicit deformed expressions, for elements $e_1 = V + \lambda$, $e_2 = Y + \sigma$. From the deformed inner product (2.79),

$$\begin{aligned} \langle e^{\mathcal{G}}(V+\lambda), e^{\mathcal{G}}(Y+\sigma)\rangle &= \langle V + \mathcal{G}(V,\cdot) + \lambda, Y + \mathcal{G}(Y,\cdot) + \sigma\rangle & (2.83)\\ &= i_V\sigma + i_Y\lambda + \mathcal{G}(V,Y) + \mathcal{G}(Y,V) & (2.84)\\ &= i_V\sigma + i_Y\lambda + (g+B)(V,Y) + (g+B)(Y,V) & (2.85)\\ &= \langle V+\lambda, Y+\sigma\rangle + 2g(V,Y) \ , & (2.86) \end{aligned}$$

as in (2.81). We are using the convention $i_V\mathcal{G} := \mathcal{G}(V,\cdot)$, where the second slot is open for contraction. For short-hand notation, $\mathcal{G}(V,\cdot)$ can be written as $\mathcal{G}(V)$.

From the deformed Dorfman bracket (2.80),

$$\begin{aligned} e^{-\mathcal{G}}[e^{\mathcal{G}}(V+\lambda), e^{\mathcal{G}}(Y+\sigma)]_{\mathrm{D}} &= e^{-\mathcal{G}}[V + \mathcal{G}(V,\cdot) + \lambda, Y + \mathcal{G}(Y,\cdot) + \sigma]_{\mathrm{D}} & (2.87)\\ &= e^{-\mathcal{G}}([V+\lambda, Y+\sigma]_{\mathrm{D}} + [\mathcal{G}(V,\cdot), Y]_{\mathrm{D}} & (2.88)\\ &\quad +[V, \mathcal{G}(Y,\cdot)]_{\mathrm{D}}) & \\ &= e^{-\mathcal{G}}([V+\lambda, Y+\sigma]_{\mathrm{D}} - i_Y d\mathcal{G}(V,\cdot) & (2.89)\\ &\quad +\mathcal{L}_V\mathcal{G}(Y,\cdot)) & \\ &= [V+\lambda, Y+\sigma]_{\mathrm{D}} - \mathcal{G}([V,Y]_{\mathrm{Lie}},\cdot) - i_Y d\mathcal{G}(V,\cdot) & (2.90)\\ &\quad +\mathcal{L}_V\mathcal{G}(Y,\cdot) & \\ &= [V+\lambda, Y+\sigma]_{\mathrm{D}} - g([V,Y]_{\mathrm{Lie}},\cdot) - i_Y dg(V,\cdot) & (2.91)\\ &\quad +\mathcal{L}_V g(Y,\cdot) - B([V,Y]_{\mathrm{Lie}},\cdot) - i_Y dB(V,\cdot) & \\ &\quad +\mathcal{L}_V B(Y,\cdot) & \\ &= [V+\lambda, Y+\sigma]_{\mathrm{D}} + A(V,Y) \ , & (2.92) \end{aligned}$$

where we have defined the 1-form $A(V,Y)$. We can check what structure the 1-form can be. For the g part in A, when it is contracted with a vector Z[5], it becomes

$$-g([V,Y]_{\mathrm{Lie}}, Z) - \langle Z, i_Y dg(V,\cdot)\rangle + \langle Z, \mathcal{L}_V g(Y,\cdot)\rangle \ . \tag{2.93}$$

Using a Lie derivative formula

$$\mathcal{L}_Y = i_Y d + d i_Y \ , \tag{2.94}$$

[5]Keep in mind that contraction is to be done with respect to the deformed inner product from here onwards.

the second term in (2.93) is

$$\begin{aligned}\langle Z, i_Y dg(V,\cdot)\rangle &= \langle Z, \mathcal{L}_Y g(V,\cdot)\rangle - \langle Z, di_Y g(V,\cdot)\rangle && (2.95)\\ &= \langle Z, (\mathcal{L}_Y g)(V,\cdot)\rangle + \langle Z, g(\mathcal{L}_Y V,\cdot)\rangle - \langle Z, di_Y g(V,\cdot)\rangle && (2.96)\\ &= (\mathcal{L}_Y g)(V,Z) + g(\mathcal{L}_Y V, Z) - Z\cdot g(V,Y) && (2.97)\\ &= \mathcal{L}_Y \cdot g(V,Z) - g(V,\mathcal{L}_Y Z) - Z\cdot g(V,Y) && (2.98)\\ &= Y\cdot g(V,Z) - g(V,[Y,Z]_{\text{Lie}}) - Z\cdot g(V,Y)\ . && (2.99)\end{aligned}$$

The second slot in the last term in (2.96), $i_Y g(V,\cdot) = g(V,Y)$ is a function.
For the third term in (2.93),

$$\begin{aligned}\langle Z, \mathcal{L}_V g(Y,\cdot)\rangle &= \langle Z, (\mathcal{L}_V g)(Y,\cdot)\rangle + \langle Z, g(\mathcal{L}_V Y,\cdot)\rangle && (2.100)\\ &= (\mathcal{L}_V g)(Y,Z) + g(\mathcal{L}_V Y, Z) && (2.101)\\ &= \mathcal{L}_V\cdot g(Y,Z) - g(Y,\mathcal{L}_V Z) && (2.102)\\ &= V\cdot g(Y,Z) - g(Y,[V,Z]_{\text{Lie}})\ . && (2.103)\end{aligned}$$

Hence the total result from (2.93) is

$$\begin{aligned}& -g([V,Y]_{\text{Lie}},Z) - \langle Z, i_Y dg(V,\cdot)\rangle + \langle Z, \mathcal{L}_V g(Y,\cdot)\rangle && (2.104)\\ =\ & -g([V,Y]_{\text{Lie}},Z) - Y\cdot g(V,Z) + g(V,[Y,Z]_{\text{Lie}}) + Z\cdot g(V,Y) && (2.105)\\ & +V\cdot g(Y,Z) - g(Y,[V,Z]_{\text{Lie}})\\ =\ & 2g(\nabla^{\text{LC}}_Z V, Y)\ , && (2.106)\end{aligned}$$

which is the formula for Levi-Civita connection ∇^{LC}.

Similarly as worked out for g above, for B part in the 1-form $A(V,Y)$ in (2.92) contracted with vector Z,

$$\begin{aligned}& -B([V,Y]_{\text{Lie}},Z) - \langle Z, i_Y dB(V,\cdot)\rangle + \langle Z, \mathcal{L}_V B(Y,\cdot)\rangle && (2.107)\\ =\ & -B([V,Y]_{\text{Lie}},Z) - Y\cdot B(V,Z) + B(V,[Y,Z]_{\text{Lie}}) + Z\cdot B(V,Y) && (2.108)\\ & +V\cdot B(Y,Z) - B(Y,[V,Z]_{\text{Lie}})\\ =\ & H(V,Y,Z) && (2.109)\end{aligned}$$

is a totally anti-symmetric 3-form. We can work out backwards to prove this, as

$$H(V,Y,Z) = i_Z i_Y i_V dB\ . \qquad (2.110)$$

Firstly[6],

$$\begin{aligned}i_Y i_V dB &= i_Y \mathcal{L}_V B - i_Y d i_V B && (2.111)\\ &= \mathcal{L}_V i_Y B - i_{[V,Y]} B - i_Y d i_V B && (2.112)\\ &= \mathcal{L}_V B(Y) - B([V,Y]) - i_Y dB(V)\ , && (2.113)\end{aligned}$$

[6]Brackets without any subscript label will now indicate it is a Lie bracket.

where the formula

$$\mathcal{L}_V i_Y - i_Y \mathcal{L}_V = i_{[V,Y]} \tag{2.114}$$

was used. As a whole,

$$\begin{aligned} H(V,Y,Z) = i_Z i_Y i_V dB &= i_Z \mathcal{L}_V B(Y) - i_Z B([V,Y]) - i_Z i_Y dB(V) && (2.115) \\ &= \mathcal{L}_V i_Z B(Y) - i_{[V,Z]} B(Y) - B([V,Y],Z) && (2.116) \\ &\quad -i_Z(\mathcal{L}_Y B(V) - d i_Y B(V)) \\ &= \mathcal{L}_V B(Y,Z) - B(Y,[V,Z]) - B([V,Y],Z) && (2.117) \\ &\quad -i_Z \mathcal{L}_Y B(V) + i_Z dB(V,Y) \\ &= V \cdot B(Y,Z) - B(Y,[V,Z]) - B([V,Y],Z) && (2.118) \\ &\quad -\mathcal{L}_Y i_Z B(V) + i_{[Y,Z]} B(V) + \mathcal{L}_Z B(V,Y) \\ &\quad -d i_Z B(V,Y) \\ &= V \cdot B(Y,Z) - Y \cdot B(V,Z) + Z \cdot B(V,Y) && (2.119) \\ &\quad -B(Y,[V,Z]) - B([V,Y],Z) + B(V,[Y,Z]) \ , \end{aligned}$$

where formula

$$\mathcal{L}_V f = V \cdot f \tag{2.120}$$

for a function f and $i_Z B(V,Y) = 0$ was used[7].
Hence, the 1-form defined in (2.92) can be written as (see also [48])

$$A(V,Y) = 2g(\nabla^{\mathrm{LC}}_{\cdot} V, Y) + H(V,Y,\cdot) \ . \tag{2.121}$$

It becomes a sum of the Levi-Civita connection and a 3-form after contracting with a vector. The open slots were taken by vector Z in the preceding discussion. One can then relate the result of $A(V,Y)$ to the second term (that is, $2g(\nabla V, Y)$) in (2.82).

There is a more compact way to approach (2.87),

$$\begin{aligned} & e^{-\mathcal{G}}[e^{\mathcal{G}}(V+\lambda), e^{\mathcal{G}}(Y+\sigma)]_{\mathrm{D}} && (2.122) \\ = \ & [V+\lambda, Y+\sigma]_{\mathrm{D}} + V\cdot\mathcal{G}(Y,\cdot) - Y\cdot\mathcal{G}(V,\cdot) + d\mathcal{G}(V,Y) && (2.123) \\ & -\mathcal{G}(Y,[V,\cdot]) - \mathcal{G}([V,Y],\cdot) + \mathcal{G}(V,[Y,\cdot]) \\ = \ & [V+\lambda, Y+\sigma]_{\mathrm{D}} + 2g(\nabla V, Y) \ , && (2.124) \end{aligned}$$

where $\mathcal{G} = g + B$. This is achieved simply by careful assignment of the vectors. We thus have generalized the Koszul formula for connection of Levi-Civita to include an anti-symmetric part which is a 2-form B [49]. To summarize,

$$A(V,Y) = 2g(\nabla V, Y) = 2g(\nabla^{\mathrm{LC}}_{\cdot} V, Y) + H(V,Y,\cdot) \ . \tag{2.125}$$

[7] Recall that the interior product $i_Z : \Omega^p(M) \to \Omega^{p-1}(M)$ for forms Ω. Thus the interior product of a function (0-form) is zero.

2.4 On Deformed Axioms

The deformations are realized under the condition that all properties of the Courant algebroid are *not* violated. Simply put, when the deformed structures, which we have chosen to only be the pairing and bracket, are inserted in the Courant algebroid axioms, the additional terms produced by deformations are examined. Furthermore, we have also chosen that the deforming map $\mathcal{G}$ be a mapping from tangent to cotangent bundle. This is how the solution of $\mathcal{G}$ was being found as $g+B$ consequently.

Leibniz rule can be regarded as the initial guide to the general form of the 1-form $A(V,Y)$ (2.121). After plugging in the axiom (2.50) the deformed Dorfman bracket, we get the additional part

$$fA(V,Y) = A(V,fY)\ . \tag{2.126}$$

This condition says that the function f is linear in the second slot of $A(V,Y)$. Besides tensor, one guess is that $A(V,Y)$ can be a connection, provided with a careful allocation of vectors into the slots, as we know of the connection properties (2.6) and (2.7).

From (2.45) which is a reminiscent of a Killing equation, we have the condition

$$A(V,Y)(Z) + A(V,Z)(Y) = 2\left(V\cdot g(Y,Z) - g([V,Y],Z) - g(Y,[V,Z])\right)\ . \tag{2.127}$$

In order to fulfill this condition, the torsion T of the torsionful connection in (2.82) is determined to be an anti-symmetric 3-form H. Since contortion

$$K(V,Y,Z) = \frac{1}{2}(g(T(V,Y),Z) + g(T(Z,V),Y) + g(T(Z,Y),V))\ , \tag{2.128}$$

given $g(T(V,Y),Z) = H(V,Y,Z)$, contortion is equal to $\frac{1}{2}H(V,Y,Z)$.

From the Jacobi identity of the Dorfman bracket, we have

$$\begin{aligned} & [V,A(Y,Z)]_{\mathrm{D}} + A(V,[Y,Z]) && (2.129)\\ = \ & [A(V,Y),Z]_{\mathrm{D}} + A([V,Y],Z) + [Y,A(V,Z)]_{\mathrm{D}} + A(Y,[V,Z])\ . && (2.130)\end{aligned}$$

Note that $A(V,Y)$ only takes in vectors to be non-vanishing and $[V,Y]_{\mathrm{D}} = [V,Y]_{\mathrm{Lie}}$. After computing with the formula of Dorfman bracket (2.33), we get

$$\begin{aligned} & \mathcal{L}_V A(Y,Z) + A(V,[Y,Z]) && (2.131)\\ = \ & -i_Z dA(V,Y) + A([V,Y],Z) + \mathcal{L}_Y A(V,Z) + A(Y,[V,Z])\ . && (2.132)\end{aligned}$$

Since this is made up of 1-form's, we can contract it with a vector X to see what the condition becomes. It gives

$$\begin{aligned} & \langle \mathcal{L}_V A(Y,Z), X\rangle + \langle A(V,[Y,Z]),X\rangle && (2.133)\\ = \ & -\langle \mathcal{L}_Z A(V,Y),X\rangle + X\cdot\langle A(V,Y),Z\rangle && (2.134)\\ & +\langle A([V,Y],Z),X\rangle + \langle \mathcal{L}_Y A(V,Z),X\rangle + \langle A(Y,[V,Z]),X\rangle\ . \end{aligned}$$

When the solution of $A(V,Y)$ (2.125) is plugged in, and a formula such as

$$\nabla^{\rm LC}_V Y - \nabla^{\rm LC}_Y V - [V,Y] = 0 \ , \tag{2.135}$$

which is a torsionless condition, is used, we arrive at

$$\begin{aligned} & dK(Y,X,V,Z) && (2.136) \\ = \ & 2g(R(V,X)Y,Z) + 2g(R(X,Y)V,Z) + 2g(R(Y,V)X,Z) = 0 \ . && (2.137) \end{aligned}$$

The last equality to zero is by the fact that it is the First Bianchi identity [48]. Note that the Riemann curvature here is torsionless. We remark that there is a useful formula for the exterior derivative acting on forms. As an example, for a 3-form K,

$$\begin{aligned} dK(X,Y,Z,W) \ = \ & X\cdot K(Y,Z,W) - K([X,Y],Z,W) \\ & -Y\cdot K(X,Z,W) + K([X,Z],Y,W) \\ & +Z\cdot K(X,Y,W) - K([X,W],Y,Z) \\ & -W\cdot K(X,Y,Z) - K([Y,Z],X,W) \\ & +K([Y,W],X,Z) - K([Z,W],X,Y) \ . \end{aligned} \tag{2.138}$$

Therefore, from the Jacobi identity axiom of the deformed Courant algebroid, not only the Bianchi identity is encoded, but contortion is found to be closed as well.

From the non-anti-symmetry property of the Dorfman bracket, after contracting with vector Z, we have

$$A(V,Y)(Z) + A(Y,V)(Z) = 2Z\cdot g(V,Y) \ . \tag{2.139}$$

From this, after using (2.125), we get metricity of the connection, easily as the anti-symmetric contortion cancels out.

The homomorphism property simply implies that the anchor map of a 1-form is zero,

$$a(A(V,Y)) = 0 \ , \tag{2.140}$$

which we have used throughout the computations.

We like to remark that, from (2.127), when the solution (2.125) is substituted,

$$2g(\nabla^{\rm LC}_Z V,Y) + 2g(\nabla^{\rm LC}_Y V,Z) = 2(V\cdot g(Y,Z) - g([V,Z],Y) - g([V,Y],Z)) \ , \tag{2.141}$$

we can rewrite the LHS,

$$\begin{aligned} & g(\nabla^{\rm LC}_V Z,Y) + g([Z,V],Y) + g(\nabla^{\rm LC}_V Y,Z) + g([Y,V],Z) && (2.142) \\ = \ & V\cdot g(Y,Z) - g([V,Z],Y) - g([V,Y],Z) && (2.143) \end{aligned}$$

and get the metricity condition

$$g(\nabla^{\mathrm{LC}}_V Z, Y) + g(\nabla^{\mathrm{LC}}_V Y, Z) = V \cdot g(Y, Z) \ . \tag{2.144}$$

We note that the metricity condition is useful in the sense that connection properties can be derived from it. For metricity

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \ , \tag{2.145}$$

(1) let $Y \to fY$,

$$X(g(fY, Z)) = g(\nabla_X(fY), Z) + g(fY, \nabla_X Z) \tag{2.146}$$

$$X(fg(Y, Z)) = g(\nabla_X(fY), Z) + g(fY, \nabla_X Z) \tag{2.147}$$

$$X(f)(g(Y, Z)) + fX(g(Y, Z)) = g(\nabla_X(fY), Z) + fg(Y, \nabla_X Z) \tag{2.148}$$

$$g((X(f))Y, Z) + f(g(\nabla_X Y, Z) + g(Y, \nabla_X Z)) = g(\nabla_X(fY), Z) + fg(Y, \nabla_X Z) \ , \tag{2.149}$$

therefore,

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y \ , \tag{2.150}$$

similarly for Z, as Y and Z are symmetric,
(2) let $X \to fX$,

$$fX(g(Y, Z)) = g(\nabla_{fX} Y, Z) + g(Y, \nabla_{fX} Z) \ , \tag{2.151}$$

LHS is

$$fX(g(Y, Z)) = fg(\nabla_X Y, Z) + fg(Y, \nabla_X Z) = g(f\nabla_X Y, Z) + g(Y, f\nabla_X Z) \tag{2.152}$$

and LHS = RHS gives

$$\nabla_{fX} = f\nabla_X \ . \tag{2.153}$$

The metricity condition has been used repeatedly.

2.5 Gravity with Kalb-Ramond Field Strength

Contrary to Courant braket invariance with respect to a closed B-form, the Dorfman bracket in our work is deformed additionally with a non-tensorial object, namely connection, and yet remains consistent with all the axioms in the algebroid. Complete contraction of the structures (with a vector) helps us to realize a more physical meaning of the deformed axioms, for instance the metricity, which means that the metric g is covariantly constant which says that the inner product of two vectors being parallel transported along any curves remains constant [33].

The main result realized from the set of axioms is

$$g(\nabla_X Y, Z) = g(\nabla_X^{\mathrm{LC}} Y, Z) + \frac{1}{2} H(X, Y, Z) \ . \tag{2.154}$$

It is nice to know that behind such a connection lies a mathematical structure. In [50], Riemannian geometry with a skew torsion has been studied extensively.

In physics, we are interested in actions. Given a connection, we can go on computing Riemann curvature, Ricci tensor and then the Ricci scalar. In this work, the symmetric metric g is used to raise and lower the indices. In coordinates, the connection coefficient is

$$\Gamma_{ij}{}^{m} = (\Gamma_{ij}{}^{m})^{\mathrm{LC}} + \frac{1}{2} H_{ij}{}^{m} \ . \tag{2.155}$$

Subsequently, the non-symmetric Ricci tensor is

$$R^{k}{}_{jkl} = R_{jl} = R_{jl}^{\mathrm{LC}} - \frac{1}{2} \nabla_i^{\mathrm{LC}} H_{jl}{}^{i} - \frac{1}{4} H_{lm}{}^{i} H_{ij}{}^{m} \ , \tag{2.156}$$

where

$$\nabla_i^{\mathrm{LC}} H_{jl}{}^{i} = \partial_i H_{jl}{}^{i} - (\Gamma_{il}{}^{m})^{\mathrm{LC}} H_{jm}{}^{i} - (\Gamma_{ij}{}^{m})^{\mathrm{LC}} H_{ml}{}^{i} + (\Gamma_{im}{}^{i})^{\mathrm{LC}} H_{jl}{}^{m} \ . \tag{2.157}$$

Considering now the Ricci tensor as vacuum field equation, that is, set $R_{jl} = 0$. These equations can be realized from the action

$$S_{\mathcal{G}=g+B}(g, B) = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \left(R^{\mathrm{LC}} - \frac{1}{12} H_{ijk} H^{ijk} \right) \ , \tag{2.158}$$

where G_N is Newton's gravitational constant in d dimensions and R^{LC} denotes the Ricci scalar for a Levi-Civita connection. To check its field equation, the explicit variation of the action terms are, with respect to g [33],

$$\begin{aligned} \delta R^{\mathrm{LC}} &= \delta(g^{ij}(R_{ij})^{\mathrm{LC}}) = \delta(g_{ij}(R^{ij})^{\mathrm{LC}}) && (2.159) \\ &= (R^{ij})^{\mathrm{LC}} \delta g_{ij} \ , && (2.160) \end{aligned}$$

where $\delta(R_{ij})^{\mathrm{LC}} = 0$ for Levi-Civita Ricci tensor,

$$\begin{aligned} \delta_g(H_{ijk} H^{ijk}) &= \delta_g(H^{ijk} H^{i'j'k'} g_{ii'} g_{jj'} g_{kk'}) && (2.161) \\ &= 3 H^{ijk} H^{i'}{}_{jk} \, \delta_g g_{ii'} \ , && (2.162) \end{aligned}$$

and with respect to B,

$$\begin{aligned} \delta_B(H_{ijk} H^{ijk}) &= \delta(H^{ijk}(\partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij})) && (2.163) \\ &= 6 H^{ijk} \delta(\partial_i B_{jk}) && (2.164) \\ &= 6 H^{ijk} \delta(\nabla_i^{\mathrm{LC}} B_{jk} + (\Gamma_{ij}{}^{l})^{\mathrm{LC}} B_{lk} + (\Gamma_{ik}{}^{l})^{\mathrm{LC}} B_{jl}) && (2.165) \\ &= 6 H^{ijk} (\nabla_i^{\mathrm{LC}} \delta B_{jk}) && (2.166) \\ &= -6 (\nabla_i^{\mathrm{LC}} H^{ijk}) \delta B_{jk} \ , && (2.167) \end{aligned}$$

as $H^{ijk}(\Gamma_{ij}{}^{l})^{\rm LC} = 0$ and a boundary term is ignored in the last equality.
Therefore, from

$$\frac{\delta S_{\mathcal{G}}}{\delta g_{ij}} = 0 \qquad \text{and} \qquad \frac{\delta S_{\mathcal{G}}}{\delta B_{ij}} = 0 \ , \tag{2.168}$$

we have

$$\frac{\delta S_{\mathcal{G}}}{\delta g_{ij}} + \frac{\delta S_{\mathcal{G}}}{\delta B_{ij}} = 0 \ , \tag{2.169}$$

which contains the non-symmetric Ricci tensor (2.156),

$$(R^{ji})^{\rm LC} - \frac{1}{4}H^{ikl}H^{j}{}_{kl} - \frac{1}{2}\nabla^{\rm LC}_{k}H^{kji} = 0 \ . \tag{2.170}$$

The Ricci tensor of the Levi-Civita connection is symmetric. However, we also note that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ij}\delta g^{ij} = \frac{1}{2}\sqrt{-g}g^{ij}\delta g_{ij} \ , \tag{2.171}$$

which in principle contributes two additional terms,

$$\frac{1}{2}g^{ij}\left(R^{\rm LC} - \frac{1}{12}H_{i'j'k'}H^{i'j'k'}\right)\delta g_{ij} \ , \tag{2.172}$$

which are part of the field equation of the action $S_{\mathcal{G}}$. Before we comment about these additional terms, which are to be related to the dilaton field that we have not considered in the work here, let us remark that (2.156) is equal to the beta functions of g and B.

The non-linear sigma model describing string propagation on a 2-dimensional worldsheet Σ with background fields g, Kalb-Ramond B and dilaton ϕ is given by [51]

$$\begin{aligned} S_{\rm nlsm} &= \frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma\sqrt{\gamma}\,(\,\gamma^{\mu\nu}g_{mn}(X)\,\partial_\mu X^m\partial_\nu X^n + i\epsilon^{\mu\nu}B_{mn}(X)\,\partial_\mu X^m\partial_\nu X^n \\ &\qquad + \alpha'\phi(X)R\,) \ , \end{aligned} \tag{2.173}$$

where $m, n = 0, 1, \cdots, 25$, in 26-dimensional spacetime (target space), α' is coupling constant (inverse string tension), g_{mn} is curved spacetime metric, $\gamma^{\mu\nu}$ is worldsheet metric ($\mu, \nu = 0, 1$). The non-trivial beta functions for the background fields derived from the model,

$$\beta_{\mu\nu}(g) = \alpha' R_{\mu\nu} - \frac{\alpha'}{4}H_{\mu\lambda\kappa}H_{\nu}{}^{\lambda\kappa} + 2\alpha'\nabla_\mu\nabla_\nu\phi \ , \tag{2.174}$$

$$\beta_{\mu\nu}(B) = -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\phi\, H_{\lambda\mu\nu} \ , \tag{2.175}$$

$$\beta(\phi) = -\frac{\alpha'}{2}\nabla^2\phi + \alpha'\nabla_\mu\phi\nabla^\mu\phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} \ , \tag{2.176}$$

are required to vanish, in order to preserve Weyl invariance of string theory as a quantum theory. The vanishing beta functions from the non-linear sigma model, when derivable from an action as the action's field equations, gives the spacetime action,

$$S_{\rm eff} = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-g}e^{-2\phi}\left(R - \frac{1}{12}H_{abc}H^{abc} + 4g^{ab}\partial_a\phi\,\partial_b\phi\right) \ , \tag{2.177}$$

which is the low-energy effective action of the bosonic string [52], [53]. The overall constant κ scales as $\kappa^2 \sim (\alpha')^{12}$ [54]. Varying the effective action with respect to the spacetime fields gives [54]

$$\delta S_{\text{eff}} = \frac{1}{2\kappa^2\alpha'} \int d^{26}X\sqrt{-g}e^{-2\phi}(\delta g_{\mu\nu}\beta^{\mu\nu}(g) - \delta B_{\mu\nu}\beta^{\mu\nu}(B) \tag{2.178}$$

$$-(2\delta\phi + \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu})(\beta^{\lambda}{}_{\lambda}(g) - 4\beta(\phi)))\ . \tag{2.179}$$

When dilaton field is turned off, the non-symmetric Ricci tensor (2.156) is indeed equal to the beta functions (2.174) plus (2.175), that is, it corresponds to line (2.178) in the parentheses. Be cautious that the last term in $\beta(\phi)$ (2.176) is without ϕ, thus from line (2.179), the term

$$-\frac{1}{2}\alpha' g^{\mu\nu}\delta g_{\mu\nu}\left(R - \frac{1}{4}H_{\mu\lambda\kappa}H^{\mu\lambda\kappa} - 4(-\frac{1}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda})\right) \tag{2.180}$$

survives, which we have in (2.172). Note here that the connection (hence Riemann curvature and so forth) is of Levi-Civita type. Without couplings of B and ϕ, vanishing of the beta function $\beta(g)$ (2.174) implies that the background spacetime must be Ricci flat, that is, $R_{\mu\nu} = 0$. The three massless spacetime fields g, B and ϕ are the basic contents in bosonic string theory. We have so far incorporated only g and B in the Courant algebroid, with deformations involving only the tangent bundle. An alternative formulation to get to the similar action containing g and B can be found in [55].

Finally, we present another formulation in which we can contract at the level of the non-symmetric Ricci tensor (2.156) with the non-symmetric metric $\mathcal{G}$, which is $g + B$, to obtain the curvature scalar

$$R_{\mathcal{G}} := \mathcal{G}^{jl}R_{jl} = (g^{jl} + g^{jj'}B_{j'l'}g^{ll'})R_{jl} \tag{2.181}$$

$$= (g^{jl} + B^{jl})R_{jl} \tag{2.182}$$

$$= g^{jl}(R_{jl})^{\text{LC}} - \frac{1}{2}g^{jl}\nabla_i^{\text{LC}}H_{jl}{}^{i} - \frac{1}{4}g^{jl}H_{lm}{}^{i}H_{ij}{}^{m} \tag{2.183}$$

$$+B^{jl}(R_{jl})^{\text{LC}} - \frac{1}{2}B^{jl}\nabla_i^{\text{LC}}H_{jl}{}^{i} - \frac{1}{4}B^{jl}H_{lm}{}^{i}H_{ij}{}^{m}$$

$$= R^{\text{LC}} - \frac{1}{4}g^{jl}H_{lm}{}^{i}H_{ij}{}^{m} - \frac{1}{2}B^{jl}\nabla_i^{\text{LC}}H_{jl}{}^{i}\ , \tag{2.184}$$

where $R^{\text{LC}} = g^{jl}(R_{jl})^{\text{LC}}$. Note that $\mathcal{G}^{jl} \neq (\mathcal{G}_{jl})^{-1}$. The term $\frac{1}{4}B^{jl}H_{lm}{}^{i}H_{ij}{}^{m}$ is zero, as

$$B^{jl}H_{lm}{}^{i}H_{ij}{}^{m} = B^{jl}H_{ij}{}^{m}H_{lm}{}^{i} = -B^{jl}H_{ji}{}^{m}H_{lm}{}^{i} \tag{2.185}$$

and on the other hand

$$B^{jl}H_{lm}{}^{i}H_{ij}{}^{m} = B^{lj}H_{jm}{}^{i}H_{il}{}^{m} = B^{lj}H_{ji}{}^{m}H_{ml}{}^{i} = -B^{lj}H_{ji}{}^{m}H_{lm}{}^{i} = B^{jl}H_{ji}{}^{m}H_{lm}{}^{i}\ . \tag{2.186}$$

Therefore, an Einstein-Hilbert-like action is

$$S_{\mathcal{G}} = \frac{1}{16\pi G_N}\int d^dx\sqrt{-g}\,R_{\mathcal{G}}\ . \tag{2.187}$$

Let us analyze the last term in (2.184), in the action,

$$-\frac{1}{2}\int d^dx\sqrt{-g}\,B^{jl}\nabla_i^{\rm LC}H_{jl}{}^i = -\frac{1}{2}\int d^dx\sqrt{-g}\left(\nabla_i^{\rm LC}(B^{jl}H_{jl}{}^i) - (\nabla_i^{\rm LC}B^{jl})H_{jl}{}^i\right) \ . \quad (2.188)$$

The first term

$$\begin{aligned}\nabla_i^{\rm LC}(B^{jl}H_{jl}{}^i) &= \nabla_i^{\rm LC}(B_{j'l'}g^{jj'}g^{ll'}H_{jl}{}^i) & (2.189)\\ &= (\nabla_i^{\rm LC}B_{j'l'})g^{jj'}g^{ll'}H_{jl}{}^i + B_{j'l'}g^{jj'}g^{ll'}(\nabla_i^{\rm LC}H_{jl}{}^i) & (2.190)\\ &= (\nabla_i^{\rm LC}B_{j'l'})H^{j'l'i} + B^{jl}(\nabla_i^{\rm LC}H_{jl}{}^i) & (2.191)\end{aligned}$$

as $\nabla_i^{\rm LC}g^{jj'} = 0$, while the second term

$$\begin{aligned}(\nabla_i^{\rm LC}B_{jl})H^{jli} &= \frac{1}{3}((\nabla_i^{\rm LC}B_{jl})H^{jli} + (\nabla_j^{\rm LC}B_{li})H^{lij} + (\nabla_l^{\rm LC}B_{ij})H^{ijl}) & (2.192)\\ &= \frac{1}{3}((\partial_iB_{jl})H^{jli} + (\partial_jB_{li})H^{lij} + (\partial_lB_{ij})H^{ijl}) & (2.193)\\ &= \frac{1}{3}(\partial_iB_{jl} + \partial_jB_{li} + \partial_lB_{ij})H^{ijl} & (2.194)\\ &= \frac{1}{3}H_{ijl}H^{ijl} & (2.195)\end{aligned}$$

as all the Christoffel parts such as $(\Gamma_{ij}{}^m)^{\rm LC}B_{ml}H^{jli} = 0$ drop out. Therefore, for (2.188),

$$-\frac{1}{2}\int d^dx\sqrt{-g}\,B^{jl}\nabla_i^{\rm LC}H_{jl}{}^i = \frac{1}{6}\int d^dx\sqrt{-g}\,H_{ijl}H^{ijl} \ , \quad (2.196)$$

where the boundary term

$$\begin{aligned}&-\frac{1}{2}\int d^dx\sqrt{-g}\,\nabla_i^{\rm LC}(B^{jl}H_{jl}{}^i) & (2.197)\\ =\ &-\frac{1}{2}\int d^dx\sqrt{-g}\left(\partial_i(B^{jl}H_{jl}{}^i) + \Gamma_{(ia)}{}^iB^{jl}H_{jl}{}^a\right) & (2.198)\\ =\ &-\frac{1}{2}\int d^dx\sqrt{-g}\left(\partial_i(B^{jl}H_{jl}{}^i) + (\partial_a\sqrt{-g})B^{jl}H_{jl}{}^a\right) & (2.199)\\ =\ &-\frac{1}{2}\int d^dx\,\partial_i(\sqrt{-g}B^{jl}H_{jl}{}^i) & (2.200)\end{aligned}$$

is just a total derivative. We have used the fact that since

$$(\Gamma_{ia}{}^i)^{\rm LC} = \Gamma_{(ia)}{}^i = \frac{1}{2}g^{im}\partial_ag_{im} \ , \quad (2.201)$$

and we know

$$\partial\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{ab}\partial g_{ab} \ , \quad (2.202)$$

thus,

$$\Gamma_{(ia)}{}^i = \frac{1}{\sqrt{-g}}\partial_a\sqrt{-g} \ . \quad (2.203)$$

It is a well known formula for a vector V that

$$\int\sqrt{-g}\,\nabla_i^{\rm LC}V^i = \int\partial_i(\sqrt{-g}\,V^i) \quad (2.204)$$

is a total derivative. Note that this also holds for our connection $g \circ \nabla^{\mathrm{LC}} + H/2$. From (2.187), the Einstein-Hilbert-like action is hence

$$S_{\mathcal{G}} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g}\, (R^{\mathrm{LC}} - \frac{1}{4}H^2 + \frac{1}{6}H^2) \qquad (2.205)$$

$$= \frac{1}{16\pi G_N} \int d^d x \sqrt{-g}\, (R^{\mathrm{LC}} - \frac{1}{12}H^2)\ , \qquad (2.206)$$

where $H^2 = H_{ijk}H^{ijk}$. $H(= dB)$ is also known as a Kalb-Ramond field strength. This action is precisely (2.158). Again, here, only the symmetric metric g was used to raise and lower the indices, until before the contraction of Ricci tensor to get Ricci scalar for the standard form of gravity action, which is of the Einstein-Hilbert form. It is an exception to contract the Ricci tensor with the entire non-symmetric metric $g + B$, that the effective bosonic string action without dilaton can be successfully reproduced. Note that the action is for arbitrary spacetime dimensions, constructed without reference to a non-linear sigma model. Nevertheless, in string theory, the critical dimension of 26 was determined by the role of the dilaton.

