Modular double of a Quantum Group

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Abstract. The well known technical difficulties in the theory of quantum groups with $q = i\pi\tau$ appear when τ is real. It is argued that adding a second quantum group with dual parameter $1/\tau$ cures these difficulties. Moreover the new object, which we propose to call a modular double, has natural applications in conformal field models.

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As is clear from the title, I shall deal with some questions connected with the theory of Quantum Groups. If I remember right, Moshé did not like Quantum Groups after this notion was crystallized by Drinfeld [1] in a purely algebraic manner. However his own attraction to deformations (as well as the pressure of authors in LMP) made him change his mind. When I presented the subject described below at a St. Petersburg meeting in May 1998, he did not express any bad feelings. So I decided to publish it in this memorial volume.

There are several sources to my proposal. I shall give just two of them, one "mathematical" and another "physical", as is appropriate for a paper in Mathematical Physics.

1) In the definition of Quantum Group one uses the deformation of the Chevalley generators K, f, e, whereas for the construction of the universal R-matrix one needs nonpolynomial elements like $H = \ln K$. Explicit formulas will be reminded below. This unfortunate obstacle can be circumvented in several ways: one, *à la* Lusztig [2] is just not to use the explicit formulas of Drinfeld; another, followed in most texts on Quantum Groups (see e.g. [3]), is to employ formal series in $\ln q$. However the value of R-matrix as a genuine operator is too high and deserves a more friendly attitude.

2) In the applications of Quantum Groups to Conformal Field Theory one explicitly sees, that together with the attributes of Quantum Groups (i.e. eigenvalues of Laplacians) for $q = e^{i\pi\tau}$ there enter analogous objects, corresponding to $\tilde{q} = e^{-i\pi/\tau}$. This modular duality is a well-known "experimental" fact, which goes without satisfactory explanation.

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In the following an extension of Quantum Groups will be described, which will throw some light on both topics above. Roughly speaking I propose to unify the Quantum Groups for q and $\tilde{q}$ in one object having a modular structure (see e.g. [4]). Thus the combination of words “Modular double of a Quantum Group” seems to be quite relevant for it.

The element $\ln K$ will have a natural definition and the expression for the universal R-matrix will become much more meaningfull in this extension. I believe also, that it is this modular double which in fact defines the hidden symmetry in the Conformal Field Theory. There are some indications of this in the literature [5, 6, 7] and recently it was made more explicit in [8, 9]. At this moment I know well how my proposal works in all details in the case of rank 1, the $SL(2)$ group. This will be presented below. The tools for the $SL(N)$ generalizations are known [10, 11], but other series of simple groups are not treated yet.

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1. Reminder on the Quantum Group $SL(2)$

I shall use an extension of $SL_q(2)$ which is the Drinfeld double of its Borel part. There are four generators K, K', e, f with familiar relations:

$$\begin{aligned} Ke &= q^2 eK & K'e &= q^{-2} eK' \\ Kf &= q^{-2} fK & K'f &= q^2 fK' \\ ef - fe &= \frac{K - K'}{q - q^{-1}} & KK' &= K'K. \end{aligned}$$

The algebra $\mathscr{U}_q$ generated by K, K', e, f over the field $\mathbb{C}$ of complex numbers, (so that q is a complex number) has two central elements:

$$J = KK', \quad C = \frac{K - K'}{q - q^{-1}} + (q - q^{-1})^2 (ef - fe).$$

Reduction to $SL_q(2)$ is achieved if we put $J = \mathrm{id}$; however, we shall not do it here.

The universal R-matrix is affiliated with the tensor square $\mathscr{U}_q \otimes \mathscr{U}_q$ and defines the property of the Hopf multiplication $\triangle$ in $\mathscr{U}_q$, $\sigma \circ \triangle = R \triangle R^{-1}$, where σ is a permutation $\sigma(a \otimes b) = b \otimes a$. Drinfeld has given a formal expression for R:

$$R = q^{-\frac{H \otimes H'}{2}} s_q(-(q - q^{-1})^2 e \otimes f)$$

where $K = q^H$, $K' = q^{H'}$, and $s_q(w)$ is a q-exponent which can be written in several forms:

$$\begin{aligned} s_q(w) &= \prod_{n=0}^{\infty}(1+q^{2n+1}w) \\ &= 1+\sum_{k=0}^{\infty}\frac{(-1)^k q^{\frac{n(n-1)}{2}} w^k}{(q-q^{-1})\cdots(q^k-q^{-k})} \\ &= \exp\sum_{k=1}^{\infty}\frac{(-1)^k w^k}{k(q^k-q^{-k})}. \end{aligned}$$

