HELSINKI INSTITUTE OF PHYSICS INTERNAL REPORT SERIES

HIP-2023-03

# Investigations of Boundaries and Branes in the Holographic Duality

Jani Kastikainen

Helsinki Institute of Physics and
Department of Physics
University of Helsinki
Finland

Astroparticule et Cosmologie
Université Paris Cité
France

*Doctoral dissertation, to be presented for public discussion with the permission of the Faculty of Science of the University of Helsinki, in Auditorium A129, Chemicum building, on the 19th of June 2023 at 13 o'clock.*

Helsinki 2023

ISBN 978-951-51-8417-7 (paper)
ISBN 978-951-51-8418-4 (pdf)
ISSN 1455-0563 (print)
ISSN 2814-9459 (online)
http://ethesis.helsinki.fi
Unigrafia
Helsinki 2023

Université Paris Cité

UNIVERSITY OF HELSINKI

# Université Paris Cité

En cotutelle avec University of Helsinki

**Ecole doctorale** STEP'UP (560)

***Astroparticule et Cosmologie***

# Investigations of Boundaries and Branes in the Holographic Duality

Par Jani KASTIKAINEN

Thèse de doctorat de Physique de l'Univers

Dirigée par Francesco NITTI
Et par Esko KESKI-VAKKURI

Présentée et soutenue publiquement le 19 Juin 2023

Devant un jury composé de :

Francesco NITTI, Maitre de Conferences HDR, Université Paris Cité, Directeur
Esko KESKI-VAKKURI, Lecteur, University of Helsinki, Codirecteur
Ulf GRAN, Professor, Chalmers University of Technology, Rapporteur
Valentina PULETTI, Professor, University of Iceland, Rapporteur
Kimmo TUOMINEN, Professor, University of Helsinki, Président
Diego HOFMAN, Associate Professor, University of Amsterdam, Opponent
Vasilis NIARCHOS, Associate Professor, University of Crete, Membre
Cristina VOLPE, Directrice de Recherche, Université Paris Cité, Membre
Aleksi VUORINEN, Professor, University of Helsinki, Invité

J. Kastikainen: Investigations of Boundaries and Branes in the Holographic Duality
University of Helsinki, Université Paris Cité, 2023, 71 pages
Helsinki Institute of Physics, Internal Report Series, HIP-2023-03
ISBN 978-951-51-8417-7 (paper), ISBN 978-951-51-8418-4 (pdf)
ISSN 1455-0563 (print), ISSN 2814-9459 (online)

**Keywords:** Holographic duality, AdS/CFT correspondence, conformal field theory, string theory, Lovelock gravity, supergravity, ABJM theory

# Abstract

Quantum conformal field theory (CFT) is a special type of quantum field theory which is symmetric under conformal transformations of spacetime. Because of their relative simplicity, CFTs appear in many different areas of theoretical physics such as condensed matter physics and quantum gravity. For example in the study of statistical mechanical systems, CFTs appear as continuum effective descriptions of collective phenomena near a critical point where a phase transition occurs. In this context, the existence of different types of CFTs is connected to the classification of different universality classes of phase transitions.

As theories by themselves, CFTs do not describe the gravitational force which is transmitted by the graviton particle. However, CFTs are deeply connected to gravity in string theory where they determine the dynamics of quantum mechanical strings. The spectrum of vibrational modes of a string include the graviton making string theory a theory of quantum gravity which is self-consistent based on vast theoretical knowledge. Hence CFTs have played a central role in the development of string theory and in our quest to unify Einstein's theory of gravity with quantum mechanics.

The main milestone in the modern understanding of quantum gravity was the discovery of the holographic duality in string theory. Simply put, the duality says that there exist special CFTs which contain rules to describe gravity hidden in them. However, these rules are holographic, because they describe gravity in a spacetime with one extra dimension: the duality is like a hologram where a three-dimensional image (gravity) is encoded on a two-dimensional surface (CFT). Not only is the holographic duality useful in understanding the structure of quantum gravity, it is also extremely powerful in predicting dynamics of strongly interacting quantum fields that occur inside neutron stars for example.

This thesis is devoted to the study of the holographic duality and it is based on four research articles on the topic. The focus is on how extended objects, namely boundaries and branes, behave on both sides of the duality. In string theory, boundaries of open strings describe dynamics of D-branes whose understanding was crucial for the discovery of the holographic duality in the first place. Similarly on the CFT side, boundaries give raise to observable effects such as the Casimir effect when the system is confined between two parallel plates. The goal of the thesis is to give an introduction to these concepts in the context of the holographic duality.

# Acknowledgments

First and foremost I would like to thank my supervisors. I thank Esko Keski-Vakkuri, for guiding me through the world of theoretical physics since my undergraduate days, and Francesco Nitti, for his support and assistance during my time in Paris.

I am thankful to everyone who contributed to making my PhD defence possible. First I want to thank my pre-examiners Ulf Gran and Valentina Puletti for reading and giving comments on my dissertation. I thank Diego Hofman, for agreeing to be my opponent, and Kimmo Tuominen, for being the custos during the defence. I would also like to thank Vasilis Niarchos, Cristina Volpe and Aleksi Vuorinen for agreeing to be part of my jury.

I am thankful to my collaborators, without whom the articles featured in this thesis would not have come to fruition. First, I want to thank Niko Jokela who presented me with the life changing opportunity to pursue a cotutelle degree in Paris. I want to thank Jan de Boer and Elias Kiritsis who assisted in securing me a post-doctoral position after my PhD. I also want to thank Victor Godet and Sanjit Shashi for our long and close collaboration which I really enjoyed. I am grateful to Shovon Biswas and James Sully for their interest in my work, as it resulted in our fruitful collaboration.

In addition, I want to extend my gratitude to my university friends, many of whom have become dear friends outside academia. I am also thankful to the regular participants of the 12:15 lunch train in Paris and the HIP coffee breaks in Helsinki. Lastly, I want to thank my friends outside of academia, for keeping me grounded during my PhD journey, and my family, for their love and support.

# List of included articles

This thesis is based on the following four publications [A1–A4].

[A1] **Holographic BCFT spectra from brane mergers**
Shovon Biswas, Jani Kastikainen, Sanjit Shashi and James Sully
JHEP 11 (2022) 158

[A2] **Flavored ABJM theory on the sphere and holographic F-functions**
Niko Jokela, Jani Kastikainen, Elias Kiritsis and Francesco Nitti
JHEP 03 (2022) 091

[A3] **Structure of holographic BCFT correlators from geodesics**
Jani Kastikainen and Sanjit Shashi
Phys.Rev.D 105 (2022) 4, 046007

[A4] **Conical defects and holography in topological AdS gravity**
Jani Kastikainen
Class.Quant.Grav. 37 (2020) 19, 195010

# Contributions of the author

[A1] The author did majority of the main computations and wrote sections including appendices.

[A2] The author did all of the analytical calculations, all of the numerics, and wrote the main body of the paper including appendices.

[A3] The author came up with the project together with the co-author, did multiple calculations, and wrote sections including appendices.

[A4] Single authored. The author came up with the idea, did all the computations, and wrote the final article from start to finish.

# Other work by the author

In addition to the four articles [A1–A4] included in this thesis, the author has published two other articles [A5, A6] which are not included due to lacking a direct connection to the holographic duality.

[A5] **Quantum hypothesis testing in many-body systems**
Jan de Boer, Victor Godet, Jani Kastikainen and Esko Keski-Vakkuri
SciPost Phys.Core 4 (2021) 019

[A6] **Quasi-local energy and ADM mass in pure Lovelock gravity**
Jani Kastikainen
Class.Quant.Grav. 37 (2020) 2, 025001

# Contents


**1 Introduction** **1**

**2 Aspects of conformal field theory** **5**

2.1 Conformal symmetry on curved manifolds . . . . . 5

2.1.1 Examples of conformal groups . . . . . 8

2.2 Conformal field theory on a curved manifold . . . . . 10

2.3 Quantization of conformal field theory . . . . . 11

2.3.1 Stress tensor and the Weyl anomaly . . . . . 12

2.3.2 Scalar correlation functions . . . . . 13

2.4 Relation to the articles . . . . . 14

**3 Conformal field theory in two dimensions** **15**

3.1 Operator formulation . . . . . 15

3.1.1 Structure of the Hilbert space . . . . . 17

3.1.2 The state operator correspondence . . . . . 18

3.2 Stress tensor correlators from the Polyakov action . . . . . 19

3.3 2D CFT with two boundaries . . . . . 21

3.3.1 CFT on the Lorentzian strip . . . . . 21

3.3.2 Open closed string duality . . . . . 23

3.4 Relation to the articles . . . . . 26

**4 Aspects of bottom-up holography** **27**

4.1 Classical theories of gravity . . . . . 28

4.1.1 Lovelock gravity . . . . . 28

4.1.2 Topological Lovelock gravity . . . . . 29

4.1.3 Boundary and corner terms . . . . . 30

4.2 Asymptotic symmetries of AdS space . . . . . 32

4.2.1 Subleading terms in the FG expansion . . . . . 34

4.2.2 Asymptotic symmetries in three bulk dimensions . . . . . 35

4.3 The holographic Weyl anomaly . . . . . 36

4.4 Holographic models of 2D boundary CFTs . . . . . 39

4.5 Relation to the articles . . . 40

**5 Aspects of top-down holography** **42**

5.1 Holographic dual of $\mathcal{N} = 4$ super Yang–Mills theory . . . 42

5.1.1 Effective actions of closed and open strings . . . 42

5.1.2 Supergravity description . . . 47

5.1.3 The AdS/CFT conjecture . . . 48

5.2 Holographic dual of ABJM theory . . . 49

5.2.1 Effective action of M-theory . . . 50

5.2.2 Supergravity solutions . . . 52

5.2.3 Free energy on a sphere . . . 54

5.3 Holographic dual of flavor in ABJM theory . . . 55

5.3.1 Adding flavor in M-theory . . . 56

5.3.2 Adding flavor in type IIA supergravity . . . 57

5.4 Relation to the articles . . . 58

**6 Summary and conclusion** **59**

# Chapter 1

# Introduction

When an object looks the same at small and large scales it is said to exhibit self-similarity or scale invariance meaning that the physical characteristics of the object are independent of scale. Approximate scale invariance is ubiquitous in nature, for example, the ocean coastline of Europe seen from space looks to be the same wiggly line at all zoom levels. When scale invariance is exact, the object is a fractal with endless self-similarity.

Scale symmetry is realized in theories of physics that describe phase transitions in systems with many degrees of freedom. A simple example is the heating of a permanent magnet that looses its magnetism in a phase transition at the Curie temperature. At small scales, the magnet consists of a large number of tiny spins that point up or down, and near the phase transition (the critical point), the up-down distribution of the spins exhibits self-similarity. At the critical point, the magnet is thus described by a field theory with exact scale invariance.

In addition to phase transitions driven by thermal fluctuations, there are also quantum phase transitions occurring at absolute zero temperature that are driven by quantum mechanical fluctuations. It is widely believed that the laws of quantum mechanics (combined with causality and few other assumptions) require the enhancement of scale invariance at a quantum critical point to conformal invariance [1, 2]. In addition to changes of scale, conformal transformations include all transformations of space that preserve angles at each point. Field theories exhibiting conformal invariance are even more symmetric than scale invariant theories which leads to strong constraints on the properties of the theory. The study of these so called conformal field theories (CFT) has a long history starting from their applications in condensed matter physics and in string theory.

Quantum field theories (QFT) can be coupled to different types of extended objects such as boundaries or defects that affect the behaviour of the system. In particular, the presence of boundaries leads to drastic observable effects such as the Casimir effect [3] and the divergence of energy density near a boundary [4]. The effects of boundaries are simplest to study in CFTs where conformal symmetry is a strong constraint on the spectrum of the theory. In such a CFT with a boundary (BCFT), the original conformal symmetry is broken to a subgroup that preserves the location of the boundary and its boundary conditions. However, the remaining subgroup is still very large especially in two dimensions where its constraints can be implemented in astonishing detail [5–7].

In the early days of the development of CFT, the motivation was string theory where both CFTs with and without boundaries appear. In this context, CFTs live on worldsheets of closed

strings while BCFTs live on worldsheets of open strings which are anchored on extended surfaces in spacetime. An important development in string theory was the realization that the anchor surfaces are dynamical quantum mechanical objects in their own right [8, 9]. These objects became known as D-branes, and just like strings, D-branes vibrate and move in spacetime. Their dynamics at low-energies are described by a generalization of the Dirac–Born–Infeld (DBI) action [10] so that classically the brane attempts to minimize its worldvolume.

As a result, string theory is not only a quantum theory of strings, but a quantum theory of extended objects that include D-branes. These objects respect the laws of quantum mechanics and relativity while interacting with each other in complicated ways making general calculations difficult. Hence it is common in string theory to study its building blocks in corners of parameter space where calculations are under control and progress can be made. A standard example is closed string perturbation theory valid at the corner where the string coupling is weak so that Feynman diagram methods are applicable. Another example is the limit where a condensate of a large number of closed strings behaves as a curved spacetime governed by Einstein's theory of gravity, or more precisely, by equations of supergravity [11, 12].

Supergravity contains various classical solutions sourced by extended strongly gravitating objects known as $p$-branes that are similar to extremal black holes [13]. Another important discovery was that these sources are simply D-branes in disguise [14]. This is possible, because open strings attached to D-branes can be reinterpreted as emission of closed strings so that a large number of D-branes can source supergravity fields. Hence D- and $p$-branes give two descriptions of the same physical object in two different regimes of parameter space.

Above types of non-trivial relations between different corners of parameter space are known as dualities. A famous example of a duality in physics is the Kramers–Wannier (KM) duality of the Ising model in two dimensions [15, 16] that relates low-temperature properties of the model to its high-temperature physics. There a variety of different dualities relating different string theories to each other [17]. For example, the analog of the KM duality is the T-duality [18–20] relating strings living on different target spaces. On the other hand, the duality behind the identification of D- and $p$-branes is the open closed string duality on the string worldsheet [21–23] which is a statement about two-dimensional CFTs. What makes stringy dualities so remarkable is that they often involve different theories that can even live in different dimensions.

The most recent example of a duality is the holographic duality which is also called the AdS/CFT conjecture. In its original formulation [24], it conjectures that physics of a specific supersymmetric CFT at strong coupling is equivalently described by a subsector of string theory at weak coupling. This duality is called holographic, because the dimensionality of spacetime in the two descriptions differs by one: the CFT lives on a $d$-dimensional spacetime while the string theory lives effectively in $d+1$ dimensions. More precisely, the string theory lives on an asymptotically $\mathrm{AdS}_{d+1} \times M_q$ spacetime called the bulk where $\mathrm{AdS}_{d+1}$ denotes anti-de Sitter space and $M_q$ is a compact manifold of dimension $q$ (usually a sphere). The holographic dual CFT then lives on the conformal boundary of the $\mathrm{AdS}_{d+1}$ bulk [25, 26].

The holographic duality follows from the identification of D- and $p$-branes in a decoupling limit [24]. By taking the limit using different types of branes in various corners of string theory produces different supersymmetric CFTs as holographic duals. For example, D3-branes in type IIB string theory and M2-branes on an orbifold singularity in M-theory lead to a Yang–Mills theory in four dimensions [24] and a Chern–Simons theory in three dimensions [27] respectively. The duality has been generalized by considering increasingly more complicated brane configu-

rations, for example, the inclusion of D7-branes in the D3-brane system leads to the addition of flavor degrees of freedom to the dual Yang–Mills theory [28].

One of the goals of these generalizations was to construct dual QFTs that are more similar to theories describing nature such as quantum chromodynamics (QCD) (see [29] for early work in this direction). The string theory dual of QCD would allow the use of weakly coupled gravitational calculations to study the strongly coupled regime of QCD which is otherwise inaccessible for traditional perturbative methods. However, top-down string theory constructions are often complicated so simpler gravitational models that capture necessary QCD physics are preferred. This lead to the development of the bottom-up approach to holography where holographic gravity models are built on top of a simple classical theory of gravity. Bottom-up models of holography have been successful in describing strongly coupled properties of QCD such as confinement [30], the meson mass spectrum [31] and the phase diagram at finite density [32].

In the holographic duality, quantum gravitational degrees of freedom are encoded in an ordinary QFT without gravity in a non-trivial manner. The bottom-up approach has been useful for breaking down this encoding, for example, it was understood early on that quantum entanglement plays an important role in the formation of a connected spacetime on the gravity side [33]. The role of entanglement could be made quantitative in [34, 35] after the discovery of the HRT formula [36–38] which says that entanglement entropy of quantum fields in the CFT is equal to the area of a minimal surface in gravity. More recently, these bottom-up developments lead to a proposal for a mechanism that explains how quantum gravity preserves unitarity in an evaporation process of a black hole [39–42].

It is also natural to ask what type of special features make a CFT enjoy a gravitational dual description. An example of a simple bottom-up model to study this question is three-dimensional Einstein gravity which does not contain any propagating degrees of freedom (gravitons). On a locally $AdS_3$ manifold, this theory is expected to be holographically dual to a two-dimensional CFT with infinite-dimensional conformal symmetry. Early evidence for the existence of a dual CFT was the finding that the asymptotic symmetry group of 3D gravity is centrally extended [43], but whether the dual theory actually exists is still an open question (see [44] for an early proposal). Progress can be made by constraining the spectrum of the dual CFT: modular invariance of the CFT combined with the presence of black hole solutions in 3D gravity requires the CFT to have a large central charge and a large gap [45]. It is also possible that the dual theory is not a single CFT, but instead an ensemble [46].

As mentioned earlier, the presence of boundaries in the CFT leads to various effects that should also be realized on the gravity side of the duality. A simple way to incorporate boundaries is to couple the gravity theory to dynamical end-of-the-world (EOW) branes [47, 48]. The dynamics of EOW branes are determined by the boundary Einstein equation which follows after imposing a Neumann boundary condition for the metric at the brane. In 3D gravity, such an EOW brane model reproduces expected properties of two-dimensional BCFTs such as boundary entropies and transitions between open and closed string channels [47]. Gaining a more detailed understanding of holographic BCFTs still requires further investigation.

This thesis is structured as follows. In chapter 2, we begin by giving a detailed introduction to conformal symmetry and CFTs on curved manifolds. Then in chapter 3, we restrict to CFTs in two dimensions with and without boundaries and study their operator structure. In chapter 4, we describe aspects of bottom-up holography based on simple theories of classical gravity. In chapter 5, we review string theory constructions of various holographic dual pairs and describe

how they can be extended to include flavor on the field theory side. Lastly in chapter 6, we give a summary and conclude.

# Chapter 2

# Aspects of conformal field theory

In this chapter, we will present the exact definition of a (quantum) conformal field theory and study its properties. We begin by introducing the group of conformal diffeomorphisms of a general curved manifold equipped with a Riemannian metric. The conformal group is the classical symmetry group of a CFT which we will quantize using path integral methods. We show how quantum effects can lead to the break down of conformal symmetry due to the Weyl anomaly and also review constraints imposed by conformal invariance on simple correlation functions.

## 2.1 Conformal symmetry on curved manifolds

Consider a $d$-dimensional differential manifold $(\mathcal{M}, g)$ equipped with a metric $g$ of either Lorentzian or Euclidean signature. An *active* diffemorphism $\psi\colon \mathcal{M} \to \mathcal{M}$ is a smooth bijective map from the manifold to itself, in other words, $\psi$ is a permutation of points of the manifold. The action of an active diffeomorphism on the metric $g$ produces a new metric $\widetilde{g} = \psi^* g$ as the pull-back of the old metric. The pull-back is defined in the standard way via the action of the push-forward of $\psi$ on tangent vectors.

Consider a local coordinate system $x^a$, with $a = 1, \ldots, d$, in which $p, \psi(p) \in \mathcal{M}$ correspond to $x, F^{-1}(x) \in \mathbb{R}^d$ respectively where $F\colon \mathbb{R}^d \to \mathbb{R}^d$ is a *passive* diffeomorphism. The action of $\psi$ in the coordinate $x$ is given by

$$\widetilde{g}_{ab}(x) = \frac{\partial F^c}{\partial x^a} \frac{\partial F^d}{\partial x^b}\, g_{cd}(F(x)), \tag{2.1}$$

where $g_{ab}\colon \mathbb{R}^d \to \mathbb{R}$ are component functions of the metric $g$ and $a, b, \ldots = 1, \ldots, d$. We will only be interested in orientation-preserving diffeomorphisms whose Jacobian determinant is positive:

$$\det \frac{\partial F^a}{\partial x^b} > 0. \tag{2.2}$$

Usually passive diffeomorphisms are called coordinate transformations and active diffeomorphisms simply diffeomorphisms which is the terminology we will adopt in this thesis. The distinction between these two types of diffeomorphisms played an important role in the conceptual development of general relativity by Einstein [49].

Orientation-preserving diffeomorphisms $\psi$ of $\mathcal{M}$ generate the diffeomorphism group $\mathrm{Diff}_+\mathcal{M}$

with group operations being composition $\psi_1 \circ \psi_2$ and inversion $\psi^{-1}$. Consider now the subgroup

$$\mathrm{Iso}\,(\mathcal{M}, g) \subset \mathrm{Diff}_+\mathcal{M} \tag{2.3}$$

whose elements keep the metric $g$ invariant $\psi^* g = g$ which in components reads

$$\widetilde{g}_{ab}(x) = g_{ab}(x). \tag{2.4}$$

The subgroup (2.3) is called the isometry group of the metric $g$.

The diffeomorphism group also has the subgroup

$$\mathrm{Conf}\,(\mathcal{M}, g) \subset \mathrm{Diff}_+\mathcal{M} \tag{2.5}$$

whose action keeps the metric $g$ invariant up to a conformal factor:

$$\widetilde{g}_{ab}(x) = \Omega_F(x)^2\, g_{ab}(x). \tag{2.6}$$

We will refer to the subgroup (2.5) as the conformal group of the metric $g$ and to its elements as conformal diffeomorphisms. By computing the determinant of both sides of (2.6), we get

$$\Omega_F(x) = \left(\frac{\widetilde{g}(x)}{g(x)}\right)^{1/(2d)} = \left[\left(\frac{g(F(x))}{g(x)}\right)^{1/2} \det\left(\frac{\partial F^a}{\partial x^b}\right)\right]^{1/d} \tag{2.7}$$

where

$$g(x) \equiv \det g_{ab}(x) = \frac{1}{d!}\, \varepsilon^{a_1\ldots a_d} \varepsilon^{b_1\ldots b_d}\, g_{a_1 b_1}(x)\, g_{a_2 b_2}(x) \cdots g_{a_d b_d}(x) \tag{2.8}$$

is the determinant of the metric components and $\varepsilon_{a_1\ldots a_d} = \varepsilon^{a_1\ldots a_d}$ is the completely anti-symmetric Levi–Civita symbol.[1] In equation (2.7), we also used that

$$\widetilde{g}(x) = \left[\det\left(\frac{\partial F^a}{\partial x^b}\right)\right]^2 g(F(x)). \tag{2.9}$$

The Jacobian matrix of a conformal diffeomorphism can be decomposed as

$$\frac{\partial F^a}{\partial x^b} = \Omega_F(x)\, M^a{}_b(x) \tag{2.10}$$

where we have defined the matrix $M^a{}_b(x)$ that satisfies

$$M^c{}_a(x)\, M^d{}_b(x)\, g_{cd}(F(x)) = g_{ab}(x). \tag{2.11}$$

This can be checked by substituting to (2.1) which gives (2.6). The decomposition (2.10) exists at least when the metric is flat in which case $M^a{}_b(x) = \frac{\partial h^a}{\partial x^b}$ where $h$ is an isometry (2.4) of the flat metric.