The Kalb-Ramond field strength, coupled conformally to a real scalar field in a four-dimensional model is investigated in [56]. The main interest in this work is in the contribution of the Kalb-Ramond action to inflation. Some cosmological driven interests are also seen in [57] and [58], where the Einstein-Kalb-Ramond cosmology is discussed in [57]. While in [58], a d-dimensional gravity coupled to a 3-form anti-symmetric field, which is not identified as the Kalb-Ramond field strength, but instead as the Maxwell field strength is investigated.

2.6 Discussion

As mentioned just previously, we remark again that, while the symmetric metric g is used to raise and lower the indices, the "non-symmetric" Ricci scalar in our case, which we mean by $R_{\mathcal{G}} = \mathcal{G}^{jl}R_{jl}$ (2.181), was obtained using the non-symmetric metric $\mathcal{G}$. Moreover, $\mathcal{G}^{jl}$ is not the inverse of $\mathcal{G}_{jl}$. This is a crucial difference with the previous NGT proposals discussed in the beginning, recall for instance (1.11) and (1.30).

In this work, a non-symmetric metric gravity theory is realized from a deformation of a Courant algebroid. Presumably, other models of gravity can be as well investigated and described in terms of generalized geometry. Deeper implications of such particular systematic approach are foreseeable. From these intricate connections between gravitational physics and mathematical objects, we are led to possible ideas in approaching the long-standing problem of computing higher order corrections in gravity/string theories. As an outlook, (2.172) could serve as a starting point to backward engineer for dilaton field inclusion in the deformation of the algebroid. Similar work has appeared in [59]. There is also a proposal to extend the generalized bundle to discuss the dilaton [60]. Alternatively and naively, one can as well try replacing g_{ij} with $e^{-2\phi}g_{ij}$ and similarly for B_{ij}, and check if beta functions can be fully worked out. There is a construction which combines metric, dilaton and a (quasi-)symplectic structure, instead of a 2-form, in an action of Einstein-Hilbert type, using algebroid approach [61]. Supergravity actions formulated in the context of generalized geometry can also be found in [62]. There have also been attempts to understand the α'-corrections to supergravity. For instance, an α'-deformed generalized geometry was proposed in [63] where the Courant bracket is α'-deformed.

3 Graded Geometry

3.1 Introduction

Before getting to graded geometry, we should mention supergeometry. Supergeometry extends the classical geometry where usual coordinates commute, allowing anti-commuting odd coordinates. Supermanifolds are global objects obtained by gluing the extended coordinate systems [64].

A super vector space is a $\mathbb{Z}_2$-graded vector space [65] over $\mathbb{R}$ or $\mathbb{C}$,

$$V = V_0 \bigoplus V_1 \ , \tag{3.1}$$

with even and odd components, hence even and odd parity for the homogeneous elements in V_0 and V_1 respectively. $\mathbb{Z}_2$-graded vector space is the same as superspace by standard terminology [64]. Using a parity reversion functor Π, the parity of a component is changed, $(\Pi V)_0 = V_1$ and $(\Pi V)_1 = V_0$. When multiplication of the homogeneous elements v, v' in V respects the grading,

$$|vv'| = |v| + |v'| \mod 2 \tag{3.2}$$

and if V is associative algebra, this is referred to as a superalgebra. $|v|$ denotes the parity of the homogenous element. When V is equipped with a Lie bracket $[\cdot, \cdot]$ of parity (degree) ℓ and satisfies

$$[v, v'] = -(-1)^{(|v|+\ell)(|v'|+\ell)}[v', v] \tag{3.3}$$

and

$$[v, [v', v'']] = [[v, v'], v''] + (-1)^{(|v|+\ell)(|v'|+\ell)}[v', [v, v'']] \ , \tag{3.4}$$

which obeys the grading $|[v, v']| = |v| + |v'| + \ell$, we have a Lie superalgebra.

Let us illustrate the discussion in local coordinates for a better understanding. Let $E \to M$ be a vector bundle with local coordinate x^i on manifold M, where

$$x^i x^j = x^j x^i \ . \tag{3.5}$$

A supermanifold ΠE is obtained by making the fiber coordinates θ^μ odd, thus anti-commuting. Local coordinates on ΠE are hence (x^i, θ^μ), where

$$\theta^\mu \theta^\nu = -\theta^\nu \theta^\mu \ . \tag{3.6}$$

Smooth function on ΠE takes the form

$$f(x, \theta) = f_0(x) + f_\mu(x)\theta^\mu + f_{\mu\nu}(x)\theta^\mu\theta^\nu + \cdots \ , \tag{3.7}$$

where the maximal number of θ is the dimension of the fiber [66]. A smooth function on the supermanifold is equivalent to a section of the ungraded (or graded [67]) cotangent bundle

$$C^\infty(\Pi E) = \Gamma(\Lambda^\bullet E^*)\ .^8 \tag{3.8}$$

This is a general fact. As an example, we can associate to a smooth manifold M an odd tangent bundle, thus having a supermanifold ΠTM, defined by gluing rule

$$\tilde{x}^\mu = \tilde{x}^\mu(x)\ , \qquad \tilde{\theta}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\theta^\nu\ , \tag{3.9}$$

where θs are glued as dx^μ [64]. A function on this supermanifold is expanded as

$$f(x,\theta) = \sum_{p=0}^{\dim M} \frac{1}{p!} f_{\mu_1\mu_2\cdots\mu_p}(x)\, \theta^{\mu_1}\theta^{\mu_2}\cdots\theta^{\mu_p}\ , \tag{3.10}$$

thus the identification of the function with the differential form,

$$C^\infty(\Pi TM) = \Omega^\bullet(M)\ . \tag{3.11}$$

For the supermanifold ΠT^*M, the gluing is done through

$$\tilde{x}^\mu = \tilde{x}^\mu(x)\ , \qquad \tilde{\theta}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^u}\theta_\nu\ , \tag{3.12}$$

where θ_μ transforms as ∂_μ. The function on ΠT^*M is

$$f(x,\theta) = \sum_{p=0}^{\dim M} \frac{1}{p!} f^{\mu_1\mu_2\cdots\mu_p}(x)\, \theta_{\mu_1}\theta_{\mu_2}\cdots\theta_{\mu_p}\ , \tag{3.13}$$

thus the identification of this function with the multivector field,

$$C^\infty(\Pi T^*M) = \Gamma(\wedge^\bullet TM)\ . \tag{3.14}$$

If E is a vector bundle, a derivation (sometimes called differential) of $\Gamma(\Lambda^\bullet E^*)$ is identified with a vector field on the supermanifold ΠE. When it is a Lie algebroid, that is, $E = A$, its derivation d_A is identified with a vector field Q_A on ΠA, satisfying the graded commutator

$$[Q_A, Q_A] = 0\ , \tag{3.15}$$

which is then called the homological vector field [69]. In addition to the previous Lie algebroid definition in Sec. (2.1), a derivation of exterior algebra of degree +1 can be defined, due to the skew-symmetry of Lie bracket [48], that is, an $\mathbb{R}$-linear map $d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$,

$$d_A(\sigma\wedge\sigma') = d_A\sigma\wedge\sigma' + (-1)^{|\sigma|}\sigma\wedge d_A\sigma'\ , \tag{3.16}$$

[8] $\Gamma(\Lambda^\bullet E^*)$ denotes the same as $\Omega^\bullet(E)$, used previously to refer to forms. Note that set of sections on M, for instance, $\Gamma(M, T^*M) = \Omega^1(M)$, and $\Gamma(M, TM)$ is identified with set of vector fields $\mathfrak{X}(M)$ [33]. In general, smooth differential forms $\Omega^p(M) := \Gamma(\Lambda^p T^*M)$ [68].

for $\sigma, \sigma' \in \Omega^\bullet(A)$, for degree $|\sigma|$. The derivations square to zero, $d_A^2 = 0$. This is equivalent to the Jacobi and Leibniz identities for $[\cdot,\cdot]_A$ in a Lie algebroid.

A supermanifold ΠE becomes a graded manifold, denoted by $E[1]$, when the coordinates on the base are assigned with degree 0 and the coordinates on the fiber are assigned with degree 1.[9] In general, fiber coordinates can have any degree N, where N is a positive integer, hence a graded manifold of $E[N]$. For the above Lie algebroid example with a 1-degree homological vector field Q_A, it becomes a 1-degree graded manifold $A[1]$, and is noted as a Q-manifold. Another example is the P-manifold. It is a graded manifold of degree 1, $T^*[1]M$, with a canonical symplectic structure of degree 1, ω. This symplectic structure ω defines a Poisson bracket of degree -1 on $C^\infty(T^*[1]M) = \Gamma(\Lambda^\bullet TM)$, that is, it corresponds to the Schouten-Nijenhuis bracket of multivector fields [66]. In fact, manifolds M are in one-to-one correspondence with P-manifolds.

The Schouten-Nijenhuis bracket $[\cdot,\cdot]_A$ induced from a Lie algebroid (another induced structure was the derivation d_A), provides a multivector field algebra $\mathfrak{X}^\bullet(A)$. It defines a Gerstenhaber algebra $(\mathfrak{X}^\bullet(A), [\cdot,\cdot]_A)$, in which
(i) $[\cdot,\cdot]_A$ is a degree -1 map,

$$|[P,Q]_A| = |P| + |Q| - 1 \ , \tag{3.17}$$

that is, $[P,Q]_A \in \mathfrak{X}^{|P|+|Q|-1}(M)$,
(ii) $[P,\cdot]_A$ is a derivation of degree $(|P|-1)$ of the exterior algebra $\mathfrak{X}^\bullet(A)$,

$$[P, Q \wedge R]_A = [P,Q]_A \wedge R + (-1)^{(|P|-1)\,|Q|}\, Q \wedge [P,R]_A \ , \tag{3.18}$$

(iii) the bracket is graded skew-symmetric

$$[P,Q]_A = -(-1)^{(|P|-1)(|Q|-1)}\,[Q,P]_A \ , \tag{3.19}$$

and
(iv) satisfies graded Jacobi identity

$$[P,[Q,R]_A]_A = [[P,Q]_A, R]_A + (-1)^{(|P|-1)(|Q|-1)}[Q,[P,R]_A]_A \ , \tag{3.20}$$

for $P, Q, R \in \mathfrak{X}^\bullet(A)$ [48].

When a P-manifold has in addition a degree 1 homological vector field Q_π, and the symplectic structure in P-manifold is invariant with respect to Q_π, it is noted as a QP-manifold. The one-to-one correspondence is $(T^*[1]M, Q_\pi, \omega) \leftrightarrow (M,\pi)$, where (M,π) is a Poisson

[9] Graded geometry is $\mathbb{Z}$-refinement of supergeometry. Generalization of most definitions is straightforward [64]. Basically, the $\mathbb{Z}$-graded vector space is $V = \bigoplus_{i\in\mathbb{Z}} V_i$, for integer i which is the corresponding degree for the respective homogenous element .

manifold. Q_π on $T^*[1]M$ is the Lichnerowicz-Poisson differential acting on $\Gamma(\Lambda^\bullet TM)$, which is $d_\pi = [\pi, \cdot]$, where the bracket is the Schouten-Nijenhuis bracket of multivector fields on M.

The Lichnerowicz-Poisson differential (denoted as d below) is associated to the Lie algebroid $(A = T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$ [70], where anchor $\pi^\sharp : T^*M \to TM$ and the Lie algebroid structure $[\cdot,\cdot]_\pi$ are defined by

$$[df, dg]_\pi = d\{f, g\} \ , \tag{3.21}$$

for $f, g \in C^\infty(M)$.

Generally, given local coordinates x^i on M and a local basis e_α for sections, the Lie algebroid anchor map is

$$a(e_\alpha) = a^i_\alpha \frac{\partial}{\partial x^i} \tag{3.22}$$

and the anti-symmetric bracket is

$$[e_\alpha, e_\beta]_A = C_{\alpha\beta}{}^\gamma e_\gamma \ . \tag{3.23}$$

When $a^i_\alpha = \delta^i_\alpha$ and $C_{\alpha\beta}{}^\gamma = 0$, the Poisson structure on T^*M is the canonical symplectic structure.

A Poisson manifold (M, π) contains a bivector field $\pi \in \mathfrak{X}^2(M)$ which is a Poisson structure on M, and satisfies $[\pi, \pi] = 0$, which is equivalent to the Jacobi identity for the Poisson bracket $\{\cdot,\cdot\}$, as can be seen from

$$\frac{1}{2}[\pi, \pi](df_1, df_2, df_3) = \{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} \ , \tag{3.24}$$

for $\{f, g\} = \pi(df, dg)$ [68].

A symplectic manifold (M, ω) contains a non-degenerate, closed 2-form $\omega \in \Omega^2(M)$ which is called the symplectic structure on M. For a 2-form $\omega \in \Omega^2(M)$, there is a map $\omega^\flat : TM \to T^*M$.

There is a one-to-one correspondence between a non-degenerate symplectic structure and a non-degenerate Poisson structure, thus[10] $\omega = \pi^{-1}$ or $\pi = \omega^{-1}$. In local coordinates, a Poisson structure

$$\pi = \sum_{i<j} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{3.25}$$

is non-degenerate if the matrix $\pi^{ij}(x) (= \{x^i, x^j\}(x))$ is invertible; correspondingly the symplectic structure is

$$\omega = \sum_{i<j} \omega_{ij}(x)\, dx^i \wedge dx^j \ , \tag{3.26}$$

where $\omega_{ij}(x) = (\pi^{ij}(x))^{-1}$ [68]. Canonically on $\mathbb{R}^{2n}$ (even dimensional M), the non-degenerate Poisson structure is

$$\pi = \sum_{i=1}^n \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} \ , \qquad \pi^{-1} = \sum_{i=1}^n dq^i \wedge dp_i \ . \tag{3.27}$$

[10] For the correspondence between a non-degenerate 2-form and a non-degenerate bivector: $\omega^\flat = (\pi^\sharp)^{-1} \leftrightarrow \pi^\sharp = (\omega^\flat)^{-1}$.

One of the reasons to be interested in the graded geometry formalism is in the interpretation of structures in generalized geometry, such as the Dorfman bracket, which finds its place in the graded picture. Defined in [71], the derived bracket (denoted by $[\cdot,\cdot]_d$) is

$$[e_1, e_2]_d = [[e_1, d], e_2] \ . \tag{3.28}$$

$[\cdot,\cdot]$ is a graded commutator, which we have encountered for instance with the Schouten-Nijenhuis bracket with properties such as (3.19) and (3.20). An element $d \in \mathcal{D}$ of odd degree satisfies $[d, d] = 0$, where $\mathcal{D}$ is a space generated by several commuting elements that square to zero, which is a generalization of the 1-dimensional vector space generated by the de Rahm differential. Let $e_1, e_2 \in \mathfrak{A}$, where $\mathfrak{A}$ is an abelian Lie algebra which is a generalization of the space of interior product by X, i_X's where X is a vector field on the manifold. The derived bracket is a graded Lie bracket. A simple example can be seen from the Cartan formulas. As an example, for $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$, where X, Y are vector fields and $\mathcal{L}_X$ is Lie derivation by X, and $[\cdot,\cdot]$ is a graded commutator, except that $[X, Y]$ is a Lie bracket. In addition, $\mathcal{L}_X = [d, i_X] = di_x + i_X d$. The Cartan formula can be written as

$$i_{[X,Y]} = [\mathcal{L}_X, i_Y] = [[d, i_X], i_Y] \ , \tag{3.29}$$

hence

$$i_{[X,Y]} = [[d, i_X], i_Y] = [i_X, i_Y]_d \ , \tag{3.30}$$

which is a derived bracket. When making the correspondence $i_X \leftrightarrow X$, $[i_X, i_Y]_d$ becomes $[X, Y]$. This shows that the Lie bracket of vector fields is a derived bracket [71].

For a generalized vector $\mathbb{V} = X + \lambda$ where λ is a 1-form, let $\xi_{\mathbb{V}} := i_X + \lambda\wedge$, which defines action of generalized vectors on forms $\omega \in \Omega^\bullet(M)$,

$$\xi_{\mathbb{V}}(\omega) = i_X\omega + \lambda \wedge \omega \ . \tag{3.31}$$

We have the following relations

$$\xi_{\mathbb{V}}\xi_{\mathbb{Y}} + \xi_{\mathbb{Y}}\xi_{\mathbb{V}} = \langle \mathbb{V}, \mathbb{Y} \rangle \tag{3.32}$$

and

$$\mathcal{L}_{\mathbb{V}} = [d, \xi_{\mathbb{V}}] = d\xi_{\mathbb{V}} + \xi_{\mathbb{V}} d \ , \tag{3.33}$$

$$[\mathcal{L}_{\mathbb{V}}, \xi_{\mathbb{Y}}] = [[d, \xi_{\mathbb{V}}], \xi_{\mathbb{Y}}] = \xi_{[\mathbb{V},\mathbb{Y}]_{\mathrm{D}}} \ . \tag{3.34}$$

Hence, the Dorfman bracket (2.33) is a derived bracket. For a detailed proof, see [71].

Our work in this section is closely connected to the previous topic of study *and has been prepared for submission.* In the following, we will relate the Courant algebroid structure using graded manifolds. In generalized geometry, we have been working with the objects of a Courant algebroid such as the bracket and inner product, where the variables of the objects are vector fields and 1-forms. A distinct difference here, in the graded formalism, is that we basically deal with the coordinates of the fiber and base with degrees. There are relations between these coordinates to be satisfied and through them connections to the Courant algebroid structure are established.

Before we set up the investigation in the language of graded geometry, we remind the readers briefly about the Courant algebroid structure in generalized geometry. The foundation of our investigation has been initiated from and built around a standard example of a Courant algebroid, the one which we discussed in the previous topic. We consider a generalized tangent bundle, $E = TM \bigoplus T^*M$, where the non-anti-symmetric Dorfman bracket, $[\cdot,\cdot]_{\mathrm{D}}$ and symmetric pairing, $\langle\cdot,\cdot\rangle$ are defined on elements of the space of sections, $\Gamma(E)$. In addition, there is an anchor map $\rho : E \to TM$. The axioms of a Courant algebroid are the Leibniz rule, Jacobi identity, homomorphism, compatibility condition between pairing and Dorfman bracket, and non-anti-symmetrism of the Dorfman bracket.

In this work, we are following [72] closely and perform deformations. Consider a graded manifold $E_{\mathrm{graded}} = T^*[2]T[1]M$, where $[\#]$ indicates the (shift of) degree in the fiber coordinate. Given this graded manifold, which is a cotangent bundle with degree shifted by 2 of a tangent bundle with degree shifted by 1 of the manifold M, the generators are 0-degree function f, 1-degree anti-commuting X, Y and 2-degree v, w. Identification of the smooth function on the graded manifold with space of section of $\wedge^\bullet E^*$ results in the following graded Poisson algebra [72]:

$$\{v, f\} = v \cdot f \ , \tag{3.35}$$
$$\{X, Y\} = \langle X, Y\rangle \ , \tag{3.36}$$
$$\{v, X\} = \nabla_v X \ , \tag{3.37}$$
$$\{v, w\} = [v, w]_{\mathrm{Lie}} + R(v, w) \ , \tag{3.38}$$

where $\{\cdot,\cdot\}$ is a degree -2 Poisson bracket. It satisfies a graded Jacobi identity

$$\{e_1, \{e_2, e_3\}\} = \{\{e_1, e_2\}, e_3\} + (-1)^{(|e_1|\cdot|e_2|)}\{e_2, \{e_1, e_3\}\} \ , \tag{3.39}$$

where $|e_1|$ denotes the degree of the generator e_1. The curvature of the connection ∇ is a 2-degree R. The difference here with the examples of graded manifolds discussed before is that the fiber coordinates of the cotangent bundle include a degree 2 v.

Given a function Θ of degree 3 on the graded manifold, the structure equation

$$\{\Theta, \Theta\} = 0 \ , \tag{3.40}$$

corresponds to the axioms of Courant algebroid [72], [73]. (3.40) is also called Maurer-Cartan equation. Briefly recall that for a Lie algebra valued 1-form θ, the Maurer-Cartan equation

is $d\theta + \frac{1}{2}[\theta, \theta]_{\text{Lie}} = 0$. So, from the basic objects $([\cdot,\cdot]_{\text{D}}, \langle\cdot,\cdot\rangle, \rho)$ for specification of a Courant algebroid, it reduces to a graded Poisson bracket, or more specifically the object Θ to study. The role of Θ will come clear momentarily.
The derivation of the graded Poisson algebra is identified with a homological Hamiltonian vector field, $d_\Theta = \{\Theta, \cdot\}$ on $T^*[2]T[1]M$,

$$[d_\Theta, d_\Theta]_{\text{Lie}} = 0 \ , \qquad d_\Theta^2 = 0 \ . \tag{3.41}$$

Relations (3.40) and (3.41) are equivalent. With a Hamiltonian Θ, the Dorfman bracket is realized as a derived bracket

$$\{\{e_1, \Theta\}, e_2\} := [e_1, e_2]_{\text{D}} \tag{3.42}$$

and the anchor map ρ as

$$\{\{e_1, \Theta\}, f\} := \rho(e_1) \cdot f \ , \tag{3.43}$$

where $e_1, e_2 \in \Gamma(E = TM \bigoplus T^*M)$.

3.2 Deformation I

In local coordinates of $E_{\text{graded}} = T^*[2]T[1]M$, we have a 0-degree function x^i, two 1-degree anti-commuting χ_i, θ^i (χ_i, θ^i are dual to each other) and a 2-degree p_i (p_i is dual to x^i). The degrees of χ_i and p_i are shifted by 2 in $T^*[2]T[1]M$. The non-vanishing graded Poisson brackets are

$$\{\chi_i, \theta^j\} = \delta_i^j = \{\theta^j, \chi_i\} \ , \tag{3.44}$$

$$\{p_i, x^j\} = \delta_i^j = -\{x^j, p_i\} \ , \tag{3.45}$$

where $i, j = 1, \cdots, d$,

$$\{\xi_A, \xi_B\} = \eta_{AB} = \begin{pmatrix} 0 & \mathbb{1}^d \\ \mathbb{1}^d & 0 \end{pmatrix} \ , \tag{3.46}$$

where $\xi = (\chi, \theta)$ and $A, B = 1, \cdots, 2d$. The corresponding anchor map to the tangent bundle is

$$\rho(\chi_i) = p_i \quad \text{and} \quad \rho(\theta^i) = 0 \ . \tag{3.47}$$

We consider a deformation of the Poisson bracket with $\mathcal{G} = g + B$, which is composed of a symmetric metric g and an anti-symmetric two-form B. Hence, the deformed Poisson bracket is

$$\{e_1, e_2\}' = e^{-\mathcal{G}}\{e^{\mathcal{G}}(e_1), e^{\mathcal{G}}(e_2)\} \ , \tag{3.48}$$

where $e^{\mathcal{G}} : E_{\text{graded}} \to E_{\text{graded}} : e^{\mathcal{G}}(\chi_i) = \chi_i + \mathcal{G}_{ij}\theta^j$ in coordinates. We will omit the prime notation for deformed objects hereafter. This subsequently amounts to a deformed bracket

$$\{\xi_\alpha, \xi_\beta\} = G_{\alpha\beta} = \begin{pmatrix} 2g(x) & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ , \tag{3.49}$$

where $\alpha, \beta = 1, \cdots, 2d$ that contains

$$\{\chi_i, \chi_j\} = 2g_{ij}(x) \ . \tag{3.50}$$

Furthermore,

$$\{p_i, \chi_j\} = (\partial_i \mathcal{G}_{jk})\theta^k \ , \tag{3.51}$$

where $\mathcal{G}_{jk} = g_{jk} + B_{jk}$. (3.50) is obtained from (3.48),

$$\begin{aligned}
\{\chi_i, \chi_j\} &= e^{-\mathcal{G}}\{e^{\mathcal{G}}\chi_i, e^{\mathcal{G}}\chi_j\} && (3.52)\\
&= e^{-\mathcal{G}}\{\chi_i + \mathcal{G}_{ik}\theta^k, \chi_j + \mathcal{G}_{jk}\theta^k\} && (3.53)\\
&= \mathcal{G}_{jk}\delta_i^k + \mathcal{G}_{ik}\delta_j^k = \mathcal{G}_{ji} + \mathcal{G}_{ij} && (3.54)\\
&= 2g_{ij} \ , && (3.55)
\end{aligned}$$

while (3.51) can as well be derived from (3.48),

$$\begin{aligned}
\{p_i, \chi_j\} &= e^{-\mathcal{G}}\{p_i, e^{\mathcal{G}}\chi_j\} && (3.56)\\
&= e^{-\mathcal{G}}\{p_i, \chi_j + \mathcal{G}_{jk}\theta^k\} && (3.57)\\
&= \partial_i \mathcal{G}_{jk}\, \theta^k \ . && (3.58)
\end{aligned}$$

We can infer from the following contraction with a χ,

$$\begin{aligned}
\{\{p_i, p_j\}, \chi_k\} &= \{p_i, \{p_j, \chi_k\}\} - \{p_j, \{p_i, \chi_k\}\} && (3.59)\\
&= \{p_i, \{p_j, \chi_k\}\} + \{\{p_i, \chi_k\}, p_j\} && (3.60)\\
&= \{p_i, \partial_j \mathcal{G}_{kl}\, \theta^l\} + \{\partial_i \mathcal{G}_{kl}\, \theta^l, p_j\} && (3.61)\\
&= \partial_i \partial_j \mathcal{G}_{kl}\, \theta^l - \partial_j \partial_i \mathcal{G}_{kl}\, \theta^l && (3.62)\\
&= 0 \ , && (3.63)
\end{aligned}$$

that

$$\{p_i, p_j\} = 0 \ . \tag{3.64}$$

As a summary, the Poisson relations are

$$\begin{aligned}
&\{x^i, x^j\} = 0 \ , \quad \{x^i, \theta^j\} = 0 \ , \quad \{x^i, \chi_j\} = 0 \ , && (3.65)\\
&\{p_i, x^j\} = \delta_i^{\ j} \ , \quad \{\chi_i, \theta^j\} = \delta_i^{\ j} \ , \quad \{\theta^i, \theta^j\} = 0 \ , && (3.66)\\
&\{p_i, \theta^j\} = 0 \ , \quad \{\chi_i, \chi_j\} = 2g_{ij} \ , \quad \{p_i, \chi_j\} = \partial_i \mathcal{G}_{jk}\, \theta^k \ . && (3.67)
\end{aligned}$$

This set of graded Poisson brackets obey the graded Jacobi identity. All of the coordinates have to be checked to satisfy the Jacobi. For example, the relation

$$\{\theta^i, \{\theta^j, \chi_k\}\} = \{\{\theta^i, \theta^j\}, \chi_k\} - \{\theta^j, \{\theta^i, \chi_k\}\} \tag{3.68}$$

between coordinates θ and χ satisfies the Jacobi identity. Note that θs (as well as χs) anti-commute. Another example is between coordinates p_i and χ_j, we have the Jacobi identity

$$\{p_i, \{\chi_j, \chi_k\}\} = \{\{p_i, \chi_j\}, \chi_k\} + \{\chi_j, \{p_i, \chi_k\}\} \ , \tag{3.69}$$

as on the LHS

$$\{p_i, \{\chi_j, \chi_k\}\} = \{p_i, 2g_{jk}\} = 2\partial_i g_{jk} \tag{3.70}$$

and on the RHS

$$\begin{aligned}
\{\{p_i, \chi_j\}, \chi_k\} + \{\chi_j, \{p_i, \chi_k\}\} &= \{\partial_i \mathcal{G}_{jl}\, \theta^l, \chi_k\} + \{\chi_j, \partial_i \mathcal{G}_{kl}\, \theta^l\} &(3.71)\\
&= \partial_i \mathcal{G}_{jk} + \partial_i \mathcal{G}_{kj} &(3.72)\\
&= 2\partial_i g_{jk}\ . &(3.73)
\end{aligned}$$

We remark that, in general, by (3.35) - (3.37), the Jacobi identity between the 2-degree v and 1-degrees X, Y is easily satisfied, as

$$\begin{aligned}
\{v, \{X, Y\}\} &= \{v, \langle X, Y\rangle\} &(3.74)\\
&= v \cdot \langle X, Y\rangle = \langle \nabla_v X, Y\rangle + \langle X, \nabla_v Y\rangle &(3.75)\\
&= \{\{v, X\}, Y\} + \{X, \{v, Y\}\}\ , &(3.76)
\end{aligned}$$

by using the metricity condition.

In the order of coordinates (x, θ, χ, p), all their Poisson relations can be summarized in matrix form:

$$\omega^{-1} = \begin{pmatrix} 0 & 0 & 0 & -\delta^i{}_j \\ 0 & 0 & \delta^i{}_j & 0 \\ 0 & \delta_i{}^j & 2g_{ij} & -(\partial_j \mathcal{G}_{il})\theta^l \\ \delta_i{}^j & 0 & (\partial_i \mathcal{G}_{jl})\theta^l & 0 \end{pmatrix}\ . \tag{3.77}$$

The respective inverse matrix is

$$\omega = \begin{pmatrix} 0 & -(\partial_j \mathcal{G}_{kl})\theta^l & 0 & \delta_j{}^k \\ -(\partial_k \mathcal{G}_{jl})\theta^l & -2g_{jk} & \delta_j{}^k & 0 \\ 0 & \delta^j{}_k & 0 & 0 \\ -\delta^j{}_k & 0 & 0 & 0 \end{pmatrix}\ , \tag{3.78}$$

which has been worked out block-by-block using $\omega^{-1}\omega = \mathbb{1}$. (3.78) translates into the 2-form

$$\begin{aligned}
\omega \;=\; & -dx^j \wedge (\partial_j \mathcal{G}_{kl})\theta^l \wedge d\theta^k + dx^j \wedge \delta_j{}^k \wedge dp_k &(3.79)\\
& -d\theta^j \wedge (\partial_k \mathcal{G}_{jl})\theta^l \wedge dx^k - d\theta^j \wedge 2g_{jk} \wedge d\theta^k &(3.80)\\
& +d\theta^j \wedge \delta_j{}^k \wedge d\chi_k + d\chi_j \wedge \delta^j{}_k \wedge d\theta^k - dp_j \wedge \delta^j{}_k \wedge dx^k. &(3.81)
\end{aligned}$$

It is straightforward to check that this non-degenerate 2-degree 2-form is closed: $d\omega = 0$ and hence symplectic.

$$\begin{aligned}
d\omega &= -dx^j\, d(\partial_j \mathcal{G}_{kl}\, \theta^l)\, d\theta^k - d\theta^j\, d(\partial_k \mathcal{G}_{jl}\, \theta^l)\, dx^k - d\theta^j\, d(2g_{jk})\, d\theta^k &(3.82)\\
&= -dx^j\, (\partial_h \partial_j \mathcal{G}_{kl}\, dx^h\, \theta^l + \partial_j \mathcal{G}_{kl}\, d\theta^l)\, d\theta^k &(3.83)\\
&\quad -d\theta^j\, (\partial_h \partial_k \mathcal{G}_{jl}\, dx^h\, \theta^l + \partial_k \mathcal{G}_{jl}\, d\theta^l)\, dx^k \\
&\quad -2\, d\theta^j\, \partial_h g_{jk}\, dx^h\, d\theta^k \\
&= 0\ , &(3.84)
\end{aligned}$$

noting that

$$\begin{aligned} d\theta^j \wedge d\theta^l \wedge dx^k &= dx^k \wedge d\theta^j \wedge d\theta^l \ , && (3.85)\\ d\theta^j \wedge dx^h \wedge dx^k &= dx^h \wedge dx^k \wedge d\theta^j \ , && (3.86)\\ d\theta^j \wedge dx^h \wedge d\theta^k &= -dx^h \wedge d\theta^j \wedge d\theta^k \ , && (3.87)\end{aligned}$$

and

$$(\partial_j B_{kl})\, dx^j \wedge d\theta^l \wedge d\theta^k = 0 \qquad (3.88)$$

as B is anti-symmetric and $d\theta$s are symmetric. To summarize, the sign convention used is

$$\begin{aligned} dx^i \wedge dx^j &= -dx^j \wedge dx^i \ , && (3.89)\\ d\chi_i \wedge d\theta^j &= d\theta^j \wedge d\chi_i \ , && (3.90)\\ d\theta^j \wedge d\theta^l &= d\theta^l \wedge d\theta^j \ , && (3.91)\\ dx^i \wedge d\chi_j &= -d\chi_j \wedge dx^i \ . && (3.92)\end{aligned}$$

This is based on two separate facts: sign induced by the anti-commuting odd coordinates and by the exterior derivative.

We consider a simple Hamiltonian

$$\Theta = \theta^i p_i, \qquad (3.93)$$

it is clear that $\{\Theta, \Theta\} = 0$,

$$\{\Theta, \Theta\} = \{\theta^i p_i, \theta^j p_j\} = \{\theta^i, \theta^j p_j\} p_i + \theta^i \{p_i, \theta^j p_j\} = 0 \ , \qquad (3.94)$$

as $\{\theta^i, \theta^j\} = 0$ and $\{\theta^i, p_j\} = 0$.