The term q-exponent is the most appropriate for the second form. In the third form there enters the q-deformed dilogarithm, which was explored in particular in [12, 13]. The title q-exponent is strongly supported by the following property, first found in [14]: let u, v be a Weyl pair

$$uv = q^2 vu,$$

then

$$s_q(u)s_q(v) = s_q(u+v).$$

Let us note that in [12] it was found that one more property holds:

$$s_q(v)s_q(u) = s_q(u+v+q^{-1}uv) = s_q(u)s_q(q^{-1}uv)s_q(v),$$

which can be called a "pentagon identity" and is a quantum deformation of the corresponding property of the dilogarithm [13]. Now we see one more deficiency in the definition of the universal R-matrix, besides the necessity of using $\log K$. The function $s_q(w)$ behaves badly for q lying on the unit circle $|q| = 1$. Indeed, for example, in the third form of $s_q(w)$ we see small denominators. Thus the expression for the universal R-matrix needs some mending.

2. The main idea

The problem of defining the log of an operator appears already in a simpler example of Weyl pair u, v. It is easy to realize the defining relation for u, v via the Heisenberg pair P, Q with relation $[Q,P] = 2\pi i\hbar I$.

Indeed, the pair $u = e^{\alpha P}$, $v = e^{\beta Q}$ satisfies Weyl relations: $uv = q^2 vu$ with $\ln q = \frac{\pi\alpha\beta\hbar}{i}$. Thus the pair (P,Q) defines (u,v). However the inverse is not true, partly because $\log u$ or $\log v$ are badly defined. A more subtle fact is that the pair P, Q defines a second Weyl pair

$$\tilde{u} = e^{\tilde{\alpha}P}, \quad \tilde{v} = e^{\tilde{\beta}Q}$$

with a different phase $\tilde{q}$, $\ln\tilde{q} = \frac{\pi\tilde{\alpha}\tilde{\beta}\hbar}{i}$, which commutes with the first pair if $\alpha\tilde{\beta} = \hbar^{-1}$ and $\tilde{\alpha}\beta = \hbar^{-1}$, so that in particular

$$\hbar^2\alpha\beta = \frac{1}{\tilde{\alpha}\tilde{\beta}}.$$

What is less trivial is the fact that, together, the commuting pairs (u,v) and $(\tilde{u},\tilde{v})$ define naturally P and Q for generic q. This fact was discussed explicitly in [15], but of course could be traced to earlier literature, in particular to A. Connes' monograph on noncommutative geometry [16] and to the paper [17].

We see more concisely, that the algebra $\mathscr{B}$ generated by P, Q is factored into the product

$$\mathscr{B} = \mathscr{A}_q \otimes \mathscr{A}_{\tilde{q}}$$

of commuting factors, generated by (u,v) and $(\tilde{u},\tilde{v})$ correspondingly. I used the term "factor" with full algebraic meaning: Indeed, neither $\mathscr{A}_q$ nor $\mathscr{A}_{\tilde{q}}$ have non-trivial center. However, I did not introduce any $*$-structure, so the connection with von Neumann theory (see e.g. [16]) is incomplete. In particular, the last formula must contain some closure. In other words, the use of tensor product in this formula is somewhat loose. Indeed, $\mathscr{A}_q$ and $\mathscr{A}_{\tilde{q}}$ are factors of type II_1 for generic q because they are infinite dimensional but allow for trace

$$\mathrm{tr}\left(\sum a_{mn} u^m v^n\right) = a_{00},$$

which is equal to 1 for the unit operator. On the other hand, $\mathscr{B}$ is a factor I_∞ since it can be realized as the algebra of all operators in $L_2(\mathbb{R})$.