In addition to diffeomorphisms, we define a Weyl transformation whose action on the components of the metric $g$ at a point $p \in \mathcal{M}$ is given by

$$\widehat{g}_{ab}(x) = e^{2\omega(x)} g_{ab}(x), \tag{2.12}$$

[1] In our conventions, $\varepsilon_{012\ldots d} = 1$.

where $\omega\colon \mathbb{R}^d \to \mathbb{R}$. In other words, a Weyl transformation maps a metric $g$ of $\mathcal{M}$ to another metric $\widehat{g}$ by pointwise rescaling. Weyl transformations on the space of metrics of $\mathcal{M}$ generate the abelian group $\mathrm{Weyl}\,\mathcal{M}$ where group operations are pointwise summation $\omega_1+\omega_2$ and subtraction $\omega_1-\omega_2$ of functions. Together, diffeomorphisms and Weyl transformations generate the non-Abelian group

$$\mathrm{Diff} \ltimes \mathrm{Weyl} \equiv \mathrm{Diff}_+\mathcal{M} \ltimes \mathrm{Weyl}\,\mathcal{M} \tag{2.13}$$

with group multiplication in local coordinates

$$(F_1,\omega_1)\cdot(F_2,\omega_2) = (F_1\circ F_2, \omega_1+\omega_2\circ F_1). \tag{2.14}$$

It is a semi-direct product since diffeomorphisms do not commute with Weyl transformations. The action[2] of $(F,\omega)$ on $g$ gives a new metric whose components are given by

$$\widehat{\widehat{g}}_{ab}(x) = e^{2\omega(x)}\,\frac{\partial F^c}{\partial x^a}\frac{\partial F^d}{\partial x^b}\,g_{cd}(F(x)). \tag{2.15}$$

Consider now the subgroup

$$\mathrm{CIso}\,(\mathcal{M},g) \subset \mathrm{Diff} \ltimes \mathrm{Weyl} \tag{2.16}$$

whose action keeps the metric invariant:

$$\widehat{\widehat{g}}_{ab}(x) = g_{ab}(x). \tag{2.17}$$

We will call the subgroup (2.16) the conformal isometry group of the metric $g$ and it contains the isometry group $\mathrm{Iso}\,(\mathcal{M},g)$ as its subgroup. $\mathrm{CIso}\,(\mathcal{M},g)$ is generated by elements $(F,\omega_F)$ where the conformal factor (2.7) arising from the conformal diffeomorphism is cancelled by the Weyl transformation:

$$\omega_F(x) = -\log\Omega_F(x). \tag{2.18}$$

The conformal isometry group is isomorphic to the conformal diffeomorphism group

$$\mathrm{CIso}\,(\mathcal{M},g) \cong \mathrm{Conf}\,(\mathcal{M},g) \tag{2.19}$$

via the map $(F,\omega_F)\mapsto F$.

We will now study actions of infinitesimal group elements. Consider an infinitesimal diffeomorphism given by

$$F^a(x) = x^a + \xi^a(x) \tag{2.20}$$

where $\xi^a(x) \ll 1$ are components of a vector field $\xi$ on $\mathcal{M}$. By directly expanding (2.1) it follows that the metric transforms as

$$\delta g_{ab}(x) \equiv \widetilde{g}_{ab}(x) - g_{ab}(x) = \nabla_a\xi_b(x) + \nabla_b\xi_a(x) + \mathcal{O}(\xi^2) \tag{2.21}$$

where $\nabla_a$ is the covariant derivative constructed with the Levi–Civita connection of $g$. On the other hand, the conformal factor (2.7) expands as

$$\Omega_F(x)^2 = 1 + \frac{2}{d}\,\nabla^a\xi_a + \mathcal{O}(\xi^2) \tag{2.22}$$

[2]The action is understood to be done from the left so that $(F_1,\omega_1)\cdot(F_2,\omega_2)$ is implemented by first applying $(F_2,\omega_2)$ and then $(F_1,\omega_1)$. As a result (2.14) is consistent.

where we used $\delta \det(g_{ab}) = \det(g_{ab})\, g^{ab}\, \delta g_{ab}$ for the infinitesimal transformation of the determinant (2.8). Hence the definition of a conformal isometry (2.17) becomes at the infinitesimal level

$$\nabla_a \xi_b + \nabla_b \xi_a - \frac{2}{d} (\nabla^c \xi_c)\, g_{ab} = 0. \tag{2.23}$$

Vector fields $\xi$ obeying this equation are known as conformal Killing vectors and they are the generators of the Lie algebra of $\mathrm{CIso}(\mathcal{M}, g)$ and also of $\mathrm{Conf}(\mathcal{M}, g)$ due to the isomorphism (2.19).

### 2.1.1 Examples of conformal groups

We will start by considering the manifold $\mathbb{R}^{1,d} \equiv (\mathbb{R}^d, g^{\text{flat}})$ with a flat Lorentzian metric $g^{\text{flat}}$ (Minkowski space). We will choose $x^a$ to be Cartesian coordinates such that the metric components are simply

$$g^{\text{flat}}_{ab}(x) = \eta_{ab} \equiv \mathrm{diag}(-1, 1, \ldots, 1). \tag{2.24}$$

In these coordinates, a conformal diffeomorphism $F$ satisfies (equation (2.6))

$$\frac{\partial F^c}{\partial x^a} \frac{\partial F^d}{\partial x^b}\, \eta_{cd} = \Omega_F(x)^2\, \eta_{ab}, \quad \Omega_F(x)^d = \det\left(\frac{\partial F^a}{\partial x^b}\right). \tag{2.25}$$

In other words, a conformal diffeomorphism defines a curvilinear coordinate system $x' = F^{-1}(x)$ on $\mathbb{R}^d$ in which the metric components are proportional to the diagonal Minkowski metric up to a multiplicative factor. A general conformal diffeomorphism is obtained by composing rotations, scale transformations (dilatations) and special conformal transformations [50, 51]

$$F^a(x) = \Lambda^a{}_b x^b, \quad F^a(x) = \lambda x^a, \quad F^a(x) = \frac{x^a + v^a |x|^2}{1 + 2(v \cdot x) + |v|^2 |x|^2} \tag{2.26}$$

where $\Lambda^a{}_b \in SO(d-1, 1)$, $v \in \mathbb{R}^d$ and $v \cdot x \equiv \eta_{ab}\, v^a x^b$. The conformal factor associated to each diffeomorphism is

$$\Omega_F(x) = 1, \quad \Omega_F(x) = \lambda, \quad \Omega_F(x) = \frac{1}{1 + 2(v \cdot x) + |v|^2 |x|^2}. \tag{2.27}$$

Together, the diffeomorphisms (2.26) generate $SO(d, 2)$ so that the Lorentzian conformal group of Minkowski space is given by

$$\mathrm{Conf}\, \mathbb{R}^{1,d} \cong SO(d, 2), \quad d > 2. \tag{2.28}$$

This isomorphism can be seen by embedding $\mathbb{R}^{1,d}$ into $\mathbb{R}^{2,d}$ with a metric of signature $(-1, -1, 1, \ldots, 1)$. Under this embedding, conformal diffeomorphisms (2.26) become Lorentz rotations of $\mathbb{R}^{2,d}$ [52].

In $d = 2$, the Lorentzian conformal group is infinite-dimensional and the group $SO(2, 2)$ (2.28) is only a subgroup in this case [53, 54]. Let us work in light-ray coordinates $x^{\pm} = x^1 \pm x^2 \in \mathbb{R}$ in which the flat metric (2.24) takes the form

$$g^{\text{flat}}_{ab}\, dx^a dx^b = dx^- dx^+. \tag{2.29}$$

Consider the diffeomorphism

$$F^{\pm}(x^+, x^-) = f_{\pm}(x^{\pm}), \tag{2.30}$$

which only acts on each light-ray coordinate separately. Here $f_\pm \in \mathrm{Diff}_+\mathbb{R}$ are diffeomorphisms of $\mathbb{R}$ with a positive derivative (orientation-preserving)

$$\mathrm{Diff}_+\mathbb{R} = \{f\colon \mathbb{R} \to \mathbb{R} \,|\, f'(x) > 0, f \text{ is smooth}\} \tag{2.31}$$

whose group operations are composition and inversion of diffeomorphisms. The condition $\partial_\pm f_\pm > 0$ ensures that the two-dimensional diffeomorphism (2.30) is also orientation-preserving. One can see that (2.30) is a conformal diffeomorphism with

$$\Omega_F(x^-, x^+)^2 = (\partial_- f^-)\,(\partial_+ f^+). \tag{2.32}$$

Hence the conformal group of two-dimensional Minkowski space is isomorphic to the direct product [54]

$$\mathrm{Conf}\,\mathbb{R}^{1,1} \cong \mathrm{Diff}_+\mathbb{R} \times \mathrm{Diff}_+\mathbb{R}. \tag{2.33}$$

Now the group (2.28) is the subgroup[3]

$$SO(2,2) \cong PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}) \subset \mathrm{Diff}_+\mathbb{R} \times \mathrm{Diff}_+\mathbb{R} \tag{2.34}$$

where each factor is embedded separately $PSL(2,\mathbb{R}) \subset \mathrm{Diff}_+\mathbb{R}$ and contain diffeomorphisms of the type

$$f(y) = \frac{ay+b}{cy+d}, \quad ad - bc = 1, \quad a,b,c,d \in \mathbb{R}, \quad y \in \mathbb{R}. \tag{2.35}$$

We can also consider the Lorentzian cylinder $(S^1 \times \mathbb{R}, g^{\text{flat}})$ where the time-like direction is $\mathbb{R}$ while $S^1$ is the space-like direction. We will denote light-ray coordinates of $S^1 \times \mathbb{R}$ by

$$x^\pm = \phi \pm t \tag{2.36}$$

where $\phi \sim \phi + 2\pi$ and $t \in \mathbb{R}$. They satisfy (see also [55])

$$(x^-, x^+) \sim (x^- + 2\pi, x^+ + 2\pi) \tag{2.37}$$

which has to be preserved by diffeomorphisms. Hence conformal diffeomorphisms of $S^1 \times \mathbb{R}$ are given by the same formula (2.30), but with the extra condition

$$f_\pm(x^\pm + 2\pi) = f_\pm(x^\pm) + 2\pi. \tag{2.38}$$

Diffeomorphisms of $\mathbb{R}$ with this property (called $2\pi$-equivariance) form the group

$$\widetilde{\mathrm{Diff}}_+S^1 = \{f\colon \mathbb{R} \to \mathbb{R} \,|\, f(x+2\pi) = f(x) + 2\pi,\ f'(x) > 0, f \text{ is smooth}\} \tag{2.39}$$

which is the universal-covering group of the group $\mathrm{Diff}_+S^1$ of orientation-preserving diffeomorphisms of $S^1$ (see for example [56]). Hence we get [53, 54]

$$\mathrm{Conf}\,(S^1 \times \mathbb{R}, g^{\text{flat}}) \cong \widetilde{\mathrm{Diff}}_+S^1 \times \widetilde{\mathrm{Diff}}_+S^1. \tag{2.40}$$

It is possible to map conformal diffeomorphisms of Minkowski space and the Lorentzian cylinder to each other via an embedding $\mathbb{R}^{1,1} \subset (S^1 \times \mathbb{R}, g^{\text{flat}})$ which can be found in [57].

[3] Here $PSL(2,\mathbb{R}) \cong SL(2,\mathbb{R})/\mathbb{Z}_2$.

The result (2.40) also extends to $(S^1 \times \mathbb{R}, g)$ where $g$ is an arbitrary curved metric. This is because in two dimensions, the flat metric $g^{\text{flat}}$ can be mapped to any metric $g$ by the action of Diff $\ltimes$ Weyl: the number of independent components of $g$ (three) is equal to the number of functions that define the action of Diff $\ltimes$ Weyl. In other words, the equation

$$e^{2\omega(x)} \frac{\partial D^c}{\partial x^a} \frac{\partial D^d}{\partial x^b} g^{\text{flat}}_{cd}(D(x)) = g_{ab}(x) \tag{2.41}$$

written in light-ray coordinates fixes the three functions $\omega$ and $D^{\pm}$ in terms of the three components $g_{+-}$ and $g_{\pm\pm}$ of the metric. The conformal group of $(S^1 \times \mathbb{R}, g)$ is then generated by diffeomorphisms $D \circ F \circ D^{-1}$ where $F$ is a conformal diffeomorphism (2.30) of the flat metric.

## 2.2 Conformal field theory on a curved manifold

Consider a spin-$s$ matter tensor field $\Phi$ on a Lorentzian manifold $(\mathcal{M}, g)$ described by a classical action

$$I[g, \Phi] = \int_{\mathcal{M}} d^d x \sqrt{-g(x)}\, \mathcal{L}\big(g_{ab}(x), \Phi_{a_1 \dots a_s}(x), \dots\big) \tag{2.42}$$

where the Lagrangian density involves up to second derivatives of $g_{ab}(x)$ and first derivatives of $\Phi_{a_1 \dots a_s}(x)$. Under a diffeomorphism $F$ the matter field transforms by pull-back as

$$\widetilde{\Phi}_{a_1 \dots a_s}(x) = \frac{\partial F^{b_1}}{\partial x^{a_1}} \cdots \frac{\partial F^{b_s}}{\partial x^{a_s}} \Phi_{b_1 \dots b_s}(F(x)). \tag{2.43}$$

Using (2.10), the action of a conformal diffeomorphism is thus given by

$$\widetilde{\Phi}_{a_1 \dots a_s}(x) = \Omega_F(x)^s\, M^{b_1}{}_{a_1}(x) \cdots M^{b_s}{}_{a_s}(x)\, \Phi_{b_1 \dots b_s}(F(x)). \tag{2.44}$$

The action of Weyl $\mathcal{M}$ can also be extended to the matter fields as (note the minus sign in the exponent [58, 59])

$$\widehat{\Phi}_{a_1 \dots a_s}(x) = e^{-\Delta_{\text{W}}\, \omega(x)}\, \Phi_{a_1 \dots a_s}(x), \tag{2.45}$$

where $\Delta_{\text{W}} > 0$ is the Weyl dimension of $\Phi$ which is independent of the spin $s$. Finally the action of CIso $(\mathcal{M}, g)$ is given by

$$\widehat{\widetilde{\Phi}}_{a_1 \dots a_s}(x) = \Omega_F(x)^{-\Delta}\, M^{b_1}{}_{a_1}(x) \cdots M^{b_s}{}_{a_s}(x)\, \Phi_{a_1 \dots a_s}(F(x)), \tag{2.46}$$

where $\Delta = \Delta_{\text{W}} - s$ is the scaling dimension of $\Phi$. Notice that the scaling and Weyl dimensions differ by the spin of the field. The transformation law (2.46) can also be generalized beyond spin-$s$ tensor fields by replacing $M^a{}_b$ with an appropriate representation of the group (2.3) (at least in flat space [60–62]).

We will assume that the theory defined by (2.42) is diffeomorphism invariant

$$I[\widetilde{g}, \widetilde{\Phi}] = I[g, \Phi] \tag{2.47}$$

which is achieved if the Lagrangian density transforms a scalar field $\widetilde{\mathcal{L}}(x) = \mathcal{L}(F(x))$. In addition, we assume that theory is Weyl invariant

$$I[\widehat{g}, \widehat{\Phi}] = I[g, \Phi] \tag{2.48}$$

which requires that the Lagrangian density $\mathcal{L}(x)$ has Weyl dimension $\Delta_{\rm W} = d$. As a result, the theory is Diff $\ltimes$ Weyl invariant and it is called a conformal field theory (CFT):

$$\text{Conformal Field Theory} = \text{Diff} \ltimes \text{Weyl invariant field theory}.$$

Due to Weyl invariance, the CFT is only sensitive to the conformal class of the metric (the metric modulo Weyl transformations). Usually we want to fix the metric $g$ itself (not only the conformal class) which determines the "background" on which the CFT lives. This breaks the Diff $\ltimes$ Weyl symmetry group to the subgroup that keeps the metric invariant: the conformal isometry group $\text{CIso}(\mathcal{M}, g)$. As a result, CFTs on a fixed background have more symmetry than ordinary field theories that are only invariant under the isometry group $\text{Iso}(\mathcal{M}, g)$.

Note that one always obtains a $\text{CIso}(\mathcal{M}, g)$ invariant theory from a Diff $\ltimes$ Weyl invariant theory upon fixing the background metric, but the reverse is not true upon coupling to a background metric [63]. There is evidence that the lift can be done for unitary theories, but for higher-derivative non-unitary theories there are known counterexamples [64, 65]. Hence the distinction between CIso and Diff $\ltimes$ Weyl is subtle and we take (2.49) as our definition of a CFT. For an early discussion on the relation between CIso and Diff $\ltimes$ Weyl symmetry for local fields, see [66].

## 2.3 Quantization of conformal field theory

Let us quantize the classical CFT using path integrals. For this, we assume that the manifold $\mathcal{M}$ admits a time-like direction parametrized by $t \in \mathbb{R}$ that can be used to foliate the spacetime with codimension-1 space-like slices. Observables of the CFT are time-ordered correlation functions defined as

$$\langle \cdots \rangle_g \equiv \frac{1}{Z[g]} \int [d\Phi]_g \, \mathcal{T}\{\cdots\} \, e^{iI[g,\Phi]}, \quad Z[g] = e^{iW[g]} = \int [d\Phi]_g \, e^{iI[g,\Phi]}, \tag{2.49}$$

where $\mathcal{T}$ denotes time-ordering in $t$ and the integration is performed over all field configurations on $\mathcal{M}$.[4] The integration measure $[d\Phi]_g$ necessarily depends on the background metric, because it is needed for constructing a coordinate invariant inner-product on the space of field configurations $\Phi$ (see for instance [67]). For us the path integral (2.49) is a formal object that we use as a rule to do formal manipulations on correlation functions. Its definition also requires regularization and renormalization which we will keep implicit.

The integrals over spatial directions do not converge in general due to oscillatory behaviour of the complex exponential. The integrals can be made to converge by giving the time-variable a small imaginary part

$$t \to t(1 - i\epsilon) = te^{-i\epsilon}, \quad \epsilon \to 0^+ \tag{2.50}$$

which is assumed implicitly in (2.49) and which is consistent with time-ordering. This requires that the domain of all fields is analytically continued to the neighbourhood of the real axis in the complex-$t$. The analytic continuation can be assumed to extend such that we can take $t$ to be completely imaginary $t = -i\tau$ so that (2.49) becomes a Euclidean path integral.

[4] The theory is not gravitational so the metric is not integrated over.

### 2.3.1 Stress tensor and the Weyl anomaly

Symmetries of the classical theory can be broken by the integration measure $[d\Phi]_g$, because to properly define it, it has to be regulated and the regulator might break these symmetries. We will assume that the regularization method is chosen such that diffeomorphism invariance is respected which is achieved for example using dimensional regularization. However, it turns out that depending on the dimension $d$ and on the presence of boundaries, Weyl invariance cannot be simultaneously respected. Thus generically we have the relations

$$[d\widetilde{\Phi}]_{\widetilde{g}} = [d\Phi]_g, \quad [d\widehat{\Phi}]_{\widehat{g}} = [d\Phi]_g \, e^{i\Gamma_{\rm W}[g,\widehat{g}]}, \tag{2.51}$$

where $\Gamma_{\rm W}[g,\widehat{g}] \neq 0$ so that Weyl invariance is broken. As a result, the generating functional $W[g]$ defined in (2.49) is diffeomorphism invariant, but not Weyl invariant:

$$W[\widetilde{g}] = W[g], \quad W[\widehat{g}] = W[g] + \Gamma_{\rm W}[g,\widehat{g}]. \tag{2.52}$$

Hence we can write

$$W[g] = \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\,\mathcal{L}_{\rm W}(g_{ab}(x),\ldots), \tag{2.53}$$

where ellipsis denote derivatives of the metric and the function $\mathcal{L}_{\rm W}$ transforms as a scalar under diffeomorphisms since $W[g]$ is diffeomorphism invariant. The expression for $\mathcal{L}_{\rm W}$ can be found for example in two-dimensional CFTs [67] (see the next chapter) and in some higher-dimensional cases [58, 59]. In general, it involves derivatives of infinitely high order in $g_{ab}(x)$ and is thus a non-local.

The non-invariance of the integration measure under Weyl transformations (2.51) appears also in correlation functions of the stress tensor of the CFT which is defined from the classical action as

$$T^g_{ab}(x) = \frac{2}{\sqrt{-g}}\frac{\delta I[g,\Phi]}{\delta g^{ab}(x)} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial g^{ab}(x)}. \tag{2.54}$$

Now the invariance of the classical action $I[g,\Phi]$ under an infinitesimal element of Diff $\ltimes$ Weyl gives the equation

$$\int_{\mathcal{M}} d^dx\sqrt{-g}\,T^g_{ab}\,(\nabla^a\xi^b - \delta\omega\, g^{ab}) = 0 \tag{2.55}$$

which after integration by parts and dropping boundary terms gives

$$\nabla^a T^g_{ab}(x) = 0, \quad g^{ab}(x)\,T^g_{ab}(x) = 0. \tag{2.56}$$

We see that diffeomorphism invariance implies conservation of the classical stress tensor while Weyl invariance implies that it is traceless with respect to the metric $g$. However, due to the Weyl anomaly (2.51), this is no longer true at the quantum level. To see this, consider the 1-point function of (2.54) obtained as

$$\langle T^g_{ab}(x)\rangle_g = \frac{2}{\sqrt{-g}}\frac{\delta W[g]}{\delta g^{ab}(x)} \tag{2.57}$$

and due to (2.52) it is conserved but not traceless

$$\nabla^a\langle T^g_{ab}(x)\rangle_g = 0, \quad g^{ab}(x)\,\langle T^g_{ab}(x)\rangle_g = A^g_{\rm W}(x), \tag{2.58}$$

where the Weyl anomaly is explicitly

$$A^g_{\mathrm{W}}(x) = g^{ab}(x)\,\frac{\partial(\sqrt{-g}\,\mathcal{L}_{\mathrm{W}})}{\partial g^{ab}(x)} \tag{2.59}$$

In general, $A^g_{\mathrm{W}}$ is a local curvature invariant constructed from derivatives of the metric $g$. The possible curvature invariants appearing can be divided into types A and B depending on how the transform under Weyl transformations [68].