To investigate the Dorfman bracket (3.42) that will arise from the deformation of the graded Poisson algebra, let us work with the 1-degree X, Y only in the χ basis, that is, $X = X^i(x)\chi_i$ and $Y = Y^i(x)\chi_i$, as χ is the corresponding vector basis in the picture of generalized geometry. Firstly,

$$\begin{aligned} \{X, \Theta\} &= \{X^i \chi_i, \theta^j p_j\} && (3.95)\\ &= \{X^i(x)\chi_i, \theta^j\} p_j - \theta^j \{X^i(x)\chi_i, p_j\} && (3.96)\\ &= X^i p_i + \theta^j (\partial_j X^i)\chi_i + X^i \theta^j (\partial_j \mathcal{G}_{il})\theta^l \ . && (3.97)\end{aligned}$$

Then,

$$\begin{aligned} \{\{X, \Theta\}, Y\} = \; & X^i(\partial_i Y^k)\chi_k + X^i Y^k (\partial_i \mathcal{G}_{kl})\theta^l - (\partial_k X^i) Y^k \chi_i && (3.98)\\ & + 2 g_{ik}\theta^j (\partial_j X^i) Y^k - X^i (\partial_k \mathcal{G}_{il}) Y^k \theta^l + X^i \theta^j (\partial_j \mathcal{G}_{il}) Y^l \ . \end{aligned}$$

Lastly, contracting the result with $Z = Z^i(x)\chi_i$,

$$\begin{aligned} \{Z, \{\{X, \Theta\}, Y\}\} &= 2g_{lk}\left(X^i(x)(\partial_i Y^k) - Y^i(x)(\partial_i X^k)\right) Z^l(x) \\ &\quad + Z^l(x)X^i(x)Y^k(x)(\partial_i \mathcal{G}_{kl} - \partial_k \mathcal{G}_{il} + \partial_l \mathcal{G}_{ik}) \\ &\quad + Z^i(x)(\partial_i X^l) 2g_{lk} Y^k(x) \qquad (3.99) \\ &= 2g(Z, [X,Y]_{\text{Lie}}) + 2g(\nabla_Z X, Y)\ . \qquad (3.100) \end{aligned}$$

Recall that the result (3.100) is equal to the result we obtained in the previous topic of generalized geometry: $\langle Z, [X,Y]_{\text{D}}\rangle = \langle Z, [X,Y]_{\text{Lie}}\rangle + 2g(\nabla_Z X, Y)$. From the first to second equality (3.100), the connection coefficient can be read off as

$$g_{jk}\Gamma_{li}^{\ \ j} = \frac{1}{2}\left(\partial_i \mathcal{G}_{kl} - \partial_k \mathcal{G}_{il} + \partial_l \mathcal{G}_{ik}\right)\ . \qquad (3.101)$$

Plugging in $\mathcal{G} = g + B$, we have

$$g_{jk}\Gamma_{li}^{\ \ j} = \frac{1}{2}\left(\partial_i g_{kl} - \partial_k g_{il} + \partial_l g_{ik}\right) + \frac{1}{2}H_{ikl}\ , \qquad (3.102)$$

with the totally anti-symmetric 3-form $H = dB$.
To put into (3.100), note that the Lie bracket in coordinates is

$$[X,Y]_{\text{Lie}} = X^i(\partial_i Y^j)\chi_j - Y^i(\partial_i X^j)\chi_j\ , \qquad (3.103)$$

and due to the relation $\langle X, Y\rangle = \{X, Y\}$, we can contract the bracket with a 1-degree Z which we have chosen to expand in only χ basis,

$$\begin{aligned} \langle Z, [X,Y]_{\text{Lie}}\rangle &= \{Z, [X,Y]_{\text{Lie}}\} \qquad (3.104) \\ &= 2g_{lj}(Z^l X^i(\partial_i Y^j) - Z^l Y^i(\partial_i X^j))\ . \qquad (3.105) \end{aligned}$$

Moreover, since

$$\nabla_Z X = Z^i(\partial_i X^j + \Gamma_{il}^{\ \ j} X^l)\partial_j\ , \qquad (3.106)$$

the connection is

$$\begin{aligned} 2g(\nabla_Z X, Y) &= 2g(Z^i(\partial_i X^j)\partial_j + Z^i\Gamma_{il}^{\ \ j} X^l \partial_j, Y^k \partial_k) \qquad (3.107) \\ &= 2g(\partial_j, \partial_k)\, Z^i(\partial_i X^j)Y^k + 2g(\partial_j, \partial_k)\, Z^i \Gamma_{il}^{\ \ j} X^l Y^k \qquad (3.108) \\ &= 2g_{jk} Z^i(\partial_i X^j)Y^k + 2g_{jk}\Gamma_{il}^{\ \ j} Z^i X^l Y^k\ , \qquad (3.109) \end{aligned}$$

where the connection coefficient is (3.101).
On the other hand, from the definition (3.43), given (3.93), we have

$$\{X, \Theta\} = X^i p_i + \theta^j(\partial_j X^i)\chi_i + X^i\theta^j(\partial_j \mathcal{G}_{il})\theta^l\ , \qquad (3.110)$$

hence,

$$\{\{X, \Theta\}, f\} = \rho(X)\cdot f = X^i \partial_i f\ , \qquad (3.111)$$

where the anchor map is undeformed, as $\rho(\chi_i) = p_i$, and $\{p_i, f(x)\} = \partial_i f$.
An important note from the correspondence is in the following picture, for the graded manifold on the left and the generalized tangent bundle from generalized geometry on the right,

$$\begin{array}{ccc} T^*[2]T[1]M & \qquad & TM\bigoplus T^*M \\ \downarrow \quad \downarrow & & \downarrow \qquad \downarrow \\ p_i, \chi_i \quad x^i, \theta^i & & V^i(x)\chi_i \quad \lambda_i(x)\theta^i \end{array}$$

we notice that the notion of vector field and 1-form gets interchanged.

3.3 Discussion I

We recall our previous deformed Courant algebroid formulation that has led us to a non-symmetric gravity theory [49]. By deforming the pairing and Dorfman bracket in our Courant algebroid in such a way that the axioms are preserved,

$$\langle e_1, e_2\rangle' = \langle e^{\mathcal{G}}(e_1), e^{\mathcal{G}}(e_2)\rangle = \langle e_1, e_2\rangle + 2g\left(a(e_1), a(e_2)\right) \tag{3.112}$$

and

$$[e_1, e_2]'_{\mathrm{D}} = e^{-\mathcal{G}}[e^{\mathcal{G}}(e_1), e^{\mathcal{G}}(e_2)]_{\mathrm{D}} = [e_1, e_2]_{\mathrm{D}} + 2g(\nabla a(e_1), a(e_2)) \ , \tag{3.113}$$

we found a general connection ∇ in the deformed Dorfman bracket that comprises Levi-Civita connection and a contortion which is a totally anti-symmetric 3-form H:

$$g \circ \nabla = g \circ \nabla^{\mathrm{LC}} + \frac{1}{2}H \ . \tag{3.114}$$

Note that the non-symmetric metric (as we named it in [49]), $\mathcal{G} = g + B$ in (3.112) and (3.113) is a map $\mathcal{G} : TM \to T^*M$ for a Courant algebroid that is defined on $TM \bigoplus T^*M$. The resulting connection (3.114) is an object on TM. In the graded context, the map is $e^{\mathcal{G}} : E_{\text{graded}} \to E_{\text{graded}}$. A connection is realized through the derived bracket, provided with a 3-form function Θ.

In [49], using the connection (3.114), we computed the curvature tensor in a coordinate basis. In the current formulation, we find from the deformation (3.48) that $\{p_i, p_j\} = 0$. This means that the deformed graded Poisson algebra has no curvature (see (3.38)). The result reminds us of a Weitzenböck connection which is constructed in terms of vielbeins. It is a torsion connection without curvature.
It is well known that any metric

$$G_{\alpha\beta} = E_{\alpha}{}^{A} \cdot \eta_{AB} \cdot E^{B}{}_{\beta} \tag{3.115}$$

can be locally expressed in terms of vielbeins E acting on a constant metric η. In the picture of Courant algebroids, there is an $O(d,d)$-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$, where d

is the dimension of TM and T^*M respectively. From (3.112), we have a decomposition in block matrices

$$\begin{pmatrix} \mathbb{1} & \mathcal{G}^T \\ 0 & \mathbb{1} \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1} & 0 \\ \mathcal{G} & \mathbb{1} \end{pmatrix} = \begin{pmatrix} 2g(x) & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{3.116}$$

All the indices in (3.115) run from $1, \cdots, 2d$. Note that

$$\mathcal{G}^T = (g+B)^T = g^T + B^T = g - B . \tag{3.117}$$

Due to the fact that a g-transformation is not an $O(d,d)$ transformation, it changes the pairing. Now we take the point of view to treat the matrix $\begin{pmatrix} \mathbb{1} & 0 \\ \mathcal{G} & \mathbb{1} \end{pmatrix}$ as a non-degenerate vielbein of $GL(2d, \mathbb{R})$. The Weitzenböck connection coefficient is defined as

$$W^{\beta}{}_{\alpha i} = E_A{}^{\beta} \cdot \partial_i E^A{}_{\alpha} \tag{3.118}$$

with vielbeins $E_A{}^{\beta}$ [74] with indices $\alpha, \beta, A = 1, \cdots, 2d$ and $i = 1, \cdots, d$. We compute our Weitzenböck connection with the vielbein from (3.116) to obtain

$$W_i = \begin{pmatrix} \mathbb{1} & 0 \\ -\mathcal{G} & \mathbb{1} \end{pmatrix} \cdot \partial_i \begin{pmatrix} \mathbb{1} & 0 \\ \mathcal{G} & \mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \partial_i \mathcal{G} & 0 \end{pmatrix} . \tag{3.119}$$

The only non-vanishing component is $\partial \mathcal{G}$, which appeared in (3.51). A useful formula for inverting block matrices, as we need for the inverse vielbein in (3.118), is

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{pmatrix} , \tag{3.120}$$

where block matrices $\mathbf{A}$ and $\mathbf{D}$ must be square matrices, and $\mathbf{A}$ and $\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ must be non-singular. Again recall that the curvature formula is $R^i_{\ jkl} = \partial_k \Gamma_{lj}{}^i - \partial_l \Gamma_{kj}{}^i + \Gamma_{ka}{}^i \Gamma_{lj}{}^a - \Gamma_{la}{}^i \Gamma_{kj}{}^a$. The curvature of connection (3.119) vanishes,

$$\mathsf{R}_{kl} = \begin{pmatrix} 0 & 0 \\ \partial_k \partial_l \mathcal{G} - \partial_l \partial_k \mathcal{G} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . \tag{3.121}$$

It is a Weitzenböck connection. The combination $W_{lj}{}^a W_{ka}{}^i$, $W_{kj}{}^a W_{la}{}^i$ always vanishes. The non-vanishing torsion of Weitzenböck connection is

$$\mathsf{T}^k{}_{ij} = W_{ij}{}^k - W_{ji}{}^k = \partial_i \mathcal{G}_j{}^k - \partial_j \mathcal{G}_i{}^k . \tag{3.122}$$

The contortion $\mathsf{K}^k{}_{ij}$, which is equal to $\frac{1}{2}(\mathsf{T}^k{}_{ij} + \mathsf{T}_i{}^k{}_j + \mathsf{T}_j{}^k{}_i)$, is given by

$$\mathsf{K}^k{}_{ij} = \frac{1}{2}\left(\partial_i \mathcal{G}_j{}^k - \partial_j \mathcal{G}_i{}^k + \partial_i \mathcal{G}^k{}_j - \partial_j \mathcal{G}^k{}_i + \partial_j \mathcal{G}^k{}_i - \partial_i \mathcal{G}^k{}_j\right) = \frac{1}{2}\left(\partial_i \mathcal{G}_j{}^k - \partial_j \mathcal{G}_i{}^k\right) , \tag{3.123}$$

where we have used the anti-symmetry of B, $B_j{}^k = g^{kk'} B_{jk'} = -B^k{}_j$. Hence, $\mathsf{K}^k{}_{ij} = \frac{1}{2}\mathsf{T}^k{}_{ij}$. The contortion is anti-symmetric in the lower two indices.

Let us compute the curvature of the contortion part,

$$\begin{aligned}
\mathcal{R}^{i}{}_{jkl} &= \frac{1}{2}\left(\partial_k(\partial_l \mathcal{G}_j{}^i - \partial_j \mathcal{G}_l{}^i) - \partial_l(\partial_k \mathcal{G}_j{}^i - \partial_j \mathcal{G}_k{}^i)\right) && (3.124)\\
&= \frac{1}{2}\partial_j(\partial_l \mathcal{G}_k{}^i - \partial_k \mathcal{G}_l{}^i) && (3.125)\\
&= \frac{1}{2}\partial_j(W_{lk}{}^i - W_{kl}{}^i) && (3.126)\\
&= \frac{1}{2}\partial_j \mathsf{T}^i{}_{lk}\ . && (3.127)
\end{aligned}$$

Consequently, its Ricci tensor is

$$\mathcal{R}_{jl} = \frac{1}{2}\partial_j \mathsf{T}^i{}_{li} \tag{3.128}$$

and Ricci scalar, contracted with the metric g is

$$\mathcal{R} = \frac{1}{2} g^{kl}\partial_k \mathsf{T}^j{}_{lj} = \frac{1}{2}\partial^l \mathsf{T}^j{}_{lj}\ . \tag{3.129}$$

Next, let us compute the curvature of a connection which is composed of a Levi-Civita connection plus the contortion above, and then the Ricci tensor,

$$R_{jl} = R^{\mathrm{LC}}_{jl} + \mathcal{R}_{jl} + \Gamma_{(lj)}{}^a \mathsf{K}^k{}_{ka} + \mathsf{K}^a{}_{lj}\Gamma_{(ka)}{}^k - \Gamma_{(kj)}{}^a \mathsf{K}^k{}_{la} - \mathsf{K}^a{}_{kj}\Gamma_{(la)}{}^k\ , \tag{3.130}$$

where R^{LC}_{jl} is the usual Ricci tensor of the Levi-Civita connection. We contract the Ricci tensor (3.130) with g to get the corresponding Ricci scalar,

$$\begin{aligned}
R &= R^{\mathrm{LC}} + \frac{1}{2}\partial^l \mathsf{T}^j{}_{lj} + \frac{1}{2} g^{jl}\Gamma_{(lj)}{}^a T^k{}_{ka} && (3.131)\\
&= R^{\mathrm{LC}} + \frac{1}{2}(\partial^l - (\Gamma^j{}_j{}^l)^{\mathrm{LC}})\mathsf{T}^h{}_{lh} && (3.132)\\
&= R^{\mathrm{LC}} + \frac{1}{2}\nabla^l_{\mathrm{LC}} \mathsf{T}^j{}_{lj}\ . && (3.133)
\end{aligned}$$

Therefore, for the Ricci scalar (3.133), we can have an action

$$S_{\mathsf{T}} = \frac{1}{16\pi G_N}\int d^d x\,\sqrt{-g}\left(R^{\mathrm{LC}} + \frac{1}{2}\nabla^l_{\mathrm{LC}}\mathsf{T}^j{}_{lj}\right)\ , \tag{3.134}$$

where G_N is the gravitational constant. This action is certainly equivalent to the Einstein-Hilbert action, as the second term is just a total derivative (recall identity (2.204)).
We know that the ordinary widely known teleparallel gravity action is given by [75]

$$S_{\text{teleparallel grav.}} = \frac{1}{16\pi G_N}\int d^4 x\,\sqrt{-g}\left(R^{\mathrm{LC}} + 2\nabla^l_{\mathrm{LC}}\mathsf{T}^j{}_{lj}\right)\ . \tag{3.135}$$

This teleparallel gravity is an alternative formulation of general relativity, with torsion but with vanishing curvature.

In summary: By deforming the pairing between the "vector" coordinates ($\{\chi_i, \chi_j\} = \langle \chi_i, \chi_j \rangle$) with the metric g, the consequent outcome in the graded Poisson algebra is rich. From connection (3.102) which is a sum of Levi-Civita connection plus contortion, there is nothing to forbid us from computing its curvature in the standard GR way. On the other hand, we can deduce a Weitzenböck connection after we learn purely from the graded algebra that it contains a vanishing curvature. In spite of having a strict correspondence between a standard Courant algebroid and its established graded manifold construction, we arrive at two separate pictures in the current formulation. One is a curvature-less connection on the graded manifold, another is a torsion connection derived through the Hamiltonian Θ.

3.4 Conclusion I

By respecting the Jacobi identity in the graded framework, we have derived a Weitzenböck connection and a Levi-Civita connection with contortion (Kalb-Ramond field strength). One may as well turn off the 2-form Kalb-Ramond B-field so that the connection is of Levi-Civita type only. Our interest in the B-field is due to its natural realization in generalized geometry and string theory. Property (3.43) that involves the anchor map in a Courant algebroid is easily satisfied. Having the condition $\{\Theta, \Theta\} = 0$, the quantization of the structure will be the next thing to investigate. We note that the curvature that is computed from having the connection from the derived backet, is certainly not an object that is contained in the mathematical framework itself. In the next section, we will attempt to incorporate the curvature directly in the framework.

3.5 Deformation II

In this section, we would like to discuss another possible deformation. In a coordinate basis, consider the following set of graded Poisson relations

$$\{\chi_i, \chi_j\} = 2g_{ij} \tag{3.136}$$

$$\{\theta^i, \theta^j\} = 0 \tag{3.137}$$

$$\{p_i, \chi_j\} = \Gamma_{ij}{}^{k}\, \chi_k \tag{3.138}$$

$$\{p_i, \theta^j\} = -\Gamma_{ik}{}^{j}\, \theta^k \tag{3.139}$$

$$\{p_i, p_j\} = R^{\ell}{}_{kij}\chi_\ell\theta^k - R_{a\ell ij}\theta^a\theta^\ell := R_{ij} \tag{3.140}$$

with a general connection coefficient and its curvature[11]. The connection $\Gamma_{ij}{}^{k}$ can in general be expressed as $\Gamma_{(ij)}{}^{k} + \Gamma_{[ij]}{}^{k}$. It is our guiding principle that the set of brackets are consistent with the Jacobi identity. Let us go through a few of these Jacobi relations. For one that involves p, θ, χ, we have

$$\{p_i, \{\theta^j, \chi_k\}\} = \{\{p_i, \theta^j\}, \chi_k\} + \{\theta^j, \{p_i, \chi_k\}\}\ . \tag{3.141}$$

[11]We note that two of the Poisson relations can generally be written in $\{p_i, \chi_j\} = \nabla_i \chi_j$ and $\{p_i, \theta^j\} = \nabla_i \theta^j$.

Since the LHS is zero, if one assumed a different connection $\widetilde{\Gamma}$ for $\{p, \theta\}$, different from the connection Γ in $\{p, \chi\}$, the Jacobi relation

$$0 = -\widetilde{\Gamma}_{ik}{}^{j} + \Gamma_{ik}{}^{j} \tag{3.142}$$

implies that $\widetilde{\Gamma} = \Gamma$. Hence there is only one type of connection Γ in the set of Poisson relations. From the following Jacobi identity,

$$\{p_i, \{\chi_j, \chi_k\}\} = \{\{p_i, \chi_j\}, \chi_k\} + \{\chi_j, \{p_i, \chi_k\}\} \ , \tag{3.143}$$

we obtain the metricity condition

$$\partial_i g_{jk} - \Gamma_{ij}{}^{l} g_{lk} - \Gamma_{ik}{}^{l} g_{jl} = 0 \ . \tag{3.144}$$

For coordinates p, p, θ, the Jacobi

$$\{p_i, \{p_j, \theta^k\}\} = \{\{p_i, p_j\}, \theta^k\} + \{p_j, \{p_i, \theta^k\}\} \tag{3.145}$$

tells us that on the LHS

$$\{p_i, -\Gamma_{jl}{}^{k}\theta^l\} = -\partial_i \Gamma_{jl}{}^{k}\theta^l + \Gamma_{jl}{}^{k}\Gamma_{ih}{}^{l}\theta^h \ , \tag{3.146}$$

while from the term $\{p_j, \{p_i, \theta^k\}\}$ on the RHS, it gives

$$\{p_j, -\Gamma_{il}{}^{k}\theta^l\} = -\partial_j \Gamma_{il}{}^{k}\theta^l + \Gamma_{il}{}^{k}\Gamma_{jh}{}^{l}\theta^h \ , \tag{3.147}$$

thus

$$\{\{p_i, p_j\}, \theta^k\} = R^k{}_{hji}\theta^h \ , \tag{3.148}$$

where we have a curvature. Similarly, we have curvature from Jacobi relation between p, p, χ,

$$\{\{p_i, p_j\}, \chi_k\} = R^l{}_{kij}\chi_l \ . \tag{3.149}$$

For a sign reminder, note that $\{\theta^i, p_j\} = \Gamma_{jl}{}^{i}\theta^l$ and $\{\chi_i, p_j\} = -\Gamma_{ji}{}^{l}\chi_l$. Extracting the precise expression for $\{p, p\}$ requires a bit of effort. (3.140) was found to reproduce correctly (3.148) and (3.149):

$$\begin{aligned} \{\{p_i, p_j\}, \theta^k\} &= \{R^l{}_{hij}\chi_l\theta^h, \theta^k\} - \{R_{alij}\theta^a\theta^l, \theta^k\} && (3.150) \\ &= -R^l{}_{hij}\theta^h\delta^k_l && (3.151) \\ &= R^k{}_{hji}\theta^h \ , && (3.152) \end{aligned}$$

$$\begin{aligned} \{\{p_i, p_j\}, \chi_k\} &= \{R^l{}_{hij}\chi_l\theta^h, \chi_k\} - \{R_{alij}\theta^a\theta^l, \chi_k\} && (3.153) \\ &= -2g_{lk}R^l{}_{hij}\theta^h + R^l{}_{kij}\chi_l + R_{klij}\theta^l - R_{akij}\theta^a && (3.154) \\ &= R^l{}_{kij}\chi_l \ , && (3.155) \end{aligned}$$

where we have used the anti-symmetry property of the curvature such as $R^l{}_{hij} = -R^l{}_{hji}$. Another interesting Jacobi relation is

$$\{p_i, \{p_j, p_k\}\} = \{\{p_i, p_j\}, p_k\} + \{p_j, \{p_i, p_k\}\} \ . \tag{3.156}$$

On the LHS, it is

$$\begin{aligned}\{p_i, R^l{}_{bjk}\chi_l\theta^b - R_{abjk}\theta^a\theta^b\} &= (\partial_i R^l{}_{bjk})\chi_l\theta^b + R^l{}_{bjk}\Gamma_{il}{}^c\chi_c\theta^b - R^l{}_{bjk}\Gamma_{ic}{}^b\chi_l\theta^c \qquad (3.157)\\ &\quad -(\partial_i R_{abjk})\theta^a\theta^b + R_{abjk}\Gamma_{ic}{}^a\theta^c\theta^b + R_{abjk}\Gamma_{ic}{}^b\theta^a\theta^c \\ &= \nabla_i R_{jk} \ , \qquad (3.158)\end{aligned}$$

on the RHS, we have

$$\begin{aligned}&-\{p_k, \{p_i, p_j\}\} + \{p_j, \{p_i, p_k\}\} \qquad (3.159)\\ =\ & -(\partial_k R^l{}_{bij})\chi_l\theta^b - R^l{}_{bij}\Gamma_{kl}{}^c\chi_c\theta^b + R^l{}_{bij}\Gamma_{kc}{}^b\chi_l\theta^c \qquad (3.160)\\ &+(\partial_k R_{abij})\theta^a\theta^b - R_{abij}\Gamma_{kc}{}^a\theta^c\theta^b - R_{abij}\Gamma_{kc}{}^b\theta^a\theta^c \\ &+(\partial_j R^l{}_{bik})\chi_l\theta^b + R^l{}_{bik}\Gamma_{jl}{}^c\chi_c\theta^b - R^l{}_{bik}\Gamma_{jc}{}^b\chi_l\theta^c \\ &-(\partial_j R_{abik})\theta^a\theta^b + R_{abik}\Gamma_{jc}{}^a\theta^c\theta^b + R_{abik}\Gamma_{jc}{}^b\theta^a\theta^c \\ =\ & -\nabla_k R_{ij} + \nabla_j R_{ik} \ , \qquad (3.161)\end{aligned}$$

in total from the Jacobi (3.156), we get the second Bianchi identity,

$$\nabla_i R_{jk} + \nabla_k R_{ij} + \nabla_j R_{ki} = 0 \ . \tag{3.162}$$

The graded Poisson relations are summarized in

$$\omega^{-1} = \begin{pmatrix} 0 & 0 & 0 & -\delta^i{}_j \\ 0 & 0 & \delta^i{}_j & \Gamma_{j\ell}{}^i\,\theta^\ell \\ 0 & \delta_i{}^j & 2g_{ij} & -\Gamma_{ji}{}^\ell\,\chi_\ell \\ \delta_i{}^j & -\Gamma_{i\ell}{}^j\,\theta^\ell & \Gamma_{ij}{}^\ell\,\chi_\ell & R_{ij} \end{pmatrix} \ . \tag{3.163}$$

Its inverse gives

$$\omega = \begin{pmatrix} C_{jk} & A_{jk} & \Gamma_{j\ell}{}^k\,\theta^\ell & \delta_j{}^k \\ B_{jk} & -2g_{jk} & \delta_j{}^k & 0 \\ \Gamma_{k\ell}{}^j\,\theta^\ell & \delta^j{}_k & 0 & 0 \\ -\delta^j{}_k & 0 & 0 & 0 \end{pmatrix} , \tag{3.164}$$

where

$$\begin{aligned} A_{jk} &= -2g_{ak}\Gamma_{j\ell}{}^a\,\theta^\ell - \Gamma_{jk}{}^\ell\,\chi_\ell \ , \qquad (3.165)\\ B_{jk} &= -2g_{ja}\Gamma_{k\ell}{}^a\,\theta^\ell - \Gamma_{kj}{}^\ell\,\chi_\ell \ , \qquad (3.166)\\ C_{jk} &= -2g_{ab}\Gamma_{j\ell}{}^a\Gamma_{k\ell'}{}^b\,\theta^\ell\theta^{\ell'} - \Gamma_{j\ell}{}^b\Gamma_{kb}{}^{\ell'}\,\theta^\ell\chi_{\ell'} - \Gamma_{jb}{}^\ell\Gamma_{k\ell'}{}^b\,\chi_\ell\theta^{\ell'} + R_{jk} \ . \qquad (3.167)\end{aligned}$$

The symplectic 2-form corresponding to (3.164) is

$$\begin{aligned}
\omega &= dx^j \wedge C_{jk} \wedge dx^k + dx^j \wedge A_{jk} \wedge d\theta^k && (3.168)\\
&\quad + dx^j \wedge \Gamma_{j\ell}{}^k\, \theta^\ell \wedge d\chi_k + 2\, dx^k \wedge dp_k && (3.169)\\
&\quad + d\theta^j \wedge B_{jk} \wedge dx^k - d\theta^j \wedge 2g_{jk} \wedge d\theta^k && (3.170)\\
&\quad + 2\, d\theta^k \wedge d\chi_k + d\chi_j \wedge \Gamma_{k\ell}{}^j\, \theta^\ell \wedge dx^k\ . && (3.171)
\end{aligned}$$

Remarks regarding the signs, when the order of the coordinates is exchanged, are for instances,

$$dx\,\theta = \theta\, dx\ , \qquad d\theta\,\theta = -\theta\, d\theta\ , \qquad (3.172)$$

$$dx\, d\theta = -d\theta\, dx\ , \qquad d\chi\, d\theta = d\theta\, d\chi\ , \qquad (3.173)$$

additionally,

$$d(\theta^i\theta^j) = d\theta^i\, \theta^j + \theta^i\, d\theta^j\ . \qquad (3.174)$$

More importantly, we have used the form of $dx^I\, \Omega_{IJ}\, dx^J$, doing the computations, with Ω_{IJ} positioned in the middle, contains the matrix elements of ω (3.164), and I, J denote all the type of coordinates there are. Therefore $d(dx^I\, \Omega_{IJ}\, dx^J) = -dx^I\, d\Omega_{IJ}\, dx^J$.

Using the standard curvature formula

$$R^\ell{}_{ajk} = \partial_j \Gamma_{ka}{}^\ell - \partial_k \Gamma_{ja}{}^\ell + \Gamma_{jb}{}^\ell \Gamma_{ka}{}^b - \Gamma_{kb}{}^\ell \Gamma_{ja}{}^b\ , \qquad (3.175)$$

we have

$$\begin{aligned}
& R_{aljk}\theta^a && (3.176)\\
= \;& -g_{ll'} R^{l'}{}_{ajk}\theta^a && (3.177)\\
= \;& -g_{ll'}(\partial_j \Gamma_{kk'}{}^{l'})\theta^{k'} + g_{ll'}(\partial_k \Gamma_{jk'}{}^{l'})\theta^{k'} - g_{ll'}\Gamma_{jk'}{}^{l'}\Gamma_{kh}{}^{k'}\theta^h + g_{ll'}\Gamma_{kk'}{}^{l'}\Gamma_{jh}{}^{k'}\theta^h && (3.178)\\
= \;& -\partial_j(g_{ll'}\Gamma_{kk'}{}^{l'})\theta^{k'} + \partial_k(g_{ll'}\Gamma_{jk'}{}^{l'})\theta^{k'} + g_{hl'}\Gamma_{jl}{}^h\Gamma_{kk'}{}^{l'}\theta^{k'} + g_{hl}\Gamma_{jl'}{}^h\Gamma_{kk'}{}^{l'}\theta^{k'} && (3.179)\\
& -g_{hl'}\Gamma_{kl}{}^h\Gamma_{jk'}{}^{l'}\theta^{k'} - g_{hl}\Gamma_{kl'}{}^h\Gamma_{jk'}{}^{l'}\theta^{k'} - g_{ll'}\Gamma_{jk'}{}^{l'}\Gamma_{kh}{}^{k'}\theta^h + g_{ll'}\Gamma_{kk'}{}^{l'}\Gamma_{jh}{}^{k'}\theta^h &&\\
= \;& -\partial_j(g_{ll'}\Gamma_{kk'}{}^{l'})\theta^{k'} + \partial_k(g_{ll'}\Gamma_{jk'}{}^{l'})\theta^{k'} + g_{hl'}\Gamma_{jl}{}^h\Gamma_{kk'}{}^{l'}\theta^{k'} - g_{hl'}\Gamma_{kl}{}^h\Gamma_{jk'}{}^{l'}\theta^{k'}\ , && (3.180)
\end{aligned}$$

where the metricity condition

$$\partial_j g_{ll'} = \Gamma_{jl}{}^h g_{hl'} + \Gamma_{jl'}{}^h g_{hl} \qquad (3.181)$$

has been used in the third equality. Hence,

$$R_{a\ell jk} = -\partial_j(g_{\ell b}\Gamma_{ka}{}^b) + \partial_k(g_{\ell b}\Gamma_{ja}{}^b) + g_{hb}\Gamma_{j\ell}{}^h\Gamma_{ka}{}^b - g_{hb}\Gamma_{k\ell}{}^h\Gamma_{ja}{}^b\ . \qquad (3.182)$$

This explicit expression is useful for later determination of some results. After careful computations utilizing (3.175) and (3.182), we find indeed $d\omega = 0$. The 2-degree 2-form symplectic structure is proven closed. We remark that the metricity condition can also be obtained from the vanishing component of $d\theta\wedge dx\wedge d\theta$ from $d\omega$, that is, from $d\theta^\ell\wedge dx^j\wedge d\theta^k$, we get $-2\partial_j g_{\ell k}+2g_{j'k}\Gamma_{j\ell}{}^{j'}+2g_{\ell j'}\Gamma_{jk}{}^{j'}$. Note that $d\omega=0$ is done by checking all the components to be vanishing. For instance, we get from the components

$$d\chi_\ell\wedge dx^j\wedge d\theta^k : -2\,\Gamma_{jk}{}^{\ell}+2\,\Gamma_{jk}{}^{\ell}\ , \tag{3.183}$$

which is trivially zero,

$$d\chi_\ell\wedge dx^j\wedge dx^k : \Gamma_{jm}{}^{b}\Gamma_{kb}{}^{\ell}\theta^m-\Gamma_{jb}{}^{\ell}\Gamma_{km}{}^{b}\theta^m+R^\ell{}_{mjk}\theta^m-2\,\partial_j\Gamma_{km}{}^{\ell}\theta^m\ , \tag{3.184}$$

which is zero, knowing the formula of curvature tensor. The same occurs with the other components.

In local coordinates, the function Θ can in general take the form

$$\Theta=\theta^ip_i+f_{ijk}(x)\,\theta^i\theta^j\theta^k+\dots \tag{3.185}$$

For the set of deformations proposed here, we define the 3-degree Hamiltonian as

$$\begin{aligned}\Theta &= \theta^ip_i+\Gamma_{[ij]}{}^{k}\theta^i\theta^j\chi_k-\Gamma_{[ijk]}\theta^i\theta^j\theta^k && (3.186)\\ &= \theta^ip_i+\frac{1}{2}T_{ij}{}^{k}\theta^i\theta^j\chi_k-\Gamma_{[ijk]}\theta^i\theta^j\theta^k, && (3.187)\end{aligned}$$

where $\Gamma_{[ij]}{}^{k}=\frac{1}{2}T_{ij}{}^{k}$ in coordinate basis with torsion T. After extensive computations, we find that

$$\{\Theta,\Theta\}=0\ . \tag{3.188}$$

To get to the above result (3.186) that leads to (3.188), we begin with a Hamiltonian in $\Theta=\theta^ip_i+C_{ij}{}^{k}\theta^i\theta^j\chi_k-C_{ijk}\theta^i\theta^j\theta^k$, to solve for two distinct (independent) functions $C_{ij}{}^{k}$ and C_{ijk}. Firstly we have from

$$\{\theta^ip_i,\theta^jp_j\}=-2\,\theta^i\theta^j\Gamma_{ij}{}^{m}p_m+\theta^i\theta^j(R^m{}_{kij}\chi_m\theta^k-R_{amij}\theta^a\theta^m)\ . \tag{3.189}$$

For other components in $\{\Theta,\Theta\}$, we get

$$\{C_{ijk}\theta^i\theta^j\theta^k,C_{i'j'k'}\theta^{i'}\theta^{j'}\theta^{k'}\}=0 \tag{3.190}$$

as $\{\theta^i,\theta^j\}=0$ and $\{f(x),g(x)\}=0$ for functions f,g,

$$\begin{aligned}2\,\{\theta^mp_m,C_{ij}{}^{k}\theta^i\theta^j\chi_k\} &= 2\,C_{ij}{}^{k}\theta^i\theta^jp_k+2\,\theta^m(\partial_mC_{ij}{}^{k})\theta^i\theta^j\chi_k && (3.191)\\ &\quad -2\,\theta^mC_{ij}{}^{k}\Gamma_{m\ell}{}^{i}\theta^\ell\theta^j\chi_k-2\,\theta^mC_{ij}{}^{k}\Gamma_{m\ell}{}^{j}\theta^i\theta^\ell\chi_k\\ &\quad +2\,\theta^mC_{ij}{}^{k}\Gamma_{mk}{}^{\ell}\theta^i\theta^j\chi_\ell\ ,\end{aligned}$$

$$\begin{aligned}2\,\{\theta^mp_m,-C_{ijk}\theta^i\theta^j\theta^k\} &= -2\,\theta^m(\partial_mC_{ijk})\theta^i\theta^j\theta^k+2\,\theta^mC_{ijk}\Gamma_{mh}{}^{i}\theta^h\theta^j\theta^k && (3.192)\\ &\quad -2\,\theta^iC_{ijk}\Gamma_{mh}{}^{j}\theta^m\theta^h\theta^k-2\,\theta^iC_{ijk}\Gamma_{mh}{}^{k}\theta^m\theta^j\theta^h\ ,\end{aligned}$$

$$
\begin{aligned}
2\{C_{ij}{}^{k}\theta^i\theta^j\chi_k, -C_{i'j'k'}\theta^{i'}\theta^{j'}\theta^{k'}\} &= -2\,C_{ij}{}^{k}C_{kj'k'}\theta^i\theta^j\theta^{j'}\theta^{k'} \\
&\quad +2\,C_{ij}{}^{k}C_{k'kj'}\theta^i\theta^j\theta^{k'}\theta^{j'} \\
&\quad -2\,C_{ij}{}^{k}C_{k'j'k}\theta^i\theta^j\theta^{k'}\theta^{j'}\ ,
\end{aligned}
\tag{3.193}
$$

$$
\begin{aligned}
\{C_{ij}{}^{k}\theta^i\theta^j\chi_k, C_{i'j'}{}^{k'}\theta^{i'}\theta^{j'}\chi_{k'}\} &= C_{ij}{}^{k}C_{kj'}{}^{k'}\theta^i\theta^j\theta^{j'}\chi_{k'} - C_{ij}{}^{k}C_{j'k}{}^{k'}\theta^i\theta^j\theta^{j'}\chi_{k'} \\
&\quad +C_{ij}{}^{k}C_{k'j'}{}^{i}\theta^{k'}\theta^{j'}\theta^j\chi_k - C_{ij}{}^{k}C_{k'j'}{}^{j}\theta^{k'}\theta^{j'}\theta^i\chi_k \\
&\quad +2g_{ki'}C_{ij}{}^{k}C_{k'j'}{}^{i'}\theta^{k'}\theta^{j'}\theta^i\theta^j\chi_k\ .
\end{aligned}
\tag{3.194}
$$

After some manipulations, we obtain $C_{ij}{}^{k} = \Gamma_{[ij]}{}^{k}$ and $C_{ijk} = \Gamma_{[ijk]}$, in order to have $\{\Theta, \Theta\} = 0$.