I hope that now the idea of what to do in the Quantum Group case is clear — to define the $\log K$ one has to extend the algebra $\mathscr{A}_q$, adding to it some dual generators. The definition of "dual" is clear for the Weyl type operators

$$\tilde{K} = (K)^{1/\tau}$$

if we put $q = e^{i\pi\tau}$. However not all generators of a Quantum Group are of Weyl type. So we have to seek a new set of generators which have this property. Fortunately, the theory of integrable models, which previously led to main relations of Quantum Groups [18], produces also this relevant set of generators. Indeed, the Quantum Group generators appeared first as the elements of the Lax operator of the XXZ model. The lattice Sine-Gordon model introduced in [19] belongs to this class (see e.g. [20]) and, in its turn, naturally uses the Weyl type generators. In the next section we shall give the corresponding formulas in somewhat purified form.

3. Explicit construction

Consider the algebra $\mathscr{C}$ generated by four generators w_1, w_2, w_3, w_4, which is convenient to label by an index $n \in \mathbb{Z}_4$, so that $w_{n+4} = w_n$. We impose Weyl type relations on the nearest neighbours

$$w_n w_{n+1} = q^2 w_{n+1} w_n,$$

whereas

$$w_n w_m = w_m w_n \quad |m-n| > 1.$$

The algebra $\mathscr{C}_q$ (supplied by label q) has two central elements:

$$Z_1 = w_1 w_3, \quad Z_2 = w_2 w_4.$$

It is a simple exercise to check that

$$e = i\frac{w_1 + w_2}{q - q^{-1}} \qquad f = i\frac{w_3 + w_4}{q - q^{-1}}$$
$$K = q^{-1} w_2 w_3 \qquad K' = q^{-1} w_4 w_1$$

satisfy the defining relations of a Quantum Group. The relation between the central elements is as follows:

$$J = Z_1 Z_2, \quad C = Z_1 + Z_2,$$

and so $\mathscr{C}_q$ is a double cover of $\mathscr{A}_q$.

Following the reasoning of Section 2, I introduce a second algebra $\mathscr{C}_{\tilde{q}}$ with generators $\tilde{w}_n = w_n^{1/\tau}$. Both can be described in terms of the Heisenberg type generators p_n with relations[1]

$$[p_n, p_{n+1}] = -2\pi i I,$$

if we put $w_n = e^{bp_n}$, $\tilde{w}_n = e^{p_n/b}$, where $q = e^{i\pi b^2}$, $\tau = b^2$. In particular, we see that

$$K = e^{b(p_2+p_3)} \qquad K' = e^{b(p_1+p_4)}$$
$$\tilde{K} = e^{\frac{(p_2+p_3)}{b}} \qquad \tilde{K}' = e^{\frac{(p_1+p_4)}{b}}$$

and the first factor in the universal R-matrix is expressed as

$$q^{-\frac{H\otimes H'}{2}} = e^{\frac{\pi}{2i}(p_2+p_3)\otimes(p_1+p_4)} = \tilde{q}^{-\frac{\tilde{H}\otimes\tilde{H}'}{2}},$$

and does not depend on q, serving both quantum groups $\mathscr{A}_q$ and $\mathscr{A}_{\tilde{q}}$.

Let us turn now to the second factor in the universal R-matrix. We have

$$\begin{aligned} s_q(-(q-q^{-1})^2 e\otimes f) &= s_q((w_1+w_2)\otimes(w_3+w_4)) \\ &= s_q(w_1\otimes w_3)s_q(w_1\otimes w_4)s_q(w_2\otimes w_3)s_q(w_2\otimes w_4), \end{aligned}$$

(the last equality follows from Schützenberger relation) so that only Weyl type combinations enter here. Now I use the observation from [21, 15]: Consider the function

$$\psi(p) = \exp\frac{1}{4}\oint_{-\infty}^{\infty} \frac{e^{ip\xi/\pi}}{\operatorname{sh} b\xi \operatorname{sh}\xi/b}\frac{d\xi}{\xi},$$

where the singularity at $\xi = 0$ in the integral is circled from above. It is easy to see that

$$\psi(p) = \frac{s_q(w)}{s_{\tilde{q}}(\tilde{w})},$$

[1] We put now the Planck constant $\hbar$ equal to 1.