It is possible in some cases to remove the Weyl anomaly by the addition of local (in components of the background metric) counterterms to the action of the CFT, but these counterterms violate diffeomorphism invariance so that $\langle T^g_{ab}\rangle_g$ is no longer conserved [69]. As a result, the Weyl anomaly can be exchanged for a diffeomorphism anomaly. There does not exists a scheme in which both the Weyl and diffeomorphism anomaly can be removed simultaneously which is due to the fact that the CFT actually exhibits a combined Diff $\ltimes$ Weyl anomaly.

### 2.3.2 Scalar correlation functions

From the fundamental fields $\Phi$ one can construct composite fields $\mathcal{O}_i(x) \equiv \mathcal{O}_i(\Phi(x))$ which for simplicty we assume to transform as scalars under diffeomorphisms with equal Weyl and scaling dimensions $\Delta_i = \Delta_{\mathrm{W},i}$.[5] We want to compute correlation functions (2.49) with operator insertions as

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_g = \frac{1}{Z[g]}\int [d\Phi]_g\,\mathcal{O}_1(\Phi(x))\cdots\mathcal{O}_n(\Phi(x))\,e^{iI[g,\Phi]}. \tag{2.60}$$

We will consider the action of an element $\psi \equiv (F,\omega) \in$ Diff $\ltimes$ Weyl on these correlation functions. The integration in (2.60) is over all field configurations and we are free to replace the dummy integration variable by $\Phi \to \psi^{-1}\Phi$. Then using Diff $\ltimes$ Weyl invariance of the action $I[g,\psi^{-1}\Phi] = I[\psi g,\Phi]$ and transformation properties (2.51) of the integration measure $[d(\psi^{-1}\Phi)]_g = e^{i\Gamma_{\mathrm{W}}[\psi g,g]}\,[d\Phi]_{\psi g}$, we get (see also [70])

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_g = e^{i\Gamma_{\mathrm{W}}[\psi g,g]}\,\langle(\psi^{-1}\mathcal{O}_1)(x_1)\cdots(\psi^{-1}\mathcal{O}_n)(x_n)\rangle_{\psi g} \tag{2.61}$$

which is a relation between correlation functions of the CFT living on manifolds $(\mathcal{M},g)$ and $(\mathcal{M},\psi g)$ with two different metrics. If we specialize to the action of a conformal isometry $\psi = (F,\omega_F) \in \mathrm{CIso}\,(\mathcal{M},g)$ under which the metric $g$ is invariant, we get a constraint for the correlations function on the same background metric

$$\Omega_F(x_1)^{\Delta_1}\cdots\Omega_F(x_n)^{\Delta_n}\,\langle\mathcal{O}_1(F(x_1))\cdots\mathcal{O}_n(F(x_n))\rangle = \langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_g, \tag{2.62}$$

where we used (2.18), $\Omega_{F^{-1}}(F(x)) = \Omega_F(x)^{-1}$ and that $\Gamma_{\mathrm{W}}[g,g] = 0$.

Expanding the left-hand side of (2.62) for infinitesimal diffeomorphisms gives a differential equation for the correlation function which fixes its functional form completely in certain cases such as in Minkowski space $\mathbb{R}^{1,1}$. Let $x^a$ be Cartesian coordinates on $\mathbb{R}^{1,1}$ where the metric is diagonal (2.24). The one- and two-point functions are fixed to [50]

$$\langle\mathcal{O}_i(x)\rangle_{\mathbb{R}^{1,1}} = 0, \quad \langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle_{\mathbb{R}^{1,1}} = \frac{\delta_{ij}}{|x_1-x_2|^{\Delta_i+\Delta_j}} \tag{2.63}$$

[5] The Lagrangian density is an example of such an operator.

where the Lorentzian norm

$$|x_1 - x_2| \equiv \eta_{ab}\,(x_1 - x_2)^a (x_1 - x_2)^b \tag{2.64}$$

and we have used the freedom in the coefficients of $\mathcal{O}_i$ to normalize the coefficient of the two-point functions. The three-point function is fixed up to a single real parameter $c_{ijk}$ as

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle_{\mathbb{R}^{1,1}} = \frac{c_{ijk}}{|x_1 - x_2|^{\Delta_i+\Delta_j-\Delta_k}|x_2 - x_3|^{\Delta_j+\Delta_k-\Delta_i}|x_1 - x_3|^{\Delta_k+\Delta_i-\Delta_j}} \tag{2.65}$$

while four-point functions are fixed up to an arbitrary function of two cross ratios [50]. Finally these results can be extended to any conformally flat metric by using (2.61).

The state-operator correspondence of the CFT implies reduction of higher $n$-point functions to sums of products of three- and two-point functions via the operator product expansion (OPE) (see for example [51]). The coefficients $c_{ijk}$ appearing in the three-point functions are known as OPE coefficients and they determine correlation functions of the theory. Hence a set of operators $\{\mathcal{O}_i(x)\}$ with associated with scaling dimensions, spins and OPE coefficients $\{\Delta_i, s_i, c_{ijk}\}$ completely determine a CFT in the sense that they can be used to construct all $n$-point correlation functions that satisfy conformal Ward identities. Such a construction does not require reference to an underlying Lagrangian which does not necessarily exist (it might not be possible to interpret the correlation functions as a path integral). For a CFT defined by a Lagrangian, one has to a specify the set of fundamental fields from which one can construct a set composite operators (usually an infinite set). Determining scaling dimensions and OPE coefficients using the path integral then determines the CFT.

## 2.4 Relation to the articles

The basics of CFTs reviewed in this chapter form the basis of all four articles [A1–A4]. Understanding properties of CFTs on curved manifolds was the focus of the article [A2] where we considered a three-dimensional CFT on an $S^3$ (ABJM theory). In the article [A4], we used the Weyl anomaly presented above to compute Euclidean path integrals of CFTs on a manifold with a curvature singularity.

# Chapter 3

# Conformal field theory in two dimensions

In the previous chapter, we studied CFTs in general dimensions, but in many condensed matter systems, the CFTs that arise are actually only two-dimensional. A traditional example is a one-dimensional quantum many-body system consisting of qubits (spins) on a chain with nearest neighbour interactions [71]. These spin chains can exhibit a quantum phase transition[1] in the thermodynamic limit where the number of spins is taken to infinity. At the critical point where the phase transition occurs, the low-energy vacuum fluctuations of the spin chain are described by a two-dimensional continuum CFT on Minkowski space [72].

In addition, 2D CFTs in Euclidean signature appear as continuum descriptions of 2D classical statistical mechanical systems near a thermal phase transition. For example, the partition function of a 2D classical Ising model at the critical temperature can be written as a Euclidean path integral in a 2D CFT of central charge $c = 1/2$ [70]. Hence Euclidean CFT methods are also important for condensed matter applications.

In this chapter, we will focus our attention to two-dimensional CFTs on a flat background and consider their operator formulation. The fact that the conformal group in two dimensions is infinite-dimensional leads to a rich mathematical structure. Our main application is in CFTs with boundaries where the large amount of conformal symmetry can be translated into constraints on the spectrum of the theory. The purpose of this chapter is not to be exhaustive, but present the necessary background for the articles [A1, A3] appearing in this thesis. In contrary to previous introductory reviews of CFTs that have appeared in the literature, we will mostly work in Lorentzian signature instead of the Euclidean plane.

## 3.1 Operator formulation

Consider a two-dimensional CFT on the Lorentzian cylinder $(S^1 \times \mathbb{R}, g^{\text{flat}})$ with a fixed flat background metric and coordinates $x = (\phi, t)$. The conformal isometry group is given by (2.40) so that the CFT has an infinite-dimensional $\widetilde{\text{Diff}}_+ S^1 \times \widetilde{\text{Diff}}_+ S^1$ symmetry at the classical level. In the previous section, we considered quantization of the CFT using path integrals which is equivalent to canonical quantization. In the canonically quantized theory, classical fields are replaced by local operators acting on a Hilbert space.

[1] In contrast to thermal phase transitions that occur as a function of temperature, quantum phase transitions occur at zero temperature as a function of a parameter (for example the strength of an external magnetic field) appearing in the Hamiltonian operator of the system.

To every classical symmetry there is a corresponding Noether charge which becomes an operator that implements these symmetries. Given a conformal Killing vector $\xi$ (2.23), infinitesimal conformal diffeomorphisms (2.20) are implemented by a Hermitian operator $H_\xi$ as

$$\delta\mathcal{O}(x) = -i\,[H_\xi, \mathcal{O}(x)]. \tag{3.1}$$

Denoting the stress tensor (2.54) restricted to the flat background by $T_{ab}(\phi, t)$, the generator is given by

$$H_\xi = \int_0^{2\pi} d\phi\, \xi^a(\phi, 0)\, T_{at}(\phi, 0). \tag{3.2}$$

In light-ray coordinates $x^\pm = \phi \pm t$ introduced in (2.36), conservation and tracelessness (2.58) in flat space imply that the components of the stress tensor satisfy

$$T_{--}(x^-, x^+) = T_{--}(x^-), \quad T_{++}(x^-, x^+) = T_{++}(x^+), \quad T_{-+}(x^-, x^+) = 0. \tag{3.3}$$

Hence there are only two independent components $T_{\pm\pm}(x^\pm + 2\pi) = T_{\pm\pm}(x^\pm)$ and they respect the identification (2.37). Since conformal Killing vectors in two-dimensions are given by $\xi^\pm(\phi, t) = \xi^\pm(x^\pm)$, the generator of conformal diffeomorphisms is simply

$$H_\xi = \int_0^{2\pi} d\phi\, [\xi^+(\phi)\, T_{++}(\phi) - \xi^-(\phi)\, T_{--}(\phi)]. \tag{3.4}$$

We can now use these equations to determine the commutation relation of the stress tensor with itself. For a two-dimensional conformal diffeomorphism (2.30), the only non-zero components of the Jacobian matrix are $\partial_\pm F^\pm = \partial_\pm f_\pm$. From the transformation law (2.43) of a spin $s = 2$ tensor under diffeomorphisms, we get that the stress tensor transforms under conformal diffeomorphisms as ($T_{-+}$ remains zero)

$$\widetilde{T}_{\pm\pm}(x^\pm) = (\partial_\pm f_\pm)^2\, T_{\pm\pm}(f_\pm(x^\pm)) \tag{3.5}$$

which at the infinitesimal level $f_\pm(x^\pm) = x^\pm + \xi^\pm(x^\pm)$ corresponds to

$$\delta T_{\pm\pm}(x^\pm) = 2\,(\partial_\pm \xi^\pm)\, T_{\pm\pm}(x^\pm) + \xi^\pm(x^\pm)\, \partial_\pm T_{\pm\pm}(x^\pm). \tag{3.6}$$

This is implemented by (3.1) when the stress tensors are given by[2]

$$T_{--}(x^-) = T(x^-) \otimes \mathbf{1}, \quad T_{++}(x^+) = \mathbf{1} \otimes T(-x^+) \tag{3.7}$$

where the operator $T(\phi)$ satisfies the commutation relation[3]

$$[T(\phi_1), T(\phi_2)] = -i\,\big(T(\phi_1) + T(\phi_2)\big)\, \delta'_{2\pi}(\phi_1 - \phi_2) + \text{c-number}. \tag{3.8}$$

Notice that the tensor product structure implies that $[T_{\pm\pm}(\phi_1), T_{\mp\mp}(\phi_2)] = 0$.

Neglecting the additive c-number (the Schwinger term [74]) in (3.8), this is the algebra $\mathrm{Vect}\, S^1$ of real valued vector fields on the circle which is the Lie algebra of $\widetilde{\mathrm{Diff}}_+ S^1$ (see for

[2]Notice the minus sign in $T(-x^+)$ which is in agreement for example with [73]. We will show below how the standard Euclidean stress tensors are obtained from Wick rotation given this sign.

[3]Here $\delta_{2\pi}(\phi)$ is the $2\pi$-periodic delta function defined as the Fourier series $\delta_{2\pi}(\phi) = \sum_{n=-\infty}^{\infty} e^{in\phi}$.

example [56]). Hence the stress tensor gives a representation of the conformal algebra Vect $S^1 \times$ Vect $S^1$ which is the infinitesimal classical symmetry of the CFT. The Schwinger term is a quantum correction that extends the classical symmetry algebra. It turns out that the Schwinger term is *required* due to the presence of the Weyl anomaly in correlation functions of the stress tensor, and in the next section, we will show that the Schwinger term is completely fixed by the Weyl anomaly. It is also fixed by mathematical fact that the central extension of Vect $S^1$ is unique and isomorphic to the Virasoro algebra $\mathfrak{vir}$. The result is

$$[T(\phi_1), T(\phi_2)] = -i\left(T(\phi_1) + T(\phi_2)\right)\delta'_{2\pi}(\phi_1 - \phi_2) + \frac{ic}{24\pi}\delta'''_{2\pi}(\phi_1 - \phi_2) \tag{3.9}$$

which is a local presentation of $\mathfrak{vir}$. The algebra can be written in a more familiar form by expanding

$$T(\phi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(L_n - \frac{c}{24}\delta_{n,0}\right)e^{in\phi}, \tag{3.10}$$

where $L_n^\dagger = L_{-n}$. Substituting to (3.9) gives

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{c}{12}n(n^2-1)\,\delta_{n,-m}. \tag{3.11}$$

The Virasoro algebra $\mathfrak{vir}$ (3.11) is the Lie algebra of the Virasoro group Vir which is the central extension of the Lie group $\widetilde{\mathrm{Diff}}_+ S^1$ (see for example [56]). Hence the presence of the Weyl anomaly in the CFT centrally extends the classical symmetry group. This is a manifestation of the fact that in quantum mechanics, classical symmetries are represented by projective representations acting on the Hilbert space and they satisfy the group multiplication law only up to a phase. To each projective representation of $\widetilde{\mathrm{Diff}}_+ S^1$ corresponds an ordinary representation of Vir which is why Vir $\times$ Vir is the relevant group for CFTs at the quantum level.

Now using the centrally extended algebra (3.9) in (3.1) leads to an additional term $-\frac{c}{24\pi}\partial_\pm^3\xi^\pm$ in the infinitesimal transformation (3.6) of the stress tensor. The result can be integrated which gives the quantum modification of (3.5) as

$$\widetilde{T}_{\pm\pm}(x^\pm) = (\partial_\pm f_\pm)^2\, T_{\pm\pm}(f_\pm(x^\pm)) - \frac{c}{24\pi}\{f_\pm(x^\pm), x^\pm\} \tag{3.12}$$

where the Schwarzian derivative is defined as

$$\{f(\phi), \phi\} = \left(\frac{f''(\phi)}{f'(\phi)}\right)' - \frac{1}{2}\left(\frac{f''(\phi)}{f'(\phi)}\right)^2. \tag{3.13}$$

Equation (3.12) implies that the stress tensor transforms in the coadjoint representation of Vir $\times$ Vir [56].

### 3.1.1 Structure of the Hilbert space

Due to the presence of Vir $\times$ Vir symmetry, the Hilbert space of the CFT decomposes to a direct sum of its irreducible representations $\mathcal{H}_h \otimes \mathcal{H}_{\bar{h}}$ parametrized by two positive real numbers $(h, \bar{h})$ as

$$\mathcal{H}_{\mathrm{CFT}} = \bigoplus_{h,\bar{h}} d_{h\bar{h}}\,\mathcal{H}_h \otimes \mathcal{H}_{\bar{h}} \tag{3.14}$$

where $d_{h\bar{h}}$ are integers characterizing the degeneracy of each representation. Each irreducible representation $\mathcal{H}_h$ is spanned by basis states

$$|h, \{m_1, \ldots, m_n\}\rangle = \prod_{k=1}^{n} L_{-m_k}|h\rangle, \quad L_0|h\rangle = h|h\rangle, \quad L_{n>0}|h\rangle = 0. \tag{3.15}$$

The Hamiltonian $H_{\rm CFT}$ of the CFT is given by

$$H_{\rm CFT} = \int_0^{2\pi} d\phi\, T_{tt}(\phi, 0) = \int_0^{2\pi} d\phi\, [T_{--}(\phi) + T_{++}(\phi)] \tag{3.16}$$

which generates time-translations $t + \delta t$ corresponding to $\xi^{\pm}(x^{\pm}) = \pm 1$ in (3.4). The integral simply picks up the zero mode of the Fourier expansions (3.10) with the result

$$H_{\rm CFT} = L_0 \otimes \mathbf{1} + \mathbf{1} \otimes L_0 - \frac{c}{12}. \tag{3.17}$$

Thus the energy of the highest-weight state

$$|h, \bar{h}\rangle = |h\rangle \otimes |\bar{h}\rangle \tag{3.18}$$

is given by $E_{h,\bar{h}} = h + \bar{h} - \frac{c}{12}$ and the ground state energy is $E_{\rm gs} \equiv E_{0,0} = -\frac{c}{12}$ which is the Casimir energy due to periodic boundary conditions on the circle.

### 3.1.2 The state operator correspondence

In addition to the stress tensor, the 2D CFT includes other fields whose components in light-ray coordinates transform under conformal diffeomorphisms (2.30) as

$$\widetilde{\mathcal{O}}_{h,\bar{h}}(x^-, x^+) = (\partial_- f_-)^h\, (\partial_+ f_+)^{\bar{h}}\, \mathcal{O}_{h,\bar{h}}(f_-(x^-), f_+(x^+)). \tag{3.19}$$

These fields are called conformal tensors of weight $(h, \bar{h}) \in \mathbb{R}^2$ that are also known as primary operators. An example is a 2-form $\Phi_{ab}$ with only one independent component $\Phi_{-+} = -\Phi_{+-}$ that transforms as $(1, 1)$ conformal tensor. Similarly, the components of the stress tensor are $(2, 0)$ and $(0, 2)$ conformal tensors at the classical level. There are also conformal tensors with negative weights.

We can write the conformal tensor as a tensor product operator

$$\mathcal{O}_{h,\bar{h}}(x^-, x^+) \equiv \mathcal{O}_h(x^-) \otimes \mathcal{O}_{\bar{h}}(-x^+) \tag{3.20}$$

where the factors transform with weights $(h, 0)$ and $(0, \bar{h})$ respectively. Defining the mode expansion

$$\mathcal{O}_h(\phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathcal{O}_{h,n}\, e^{in\phi}, \tag{3.21}$$

with $\mathcal{O}_{h,n}^{\dagger} = \mathcal{O}_{h,-n}$, the transformation law (3.19) is reproduced by (3.1) at the infinitesimal level when we have the commutation relations [50]

$$[L_n, \mathcal{O}_{h,m}] = ((h-1)\, n - m - h)\, \mathcal{O}_{h,n+m}. \tag{3.22}$$

Two-dimensional CFTs have a state-operator correspondence which means that to each state in $\mathcal{H}_{\text{CFT}}$ there is a corresponding local operator acting on $\mathcal{H}_{\text{CFT}}$. Local operators corresponding to highest-weight states (3.18) are called primary operators and they are given by

$$|h,\bar{h}\rangle = \mathcal{O}_{h,\bar{h}}|0\rangle. \tag{3.23}$$

One can check that the conditions (3.15) for a highest-weight states are satisfied due to (3.22) and $\mathcal{O}_{h,n>0}|0\rangle = 0$ (which is required for consistency with the fact that the one-point function of a primary operator on the circle vanishes). As a result, we get

$$|h,\bar{h}\rangle = \lim_{\tau\to-\infty} \mathcal{O}_{h,\bar{h}}(\phi+i\tau,\phi-i\tau)|0\rangle \tag{3.24}$$

which can be understood as an insertion of the local operator (3.19) at $\tau = -\infty$ in Euclidean time. More generally, any local operator defines a unique state in the Hilbert space in this manner and the reverse is also true: every state determines a local operator. This is the state-operator correspondence.

## 3.2 Stress tensor correlators from the Polyakov action

In this section, we will study Euclidean correlation functions of the stress tensor. The Euclidean CFT is obtained by a Wick rotation $t = -i\tau$ after which the light-ray coordinates become complex:

$$w = \phi + i\tau = x^-|_{t=-i\tau}, \quad \overline{w} = \phi - i\tau = x^+|_{t=-i\tau} \tag{3.25}$$

and the flat Lorentzian metric becomes Euclidean. Assuming $\tau \in \mathbb{R}$, we obtain an infinite Euclidean cylinder which can be mapped to the complex plane via

$$z = e^{-iw} = e^{\tau-i\phi}, \quad \bar{z} = e^{i\overline{w}} = e^{\tau+i\phi}. \tag{3.26}$$

Stress tensors on the complex plane are defined as

$$T_{zz}(z) = (\partial_w z)^2\, T_{--}(w(z)) - \frac{c}{24\pi}\{w(z),z\}, \quad T_{\bar{z}\bar{z}}(\bar{z}) = (\partial_{\overline{w}}\bar{z})^2\, T_{++}(\overline{w}(\bar{z})) - \frac{c}{24\pi}\{\overline{w}(\bar{z}),\bar{z}\} \tag{3.27}$$

and substituting the expansions (3.10) gives

$$T_{zz}(z) = -\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(L_0\otimes\mathbf{1})\, z^{-n-2}, \quad T_{\bar{z}\bar{z}}(\bar{z}) = -\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(\mathbf{1}\otimes L_0)\,\bar{z}^{-n-2} \tag{3.28}$$

which are the standard expansions obtained from radial quantization.[4] We want to compute Euclidean correlation functions on the complex plane defined as

$$\langle T_{zz}(z_1)\cdots T_{zz}(z_n)\rangle \equiv \langle 0|\mathcal{R}\{T_{zz}(z_1)\cdots T_{zz}(z_n)\}|0\rangle \tag{3.29}$$

where $\mathcal{R}$ denotes radial ordering in increasing $|z| = e^\tau$ which is what time-ordering $\mathcal{T}$ in the Lorentzian time $t$ becomes under the mappings above.

[4] It is common in the literature to adopt the notation $L_n \equiv L_n\otimes\mathbf{1}$ and $\bar{L}_n \equiv \mathbf{1}\otimes L_n$, but we will not do that here.

In two-dimensions, the infinite dimensional conformal symmetry is enough to fix the generating functional $W[g]$ of Euclidean stress tensor correlation functions up to a single coefficient. $W[g]$ can be found by analyzing the Weyl anomaly which has fixed the one-point function as[5]

$$g^{ab}(x)\,\langle T^{g}_{ab}(x)\rangle_{g} = \frac{c}{24\pi}R(x). \tag{3.30}$$

where $R$ is the Ricci scalar of $g$ and we will see below that the constant $c$ matches with the central charge of the Virasoro algebra.