The Hamiltonian (3.186) in this case contains additional terms that involve the connection coefficient with certain symmetries. The fully contracted derived bracket is

$$
\begin{aligned}
\{Z, \{\{X, \Theta\}, Y\}\} &= 2g(Z, [X, Y]_{\text{Lie}}) && (3.195)\\
&\quad +2g(\nabla^{\text{LC}}_Z X, Y) && (3.196)\\
&\quad +Z^h X^b Y^c (g_{ac}T_{bh}{}^{a} + 2g_{ba}T_{hc}{}^{a}) && (3.197)\\
&\quad +6Z^h X^b Y^c \Gamma_{[bhc]}\ , && (3.198)
\end{aligned}
$$

where $\Gamma_{[abc]} = g_{c'[c}\Gamma_{ab]}{}^{c'}$.

3.6 Discussion II

In this deformation, we have similarly begun with the same metric (3.136) in block matrices

$$
\begin{pmatrix} 2g(x) & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.199}
$$

Recall again that the $O(d, d)$ metric in generalized geometry is $\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. Instead of a Weitzenböck connection that appeared in (3.51) previously, we have started with a general connection coefficient in (3.138) - (3.140) and consequently its curvature in the algebra explicitly. As usual, the general connection is composed of a Levi-Civita connection and contortion. From (3.198), the resulting Dorfman bracket is found to be deformed with a Levi-Civita connection, torsions and a totally anti-symmetric connection coefficient. This totally anti-symmetric connection coefficient which first shows up in the Hamiltonian (3.186) is indeed an additional piece of information to the graded Poisson algebra. The corresponding property (3.43) is the same as in Deformation I.

We remark that in this deformation, we have not used any raising metric g^{-1} throughout the computations. A similar outlook as in the Sec. (3.2) (Deformation I) is to study the quantization of $\{\Theta, \Theta\} = 0$. This may serve as an alternative window to the quantization of gravity. Apart from that, one can as well attempt to compute the curvature using the connection coefficients and torsion in (3.198), thus obtaining the action for such deformation.

4 Galileons

4.1 Introduction

A special symmetry, called Galileon symmetry

$$\pi \to \pi + b_\mu x^\mu + c \tag{4.1}$$

is a transformation of a scalar field $\pi(x)$ which contains a linear piece with a vectorial parameter b_μ and a constant piece c in its shift. Both b_μ and c are constants. The symmetry is a generalization of the Galilean symmetry, $\dot{x} \to \dot{x} + v$ in non-relativistic mechanics, in which "dot" denotes time derivative and v is the velocity. The field π is hence coined as Galileon. The gradient of (4.1) gives $\partial_\mu \pi \to \partial_\mu \pi + b_\mu$.

In the interest of gravity theories, besides the requirement of general covariance, having extra special symmetry in the theory can strongly constrain the theory and reduce the parameter space [18]. The Galileon symmetry (4.1) was imposed on a Minkowski background in [22]. The possible Lagrangians which are invariant under (4.1) are

$$\begin{aligned}
\mathcal{L}_1 &= \pi \;, && (4.2)\\
\mathcal{L}_2 &= -\frac{1}{2}\,\partial_\mu\pi\,\partial^\mu\pi \;, && (4.3)\\
\mathcal{L}_3 &= -\frac{1}{2}\,\partial^2\pi\,\partial_\mu\pi\,\partial^\mu\pi \;, && (4.4)\\
\mathcal{L}_4 &= -\frac{1}{2}\left((\partial^2\pi)^2 - (\partial_\mu\partial_\nu\pi)\,(\partial^\mu\partial^\nu\pi)\right)\partial_\mu\pi\,\partial^\mu\pi \;, && (4.5)\\
\mathcal{L}_5 &= -\frac{1}{2}\left((\partial^2\pi)^3 - 3(\partial^2\pi)\,(\partial_\mu\partial_\nu\pi)\,(\partial^\mu\partial^\nu\pi) + 2(\partial_\mu\partial^\nu\pi)\,(\partial_\nu\partial^\rho\pi)\,(\partial_\rho\partial^\mu\pi)\right)\times\\
&\quad \times\,\partial_\mu\pi\,\partial^\mu\pi \;, && (4.6)
\end{aligned}$$

in 4-dimensional Minkowski spacetime [22], where $\partial^2 = \partial_\mu\partial^\mu$. Derivative self-interacting non-linear terms are introduced in the Lagrangian. The corresponding equations of motion $\sum_i \mathcal{E}_i = 0$ are

$$\begin{aligned}
\mathcal{E}_1 &= 1 \;, && (4.7)\\
\mathcal{E}_2 &= \partial^2\pi \;, && (4.8)\\
\mathcal{E}_3 &= (\partial^2\pi)^2 - (\partial_\mu\partial_\nu\pi)\,(\partial^\mu\partial^\nu\pi) \;, && (4.9)\\
\mathcal{E}_4 &= (\partial^2\pi)^3 - 3(\partial^2\pi)(\partial_\mu\partial_\nu\pi)\,(\partial^\mu\partial^\nu\pi) + 2(\partial_\mu\partial^\nu\pi)\,(\partial_\nu\partial^\rho\pi)\,(\partial_\rho\partial^\mu\pi) \;, && (4.10)\\
\mathcal{E}_5 &= (\partial^2\pi)^4 - 6(\partial^2\pi)^2(\partial_\mu\partial_\nu\pi)\,(\partial^\mu\partial^\nu\pi) + 8(\partial^2\pi)(\partial_\mu\partial^\nu\pi)\,(\partial_\nu\partial^\rho\pi)\,(\partial_\rho\partial^\mu\pi)\\
&\quad +3(\partial_\mu\partial_\nu\pi)\,(\partial^\mu\partial^\nu\pi)(\partial_\rho\partial_\sigma\pi)\,(\partial^\rho\partial^\sigma\pi) - 6(\partial_\mu\partial_\nu\pi)\,(\partial^\nu\partial^\rho\pi)(\partial_\rho\partial_\sigma\pi)\,(\partial^\sigma\partial^\mu\pi)\;. && (4.11)
\end{aligned}$$

In general, the variation of the action gives [76]

$$\frac{\delta}{\delta\pi}\int d^4x\,\mathcal{L}_i(\pi,\partial\pi,\partial\partial\pi) = \mathcal{E}_i(\partial\partial\pi) \;, \tag{4.12}$$

where

$$\mathcal{E}_i(\partial\partial\pi) = (i-1)!\,\delta^{\mu_1}_{[\nu_1}\cdots\delta^{\mu_{i-1}}_{\nu_{i-1}]}\,(\partial_{\mu_1}\partial^{\nu_1}\pi)\,\cdots\,(\partial_{\mu_{i-1}}\partial^{\nu_{i-1}}\pi)\ . \tag{4.13}$$

The equations of motion of π are also invariant under (4.1), and are at most of second order in derivatives.

The promotion of the partial derivatives to covariant derivatives, thus generalizing the flat Lagrangian to a Lagrangian in curved spacetime was done in [77]. For Lagrangians $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$, the replacement of partial derivatives by covariant derivatives still gives equations of motion with terms of not higher than second derivatives, both in the fields and metric. While for Lagrangians $\mathcal{L}_4$ and $\mathcal{L}_5$, non-minimal couplings between the field π and Riemann tensor are introduced to play the role of keeping the equations of motion free from higher derivatives. In curved spacetime, they become

$$\mathcal{L}_4 = (\nabla_\lambda\pi\,\nabla^\lambda\pi)\left(2(\Box\pi)^2 - 2(\nabla_\nu\nabla_\mu\pi)(\nabla^\nu\nabla^\mu\pi) - \frac{1}{2}R\,(\nabla_\rho\pi\,\nabla^\rho\pi)\right)\ , \tag{4.14}$$

$$\begin{aligned}\mathcal{L}_5 = {} & (\nabla_\lambda\pi\,\nabla^\lambda\pi)[(\Box\pi)^3 - 3\,\Box\pi(\nabla_\nu\nabla_\mu\pi)(\nabla^\nu\nabla^\mu\pi) \\ & +2(\nabla_\mu\nabla^\nu\pi)\,(\nabla_\nu\nabla^\rho\pi)\,(\nabla_\rho\nabla^\mu\pi) - 6(\nabla_\mu\pi)(\nabla^\mu\nabla^\nu\pi)(\nabla^\rho\pi)\,G_{\nu\rho}]\ ,\end{aligned} \tag{4.15}$$

where $G_{\nu\rho}$ is the Einstein tensor, which is equal to $R_{\nu\rho} - \frac{1}{2}Rg_{\nu\rho}$. $\Box = \nabla_\lambda\nabla^\lambda$, where the covariant derivative ∇_λ is associated with the symmetric metric g. When g is replaced with the Minkowski metric, they reduce back to the Lagrangians in flat spacetime. The Galileon symmetry is broken at the level of the equations of motion of the covariant version of the Lagrangians. This is due to the appearence of first derivatives in the field π, as can be inspected from the Lagrangians, which arises from the coupling term with curvature. For explicit expressions of the equations of motion of $\mathcal{L}_4$ and $\mathcal{L}_5$, see [77]. The more crucial demand to preserve is that the equations of motion contain only up to second order in derivatives. This is based on the Ostrogradsky's theorem which states that higher-derivative theories usually contain ghost degrees of freedom [19].

The extension of the theory to arbitrary dimensions d was worked out in [23]. The action is

$$S = \int d^dx\,\sqrt{-g}\,\sum_{p=0}^{p_{\max}}\mathcal{C}_{(n+1,p)}\mathcal{L}_{(n+1,p)}\ , \tag{4.16}$$

where $p_{\max}$ is the maximal integer part of $\frac{n-1}{2}$ with $n \le d$. The coefficient is

$$\mathcal{C}_{(n+1,p)} = \left(-\frac{1}{8}\right)^p\frac{(n-1)!}{(n-1-2p)!(p!)^2} \tag{4.17}$$

and the Lagrangian form for $(n+1)$ number of occurences of field π is

$$\begin{aligned}\mathcal{L}_{(n+1,p)} = {} & -\frac{1}{(d-n)!}\,\varepsilon^{\mu_1\mu_3\cdots\mu_{2n-1}\nu_1\cdots\nu_{d-n}}\varepsilon^{\mu_2\mu_4\cdots\mu_{2n}}{}_{\nu_1\cdots\nu_{d-n}}\pi_{;\mu_1}\pi_{;\mu_2}(\pi^{;\lambda}\pi_{;\lambda})^p\,\times \\ & \times\prod_{i=1}^{p}R_{\mu_{4i-1}\mu_{4i+1}\mu_{4i}\mu_{4i+2}}\prod_{j=0}^{n-2-2p}\pi_{;\mu_{2n-1-2j}\mu_{2n-2j}}\ ,\end{aligned} \tag{4.18}$$

where $\varepsilon^{\mu_1\mu_3\cdots}$ is the Levi-Civita tensor. When p is zero, it just means that the Riemann tensor part is removed from the Lagrangian.

4.2 Preliminaries on the Graded Formalism and Mixed-Symmetry Tensors

The Galileon actions known are intricately decorated with indices, structured by two Levi-Civita tensors, carefully contracting the fields. Scalar fields π and p-forms have been well investigated in the literature [23], [78]. Here, we are going to reformulate these known Galileon actions using the two set of graded variables, which we have come across in the previous topic of graded geometry. Furthermore, we are going to play the same trick in considering mixed-symmetry tensor fields, which has not appeared in literature. *Our work* [79] *is expected to appear on arXiv in a few weeks time. Certain materials here are quoted from it.* The two independent graded variables we are using are the degree 1, self-anti-commuting χ and θ,

$$\theta^i\theta^j = -\theta^j\theta^i \ , \quad \chi^i\chi^j = -\chi^j\chi^i \ , \quad \theta^i\chi^j = \chi^j\theta^i \ . \tag{4.19}$$

They can be simply called Grassmann variables. We have chosen the χ and θ to be mutually commuting, as seen from the last expression. If they were anti-commuting with respect to each other, it only amounts to an overall sign difference in the flat spacetime. When we covariantize the flat-spacetime actions with the Levi-Civita connection, this convention is still consistent. The graded manifold in this case can be denoted as $(T[1]\bigoplus T[1])M$.

Using these graded variables, we define the degree 1 exterior super derivatives

$$\mathbf{d} = \theta^i\partial_i \qquad \text{and} \qquad \widetilde{\mathbf{d}} = \chi^i\partial_i \ . \tag{4.20}$$

As χ^i commutes with θ^i, $\mathbf{d}\widetilde{\mathbf{d}} = \widetilde{\mathbf{d}}\mathbf{d}$. The two derivatives are nilpotent,

$$\mathbf{d}^2 = \theta^i\theta^j\partial_i\partial_j = 0 \qquad \text{and} \qquad \widetilde{\mathbf{d}}^2 = \chi^i\chi^j\partial_i\partial_j = 0 \ . \tag{4.21}$$

Besides derivatives, we need to define the integration, in order to build the actions. Let us recall the basic formula for the Berezin integral,

$$\int \mathrm{d}\theta\,\theta = 1 \ , \tag{4.22}$$

in the flat spacetime, which implies

$$\int \mathrm{d}^D\theta\ \theta^{j_p}\theta^{j_{p-1}}\ldots\theta^{j_1} = \varepsilon^{j_pj_{p-1}\cdots j_1} \ , \tag{4.23}$$

in the curved spacetime, for Levi-Civita tensor $\varepsilon^{j_pj_{p-1}\cdots j_1}$.

For the curved-spacetime version of the theory, similarly, the covariant exterior derivatives in terms of the graded variables are

$$\boldsymbol{\nabla} = \theta^i \nabla_i \qquad \text{and} \qquad \widetilde{\boldsymbol{\nabla}} = \chi^i \nabla_i \ . \tag{4.24}$$

The covariant derivatives act on the variables (treated as bases) as

$$\nabla_i \theta^k = -\theta^j \Gamma^k_{ij} \qquad \text{and} \qquad \nabla_i \chi^k = -\chi^j \Gamma^k_{ij} \ . \tag{4.25}$$

Hence, we obtain

$$\boldsymbol{\nabla}\theta^k = -\theta^i\theta^j\Gamma^k_{ij} = -\frac{1}{2}\theta^i\theta^j\mathbb{T}^k_{ij} = 0 \tag{4.26}$$
$$\boldsymbol{\nabla}\chi^k = -\theta^i\chi^j\Gamma^k_{ij} =: -\boldsymbol{\Gamma}^k \tag{4.27}$$
$$\widetilde{\boldsymbol{\nabla}}\theta^k = -\chi^i\theta^j\Gamma^k_{ij} =: -\boldsymbol{\Gamma}^k \tag{4.28}$$
$$\widetilde{\boldsymbol{\nabla}}\chi^k = -\frac{1}{2}\chi^i\chi^j\mathbb{T}^k_{ij} = 0 \ . \tag{4.29}$$

We shall assume a Levi-Civita connection, with vanishing torsion $\mathbb{T}^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} = 0$. We have only one type of connection, where each of the two lower indices in the component of $\boldsymbol{\Gamma}^k$ contracts separately with χ and θ variables, as can be seen in (4.27) and (4.28).
Obviously, the square of covariant derivatives is not zero. As it involves the commutator of the covariant derivatives, it gives us curvature,

$$\boldsymbol{\nabla}^2 = \frac{1}{2}\theta^i\theta^j[\nabla_i, \nabla_j] \ , \qquad \widetilde{\boldsymbol{\nabla}}^2 = \frac{1}{2}\chi^i\chi^j[\nabla_i, \nabla_j] \ , \tag{4.30}$$

which also means

$$\boldsymbol{\nabla}^2\chi^j = -\boldsymbol{\nabla}\boldsymbol{\Gamma}^j = -\frac{1}{2}\chi^i\theta^k\theta^\ell R^j{}_{ik\ell} \ , \tag{4.31}$$
$$\widetilde{\boldsymbol{\nabla}}^2\theta^j = -\widetilde{\boldsymbol{\nabla}}\widetilde{\boldsymbol{\Gamma}}^j = -\frac{1}{2}\theta^i\chi^k\chi^\ell R^j{}_{ik\ell} \ . \tag{4.32}$$

Recalling the basic formula from Riemannian geometry on a vector field V, we have

$$[\nabla_i, \nabla_j]V^k = R^k{}_{\ell ij}V^\ell \ . \tag{4.33}$$

Acting on the graded variables, the commutator between $\boldsymbol{\nabla}, \widetilde{\boldsymbol{\nabla}}$ gives

$$[\boldsymbol{\nabla}, \widetilde{\boldsymbol{\nabla}}]\chi^j = \theta^i\chi^k\chi^\ell R^j{}_{\ell ki} \ , \qquad [\boldsymbol{\nabla}, \widetilde{\boldsymbol{\nabla}}]\chi_j = \theta^i\chi^k\chi_\ell R^\ell{}_{jik} \ , \tag{4.34}$$

$$[\boldsymbol{\nabla}, \widetilde{\boldsymbol{\nabla}}]\theta^j = \chi^i\theta^k\theta^\ell R^j{}_{\ell ik} \ , \qquad [\boldsymbol{\nabla}, \widetilde{\boldsymbol{\nabla}}]\theta_j = \chi^i\theta^k\theta_\ell R^\ell{}_{jki} \ . \tag{4.35}$$

Let us conveniently define

$$\boldsymbol{Riem} = R_{ijkl}\theta^i\theta^j\chi^k\chi^l \ , \tag{4.36}$$

with the components of the Riemann tensor. The symmetries of the curvature tensor, that is, the anti-symmetrization of the first two and the last two indices are naturally incorporated. We like to mention that in this work, the Bianchi identities are heavily used and have served a very important role in the simplification, especially the second Bianchi identity, which can be rephrased as

$$\boldsymbol{\nabla Riem} = 0 \ , \qquad \widetilde{\boldsymbol{\nabla}}\boldsymbol{Riem} = 0 \ , \tag{4.37}$$

besides the first Bianchi identity, $R^m{}_{ijk}\theta^i\theta^j\theta^k = 0$, $R^m{}_{ijk}\chi^i\chi^j\chi^k = 0$.
We further note that

$$\chi^i \boldsymbol{\nabla}^2 \chi_i = -\frac{1}{2}\boldsymbol{Riem} = \theta^i \widetilde{\boldsymbol{\nabla}}^2 \theta_i \ , \tag{4.38}$$

where $\chi_i = g_{ii'}\chi^{i'}$, $\theta_i = g_{ii'}\theta^{i'}$.

Next, let us present the necessary graded definitions needed for the mixed-symmetry fields. As it is known, symmetries of a tensor can be visualized in Young tableau diagrams. A row diagram, for instance for a rank-two tensor $\boxed{i \,|\, j}$ represents a totally symmetric tensor $T_{(ij)}$, while a totally anti-symmetric tensor $T_{[ij]}$ is represented by a column $\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$.
Mixed-symmetry tensor will be represented by a combination of these two types of diagram, that is, in the one diagram, it merges both row and column. For some examples of Young tableaux, see [80], where the interest is in actions for general tensor gauge fields.

A (p,q) mixed-symmetry tensor can be represented in a local coordinate basis (x^i) on spacetime M as

$$T^{(p,q)} = \frac{1}{p!q!} T_{[i_1 \dots i_p][j_1 \dots j_q]} \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_p} \otimes \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_q} \ , \tag{4.39}$$

where p, q are positive integers. Equivalently, it is

$$T^{(p,q)} = \frac{1}{p!q!} T_{i_1 \dots i_p j_1 \dots j_q} \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p} \otimes \mathrm{d}x^{j_1} \wedge \dots \wedge \mathrm{d}x^{j_q} \ . \tag{4.40}$$

The tensor is separately anti-symmetric in the i indices and j indices. Therefore, components of such tensors satisfy

$$T_{[i_1 \dots i_p j_1] \dots j_q} = 0 \ , \tag{4.41}$$

$$T_{[i_1 \dots i_p][j_1 \dots j_q]} = T_{[j_1 \dots j_q][i_1 \dots i_p]} \ , \text{ for } p = q \ . \tag{4.42}$$

There are no additional anti-symmetries in the tensor, as stated by (4.41). For $p = q$, there is no distinction between $T^{(p,q)}$ and $T^{(q,p)}$, as meant by (4.42). In our case, $T^{(p,q)}$ can be represented by a Young tableaux consisting of a column of length p and a column of length q [81]. In graded variables, the mixed-symmetry tensors[12] are expressed as

$$\boldsymbol{T}^{(p,q)} = \frac{1}{p!q!} T_{i_1 \dots i_p j_1 \dots j_q} \theta^{i_1} \dots \theta^{i_p} \chi^{j_1} \dots \chi^{j_q} \ , \tag{4.43}$$

$$\widetilde{\boldsymbol{T}}^{(p,q)} = \frac{1}{p!q!} T_{i_1 \dots i_p j_1 \dots j_q} \chi^{i_1} \dots \chi^{i_p} \theta^{j_1} \dots \theta^{j_q} \ . \tag{4.44}$$

[12]Sometimes denoted in the work as mixed-tensors or mixed-fields in short.

Apart from the fact that mixed-symmetry tensors include scalars and differential forms as special cases (as we will see later in Sec. (4.6)), let us also mention that in string theory, mixed-symmetry fields were found to be coupled to non-standard branes [82].

Let us now explain the operation of the previously defined structures on the tensor. The exterior derivatives are maps of

$$\mathrm{d}:\Omega^{(p,q)}(M)\to\Omega^{(p+1,q)}(M) \qquad \text{and} \qquad \widetilde{\mathrm{d}}:\Omega^{(p,q)}(M)\to\Omega^{(p,q+1)}(M)\ , \tag{4.45}$$

acting accordingly as

$$\begin{aligned}\mathrm{d}T &= \frac{1}{p!q!}\partial_{i_0}T_{i_1\ldots i_p j_1\ldots j_q}\mathrm{d}x^{i_0}\wedge\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}\otimes\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_q}\ , &(4.46)\\ \widetilde{\mathrm{d}}T &= \frac{1}{p!q!}\partial_{j_0}T_{i_1\ldots i_p j_1\ldots j_q}\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}\otimes\mathrm{d}x^{j_0}\wedge\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_q}\ . &(4.47)\end{aligned}$$

It is straightforward to see that

$$\mathbf{d}\boldsymbol{T}^{(1,1)}=\theta^k(\partial_k T_{ij})\theta^i\chi^j \tag{4.48}$$

for a $(1,1)$ mixed tensor $\boldsymbol{T}^{(1,1)}=T_{ij}\theta^i\chi^j$ as an example. For the covariant version, it is

$$\boldsymbol{\nabla}\boldsymbol{T}^{(1,1)}=(\mathbf{d}T_{ij})\theta^i\chi^j-T_{ij}\Gamma^j_{k\ell}\,\theta^k\theta^i\chi^\ell=(\partial_k T_{i\ell}-T_{ij}\Gamma^j_{k\ell})\theta^k\theta^i\chi^\ell=\mathbf{d}\boldsymbol{T}^{(1,1)}-\boldsymbol{\Gamma}^j T_{ij}\theta^i\ . \tag{4.49}$$

For a general mixed-symmetry tensor $\widetilde{\boldsymbol{T}}^{(p,q)}$, it is

$$\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}=\mathbf{d}\widetilde{\boldsymbol{T}}-p\,\boldsymbol{\Gamma}^\ell T_{\ell i_2\ldots i_p j_1\ldots j_q}\chi^{i_2}\ldots\chi^{i_p}\theta^{j_1}\ldots\theta^{j_q}\ . \tag{4.50}$$

Finally, we will need the symmetric metric for the flat and curved spacetime, which we define as

$$\boldsymbol{\eta}=\eta_{ij}\theta^i\chi^j\ , \qquad \boldsymbol{g}=g_{ij}\theta^i\chi^j\ . \tag{4.51}$$

4.3 Scalar Galileons and Differential Forms in Flat Spacetime

The action for scalar Galileons in $D\geq n$ dimensions with $n+1$ occurences of the scalar field π may be written as

$$\begin{aligned}S_{n+1}[\pi] &= \frac{1}{(D-n)!}\int\mathrm{d}^D x\int d^D\theta\,d^D\chi\,\boldsymbol{\eta}^{D-n}\,\mathbf{d}\pi\,\widetilde{\mathbf{d}}\pi\,(\mathbf{d}\widetilde{\mathbf{d}}\pi)^{n-1} &(4.52)\\ &= -\frac{1}{(D-n)!}\int\mathrm{d}^D x\int d^D\theta\,d^D\chi\,\boldsymbol{\eta}^{D-n}\,\pi\,(\mathbf{d}\widetilde{\mathbf{d}}\pi)^n\ , &(4.53)\end{aligned}$$

where $\mathrm{d}^D\theta = \mathrm{d}\theta^1 \dots \mathrm{d}\theta^D$ and similarly for $\mathrm{d}^D\chi$. Using the definitions from the last section, the action (4.53) becomes

$$\begin{aligned} S_{n+1}[\pi] &= \frac{1}{(D-n)!}\int \mathrm{d}^D x \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, (\eta_{kl}\theta^k\chi^l)^{D-n}(\theta^{i_1}\partial_{i_1}\pi)(\chi^{j_1}\partial_{j_1}\pi)(\theta^i\chi^j\partial_i\partial_j\pi)^{n-1} \\ &= \frac{1}{(D-n)!}\int \mathrm{d}^D x \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \eta_{k_{n+1}l_{n+1}}\dots\eta_{k_D l_D}\theta^{k_{n+1}}\dots\theta^{k_D}\chi^{l_{n+1}}\dots\chi^{l_D}\, \times \\ &\qquad \times\, \theta^{i_1}\dots\theta^{i_n}\chi^{j_1}\dots\chi^{j_n}\pi_{i_1}\pi_{j_1}\pi_{i_2j_2}\dots\pi_{i_nj_n} \\ &= \frac{1}{(D-n)!}\int \mathrm{d}^D x \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \theta^1\dots\theta^D\chi^1\dots\chi^D\varepsilon^{i_1\dots i_n k_{n+1}\dots k_D}\varepsilon^{j_1\dots j_n l_{n+1}\dots l_D}\, \times \\ &\qquad \times\, \eta_{k_{n+1}l_{n+1}}\dots\eta_{k_D l_D}\pi_{i_1}\pi_{j_1}\pi_{i_2j_2}\dots\pi_{i_nj_n} \\ &= \frac{1}{(D-n)!}\int \mathrm{d}^D x\, \varepsilon^{i_1\dots i_n k_{n+1}\dots k_D}\varepsilon^{j_1\dots j_n}{}_{k_{n+1}\dots k_D}\pi_{i_1}\pi_{j_1}\pi_{i_2j_2}\dots\pi_{i_nj_n}\ , \end{aligned} \tag{4.54}$$

in the notation $\pi_i = \partial_i\pi$ and $\pi_{ij} = \partial_j\partial_i\pi$ [83]. We have used the following properties,

$$\theta^{i_1}\dots\theta^{i_D} = \varepsilon^{i_1\dots i_D}\theta^1\dots\theta^D\ , \tag{4.55}$$

$$\int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \theta^1\dots\theta^D\chi^1\dots\chi^D = 1\ . \tag{4.56}$$

(Recall that $\varepsilon^{i_1\dots i_D} = \frac{1}{\sqrt{|g|}}\delta^{i_1\dots i_D}_{1\dots D}$ and $\varepsilon_{i_1\dots i_D} = \sqrt{|g|}\delta^{1\dots D}_{i_1\dots i_D}$ are the Levi-Civita tensors in the case of curved spacetime.) Due to $\mathbf{d}\mathbf{d}\pi = \widetilde{\mathbf{d}}\widetilde{\mathbf{d}}\pi = 0$, the field equations from this scalar action are

$$E_{n+1} = \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \boldsymbol{\eta}^{D-n}(\mathbf{d}\widetilde{\mathbf{d}}\pi)^n = 0\ , \tag{4.57}$$

which are obviously second order in derivatives.

A generalization of Galileons for an arbitrary degree of differential forms is easily manifested in the index-free formalism using graded variables. For an abelian 1-form A, which we will see it appears in a physical theory, let us define

$$\boldsymbol{A} = A_i\theta^i \qquad \text{and} \qquad \widetilde{\boldsymbol{A}} = A_i\chi^i\ , \tag{4.58}$$

with the field strength $F = \mathrm{d}A$, which then gives rise to

$$\boldsymbol{F} = \mathbf{d}\boldsymbol{A} = \frac{1}{2}F_{ij}\theta^i\theta^j \qquad \text{and} \qquad \widetilde{\boldsymbol{F}} = \widetilde{\mathbf{d}}\widetilde{\boldsymbol{A}} = \frac{1}{2}F_{ij}\chi^i\chi^j\ . \tag{4.59}$$

They satisfy the Bianchi identities

$$\mathbf{d}\boldsymbol{F} = 0 \qquad \text{and} \qquad \widetilde{\mathbf{d}}\widetilde{\boldsymbol{F}} = 0\ . \tag{4.60}$$

The corresponding Galileon action is given by

$$\begin{aligned} S_{2n}[A] &= \frac{1}{(D-3n+1)!}\int \mathrm{d}^D x \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \boldsymbol{\eta}^{D-3n+1}\, \mathbf{d}\boldsymbol{A}\, \widetilde{\mathbf{d}}\widetilde{\boldsymbol{A}}\, (\mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{A}})^{n-1}\, (\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{A})^{n-1} \\ &= \frac{1}{(D-3n+1)!}\int \mathrm{d}^D x \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \boldsymbol{\eta}^{D-3n+1}\, \boldsymbol{F}\, \widetilde{\boldsymbol{F}}\, (\mathbf{d}\widetilde{\boldsymbol{F}})^{n-1}\, (\widetilde{\mathbf{d}}\boldsymbol{F})^{n-1}\ . \end{aligned} \tag{4.61}$$

Following the same steps as in (4.54), the action expressed with two Levi-Civita tensors (here, Levi-Civita symbols as we are in flat spacetime) is recovered. The 1-form Galileon action in the flat spacetime is a total derivative, for $n > 1$ [78], since it can be written in

$$S_{2n}[A] = \frac{1}{(D-3n+1)!}\int \mathrm{d}^D x \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \mathbf{d}\left(\frac{1}{2}\boldsymbol{\eta}^{D-3n+1}\,\boldsymbol{F}\,\widetilde{\boldsymbol{F}}^2\,(\mathbf{d}\widetilde{\boldsymbol{F}})^{n-2}\,(\widetilde{\mathbf{d}}\boldsymbol{F})^{n-1}\right)\ , \tag{4.62}$$

for $2n \geq 4$. In the language of graded formalism, this is simply because $\boldsymbol{A}$ and $\widetilde{\boldsymbol{A}}$ are degree 1 fields (4.58) and thus

$$(\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{A})^2 = (\mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{A}})^2 = 0\ . \tag{4.63}$$

This is easily proven as follows,

$$\begin{aligned}
(\mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{A}})^2 &= (\theta^i\chi^j\chi^k\partial_i\partial_jA_k)(\theta^l\chi^m\chi^n\partial_l\partial_mA_n) \\
&= \theta^i\theta^l\chi^j\chi^k\chi^m\chi^n\partial_i\partial_jA_k\partial_l\partial_mA_n \\
&= \frac{1}{2}\theta^i\theta^l\chi^j\chi^k\chi^m\chi^n\partial_i\partial_jA_k\partial_l\partial_mA_n + \frac{1}{2}\theta^l\theta^i\chi^m\chi^n\chi^j\chi^k\partial_l\partial_mA_n\partial_i\partial_jA_k \\
&= \frac{1}{2}(\theta^i\theta^l + \theta^l\theta^i)\chi^j\chi^k\chi^m\chi^n\partial_i\partial_jA_k\partial_l\partial_mA_n = 0\ ,
\end{aligned} \tag{4.64}$$

and similarly for $\boldsymbol{A}$. For $n = 1$, the abelian 1-form action (4.61) is not a total derivative, but an action for electromagnetism

$$\begin{aligned}
\int \mathrm{d}^D\theta\,\mathrm{d}^D\chi\,\boldsymbol{\eta}^{D-2}\boldsymbol{F}\widetilde{\boldsymbol{F}} &= \int \mathrm{d}^D\theta\,\mathrm{d}^D\chi\,(\eta_{kl}\theta^k\chi^l)^{D-2}\left(\frac{1}{2}F_{i_1i_2}\theta^{i_1}\theta^{i_2}\right)\left(\frac{1}{2}F_{j_1j_2}\chi^{j_1}\chi^{j_2}\right) && (4.65)\\
&= \frac{1}{4}\varepsilon^{i_1i_2k_3\ldots k_D}\varepsilon^{j_1j_2}{}_{k_3\ldots k_D}F_{i_1i_2}F_{j_1j_2} && (4.66)\\
&= \frac{1}{2}F_{ij}F^{ij}\ , && (4.67)
\end{aligned}$$

which indeed has second order field equations only. Hence, the standard 4-dimensional electrodynamics may be written in this graded formalism as

$$S_{\mathrm{e.m.}}[A] = -\frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4\theta\,\mathrm{d}^4\chi\,\boldsymbol{\eta}^2\,\mathbf{d}\boldsymbol{A}\,\widetilde{\mathbf{d}}\widetilde{\boldsymbol{A}}\ . \tag{4.68}$$

For the general case of a p-form ω with field strength $F = \mathrm{d}\omega$, let us define for a p-form $\omega^{(p)}$ as

$$\boldsymbol{\omega}^{(p)} = \frac{1}{p!}\omega_{i_1\ldots i_p}\theta^{i_1}\ldots\theta^{i_p}\ . \tag{4.69}$$

Note that we are using the boldface notation to denote objects associated with the anti-commuting graded variables, which in (4.69) is the θs. There is similarly a second object with the same components,

$$\widetilde{\boldsymbol{\omega}}^{(p)} = \frac{1}{p!}\omega_{i_1\ldots i_p}\chi^{i_1}\ldots\chi^{i_p}\ , \tag{4.70}$$

in the set of graded variables χ. For 0-forms (scalar fields), we have that $\boldsymbol{\omega}^{(0)} = \widetilde{\boldsymbol{\omega}}^{(0)} = \omega^{(0)}$. The $(p+1)$-form field strength leads to the expressions

$$\boldsymbol{F} = \mathbf{d}\boldsymbol{\omega} = \tfrac{1}{(p+1)!}F_{i_1\ldots i_{p+1}}\theta^{i_1}\ldots\theta^{i_{p+1}} \quad \text{and} \quad \widetilde{\boldsymbol{F}} = \widetilde{\mathbf{d}}\widetilde{\boldsymbol{\omega}} = \tfrac{1}{(p+1)!}F_{i_1\ldots i_{p+1}}\chi^{i_1}\ldots\chi^{i_{p+1}}\ . \tag{4.71}$$

The p-form Galileon action is

$$S_{2n}[\omega] = \frac{1}{(D-(p+2)n+1)!} \int \mathrm{d}^D x \int \mathrm{d}^D\theta \, \mathrm{d}^D\chi \, \boldsymbol{\eta}^{D-(p+2)n+1} \, \boldsymbol{F} \, \widetilde{\boldsymbol{F}} \, (\mathbf{d}\widetilde{\boldsymbol{F}})^{n-1} \, (\widetilde{\mathbf{d}}\boldsymbol{F})^{n-1} \, . \tag{4.72}$$

The 0-form case (4.53) with $\boldsymbol{F} = \widetilde{\boldsymbol{F}} = \mathrm{d}\pi$, is implemented via the redefinition $n \to \frac{n+1}{2}$. For $n = 1$ in (4.72), one obtains the standard kinetic term $F_{i_1 \ldots i_{p+1}} F^{i_1 \ldots i_{p+1}}$ for an abelian p-form. For an odd p-form $\omega^{(\mathrm{odd})}$, the action (4.72) for $n > 1$ is a total derivative for the same reason as for $p = 1$ [78], that is,

$$(\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{\omega}^{(\mathrm{odd})})^2 = (\mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{\omega}}^{(\mathrm{odd})})^2 = 0 \, . \tag{4.73}$$

4.4 Mixed-Symmetry Tensor in Flat Spacetime

Our goal is to write a general higher-derivative action for a (p, q) mixed-symmetry tensor field such that it leads to exactly second order field equations. Consider a (p, q) mixed-symmetry tensor field $T^{(p,q)}$ and its graded counterparts given in Eqs. (4.43) and (4.44), the mixed-symmetry Galileon action is

$$S_{2n}[T^{(p,q)}] = \frac{1}{(D-k)!} \int \mathrm{d}^D x \int \mathrm{d}^D\theta \, \mathrm{d}^D\chi \, \boldsymbol{\eta}^{D-k} \mathbf{d}\boldsymbol{T} \, \widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}} \, (\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})^{n-1} \, (\mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}})^{n-1} \, , \tag{4.74}$$

with $k = (p + q + 2)n - 1$. Variation with respect to $\boldsymbol{T}$ shows that the field equations are strictly second order in derivatives. This action is gauge invariant provided that

$$\mathbf{d}\widetilde{\mathbf{d}}(\delta\boldsymbol{T}) = 0 \, , \tag{4.75}$$

which is when

$$\delta\boldsymbol{T}^{(p,q)} = \mathbf{d}\boldsymbol{\lambda}^{(p-1,q)} + \widetilde{\mathbf{d}}\boldsymbol{\lambda}'^{(p,q-1)} \, . \tag{4.76}$$

For the mixed-symmetry field, once again it holds that

$$(\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T}^{(p,q)})^2|_{p+q=\mathrm{odd}} = (\mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}}^{(p,q)})^2|_{p+q=\mathrm{odd}} = 0 \, . \tag{4.77}$$

Therefore, the action (4.74) for $n > 1$ with $p + q = $ odd is a total derivative. The first non-trivial single-field case with $p \neq q$ arises for $p = 3, q = 1$. The action in 11 dimensions for $n = 2$ (that is, action containing 4 fields $T^{(3,1)}$) is

$$S_4[T^{(3,1)}] = \int \mathrm{d}^{11}x \int \mathrm{d}^{11}\theta \, \mathrm{d}^{11}\chi \; \mathbf{d}\boldsymbol{T}^{(3,1)} \widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}}^{(3,1)} \mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T}^{(3,1)} \mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}}^{(3,1)} \tag{4.78}$$

$$= \int \mathrm{d}^{11}x \; \varepsilon^{i_1 \ldots i_{11}} \varepsilon^{j_1 \ldots j_{11}} \partial_{i_1} T_{i_2 i_3 i_4 j_1} \partial_{j_2} T_{j_3 j_4 j_5 i_5} \partial_{i_6} \partial_{j_6} T_{j_7 j_8 j_9 i_7} \partial_{i_8} \partial_{j_{10}} T_{i_9 i_{10} i_{11} j_{11}} \, . \tag{4.79}$$

Using graded variables, it is straightforward to translate into the index formalism widely used in the literature.