where $w = e^{bp}$, $\tilde{w} = e^{p/b}$. Thus the integral $\psi(p)$ unifies both q-exponents $s_q(w)$ and $s_{\tilde{q}}(\tilde{w})$, dual to each other. On the other hand, it definitely does not have their deficiencies; in particular, it does not suffer from the problem of small denominators. The pentagon relation for it takes the form

$$\psi(P)\psi(Q) = \psi(Q)\psi(P+Q)\psi(P)$$

if $[P,Q] = -2\pi iI$.

The function $\psi(p)$ was used extensively in [15, 21] and later by Kashaev in his new proposal for the knot invariants [22] and quantizing of the Teichmüller space [23], which was done also independently by Chekhov and Fock [24].

The relation of $\psi(p)$ to q-exponents makes our proposal quite clear: Consider the algebra

$$\mathscr{D} = \mathscr{C}_q \otimes \mathscr{C}_{\tilde{q}},$$

generated by the generators w_n, $\tilde{w}_n$, $n = 1,2,3,4$. The quantum groups $\mathscr{U}_q$ and $\mathscr{U}_{\tilde{q}}$ are naturally imbedded into $\mathscr{D}$ by means of defining relations between Chevalley generators and w-s. The algebra $\mathscr{D}$ can also be considered as being generated by the generators p_n, $n = 1,2,3,4$.

The element $\mathscr{R}$ in $\mathscr{D} \otimes \mathscr{D}$:

$$\mathscr{R} = \exp\left\{\frac{\pi}{2i}(p_2+p_3)\otimes(p_1+p_4)\right\}\psi(p_{13})\psi(p_{14})\psi(p_{23})\psi(p_{24}),$$

where $p_{ik} = p_i \otimes I + I \otimes p_k$, plays the role of the universal R-matrix for both $\mathscr{U}_q$ and $\mathscr{U}_{\tilde{q}}$. The Yang-Baxter relation $\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12}$ is an easy consequence of the pentagon relation, as was shown by R. Kashaev and A. Volkov [11]. They also have obtained corresponding construction for the $SL_q(N)$ case.

There is a natural way to introduce the $*$-structure into $\mathscr{D}$: one is to consider p_n to be selfadjoint $p_n^* = p_n$. The corresponding formula in terms of w looks as follows:

$$w^* = w^{\bar{b}/b}, \quad \tilde{w}^* = \tilde{w}^{b/\bar{b}},$$

and does not respect the tensor structure of $\mathscr{D}$ in general.

However there are several particular cases when a $*$-involution can be related to this structure:

1) $\tau > 0$, so that b is real. The $*$ takes the form: $w^* = w$, $\tilde{w}^* = \tilde{w}$, and corresponds to the $SL_q(2,\mathbb{R})$ reduction.

2) $\tau < 0$, so that b is imaginary and $w^* = w^{-1}$, $\tilde{w}^* = \tilde{w}^{-1}$, which corresponds to the $SU_q(2)$ reduction.

3) $\tau = e^{i\theta}$, so that $\tau = e^{i\theta/2} = \bar{\bar{b}}$. The involution takes the form $w^* = \tilde{w}$, and so interchanges the factors in the modular double.

For all three cases the parameter of the central extension of the class mapping group

$$C = 1 + 6(\tau + 1/\tau + 2) = 1 + 6(b + 1/b)^2$$

is real. The values of C are $C \geq 25$, $C \leq 1$ and $1 \leq C \leq 25$ correspondingly for the case 1, 2 and 3. The relevance of this $*$ operation to Conformal Field Theory is discussed in [9].

In view of the modular duality relation $\tau \to -1/\tau$, it is natural to call $\mathscr{D}$ modular double.

I conclude by proposing several problems:

1) Give the generalization of our construction to other series of Quantum Groups;

2) Describe the coproduct in $\mathscr{U}_q$ in terms of w-generators;

3) Find a natural definition of the closure entering in formal tensor products like those for algebras $\mathscr{B}$ or $\mathscr{D}$.

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