Equation (3.30) contains enough information to fix all higher-order correlation functions (3.29) of the stress tensor, because it fixes the generating functional $W[g]$. Now $W[g]$ is an integral of $\mathcal{L}_{\mathrm{W}}$ as defined in (2.53). Using (2.58) combined with (3.30), we get the equation

$$g^{ab}\,\frac{\partial(\sqrt{g}\,\mathcal{L}_{\mathrm{W}})}{\partial g^{ab}} = \frac{c}{24\pi}\sqrt{g}\,R. \tag{3.31}$$

In two-dimensions, we can always find a coordinate system $x$ such that the metric is diagonal up to a conformal factor

$$g_{ab}(x) = e^{2\varphi(x)}\,\delta_{ab} \tag{3.32}$$

where $\varphi(x)$ is a function. This coordinate system is known to exist globally on $\mathbb{R}^2$ [75, 76]. In these coordinates $\sqrt{g} = e^{2\varphi}$ and $R = 2e^{-2\varphi}\,\Box\,\varphi$ so that equation (3.31) becomes

$$\frac{\partial(e^{2\varphi}\mathcal{L}_{\mathrm{W}})}{\partial\varphi} = \frac{c}{24\pi}\,\Box\,\varphi \tag{3.33}$$

where $\Box = -\delta^{ab}\partial_a\partial_b$ is the Laplacian of the flat Euclidean metric in Cartesian coordinates and $\mathcal{L}_{\mathrm{W}} = \mathcal{L}_{\mathrm{W}}(\varphi)$ is only a function of $\varphi$ and its derivatives due to (3.32).

Equation (3.33) can be integrated and a Lagrangian $\mathcal{L}_{\mathrm{W}}$ constructed from derivatives of $\varphi(x)$ exists, because the right-hand side satisfies the necessary integrability conditions [77]. The integrability conditions also ensure that the equation can be integrated along any curve so we can simply integrate along $s\varphi(x)$ with $s \in (0,1)$ which gives the result [59, 77]

$$e^{2\varphi}\mathcal{L}_{\mathrm{W}}(\varphi) - \mathcal{L}_{\mathrm{W}}(0) = \frac{c}{24\pi}\,\varphi\,\Box\,\varphi\int_0^1 ds\,s. \tag{3.34}$$

When $\varphi(x) = 0$, the metric becomes exactly flat and the Weyl anomaly vanishes. Imposing $\mathcal{L}(0) = 0$ and performing the integral over $s$ gives

$$\mathcal{L}_{\mathrm{W}}(\varphi) = \frac{c}{48\pi}\,e^{-2\varphi}\,\varphi\,\Box\,\varphi \tag{3.35}$$

which is valid in the coordinate system where the metric is of the form (3.32). Inverting the expression $R = 2e^{-2\varphi}\,\Box\,\varphi$ gives

$$\varphi(x) = \frac{1}{2}\int_{\mathbb{R}^2} d^2y\,e^{2\varphi(y)}\,G(x,y)\,R(y) \tag{3.36}$$

[5] This is the only possibility, because the 1-point function has to satisfy the WZ consistency conditions. It can also be derived directly from the integration measure [70].

where $G(x, x')$ is the Green's function of the Laplacian of the flat metric:

$$\Box\, G(x, x') = \delta^{(2)}(x - x'). \tag{3.37}$$

Combining (3.36) with (3.35) and substituting to (2.53) gives

$$W[g] = \frac{c}{192\pi} \int_{\mathbb{R}^2} d^2x \sqrt{g(x)} \int_{\mathbb{R}^2} d^2y \sqrt{g(y)}\, G(x, y)\, R(x)\, R(y) \tag{3.38}$$

which is the Polyakov action on the Euclidean plane [78]. This expression is diffeomorphism invariant so that it holds in any coordinate system beyond of (3.32). The generalization of (3.38) to higher-genus Riemann surfaces can also be found [79, 80].

For the holomorphic components of the stress tensor, the Euclidean one- and two-point functions obtained from (3.38) are given by [81]

$$\langle T_{zz}(z) \rangle = 0, \quad \langle T_{zz}(z_1)\, T_{zz}(z_2) \rangle = \frac{c}{8\pi} \frac{1}{(z_1 - z_2)^4}. \tag{3.39}$$

One can check this two-point produces the Schwinger term in the commutator of two stress tensors (3.9) (see for example [57]). This proves that the Weyl anomaly fixes the central extension of the conformal algebra and that the constant $c$ coincides with the central charge.

## 3.3 2D CFT with two boundaries

In this section, we will introduce boundaries to the two-dimensional CFT. This breaks conformal symmetry of the original CFT to a single $\widetilde{\mathrm{Diff}}_+ S^1$ which affects the structure of the Hilbert space and of correlation functions.

### 3.3.1 CFT on the Lorentzian strip

In the previous section we considered a CFT on the Lorentzian cylinder $(S^1 \times \mathbb{R}, g^{\mathrm{flat}})$ where each spatial slice is a circle. We will now introduce two boundaries at $\phi = 0$, $\phi = \pi$ so that the CFT lives on the Lorentzian strip $((0, \pi) \times \mathbb{R}, g^{\mathrm{flat}})$ whose spatial slices are intervals of length $\pi$ (half of the original circle).

The presence of the two boundaries breaks the conformal group $\widetilde{\mathrm{Diff}}_+ S^1 \times \widetilde{\mathrm{Diff}}_+ S^1$ to the subgroup of diffeomorphisms that do not move the boundaries, in other words, diffeomorphisms that preserve the strip. This subgroup is known as the boundary conformal group. We can see that under $F \in \widetilde{\mathrm{Diff}}_+ S^1 \times \widetilde{\mathrm{Diff}}_+ S^1$, the $\phi = 0$ line is mapped to

$$F(0, t) = \frac{1}{2}(f_+(t) + f_-(-t)), \quad \forall t \in \mathbb{R}. \tag{3.40}$$

Imposing $F(0, t) = 0$ then gives

$$f_-(t) = f_+(-t), \quad t \in \mathbb{R}. \tag{3.41}$$

Hence only conformal diffeomorphisms of the form $F(x^-, x^+) = (f(x^-), f(-x^+))$ with $f \in \widetilde{\mathrm{Diff}}_+ S^1$ preserve the boundary at $\phi = 0$. Requiring that $\phi = \pi$ is fixed does not impose any extra condition since $F(\pi, t) = \pi$ follows automatically from $F(0, t) = 0$ and the periodicity conditions. Thus the boundary conformal group is isomorphic to a single $\widetilde{\mathrm{Diff}}_+ S^1$.

To completely define the CFT on the strip, we need to impose boundary conditions at the two boundaries. At $\phi = 0$, we will impose the condition

$$T_{\phi t}(0,t) = 0, \quad \forall t \in \mathbb{R}. \tag{3.42}$$

We also impose this on the second boundary $\phi = \pi$, but it is not an independent condition, because $T_{\phi t}(\pi, t+\pi) = 0$ follows automatically from (3.42) due to $2\pi$-periodicity of $T_{++}(x^+)$.

Boundary conditions satisfying (3.42) are known as a conformal boundary conditions and we will denote the two (in general different) boundary conditions at $\phi = 0, \pi$ by $A$ and $B$ respectively. Dirichlet and Neumann conditions for free fields are examples of conformal boundary conditions, but (3.42) can be imposed without reference to fundamental fields and is thus valid even in CFTs without Lagrangians.

By using $T_{\phi t}(\phi, t) = T_{++}(x^+) - T_{--}(x^-)$, the condition (3.42) implies

$$T_{--}(t) = T_{++}(-t), \quad t \in \mathbb{R} \tag{3.43}$$

so that the two components of the classical stress tensor on the Lorentzian strip $(0,\pi) \times \mathbb{R}$ are not independent. Upon quantization, there is thus only one stress tensor operator

$$T(\phi) \equiv T_{\pm\pm}(\mp\phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left( L_n - \frac{c}{24} \delta_{n,0} \right) e^{in\phi} \tag{3.44}$$

where $L_n$ satisfy the Virasoro algebra (3.11). In particular, there is no tensor product structure as in (3.7). The corresponding Virasoro group provides a projective representation for the boundary conformal diffeomorphisms $f_\pm(\phi) = f(\mp\phi)$ that are classical symmetries of the boundary CFT.

Due to the presence of only one copy of the Virasoro symmetry, the Hilbert space of the BCFT decomposes as

$$\mathcal{H}_{\text{BCFT}} = \bigoplus_h n_{AB}^h \mathcal{H}_h \tag{3.45}$$

where $n_{AB}^h \in \mathbb{N}_0$ gives the degeneracy of the representation $\mathcal{H}_h$ of Vir and it depends on the boundary conditions.

The Hamiltonian operator of the BCFT is given by

$$H_{\text{BCFT}} = \int_0^\pi d\phi\, T_{tt}(\phi, 0) = \int_{-\pi}^{\pi} d\phi\, T(\phi) \tag{3.46}$$

where we used $T_{tt}(\phi, t) = T_{--}(x^-) + T_{++}(x^+)$ and equation (3.44) to extend the range of the integral to $(-\pi, \pi)$. By using the mode expansion (3.44), the integral picks up the zero mode as

$$H_{\text{BCFT}} = L_0 - \frac{c}{24} \tag{3.47}$$

so that primary states are energy eigenstates with energy $E_h = h - \frac{c}{24}$.

In a unitary boundary CFT the vacuum representation satisfies $n_{AB}^0 = \delta_{AB}$ so that it only appears when $A = B$ in which case the ground state energy $E_{\text{gs}} = E_0 = -\frac{c}{24}$ is the Casimir energy between two parallel plates. When $A \neq B$, the vacuum state does not appear in the Hilbert space (3.45) so that different boundary conditions lead to an increase in the ground

state energy. Denoting the smallest non-zero weight appearing in (3.45) by $h_{\rm bcc}$, the increase in the ground state energy due to boundary conditions is $\Delta E_{\rm gs} = h_{\rm bcc}$.

The CFT on the strip also enjoys state-operator correspondence which works in the same way as for the CFT on a circle. The main difference is that primary operators no longer have the tensor product structure (3.20), because the Hilbert space of the BCFT does not involve tensor product representations. Instead, the operator corresponding to the highest weight state $|h\rangle \in \mathcal{H}_h$ is simply the chiral operator $\mathcal{O}_h(x^-)$ and it is known as a boundary primary. When the boundary conditions are different $A = B$, the boundary primary operator $\mathcal{O}_{\rm bcc}$ corresponding to $|h_{\rm bcc}\rangle$ is called the boundary condition changing (BCC) operator:

$$|h_{\rm bcc}\rangle = \lim_{\tau\to-\infty} \mathcal{O}_{\rm bcc}(w)|0\rangle. \tag{3.48}$$

The name comes from the fact that when the Wick rotated Lorentzian strip is mapped to the upper-half plane by (3.26), the operator $\mathcal{O}_{\rm bcc}$ is inserted at the point $w = \overline{w} = 0$ where the boundary condition jumps from $A$ to $B$.

The BCFT also includes operators of the form

$$\mathcal{O}_h(x^-, x^+) \equiv \mathcal{O}_h(x^-)\,\mathcal{O}_h(-x^+) \tag{3.49}$$

where $\mathcal{O}_h(\phi)$ is defined by the expansion (3.21). Under a boundary conformal diffeomorphism $F(x^-, x^+) = (f(x^-), f(-x^+))$ with $f \in \widetilde{\rm Diff}_+ S^1$, it transforms as

$$\widetilde{\mathcal{O}}_h(x^-, x^+) = (\partial_- f(x^-)\,\partial_+ f(x^+))^h\,\mathcal{O}_h(f(x^-), f(-x^+)) \tag{3.50}$$

which coincides with the transformation law of a $h = \bar{h}$ primary operator of a CFT on the full cylinder. The operator (3.49) is a local operator on the Lorentzian strip $\phi \in (0, \pi)$, but a bilocal operator on the full cylinder $\phi \in (0, 2\pi)$ [82].[6] It is local on the strip, because the factors in the product never coincide $x^- \neq -x^+$ unless the operator is placed at a boundary $\phi = 0, \pi$. Taking (3.49) to the boundary, the two factors coincide and the one-point function will have a divergence which coincides with the divergence of a two-point function on the full cylinder. This is the doubling trick on the Lorentzian cylinder.

### 3.3.2 Open closed string duality

We will now consider a CFT on a finite cylinder $S^1_\beta \times (0, W)$ of circumference $\beta$ and width $W$ equipped with the flat Euclidean metric. The object of interest is the Euclidean path integral of the CFT with conformal boundary conditions $A$ and $B$:

$$Z_{AB}(W/\beta) \equiv \int_{A,B} [d\Phi]\, e^{-I[\Phi]} \tag{3.51}$$

which is a function of the dimensionless modulus $W/\beta$ only. There are two ways to slice the path integral and interpret it from the point of view of the operator formulation of the CFT.

The first way is to slice by intervals in which case the Hilbert space on each slice is that of a boundary CFT (3.45). The Euclidean path integral over the cylinder then computes a thermal trace $\mathrm{Tr}\, e^{-\beta H^{\rm op}}$ over $\mathcal{H}_{\rm BCFT}$ where $H^{\rm op}$ is the generator of translations along the circular direction. The second way is to slice by circles giving the Hilbert space $\mathcal{H}_{\rm CFT}$ of a CFT on a circle. In this case, the Euclidean path integral computes a matrix element[7] $\langle A|e^{-WH^{\rm cl}}|B\rangle$ of

[6] By local we mean local from the point of view of the one-dimensional algebra living on a single light-ray. To obtain local operators on the full cylinder one has to take a tensor product as in (3.20).

[7] It is the Wick rotation $t = -i\tau$ of a Lorentzian transition amplitude to Euclidean signature.

the operator generating the Euclidean evolution from $\tau = 0$ to $\tau = W$. We started from the same path integral so that in principle the thermal trace and the matrix element should give the same result:

$$Z_{AB} = \langle A|e^{-WH^{\rm cl}}|B\rangle = \mathrm{Tr}\, e^{-\beta H^{\rm op}} \tag{3.52}$$

which is the open closed string duality.

The states $|A\rangle, |B\rangle \in \mathcal{H}_{\rm CFT}$ appearing in the closed string transition amplitude are special states called boundary states and they are determined by the boundary conditions $A, B$. A generic boundary state $|\mathrm{BS}\rangle \in \mathcal{H}_{\rm CFT}$ satisfies the equation [83]

$$(T_{--}(x^-) - T_{++}(x^+))|\mathrm{BS}\rangle = 0 \tag{3.53}$$

where $T_{\pm\pm}(x^\pm)$ are stress tensor operators (3.9) of the full CFT without a boundary. Using the mode expansions (3.10), we get the conditions

$$L_n|\mathrm{BS}\rangle = \bar{L}_{-n}|\mathrm{BS}\rangle, \quad \forall n \in \mathbb{Z}. \tag{3.54}$$

Setting $n = 0$ shows that $|\mathrm{BS}\rangle$ can only have components on representations $\mathcal{H}_h \otimes \overline{\mathcal{H}}_{\bar{h}}$ with $\bar{h} = h$. Hence boundary states (3.14) belong to the subspace of the Hilbert space containing only spinless irreducible representations of Vir $\times$ Vir. When the CFT is diagonal $d_{h\bar{h}} = \delta_{h\bar{h}}$, one can show that (3.54) is solved by an Ishibashi state [22, 23, 83],

$$|I_h\rangle = \sum_{\{m\}} |h, \{m\}\rangle \otimes \overline{|h, \{m\}\rangle}, \tag{3.55}$$

for any $h$ and it turns out that the set of Ishibashi states for different $h$ span the solution space of the equation (3.54) (at least in diagonal CFTs or in theories with a current algebra symmetry). Hence a general solution of (3.54) is a linear combination of Ishibashi states:

$$|\mathrm{BS}\rangle = \sum_h \langle I_h|\mathrm{BS}\rangle |I_h\rangle. \tag{3.56}$$

Hence for the boundary states $|A\rangle, |B\rangle$ appearing in (3.52), the boundary condition is encoded in the overlaps $\langle I_h|A\rangle$, $\langle I_h|B\rangle$ between Ishibashi states.

The open closed string duality (3.52) imposes constraints on the open string $\mathcal{H}_{\rm BCFT}$ and closed string $\mathcal{H}_{\rm CFT}$ spectra of the CFT. To derive the constraints, we will expand (3.52) in terms of characters $\chi_h(q)$ of the irreducible representations $\mathcal{H}_h$ of the Virasoro algebra. For this, we need to write the open and closed string Hamiltonians in terms of the Virasoro generators. The Hamiltonians for general $\beta$ and $W$ (instead of $\beta = 2\pi$ and $W = \pi$ as above) are given by

$$H^{\rm op} = \frac{\pi}{W} H_{\rm BCFT} = \frac{\pi}{W}\left(L_0 - \frac{c}{24}\right), \quad H^{\rm cl} = \frac{2\pi}{\beta} H_{\rm CFT} = \frac{2\pi}{\beta}\left(L_0 \otimes \mathbf{1} + \mathbf{1} \otimes L_0 - \frac{c}{12}\right). \tag{3.57}$$

Hence we get

$$\mathrm{Tr}\, e^{-\beta H^{\rm op}} = \mathrm{Tr}\, q^{L_0 - \frac{c}{24}}, \quad \langle A|e^{-WH^{\rm cl}}|B\rangle = \langle A|\tilde{q}^{\frac{1}{2}(L_0\otimes\mathbf{1} + \mathbf{1}\otimes L_0 - \frac{c}{12})}|B\rangle, \tag{3.58}$$

where we have defined

$$q = e^{-\pi\beta/W}, \quad \tilde{q} = e^{-4\pi W/\beta}, \tag{3.59}$$

which take values between zero and one. These parameters are related via a modular $S$-transformation $\omega \to -1/\omega$ if we write them in the form $q = e^{2\pi i\omega}$ and $\tilde{q} = e^{-2\pi i/\omega}$ with $\omega = i\beta/(2W)$.

By using the decomposition (3.45) of $\mathcal{H}_{\rm BCFT}$, we get

$$\mathrm{Tr}\, e^{-\beta H^{\rm op}} = \sum_h n_{AB}^h\, \chi_h(q), \tag{3.60}$$

where the character of an irreducible representation $\mathcal{H}_h$ is given by[8]

$$\chi_h(q) = \mathrm{Tr}_{\mathcal{H}_h}\, q^{L_0 - \frac{c}{24}} = \frac{q^{h-\frac{c}{24}}}{\prod_{n=1}^{\infty}(1-q^n)}. \tag{3.61}$$

By expanding the boundary states $|A\rangle, |B\rangle$ in the basis of Ishibashi states (3.56), we get

$$\langle A|e^{-WH^{\rm cl}}|B\rangle = \sum_{h,h'} \langle A|I_h\rangle\langle I_{h'}|B\rangle\, \langle I_h|\tilde{q}^{\frac{1}{2}(L_0\otimes\mathbf{1}+\mathbf{1}\otimes L_0-\frac{c}{12})}|I_{h'}\rangle. \tag{3.62}$$

The Ishibashi states satisfy the orthogonality relation [83]

$$\langle I_h|\tilde{q}^{\frac{1}{2}(L_0\otimes\mathbf{1}+\mathbf{1}\otimes L_0-\frac{c}{12})}|I_{h'}\rangle = \delta_{hh'}\chi_h(\tilde{q}). \tag{3.63}$$

so that (3.62) becomes

$$\langle A|e^{-WH^{\rm cl}}|B\rangle = \sum_h \langle A|I_h\rangle\langle I_h|B\rangle\, \chi_h(\tilde{q}). \tag{3.64}$$

Under modular S-transformations the Virasoro characters (3.61) satisfy

$$\chi_h(\tilde{q}) = \sum_{h'} S_{hh'}\, \chi_{h'}(q) \tag{3.65}$$

where $S_{hh'}$ is known as the modular $S$-matrix which represents the modular $S$-transformation $\omega \to -1/\omega$ on the space of characters. Hence we get

$$\langle A|e^{-WH^{\rm cl}}|B\rangle = \sum_{h,h'} S_{hh'}\, \langle A|I_h\rangle\langle I_h|B\rangle\, \chi_{h'}(q). \tag{3.66}$$

Since the characters $\chi_h(q)$ are linearly independent, the open closed string duality (3.52) implies

$$n_{AB}^h = \sum_{h'} S_{hh'}\, \langle A|I_{h'}\rangle\langle I_{h'}|B\rangle \tag{3.67}$$

which is known as the Cardy equation. The Cardy equation puts constraints on possible boundary conditions $\langle I_h|\mathrm{BS}\rangle$, degeneracy factors $n_{AB}^h$ and $S$-matrices that can appear in a CFT obeying open closed string duality. When the CFT is diagonal and contains a finite number of primaries, a solution was found by Cardy [7].

[8] This is for representations $h > 0$ that do not contain null-vectors. For the character of the vacuum representation, the product in the denominator starts from $n = 2$.

## 3.4 Relation to the articles

Boundary CFTs introduced in this chapter are studied from a holographic point of view in the two articles [A1, A3] where we reproduce some properties of BCFTs presented above from the gravity dual. In the article [A3], the focus was on correlation functions of a scalar operator in the presence of a single boundary in general higher dimensional CFTs. The simplest example we considered was the one-point function which can be non-vanishing when a boundary is present.

In [A1], we considered a two-dimensional holographic BCFT and studied its spectrum of operators using the dual gravity description. To extract the spectrum of the BCFT, we considered its Euclidean path integral on a cylinder with two boundaries as presented above. The gravity dual is able to describe various open and closed string excited states whose scaling dimensions we computed from gravity. We found that boundary condition changing operators also have a dual gravity description. More details on the holographic aspects of [A1, A3] are given in section 4.4 of chapter 4.

# Chapter 4

# Aspects of bottom-up holography

After reviewing aspects of CFTs in the previous two chapters, we move on to study the gravitational side of the holographic duality. In this chapter, we will take the bottom-up approach and work with theories of classical gravity in anti-de Sitter space with the goal of motivating existence of the holographic duality without reference to string theory. The detailed top-down derivation of the duality will be presented in the next chapter.

A simple premise of holography is that the symmetries on both sides of the duality have to match. In top-down string theory examples of the duality, global R-symmetry of the CFT is reflected in isometries of the compact space $M_q$ that appears in the asymptotic expansion of the bulk spacetime in the form $\mathrm{AdS}_{d+1} \times M_q$. We will show how similar matching holds for the spacetime symmetry group when the dual CFT lives on a curved background: the asymptotic symmetry group of $\mathrm{AdS}_{d+1}$ is isomorphic to Diff $\ltimes$ Weyl and its action on the background metric of the CFT is via diffeomorphisms and Weyl transformation. In particular, asymptotic symmetries that preserve the metric on the conformal boundary are dual to conformal isometries of the dual CFT.