Let us now discuss the special case of $p = q$, where we can have $\boldsymbol{T} = \widetilde{\boldsymbol{T}}$, which then allows also the following form of the action,

$$S_{n+1}[T^{(p,p)}] = \frac{1}{(D-k)!}\int \mathrm{d}^D x \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \boldsymbol{\eta}^{D-k} \mathbf{d}\boldsymbol{T}\, \widetilde{\mathbf{d}}\boldsymbol{T}\, (\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})^{n-1} , \tag{4.80}$$

with $k = (p+1)n+p$. If we consider $h^{(1,1)}$ as a symmetric tensor fluctuation to the Minkowski metric such that $h_{ij}(x) \ll 1$ as in linearised gravity, we can express the linearised Einstein-Hilbert action in four dimensions as

$$\begin{aligned} S_{\mathrm{LEH}}[h] &= -\frac{1}{4}\int \mathrm{d}^4x \int \mathrm{d}^4\theta\, \mathrm{d}^4\chi\, \boldsymbol{\eta}\, \boldsymbol{h}\, \mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{h} & (4.81)\\ &= -\frac{1}{4}\int \mathrm{d}^4x \int \mathrm{d}^4\theta\, \mathrm{d}^4\chi\, \eta_{i_1j_1}\theta^{i_1}\chi^{j_1}\, h_{i_2j_2}\theta^{i_2}\chi^{j_2} \times & (4.82)\\ &\qquad \times \theta^{i_3}\chi^{j_3}\theta^{i_4}\chi^{j_4}\partial_{i_3}\partial_{j_3}h_{i_4j_4} & \\ &= -\frac{1}{8}\int \mathrm{d}^4x \int \mathrm{d}^4\theta\, \mathrm{d}^4\chi\, \eta_{i_1j_1}\theta^{i_1}\chi^{j_1}\, h_{i_2j_2}\theta^{i_2}\chi^{j_2}\, (\theta^{i_3}\chi^{j_3}\theta^{i_4}\chi^{j_4}R_{i_4i_3j_3j_4}) , & (4.83) \end{aligned}$$

where $R_{i_4i_3j_3j_4}$ is the linearised Riemann curvature. Recall that the linearised Riemann curvature is expressed in terms of the linearised metric h as

$$R_{i_4i_3j_3j_4} = \frac{1}{2}\left(\partial_{i_3}\partial_{j_3}h_{i_4j_4} + \partial_{i_4}\partial_{j_4}h_{i_3j_3} - \partial_{i_3}\partial_{j_4}h_{i_4j_3} - \partial_{i_4}\partial_{j_3}h_{i_3j_4}\right) . \tag{4.84}$$

We proceed to obtain

$$\begin{aligned} S_{\mathrm{LEH}}[h] &= -\frac{1}{2}\int \mathrm{d}^4x \left(h^{ij}R_{ij} - \frac{1}{2}hR\right) & (4.85)\\ &= -\frac{1}{2}\int \mathrm{d}^4x\; h^{ij}\left(R_{ij} - \frac{1}{2}\eta_{ij}R\right) , & (4.86) \end{aligned}$$

where $R_{ij} = R^k{}_{ikj}$ is the linearised Ricci tensor and $R = \eta^{ij}R_{ij}$ is the linearised Ricci scalar. Recall that from the linearised graviton, $g_{ij} = \eta_{ij} + \kappa h_{ij}$, where $\kappa^2 = 8\pi G_N = 8\pi\frac{\hbar c}{M^2_{\mathrm{Planck}}}$ for Newton's gravitational constant G_N. We have taken $\hbar = 1 = c$ in the computations.

With more occurrences of h, we expect to find non-trivial Lovelock invariants. We have from the action (4.80) for instance at 5 dimensions,

$$\begin{aligned} S_{\mathrm{LGB}}[h] &= -\frac{1}{4}\int \mathrm{d}^5x \int \mathrm{d}^5\theta\, \mathrm{d}^5\chi\, \boldsymbol{h}\, \mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{h}\, \mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{h} & (4.87)\\ &= -\frac{1}{16}\int \mathrm{d}^5x\, \varepsilon^{i_1i_2i_3i_4i_5}\varepsilon^{j_1j_2j_3j_4j_5}\, h_{i_1j_1}\, R_{i_2i_3j_2j_3}\, R_{i_4i_5j_4j_5} . & (4.88) \end{aligned}$$

Out of the 5! possibilities, let us work on the following permutations to get

$$\begin{aligned} &-\frac{1}{16}\int \mathrm{d}^5x\, (\delta^{i_1j_1}\delta^{i_2j_2}\delta^{i_3j_3}\delta^{i_4j_4}\delta^{i_5j_5} + \delta^{i_1j_1}\delta^{i_2j_4}\delta^{i_3j_5}\delta^{i_4j_2}\delta^{i_5j_3} & (4.89)\\ &\qquad -\delta^{i_1j_1}\delta^{i_2j_2}\delta^{i_3j_5}\delta^{i_4j_4}\delta^{i_5j_3} - \delta^{i_1j_1}\delta^{i_2j_4}\delta^{i_3j_3}\delta^{i_4j_2}\delta^{i_5j_5} & \\ &\qquad -\delta^{i_1j_1}\delta^{i_2j_3}\delta^{i_3j_4}\delta^{i_4j_5}\delta^{i_5j_2} - \delta^{i_1j_1}\delta^{i_2j_5}\delta^{i_3j_2}\delta^{i_4j_3}\delta^{i_5j_4}) \times & \\ &\qquad \times h_{i_1j_1}\, R_{i_2i_3j_2j_3}\, R_{i_4i_5j_4j_5} & \\ = &-\frac{1}{16}\int \mathrm{d}^5x\, h(R^2 + R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij}) . & (4.90) \end{aligned}$$

We find that it contains the familiar looking Gauss-Bonnet term, in linearised Riemann curvature. Starting from the Galileon type of action, we have managed to derive the linearised gravity action and also got a clue of the correction term.

Using the graded formalism, it is simpler to generalize the mixed-symmetry Galileon actions to multiple species, with field equations not more than second order in derivatives. For a tower of n mixed-symmetry tensor fields, an action which continues to possess the characteristic of being polynomial in the fields, is given by

$$\begin{aligned} S[\mathrm{T}^{(p_1,q_1)},\ldots,\mathrm{T}^{(p_n,q_n)}] &= \frac{1}{(D-k)!}\int \mathrm{d}^D x \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \boldsymbol{\eta}^{D-k} \times \\ &\times \prod_{r_1=1}^{n_1} \mathbf{d}\boldsymbol{T}^{(p_{r_1},q_{r_1})} \prod_{r_2=n_1+1}^{n_2} \widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}}^{(p_{r_2},q_{r_2})} \times \\ &\times \prod_{s_1=n_2+1}^{n_3} \mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T}^{(p_{s_1},q_{s_1})} \prod_{s_2=n_3+1}^{n} \mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}}^{(p_{s_2},q_{s_2})} , \end{aligned} \tag{4.91}$$

for some ordered integers $n_1 < n_2 < n_3 < n$. The action has field equations of only second order in derivatives or less. It is non-vanishing only when it contains an equal number of anti-commuting variables χ, θ. This means

$$2n_1 - n_2 + \sum_{r=1}^{n_1}(p_r - q_r) - \sum_{r=n_1+1}^{n_2}(p_r - q_r) + \sum_{r=n_2+1}^{n_3}(p_r - q_r) - \sum_{r=n_3+1}^{n}(p_r - q_r) = 0 \ . \tag{4.92}$$

In addition, the action is defined for $D \geq k$ dimensions, where

$$k = n + n_1 - n_2 + \sum_{r=1}^{n_1} p_r + \sum_{r=n_1+1}^{n_2} q_r + \sum_{r=n_2+1}^{n_3} p_r + \sum_{r=n_3+1}^{n} q_r \ . \tag{4.93}$$

4.5 Covariant Mixed-Symmetry Tensor

The covariantization procedure as proposed by [77] is to replace all partial derivatives with covariant derivatives and to add compensating actions to remove all higher than second derivative field equations from the covariantized action. If we ask the question of what is the most general covariant Galileon action that will prevent us from getting higher than second order field equations both for the field $T^{(p,q)}$ and metric g, it is clearly and simply

$$S_2[T^{(p,q)}, g] = \frac{1}{(D-k)!}\int \mathrm{d}^D x \int d^D\theta\, d^D\chi\, \sqrt{-g}\, \boldsymbol{g}^{D-k}\, \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \tag{4.94}$$

with $k = p + q + 1$.

To be in coherence with the previous forms of actions, for the mixed-symmetry field, we have the action as

$$S_{2(n+1)}[T^{(p,q)},g] = \frac{1}{(D-k)!}\int \mathrm{d}^D x \int d^D\theta\, d^D\chi\, \sqrt{-g}\, \boldsymbol{g}^{D-k}\, \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\; \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\; (\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})^n\; (\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})^n\,, \tag{4.95}$$

with $k = (p+q+1)(n+1)+1$, which is more complicated compared to (4.94) as it contains polynomial of second derivatives. There are a total number of $(2(n+1))$ fields in the action. For simplicity, we will work with $n=1$ in the following[13]. The goal is again to have equations of motion of at most second order in derivatives for the field and the metric. If there are terms with higher derivatives, which we call "dangerous"[14], compensating actions are needed to cancel them in the equations of motion. Note that it is ensured by the action that the equations of motion of the mixed-field is at most of second order only. Obviously this is because the derivatives that actually act on the mixed-field are the partial derivatives, as the connection part of the covariant derivatives only acts as a multiplication to the field. The danger of having more than second derivatives in the metric is however inevitable, as there will be instances where the Riemannian curvature gets differentiated. On the other hand, from the covariant action (4.95), the variation with respect to the metric will render *both* the mixed-field and metric to have higher than second derivatives in the field equations, in fact the variation with respect to the metric is more cumbersome.

From the action (4.95), there are 5 dangerous terms from the field equation for $\widetilde{\boldsymbol{T}}$,

$$\begin{aligned}
&(1+(-1)^{p+q})(\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\; \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\; \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T} - \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\; (\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}^2\boldsymbol{T})\; \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \\
&-\boldsymbol{\nabla}^2\widetilde{\boldsymbol{T}}\; \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\; (\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}) + (-1)^q\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\; \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\; (\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}) \\
&+(-1)^{p+q}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\; \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\; (\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,.
\end{aligned} \tag{4.96}$$

Similarly for $\boldsymbol{T}$, there are 5 dangerous terms

$$\begin{aligned}
&(1+(-1)^{p+q})(\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\; \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\; \widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\; (\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}^2\widetilde{\boldsymbol{T}})\; \widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}} \\
&-\widetilde{\boldsymbol{\nabla}}^2\boldsymbol{T}\; \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\; (\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}) + (-1)^q\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\; \widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\; (\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}) \\
&+(-1)^{p+q}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\; \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\; (\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})
\end{aligned} \tag{4.97}$$

in the equation of motion. The field equations (4.96) and (4.97) are the same under $\chi \leftrightarrow \theta$. Hence the covariant action (4.95), which is invariant under $\chi \leftrightarrow \theta$, precisely describes a single degree of freedom $T^{(p,q)}$. In what follows, discussions will be based on the copy (4.96), bearing in mind that results are to be multiplied by 2. The way to proceed now is to express these terms in curvature, before we can introduce the necessary compensating actions (compensators).

[13] To avoid bulkiness in the analysis, we will also leave out the normalization factors.

[14] Always, when we say dangers or dangerous terms, we are referring to terms higher than second order in derivatives, which we sometimes also say they are the higher derivatives here.

From the Jacobi identity of covariant derivatives

$$\chi^{j_1}\theta^i\chi^{j_2}([\nabla_{j_1},[\nabla_i,\nabla_{j_2}]]+[\nabla_i,[\nabla_{j_2},\nabla_{j_1}]]+[\nabla_{j_2},[\nabla_{j_1},\nabla_{i_1}]])=0\ , \tag{4.98}$$

we obtain

$$2\chi^{j_1}\theta^i\chi^{j_2}[\nabla_{j_1},[\nabla_i,\nabla_{j_2}]]=-\chi^{j_1}\theta^i\chi^{j_2}[\nabla_i,[\nabla_{j_2},\nabla_{j_1}]]\ . \tag{4.99}$$

Since

$$([\nabla_a,[\nabla_b,\nabla_c]])T=(\nabla_a[\nabla_b,\nabla_c])T \tag{4.100}$$

for any tensor T, thus

$$\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \ = \ \frac{1}{2}\boldsymbol{\nabla}[\widetilde{\boldsymbol{\nabla}},\widetilde{\boldsymbol{\nabla}}]\boldsymbol{T}+\widetilde{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}\boldsymbol{T}\ . \tag{4.101}$$

To the field $\boldsymbol{T}$, the second term on the right hand side is safe, as it is a Riemannian curvature multiplying the covariant derivative of the field. Hence, in indices, the danger on the field $\boldsymbol{T}$ from $\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}$ is given by

$$\begin{aligned}
& \frac{1}{2}\,\chi^{j_2'}\theta^i\chi^{j_1'}(\nabla_i[\nabla_{j_2'},\nabla_{j_1'}])\boldsymbol{T} && (4.102)\\
= \ & -\frac{1}{2}p\,\chi^{j_2'}\theta^i\chi^{j_1'}(\nabla_i R^{\ell}{}_{i_1 j_2' j_1'})T_{\ell i_2\cdots i_p j_1\cdots j_q}\theta^{i_1}\cdots\theta^{i_p}\chi^{j_1}\cdots\chi^{j_q}\\
= \ & -\frac{1}{4}p\,\chi^{j_2'}\theta^i\chi^{j_1'}(\nabla_i R_{j_2' j_1' \ell i_1}-\nabla_{i_1}R_{j_2' j_1' \ell i})T^{\ell}{}_{i_2\cdots i_p j_1\cdots j_q}\theta^{i_1}\cdots\theta^{i_p}\chi^{j_1}\cdots\chi^{j_q}\\
= \ & \frac{1}{4}p\,\chi^{j_2'}\theta^i\chi^{j_1'}(\nabla_\ell R_{j_2' j_1' i_1 i})T^{\ell}{}_{i_2\cdots i_p j_1\cdots j_q}\theta^{i_1}\cdots\theta^{i_p}\chi^{j_1}\cdots\chi^{j_q}\\
= \ & -\frac{1}{4}p\,(\nabla^\ell \boldsymbol{Riem})T_{\ell i_2\cdots i_p j_1\cdots j_q}\theta^{i_2}\cdots\theta^{i_p}\chi^{j_1}\cdots\chi^{j_q}\ , && (4.103)
\end{aligned}$$

where we have used the second Bianchi identity from the second to third equality. Note that $\nabla g=0$. For $\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}$, the dangerous part is

$$-\frac{1}{4}p\,(\nabla^\ell\boldsymbol{Riem})T_{\ell i_2\cdots i_p j_1\cdots j_q}\chi^{i_2}\cdots\chi^{i_p}\theta^{j_1}\cdots\theta^{j_q}\ , \tag{4.104}$$

and from $\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}^2\boldsymbol{T}$, it is

$$-\frac{1}{4}p\,(\nabla^\ell\boldsymbol{Riem})T_{\ell i_2\cdots i_p j_1\cdots j_q}\theta^{i_2}\cdots\theta^{i_p}\chi^{j_1}\cdots\chi^{j_q}\ . \tag{4.105}$$

Finally, from the four derivatives $\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}$, we get a derivative on Riemann curvature as well, which implies a third derivative on the metric, as

$$-\frac{1}{4}(\nabla^\ell\boldsymbol{Riem})\nabla_\ell\boldsymbol{T}+\frac{1}{4}q\,(\nabla^\ell\boldsymbol{Riem})\widetilde{\boldsymbol{\nabla}}T_{i_1\cdots i_p\ell j_2\cdots j_q}\theta^{i_1}\cdots\theta^{i_p}\chi^{j_2}\cdots\chi^{j_q}\ . \tag{4.106}$$

For convenient short-hand notations in what follows, let

$$\begin{aligned}
\boldsymbol{T}_\ell &:= T_{\ell i_2\cdots i_p j_1\cdots j_q}\theta^{i_2}\cdots\theta^{i_p}\chi^{j_1}\cdots\chi^{j_q}\ , && (4.107)\\
\widetilde{\boldsymbol{T}}_\ell &:= T_{\ell i_2\cdots i_p j_1\cdots j_q}\chi^{i_2}\cdots\chi^{i_p}\theta^{j_1}\cdots\theta^{j_q}\ , && (4.108)\\
\boldsymbol{T}_{\tilde\ell} &:= T_{i_1\cdots i_p\ell j_2\cdots j_q}\theta^{i_1}\cdots\theta^{i_p}\chi^{j_2}\cdots\chi^{j_q}\ , && (4.109)\\
\widetilde{\boldsymbol{T}}_{\tilde\ell} &:= T_{i_1\cdots i_p\ell j_2\cdots j_q}\chi^{i_1}\cdots\chi^{i_p}\theta^{j_2}\cdots\theta^{j_q}\ . && (4.110)
\end{aligned}$$

The total 5 dangerous terms from (4.96) expressed in terms of the Riemann tensor are

$$-\frac{1}{4}p\,(1+(-1)^{p+q})\,(\nabla^{\ell}\boldsymbol{Riem})\,\widetilde{\boldsymbol{T}}_{\ell}\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \tag{4.111}$$

$$-\frac{1}{4}p\,(-1)^{q}(1+(-1)^{p+q})\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{T}_{\ell}\,(\nabla^{\ell}\boldsymbol{Riem}) \tag{4.112}$$

$$+\frac{1}{4}p\,\boldsymbol{\nabla}^{2}\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,(\nabla^{\ell}\boldsymbol{Riem})\boldsymbol{T}_{\ell} \tag{4.113}$$

$$-\frac{1}{4}(-1)^{p+q}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,(\nabla^{\ell}\boldsymbol{Riem})\nabla_{\ell}\boldsymbol{T} \tag{4.114}$$

$$+\frac{1}{4}q\,(-1)^{p+q}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,(\nabla^{\ell}\boldsymbol{Riem})\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}_{\tilde{\ell}}\,. \tag{4.115}$$

Note that when $p+q$ is odd, (4.111) and (4.112) are zero.

The compensation procedure is very lengthy, in the sense that, compensators introduced are inducing back dangerous higher derivative terms in their equations of motion. It is then necessary to re-compensate the compesators. Here we propose the compensating actions such that that they produce the same equations of motion (with the same sign) as the dangerous ones from the covariant action (4.95). Therefore, the covariant action (4.95) is to substract all the compensating actions, to give non-higher-than-second-order field equation for the mixed-field. We will ignore the safe terms and discuss only the dangerous terms.

For even $p+q$, we can introduce a compensator,

$$\begin{aligned} S_{4,r=1}[T^{(p+q=\text{even})},g] \;=\; & -\frac{1}{2}p\,(-1)^{q}\int \mathrm{d}^{D}x\int d^{D}\theta\,d^{D}\chi\,\sqrt{-g}\,\boldsymbol{g}^{D-k}\;\times \\ & \times\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\nabla^{\ell}\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_{\ell}\,\boldsymbol{Riem}\;, \end{aligned} \tag{4.116}$$

to give (4.111). That is to say, with the original action (4.95) minus this compensator, (4.111) is cancelled. r counts the number of curvature terms in the action. However, (4.116) produces simultaneously unwanted dangerous equations of motion from the mixed-field variation. They are[15]

$$p\,\delta\boldsymbol{T}\,(\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,\nabla^{\ell}\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_{\ell}\,\boldsymbol{Riem}\;,\quad \frac{1}{2}p\,\delta\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,(\nabla^{\ell}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,\widetilde{\boldsymbol{T}}_{\ell}\,\boldsymbol{Riem}\;, \tag{4.118}$$

$$-\frac{1}{2}p\,\delta\boldsymbol{T}\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,(\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\nabla^{\ell}\widetilde{\boldsymbol{T}})\,\widetilde{\boldsymbol{T}}_{\ell}\,\boldsymbol{Riem}\,. \tag{4.119}$$

This is where compensations are again needed. Although it seems to be finitely achievable, that a total of 21 compensating actions are found for even $p+q$ in order to keep all the terms

[15] We note that by equations of motion or field equations, we mean

$$\frac{\delta S}{\delta T^{(p,q)}}=0 \tag{4.117}$$

as the field equation for the field $T^{(p,q)}$, obtained from the variation of the action S with respect to the field $T^{(p,q)}$. Equating it to zero is understood. Similarly for the metric g. In order to be explicit with the component of the field that the action is varied with respect to, as we will soon notice it is necessary especially in the case for metric field equations, we often leave δ(field) in the expressions.

in the equations of motion lower than third order derivatives in the field and metric; this is only a partial result from the side of $\frac{\delta S}{\delta T^{(p,q)}}$, that is, variation of the action with respect to the field $T^{(p,q)}$. Notice that in the process of compensation, we have also simplified the analysis in certain terms to express in only the exterior derivative parts (hence not covariant). Therefore, additional dangers may appear after taking into account the respective full covariant expressions. We will come to the variation with respect to metric after we state the result of the 21 integrands of the *compensators* for even $p+q$, in which after the first four, are all resultants from re-compensations:

$$-\tfrac{1}{2}p\,(-1)^q\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{4.120}$$

$$\tfrac{1}{2}p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{4.121}$$

$$-\tfrac{1}{4}p\,(-1)^q\,\boldsymbol{\nabla}^2\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{4.122}$$

$$\tfrac{1}{8}\,(-1)^{p+q}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;(\nabla^\ell\boldsymbol{T}-q\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}^{\widetilde{\ell}})\,(\nabla_\ell\widetilde{\boldsymbol{T}}-q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}_{\widetilde{\ell}})\;\boldsymbol{Riem} \tag{4.123}$$

$$-\tfrac{2}{4^3}p^2\,(-1)^p\,\nabla^m\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_m\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem}^2 \tag{4.124}$$

$$-\tfrac{1}{2}p^3\,(-1)^q\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g^{\ell m}(\mathbf{d}\widetilde{\boldsymbol{R}}_{nm})g^{n\ell'}\;\boldsymbol{T}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{4.125}$$

$$-\tfrac{1}{2}p\,(-1)^p\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\mathrm{d}\partial^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{4.126}$$

$$-\tfrac{1}{4}p^2\,(-1)^p\,\mathbf{d}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g^{\ell k}\widetilde{\boldsymbol{R}}^m{}_k\;\widetilde{\boldsymbol{T}}_m\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{4.127}$$

$$\tfrac{1}{4}p^2\,(-1)^p\,\widetilde{\boldsymbol{T}}\;\mathbf{d}\boldsymbol{T}\;g^{\ell k}(\boldsymbol{\nabla}\widetilde{\boldsymbol{R}}^m{}_k)\;\widetilde{\boldsymbol{T}}_m\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{4.128}$$

$$\tfrac{1}{4}pq\,(-1)^p\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g^{\ell k}\boldsymbol{R}^m{}_k\;\widetilde{\boldsymbol{T}}_{\widetilde{m}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{4.129}$$

$$\tfrac{1}{8}p^2\,\nabla^m\boldsymbol{T}\;\boldsymbol{T}_m\;\boldsymbol{\nabla}^\ell\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem}^2 \tag{4.130}$$

$$-p\,\mathbf{d}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\partial^\ell\widetilde{\mathbf{d}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{4.131}$$

$$-\tfrac{1}{2}p^2\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\mathbf{d}\widetilde{\boldsymbol{T}}\;g^{\ell m}\boldsymbol{R}_{nm}\,g^{n\ell'}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{4.132}$$

$$\tfrac{1}{2}p\,\mathbf{d}\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\mathrm{d}}\partial^\ell\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{4.133}$$

$$-\tfrac{1}{4}p^2(-1)^p\,\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}}\;\mathbf{d}\boldsymbol{T}\;g^{\ell m}\boldsymbol{R}_{nm}\,g^{n\ell'}\;\boldsymbol{T}_{\ell'}\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{4.134}$$

$$-\tfrac{1}{4}p^2\,\mathbf{d}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g^{\ell m}\widetilde{\boldsymbol{R}}_{nm}\,g^{n\ell'}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{4.135}$$

$$\tfrac{1}{4}p\,\mathbf{d}\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g^{\ell k}\widetilde{\boldsymbol{R}}^m{}_k\;\widetilde{\boldsymbol{T}}_m\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{4.136}$$

$$-\tfrac{1}{4}pq\,\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g^{\ell k}(\widetilde{\mathbf{d}}\boldsymbol{R}^m{}_k)\;\widetilde{\boldsymbol{T}}_{\widetilde{m}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{4.137}$$

$$\tfrac{1}{2^6}p^2(-1)^p\,\nabla^m\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_m\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem}^2 \tag{4.138}$$

$$-\tfrac{1}{2^3}p^2(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g^{n\ell'}\widetilde{\boldsymbol{R}}_{nm}\,g^{m\ell}\;\widetilde{\boldsymbol{T}}_{\ell'}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem}\;, \tag{4.139}$$

where $\widetilde{\boldsymbol{R}}_{nm} := R_{njim}\chi^j\theta^i$, $\boldsymbol{R}_{nm} := R_{nijm}\theta^i\chi^j$, $\widetilde{\boldsymbol{R}}^m{}_k := R^m{}_{kj_1j_2}\chi^{j_1}\chi^{j_2}$, and $\boldsymbol{R}^m{}_k := R^m{}_{ki_1i_2}\theta^{i_1}\theta^{i_2}$, all with curvature component. For odd $p+q$, the compensators required are only the first four and last two out of this set of 21. The integrands above turn into actions, after inserting them into $\int \mathrm{d}^Dx \int d^D\theta\, d^D\chi\,\sqrt{-g}\,\boldsymbol{g}^{D-k}$.

From the mixed-field variation of the covariant action, the dangerous terms are mostly due to higher derivatives in the metric. A significant difference with the literature is that here for covariant mixed-tensor, not only that it requires more rounds of compensations, it is also manifested at the expense of two Riemann tensors, instead of one in the literature. This is reasonable as the factors $\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}$ and $\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}$ in the covariant action each contains Riemann curvature. We remark again that the four-derivative dangers only need to be compensated once (4.123), while the third derivatives in the field equations are requiring more compensations, compensating the compensators.

Now we move on to the metric variation of the action. Let us rearrange the form of the action terms to show explicit $\boldsymbol{\Gamma}$, hence metric component g_{ij} dependence. From the metric part of $\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}$ in the covariant action (4.95), we get $-p\,\boldsymbol{\Gamma}^\ell\widetilde{\boldsymbol{T}}_\ell$. From the term $\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}$, its metric parts are

$$-p\,(\mathbf{d}\boldsymbol{\Gamma}^\ell)\,\boldsymbol{T}_\ell\ ,\ \ -\boldsymbol{\Gamma}^\ell\,\nabla_\ell\boldsymbol{T}\ ,\ \ q\,\boldsymbol{\Gamma}^\ell\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}_{\widetilde{\ell}}\ ,\ \ -p\,\boldsymbol{\Gamma}^\ell_m\boldsymbol{\Gamma}^m\boldsymbol{T}_\ell\ ,\ \ -pq\,\boldsymbol{\Gamma}^\ell\boldsymbol{\Gamma}^m\boldsymbol{T}_{\ell\widetilde{m}}\ ,\ \ p\,\boldsymbol{\Gamma}^\ell\,\mathbf{d}\boldsymbol{T}_\ell\ , \tag{4.140}$$

where $\boldsymbol{\Gamma}^\ell_m := \theta^i\Gamma^\ell_{mi}$ and $\boldsymbol{T}_{\ell\widetilde{m}} := T_{\ell i_2\cdots i_p m j_2\cdots j_q}\theta^{i_2}\cdots\theta^{i_p}\chi^{j_2}\cdots\chi^{j_q}$. While from $\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}$, there are

$$-p\,(\widetilde{\mathbf{d}}\boldsymbol{\Gamma}^\ell)\,\widetilde{\boldsymbol{T}}_\ell\ ,\ \ -\boldsymbol{\Gamma}^\ell\,\nabla_\ell\widetilde{\boldsymbol{T}}\ ,\ \ q\,\boldsymbol{\Gamma}^\ell\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}_{\widetilde{\ell}}\ ,\ \ -p\,\widetilde{\boldsymbol{\Gamma}}^\ell_m\boldsymbol{\Gamma}^m\widetilde{\boldsymbol{T}}_\ell\ ,\ \ -pq\,\boldsymbol{\Gamma}^\ell\boldsymbol{\Gamma}^m\widetilde{\boldsymbol{T}}_{\ell\widetilde{m}}\ ,\ \ p\,\boldsymbol{\Gamma}^\ell\,\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}}_\ell\ , \tag{4.141}$$

where $\widetilde{\boldsymbol{\Gamma}}^\ell_m := \chi^j\Gamma^\ell_{mj}$. Note that (4.141) is a copy of (4.140) under exchange of χs and θs.

From the action (4.95), for $n = 1$, we have 12 dangerous occurences, involving the derivatives on metric and on the mixed-field, stated in integrand of the action:

$$-p\,\widetilde{\boldsymbol{T}}_\ell\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ (\boldsymbol{\Gamma}^\ell\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ , \tag{4.142}$$
$$-p\,\widetilde{\boldsymbol{T}}_\ell\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ (\boldsymbol{\Gamma}^\ell\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ , \tag{4.143}$$
$$p(-1)^{p+q}\,((\mathbf{d}\boldsymbol{\Gamma}^\ell)\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \boldsymbol{T}_\ell\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ , \tag{4.144}$$
$$p(-1)^{p+q}\,((\widetilde{\mathbf{d}}\boldsymbol{\Gamma}^\ell)\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \widetilde{\boldsymbol{T}}_\ell\ \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ , \tag{4.145}$$
$$-(-1)^{p+q}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \partial_\ell\boldsymbol{T}\ (\boldsymbol{\Gamma}^\ell\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ , \tag{4.146}$$
$$q(-1)^{p+q}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \widetilde{\mathbf{d}}\boldsymbol{T}_{\widetilde{\ell}}\ (\boldsymbol{\Gamma}^\ell\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ , \tag{4.147}$$
$$-p(-1)^{p}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \boldsymbol{\Gamma}^m\boldsymbol{T}_\ell\ (\boldsymbol{\Gamma}^\ell_m\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ , \tag{4.148}$$
$$p(-1)^{p+q}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \boldsymbol{\Gamma}^\ell_m\boldsymbol{T}_\ell\ (\boldsymbol{\Gamma}^m\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ , \tag{4.149}$$
$$-pq(-1)^{p+q}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \boldsymbol{\Gamma}^\ell\boldsymbol{T}_{\ell\widetilde{m}}\ (\boldsymbol{\Gamma}^m\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ , \tag{4.150}$$
$$-pq(-1)^{p+q}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \boldsymbol{\Gamma}^m\boldsymbol{T}_{\ell\widetilde{m}}\ (\boldsymbol{\Gamma}^\ell\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ , \tag{4.151}$$
$$-p(-1)^{q}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \boldsymbol{T}_\ell\ ((\mathbf{d}\boldsymbol{\Gamma}^\ell)\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ , \tag{4.152}$$
$$p(-1)^{p+q}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \mathbf{d}\boldsymbol{T}_\ell\ (\boldsymbol{\Gamma}^\ell\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}})\ . \tag{4.153}$$

There is as well another copy of the above 12 terms with χ, θ interchanged. Hence, results in the following are to be multiplied by 2 wherever applicable. As was mentioned earlier and intended, the terms have been written in connection coefficients that cause higher-derivative danger on other terms.

An explicit computation of (4.143) shows the danger of having third derivatives in metric,

$$\frac{1}{4}p^2\, g^{mm'}\delta \boldsymbol{g}\, (\partial^\ell \widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{g}}_{m'})\, \widetilde{\boldsymbol{T}}_m\, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\, \widetilde{\boldsymbol{T}}_\ell\, \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ , \tag{4.154}$$

$$\frac{1}{4}p^2\, g^{\ell\ell'}\delta \widetilde{\boldsymbol{g}}_{\ell'}\, (\mathbf{d}\widetilde{\mathbf{d}}\partial^m \boldsymbol{g})\, \widetilde{\boldsymbol{T}}_m\, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\, \widetilde{\boldsymbol{T}}_\ell\, \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ , \tag{4.155}$$

$$\frac{1}{4}p^2(-1)^{p+q}\, \delta \boldsymbol{g}\, (\partial^\ell \widetilde{\mathbf{d}}\partial^m \boldsymbol{g})\, \widetilde{\boldsymbol{T}}_\ell\, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\, \widetilde{\boldsymbol{T}}_m\, \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ , \tag{4.156}$$

where $\widetilde{\boldsymbol{g}}_{m'} := g_{m'j'}\chi^{j'}$. Together from (4.143) and (4.145), when $p+q$ is odd, the dangerous field equations are

$$\frac{1}{2}p^2\, g^{mm'}\delta \boldsymbol{g}\, (\partial^\ell \widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{g}}_{m'})\, \widetilde{\boldsymbol{T}}_m\, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\, \widetilde{\boldsymbol{T}}_\ell\, \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ , \tag{4.157}$$

$$\frac{1}{2}p^2\, g^{\ell\ell'}\delta \widetilde{\boldsymbol{g}}_{\ell'}\, (\mathbf{d}\widetilde{\mathbf{d}}\partial^m \boldsymbol{g})\, \widetilde{\boldsymbol{T}}_m\, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\, \widetilde{\boldsymbol{T}}_\ell\, \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ , \tag{4.158}$$

$$\frac{1}{2}p^2(-1)^{p+q}\, \delta \boldsymbol{g}\, (\partial^\ell \widetilde{\mathbf{d}}\partial^m \boldsymbol{g})\, \widetilde{\boldsymbol{T}}_\ell\, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\, \widetilde{\boldsymbol{T}}_m\, \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ , \tag{4.159}$$

while when $p+q$ is even, there is only

$$\frac{1}{2}p^2(-1)^{p+q}\, \delta \boldsymbol{g}\, (\partial^\ell \widetilde{\mathbf{d}}\partial^m \boldsymbol{g})\, \widetilde{\boldsymbol{T}}_\ell\, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\, \widetilde{\boldsymbol{T}}_m\, \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ . \tag{4.160}$$

Following the spirit of [77], we hope for the compensators previously introduced which have successfully and *fully* compensated all the dangers arising from $\frac{\delta S}{\delta T}$, to be able to compensate the dangers from $\frac{\delta S}{\delta g}$. That is, when we vary our action (4.95) with respect to the metric, the dangers (higher derivatives in the metric and the field) can be completely cancelled by the results from metric variation of all the compensators. Besides that, the metric variation of these compensators themselves *should not* induce further dangers. If they do, recompensations commence again, like what has happened in $\frac{\delta S}{\delta T}$. We should be aware that the dangers from $\frac{\delta S}{\delta T}$ are solely due to the higher derivatives in metric that require cancellation. From $\frac{\delta S}{\delta g}$, both higher derivatives in metric and field occur. We have resolved (to a certain degree) the higher-derivative issue at $\frac{\delta S}{\delta T}$ with the expense of a goood amount of compensators. If extra dangers should show up in $\frac{\delta S}{\delta g}$, which means the previous compensators found from the side of $\frac{\delta S}{\delta T}$ fail to fully cancel the dangers from $\frac{\delta S}{\delta g}$, the computation gets cumbersome, as there are two types of higher-derivative dangers to take care of, in the metric and in the field.

Here is an example of a set of dangerous terms worked out from the compensating action

S^1 (4.116). The integrands are

$$-\tfrac{1}{2}p^2(-1)^p\,\boldsymbol{T}_m\,(\boldsymbol{\Gamma}^m\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,\boldsymbol{Riem}\;, \tag{4.161}$$
$$-\tfrac{1}{2}p^2(-1)^q\,\boldsymbol{T}_m\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,(\boldsymbol{\Gamma}^m\boldsymbol{Riem})\;, \tag{4.162}$$
$$-\tfrac{1}{2}p^2\,((\mathbf{d}\boldsymbol{\Gamma}^m)\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,\boldsymbol{T}_m\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,\boldsymbol{Riem}\;, \tag{4.163}$$
$$\tfrac{1}{2}p^2\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{T}_m\,((\mathbf{d}\boldsymbol{\Gamma}^m)\nabla^\ell\widetilde{\boldsymbol{T}})\,\widetilde{\boldsymbol{T}}_\ell\,\boldsymbol{Riem}\;, \tag{4.164}$$
$$-\tfrac{1}{2}p(-1)^p\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\partial_m\boldsymbol{T}\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,(\boldsymbol{\Gamma}^m\boldsymbol{Riem})\;, \tag{4.165}$$
$$\tfrac{1}{2}pq(-1)^p\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\widetilde{\mathbf{d}}\boldsymbol{T}_{\widetilde{m}}\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,(\boldsymbol{\Gamma}^m\boldsymbol{Riem})\;, \tag{4.166}$$
$$\tfrac{1}{2}p^2(-1)^{p+q}\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{\Gamma}^m\boldsymbol{T}_n\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,(\boldsymbol{\Gamma}^n_m\boldsymbol{Riem})\;, \tag{4.167}$$
$$\tfrac{1}{2}p^2(-1)^p\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{\Gamma}^n_m\boldsymbol{T}_n\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,(\boldsymbol{\Gamma}^m\boldsymbol{Riem})\;, \tag{4.168}$$
$$-p^2q(-1)^p\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{\Gamma}^n\boldsymbol{T}_{n\widetilde{m}}\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,(\boldsymbol{\Gamma}^m\boldsymbol{Riem})\;, \tag{4.169}$$
$$-\tfrac{1}{2}p^2\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{T}_m\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,((\mathbf{d}\boldsymbol{\Gamma}^m)\boldsymbol{Riem})\;, \tag{4.170}$$
$$\tfrac{1}{2}p^2(-1)^p\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\mathbf{d}\boldsymbol{T}_m\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,(\boldsymbol{\Gamma}^m\boldsymbol{Riem})\;, \tag{4.171}$$
$$-\tfrac{1}{2}p^2\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,g^{\ell k}(\widetilde{\boldsymbol{\Gamma}}^m_k\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,\widetilde{\boldsymbol{T}}_m\,\widetilde{\boldsymbol{T}}_\ell\,\boldsymbol{Riem}\;, \tag{4.172}$$
$$-\tfrac{1}{2}pq(-1)^{p+q}\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,g^{\ell k}(\boldsymbol{\Gamma}^m_k\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,\widetilde{\boldsymbol{T}}_{\widetilde{m}}\,\widetilde{\boldsymbol{T}}_\ell\,\boldsymbol{Riem}\;, \tag{4.173}$$
$$\tfrac{1}{2}p^2(-1)^q\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\widetilde{\boldsymbol{T}}_m\widetilde{\boldsymbol{T}}_\ell\,g^{\ell k}(\widetilde{\boldsymbol{\Gamma}}^m_k\boldsymbol{Riem})\;, \tag{4.174}$$
$$-\tfrac{1}{2}pq(-1)^q\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\widetilde{\boldsymbol{T}}_{\widetilde{m}}\,\widetilde{\boldsymbol{T}}_\ell\,g^{\ell k}(\boldsymbol{\Gamma}^m_k\boldsymbol{Riem})\;, \tag{4.175}$$
$$-p(-1)^q\,g_{aa'}\chi^a((\mathbf{d}\boldsymbol{\Gamma}^{a'})\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\;, \tag{4.176}$$
$$-p(-1)^q\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,g_{aa'}\chi^a((\mathbf{d}\boldsymbol{\Gamma}^{a'})\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\;, \tag{4.177}$$
$$p(-1)^p\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,g_{aa'}\chi^a(\boldsymbol{\Gamma}^{a'}_k\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,\boldsymbol{\Gamma}^k\;, \tag{4.178}$$
$$p(-1)^p\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,(\boldsymbol{\Gamma}^k\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})\,\nabla^\ell\widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{T}}_\ell\,g_{aa'}\chi^a\boldsymbol{\Gamma}^{a'}_k\;, \tag{4.179}$$
$$-p(-1)^q\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,g_{aa'}\chi^a((\mathbf{d}\boldsymbol{\Gamma}^{a'})\nabla^\ell\widetilde{\boldsymbol{T}})\,\widetilde{\boldsymbol{T}}_\ell\;. \tag{4.180}$$

Out of the 21 compensating actions proposed for even $p+q$ case, for one of them (S^1), the resulting dangers are the 20 terms above. This looks unfeasible to tackle. Let us instead try to study a more generic term, (4.146), as it exists independently of the prefactor p or q.

Before that, know that the Christoffel symbol

$$\Gamma^i_{jk} = \frac{1}{2}g^{\ell i}(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk}) \tag{4.181}$$

has three different derivatives of the metric g, where the last g is being acted upon with a contracted partial derivative. This partial derivative will thus *always* cause higher derivatives on the field and metric. For the Riemann curvature,

$$\boldsymbol{Riem} = 2\,g_{aa'}\chi^a(\mathbf{d}\boldsymbol{\Gamma}^{a'}) + 2\,g_{aa'}\chi^a\boldsymbol{\Gamma}^{a'}_k\boldsymbol{\Gamma}^k \tag{4.182}$$
$$= -2\,\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g} + 2\,g_{aa'}\chi^a\boldsymbol{\Gamma}^{a'}_k\boldsymbol{\Gamma}^k\;. \tag{4.183}$$

Since the first term in ***Riem*** contains two derivatives of metric, it is a potential spot to give rise to higher than two derivatives in metric, when it is hit by the contracted partial derivative in the connection. For instance, from the term $\mathbf{\Gamma}^m \boldsymbol{Riem}$, varying with respect to the metric in $\mathbf{\Gamma}^m$ will give third order derivatives on the metric in term $-2\,\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g}$ in $\boldsymbol{Riem}$.

A few useful computational notes are:
(1)

$$\begin{aligned}
\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T} &= \theta^m\chi^k\partial_m(\nabla_k\boldsymbol{T}) - \theta^m\chi^k\Gamma^\ell_{mk}\nabla_\ell\boldsymbol{T} && (4.184)\\
&\quad -q\,\theta^m\chi^k\Gamma^\ell_{mj_1}\nabla_k T_{i_1\ldots i_p\ell j_2\ldots j_q}\theta^{i_1}\ldots\theta^{i_p}\chi^{j_1}\ldots\chi^{j_q}\\
&= \theta^m\chi^k\partial_m\partial_k\boldsymbol{T} - p\,\theta^m\chi^k\partial_m(\Gamma^\ell_{ki_1}T_{\ell i_2\ldots i_p j_1\ldots j_q})\theta^{i_1}\ldots\theta^{i_p}\chi^{j_1}\ldots\chi^{j_q} && (4.185)\\
&\quad -\theta^m\chi^k\Gamma^\ell_{mk}\partial_\ell\boldsymbol{T} + p\,\theta^m\chi^k\Gamma^h_{mk}\Gamma^\ell_{hi_1}\theta^{i_1}\boldsymbol{T}_\ell\\
&\quad -q\,\theta^m\chi^k\Gamma^\ell_{mj_1}\chi^{j_1}\partial_k\boldsymbol{T}_{\widetilde{\ell}} + pq\,\theta^m\chi^k\Gamma^\ell_{ki_1}\Gamma^h_{mj_1}\theta^{i_1}\chi^{j_1}\boldsymbol{T}_{\ell\widetilde{h}}\\
&= \mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T} - \frac{1}{2}p\,\theta^m\chi^k\theta^{i_1}R^\ell{}_{kmi_1}\boldsymbol{T}_\ell + p\,\mathbf{\Gamma}^\ell\mathbf{d}\boldsymbol{T}_\ell - \mathbf{\Gamma}^\ell\partial_\ell\boldsymbol{T} && (4.186)\\
&\quad +q\,\mathbf{\Gamma}^\ell\widetilde{\mathbf{d}}\boldsymbol{T}_{\widetilde{\ell}} + pq\,\theta^m\chi^k\Gamma^\ell_{ki_1}\Gamma^h_{mj_1}\theta^{i_1}\chi^{j_1}\boldsymbol{T}_{\ell\widetilde{h}}\ ,
\end{aligned}$$

(2)

$$\nabla^\ell\widetilde{\boldsymbol{T}} = \partial^\ell\widetilde{\boldsymbol{T}} - pg^{\ell k}\Gamma^m_{ki_1}\chi^{i_1}\widetilde{\boldsymbol{T}}_m - qg^{\ell k}\Gamma^m_{kj_1}\theta^{j_1}\widetilde{\boldsymbol{T}}_{\widetilde{m}}\ , \tag{4.187}$$

Varying (4.146) with respect to metric, all the possible dangers we get are

$$\begin{aligned}
&\tfrac{1}{4}p(-1)^{p+q}\,\delta\boldsymbol{g}\ g^{mm'}(\partial^\ell\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{g}}_{m'})\ \widetilde{\boldsymbol{T}}_m\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \partial_\ell\boldsymbol{T}\ , && (4.188)\\
-&\tfrac{1}{4}p(-1)^{p+q}\,\delta\boldsymbol{g}\ (\partial^\ell\widetilde{\mathbf{d}}\partial^m\boldsymbol{g})\ \widetilde{\boldsymbol{T}}_m\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \partial_\ell\boldsymbol{T}\ , && (4.189)\\
&\tfrac{1}{4}p(-1)^{p+q}\,\delta\widetilde{\boldsymbol{g}}_{\ell'}\ g^{\ell\ell'}(\mathbf{d}\widetilde{\mathbf{d}}\partial^m\boldsymbol{g})\ \widetilde{\boldsymbol{T}}_m\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \partial_\ell\boldsymbol{T}\ , && (4.190)\\
-&\tfrac{1}{2}(-1)^{p+q}\,\delta\boldsymbol{g}\ (\partial^\ell\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{T}})\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \partial_\ell\boldsymbol{T}\ , && (4.191)
\end{aligned}$$

with third-derivatives on metric. Note that only (4.191) is free of the prefactor p.
Another generic term without the prefactor p or q is from (4.123). From the variation with respect to metric, we have

$$\begin{aligned}
-&\tfrac{1}{2}(-1)^{p+q}\,\delta\boldsymbol{g}\ (\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \nabla_\ell\boldsymbol{T}\ , && (4.192)\\
&\tfrac{1}{4}p(-1)^{p+q}\,\delta\boldsymbol{g}\ g^{mk}(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{g}}_k)\ \widetilde{\boldsymbol{T}}_m\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \nabla_\ell\boldsymbol{T}\ , && (4.193)\\
-&\tfrac{1}{4}p(-1)^{p+q}\,\delta\boldsymbol{g}\ g^{\ell k}(\mathbf{d}\widetilde{\mathbf{d}}\partial^m\widetilde{\boldsymbol{g}}_k)\ \widetilde{\boldsymbol{T}}_m\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \nabla_\ell\boldsymbol{T}\ , && (4.194)\\
&\tfrac{1}{4}q(-1)^{p+q}\,\delta\boldsymbol{g}\ g^{mk}(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\boldsymbol{g}_k)\ \widetilde{\boldsymbol{T}}_{\widetilde{m}}\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \nabla_\ell\boldsymbol{T}\ , && (4.195)\\
-&\tfrac{1}{4}q(-1)^{p+q}\,\delta\boldsymbol{g}\ g^{\ell k}(\mathbf{d}\widetilde{\mathbf{d}}\partial^m\boldsymbol{g}_k)\ \widetilde{\boldsymbol{T}}_{\widetilde{m}}\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \nabla_\ell\boldsymbol{T}\ , && (4.196)\\
&\tfrac{1}{2}p\,\delta\boldsymbol{g}\ (\partial^m\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g})\ \nabla^\ell\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{T}}_m\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \nabla_\ell\boldsymbol{T}\ , && (4.197)\\
-&\tfrac{1}{4}p(-1)^{p+q}\,\delta\widetilde{\boldsymbol{g}}_k\ g^{mk}(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\boldsymbol{g})\ \widetilde{\boldsymbol{T}}_m\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \nabla_\ell\boldsymbol{T}\ , && (4.198)\\
&\tfrac{1}{4}p(-1)^{p+q}\,\delta\widetilde{\boldsymbol{g}}_k\ g^{\ell k}(\mathbf{d}\widetilde{\mathbf{d}}\partial^m\boldsymbol{g})\ \widetilde{\boldsymbol{T}}_m\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \nabla_\ell\boldsymbol{T}\ , && (4.199)
\end{aligned}$$

where $\boldsymbol{g}_k := g_{ki}\theta^i$. Only (4.192) is without an overall p, q factor.

We have found that (4.188), (4.190) and (4.191) (of (4.146) combined with (4.149) for the connection part of ∇_ℓ) can be compensated by (4.193), (4.199) and (4.192) respectively. (4.189) is cancelled by one of the dangerous terms given by (4.152). Thus, (4.146) and (4.149) combined, are cured. The combination of (4.147) and (4.150) is also cured in a similar way, which involves q component part in compensator (4.123).

Intuitively, we expect that the metric field equations from (4.146) - (4.153) that contain dangerous terms can be compensated by those from the compensator (4.123). The reason behind the expectation is because (4.146) - (4.153) are the instances interplayed between $\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}$ and $\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}$ in the action, like (4.123) which was proposed to cure dangers from $\frac{\delta S}{\delta T}$ arisen from derivative impact between $\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}$ and $\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}$. After extensive computations checking both sides to see if full cancellations happen, we found that (4.188), (4.190) and (4.191) (of (4.146) combined with (4.149) for the connection part of ∇_ℓ) can be compensated by (4.193), (4.199) and (4.192) respectively. (4.189) is cancelled by one of the dangerous terms given by (4.152). Thus, (4.146) and (4.149) combined are cured. The combination of (4.147) and (4.150) is also cured in a similar way, which involves the q component part in compensator (4.123). However, we are still left with 27 dangerous equations of motion for odd $p+q$ case ((A.1) - (A.27) in Appendix) and 25 for even $p+q$ ((A.1) - (A.25) in Appendix) from the side of the covariant action ((4.146) - (4.153)). Out of them, there are third derivatives on metric and on the mixed-field, and one of them is a 4-derivative on metric. While from the compensator (4.123), there are in total 9 dangerous equations of motion left, which contain only third derivatives on metric (see Appendix (A.28) - (A.36)). The conclusion from here is, dangers from (4.146) - (4.153) are not fully compensated. Furthermore, the compensator (4.123) contains additional dangers.

4.6 Result and Discussion

Let us study the computational results from the covariant analysis in special cases:

(i) $p = 0 = q$:
This is the case of a scalar field π. The dangerous term from $\frac{\delta S}{\delta \pi}$ is (4.114), hence the compensator needed is (4.123). From $\frac{\delta S}{\delta g}$, the dangerous term is from (4.146), hence (4.191). This is compensated by the metric field equation (4.192) from (4.123), noting that $\boldsymbol{\nabla}\pi = \mathbf{d}\pi$. This has been proven in [23], [77]. The proof is as follows, presented in the graded formalism.

The scalar Galileon is

$$S_{n+1}[\pi, g] = S_{\rm EH} - \frac{1}{(D-n)!}\int \mathrm{d}^D x \int d^D\theta\, d^D\chi\, \sqrt{-g}\, \boldsymbol{g}^{D-n}\, \pi\, (\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\pi)^n \;, \tag{4.200}$$

where the first term is the Einstein-Hilbert action. Varying with respect to the scalar field

π, the following higher-than-second-order terms are resulted:

$$\widetilde{\boldsymbol{\nabla}}\pi(\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\pi)^{n-2}\boldsymbol{\nabla}^2\widetilde{\boldsymbol{\nabla}}\pi\ , \quad \boldsymbol{\nabla}\pi(\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\pi)^{n-2}\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\pi\ , \quad \boldsymbol{\nabla}\pi\widetilde{\boldsymbol{\nabla}}\pi(\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\pi)^{n-3}\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}^2\widetilde{\boldsymbol{\nabla}}\pi\ . \tag{4.201}$$

The first two are actually safe since $\boldsymbol{\nabla}^2\widetilde{\boldsymbol{\nabla}}\pi$ and $\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\pi$ $(=\widetilde{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}\pi)$ correspond to a curvature tensor (second derivatives on g) multiplying a covariant scalar field (first derivative on π). For a Levi-Civita connection on a scalar, $\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\pi = \widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\pi$. However, the third resultant in (4.201) is indeed a higher-derivative danger in the field equation. The necessary compensation is implemented via the coupling to curvature,

$$\begin{aligned} S_{n+1,r}[\pi,g] &= \frac{1}{(D-n)!}\int \mathrm{d}^D x \int d^D\theta\, d^D\chi\, \sqrt{-g}\, \boldsymbol{g}^{D-n} \times \\ &\quad \times \boldsymbol{\nabla}\pi\, \widetilde{\boldsymbol{\nabla}}\pi\, (\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\pi)^{n-2r-1}(\nabla_i\pi\nabla^i\pi\, \boldsymbol{Riem})^r\ . \end{aligned} \tag{4.202}$$

Therefore, as proven in [23], the full expression of the compensating action is

$$S[\pi,g] = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathcal{C}_{(n+1,r)} S_{n+1,r}[\pi,g]\ , \tag{4.203}$$

with $S_{n+1,0} := S_{n+1}$ and coefficients given as

$$\mathcal{C}_{(n+1,r)} = \left(-\frac{1}{8}\right)^r \frac{(n-1)!}{(n-1-2r)!(r!)^2}\ , \tag{4.204}$$

where $\mathcal{C}_{(n+1,0)} = 1$.[16] When (4.203) is added to the original action (4.200), we will have in overall only up to second order field equations for both π and g.

(ii) $p = 0, q \neq 0$:
This is the case of p-form[17]. The compensator is (4.123) for field equations (4.114) and (4.115). The dangerous terms from metric variation are (4.191) and

$$\frac{1}{2}q(-1)^{p+q}\,\delta\boldsymbol{g}\,\partial^\ell \mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}}\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \widetilde{\mathbf{d}}\boldsymbol{T}_{\widetilde{\ell}}\ , \tag{4.205}$$

which is extracted from (4.147). As mentioned in case (i), (4.191) is compensated with (4.192), while (4.205) is cancelled by a same term from (4.123). For $q = 2$, it reduces to the 2-form case proven in [78], noting that $\boldsymbol{\nabla B} = \mathbf{d}\boldsymbol{B}$, assuming a Levi-Civita connection.

The covariant action for a 2-form at 7 dimensions is

$$S_4[B^{(0,2)},g] = \int \mathrm{d}^7x \int d^7\theta\, d^7\chi\, \sqrt{-g}\ \boldsymbol{\nabla B}\ \widetilde{\boldsymbol{\nabla}}\widetilde{\boldsymbol{B}}\ \boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\widetilde{\boldsymbol{B}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla B}\ . \tag{4.206}$$

[16] We have kept all the derivatives in the discussion as covariant derivatives,, although covariant derivative of a scalar is equal to partial derivative of the scalar. We will make precise between covariant and partial derivatives for the discussion on 2-form.

[17]The convention used when constructing the covariant action (4.95) has rendered such correspondence. Notion of p, q can be exchanged once the fields $\boldsymbol{T}, \widetilde{\boldsymbol{T}}$ are exchanged in the action.

Assuming a Levi-Civita connection implies $\boldsymbol{\nabla B} = \mathbf{d}\boldsymbol{B}$. The action becomes

$$S_4[B^{(0,2)}, g] = \int \mathrm{d}^7x \int d^7\theta\, d^7\chi \sqrt{-g}\; \mathbf{d}\boldsymbol{B}\; \widetilde{\mathbf{d}}\widetilde{\boldsymbol{B}}\; \boldsymbol{\nabla}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{B}}\; \widetilde{\boldsymbol{\nabla}}\mathbf{d}\boldsymbol{B}\;. \tag{4.207}$$

The only higher-than-second-order term resulting from the variation of the action (4.207) with respect to the 2-form field is

$$2\,\mathbf{d}\boldsymbol{B}\;\widetilde{\mathbf{d}}\widetilde{\boldsymbol{B}}\;\widetilde{\mathbf{d}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\mathbf{d}\boldsymbol{B}\;. \tag{4.208}$$

It contains four derivatives, thus third derivatives on g. The compensation to introduce to the action (4.207) is

$$S_{4,1}[B^{(0,2)}, g] = -\frac{9}{4}\int \mathrm{d}^7x \int d^7\theta\, d^7\chi \sqrt{-g}\; \boldsymbol{H}\;\widetilde{\boldsymbol{H}}\;\widetilde{\boldsymbol{H}}^{\ell}\boldsymbol{H}_{\ell}\;\boldsymbol{Riem}\;, \tag{4.209}$$

where $\boldsymbol{H}_\ell = H_{\ell i_1 i_2}\theta^{i_1}\theta^{i_2}$ and $\widetilde{\boldsymbol{H}}_\ell = H_{\ell j_1 j_2}\chi^{j_1}\chi^{j_2}$.[18] A similar computation assumed as previously. We can form a curvature out of the four derivatives and the curvature is being differentiated, multiplying $\mathbf{d}\boldsymbol{B}$.

(iii) $p \neq 0, q = 0$:
Let us take an example of $p = 2$ at 7 dimensions. This gives the action

$$S_4[B^{(2,0)}, g] = \int \mathrm{d}^7x \int d^7\theta\, d^7\chi \sqrt{-g}\; \boldsymbol{\nabla}\widetilde{\boldsymbol{B}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{B}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{B}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{B}}\;. \tag{4.210}$$

Even though it is technically a 2-form $B^{(2,0)}$, $\boldsymbol{\nabla}\widetilde{\boldsymbol{B}} \neq \mathbf{d}\widetilde{\boldsymbol{B}}$. This is due to our anti-commuting χs and θs. Compared to action (4.206), the fields $\boldsymbol{B}$, $\widetilde{\boldsymbol{B}}$ are swapped. The dangerous terms incurred can be read off from the previous sections by setting $q = 0$. This case poses certain degree of complication if one attempts to fix the dangers. We will proceed to the discussion of the most generic case later, to have a more general statement about the covariantization of mixed-symmetry Galileon.

(iv) $p = q\ \forall\ p, q \neq 0$:
Despite $\boldsymbol{T} = \widetilde{\boldsymbol{T}}$, the field equations (4.111) - (4.115) do not simplify much nor cancel each other. Covariantizing[19] (4.80), we have

$$S_{n+1}[T^{(p,p)}, g] = \frac{1}{(D-k)!}\int \mathrm{d}^Dx \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \boldsymbol{g}^{D-k}\boldsymbol{\nabla T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,(\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T})^{n-1} \tag{4.212}$$

with $k = (p+1)n + p$. Let us take a simple example of $n = 2$. The dangers from $\frac{\delta S}{\delta T}$ are

$$-2\,\boldsymbol{\nabla T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;,\qquad \widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}^2\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;. \tag{4.213}$$

[18] A normalized totally anti-symmetric 3-form, $\boldsymbol{H} := \mathbf{d}\boldsymbol{B}$, or equivalently $\widetilde{\boldsymbol{H}} := \widetilde{\mathbf{d}}\widetilde{\boldsymbol{B}}$ is defined.

[19] An action with the double covariant derivatives in $\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}$ interchanged,

$$S_{n+1}[T^{(p,p)}, g] = \frac{1}{(D-k)!}\int \mathrm{d}^Dx \int \mathrm{d}^D\theta\, \mathrm{d}^D\chi\, \boldsymbol{g}^{D-k}\boldsymbol{\nabla T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\,(\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\boldsymbol{T})^{n-1}\;, \tag{4.211}$$

is just the same as (4.212) when $\chi \leftrightarrow \theta$.

We can compensate these with the field equation $\frac{\delta S}{\delta T}$ from the action

$$S_3[T^{(p,p)}, g] = -\frac{3}{4}p \int \mathrm{d}^D x \int \mathrm{d}^D\theta \, \mathrm{d}^D\chi \, \boldsymbol{g}^{D-k} \, \nabla^\ell \boldsymbol{T} \, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \, \widetilde{\boldsymbol{T}}_\ell \, \boldsymbol{Riem} \ , \tag{4.214}$$

where $k = (p+1)2 + p$. We also manage to cancel the unwanted third derivatives on T which result from $\frac{\delta S}{\delta g}$ of (4.212), by utilizing the compensator (4.214). However, due to a discrepancy of a factor 3 in the third derivatives on metric, which are resulting as well from $\frac{\delta S}{\delta g}$, these dangers fail to be fixed altogether at this stage. More explicitly, the variation of the action (4.212) with respect to the metric gives the following third derivatives,

$$\tfrac{1}{4}p^2 \, g^{\ell\ell'} \, \delta \boldsymbol{g}_{\ell'} (\mathbf{d}\widetilde{\mathbf{d}}\partial^m \boldsymbol{g}) \, \boldsymbol{T}_\ell \, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \, \widetilde{\boldsymbol{T}}_m \tag{4.215}$$

$$-\tfrac{1}{4}p^2 \, g^{mm'} \, \delta \boldsymbol{g}_{m'} \, (\widetilde{\mathbf{d}}\mathbf{d}\partial^\ell \boldsymbol{g}) \, \boldsymbol{T}_\ell \, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \, \widetilde{\boldsymbol{T}}_m \tag{4.216}$$

$$\tfrac{1}{2}p^2 \, g^{mm'} \, \delta \widetilde{\boldsymbol{g}}_{m'} \, (\widetilde{\mathbf{d}}\mathbf{d}\partial^\ell \boldsymbol{g}) \, \widetilde{\boldsymbol{T}}_m \, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \, \widetilde{\boldsymbol{T}}_\ell \tag{4.217}$$

$$-\tfrac{1}{4}p^2 \, g^{\ell\ell'} \, \delta \boldsymbol{g} \, (\widetilde{\mathbf{d}}\mathbf{d}\partial^m \boldsymbol{g}_{\ell'}) \, \boldsymbol{T}_\ell \, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \, \widetilde{\boldsymbol{T}}_m \tag{4.218}$$

$$\tfrac{1}{4}p^2 \, g^{mm'} \, \delta \boldsymbol{g} \, (\widetilde{\mathbf{d}}\mathbf{d}\partial^\ell \boldsymbol{g}_{m'}) \, \boldsymbol{T}_\ell \, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \, \widetilde{\boldsymbol{T}}_m \tag{4.219}$$

$$\tfrac{1}{2}p^2 \, g^{\ell\ell'} \, \delta \boldsymbol{g} \, (\widetilde{\mathbf{d}}\mathbf{d}\partial^m \widetilde{\boldsymbol{g}}_{\ell'}) \, \widetilde{\boldsymbol{T}}_m \, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \, \widetilde{\boldsymbol{T}}_\ell \ . \tag{4.220}$$

They can be compensated by the compensator (4.214). The discrepancy of a factor 3 means there are 2 additional copies of the dangers stated above which come from the compensator itself.

Let us try to look into the possible cycle of compensation involving only the metric part (thus not fully covariant at the moment). For one of the third-derivative

$$\frac{1}{2}p^2 \, g^{\ell\ell'} \, \delta \boldsymbol{g}_{\ell'} (\mathbf{d}\widetilde{\mathbf{d}}\partial^m \boldsymbol{g}) \, \boldsymbol{T}_\ell \, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \, \widetilde{\boldsymbol{T}}_m \ , \tag{4.221}$$

the following action (in its integrand)

$$\frac{1}{2}p^2 \, g^{\ell\ell'} \, \mathbf{d}\boldsymbol{g}_{\ell'} \, \widetilde{\mathbf{d}}\partial^m \boldsymbol{g} \, \boldsymbol{T}_\ell \, \widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \, \widetilde{\boldsymbol{T}}_m \tag{4.222}$$

can fix (4.221). However it gives a third-derivative in g

$$\frac{1}{4}p^3 (-1)^p \, g^{\ell\ell'} \, g^{kk'} \mathbf{d}\boldsymbol{g}_{\ell'} \, \delta \boldsymbol{g} \, \boldsymbol{T}_\ell \, (\widetilde{\mathbf{d}}\mathbf{d}\partial^m \widetilde{\boldsymbol{g}}_{k'}) \, \boldsymbol{T}_k \, \widetilde{\boldsymbol{T}}_m \ , \tag{4.223}$$

when the third g in (4.222) is varied and its derivatives $\widetilde{\mathbf{d}}\partial^m$ hit the metric part of $\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}$. We can fix (4.223) with an action in its integrand

$$\frac{1}{4}p^3 \, g^{\ell\ell'} \, g^{kk'} \mathbf{d}\boldsymbol{g}_{\ell'} \, \widetilde{\mathbf{d}}\boldsymbol{g} \, \boldsymbol{T}_\ell \, \mathbf{d}\partial^m \widetilde{\boldsymbol{g}}_{k'} \, \boldsymbol{T}_k \, \widetilde{\boldsymbol{T}}_m \ . \tag{4.224}$$

However, (4.224) gives an equation of motion

$$\frac{1}{4}p^3 (-1)^p \, g^{\ell\ell'} \, g^{kk'} \mathbf{d}\boldsymbol{g}_{\ell'} \, (\mathbf{d}\partial^m \widetilde{\mathbf{d}}\boldsymbol{g}) \, \boldsymbol{T}_\ell \, \delta \widetilde{\boldsymbol{g}}_{k'} \, \boldsymbol{T}_k \, \widetilde{\boldsymbol{T}}_m \ , \tag{4.225}$$

that contains a third-derivative. Recall that for every compensating action introduced, their field equations of g and T need to be checked for higher derivative terms. Naively but straightforwardly, if we try to fix (4.225) with a compensator (in its integrand)

$$-\frac{1}{4}p^3(-1)^p\, g^{\ell\ell'}\, g^{kk'}\mathbf{d}\boldsymbol{g}_{\ell'}\ \mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g}\ \boldsymbol{T}_\ell\ \partial^m\widetilde{\boldsymbol{g}}_{k'}\ \boldsymbol{T}_k\ \widetilde{\boldsymbol{T}}_m\ , \tag{4.226}$$

the unwanted (4.223) is as well returned in another field equation of this compensator. This cycle of inspection is played between a one-derivative and two-derivative g in the action terms. One ends up with (4.223) no matter how the derivatives attached to these relevant gs are distributed to make a *compensator*.