On the gravity side, our focus will be on a theory of gravity called Lovelock gravity that forms a subclass of general higher-curvature theories that exist in higher dimensions. Lovelock gravity is the most general theory of the metric tensor whose equations of motion are second order in derivatives of the components of the metric. In four dimensions, Lovelock gravity reduces to general relativity (GR) which leads to the Lovelock's theorem: any modification of GR which has second-order equations of motion in four dimensions contains extra degrees of freedom in addition to the metric. Lovelock gravity also contains a subset of topological theories of gravity in odd dimensions that do not have propagating degrees of freedom and contain 3D Einstein gravity as a special case. These theories are simple toy-models that can be studied from a holographic perspective.

In the case of three-dimensional Einstein gravity, we will explain how the asymptotic symmetry group of $\mathrm{AdS}_{d+1}$ acts on the stress tensor of the CFT and how it is centrally extended. Namely, we are able to reproduce the Schwarzian transformation law generated by the Schwinger term in the stress tensor operator commutation relation studied in chapter 3. Hence classical 3D gravity reproduces inherently quantum mechanical properties of the putative 2D CFT living on the conformal boundary. Similar ideas play a central role in the attempts to extend AdS/CFT duality to flat space in the celestial holography program.

The 3D gravity model of a holographic CFT can also be extended to accommodate boundaries

on both sides of the duality using end-of-the-world branes. To this end, we will present a model which is able to incorporate boundary condition changing operators (see chapter 3) on the gravity side. The model requires the inclusion of Gibbons–Hawking–York [84, 85] and Hayward terms [86] to the action of 3D gravity. We will provide complete presentation of such boundary and corner terms in the broad setting of Lovelock gravity.

## 4.1 Classical theories of gravity

Consider a manifold $(\mathcal{M}, g)$ of dimension $D = d+1$ equipped with a Euclidean metric $g$, and in this section (and chapter), $a, b, \ldots = 1, \ldots D$ label bulk dimensions. An $F$(Riemann) theory of gravity in Euclidean signature has the action

$$I[g] = \int_{\mathcal{M}} d^D x \sqrt{g}\, F(g_{ab}, R_{abcd}) \tag{4.1}$$

where $R_{abcd}$ is the Riemann tensor of $g$ and the action is assumed to be diffeomorphism invariant $I[\widetilde{g}] = I[g]$. The variation of the action is given by

$$\delta I = \int_{\mathcal{M}} d^D x \sqrt{g}\, E_{ab}\, \delta g^{ab} + \int_{\partial\mathcal{M}} d^d x \sqrt{h}\, n_a \delta v^a \tag{4.2}$$

where the equation of motion tensor and the boundary term are given by [87]

$$E_{ab} = P_{acde} R_b{}^{cde} - \frac{1}{2} g_{ab} F + 2\nabla^c \nabla^d P_{acbd}, \quad n_a \delta v^a = 2 n_c\, P_a{}^{bcd} \delta \Gamma^a_{bd} + 2 n_c\, \delta g_{bd} \nabla_a P^{abcd}. \tag{4.3}$$

Here $h_{ab}$ is the induced metric of the boundary $\partial\mathcal{M}$, $n^a$ is its outward-pointing unit normal vector and we have defined the tensor

$$P_{abcd} = \frac{\partial F}{\partial R^{abcd}}. \tag{4.4}$$

The notation $\delta$ indicates that $\delta v^a$ is not a total variation of any quantity (it cannot be written as $\delta V^a$ for some vector $V^a$). The boundary term arises, because the action involves second-order derivatives of $g_{ab}$ that lead to a total derivative term $\nabla_a \delta v^a$ in the variation.

In general, the equations of motion $E_{ab} = 0$ of $F$(Riemann) gravity depend on higher-order derivatives of the metric which can lead to various unphysical effects. Hence it is natural to consider a subtheory whose equations are of second order in derivatives which is known as Lovelock gravity.

### 4.1.1 Lovelock gravity

Lovelock theories of gravity are the most general diffeomorphism invariant theories of the metric tensor $g$ whose equations of motion are second order in $g$. They are a subset of $F$(Riemann) theories (4.1) with the function [87]

$$F = \sum_{m=0}^{\lfloor D/2 \rfloor} c_{(m)}\, \mathcal{R}_{(m)} \tag{4.5}$$

where $c_{(m)}$ are arbitrary constants and $\mathcal{R}_{(m)}$ is a *Lovelock scalar* of order $m$:

$$\mathcal{R}_{(m)} = \frac{1}{2^m} \delta^{a_1 b_1 \ldots a_m b_m}_{c_1 d_1 \ldots c_m d_m} R^{c_1 d_1}_{a_1 b_1} \cdots R^{c_m d_m}_{a_m b_m} \tag{4.6}$$

such that $\mathcal{R}_{(0)} \equiv 1$ and $\lfloor D/2 \rfloor$ denotes the smallest integer $k$ that satisfies $D/2 \geq k$. Here the generalized Kronecker delta symbol is defined as

$$\delta^{a_1 \dots a_n}_{b_1 \dots b_n} = n!\, \delta^{a_1}_{[b_1} \cdots \delta^{a_n}_{b_n]}. \tag{4.7}$$

where the anti-symmetrization operation contains a factor of $1/n!$ in our conventions.

The Lovelock scalar $\mathcal{R}_{(m)}$ becomes a topological invariant (the Euler characteristic) in even dimension $D = 2m$ and vanishes identically in dimensions $D < 2m$ (since there are not enough indices to be summed over in the generalized Kronecker delta). Already in dimension $D = 2m$ the Lovelock scalar does not contribute to the equations of motion, because the variation of the Euler character is a total derivative.

Let us consider examples. For $m = 1$ the Lovelock scalar is equal to the Ricci scalar and for $m = 2$ it is the Gauss–Bonnet term:

$$\mathcal{R}_{(1)} = R, \quad \mathcal{R}_{(2)} = R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}. \tag{4.8}$$

Hence in $D = 4$, the action (4.5) reduces to general relativity (since $\mathcal{R}_{(2)}$ is a total derivative) while in $D = 5$ it is Gauss–Bonnet gravity. In high enough dimensions, multiple Lovelock scalars appear and one has the freedom to choose the coefficients $c_{(m)}$. The theory in dimension $D \geq 2m + 1$ that involves a single Lovelock scalar $\mathcal{R}_{(m)}$ is known as pure Lovelock gravity of order $m$.

For each term in the action (4.5), the tensor (4.4) is given by

$$P^{ab}_{cd(m)} = \frac{m}{2^m} \delta^{aba_2b_2 \dots a_m b_m}_{cdc_2d_2 \dots c_m d_m} R^{c_2 d_2}_{a_2 b_2} \cdots R^{c_m d_m}_{a_m b_m} \tag{4.9}$$

and due to the Bianchi identity for the Riemann tensor, it follows that

$$\nabla_a P^{ab}_{cd(m)} = \frac{m(m-1)}{2^m} \delta^{aba_2b_2 \dots a_m b_m}_{cdc_2d_2 \dots c_m d_m} \left( \nabla_{[a_1} R^{c_2 d_2}_{a_2 b_2]} \right) \cdots R^{c_m d_m}_{a_m b_m} = 0. \tag{4.10}$$

As a result, the equation of motion tensor of pure Lovelock gravity takes the form (see [A6] for an explicit proof)

$$E^a_{b(m)} = -\frac{1}{2} \frac{1}{2^m} \delta^{aa_1b_1 \dots a_m b_m}_{bc_1d_1 \dots c_m d_m} R^{c_1 d_1}_{a_1 b_1} \cdots R^{c_m d_m}_{a_m b_m}. \tag{4.11}$$

which depends only on second derivatives of the metric due to (4.10). As a result, equations of motion of (4.5) are second order. The argument can also be reversed: Lovelock gravity is the only $F$(Riemann) theory for which $\nabla^a P_{abcd} = 0$ and which has second-order equations of motion [87, 88].

### 4.1.2 Topological Lovelock gravity

Lovelock actions can be used to construct theories of gravity in odd dimensions $D = 2m + 1$ that contain no propagating degrees of freedom and that are higher-dimensional generalizations of three-dimensional Einstein gravity. These topological Lovelock gravities are also known as Lovelock–Chern–Simons (LCS) gravities since their first-order formulation is a Chern–Simons action for a gauge group which in our case will be the isometry group of AdS space $SO(2, d)$ [89] (see also [90]).[1] The Euclidean action is given by

$$I_{(m)} = \int_{\mathcal{M}} d^{2m+1}x \sqrt{g}\, F_{(m)} \tag{4.12}$$

[1] The first-order formulation also allows for non-zero torsion, but we will focus on the torsionless case in which the theory can be formulated solely with a metric.

where the Lagrangian is

$$F_{(m)} = \frac{1}{2^m}\int_0^1 ds\, \delta^{a_1b_1\ldots a_mb_m}_{c_1d_1\ldots c_md_m}\prod_{n=1}^{m}\left(R^{c_nd_n}_{a_nb_n}+\frac{s^2}{\ell^2}\delta^{c_nd_n}_{a_nb_n}\right). \tag{4.13}$$

where $\ell$ is the only free parameter of the theory and it has dimension of length. The action (4.12) is an example of a Lovelock gravity action (4.5) with a special choice of coefficients. By expanding the product in (4.13) as a binomial series, performing the Kronecker contractions and integrating term by term over $s$, we get

$$I_{(m)} = \int_{\mathcal{M}} d^{2m+1}x\sqrt{g}\sum_{n=0}^{m} c_{(m,n)}\mathcal{R}_{(n)} \tag{4.14}$$

where [A4, 91]

$$c_{(m,n)} = (2m-2n)!\binom{m}{n}\left(\frac{1}{\ell^2}\right)^{m-n}. \tag{4.15}$$

Using (4.11), the equation of motion tensor of (4.13) is given by [A4, 91]

$$\mathcal{E}^a_{b(m)} \equiv \sum_{n=0}^{m} c_{(m,n)}E^a_{b(n)} = -\frac{1}{2}\frac{1}{2^m}\delta^{aa_1b_1\ldots a_mb_m}_{bc_1d_1\ldots c_md_m}\prod_{n=1}^{m}\left(R^{c_nd_n}_{a_nb_n}+\frac{1}{\ell^2}\delta^{c_nd_n}_{a_nb_n}\right). \tag{4.16}$$

Interestingly, if we define the tensor [92]

$$\mathcal{F}^{cd}_{ab} = R^{cd}_{ab}+\frac{1}{\ell^2}\delta^{cd}_{ab}, \tag{4.17}$$

we see that (4.16) is identical to the equation of motion of pure Lovelock gravity (4.11) after the replacement $R^{cd}_{ab}\to\mathcal{F}^{cd}_{ab}$. Now the equations of motion $\mathcal{E}^a_{b(m)}=0$ of topological Lovelock gravity are trivial in the sense that all solutions satisfy

$$\mathcal{F}^{[c_1d_1}_{[a_1b_1}\cdots\mathcal{F}^{c_md_m]}_{a_mb_m]} = 0 \tag{4.18}$$

in an open neighbourhood around any point (see [A4] for a proof). In the first order formulation, (4.18) is the statement that all solutions of the theory correspond to flat connections of the AdS isometry group. For 3D Einstein gravity corresponding to $m=1$, the equation of motion (4.18) reduces to

$$R^{cd}_{ab} = -\frac{1}{\ell^2}\delta^{cd}_{ab} \tag{4.19}$$

so that that all solutions are locally $AdS_3$. This is the familiar fact that the Einstein tensor in three dimensions has as many independent components as the Riemann tensor.

### 4.1.3 Boundary and corner terms

We have seen that the variation of the action of $F$(Riemann) gravity involves a boundary variation $n_a\delta v^a$ (4.3) which can be written as

$$n_a\delta v^a = n_a\,\delta U^a + D_a\,\delta Y^a \tag{4.20}$$

where the two terms in Euclidean signature are [93]

$$n_a \delta U^a = -4\Psi_{ab}\,\delta K^{ab} + (2n^a \nabla^d P_{acbd} + 6\Psi_{ab}\,K^a_c)\,\delta h^{bc} - 2n^d P_{acbd} D^a \delta h^{bc} \tag{4.21}$$

$$\delta Y^a = -2\Psi^a_b\,\delta A^b. \tag{4.22}$$

Here $K_{ab} = h^c_a h^d_b\,\nabla_c n_d$ is the extrinsic curvature of $\partial\mathcal{M}$, $D_a$ is the covariant derivative compatible with the induced metric $h_{ab}$, $\delta A^a = -h^a_b\,n_c\,\delta g^{bc}$ and the symmetric tensor

$$\Psi_{ab} \equiv P_{acbd}\,n^c n^d. \tag{4.23}$$

Equation (4.20) involves a total derivative term which integrates to the boundary of the boundary $\partial(\partial\mathcal{M})$.

We will consider the situation where the boundary $\partial\mathcal{M}$ has two components $\partial\mathcal{M} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ that intersect non-smoothly at a corner $\mathcal{C} = \mathcal{Q}_1 \cap \mathcal{Q}_2$. In this case, the boundaries of the two boundary components coincide $\partial\mathcal{Q}_{1,2} = \mathcal{C}$ and are non-empty. Using above formulae, the variation (4.2) becomes

$$\delta I = \int_{\mathcal{M}} d^{d+1}x\sqrt{g}\,E_{ab}\,\delta g^{ab} + \sum_{s=1}^{2}\int_{\mathcal{Q}_s} d^d x\sqrt{h_s}\,n_{sa}\,\delta U^a_s + \sum_{s=1}^{2}\int_{\mathcal{C}} d^{d-1}x\sqrt{\sigma}\,r_{sa}\,\delta Y^a_s \tag{4.24}$$

where $n^a_s$ and $r^a_s$ are the outward-pointing unit normal and tangent vectors of $\mathcal{Q}_s$ respectively, and $\sigma_{ab}$ is the induced metric of the corner $\mathcal{C}$.

To obtain the equations of motion $E_{ab} = 0$ from the variational principle $\delta I = 0$, both the codimension-one boundary term and the codimension-two corner term have to vanish in (4.24). Usually, this is done by imposing Dirichlet boundary conditions

$$\delta h^{ab}_s|_{\mathcal{Q}_s} = \delta\sigma^{ab}|_{\mathcal{C}} = 0 \tag{4.25}$$

for the bulk metric $g$, in other words, the induced metrics of $\mathcal{Q}_{1,2}$ and $\mathcal{C}$ are kept fixed under the variation. However, the Dirichlet condition for $h_{ab}$ does not remove the boundary term, because $\delta U^a$ depends on $\delta K_{ab}$ which involves normal derivatives of $h_{ab}$. The corner term does not vanish either, because it involves the mixed component $r_a n_b\,\delta g^{ab}$ of the variation.

The Dirichlet problem (4.25) can only be made viable if the original action is modified by supplementing it with additional boundary and corner terms. The new action takes the form

$$I = \int_{\mathcal{M}} d^{d+1}x\sqrt{g}\,F + \int_{\partial\mathcal{M}} d^d x\sqrt{h}\,B + \int_{\mathcal{C}} d^{d-1}x\sqrt{\sigma}\,C \tag{4.26}$$

where we require that

$$\delta B = -n_a\,\delta U^a, \quad \delta C = -r_a\,\delta Y^a \tag{4.27}$$

under variations $\delta g^{ab}$ that satisfy the boundary condition (4.25). The functions $B, C$ are constructed from intrinsic and extrinsic data of $\partial\mathcal{M}$ and $\mathcal{C}$ respectively.

The exact expressions for $B, C$ in Lovelock gravity are known. The boundary term is given by [94, 95]

$$B_{(m)} = 2m\int_0^1 ds\,\delta^{aca_1b_1\ldots a_{m-1}b_{m-1}}_{bdc_1d_1\ldots c_{m-1}d_{m-1}}\,n_c n^d K^b_a \prod_{k=1}^{m-1}\left(\frac{1}{2}\widetilde{R}^{c_k d_k}_{a_k b_k} - s^2 K^{c_k}_{a_k} K^{d_k}_{b_k}\right) \tag{4.28}$$

where $\widetilde{R}^{cd}_{ab}$ is the Riemann tensor of the induced metric $h_{ab}$ of one boundary component. The corner term is [96]

$$C_{(m)} = 2m\,\Theta\,\widehat{\mathcal{R}}_{(m-1)} + (\text{extrinsic curvature terms}) \tag{4.29}$$

where $\Theta = \arccos\left(g_{ab}\,n_1^a n_2^b\right)$ is the intersection angle of the two boundary components $\mathcal{Q}_{1,2}$ and $\widehat{\mathcal{R}}_{(m)}$ is the Lovelock scalar (4.6) of the induced metric $\sigma_{ab}$. For $m = 1$, these reduce to the original Gibbons–Hawking–York term [84, 85] and to the Hayward term [86] of Einstein gravity

$$B_{(1)} = 2K, \quad C_{(1)} = 2\Theta \tag{4.30}$$

where $K = g^{ab}K_{ab}$. In general, a given Lagrangian $F$ might not admit boundary terms that make the Dirichlet problem (4.25) viable. But if one relaxes the Dirichlet condition then it is known that boundary and corner terms can be obtained for any $F$(Riemann) theory [93, 97]. Another example where the boundary term for the Dirichlet problem is known is Pontryagin gravity in $D = 4$ [98], but it does not technically belong to the class of $F$(Riemann) theories due to also involving Levi–Civita tensor in the definition of the action.

## 4.2 Asymptotic symmetries of AdS space

A Lorentzian or Euclidean manifold $(\mathcal{M}, g)$ is locally $\mathrm{AdS}_{d+1}$ around a point if the Riemann tensor satisfies (4.19) in an open neighbourhood of the point. An asymptotically locally $\mathrm{AdS}_{d+1}$ (ALAdS) manifold approaches a locally $\mathrm{AdS}_{d+1}$ manifold when the point is taken to infinite proper distance from any other point. The manifold $(\mathcal{M}, g)$ can be equipped with the Fefferham–Graham coordinates system in which the metric takes the form (we will set $\ell = 1$ in this section) [99]

$$ds^2 = g_{ab}(X)\,dX^a dX^b = \frac{1}{z^2}(dz^2 + \gamma_{ij}(z,x)\,dx^i dx^j), \tag{4.31}$$

where the Fefferham–Graham coordinate $z > 0$ and $\gamma_{ij}$ has the expansion

$$\gamma_{ij}(z,x) = g_{(0)ij}(x) + g_{(2)ij}(x)\,z^2 + g_{(4)ij}(x)\,z^4 + \ldots, \quad z \to 0. \tag{4.32}$$

In this section, $i, j, \ldots = 1, \ldots, d$ label the $d$ boundary directions while $a, b, \ldots = 1, \ldots, D$ label bulk directions as in the previous section.

We can see that the FG coordinate system $(z, x)$ is defined by the conditions

$$g_{zi}(z,x) = 0, \quad g_{zz}(z,x) = z^{-2}, \quad g_{ij}(z,x) = z^{-2}\,g_{(0)ij} + g_{(2)ij} + \ldots \tag{4.33}$$

The condition that the metric is locally $\mathrm{AdS}_{d+1}$ as $z \to 0$ gives relations between the coefficients $g_{(n)}$ so that they are not all independent.

One should not expect the FG coordinate system to cover the whole manifold, but only some neighbourhood around the codimension-one surface $\mathcal{B}$ at $z = 0$ which is called the conformal boundary. Note that $\mathcal{B}$ is not an ordinary boundary, because it is at an infinite proper distance from all bulk points and only light-rays can reach it in finite affine parameter. The leading term in the expansion (4.32) defines a metric on $\mathcal{B}$ via

$$g_{(0)ij}(x) = \lim_{z\to 0} z^2\,g_{ij}(z,x). \tag{4.34}$$

The manifold $(\mathcal{B}, g_{(0)})$ defines the background on which the putative dual $\text{CFT}_d$ lives.

Equation (4.31) constructs an $\text{ALAdS}_{d+1}$ metric $g$ from data $g_{(n)}$. However, the construction is ambiguous in the sense that there exists another metric $\widetilde{g}$, obtained by acting with a diffeomorphism $F$ on $g$, which also satisfies the conditions (4.33). In fact, there are an infinite number of such diffeomorphisms $F$ and they form the asymptotic symmetry group of ALAdS metrics. In general, asymptotic symmetries are defined to be diffeomorphisms that map different metrics with the same asymptotics to each other. In our case, asymptotic symmetries map $\{g_{(n)}\}$ to $\{\widetilde{g}_{(n)}\}$ such that both sets of data determine an ALAdS metric.

We will now determine the asymptotic symmetry group $\text{Asym AdS}_{d+1}$ of $\text{ALAdS}_{d+1}$ metrics. An asymptotic symmetry $F(z,x)$ is a diffeomorphism such that the components of the new metric $\widetilde{g}_{ab}(z,x)$ (given by (2.1)) satisfy the FG conditions (4.33) that are easiest to solve at the infinitesimal level [100].[2] We will consider an infinitesimal diffeomorphism of the form

$$F^z(z,x) = z\,(1-\omega(x)), \quad F^i(z,x) = x^i + \xi^i(z,x) \tag{4.35}$$

where both $\omega$ and $\xi^i$ are taken to be small. Then the FG condition $\widetilde{g}_{zi}(z,x) = 0$ gives at leading order in $(\omega, \xi^i)$ the equation [100]

$$\partial_z \xi^i = \frac{1}{2}\gamma^{ij}\partial_j \omega. \tag{4.36}$$

Substituting the expansion

$$\xi^a(z,x) = \xi^i_{(0)}(x) + \xi^i_{(2)}(x)\, z^2 + \ldots, \quad z \to 0 \tag{4.37}$$

together with (4.32), all the coefficients $\xi^i_{(n)}(x)$ for $n \geq 2$ are fixed in terms of $\{\omega, g_{(n)}\}$. For example, the $n = 2$ term is explicitly

$$\xi^i_{(2)} = \frac{1}{2}\, g^{ij}_{(0)}\partial_j \omega. \tag{4.38}$$

The leading term $\xi_{(0)}$ remains free. Hence the resulting diffeomorphism is completely fixed by $\{\omega, \xi_{(0)}\}$ and it is called an infinitesimal Penrose–Brown–Henneaux (PBH) diffeomorphism. The group of PBH diffeomorphisms make up the asymptotic symmetry group of $\text{AdS}_{d+1}$.

Let us now consider the effect of an infinitesimal PBH diffeomorphism on the functions $g_{(n)}(x)$. We define $\widetilde{\gamma}_{ij}(z,x) = z^2\,\widetilde{g}_{ij}(z,x)$ which gives at leading order in $\omega, \xi^i$,

$$\delta\gamma_{ij}(z,x) = \omega(x)\,(2 - z\partial_z)\,\gamma_{ij}(z,x) + \nabla_{(\gamma)i}\xi_j(z,\sigma) + \nabla_{(\gamma)j}\xi_i(z,x), \tag{4.39}$$

where $\nabla_{(\gamma)i}$ denotes the covariant derivative of $\gamma_{ij}(z,x)$ with $z$ treated as a parameter. By substituting the expansions (4.32) and (4.37), we get at leading order in $z \to 0$

$$\delta g_{(0)ij} = 2\,\omega\, g_{(0)ij} + \nabla_{(0)i}\xi_{(0)j} \tag{4.40}$$

where $\nabla_{(0)i}$ is the covariant derivative compatible with $g_{(0)}$ and we used (4.38) in the second equation. At non-linear level, the transformation of $g_{(0)}$ integrates to

$$\widetilde{g}_{(0)ij}(x) = e^{2\omega(x)}\,\frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j}\, g_{(0)ij}(x) \tag{4.41}$$

[2] The radial coordinate $\rho$ in [100] is given by $\rho = -\frac{1}{2}z^2$. This radial coordinate transformation can also be found in [99].

where $f(x) \in \text{Diff}_+\mathcal{B}$ is the diffeomorphism of $\mathcal{B}$ generated by the vector field $\xi_{(0)}$:

$$f^i(x) = x^i + \xi^j_{(0)}(x). \tag{4.42}$$

We can identify (4.41) as the action of $\text{Diff}_d \ltimes \text{Weyl}_d$ on $(\mathcal{B}, g_{(0)})$ so that the asymptotic symmetry group of $\text{AdS}_{d+1}$ is given by

$$\text{Asym}\,\text{AdS}_{d+1} \cong \text{Diff}_d \ltimes \text{Weyl}_d. \tag{4.43}$$

This isomorphism describes how to embed $\text{Diff}_d \ltimes \text{Weyl}_d$ into the diffeomorphism group $\text{Diff}_{d+1}$ of a higher-dimensional spacetime $\mathcal{M}$. It is first evidence for the possible existence of a holographic duality between a gravitational diffeomorphism invariant theory in the bulk and a CFT on the conformal boundary.

The group $\text{Asym}\,\text{AdS}_{d+1}$ has a subgroup that keeps the metric (4.34) on the conformal boundary fixed $\widetilde{g}_{(0)}(x) = g_{(0)}(x)$, but that can change subleading terms $\widetilde{g}_{(n\geq 2)}(x) \neq g_{(n\geq 2)}(x)$ in the FG expansion. This subgroup is isomorphic to the conformal isometry group of the conformal boundary $\text{CIso}(\mathcal{B}, g_{(0)})$ introduced in equation (2.16). Hence the symmetry group of a CFT on a fixed background metric is realized in terms of bulk diffeomorphisms that preserve $g_{(0)}$. We will see below that the change in the subleading term $g_{(d)}$ describes the transformation of the CFT stress tensor one-point function under conformal isometries.

### 4.2.1 Subleading terms in the FG expansion

The subleading terms in the FG expansion can also be written in terms of boundary data. For simplicity, we will consider the case $\xi_{(0)} = 0$ corresponding to a pure Weyl transform on the conformal boundary. In this case, equation (4.39) gives [100]

$$\delta g_{(2)ij} = \nabla_{(0)i}\xi_{(2)j} + \nabla_{(0)j}\xi_{(2)i} = \nabla_{(0)i}\nabla_{(0)j}\,\omega, \tag{4.44}$$

where we used (4.38). For $d > 2$, equation (4.40) coincides with the infinitesimal transformation of the (minus) Schouten tensor $-S_{(0)ij}$ under Weyl transformations of $g_{(0)}$ so that[3]

$$g_{(2)ij} = -S_{(0)ij} = -\frac{1}{d-2}\left(R_{(0)ij} - \frac{1}{2(d-1)}R_{(0)}g_{(0)ij}\right), \quad d > 2, \tag{4.45}$$

where the curvature tensors are of the conformal boundary metric $g_{(0)}$ [99–101].[4] This expression is not valid in $d = 2$ which is analysed separately in more detail below.

The expression for $g_{(4)}$ is also determined in terms of $g_{(0)}$ up to *two* free parameters [100] which are not determined by boundary data. Regardless, it is quite remarkable that an infinite number of asymptotically locally AdS metrics are determined by only two parameters to first order in the FG expansion. Higher-order functions $g_{(n)}$ will contain more free parameters. However, there is a special class of bulk metrics for which all $g_{(n)}$ are determined uniquely: these are locally conformally flat Einstein metrics that satisfy $R_{ab} = -d\,g_{ab}$ and whose Weyl or

[3] Since the non-linear form of $\delta g_{(0)}$ is a Weyl transformation, the transformation of $g_{(2)} = g_{(2)}(g_{(0)})$ must be produced by Weyl scalings of $g_{(0)}$.

[4] The equation corresponding to (4.45) in [100, 101] does not have a minus sign. The reason is the minus sign in the relations between the coordinates $\rho = -\frac{1}{2}z^2$ [99].

Cotton tensor vanishes (depending on whether $D \geq 4$ or $D = 3$). In these cases, the Fefferham–Graham expansion truncates at order $z^4$ as [99, 101]

$$\gamma_{ij} = g_{(0)ij} + g_{(2)ij}\, z^2 + g_{(4)ij}\, z^4 = \left(\delta_i^k + \frac{1}{2}\, g_{(2)i}^k\, z^2\right)\left(\delta_j^l + \frac{1}{2}\, g_{(2)j}^l\, z^2\right)\, g_{(0)kl}\ , \tag{4.46}$$

where we used that in this case [101]

$$g_{(4)ij} = \frac{1}{4}\, g_{(0)}^{kl}\, g_{(2)ik}\, g_{(2)lj}\ . \tag{4.47}$$

By (4.45), the form of the bulk metric is then completely fixed in terms of $g_{(0)}$ and its Schouten tensor $S_{(0)}$.

In dimensions $D \geq 5$, the FG expansion can truncate even if the bulk metric is not locally conformally flat. For example in odd dimensions $D = 2m + 1$ with $m = 1, 2, \ldots$, the expansion truncates when the bulk metric is a solution of topological Lovelock gravity (4.18), see [90, 102].

### 4.2.2 Asymptotic symmetries in three bulk dimensions

We will now focus on the case of three-dimensional bulk and two-dimensional boundary $D = d + 1 = 3$. Consider an asymptotically locally $AdS_3$ metric which is automatically Einstein $R_{ab} = 2g_{ab}$ in three dimensions so that the FG expansion truncates as in (4.46). However, $g_{(2)}$ can no longer be given by the Schouten tensor, because (4.45) is clearly not well defined for $d = 2$ so we have to look for another expression. We see that (4.44) implies

$$g_{(0)}^{ij} g_{(2)ij} = -\Box_{(0)}\delta\omega, \quad \nabla_{(0)}^i g_{(2)ij} = -\partial_j\, \Box_{(0)}\delta\omega. \tag{4.48}$$

Notice that $\Box_{(0)}\delta\omega = \delta R_{(0)}$ is the transformation of the Ricci scalar of $g_{(0)}$ under infinitesimal Weyl transformations. Hence we get at the non-perturbative level

$$g_{(0)}^{ij} g_{(2)ij} = -\frac{1}{2} R_{(0)}, \quad \nabla_{(0)}^i g_{(2)ij} = -\frac{1}{2}\, \partial_j R_{(0)}. \tag{4.49}$$

It is useful to define the tensor

$$\tau_{ij} \equiv g_{(2)ij} + \frac{1}{2} R_{(0)} g_{(0)ij} \tag{4.50}$$

which by (4.49) is conserved, but not traceless

$$g_{(0)}^{ij} \tau_{ij} = \frac{1}{2} R_{(0)}, \quad \nabla_{(0)}^i \tau_{ij} = 0. \tag{4.51}$$

We recognize these as the Weyl anomaly and conservation equations of a stress tensor one-point function in a two-dimensional CFT with central charge $c = 12\pi$.

Let us consider $\mathcal{B} = S^1 \times \mathbb{R}$ equipped with a Lorentzian metric $g_{(0)}$ given by

$$g_{(0)ij}\, dx^i dx^j = e^{2\varphi}\, dx^- dx^+. \tag{4.52}$$

In these coordinates, we get from (4.51) and (4.50) that

$$g_{(2)\pm\pm} = \tau_{\pm\pm}(x^\pm), \quad g_{(2)-+} = -\partial_-\partial_+\varphi \tag{4.53}$$

which allow us to extend (4.52) into the bulk as a locally $AdS_3$ metric using (4.46). Let us focus on the simple case of a flat boundary metric $\varphi = 0$ in which case we get

$$g_{ab}\, dX^a dX^b = \frac{1}{z^2}[dz^2 + (dx^- - z^2\, \tau_{--}(x^-)\, dx^+)\, (dx^+ - z^2\, \tau_{++}(x^+)\, dx^-)]. \tag{4.54}$$

This is a general solution of 3D Einstein gravity which asymptotes to the flat metric $dx^- dx^+$ on the conformal boundary known as the Banados metric [103]. The subgroup of PBH diffeomorphisms of (4.54) that preserve the flat metric on the conformal boundary can be written explicitly everywhere in the bulk at the infinitesimal level [104]. Near the conformal boundary, they can also be written at full non-perturbative level as [104]

$$\begin{aligned} F^z(z, x^-, x^+) &= \sqrt{(\partial_- f_-)\, (\partial_+ f_+)}\, z + \mathcal{O}(z^2) \\ F^\pm(z, x^-, x^+) &= f_\pm(x^\pm) + \frac{1}{2}\frac{\partial_\mp^2 f_\mp}{\partial_\mp f_\mp}\frac{\partial_\pm f_\pm}{f_\pm}\, z^2 + \mathcal{O}(z^4), \end{aligned} \tag{4.55}$$

where $f_\pm = f_\pm(x^\pm)$. Comparing with (4.35), the diffeomorphism (4.55) implements a Weyl transformation $e^{2\omega} = [(\partial_- f_-)(\partial_+ f_+)]^{-1}$ and a boundary conformal diffeomorphism with conformal factor $\Omega_F^2 = (\partial_- f_-)(\partial_+ f_+)$ that cancel each other. Hence (4.55) is the action of the conformal isometry group $\widetilde{\text{Diff}}_+ S^1 \times \widetilde{\text{Diff}}_+ S^1$ on the conformal boundary extended into the bulk.

Now the metric $\widetilde{g}$ obtained from the action of (4.55) on (4.54) has the same form, but with $\tau_{\pm\pm}(x^\pm)$ replaced by [104]

$$\widetilde{\tau}_{\pm\pm}(x^\pm) = (\partial_\pm f_\pm)^2\, \tau_{\pm\pm}(f_\pm(x^\pm)) - \frac{1}{2}\{f_\pm(x^\pm), x^\pm\} \tag{4.56}$$

which can be identified as the anomalous transformation law of the CFT stress tensor under the conformal isometry group $\widetilde{\text{Diff}}_+ S^1 \times \widetilde{\text{Diff}}_+ S^1$. It is remarkable that a purely classical transformation of a metric under diffeomorphisms reproduces a centrally extended algebra which can be thought to have a quantum mechanical origin.

## 4.3 The holographic Weyl anomaly

In the holographic duality, the dual CFT lives on the conformal boundary $(\mathcal{B}, g_{(0)})$ of an $\text{ALAdS}_{d+1}$ spacetime $(\mathcal{M}, g)$. A basic entry in the holographic dictionary is the equivalence of partition functions of the two dual theories. In the bottom-up context where the bulk theory is a classical theory of gravity this equivalence can be written as

$$Z_{\text{CFT}}[(\mathcal{B}, g_{(0)})] = Z_{\text{gravity}}[(\mathcal{M}, g)]. \tag{4.57}$$

Here the partition function of the CFT $Z_{\text{CFT}}$ is defined as a Euclidean path integral so that both $g_{(0)}$ and $g$ are assumed to be Euclidean in this equation. The definition of the Euclidean path integral of gravity is an open question, but in its traditional formulation in the context of AdS/CFT, it is given by the sum over all Euclidean metrics $g$ that asymptote to the same $g_{(0)}$ on the conformal boundary weighted by the action of Euclidean gravity:

$$Z_{\text{gravity}}[(\mathcal{M}, g)] \cong \int_{g \to g_{(0)}} [dg]\, e^{-I[g]}. \tag{4.58}$$

The definition of the integration measure $[dg]$ is very subtle and is best understood in two-dimensional theories of gravity (such as the theory on the worldsheet of a string). However,

in the limit where the coefficient of the classical action diverges, we can calculate the path integral in the saddle-point approximation as a sum over solutions of the gravitational equations of motion

$$Z_{\text{gravity}}[(\mathcal{M}, g)] \cong \sum_{g=\text{solution}} e^{-I_{\text{on-shell}}[g]}. \tag{4.59}$$

For AlAdS metrics, the coefficient of the gravitational action will be proportional to $\ell^{d-2}/G_{\text{N}}$ where $\ell$ is the radius curvature of the metric and $G_{\text{N}}$ is the $D$-dimensional Newton's constant. Hence the saddle-point approximation (4.59) is valid in the limit where the curvature radius is much larger than Planck length $\ell/\ell_{\text{p}D} \gg 1$. On the CFT side, this corresponds to the limit of large central charge $c \to \infty$ (see equation (4.67) below), or more specifically in string theory examples, to the large rank limit $N \to \infty$ of the gauge symmetry group (see the next chapter).

At leading order in the saddle-point approximation, the solution with the smallest on-shell action $I_{\text{on-shell}}[g]$ dominates the sum (4.59) leading to the formula (we will denote the dominating solution by $g$ for simplicity)

$$Z_{\text{CFT}}[(\mathcal{B}, g_{(0)})] = e^{-I_{\text{on-shell}}[g]}. \tag{4.60}$$

The gravitational equations of motion can be integrated from the conformal boundary by using the FG expansion and by starting with the initial condition $g_{(0)}$. Assuming that the equations are second order (as in Lovelock gravity), they determine all functions $g_{(n)}$ in terms of $g_{(0)}$ except for the traceless part of $g_{(d)}$. The remaining freedom in $g_{(d)}$ is fixed in terms of $g_{(0)}$ by a boundary condition deep in the bulk $z > 0$ which in Euclidean signature is a regularity condition for the bulk metric $g$. As a result, the on-shell gravity action is only a function of $g_{(0)}$ and equation (4.60) makes sense.

However, there is a subtlety which we have neglected: the on-shell gravity action has an IR divergence due to infinite volume of the ALAdS space which has to be regularized and renormalized. In the dual field theory, this corresponds to regularization and renormalization of UV divergences. The regularized action is defined by integrating the action of a ALAdS metric (4.31) up to a cut-off surface $z = \epsilon > 0$:

$$I^{\text{reg}}[g] = -\frac{1}{2\kappa^2}\int_{z>\epsilon} d^{d+1}x\sqrt{g}\,F - \frac{1}{2\kappa^2}\int_{z=\epsilon} d^d x\sqrt{\gamma}\,\frac{1}{z^d}\,B \tag{4.61}$$

where we have to include the Gibbons–Hawking term at $z = \epsilon$ (since it is a boundary with a Dirichlet boundary condition) and we have included the prefactor $2\kappa^2 = 16\pi G_{\text{N}}$ to the definition of the action. The regularized on-shell action $I^{\text{reg}}_{\text{on-shell}}$ is given by (4.61) evaluated on a solution of the equations of motion which is divergent in the limit $\epsilon \to 0$. Hence we define the renormalized on-shell action

$$I^{\text{ren}}_{\text{on-shell}} = \lim_{\epsilon\to 0}\left(I^{\text{reg}}_{\text{on-shell}} + I^{\text{ct}}\right) \tag{4.62}$$

where the counterterms $I^{\text{ct}}$ are local integrals of covariant quantities constructed from the induced metric of the cut-off surface $z = \epsilon$. For Dirichlet boundary conditions at $z = \epsilon$, the variation of these counterterms vanish so that they do not modify equations of motion. In $D = 3$ only the area term $\int_{\mathcal{B}}\sqrt{\gamma}$ is enough while in higher dimensions also curvature tensors of $\gamma_{ij}$ are needed. Thus finally the holographically renormalized version of (4.60) is given by

$$W[g_{(0)}] = I^{\text{ren}}_{\text{on-shell}}[g_{(0)}] \tag{4.63}$$

where $W[g_{(0)}]$ is the generating functional of renormalized stress tensor correlation functions (2.49) on $\mathcal{B}$.

Taking the functional derivative of (4.63) gives the formula

$$\langle T_{(0)ij}(x)\rangle_{(0)} = \frac{2}{\sqrt{g_{(0)}}}\frac{\delta I^{\text{ren}}_{\text{on-shell}}}{\delta g^{ij}_{(0)}(x)} = \lim_{\epsilon\to 0}\frac{2}{\sqrt{\gamma(\epsilon,x)}}\frac{\delta I^{\text{ren}}_{\text{on-shell}}}{\delta \gamma^{ij}(\epsilon,x)}. \tag{4.64}$$

which is the Euclidean one-point function of the CFT stress tensor on the background $g_{(0)}$. For Einstein gravity with $F = R$, it can be shown that the one-point function is given by the subleading term $g_{(d)}$ in the FG expansion of the bulk metric [105]

$$\langle T_{(0)ij}(x)\rangle_{(0)} = \frac{d\ell^{d-1}}{2\kappa^2}\, g_{(d)ij}(x) + Y^{(d)}_{ij} \tag{4.65}$$

where the tensor $Y^{(d)}_{ij}$ is determined by $g_{(n)}$ with $n < d$. This is a general property of the relationship between bulk fields and boundary operators of scaling dimension $\Delta$: the term of order $z^\Delta$ in an expansion of the bulk field in $z \to 0$ computes the one-point function of a boundary operator sourced by that field [25, 26].

From (4.65) we can compute the trace which turns out to be completely fixed in terms of $g_{(0)}$. The traceless part is also fixed after imposing the bulk IR boundary condition discussed below equation (4.60). Hence holography gives a prediction for the renormalized one-point function in the presence of a background metric (source) that depends on the details (how its solutions behave deep in the bulk) of gravitational theory considered.

Now the trace $g^{ij}_{(0)}\langle T_{(0)ij}\rangle_{(0)}$ obtained from (4.65) is non-zero which is the holographic description of the Weyl anomaly of the dual CFT. For $D = 3$ Einstein gravity, the result is [105]

$$g^{ij}_{(0)}\langle T_{(0)ij}\rangle_{(0)} = \frac{\ell}{16\pi G_{\text{N}}}\, R \tag{4.66}$$

and by comparing with (3.30) we can identify the Brown–Henneaux central charge [43]

$$c = \frac{3\ell}{2G_{\text{N}}}. \tag{4.67}$$

As shown in section 3.2, equation (4.66) completely determines the generating functional $W[g_{(0)}]$ on a genus zero Riemann surface with the result being the Polyakov action. One can also obtain the Polyakov action by a direct computation of the bulk on-shell action (4.63) which was done in the first-order formulation of 3D gravity in [106] and also in [81].

We can now use the formula (4.65) in $D = 3$ to obtain

$$\langle T_{(0)\pm\pm}(x^-,x^+)\rangle_{(0)} = \frac{\ell}{8\pi G_{\text{N}}}\, g_{(2)ij}(x) = \frac{c}{12\pi}\,\tau_{\pm\pm}(x^\pm). \tag{4.68}$$

Due to (4.56), we see that it transforms by the Schwarzian derivative under the action of conformal diffeomorphisms (as originally shown in [107]). This is what we expected based on the CFT computations.

The holographic Weyl anomaly has also been computed in various higher-curvature theories of gravity. In topological Lovelock gravity in $D = 2m + 1$ dimensions, it has been computed using the first order formulation and the result is [90, 102, 108, 109]

$$g^{ij}_{(0)}\langle T_{(0)ij}\rangle_{(0)} = \frac{\ell}{2\kappa^2}\,\mathcal{R}_{(m)} \tag{4.69}$$

where $\mathcal{R}_{(m)}$ is the Lovelock scalar of $g_{(0)}$. For $m = 1$, it reduces to the Einstein value (4.66). In the classification of Weyl anomalies [68], the anomaly is purely of type A and transforms as a total derivative under Weyl transformations.

## 4.4 Holographic models of 2D boundary CFTs

In [47] a holographic model for a boundary CFT was proposed where the boundary of the BCFT is extended into the bulk as an end-of-the-world (EOW) brane. In this case, the Euclidean gravity action is given by

$$I = -\frac{1}{2\kappa}\int_{\mathcal{M}}\sqrt{g}\left(R + \frac{d(d-1)}{\ell^2}\right) - \frac{1}{\kappa}\int_{\mathcal{Q}}\sqrt{h}\,(K - T), \tag{4.70}$$

where $T$ is a constant characterizing the brane tension. The brane tension encodes overlaps of boundary states $\langle B|0\rangle$ with the vacuum state of the BCFT.

This model can be extended by allowing for the branes to intersect and by including matter fields to the theory. In this case, the extended action becomes [A1, 110]

$$I = -\frac{1}{2\kappa}\int_{\mathcal{M}}\sqrt{g}\left(R + \frac{d(d-1)}{\ell^2} - \mathcal{L}_{\mathcal{M}}\right) - \frac{1}{\kappa}\int_{\mathcal{Q}}\sqrt{h}\,(K - T) - \frac{1}{\kappa}\int_{\mathcal{C}}\sqrt{\sigma}\,(\Theta - M), \tag{4.71}$$

where we have included the Hayward term at brane intersections and $M$ is a constant characterizing the tension of the intersection. One can show that the variation of the action is given by [A1]

$$\begin{aligned}\delta I = & -\frac{1}{2\kappa}\int_{\mathcal{M}}\sqrt{g}\left(G_{ab} + \Lambda g_{ab} + \frac{1}{2}T^{\mathcal{M}}_{ab}\right)\delta g^{ab} \\ & -\frac{1}{2\kappa}\int_{\mathcal{Q}}\sqrt{h}\,\left(K_{ab} - (K-T)\,h_{ab}\right)\delta h^{ab} \\ & +\frac{1}{2\kappa}\int_{\mathcal{C}}\sqrt{\sigma}\,(\Theta - M)\,\sigma_{ab}\,\delta\sigma^{ab},\end{aligned} \tag{4.72}$$

where we have not imposed any boundary conditions on the metrics. In the spirit of holography, we want to keep the bulk metric completely dynamical so we will not impose Dirichlet boundary conditions for $h_{ab}$ and $\sigma_{ab}$, but set the coefficients of their variations to vanish in (4.72). As a result, the variational principle $\delta I = 0$ gives three equations

$$R_{ab} - \frac{1}{2}Rg_{ab} - g_{ab} + \frac{1}{2}T^{\mathcal{M}}_{ab} = 0, \quad K_{ab} - (K-T)\,h_{ab} = 0, \quad \Theta - M = 0 \tag{4.73}$$

which are the bulk, boundary and corner Einstein's equations respectively [A1]. The boundary Einstein's equation can be seen as a modified Neumann boundary condition for the bulk metric at the branes.