For case (i) and (ii), the higher derivatives resulting from the field and metric equations of motion both can be cancelled by the set of compensators found from compensating the dangerous equation of motion of the field. On the other hand, for case (iv), in order to cancel all the higher derivatives, more compensators will be required.

For the generic p, q case ($p \neq q \neq 0$), due to the massive number of dangerous terms resulting from $\frac{\delta S}{\delta g}$, which are mostly higher derivatives in the metric, let us work on a subset of the problematic terms, consisting the third derivatives in the mixed-symmetry field T and inspect whether they can be compensated. Without simplification or factorization among the terms, direct computation has shown 37 dangerous equations of motion ((A.37) - (A.73) in Appendix) for compensators S^1 (4.116), (4.130) - (4.137); 46 ((A.74) - (A.119) in Appendix) from compensators (4.121), (4.124) - (4.129); and 3 ((A.120) - (A.122) in Appendix) from compensators (4.122), (4.138), (4.139). Recall that third derivatives in T from (4.123) have been found to be fully compensated.
Since there is obviously no $\boldsymbol{Riem}$ term in the expressions from the action's side ((4.142) - (4.153)), we can safely exclude the checks on compensators that come with the explicit $\boldsymbol{Riem}$ term. The result is, from (4.148), there is

$$\frac{1}{2}p\,(-1)^p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \boldsymbol{\Gamma}^m\ \boldsymbol{T}_\ell\ g^{\ell\ell'}\,\delta\boldsymbol{g}_{\ell'}\ \partial_m\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{T}}\ , \tag{4.227}$$

and from (4.151) and (4.153) combined, there is

$$\frac{1}{2}pq\,(-1)^{p+q}\,\delta\boldsymbol{g}\ \partial^\ell\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{T}}\ \boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \boldsymbol{\nabla}\boldsymbol{T}_\ell\ , \tag{4.228}$$

which have no compensating partners.

Therefore, at this order, apart from the set of compensators we have introduced previously for any $T^{(p,q)}$, a larger number of compensators seem to be needed to cure the dangers resulting from the metric variation of the action $\frac{\delta S}{\delta g}$.

5 Quasi-Normal Modes

5.1 Gravitational Waves

Let us begin with a review on the linearised gravity in general relativity. From the Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, R \ , \tag{5.1}$$

we get the vaccum Einstein field equation,

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \ , \tag{5.2}$$

where the Einstein tensor, $G_{\mu\nu}$ is symmetric. Contracting (5.2) with metric $g^{\mu\nu}$ gives $R = 0$. Hence, the compact form of the vacuum field equation is $R_{\mu\nu} = 0$. Gravitational waves travel in empty space as wave-like solutions of Einstein's vacuum field equation $R_{\mu\nu} = 0$. Usually, gravitational waves are discussed based on linearised Einstein gravity around flat spacetime.

Consider the curved metric g linearised around constant Minkowski spacetime η,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{5.3}$$

assuming the perturbation $h_{\mu\nu}$ to be small ($h_{\mu\nu} \ll 1$). Thus, studying terms only linear in h is a sufficient approximation. For the Christoffel symbol (Levi-Civita connection coefficient),

$$\Gamma_{\mu\nu}{}^{\lambda} = \frac{1}{2} g^{\lambda\lambda'} (\partial_\mu g_{\nu\lambda'} + \partial_\nu g_{\mu\lambda'} - \partial_{\lambda'} g_{\mu\nu}) \ , \tag{5.4}$$

the linearised version is

$$\delta\Gamma_{\mu\nu}{}^{\lambda} = \frac{1}{2} \eta^{\lambda\lambda'} (\partial_\mu h_{\nu\lambda'} + \partial_\nu h_{\mu\lambda'} - \partial_{\lambda'} h_{\mu\nu}) = \frac{1}{2} (\partial_\mu h_\nu{}^{\lambda} + \partial_\nu h_\mu{}^{\lambda} - \partial^\lambda h_{\mu\nu}) \ , \tag{5.5}$$

noting that $\partial_\mu \eta^{\lambda\lambda'} = 0$. The linearised Riemann curvature is

$$\begin{aligned} \delta R^{\lambda}{}_{\kappa\mu\nu} &= \partial_\mu (\delta\Gamma_{\nu\kappa}{}^{\lambda}) - \partial_\nu (\delta\Gamma_{\mu\kappa}{}^{\lambda}) & (5.6) \\ &= \frac{1}{2} (\partial_\mu \partial_\kappa h_\nu{}^{\lambda} + \partial_\nu \partial^\lambda h_{\kappa\mu} - \partial_\mu \partial^\lambda h_{\kappa\nu} - \partial_\nu \partial_\kappa h_\mu{}^{\lambda}) & (5.7) \end{aligned}$$

and the linearised Ricci tensor is

$$\begin{aligned} \delta R_{\mu\nu} &= -\frac{1}{2} (\partial_\mu \partial_\lambda h_\nu{}^{\lambda} + \partial_\nu \partial_\lambda h_\mu{}^{\lambda} - \partial_\lambda \partial^\lambda h_{\mu\nu} - \partial_\mu \partial_\nu h) & (5.8) \\ &= -\frac{1}{2} (-\partial_\lambda \partial^\lambda h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu) \ , & (5.9) \end{aligned}$$

where $h = h_\mu{}^{\mu}$ and $V_\mu := \partial_\lambda h_\mu{}^{\lambda} - \frac{1}{2} \partial_\mu h$. Therefore, the compact form of linearised vacuum field equation, $\delta R_{\mu\nu} = 0$, is given by

$$\partial^\lambda \partial_\lambda h_{\mu\nu} = \Box h_{\mu\nu} = 0 \ , \tag{5.10}$$

under harmonic gauge[20] $V_\mu = 0$ [84]. By (5.8), the linearised Ricci scalar curvature is found to be

$$\begin{aligned} \delta R &= -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\partial_\lambda h_\nu{}^\lambda + \partial_\nu\partial_\lambda h_\mu{}^\lambda - \partial_\lambda\partial^\lambda h_{\mu\nu} - \partial_\mu\partial_\nu h) &(5.11)\\ &= \Box h - \partial^\mu\partial^\nu h_{\mu\nu}\ . &(5.12)\end{aligned}$$

The linearised vacuum field equation becomes

$$\begin{aligned} 0 &= \delta R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,\delta R &(5.13)\\ &= \Box \mathsf{h}_{\mu\nu}\ , &(5.14)\end{aligned}$$

where redefinition $\mathsf{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ has been used, and with the condition that $\partial^\mu \mathsf{h}_{\mu\nu} = 0$. This condition is knowns as Hilbert gauge [85]. The solution for the wave equation (5.14) is plane wave solution, $\mathsf{h}_{\mu\nu} = A_{\mu\nu}e^{ik_\rho x^\rho}$ with symmetric amplitude $A_{\mu\nu}$ and $k_\rho x^\rho = k_i x^i - \omega t$ with frequency ω. In this case, the wave vector k_ρ is null, that is, the plane wave solution holds only if the wave vector is light-like with respect to Minkowski metric, $\eta^{\mu\nu}k_\mu k_\nu = k^\nu k_\nu = 0$. Therefore, a gravitational wave propagates on Minkowski spacetime with the speed of light c.

For the linearised Einstein field equation with energy-momentum tensor $T_{\mu\nu}$, we have

$$\Box \mathsf{h}_{\mu\nu} = 2\kappa T_{\mu\nu}\ , \tag{5.15}$$

where $\kappa = \frac{8\pi G}{c^4}$ with G the Newton's gravitational constant[21]. Note that the energy-momentum tensor does not comprise gravitational wave [85]. Gravitational wave is encoded in the geometry of spacetime, namely in the Christoffel symbols. Known in astrophysics, stars end their lives by supernova explosions, leaving behind a compact remnant, which can be a black hole or a neutron star, oscillating in first few seconds. Loss of energy from this source is carried away in gravitational radiation, thus the oscillations will damp out. In the far-field approximation, gravitational wave field is determined by the quadrupole moment Q^{kl} of the source, which in specific is the energy density T^{00},

$$Q^{kl}(t) = \int d^3\vec{r}\, T^{00}(x^0 = ct, \vec{r})\, x^k x^l\ , \tag{5.16}$$

where $\vec{r} = (x^1, x^2, x^3)$, $r = |\vec{r}|$ is the distance taken from the origin of the source $T_{\mu\nu} \neq 0$. The linearised field is

$$\mathsf{h}^{kl}(ct, \vec{r}) = \frac{\kappa}{4\pi r c^2}\frac{d^2Q^{kl}}{dt^2}\left(t - \frac{r}{c}\right)\ . \tag{5.17}$$

It can be viewed as a superposition of the plane wave solution propagating in radial direction.

Linearisation can be done similarly around a curved background metric, $g^{(0)}$ as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}\ , \tag{5.18}$$

[20] The harmonic gauge is also called Hilbert gauge.

[21] The metric convention is $\eta_{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, +1)$.

assuming second and higher order in $h_{\mu\nu}$ be negligible. The linearised vacuum field equation is

$$0 = \delta R_{\mu\nu} = \nabla_\mu (\delta\Gamma_{\nu\rho}{}^{\rho}) - \nabla_\rho (\delta\Gamma_{\mu\nu}{}^{\rho}) \ . \tag{5.19}$$

In this work, we have mainly worked on the background metric which is a spherically symmetric and static spacetime, with linear perturbations which are time-dependent (non-static) and not spherically symmetric.

5.2 Perturbations of Spherically Symmetric Black Holes

In the emission of gravitational waves, the frequencies of the oscillations are quasi-normal, that is, complex, in which the real part of the frequency is the actual frequency of the oscillation and the imaginary part represents damping [28]. Gravitational radiation from an oscillating black hole exhibits characteristic frequencies, the quasi-normal frequencies, which are independent of the cause of its oscillation. Therefore, quasi-normal modes contain the fingerprint of the black hole, namely, the mass, charge and angular momentum that parametrize the black hole. The wave equation for the Reissner-Nordström black hole is similar to that of Schwarzschild [86] (its quasi-normal modes calculated for instance in [87]), while for Kerr (its quasi-normal modes first determined by [88]) and Kerr-Newman black holes (spectrum studied in restricted case in [89]), the wave equations are much more complicated to solve. The catch is to be able to separate the wave equation into radial and angular parts.

Let us focus on the case of the spherically symmetric and static black hole in some gravitational theory. The spherically symmetric background metric is

$$ds^2 = g^{(0)}_{\mu\nu} dx^\mu dx^\nu = -F(r)\, dt^2 + K(r)\, dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \ , \tag{5.20}$$

hence it makes up the perturbed metric (5.18). For Einstein's general relativity (GR), the solutions are, $F(r) = 1 - \frac{2M}{r}$ and $K(r) = \left(1 - \frac{2M}{r}\right)^{-1}$, where Newton's gravitational constant G and speed of light c have been set to unity. Subsequently, we have the perturbed Einstein tensor, $\delta G_{\mu\nu} = 0$. Decomposing $h_{\mu\nu}(t, r, \theta, \varphi)$ into tensor spherical harmonics $Y_{\ell m}$ and substituting them into the linearised vacuum field equation, that is $\delta G_{\mu\nu} = 0$, the perturbation problem can be reduced to a single wave equation, using

$$\psi(t, r, \theta, \varphi) = \sum_{\ell, m} \frac{\psi_{\ell m}(r, t)}{r} Y_{\ell m}(\theta, \varphi) \ , \tag{5.21}$$

which is a product of radial and angular components [28]. The function $\psi_{\ell m}(r, t)$ is a combination of the components of $h_{\mu\nu}$. There are two decoupled families of perturbations, one is called axial perturbation where its spherical harmonic with ℓ transforms like $(-1)^{\ell+1}$, and another type is called polar perturbation which transforms like $(-1)^{\ell}$, when $\theta \to \pi - \theta$ and $\varphi \to \pi + \varphi$. Subscript m is not important here as we are dealing with a spherically symmetric spacetime, thereby it can be omitted. The radial part of ψ_ℓ satisfies the wave equation

$$\frac{\partial^2}{\partial t^2}\psi_\ell + \left(-\frac{\partial^2}{\partial r_*^2} + V_\ell(r)\right)\psi_\ell = 0 \ , \tag{5.22}$$

where tortoise coordinate

$$r_* = r + 2M \log\left(\frac{r}{2M} - 1\right) \tag{5.23}$$

for a black hole of mass M, with the effective potential barrier known as Regge-Wheeler potential [90]

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right) \tag{5.24}$$

for the axial perturbation; for the polar perturbation, it is with the effective potential

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{2n^2(n+1)r^3 + 6n^2Mr^2 + 18nM^2r + 18M^3}{r^3(nr+3M)^2}\right) \tag{5.25}$$

known as Zerilli potential [91], where $2n = (\ell-1)(\ell+2)$. By using an ansatz specific to quasi-normal modes,

$$\psi_\ell(t, r_*) = \int d\omega\, e^{-i\omega t}\psi_\ell(\omega, r_*) \ , \tag{5.26}$$

we can obtain a time-independent ordinary differential equation, the Regge-Wheeler equation

$$\frac{d^2}{dr_*^2}\psi_\ell + (\omega^2 - V_\ell(r))\psi_\ell = 0 \tag{5.27}$$

with effective potential (5.24). The same form of equation applies for Zerilli potential for polar perturbation. The potential $V_\ell(r_*)$ decays exponentially near the horizon ($r_* \to -\infty$) and decays as r_*^{-2} as $r_* \to +\infty$. There are a few important remarks, despite of (5.27) showing similar form of a Schrödinger equation: the frequency ω is quadratic in this wave equation and r_* ranges from $-\infty$ to ∞, physical boundary conditions are to be imposed on ψ_ℓ. Monopole perturbations ($\ell = 0$) and dipole perturbations ($\ell = 1$) do not describe gravitational waves in general relativity, hence usually we study quasi-normal modes starting from $\ell = 2$ mode.

There are two intrinsic characteristics of the quasi-normal modes (QNMs). First, they are the solutions of the time-independent sourceless wave equation with complex frequency $\omega = \omega_R + i\omega_I$, where ω_R gives the frequency of the mode and $\omega_I \in \mathbb{R}$, that satisfies the physical boundary conditions: purely outgoing waves at infinity and purely ingoing waves at the horizon [92], [93], [94]. Due to (5.26), the proper behaviour for the radial part of the wave ψ_ℓ is: ψ_ℓ goes as $e^{-i\omega r_*}$ as $r \sim r_h$ (ingoing at horizon), and as $e^{i\omega r_*}$ when $r \to \infty$ (outgoing at infinity) for the QNMs, where r_h is the radius of horizon. Second, indication of (in)stability of the QNMs can be judged from the factor $e^{-i\omega t} = e^{-i\omega_R t}e^{\omega_I t}$ in the decomposition (5.26). The imaginary part, ω_I signifies exponential damping if it is negative, that is, when

$$\omega_I = -\frac{1}{\tau} \ (< 0) \ , \tag{5.28}$$

where τ is the damping time. Plug this into the factor $e^{-i\omega t}$, we get $e^{-\frac{t}{\tau}}$. Perturbations decay exponentially with time. Therefore, a negative ω_I represents a stable mode. Vice versa, a positive ω_I will correspond to an unstable mode. As it happens with Schwarzschild black hole, literature [95] has shown that it is (mode) stable against linear perturbations, as ω_I's found are strictly negative, given in the following table,

n	$\ell = 2$		$\ell = 3$		$\ell = 4$	
0	0.37367	$-0.08896i$	0.59944	$-0.09270i$	0.80918	$-0.09416i$
1	0.34671	$-0.27391i$	0.58264	$-0.28130i$	0.79663	$-0.28443i$
2	0.30105	$-0.47828i$	0.55168	$-0.47909i$	0.77271	$-0.47991i$
3	0.25150	$-0.70514i$	0.51196	$-0.69034i$	0.73984	$-0.68392i$

This table shows the first four QNM frequencies (ωM) of the Schwarzschild black hole in natural units, multiplication by $2\pi \times 5142\, Hz \times M_\odot/M$ gives conversion into Hz, where $M_\odot$ is solar mass. For example, when $M = 1M_\odot$, ω is roughly $10\, kHz + i(1\, ms)$. For GR, $\omega_{\text{axial}} = \omega_{\text{polar}}$. QNM frequencies from axial and polar perturbations are approximately the same in general. The number of modes n is infinite for each ℓ [96], [97], [98], [99]. The real part of the frequency is highest at the fundamental mode $n = 0$. Typically, the real part is constant with increasing n. Meanwhile, the imaginary part is more sensitive, increasing proportionally (rapidly) with n, thus is much quickly damped. Numerical simulations have shown that fundamental modes with $\ell = 2, 3$ are easier to excite by most astrophysical relevant perturbations such as the black hole mergers.

The leading order of the decay is typically determined by the slowest damping, found at the fundamental mode $n = 0$. Equivalently, the late-time dynamic of the black hole perturbation is controlled by the fundamental mode, which is with the smallest imaginary component ω_I, that is, having the largest decay time τ [100].

Roughly speaking, from the quasi-normal mode found numerically, we can infer its damping time as in (5.28). Together with the detected frequency, parameters of the black hole can be estimated. In practise, this certainly depends on the signal to noise ratio.

5.3 Framework within Einstein-Gauss-Bonnet-dilation Theory of Gravity

In this work, we consider an alternative theory of gravity, which is a modification to the Einstein-Hilbert action with higher curvature terms and a scalar field. The modification with curvature tensors of higher order can be seen as a stringy correction. The Einstein-Gauss-Bonnet-dilaton (EGBd) gravity action is given by [101]

$$S_{\text{EGBd}}(g, \phi) = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi + \frac{1}{4} \alpha e^{\phi} R^2_{\text{GB}} \right) , \tag{5.29}$$

where ϕ is the dilaton field (scalar) and α is a coupling parameter. Dilaton field is present in the bosonic sector of the superstring theories, and is in the low-energy limit of the superstring actions, that is, the supergravity actions. The Gauss-Bonnet invariant is

$$R^2_{\rm GB} = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \ . \tag{5.30}$$

The Gauss-Bonnet term is topological here in four dimensions, thus it has no influence on the local dynamics. However, in the EGBd action (5.29), since it is coupled to the dilaton, it is no longer a topological term. The first term in the Gauss-Bonnet invariant is also known as the Kretschmann invariant. The two field equations from this theory are [102]

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\,\partial^\rho\phi\right) - \frac{1}{4}\alpha e^{\phi}\left(H_{\mu\nu} + 4(\partial^\rho\phi\,\partial^\sigma\phi + \partial^\rho\partial^\sigma\phi)P_{\mu\rho\nu\sigma}\right) \ , \tag{5.31}$$

where Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$,

$$H_{\mu\nu} = 2(RR_{\mu\nu} - 2R_{\mu\rho}R^\rho{}_\nu - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} + R_{\mu\rho\sigma\lambda}R_\nu{}^{\rho\sigma\lambda}) - \frac{1}{2}g_{\mu\nu}R^2_{\rm GB} \tag{5.32}$$

and

$$P_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + g_{\mu\sigma}R_{\rho\nu} - g_{\mu\rho}R_{\sigma\nu} + g_{\nu\rho}R_{\sigma\mu} - g_{\nu\sigma}R_{\rho\mu} + \frac{1}{2}g_{\mu\rho}g_{\sigma\nu}R - \frac{1}{2}g_{\mu\sigma}g_{\rho\nu}R \ , \tag{5.33}$$

and

$$\nabla^2\phi = \frac{1}{4}\alpha e^{\phi}R^2_{\rm GB} \ , \tag{5.34}$$

which is the dilaton field equation. Note that all the field equations, for metric and dilaton are second order. As one sees, the modified Einstein tensor (5.31) is a lot more complex, as it involves the correction part to the standard Einstein's gravity, which shows on the RHS of the field equation. Setting coupling constant α and scalar field ϕ to zero gives the standard Einstein tensor that is zero on the RHS of (5.31). According to the analysis in [103], the dilatonic black hole solutions cannot exist above a critical value, implying $\alpha/M^2 \approx 0.691$ [27].

Let us now discuss the analytical procedures. They were done using Maple, a software for symbolic calculations. We consider a spherically symmetric background with metric

$$ds^2 = g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -F(r)\,dt^2 + K(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \ . \tag{5.35}$$

As discussed earlier, we begin by perturbing the metric

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \ , \tag{5.36}$$

In addition, dilaton field is as well perturbed,

$$\phi = \phi_0(r) + \delta\phi \ , \tag{5.37}$$

where $\phi_0(r)$ is the background scalar field. We are going to study the axial and polar perturbations, in which the polar case is much more complicated than axial's.

(i) The 10 bases of tensor spherical harmonics in the order of (t, φ, r, θ) needed are [104]

$$(a_0)_{\ell m} = \begin{pmatrix} Y_{\ell m}(\theta,\varphi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.38}$$

$$(a_1)_{\ell m} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & Y_{\ell m}(\theta,\varphi) & 0 \\ 0 & 0 & 0 & 0 \\ Y_{\ell m}(\theta,\varphi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.39}$$

$$a_{\ell m} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{\ell m}(\theta,\varphi) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.40}$$

$$(b_0)_{\ell m} = \frac{1}{\sqrt{2}} n(\ell) \begin{pmatrix} 0 & \frac{\partial}{\partial\varphi} Y_{\ell m}(\theta,\varphi) & 0 & \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\varphi) \\ \frac{\partial}{\partial\varphi} Y_{\ell m}(\theta,\varphi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\varphi) & 0 & 0 & 0 \end{pmatrix} \tag{5.41}$$

$$b_{\ell m} = \frac{1}{\sqrt{2}} n(\ell) r \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial\varphi} Y_{\ell m}(\theta,\varphi) & 0 \\ 0 & \frac{\partial}{\partial\varphi} Y_{\ell m}(\theta,\varphi) & 0 & \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\varphi) \\ 0 & 0 & \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\varphi) & 0 \end{pmatrix} \tag{5.42}$$

$$f_{\ell m} = \frac{1}{\sqrt{2}} m(\ell) r^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin^2\theta\, W_{\ell m}(\theta,\varphi) & 0 & X_{\ell m}(\theta,\varphi) \\ 0 & 0 & 0 & 0 \\ 0 & X_{\ell m}(\theta,\varphi) & 0 & W_{\ell m}(\theta,\varphi) \end{pmatrix} \tag{5.43}$$

$$g_{\ell m} = \frac{1}{\sqrt{2}} r^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sin^2\theta\, Y_{\ell m}(\theta,\varphi) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_{\ell m}(\theta,\varphi) \end{pmatrix} \tag{5.44}$$

$$(c_0)_{\ell m} = \frac{1}{\sqrt{2}} in(\ell) r \begin{pmatrix} 0 & -\sin\theta \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\varphi) & 0 & \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} Y_{\ell m}(\theta,\varphi) \\ -\sin\theta \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\varphi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} Y_{\ell m}(\theta,\varphi) & 0 & 0 & 0 \end{pmatrix} \tag{5.45}$$

$$c_{\ell m} = \frac{1}{\sqrt{2}}\, in(\ell) r \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin\theta \, \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\varphi) & 0 \\ 0 & -\sin\theta \, \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\varphi) & 0 & \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} Y_{\ell m}(\theta,\varphi) \\ 0 & 0 & \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} Y_{\ell m}(\theta,\varphi) & 0 \end{pmatrix} \tag{5.46}$$

$$d_{\ell m} = \frac{1}{\sqrt{2}}\, im(\ell) r^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin\theta \, X_{\ell m}(\theta,\varphi) & 0 & -\sin\theta \, W_{\ell m}(\theta,\varphi) \\ 0 & 0 & 0 & 0 \\ 0 & -\sin\theta \, W_{\ell m}(\theta,\varphi) & 0 & \frac{1}{\sin\theta} X_{\ell m}(\theta,\varphi) \end{pmatrix} , \tag{5.47}$$

which are 4×4 matrices, where

$$X_{\ell m}(\theta,\varphi) = 2\left(\frac{\partial^2}{\partial\theta\,\partial\varphi} Y_{\ell m}(\theta,\varphi)\right) - 2\cot\theta\left(\frac{\partial}{\partial\varphi} Y_{\ell m}(\theta,\varphi)\right) \tag{5.48}$$

and

$$W_{\ell m}(\theta,\varphi) = \frac{\partial^2}{\partial\theta^2} Y_{\ell m}(\theta,\varphi) - \frac{\cos\theta \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\varphi)}{\sin\theta} - \frac{\frac{\partial}{\partial\varphi^2} Y_{\ell m}(\theta,\varphi)}{\sin^2\theta} . \tag{5.49}$$

The complete axial perturbation can be written as

$$(h_{\ell m}(t,r,\theta,\varphi))_{\text{axial}} = (c_0)_{\ell m}(Q_0)_{\ell m}(t,r) + c_{\ell m} Q_{\ell m}(t,r) + d_{\ell m} D_{\ell m}(t,r) , \tag{5.50}$$

while for polar perturbation,

$$\begin{aligned} (h_{\ell m}(t,r,\theta,\varphi))_{\text{polar}} &= (a_0)_{\ell m}(A_0)_{\ell m}(t,r) + (a_1)_{\ell m}(A_1)_{\ell m}(t,r) + a_{\ell m} A_{\ell m}(t,r) && (5.51) \\ &\quad + (b_0)_{\ell m}(B_0)_{\ell m}(t,r) + b_{\ell m} B_{\ell m}(t,r) && (5.52) \\ &\quad + g_{\ell m} G_{\ell m}(t,r) + f_{\ell m} F_{\ell m}(t,r) . && (5.53) \end{aligned}$$

The coefficients are defined as

$$\begin{aligned} (Q_0)_{\ell m}(t,r) &= \frac{i\sqrt{2}\,((h_0)_{\ell m}(t,r))_{\text{axial}}}{n(\ell) r} && (5.54) \\ Q_{\ell m}(t,r) &= \frac{i\sqrt{2}\,((h_1)_{\ell m}(t,r))_{\text{axial}}}{n(\ell) r} && (5.55) \\ D_{\ell m}(t,r) &= -\frac{i((h_2)_{\ell m}(t,r))_{\text{axial}}}{\sqrt{2}\, m(\ell) r^2} && (5.56) \\ (A_0)_{\ell m}(t,r) &= 2\, N_{\ell m}(t,r) F(r) && (5.57) \\ (A_1)_{\ell m}(t,r) &= -\sqrt{2}\,(H_1)_{\ell m}(t,r) && (5.58) \\ A_{\ell m}(t,r) &= -2\, L_{\ell m}(t,r) K(r) && (5.59) \\ (B_0)_{\ell m}(t,r) &= -\frac{\sqrt{2}\,(h_0)_{\ell m}(t,r)}{n(\ell) r} && (5.60) \\ B_{\ell m}(t,r) &= \frac{\sqrt{2}\,(h_1)_{\ell m}(t,r)}{n(\ell) r} && (5.61) \\ F_{\ell m}(t,r) &= -\frac{\sqrt{2}\, V_{\ell m}(t,r)}{m(\ell)} && (5.62) \\ G_{\ell m}(t,r) &= \sqrt{2}\,\left(\ell(\ell+1) V_{\ell m}(t,r) - 2\, T_{\ell m}(t,r)\right) . && (5.63) \end{aligned}$$

Functions $F(r)$ and $K(r)$ are of the g_{tt} and g_{rr} components in (5.35). With all these coefficients, the axial perturbation (5.50) and polar perturbation (5.51) - (5.53) are better simplified.

(ii) Next, we impose gauge-fixing by $((h_2)_{\ell m}(t,r))_{\text{axial}} = 0$ for axial perturbation. For polar perturbation, the gauge $V_{\ell m}(t,r) = 0$, $(h_0)_{\ell m}(t,r) = 0$ and $(h_1)_{\ell m}(t,r) = 0$ is used, which is numerically more stable.

(iii) We have used Maple to construct the perturbations of the metric (5.50) - (5.53), decomposed in temporal and angular parts. The metric (5.36) used to lower and raise indices has now become a complicated mixture of functions and spherical harmonics

(iv) We run the simplifying routine to calculate the corresponding Christoffel symbol, Riemann curvature, Ricci tensor and Ricci scalar, using metric (5.36). The highly complex vacuum field equation, namely the full Einstein tensor $G_{\mu\nu}$ (LHS of (5.31)) which is a sum of static and perturbed parts, is better handled with the use of a computer software like Maple. Note that $H_{\mu\nu} = 0$, after calculations.

(v) On the other hand, dilaton perturbation is only relevant to the polar perturbation (denoted as "pol" in the subscript). From (5.37), we have

$$\phi = \phi_0(r) + (\delta\phi_1)_{\text{pol}}(t,r)Y_{\ell m}(\theta,\varphi) \ . \tag{5.64}$$

Useful basic identities for the simplification routines of the equations are

$$-\frac{\cos\theta\,\frac{\partial}{\partial\theta}Y_{\ell m}(\theta,\varphi)}{\sin\theta} - \frac{\partial^2}{\partial\theta^2}Y_{\ell m}(\theta,\varphi) - \frac{\frac{\partial}{\partial\phi^2}Y_{\ell m}(\theta,\varphi)}{\sin^2\theta} = \ell(\ell+1)Y_{\ell m}(\theta,\varphi) \tag{5.65}$$

$$\frac{\partial^2}{\partial\phi^2}Y_{\ell m}(\theta,\varphi) = -m^2\,Y_{\ell m}(\theta,\varphi) \tag{5.66}$$

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}Y_{\ell m}(\theta,\varphi) = -\frac{1}{\sin^2\theta}(&\ell^2\sin^2\theta\,Y_{\ell m}(\theta,\varphi) + \ell\sin^2\theta\,Y_{\ell m}(\theta,\varphi)\\ &+\cos\theta\sin\theta\frac{\partial}{\partial\theta}Y_{\ell m}(\theta,\varphi) + \frac{\partial^2}{\partial\phi^2}Y_{\ell m}(\theta,\varphi))\end{aligned} \tag{5.67}$$

$$\begin{aligned}\frac{\partial^2}{\partial\varphi^2}Y_{\ell m}(\theta,\varphi) = &-\ell^2\sin^2\theta\,Y_{\ell m}(\theta,\varphi) - \ell\sin^2\theta\,Y_{\ell m}(\theta,\varphi)\\ &-\cos\theta\sin\theta\frac{\partial}{\partial\theta}Y_{\ell m}(\theta,\varphi) - \sin^2\theta\frac{\partial^2}{\partial\theta^2}Y_{\ell m}(\theta,\varphi) \ .\end{aligned} \tag{5.68}$$

(vi) Extract all the components from the two perturbed field equations, that is, the modified Einstein tensor and dilaton equation, separately for axial and polar perturbations. Simplify and manipulate among them.

(vii) Find the independent components and thus obtain the most minimal set of ordinary differential equations (ODEs), after letting $h_{\ell m}(t,r) = e^{-i\omega t}h_{\ell m}(r)$, and the same decomposition for all the functions of (t,r). The results are cross-checked with the Einstein's Schwarzschild case with its known values of $F(r)$ and $K(r)$, with coupling α and dilaton field ϕ set to zero.

(viii) A single ODE as (5.27) can be obtained by substitution and reparametrization. The expression is very complex. However this is not of our particular interest.

The subsequent step after having the minimal set of equations is analyzing the equations with the proper boundary conditions as mentioned in Sec. (5.2). From $e^{-i\omega r_*}$ for the radial part of the ingoing wave, concerning the imaginary part of the frequency, if $\omega_I < 0$ (which corresponds to a stable mode), we see that $e^{\omega_I r_*} \to \infty$ when $r_* \to -\infty$ (towards the horizon). At the infinity, where $r_* \to \infty$, $e^{-\omega_I r_*} \to \infty$ as well. Therefore, for a stable QNM, the radial part of the wave diverges at both horizon and infinity. At the spatial infinity, dilaton field vanishes and spacetime is flat. Boundary conditions are analyzed from both limits, at the horizon and at the infinity.

Two solutions are generated. One to satisfy the boundary conditions at the horizon and another solution to satisfy the boundary conditions at the the infinity. The two solutions are studied at some intermediate point, typically at about $r \sim 4r_h$ [100]. These solutions are only matched at this intermediate point, if *the* ω is a quasi-normal mode, which means the function and its derivative are continuous. This is the method in searching for quasi-normal modes.

The numerical technique used is "Integration of the Time Independent Wave Equation" technique [28]. For the data of this work, see reference [46] in [100], which was performed using package COLSYS [105].

5.4 Discussion and Conclusion

As discussed in Sec. (5.2), the Schwarzschild spacetime is linearly mode stable. Its fundamental $\ell = 2$ QNM mode has the frequency $M\omega_{(S)} = M\omega_{R(S)} + iM\omega_{I(S)}$, where

$$M\omega_{(S)} \approx 0.3737 - 0.08896i \; , \tag{5.69}$$

as in the Table in Sec. (5.2). On the other hand, the fundamental scalar mode has the frequency $M\omega_{(S)} \approx 0.4836 - 0.09676i$, where M is the black hole mass (reference [46] in [100]).

Let us discuss the results using the figures.

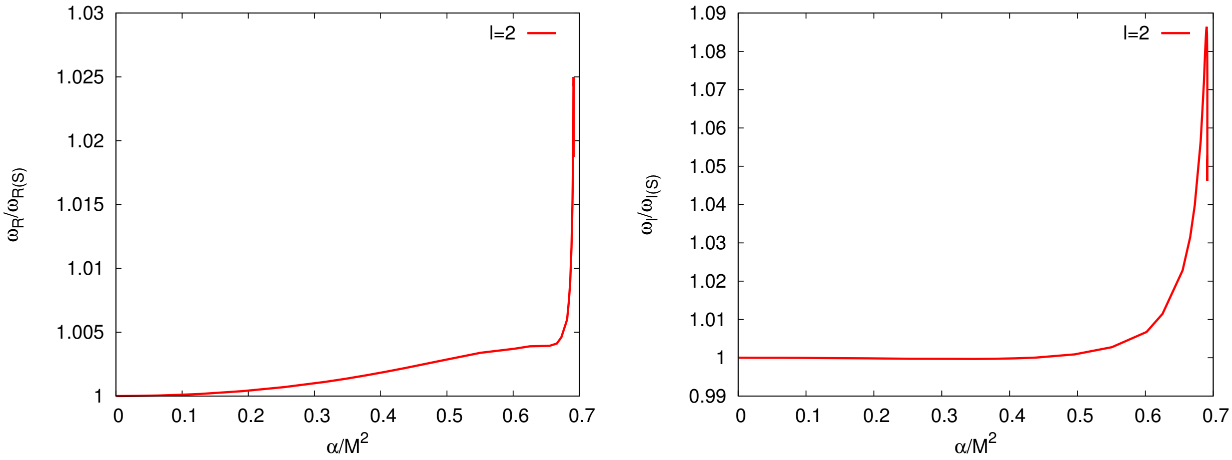

Figure 1: Real (left) and imaginary (right) part of the gravitational axial $\ell = 2$ fundamental mode, normalized by the Schwarzschild limit (5.69), as a function of the finite-coupling α/M^2.