As we have seen above, in three dimensions the bulk Einstein's equation is solved by locally $AdS_3$ metrics. Hence generic solutions of pure 3D gravity are obtained by cutting and gluing locally $AdS_3$ manifolds which can introduce boundaries (EOW branes) and conical singularities. The locations of the boundaries are fixed by the boundary Einstein's equation in terms of the brane tension. In addition, conical singularities with angular deficits $2\pi(1-\alpha)$ can be supported by adding point-particle sources of mass $4\pi(1-\alpha)$ to the theory

$$\mathcal{L}_{\mathcal{M}} = 2\pi\,(1-\alpha)\,\delta_{\mathcal{D}}. \tag{4.74}$$

The bulk Einstein's equation in (4.73) determines the locations of these singularities

One can also add scalar fields of squared mass $\Delta(2-\Delta)$ to the theory corresponding to the Lagrangian

$$\mathcal{L}_{\mathcal{M}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{g}\,\left(\nabla^a\Phi\nabla_a\Phi + m^2\Phi^2\right) \tag{4.75}$$

which can be coupled to the EOW branes by including the term [A3, 48]

$$\mathcal{L}_{\mathcal{Q}} = -\frac{1}{\kappa}\int_{\mathcal{Q}_1}\sqrt{h_1}\,\lambda_1\Phi - \frac{1}{\kappa}\int_{\mathcal{Q}_2}\sqrt{h_2}\,\lambda_2\Phi. \tag{4.76}$$

Point-particles and scalar fields are holographically dual to operators in the BCFT. If we treat the scalar field as a probe which does not backreact on the geometry, it is dual to a light operator of dimension $\Delta$ for which $\Delta/c \to 0$ as $c \to \infty$. On the other hand, point-particles are backreacting since they source a deficit angle around them. Hence they are dual to heavy operators with dimension $\Delta \to \infty$ such that $\Delta/c$ is finite. The deficit angle is fixed by the dimension via the formula (see for example [A1])

$$\alpha = \sqrt{1-\frac{12\Delta}{c}}. \tag{4.77}$$

A brane intersection is also dual to an operator in the BCFT: it is a boundary condition changing operator (see chapter 2) with dimension $\Delta_{\text{bcc}} = 2h_{\text{bcc}}$ which is fixed by the corner mass $M$ (however the exact relation is quite complicated [A1, 110]).

The outcome of these considerations is that the bottom-up gravitational model (4.71) coupled to a scalar field (4.75) allows to describe various operators appearing in two-dimensional CFTs. For example, the simple scalar coupling (4.76) is necessary to produce non-zero one-point functions for the dual operator which is allowed by the boundary conformal symmetry [A3, 48]. This can be used to narrow done the moduli space of BCFTs that are described by the action (4.71): gravity computations give predictions for observables that are only consistent in a subset of BCFTs (see [111–114] for recent work in this direction).

## 4.5 Relation to the articles

The saddle-point approximation of the gravitational path integral presented in this chapter was utilized in all four articles [A1–A4]. In addition, the holographic Weyl anomaly of topological Lovelock gravity in the presence of conical singularities was considered in the article [A4]. In [A4], we also studied effects of conical singularities to curvature invariants that appear in Lovelock gravity actions presented above. These calculations generalized previous work on the topic of conical singularities [115, 116].

The holographic model involving three-dimensional Einstein gravity coupled to EOW branes was our focus in the two articles [A1, A3]. As mentioned above, this model was first introduced in [47] and it is able to describe vacuum states of the open and closed string sectors of the BCFT. The shortcoming of the model considered in [47] is that it does not describe excited states or BCC operators in the BCFT [111]. Hence in the article [A1], we extended the model of [47] to include excited states by coupling the bulk theory to point particles as presented in the previous section. We also showed that BCC operators can be incorporated holographically by allowing branes to merge non-smoothly at a corner. The analysis in our article requires a careful treatment of boundary and corner terms on the gravity side as discussed in this chapter.

In the articles [A1, A3], we considered the holographic model containing a scalar field linearly coupled to an EOW brane as in (4.76). The implications of this coupling to correlation functions of a scalar operator was the focus of [A3] while in [A1] we focused on its contributions to the BCFT partition function.

# Chapter 5

# Aspects of top-down holography

The low energy limit of the closed string sector of string theory is described by classical supergravity which has various solutions describing extended strongly gravitating $p+1$-dimensional objects ($p$-branes). On the other hand, in the open string sector, there are open strings attached to $p+1$-dimensional surfaces (Dirichlet branes or D-branes for short) in the target space whose low energy description is a gauge theory living on the D-branes. According to the identification between D- and $p$-branes [14], the supergravity solutions and the D-brane gauge theory are descriptions of the same object in different regimes of parameter space. In a decoupling limit [24] the identification can be made precise which leads to the holographic duality.

Today, there are multiple examples of holographic dual pairs obtained by taking decoupling limits of various brane configurations. This chapter is devoted to the study of two such examples: type IIB string theory on $\mathrm{AdS}_5 \times S^5$ dual to $\mathcal{N}=4$ super Yang–Mills theory [24] and M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ dual to ABJM theory [27]. Our main motivation is to give a top-down description of flavor fields in the CFT which involves inclusion of additional D-branes in the string theory setup. In contrast to bottom-up holography, we will be able to make precise statements of the dual theory at strong coupling using gravitational calculations. The focus will be on the free energy which is relatively simple to compute from the on-shell supergravity action.

## 5.1 Holographic dual of $\mathcal{N}=4$ super Yang–Mills theory

The simplest example of the AdS/CFT correspondence (which is also the first example in the original Maldacena article [24]) is given by the duality between $\mathcal{N}=4$ super Yang–Mills (SYM) theory and type IIB string theory on a manifold with $\mathrm{AdS}_5 \times S^5$ asymptotics. Before describing the conjecture, we will first review low-energy effective actions of closed and open string theories in detail.

### 5.1.1 Effective actions of closed and open strings

In this section, we will denote worldsheet coordinates by $\sigma^\mu$ with $\mu, \nu, \ldots = 1, 2$ and target space coordinates by $X^a$ with $a, b, \ldots = 1, \ldots, D$. With these conventions, the bosonic part of the type IIB superstring has the Lorentzian worldsheet action (in this chapter, we assume that there is no background Kalb–Ramond field)

$$I_{\mathrm{cl}} = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \, \sqrt{-\gamma} \left( \gamma^{\mu\nu} g_{ab} \, \partial_\mu X^a \partial_\nu X^b - \alpha' \, R^{(2)} \, \Phi \right) \tag{5.1}$$

where $\gamma_{\mu\nu}(\sigma)$ is the worldsheet metric and $R^{(2)}$ is its Ricci scalar; $X^a(\sigma)$ is the embedding of the worldsheet to the target space; $g_{ab}(X)$ and $\Phi(X) = \Phi_0 + \phi(X)$ are the target space metric and dilaton respectively. The string action (5.1) involves only a single length scale $\ell_{\rm s} = \sqrt{\alpha'}$ which is the string length. The constant $\Phi_0$ in the dilaton contributes an overall prefactor $g_{\rm s} = e^{\Phi_0}$ to the worldsheet path integral when $\Sigma$ has genus zero so that $g_{\rm s}$ is the string coupling.

In the path integral quantization of (5.1), the worldsheet metric $\gamma_{\mu\nu}$ is integrated over (in addition to $X^a$ and the fermions), because it is an auxiliary field whose purpose is to reduce the first term in the action to the Nambu–Goto action (at the classical level). The action (5.1) has classical Diff $\ltimes$ Weyl symmetry which is required to be preserved also at the quantum level to ensure that $\gamma_{\mu\nu}$ completely decouples from the theory.[1] The first obstruction for this is the Weyl anomaly whose cancellation requires fixing the target space dimension to the critical value $D = 10$. The second obstruction is the running of the worldsheet couplings $\{g, \Phi\}$ under renormalization group flow. Imposing conformal invariance requires $\{g, \Phi\}$ to be such that their beta functions are vanishing $\beta^g = \beta^\Phi = 0$ (see [11, 12] for calculations of beta functions).

String theory is a theory of quantum gravity which at low energies should reduce to a classical field theory in the target space with a diffeomorphism invariant action (the effective action) for massless fields of the string spectrum. String theory has the interesting property that setting worldsheet beta functions to zero is equivalent to imposing equations of motion of the effective action. In the case of type IIB superstring, the beta functions produce an effective action which is type IIB supergravity in the critical dimension $D = 10$. Focusing on the relevant fields in what follows, the action in the string frame is [117]

$$I_{\rm IIB} = \frac{1}{2\kappa_{10}^2}\int d^{10}X\sqrt{-g}\,e^{-2\phi}(R + 4\,\partial_a\phi\,\partial^a\phi) - \frac{1}{4\kappa_{10}^2}\int d^{10}X\sqrt{-g}\left(F_1^2 + \frac{1}{2\cdot 5!}F_5^2\right) \tag{5.2}$$

where $F_1 = dC_0$, $F_5 = dC_4$ and a self-duality constraint $F_5 = *F_5$ is imposed by hand. The gauge fields $C_0$ and $C_4$ are related to the fermionic sector of the superstring. Linear combinations of variations of the action (5.2) with respect to $\{g, \phi\}$ reproduce beta functions of the closed string couplings.

The beta function derivation of the action (5.2) does not fix the overall coefficient (the 10-dimensional Newton's constant). To fix it, the effective action (5.2) has to be quantized perturbatively (including the graviton) and its scattering amplitudes correctly reproduce the low energy limit of string amplitudes when

$$2\kappa_{10}^2 \equiv (2\pi)^7\ell_{\rm p10}^8 = (2\pi)^7 g_{\rm s}^2\ell_{\rm s}^8 \tag{5.3}$$

where we have defined the 10-dimensional Planck length $\ell_{\rm p10} = g_{\rm s}^{1/4}\ell_{\rm s}$.

In general, type IIB string theory also includes open strings whose worldsheets $\Sigma$ have boundaries. To specify the theory, we have to impose boundary conditions for $X^a(\sigma)$ at $\partial\Sigma$. There are two options which are consistent with the variational principle of the action (5.1): Dirichlet or Neumann boundary conditions. We will consider a "Dirichlet–Neumann" theory where $9-p$ worldsheet fields $X^a$ (or rather their linear combinations) are imposed Dirichlet boundary conditions while the rest $p+1$ fields satisfy Neumann boundary conditions.

To specify the open string action, we introduce a $p+1$-dimensional hypersurface $\mathcal{Q}$ (the D-brane) with an embedding $X^a = E^a(x)$ into the target space. The worldvolume coordinates

[1] Diffeomorphism invariance is enough to decouple two degrees of freedom in $\gamma$ while Weyl invariance decouples the third and final degree of freedom.

of the brane are denoted by $x^i$ with $i, j, \ldots = 1, \ldots, p+1$. A basis of $p+1$ tangent vectors $\{e_i\}_{i=1,\ldots p+1}$ on the brane $\mathcal{Q}$ is defined as

$$e_i^a(x) = \frac{\partial E^a}{\partial x^i}. \tag{5.4}$$

There is also a basis of $9-p$ unit normal vectors $\{n_\alpha\}_{\alpha=1,\ldots,9-p}$ that are orthogonal to the brane $g_{ab}\, n_\alpha^a e_i^b|_{\partial\Sigma} = 0$ and we use $\alpha, \beta, \ldots = 1, \ldots, 9-p$ to label transverse directions of the brane. We will also define $e_a^i$ and $n_a^\alpha$ that satisfy $e_a^i e_i^b = \delta_a^b$ and $n_\alpha^a n_b^\alpha = \delta_b^a$.

At the boundary $\partial\Sigma$ of the worldsheet, we will now impose the Dirichlet boundary condition [10]

$$X^a(\sigma)|_{\partial\Sigma} = E^a(x) \tag{5.5}$$

which ensures that the string worldsheet $\Sigma$ is anchored to the brane $\partial\Sigma \subset \mathcal{Q}$. Equation (5.5) fixes boundary values of $9-p$ worldsheet degrees of freedom $X^a(\sigma)$, because the functions $E^a$ are determined only up to arbitrary reparametrizations of the $p+1$ worldvolume coordinates $x^i$.

On the worldsheet, the embedding of the open string boundary $\partial\Sigma$ is denoted by $\sigma^\mu = Y^\mu(s)$ where $s$ is the coordinate on $\partial\Sigma$. We will denote the unit normal and tangent vectors of $\partial\Sigma$ by $m^\mu$ and $u^\mu$ respectively. With these conventions, the most general Diff $\ltimes$ Weyl invariant form of the open string action is given by

$$I_{\text{op}} = I_{\text{cl}} - \frac{1}{2\pi\alpha'} \int_{\partial\Sigma} ds\, e(s)\, [\gamma^{\mu\nu}(\sigma)\, g_{ab}(X)\, l_\mu(\sigma)\, L^a(X)\, \partial_\nu X^b(\sigma) - \alpha' K^{(2)} \Phi], \tag{5.6}$$

where $I_{\text{cl}}$ is the closed string action (5.1), $e(s)^2 = \gamma_{\mu\nu}\, u^\mu u^\nu$ is the induced line element on $\partial\Sigma$ and $K^{(2)}$ is the extrinsic curvature of $\partial\Sigma$. The extrinsic curvature term is included to ensure that $g_{\text{s}} = e^{\Phi_0}$ multiplies the Euler characteristic of a worldsheet with boundaries.

The action (5.6) involves a new worldsheet coupling $L_a = L_a(X)$ which is a vector field in the target space (to ensure target space diffeomorphism invariance of (5.6)). It can be decomposed into tangential and transverse components with respect to the brane $\mathcal{Q}$ as

$$L^a(X)|_{\mathcal{Q}} = A^i(x)\, e_i^a(x) + v^\alpha(x)\, n_\alpha^a(x) \tag{5.7}$$

where the fields on the worldvolume of the brane are given by

$$A_i(x) = e_i^a(x)\, L_a(E(x)), \quad v_\alpha(x) = n_\alpha^a(x)\, L_a(E(x)). \tag{5.8}$$

Similarly on the worldsheet, we can decompose

$$l_\mu(\sigma)|_{\partial\Sigma} = l_u(s)\, u_\mu(s) + l_m(s)\, m_\mu(s). \tag{5.9}$$

Substituting these decompositions to (5.6), the cross terms involving $t_\mu v^\alpha$ and $m_\mu A^i$ vanish, because target space normal (tangent) vectors are pulled back to multiples of the normal (tangent) vector on the worldsheet. In addition, the coefficients $l_u(s)$ and $l_m(s)$ can be absorbed into $A_i$ and $v_\alpha$. The result is the action presented in [10, 118][2]

$$I_{\text{op}} = I_{\text{cl}} - \frac{1}{2\pi\alpha'} \int_{\partial\Sigma} d\tau\, [h_{ij}(x)\, A^i(x)\, \partial_\tau x^j + H_{\alpha\beta}(x)\, v^\alpha(x)\, n_a^\beta(x)\, \partial_m X^a - \alpha' K^{(2)} \Phi], \tag{5.10}$$

[2] Compared to [10, 118], we have written the action in general gauge and not in the conformal gauge.

where we have defined the proper length element $d\tau = ds\, e(s)$ on $\partial\Sigma$ and

$$h_{ij}(x) = e_i^a(x)\, e_j^b(x)\, g_{ab}(E(x)), \quad H_{\alpha\beta}(x) = n_\alpha^a(x)\, n_\beta^b(x)\, g_{ab}(E(x)) \tag{5.11}$$

are the induced metric of the brane and transverse components of the metric respectively.

The new open string couplings are thus $\{A, v\}$. To understand their physical interpretation, consider an infinitesimal target space diffeomorphism $F^a(X) = X^a + \xi^a(X)$ where the vector field decomposes at the brane as $\xi^a(X)|_{\partial\Sigma} = \zeta^i(x)\, e_i^a(x) + w^\alpha(x)\, n_\alpha^a(x)$. Then the new brane embedding is

$$F^a(E(x)) = E^a(x + \zeta) + w^\alpha(x)\, n_\alpha^a(x) \tag{5.12}$$

so that the tangential part $\zeta^i(x)$ performs a brane worldvolume diffeomorphism while the orthogonal part $w^\alpha(x)$ produces a transverse fluctuation of the embedding. From (5.8), it follows that $A_i(x)$ and $v_\alpha(x)$ respectively transform as a one-form and as a scalar under the worldvolume diffeomorphism $\zeta$. In addition, the orthogonal component $w^\alpha$ produces a shift $v^\alpha + w^\alpha$ so that physically $v^\alpha$ describes fluctuations of the brane in the $9-p$ transverse directions from the worldvolume point of view. The physical interpretation of $A_i$ is a $U(1)$ gauge field on the brane under which the end-point of the open string is charged.

Computing the variation of the action (5.6) and using $\delta X^a|_{\partial\Sigma} = e_i^a\, \delta x^i$ which follows from (5.5), the vanishing of boundary terms requires the Neumann boundary condition [10, 118]

$$[e_i^a\, \gamma^{\mu\nu} g_{ab}\, m_\mu\, \partial_\nu X^b + F_{ij}\, \partial_\tau x^i]|_{\partial\Sigma} = 0, \quad v^\alpha(x) = 0, \tag{5.13}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$. This equation fixes boundary values of $p+1$ worldsheet degrees of freedom so that together with (5.5) all ten independent degrees of freedom $X^a|_{\partial\Sigma}$ are fixed.

Beta functions of the couplings $\{A, v\}$ were studied in [8, 10, 119, 120] and they can be obtained from the low energy effective action [10]

$$I_{\rm DBI} = -T_{{\rm D}p} \int_{\mathcal{Q}} d^{p+1}x\, e^{-\phi} \sqrt{-\det\left(h_{ij} + F_{ij}\right)}, \tag{5.14}$$

where $h_{ij}$ is the induced metric (5.11) of the brane. The action (5.14) is a functional of the gauge field $A_i(x)$ and of the embedding $E^a(x)$ whose derivatives produce the beta functions $\beta^A$ and $\beta^v$ respectively. In general, the action is also supplemented with fermions to respect supersymmetry.

Again, the beta function analysis does not fix the coefficient $T_{{\rm D}p}$, but it can be fixed by considering a one-loop open string amplitude between two parallel branes which is equivalent to a tree-level closed string exchange between the branes [121] via the open closed string duality. The result is [14]

$$T_{{\rm D}p} = \frac{1}{(2\pi)^p g_{\rm s} \ell_{\rm s}^{p+1}}. \tag{5.15}$$

We will now consider the simplified situation where the D$p$-brane is embedded in a flat target space with Cartesian coordinates $X^a$ so that $g_{ab}(X) = \eta_{ab}$. We will choose the worldvolume coordinates as $x^i = X^i$ with $i = 1, \ldots, p+1$ so that the transverse directions are $X^{\alpha+p+1}$ with $\alpha = 1, \ldots, 9-p$. In addition, we take all transverse directions to be spacelike so that $g_{\alpha\beta} = \delta_{\alpha\beta}$. We will write the embedding of the brane as

$$E^i(x) = x^i, \quad E^{\alpha+p+1}(x) = 2\pi\ell_{\rm s}^2\, \Psi^\alpha(x), \quad \alpha = 1, \ldots, 9-p. \tag{5.16}$$

We will also rescale the fields $\phi, A_i$ and $v^\alpha$ by a factor of $2\pi\ell_{\rm s}^2$ so that $v^\alpha$ describes small perturbations of $\Psi^\alpha$. At leading order expanding around a flat brane, the action (5.14) has the term (taking into account the rescaling and neglecting the cosmological constant term)

$$I_{\rm YM} = -\frac{1}{4g_{\rm YM}^2}\int d^{p+1}x\left[\eta^{ij}\eta^{kl}\,F_{ik}F_{jl} + 2\,\eta^{ij}\delta_{\alpha\beta}\,\partial_i\Psi^\alpha\partial_j\Psi^\beta + \mathcal{O}(\ell_{\rm s}^2)\right] \tag{5.17}$$

where we have defined the constant

$$g_{\rm YM}^2 \equiv \frac{1}{4\pi^2\,T_{{\rm D}p}\,\ell_{\rm s}^4} = (2\pi)^{p-2}g_{\rm s}\ell_{\rm s}^{p-3}. \tag{5.18}$$

Note also that the open-closed string coupling between (rescaled) $\phi$ and $A_i$ is of order $\mathcal{O}(\ell_{\rm s}^2)$ and decouples [122]. The theory (5.17) is a Yang–Mills theory for an abelian $U(1)$ worldvolume gauge field $A_i$ with additional $9-p$ worldvolume scalars $\Psi^\alpha$ describing transverse fluctuations of the brane. The constant $g_{\rm YM}$ is the standard Yang–Mills coupling.

In general, there can be multiple D$p$-branes present in the theory that have many open strings stretching between them. Consider the case of $N$ branes that have the same embedding (5.16) (they are coincident), but that can fluctuate independently in the transverse directions. First there are strings of type (i) that start and end on the same brane each contributing a $U(1)$ gauge field to the worldvolume theory for a total of $N$ gauge fields making up a $U(1)^N$. In addition, there are strings of type (ii) that start and end on different branes. Combining strings of the two types (i) and (ii), it turns out that the $U(1)$ gauge field in the effective action (5.14) is replaced with a non-Abelian $U(N)$ gauge field [123]. In addition, the scalars $\Psi^\alpha$ describing transverse fluctuations become $N\times N$ matrices transforming in the adjoint of $U(N)$ under gauge transformations leading to a form of non-commutative geometry. The generalization of the bosonic DBI action (5.14) to the non-Abelian $A_i$ and adjoint $\Psi^\alpha$ is complicated, but has been derived using T-duality in [124].

The situation is simplified for a stack of $N$ D3-branes ($p=3$) in the $\ell_{\rm s}\to 0$ limit in which the effective action is completely fixed by supersymmetry to be the $\mathcal{N}=4$ super YM theory for the gauge group $U(N)$ which can be replaced by $SU(N)$ [123]. The same low energy theory can also be obtained by expanding the non-Abelian DBI action of [124] in the $\ell_{\rm s}\to 0$ limit [125]. The bosonic part of the action is

$$I_{\rm SYM} = -\frac{1}{4g_{\rm YM}^2}\int d^4x\ {\rm Tr}\left(\eta^{ij}\eta^{kl}\,F_{ik}F_{jl} + 2\,\eta^{ij}\delta_{\alpha\beta}\,D_i\Psi^\alpha D^i\Psi^\beta + \delta_{\alpha\gamma}\delta_{\beta\delta}\,[\Psi^\alpha,\Psi^\beta][\Psi^\gamma,\Psi^\delta]\right) \tag{5.19}$$

where $F_{ij} = \partial_iA_j - \partial_iA_j + i\,[A_i,A_j]$ is the $SU(N)$ field strength, $D_i\Psi^\alpha = \partial_i\Psi^\alpha - i\,[A_i,\Psi^\alpha]$ is the gauge covariant derivative and the YM coupling is given by (5.18). This is the generalization of the effective action (5.17) describing a stack of $N$ D3-branes in the low energy limit.