In Fig. (1), the behaviour of the axial mode is smooth in most of the range of the coupling. Small deviations from the Schwarzschild values are observed. QNMs get very sensitive with the dependence on the coupling constant, as the critical value $\alpha/M^2 \approx 0.691$ is approached [100].

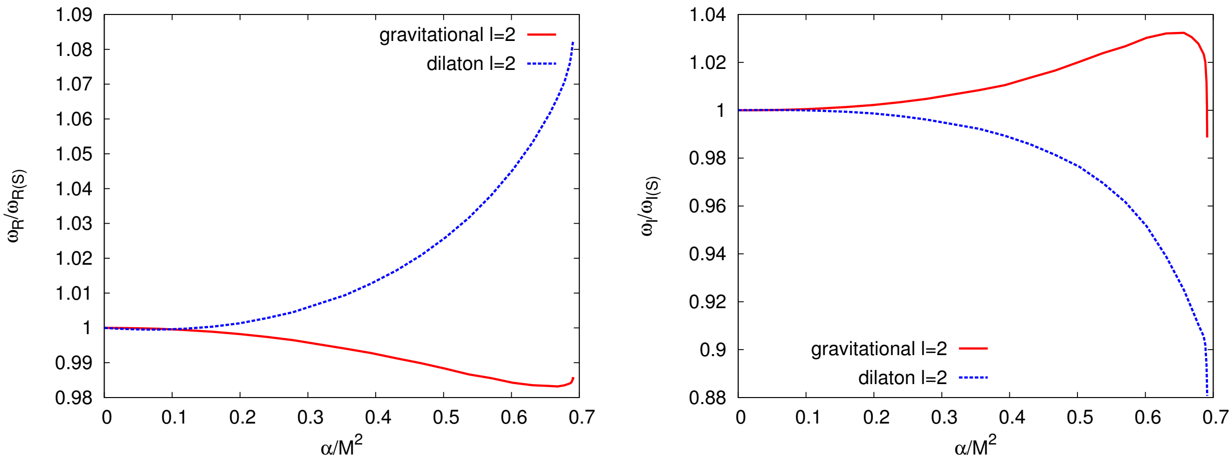

Figure 2: Real (left) and imaginary (right) part of the polar $\ell = 2$ fundamental mode, for the gravitational-led (red line) and scalar-led modes (blue dotted line), as a function of the coupling constant α/M^2, normalized by the Schwarzschild values.

For the polar case as portrayed in Fig. (2), deviation from general relativity is larger compared to the axial case. Since the dilaton is coupled to the polar perturbations, there are two family modes, one is gravitational and another one is a scalar driven mode. Setting α to zero, the gravitational mode is reduced to the QNM of Einstein's Schwarzschild black hole, while the scalar mode reduces to the QNM of a test scalar field on a Schwarzschild metric [100].

Deviations of the fundamental QNMs of an EGBd black hole from those of a GR's black hole are at most of a few percentage. Higher multipoles ($\ell \geq 2$) have been investigated. Most importantly, there are no unstable modes found in the domain of existence of the static EGBd black hole, which means that the black hole is linearly mode stable. The spectrum of the EGBd quasi-normal modes is discrete, as in the spectrum of GR (Table in Sec. (5.2)).

6 General Conclusion

In this thesis, we start our discussions with deformations of a mathematical structure in generalized geometry, a Courant algebroid in particular, with a non-$O(d,d)$-invariant, non-symmetric map $\mathcal{G}$, in a way that consistently preserve the Courant algebroid axioms. This map $\mathcal{G}$ contains a symmetric Riemannian metric and a 2-form Kalb-Ramond field, thus resulting in the structure of a torsionful metric connection which generalizes the Koszul formula. Using this connection, we compute its curvature and then the Ricci scalar, where indices are raised and lowered with the symmetric metric. On the other hand, there is a particular graded manifold which is known to correspond to a Courant algebroid. We performed similar deformations on this graded structure, carefully satisfying the identities. We found an additional piece of information — a Weitzenböck connection which is curvature-less. In another graded approach, we have also materialized explicit curvature term to contain in the graded structure. Given a connection, it is straightforward to compute to obtain the Ricci scalar and to write down an Einstein-Hilbert-like action. Our approach leads to closed string effective action without any stringy computations.

Another way to apply the mathematics we have learnt is to utilize the graded variables of graded geometry. Instead of doing tricks on the structures or axioms, we use the graded objects to help us define compact notations where we apply them to construct Galileon actions. We started with reformulation of the known Galileon actions in this compact formalism and then easily generalized these actions to coupled systems. Galileons (scalars and p-forms) in the curved spacetime are found to be coupled to curvature, to give only up to second order field equations, which means the theory is safe from ghosts. The Galileon theory can be viewed as a modified gravity theory, where the modifications are implemented through higher curvature terms.

In the last topic of the thesis, we look into another modified gravity theory — the Einstein-Gauss-Bonnet-dilaton theory, where gravity is modified with addition of curvature terms in second order. For this work, we shift our focus to the solutions beyond the theory. The physical motivation of studying the solutions is apparent, which in our case is the gravitational wave emission from the black hole solutions in this theory. Specifically, we are looking at the quasi-normal modes in the gravitational wave, where information of the source can be learned, and hence in our case justifying the existence of black holes in the universe. Furthermore, together with observations, such analysis is able to take us one step further towards the ultimate theory of gravity.

A Appendix

Here we state the list of higher derivative terms resulted from the computations, where simplifications may be possible.

From (4.146) - (4.153), for any generic p, q, the 25 equations of motion that contain dangerous terms are:

$$\tfrac{1}{4}p^2\,(-1)^p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\Gamma}^m\;\boldsymbol{T}_\ell\;g^{kk'}\;\delta\boldsymbol{g}_m\;(\partial^\ell\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{g}}_{k'})\;\widetilde{\boldsymbol{T}}_k \tag{A.1}$$
$$\tfrac{1}{4}p^2\,(-1)^p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\Gamma}^m\;\boldsymbol{T}_\ell\;g^{\ell\ell'}\;\delta\boldsymbol{g}_{\ell'}\;(\partial_m\widetilde{\mathbf{d}}\boldsymbol{\partial}^k\boldsymbol{g})\;\widetilde{\boldsymbol{T}}_k \tag{A.2}$$
$$\tfrac{1}{4}p^2\,(-1)^p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\Gamma}^m\;\boldsymbol{T}_\ell\;g^{\ell\ell'}\;\delta g_{m\ell'}\;(\partial^k\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{g})\;\widetilde{\boldsymbol{T}}_k \tag{A.3}$$
$$-\tfrac{1}{4}p^2\,(-1)^p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\Gamma}^m\;\boldsymbol{T}_\ell\;\delta\boldsymbol{g}_m\;(\partial^\ell\widetilde{\mathbf{d}}\partial^k\boldsymbol{g})\;\widetilde{\boldsymbol{T}}_k \tag{A.4}$$
$$\tfrac{1}{2}p\,(-1)^p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\Gamma}^m\;\boldsymbol{T}_\ell\;g^{\ell\ell'}\;\delta\boldsymbol{g}_{\ell'}\;(\partial_m\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{T}}) \tag{A.5}$$
$$\tfrac{1}{2}p\,(-1)^p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\Gamma}^m\;\boldsymbol{T}_\ell\;\delta\boldsymbol{g}_m\;(\partial^\ell\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{T}}) \tag{A.6}$$
$$\tfrac{1}{4}p^2q\,(-1)^{p+q}\,g^{mm'}\;\delta\boldsymbol{g}\;(\partial^\ell\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{g}}_{m'})\;\widetilde{\boldsymbol{T}}_m\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\nabla}\boldsymbol{T}_\ell \tag{A.7}$$
$$\tfrac{1}{4}p^2q\,(-1)^{p+q}\,g^{\ell\ell'}\;\delta\widetilde{\boldsymbol{g}}_{\ell'}\;(\partial^m\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{g})\;\widetilde{\boldsymbol{T}}_m\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\nabla}\boldsymbol{T}_\ell \tag{A.8}$$
$$\tfrac{1}{2}pq\,(-1)^{p+q}\,\delta\boldsymbol{g}\;(\partial^\ell\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{T}})\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\nabla}\boldsymbol{T}_\ell \tag{A.9}$$
$$\tfrac{1}{4}pq\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_\ell\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^\ell\partial^m\boldsymbol{g})\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}_{\widetilde{m}} \tag{A.10}$$
$$\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_\ell\;\boldsymbol{\Gamma}^m\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^\ell\partial_m\widetilde{\boldsymbol{g}}_{k'})\;g^{kk'}\;\widetilde{\boldsymbol{T}}_k \tag{A.11}$$
$$\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_\ell\;\boldsymbol{\Gamma}^m\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^\ell\widetilde{\mathbf{d}}g_{mk'})\;g^{kk'}\;\widetilde{\boldsymbol{T}}_k \tag{A.12}$$
$$-\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_\ell\;\boldsymbol{\Gamma}^m\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^\ell\partial^k\widetilde{\boldsymbol{g}}_m)\;\widetilde{\boldsymbol{T}}_k \tag{A.13}$$
$$\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_\ell\;\boldsymbol{\Gamma}^m\;\delta\boldsymbol{g}_{\ell'}\;(\mathbf{d}\partial^k\widetilde{\mathbf{d}}\widetilde{\boldsymbol{g}}_m)\;g^{\ell\ell'}\;\widetilde{\boldsymbol{T}}_k \tag{A.14}$$
$$-\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_\ell\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^\ell\widetilde{\mathbf{d}}\boldsymbol{g}_{m'})\;g^{m'\ell'}\;\widetilde{\boldsymbol{\nabla}}\widetilde{\boldsymbol{T}}_{\ell'} \tag{A.15}$$
$$-\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_\ell\;\delta\boldsymbol{g}_{\ell'}\;(\mathbf{d}\partial^k\widetilde{\mathbf{d}}\boldsymbol{g})\;g^{\ell\ell'}\;\widetilde{\boldsymbol{\nabla}}\widetilde{\boldsymbol{T}}_k \tag{A.16}$$
$$-\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_\ell\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^\ell\widetilde{\mathbf{d}}\partial^k\boldsymbol{g})\;\widetilde{\boldsymbol{T}}_k \tag{A.17}$$
$$-\tfrac{1}{4}p^2\,g^{mm'}\;\widetilde{\boldsymbol{T}}_\ell\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}\;(\partial^\ell\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g}_{m'})\;\boldsymbol{T}_m\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}} \tag{A.18}$$
$$-\tfrac{1}{4}p^2\,g^{\ell\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}_{\ell'}\;(\widetilde{\mathbf{d}}\mathbf{d}\partial^m\boldsymbol{g})\;\boldsymbol{T}_m\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}} \tag{A.19}$$
$$\tfrac{1}{2}p^2\,\widetilde{\boldsymbol{T}}_\ell\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}\;(\partial^\ell\mathbf{d}\partial^m\boldsymbol{g})\;\boldsymbol{T}_m\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}} \tag{A.20}$$
$$-\tfrac{1}{2}p\,\widetilde{\boldsymbol{T}}_\ell\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}\;(\partial^\ell\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}} \tag{A.21}$$
$$-\tfrac{1}{4}p^2\,g^{\ell\ell'}\;\widetilde{\boldsymbol{T}}_m\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}_{\ell'}\;(\partial^m\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g})\;\boldsymbol{T}_\ell\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}} \tag{A.22}$$
$$-\tfrac{1}{4}p^2\,g^{\ell\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}\;(\partial^m\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g}_{\ell'})\;\boldsymbol{T}_m\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{\nabla}\widetilde{\boldsymbol{T}} \tag{A.23}$$
$$\tfrac{1}{2}p^2\,(-1)^{p+q}\,\delta\boldsymbol{g}\;(\partial^\ell\widetilde{\mathbf{d}}\partial^m\boldsymbol{g})\;\widetilde{\boldsymbol{T}}_\ell\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_m\;\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \tag{A.24}$$
$$-\tfrac{1}{2}p\,(1+(-1)^{p+q})\,\widetilde{\boldsymbol{T}}_\ell\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}\;\partial^\ell\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{T}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;. \tag{A.25}$$

In addition to these are

$$\tfrac{1}{2}p^2\,g^{mm'}\,\delta\boldsymbol{g}\;(\partial^\ell\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{g}}_{m'})\;\widetilde{\boldsymbol{T}}_m\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \tag{A.26}$$
$$\tfrac{1}{2}p^2\,g^{\ell\ell'}\,\delta\widetilde{\boldsymbol{g}}_{\ell'}\;(\partial^m\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{g})\;\widetilde{\boldsymbol{T}}_m\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T} \tag{A.27}$$

for odd $(p+q)$ case.

The 9 remaining dangerous field equations from the compensator (4.123) are:

$$p\,\delta\boldsymbol{g}\;(\partial^m\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g})\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_m\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\nabla_\ell\boldsymbol{T} \tag{A.28}$$
$$-\tfrac{1}{4}q\,(-1)^p\,\nabla^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_{\widetilde{k}}\;\delta\widetilde{\boldsymbol{g}}_{k'}\;g^{kk'}\;(\partial_\ell\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g}) \tag{A.29}$$
$$\tfrac{1}{4}q\,(-1)^p\,\nabla^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_{\widetilde{k}}\;\delta\widetilde{\boldsymbol{g}}_\ell\;(\partial^k\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g}) \tag{A.30}$$
$$-\tfrac{1}{2}q^2\,(-1)^{p+q}\,\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\boldsymbol{g}_{k'})\;g^{kk'}\;\widetilde{\boldsymbol{T}}_{\widetilde{k}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}_{\widetilde{\ell}} \tag{A.31}$$
$$\tfrac{1}{2}q^2\,(-1)^{p+q}\,\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^k\boldsymbol{g}_{\ell'})\;g^{\ell\ell'}\;\widetilde{\boldsymbol{T}}_{\widetilde{k}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}_{\widetilde{\ell}} \tag{A.32}$$
$$\tfrac{1}{2}q\,(-1)^{p+q}\,\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial_\ell\boldsymbol{g}_{k'})\;g^{kk'}\;\widetilde{\boldsymbol{T}}_{\widetilde{k}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\nabla^\ell\boldsymbol{T} \tag{A.33}$$
$$-\tfrac{1}{2}q\,(-1)^{p+q}\,\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^k\boldsymbol{g}_\ell)\;g^{kk'}\;\widetilde{\boldsymbol{T}}_{\widetilde{k}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\nabla^\ell\boldsymbol{T} \tag{A.34}$$
$$\tfrac{1}{2}pq^2\,(-1)^{p+q}\,\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\boldsymbol{g})\;\widetilde{\boldsymbol{T}}_\ell\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}^{\widetilde{m}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}_{\widetilde{m}} \tag{A.35}$$
$$\tfrac{1}{2}pq^2\,(-1)^{p+q}\,g^{mm'}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}\;(\partial^k\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{g})\;\boldsymbol{T}_{k\widetilde{m}'}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}_{\widetilde{m}}\;. \tag{A.36}$$

The 37 dangerous field equations from compensators (4.116), (4.130) - (4.137), which contain third derivatives in the field $T^{(p,q)}$ for even $(p+q)$ are:

$$-\tfrac{1}{4}p^2\,(-1)^p\,\boldsymbol{T}_m\;\delta\boldsymbol{g}\;(\partial^m\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.37}$$
$$-\tfrac{1}{4}p^2\,\delta\boldsymbol{g}\;(\partial^m\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})\;\boldsymbol{T}_m\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.38}$$
$$-\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_m\;\delta\boldsymbol{g}_{m'}\;g^{mm'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.39}$$
$$\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{T}_m\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^m\partial^\ell\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.40}$$
$$\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g^{\ell\ell'}\;\delta\widetilde{\boldsymbol{g}}_{k'}\;g^{kk'}\;(\mathbf{d}\partial_{\ell'}\widetilde{\mathbf{d}}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_k\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.41}$$
$$-\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g^{\ell\ell'}\;\delta\widetilde{\boldsymbol{g}}_{\ell'}\;(\mathbf{d}\partial^k\widetilde{\mathbf{d}}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_k\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.42}$$
$$\tfrac{1}{4}pq\,(-1)^{p+q}\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g^{\ell\ell'}\;\delta\boldsymbol{g}_{h'}\;g^{hh'}\;(\mathbf{d}\partial_{\ell'}\widetilde{\mathbf{d}}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\widetilde{h}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.43}$$
$$-\tfrac{1}{4}pq\,(-1)^{p+q}\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g^{\ell\ell'}\;\delta\boldsymbol{g}_{\ell'}\;(\mathbf{d}\partial^h\widetilde{\mathbf{d}}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\widetilde{h}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.44}$$
$$p\,(-1)^q\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{T}}_\ell \tag{A.45}$$
$$\tfrac{1}{2}p\,(-1)^q\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\Gamma}^k\;g_{aa'}\chi^a\;\delta\boldsymbol{g}_m\;g^{ma'}\;(\partial_k\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_\ell \tag{A.46}$$
$$-\tfrac{1}{2}p\,(-1)^q\,\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g_{aa'}\chi^a\;\boldsymbol{\Gamma}^{a'}_k\;\delta\boldsymbol{g}\;(\partial^k\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_\ell \tag{A.47}$$
$$-\tfrac{1}{2}p^2\,\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^{\ell'}\boldsymbol{T})\;\boldsymbol{T}_{\ell'}\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.48}$$
$$-\tfrac{1}{2}p^2\,\nabla^{\ell'}\boldsymbol{T}\;\boldsymbol{T}_{\ell'}\;\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.49}$$
$$-\tfrac{1}{2}p\,\mathbf{d}\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g_{aa'}\chi^a\;\boldsymbol{\Gamma}^{a'}_k\;\delta\widetilde{\boldsymbol{g}}_{k'}\;g^{kk'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{T}}_\ell \tag{A.50}$$
$$\tfrac{1}{2}p\,\mathbf{d}\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g_{aa'}\chi^a\;\boldsymbol{\Gamma}^{a'}_k\;\delta\boldsymbol{g}\;(\partial^k\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{T}}_\ell \tag{A.51}$$
$$-\tfrac{1}{2}p\,\mathbf{d}\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\Gamma}^k\;g_{aa'}\chi^a\;\delta\boldsymbol{g}_m\;g^{ma'}\;(\partial_k\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{T}}_\ell \tag{A.52}$$
$$-\tfrac{1}{2}p\,\mathbf{d}\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\boldsymbol{\Gamma}^k\;g_{aa'}\chi^a\;\delta g_{km}\;g^{ma'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{T}}_\ell \tag{A.53}$$
$$\tfrac{1}{8}p^2\,(-1)^p\,\delta\widetilde{\boldsymbol{g}}_k\;g^{k\ell'}\;(\partial_m\mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}})\;\mathbf{d}\boldsymbol{T}\;g^{\ell m}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.54}$$
$$-\tfrac{1}{8}p^2\,(-1)^p\,\delta\boldsymbol{g}\;(\partial_m\partial^{\ell'}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}})\;\mathbf{d}\boldsymbol{T}\;g^{\ell m}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.55}$$

$$-\tfrac{1}{8}p^2\,(-1)^q\,\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}}\;\delta\boldsymbol{g}_m\;(\partial^{\ell'}\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;g^{\ell m}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.56}$$

$$\tfrac{1}{8}p^2\,(-1)^q\,\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}}\;\delta\boldsymbol{g}\;(\partial_m\partial^{\ell'}\mathbf{d}\boldsymbol{T})\;g^{\ell m}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.57}$$

$$\tfrac{1}{8}p^2\,\delta\boldsymbol{g}_k\;g^{k\ell'}\;(\partial_m\widetilde{\mathbf{d}}\mathbf{d}\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g^{\ell m}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.58}$$

$$-\tfrac{1}{8}p^2\,\delta\boldsymbol{g}\;(\partial_m\partial^{\ell'}\mathbf{d}\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;g^{\ell m}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.59}$$

$$-\tfrac{1}{8}p^2\,\mathbf{d}\widetilde{\boldsymbol{T}}\;\delta\widetilde{\boldsymbol{g}}_m\;(\partial^{\ell'}\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;g^{\ell m}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.60}$$

$$\tfrac{1}{8}p^2\,\mathbf{d}\widetilde{\boldsymbol{T}}\;\delta\boldsymbol{g}\;(\partial^{\ell'}\widetilde{\mathbf{d}}\partial_m\boldsymbol{T})\;g^{\ell m}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.61}$$

$$-\tfrac{1}{4}p\,g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_m\;g^{m\ell'}\;(\partial_k\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.62}$$

$$\tfrac{1}{4}p\,g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_k\;(\partial^{\ell'}\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.63}$$

$$-\tfrac{1}{4}pq\,g^{\ell k}\;\delta\boldsymbol{g}_m\;g^{m\ell'}\;(\partial_k\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\widetilde{\ell'}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.64}$$

$$\tfrac{1}{4}pq\,g^{\ell k}\;\delta\boldsymbol{g}_k\;(\partial^{\ell'}\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\widetilde{\ell'}}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.65}$$

$$-\tfrac{1}{4}pq\,g^{\ell k}\;\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}_m\;g^{m\ell'}\;(\widetilde{\mathbf{d}}\mathbf{d}\partial_k\widetilde{\boldsymbol{T}}_{\widetilde{\ell'}})\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.66}$$

$$\tfrac{1}{4}pq\,g^{\ell k}\;\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}_k\;(\widetilde{\mathbf{d}}\mathbf{d}\partial^{\ell'}\widetilde{\boldsymbol{T}}_{\widetilde{\ell'}})\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.67}$$

$$-\tfrac{1}{4}pq\,g^{\ell k}\;\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\widetilde{\ell'}}\;\delta\boldsymbol{g}_m\;g^{m\ell'}\;(\widetilde{\mathbf{d}}\mathbf{d}\partial_k\widetilde{\boldsymbol{T}}_\ell)\;\boldsymbol{Riem} \tag{A.68}$$

$$\tfrac{1}{4}pq\,g^{\ell k}\;\boldsymbol{T}\;\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\widetilde{\ell'}}\;\delta\boldsymbol{g}_k\;(\widetilde{\mathbf{d}}\mathbf{d}\partial^{\ell'}\widetilde{\boldsymbol{T}}_\ell)\;\boldsymbol{Riem} \tag{A.69}$$

$$-\tfrac{1}{4}p^2\,\delta\widetilde{\boldsymbol{g}}_k\;g^{k\ell'}g^{\ell m}\;(\partial_m\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})\;\mathbf{d}\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.70}$$

$$-\tfrac{1}{4}p^2\,\delta\boldsymbol{g}\;g^{\ell m}\;(\partial_m\partial^{\ell'}\widetilde{\mathbf{d}}\boldsymbol{T})\;\mathbf{d}\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.71}$$

$$-\tfrac{1}{4}p^2\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\delta\boldsymbol{g}_m\;g^{\ell m}\;(\partial^{\ell'}\mathbf{d}\widetilde{\mathbf{d}}\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem} \tag{A.72}$$

$$\tfrac{1}{4}p^2\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\delta\boldsymbol{g}\;g^{\ell m}\;(\partial^{\ell'}\mathbf{d}\partial_m\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_{\ell'}\;\widetilde{\boldsymbol{T}}_\ell\;\boldsymbol{Riem}\;. \tag{A.73}$$

From compensators (4.121), (4.124) - (4.129), the 46 field equations which have field $T^{(p,q)}$ in third derivatives for $(p+q = \text{even})$ are:

$$\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}_m\;\delta\boldsymbol{g}\;(\partial^m\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.74}$$

$$\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_m\;\delta\boldsymbol{g}_{m'}\;g^{mm'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.75}$$

$$-\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_m\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^m\partial^\ell\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.76}$$

$$-\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g^{\ell\ell'}\;\delta\widetilde{\boldsymbol{g}}_{k'}\;g^{kk'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial_{\ell'}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_k\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.77}$$

$$\tfrac{1}{4}p^2\,(-1)^q\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g^{\ell\ell'}\;\delta\widetilde{\boldsymbol{g}}_{\ell'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^k\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_k\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.78}$$

$$-\tfrac{1}{4}pq\,(-1)^p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g^{\ell\ell'}\;\delta\boldsymbol{g}_{h'}\;g^{hh'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial_{\ell'}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\widetilde{h}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.79}$$

$$\tfrac{1}{4}pq\,(-1)^p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g^{\ell\ell'}\;\delta\boldsymbol{g}_{\ell'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^h\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\widetilde{h}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.80}$$

$$-p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\;\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_\ell \tag{A.81}$$

$$\tfrac{1}{2}p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g_{aa'}\chi^a\;\boldsymbol{\Gamma}^{a'}_k\;\delta\boldsymbol{g}\;(\partial^k\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_\ell \tag{A.82}$$

$$-\tfrac{1}{2}p\,\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^k\;g_{aa'}\chi^a\;\delta\boldsymbol{g}_m\;g^{ma'}\;(\partial_k\mathbf{d}\widetilde{\mathbf{d}}\boldsymbol{T})\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_\ell \tag{A.83}$$

$$\tfrac{2}{4^2}p^2\,(-1)^p\,\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^{\ell'}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\nabla^\ell\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.84}$$

$$\tfrac{2}{4^2}p^2\,(-1)^p\,\nabla^{\ell'}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\ell'}\;\delta\boldsymbol{g}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^\ell\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.85}$$

$$\tfrac{1}{4}p^2\,(-1)^q\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_m\;\delta\boldsymbol{g}_{m'}\;g^{mm'}\;(\widetilde{\mathbf{d}}\mathbf{d}\partial^\ell\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.86}$$

$$-\tfrac{1}{4}p^2\,(-1)^q\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_m\;\delta\boldsymbol{g}\;(\partial^m\mathbf{d}\partial^\ell\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.87}$$

$$\tfrac{1}{2}p\,(-1)^p\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g_{aa'}\chi^a\;\boldsymbol{\Gamma}^{a'}_k\;\delta\boldsymbol{g}_{k'}\;g^{kk'}\;(\widetilde{\mathbf{d}}\mathbf{d}\partial^\ell\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_\ell \tag{A.88}$$

$$-\tfrac{1}{2}p\,(-1)^p\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g_{aa'}\chi^a\;\boldsymbol{\Gamma}^{a'}_k\;\delta\boldsymbol{g}\;(\partial^k\mathbf{d}\partial^\ell\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_\ell \tag{A.89}$$

$$\tfrac{1}{2}p\,(-1)^p\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^k\;g_{aa'}\chi^a\;\delta\boldsymbol{g}_m\;g^{ma'}\;(\partial_k\mathbf{d}\partial^\ell\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_\ell \tag{A.90}$$

$$-\tfrac{1}{2}p\,(-1)^p\,\widetilde{\mathbf{d}}\boldsymbol{T}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^k\;g_{aa'}\chi^a\;\delta\boldsymbol{g}_k\;(\partial^{a'}\mathbf{d}\partial^\ell\widetilde{\boldsymbol{T}})\;\boldsymbol{T}_\ell \tag{A.91}$$

$$\tfrac{1}{4}p^3\,(-1)^q\,g^{\ell m}\;\delta\boldsymbol{g}_k\;g^{k\ell'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial_m\widetilde{\boldsymbol{T}})\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.92}$$

$$-\tfrac{1}{4}p^3\,(-1)^q\,g^{\ell m}\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^{\ell'}\partial_m\widetilde{\boldsymbol{T}})\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.93}$$

$$\tfrac{1}{4}p^3\,(-1)^q\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g^{\ell m}\;\delta\boldsymbol{g}_k\;g^{k\ell'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial_m\boldsymbol{T}_{\ell'})\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.94}$$

$$-\tfrac{1}{4}p^3\,(-1)^q\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;g^{\ell m}\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^{\ell'}\partial_m\boldsymbol{T}_{\ell'})\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.95}$$

$$\tfrac{1}{4}p^3\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_{\ell'}\;g^{\ell m}\;\delta\boldsymbol{g}_k\;g^{k\ell'}(\mathbf{d}\widetilde{\mathbf{d}}\partial_m\boldsymbol{T}_\ell)\;\boldsymbol{Riem} \tag{A.96}$$

$$-\tfrac{1}{4}p^3\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\boldsymbol{T}_{\ell'}\;g^{\ell m}\;\delta\boldsymbol{g}\;(\mathbf{d}\partial^{\ell'}\partial_m\boldsymbol{T}_\ell)\;\boldsymbol{Riem} \tag{A.97}$$

$$\tfrac{1}{4}p^2\,(-1)^p\,g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_m\;g^{m\ell'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial_k\boldsymbol{T})\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.98}$$

$$-\tfrac{1}{4}p^2\,(-1)^p\,g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_k\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^{\ell'}\boldsymbol{T})\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.99}$$

$$\tfrac{1}{4}p^2\,(-1)^p\,\mathbf{d}\boldsymbol{T}\;g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_m\;g^{m\ell'}\;(\widetilde{\mathbf{d}}\mathbf{d}\partial_k\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.100}$$

$$-\tfrac{1}{4}p^2\,(-1)^p\,\mathbf{d}\boldsymbol{T}\;g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_k\;(\widetilde{\mathbf{d}}\mathbf{d}\partial^\ell\widetilde{\boldsymbol{T}})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.101}$$

$$-\tfrac{1}{4}p^2\,(-1)^q\,g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_m\;g^{m\ell'}\;(\widetilde{\mathbf{d}}\mathbf{d}\partial_k\widetilde{\boldsymbol{T}})\;\mathbf{d}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.102}$$

$$\tfrac{1}{4}p^2\,(-1)^q\,g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_k\;(\widetilde{\mathbf{d}}\mathbf{d}\partial^{\ell'}\widetilde{\boldsymbol{T}})\;\mathbf{d}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.103}$$

$$\tfrac{1}{4}p^2\,(-1)^p\,\widetilde{\boldsymbol{T}}\;\mathbf{d}\boldsymbol{T}\;g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_m\;g^{m\ell'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial_k\widetilde{\boldsymbol{T}}_{\ell'})\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.104}$$

$$-\tfrac{1}{4}p^2\,(-1)^p\,\widetilde{\boldsymbol{T}}\;\mathbf{d}\boldsymbol{T}\;g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_k\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^{\ell'}\widetilde{\boldsymbol{T}}_{\ell'})\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.105}$$

$$\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}\;\mathbf{d}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\ell'}\;g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_m\;g^{m\ell'}\;(\mathbf{d}\widetilde{\mathbf{d}}\partial_k\boldsymbol{T}_\ell)\;\boldsymbol{Riem} \tag{A.106}$$

$$-\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}\;\mathbf{d}\boldsymbol{T}\;\widetilde{\boldsymbol{T}}_{\ell'}\;g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_k\;(\mathbf{d}\widetilde{\mathbf{d}}\partial^{\ell'}\boldsymbol{T}_\ell)\;\boldsymbol{Riem} \tag{A.107}$$

$$\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^m\;g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_h\;g^{h\ell'}\;(\partial_m\partial_k\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.108}$$

$$\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^m\;g^{\ell k}\;\delta g_{kh}\;g^{h\ell'}\;(\partial_m\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.109}$$

$$-\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^m\;g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_k\;(\partial_m\partial^{\ell'}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.110}$$

$$-\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^m\;g^{\ell k}\;\delta g_{kh}\;g^{h\ell'}\;(\partial_m\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.111}$$

$$-\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^m\;\delta g_{mh}\;g^{h\ell'}\;(\widetilde{\mathbf{d}}\partial^\ell\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.112}$$

$$\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^m\;g^{\ell k}\;\delta g_{mk}\;(\partial^{\ell'}\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.113}$$

$$-\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^m_k\;g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_h\;g^{h\ell'}\;(\partial_m\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.114}$$

$$\tfrac{1}{4}p^2\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^m_k\;g^{\ell k}\;\delta\widetilde{\boldsymbol{g}}_m\;(\partial^{\ell'}\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.115}$$

$$-\tfrac{1}{2}p^2\,\widetilde{\boldsymbol{T}}\;\boldsymbol{\Gamma}^m_h\;g^{h\ell'}\;g_{mm'}\;\delta\widetilde{\boldsymbol{g}}_a\;g^{am'}\;(\partial^\ell\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.116}$$

$$\tfrac{1}{2}p^2\,\widetilde{\boldsymbol{T}}\;g^{\ell k}\;\boldsymbol{\Gamma}^m_h\;g^{h\ell'}\;\delta\widetilde{\boldsymbol{g}}_k\;(\partial_m\widetilde{\mathbf{d}}\mathbf{d}\boldsymbol{T})\;\widetilde{\boldsymbol{T}}_{\ell'}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.117}$$

$$-\tfrac{1}{4}pq\,(-1)^p\,\delta\boldsymbol{g}_m\;g^{m\ell'}\;(\widetilde{\mathbf{d}}\mathbf{d}\partial^\ell\boldsymbol{T})\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_{\widetilde{\ell'}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem} \tag{A.118}$$

$$\tfrac{1}{4}pq\,(-1)^p\,\delta\boldsymbol{g}_k\;g^{k\ell}\;(\widetilde{\mathbf{d}}\mathbf{d}\partial^{\ell'}\boldsymbol{T})\;\boldsymbol{\nabla}\widetilde{\boldsymbol{T}}\;\widetilde{\boldsymbol{T}}_{\widetilde{\ell'}}\;\boldsymbol{T}_\ell\;\boldsymbol{Riem}\;. \tag{A.119}$$

The 3 dangerous field equations from compensators (4.122), (4.138), (4.139) that contain third derivatives in the field $T^{(p,q)}$ for generic p, q are:

$$-\tfrac{1}{8}p^2\,(-1)^{p+q}\,\widetilde{\boldsymbol{T}}_{\ell'}\ \delta\boldsymbol{g}\ (\mathbf{d}\widetilde{\mathbf{d}}\partial^{\ell'}\boldsymbol{T})\ \nabla^{\ell}\widetilde{\boldsymbol{T}}\ \boldsymbol{T}_{\ell}\ \boldsymbol{Riem} \tag{A.120}$$

$$-\tfrac{1}{8}p^2\,(-1)^{p}\,\widetilde{\boldsymbol{T}}_{\ell'}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \delta\boldsymbol{g}_m\ g^{m\ell'}\ (\mathbf{d}\widetilde{\mathbf{d}}\partial^{\ell}\widetilde{\boldsymbol{T}})\ \boldsymbol{T}_{\ell}\ \boldsymbol{Riem} \tag{A.121}$$

$$\tfrac{1}{8}p^2\,(-1)^{p}\,\widetilde{\boldsymbol{T}}_{\ell'}\ \widetilde{\boldsymbol{\nabla}}\boldsymbol{T}\ \delta\boldsymbol{g}\ (\mathbf{d}\partial^{\ell'}\partial^{\ell}\widetilde{\boldsymbol{T}})\ \boldsymbol{T}_{\ell}\ \boldsymbol{Riem}\ . \tag{A.122}$$

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