In four-dimensions, the YM coupling (5.18) is dimensionless so that (5.19) is superconformally invariant which is also true at the quantum level. The quantum theory simplifies in the double scaling limit

$$N\to\infty,\quad g_{\rm YM}\to 0,\quad \lambda = g_{\rm YM}^2N = \text{fixed}, \tag{5.20}$$

taken such that the 't Hooft coupling $\lambda$ is kept fixed. In this so called planar limit, only planar Feynman diagrams contribute in perturbative calculations [126].

Consider the thermal free energy density at inverse temperature $\beta$:

$$F_\beta \equiv -\frac{\log Z[S^1_\beta \times \mathbb{R}^3]}{\operatorname{Vol}\mathbb{R}^3} \tag{5.21}$$

which is defined by the Euclidean path integral of (5.19) over the Euclidean manifold $S^1_\beta \times \mathbb{R}^3$ equipped with the flat metric. In the planar limit (5.20), the free energy density goes as

$$F_\beta - F_0 = f_\beta(\lambda)\, N^2 + \mathcal{O}(1/N) \tag{5.22}$$

for some function $f_\beta(\lambda)$ of the 't Hooft coupling. The free energy can be computed using perturbative methods in the weak coupling limit $\lambda \to 0$ with the result [127]

$$f_\beta(0) = -\frac{\pi^2}{6\beta^4}. \tag{5.23}$$

Holographic duality gives a prediction for the coefficient at strong coupling $\lambda \to \infty$ as we will see next.

### 5.1.2 Supergravity description

In the previous section, we studied weakly coupled D$p$-branes that can be described by an effective action obtained from open strings. We will now consider the strongly coupled regime where the closed string sector becomes important. To this end, we will look for solutions of the classical closed string effective action with $N$ coincident D$p$-branes

$$I = I_{\text{IIB}} + N I_{\text{D}p} \tag{5.24}$$

where $I_{\text{IIB}}$ is given by (5.2) and the D-brane action $I_{\text{D}p}$ is the DBI action (5.14) obtained from the open string calculation. For the case of $p = 3$ (D3-brane), the action (5.24) admits a solution [13]

$$ds^2_{10} = H(u)^{-1/2}\, ds^2_{\mathbb{R}^{1,3}} + H(u)^{1/2}\,(du^2 + u^2 ds^2_{S^5}) \tag{5.25}$$

$$H(u) = 1 + \frac{\ell^4}{u^4}, \quad F_5 = (1+*)\, \operatorname{Vol}\mathbb{R}^{1,3} \wedge d(H^{-1}) \tag{5.26}$$

where $\operatorname{Vol}\mathbb{R}^{1,3}$ is the volume form of $\mathbb{R}^{1,3}$ and $u > 0$. In this solution, the D3-branes act as a source for the $*F_5$ flux and their total charge $Q_{\text{D3}}$ is identified with the coefficient of the DBI action $Q_{\text{D3}} = 2\kappa^2_{10} T_{\text{D3}}\, N$ [128, 129].[3] The charge can also be computed at asymptotic infinity from the solution (5.26) as

$$Q_{\text{D3}} = 2\kappa^2_{10} T_{\text{D3}}\, N = \int_{S^5} *F_5 = 4\ell^4 \operatorname{Vol} S^5 \tag{5.27}$$

where $S^5$ is the 5-sphere at infinity $u = \infty$. This fixes the constant $\ell$ in (5.26) as

$$\ell^4 = \frac{2\kappa^2_{10} T_{\text{D3}} N}{4 \operatorname{Vol} S^5} = 4\pi N g_{\text{s}} \ell^4_{\text{s}}. \tag{5.28}$$

[3] Quantum mechanical consistency of the effective theory requires that the D3-brane charge is quantized in units of $2\kappa^2_{10} T_{\text{D3}}$ meaning that $N$ is an integer (this is the Dirac quantization condition).

The solution (5.26) describes the backreaction of a stack of $N$ D3-branes in 10-dimensional flat Minkowski space that are uniformly distributed in the directions of $S^5$.[4] Being an extremal solution (charge is equal to the ADM mass), the geometry does not contain a horizon, but an infinitely long throat with the D3-branes at the end of it.

The description of the backreaction of a stack of $N$ D3-branes in terms of classical geometry is valid as long as the radius of curvature $\ell$ of the geometry is much larger than Planck and string lengths. In other words, we take the limit $\ell_{\rm s} \to 0$ with

$$\frac{\ell}{\ell_{\rm p10}} = \text{fixed} \gg 1, \quad \frac{\ell}{\ell_{\rm s}} = \text{fixed} \gg 1. \tag{5.29}$$

Since we have

$$\frac{\ell^8}{\ell_{\rm p10}^8} = \frac{2N^2}{\pi^4}, \quad \frac{\ell^4}{\ell_{\rm s}^4} = 4\pi g_{\rm s} N, \tag{5.30}$$

the conditions (5.29) are achieved when

$$N \to \infty \quad g_{\rm s} \to 0, \quad g_{\rm s} N = \text{fixed} \gg 1. \tag{5.31}$$

### 5.1.3 The AdS/CFT conjecture

Let us then consider the double scaling limit

$$u, \ell_{\rm s} \to 0, \quad r = \frac{u}{\ell^2} = \text{fixed} \tag{5.32}$$

known as the Maldacena limit or the decoupling limit [24]. Taking further $r \to 0$, the solution (5.26) becomes

$$ds_{10}^2 = \ell^2 \left( \frac{dr^2}{r^2} + r^2 ds_{\mathbb{R}^{1,3}}^2 + ds_{S^5}^2 \right) \tag{5.33}$$

$$F_5 = -4\ell^4 \, (1 + *) \, \mathrm{Vol}\, \mathrm{AdS}_5, \tag{5.34}$$

where we used $d(H^{-1}) \approx 4\ell^4 r^3 dr$ and $\mathrm{Vol}\, \mathrm{AdS}_5$ is the volume form on unit $\mathrm{AdS}_5$:

$$\mathrm{Vol}\, \mathrm{AdS}_5 = r^3 \, dr \wedge \mathrm{Vol}\, \mathbb{R}^{1,3}. \tag{5.35}$$

This limit is known as the Maldacena limit [24] which produces the near-horizon geometry of the metric (5.26). Even though the geometry (5.34) is obtained as the $r \to 0$ limit, it is a valid solution for all values of $r$ with locally $\mathrm{AdS}_5 \times S^5$ asymptotics when $r \to \infty$. In other words, the asymptotically flat region completely decouples from the throat and we are left with a sector of supergravity on manifolds with locally AdS asymptotics.

A similar decoupling occurs in the description of D-branes from the open string perspective where the $U(N)$ SYM gauge theory on the brane decouples from supergravity away from the brane. As in the supergravity solution (5.34), the decoupled theory is supergravity on flat space while the theory in the brane region is different. Because the closed string solution (5.34) and the brane gauge theory are supposed to describe the same physical object, one is lead to the AdS/CFT conjecture: classical type IIB supergravity on a manifold with $\mathrm{AdS}_5 \times S^5$ asymptotics is an alternative description of $\mathcal{N} = 4$ $SU(N)$ SYM theory [24, 122].

[4] Solutions with D3-branes distributed arbitrarily on the $S^5$ can also be found [130].

More precisely, supergravity describes the planar limit (5.20) of SYM theory in the strongly coupled regime $\lambda \to \infty$. The relation (5.18) between the YM coupling and the string coupling in four dimensions is given by (see also [131]):

$$g_{\rm YM}^2 = 2\pi g_{\rm s}. \tag{5.36}$$

so that the planar and strong coupling limits are required by the conditions (5.31) for the validity of the supergravity description. It is natural to assume that the AdS/CFT conjecture holds beyond these limits in which case supergravity is replaced by quantum string theory (which is still weakly coupled due to $g_{\rm s} \to 0$). The strongest form of the AdS/CFT conjecture is then:

Type IIB superstring theory on a manifold with $\mathrm{AdS}_5 \times S^5$ asymptotics

$\Leftrightarrow$

$\mathcal{N} = 4$ $SU(N)$ super-Yang–Mills theory on $\mathbb{R}^{1,3}$.

In general, string theory is believed to be background independent and restricting to manifolds with $\mathrm{AdS}_5 \times S^5$ asymptotics means that we are restricting to a subsector of string theory. The duality states that this subsector (with its own Hilbert space if one wishes) is isomorphic to the Hilbert space of SYM theory.

The holographic duality gives a prediction for the thermal free energy density (5.21) of SYM theory at strong coupling $\lambda \to \infty$: it is computed by the on-shell supergravity action of the finite temperature generalization of the solution (5.34) (the black D3-brane solution). The calculation produces the expected $\sim N^2$ behaviour with the coefficient function [127]

$$f_\beta(\infty) = -\frac{\pi^2}{8\beta^4}. \tag{5.37}$$

Comparing with (5.23), we see that the weak and strong coupling free energies differ by the famous factor

$$\frac{f_\beta(\infty)}{f_\beta(0)} = \frac{3}{4}. \tag{5.38}$$

This can be seen as a prediction of the AdS/CFT conjecture that can in principle be verified by a difficult strongly coupled calculation in SYM theory.

## 5.2 Holographic dual of ABJM theory

We will now move on to study another example of a holographic duality involving the ABJM CFT. ABJM theory is a 3-dimensional $\mathcal{N} = 6$ Chern–Simons–matter theory with gauge group $U(N)_k \times U(N)_{-k}$ [27]. The subscripts $k, -k$ denote integers that appear as coefficients of the two $U(N)$ Chern–Simons terms in the action. The matter part of the theory consists of four complex scalar fields (and their fermionic partners) that transform in the bifundamental representation $(N, \bar{N})$ and $(\bar{N}, N)$ of the gauge group. ABJM theory arises as the low-energy limit of an open string effective action of intersecting D3-, NS5- and D5-branes in type IIB string theory [27]. The setup can be uplifted to M-theory where the brane configuration amounts to a single stack of $N$ M2-branes on a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold singularity (see [132, 133] for reviews). Hence ABJM theory is the low-energy description of M2-branes in M-theory, and based on this construction, ABJM theory turns out to be dual to M-theory on a manifold with $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ asymptotics.[5]

[5]The special case $k = 1$ was already considered from the gravity side in the original Maldacena article [24].

The analog of the YM coupling $g^2_{\rm YM}$ in ABJM theory is $1/k$ so that the 't Hooft coupling is $\lambda = \frac{N}{k}$ and the planar limit is

$$N, k \to \infty, \quad \lambda = \frac{N}{k} = \text{fixed}. \tag{5.39}$$

In this limit, the internal space $S^7/\mathbb{Z}_k \to \mathbb{CP}^3$ is dimensionally reduced so that M-theory reduces to type IIA superstring theory on $\text{AdS}_4 \times \mathbb{CP}^3$ [27]. Hence similarly to the previous section, the planar limit of the dual CFT is described by a string theory.

The purpose of the current section is to review basics of the ABJM duality. We start by describing the low energy effective action of M-theory, which is $D = 11$ supergravity, and review how dimensional reduction produces type IIA string theory. Then we present the relevant supergravity solutions whose decoupling limit leads to the ABJM duality.

### 5.2.1 Effective action of M-theory

M-theory is the quantum theory of the fundamental supermembrane in eleven dimensions whose worldvolume action was first written in [134]. Let $\hat{\sigma}^{\hat{\mu}}$ with $\hat{\mu} = 1, 2, 3$ index the three worldvolume coordinates of the membrane and let $\hat{X}^{\hat{a}}$ with $\hat{a} = 1, \ldots, 11$ denote the eleven target space coordinates. The bosonic part of the membrane action is then given by

$$I = -T_3 \int d^3\hat{\sigma} \left[ \frac{1}{2}\sqrt{-\hat{\gamma}}\,(G_{\hat{a}\hat{b}}(\hat{X})\,\gamma^{\hat{\mu}\hat{\nu}}\,\partial_{\hat{\mu}}\hat{X}^{\hat{a}}\partial_{\hat{\nu}}\hat{X}^{\hat{b}} - 1) + \frac{1}{3!}\,C_{\hat{a}\hat{b}\hat{c}}(\hat{X})\,\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}}\partial_{\hat{\mu}}\hat{X}^{\hat{a}}\partial_{\hat{\nu}}\hat{X}^{\hat{b}}\partial_{\hat{\rho}}\hat{X}^{\hat{c}} \right] \tag{5.40}$$

where $\hat{\gamma}^{\hat{\mu}\hat{\nu}}(\hat{\sigma})$ is the membrane metric, $\hat{g}_{\hat{a}\hat{b}}(\hat{X})$ is the target space metric, $C_{\hat{a}\hat{b}\hat{c}}(\hat{X})$ is a target space 3-form, and $T_3$ is the membrane tension (the only free parameter of the theory).

The Lorentz symmetry on the membrane determines the critical dimension of the supermembrane to be $D = 11$ [135] (see also [136] for another argument) and kappa-symmetry imposes classical equations of motion of $D = 11$ supergravity on the couplings $\{g, C\}$ [134, 137]. The action of $D = 11$ supergravity is given by [138]

$$I_{\rm 11D} = \frac{1}{2\kappa_{11}^2}\int d^{11}\hat{X}\sqrt{-\hat{g}}\left(R_{11} - \frac{1}{2\cdot 4!}F_4^2\right) - \frac{1}{12\kappa_{11}^2}\int C_3 \wedge F_4 \wedge F_4 \tag{5.41}$$

where $F_4 = dC_3$ and $2\kappa_{11}^2 \equiv (2\pi)^8\ell_{\rm p11}^9$ with $\ell_{\rm p11}$ being the 11-dimensional Planck length. The action (5.41) is interpreted as the low-energy effective action of M-theory which is the conjectured quantized theory of the supermembrane (5.40).

The Planck length $\ell_{\rm p11}$ is fixed in terms of the membrane tension $T_3$ by flux quantization conditions. Requiring the path integral of the membrane action (5.40) (the $C_3$ part) and of the effective action (5.41) (the $C_3 \wedge F_4 \wedge F_4$ part) to be single valued requires that [139]

$$T_3\int_{S^3\times S^1} F_4 = 2\pi n, \quad \frac{1}{12\kappa_{11}^2}\int_{M_{12}} F_4 \wedge F_4 \wedge F_4 = 2\pi m, \quad n, m \in \mathbb{Z}, \tag{5.42}$$

where $M_{12}$ is any closed 12-dimensional manifold. We can take $M_{12} = (S^3 \times S^1)^3$ so that using the first condition in the second gives (see also [140, 141])

$$\frac{(2\pi n)^3}{12\kappa_{11}^2 T_3^3} = 2\pi m, \quad n, m \in \mathbb{Z}. \tag{5.43}$$

We will now assume that this condition is satisfied for $m = n^3/6$ which gives

$$T_3 = \frac{1}{4\pi^2 \ell_{\rm p11}^3} \tag{5.44}$$

as in [131].

The worldsheet action of the type IIA superstring can be obtained from the supermembrane action (5.40) via double dimensional reduction [142]. In double dimensional reduction, both the membrane and the target space are reduced on the same circle to two- and ten-dimensions respectively. The reduction ansatz is

$$ds_{11}^2 = e^{-\frac{2\phi}{3}} ds_{10}^2 + e^{\frac{4\phi}{3}} (d\varphi + A_1)^2 \tag{5.45}$$

where $A_1 = A_1(X)$, $\phi = \phi(X)$, and the ten-dimensional metric $ds_{10}^2 = g_{ab}(X)\, dX^a dX^b$. We take the circle to have circumferemce $2\pi\ell_\varphi$ so that $\varphi \sim \varphi + 2\pi\ell_\varphi$.

Let us first consider the reduction of the target space effective action. Substituting to (5.41) and integrating over the fiber, the action (5.41) becomes [117]

$$I_{\rm IIA} = \frac{1}{2\kappa_{10}^2} \int d^{10}X \sqrt{-g}\, e^{-2\phi} (R_{10} + 4\partial_a\phi\partial^a\phi) - \frac{1}{4\kappa_{10}^2} \int d^{10}X \sqrt{-g} \left( \frac{1}{2!} F_2^2 + \frac{1}{4!} F_4^2 \right). \tag{5.46}$$

where $F_2 = dA_1$, and

$$2\kappa_{10}^2 = \frac{2\kappa_{11}^2}{2\pi\ell_\varphi}. \tag{5.47}$$

This is the action of type IIA supergravity where the one-form gauge field $A_1$ and the dilaton $\phi$ (not $\Phi$) are identified as components of the eleven-dimensional metric.

It is expected that type IIA supergravity arises, because it is the effective action of the type IIA superstring which is the result of the dimensional reduction of the membrane action (5.40) [136]. In other words, the supermembrane reduces to the superstring and their effective actions also reduce to each other. The coefficient of the reduced membrane action is identified with the coefficient of the string action as

$$\frac{1}{4\pi\ell_{\rm s}^2} = 2\pi\ell_\varphi \frac{T_3}{2}. \tag{5.48}$$

Combining equations (5.47) and (5.48) with (5.3) and (5.44), we get [131, 133]

$$\ell_\varphi = g_{\rm s}\ell_{\rm s}, \quad \ell_{\rm p11} = g_{\rm s}^{1/3}\ell_{\rm s} \tag{5.49}$$

or equivalently

$$g_{\rm s}^{2/3} = \frac{\ell_\varphi}{\ell_{\rm p11}}, \quad \ell_{\rm s}^4 = \frac{\ell_{\rm p11}^6}{\ell_\varphi^2}. \tag{5.50}$$

These equations relate 11-dimensional M-theory parameters to the 10-dimensional string theory parameters. In particular, the size of the compactification circle in Planck units is the string coupling which shows that M-theory is the strong coupling limit of the type IIA superstring [17].

Using the identification (5.49), the membrane tension (5.44) matches with the D2-brane tension (5.15) $T_{\rm M2} = T_{\rm D2}$. This is expected, because when the membrane is not wrapped on the circle that is being reduced, it becomes a D2-brane of type IIA supergravity (similar relation holds for other types of branes as well [133]).

### 5.2.2 Supergravity solutions

We will now review the classical solution of $D = 11$ supergravity that describes a stack of $N$ coincident M2-branes on a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. In the $k \to \infty$ limit, the solution dimensionally reduces to a ten-dimensional solution of type IIA supergravity. These solutions provide the holographic dual of strongly coupled ABJM theory in its planar limit.

Consider $\mathbb{C}^4$ with the flat Euclidean metric and consider the orbifold action

$$z^j \to e^{\frac{2\pi i}{k}} z^j, \quad k \in \mathbb{Z}, \tag{5.51}$$

with $z^j \in \mathbb{C}$ for $j = 1, \ldots, 4$. Identification under (5.51) produces the orbifold $\mathbb{C}^4/\mathbb{Z}_k$ with a conical singularity. In terms of the radial coordinate

$$u^2 = \sum_{i=1}^{4} |z^i|^2, \tag{5.52}$$

the singularity is located at $u = 0$ and the flat metric on $\mathbb{C}^4/\mathbb{Z}_k$ takes the form

$$ds^2_{\mathbb{C}^4/\mathbb{Z}_k} = du^2 + u^2 ds^2_{S^7/\mathbb{Z}_k}. \tag{5.53}$$

We will now consider the 11-dimensional metric

$$ds^2_{11} = ds^2_{\mathbb{R}^3} + ds^2_{\mathbb{C}^4/\mathbb{Z}_k} \tag{5.54}$$

and place $N$ coincident M2-brane sources at $u = 0$. We want to solve the equations of motion of the action

$$I = I_{11\mathrm{D}} + N I_{\mathrm{M2}}. \tag{5.55}$$

where $I_{11\mathrm{D}}$ is given by (5.41) and the action of an M2-brane is the same as the membrane action (5.40). The M2-branes backreact on the geometry producing a solution of 11D supergravity given by [132, 143]

$$ds^2_{11} = H(u)^{-2/3}(-dX_0^2 + dX_1^2 + dX_2^2) + H(u)^{1/3}(du^2 + u^2 ds^2_{S^7/\mathbb{Z}_k}) \tag{5.56}$$

$$H(u) = 1 + \frac{(2L)^6}{u^6}, \quad F_4 = dX_0 \wedge dX_1 \wedge dX_2 \wedge d(H^{-1}) \tag{5.57}$$

where we will keep the factor of two in $2L$ for convenience. The M2-branes are a source for the $F_4$ flux and the total charge of the solution is given by

$$Q_{\mathrm{M2}} = 2\kappa_{11}^2 T_{\mathrm{M2}}\, N = \int_{S^7/\mathbb{Z}_k} *F_4 = 6\,(2L)^6\,\mathrm{Vol}\,(S^7/\mathbb{Z}_k) \tag{5.58}$$

where the integral is performed at $u = \infty$. We get

$$L^6 = \frac{\pi^2 N k}{2}\,\ell_{\mathrm{p11}}^6 \tag{5.59}$$

where we used $\mathrm{Vol}\,(S^7/\mathbb{Z}_k) = \pi^4/(3k)$.

Let us consider the decoupling limit of the M2-brane solution (5.57) as we did for the D3-brane solution above. We will take the limit [24]

$$u, \ell_{\mathrm{p11}} \to 0, \quad r \equiv 2u^2/(2L)^3 = \text{fixed} \tag{5.60}$$

such that $r \to 0$. In this limit, the solution (5.57) becomes

$$ds_{11}^2 = L^2\left[\frac{dr^2}{r^2} + r^2(-dX_0^2 + dX_1^2 + dX_2^2) + 4ds^2_{S^7/\mathbb{Z}_k}\right] \tag{5.61}$$

$$F_4 = -3L^3 \,\mathrm{Vol}\,\mathrm{AdS}_4 \tag{5.62}$$

where the volume form of unit $\mathrm{AdS}_4$ is

$$\mathrm{Vol}\,\mathrm{AdS}_4 = r^2\, dr \wedge dX_0 \wedge dX_1 \wedge dX_2. \tag{5.63}$$

We see that (5.62) is the metric of $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ where the $\mathrm{AdS}_4$ factor has radius of curvature $L$.

Let us study the range of validity of the classical solution (5.62). The $\mathbb{Z}_k$-quotient decreases the radius of curvature of the $S^7$-factor which has to be taken into account. For this it is useful to use the fact $S^7/\mathbb{Z}_k$ is an $S^1$-fibration over $\mathbb{CP}^3$ so that the metric (5.62) can be written as

$$ds_{11}^2 = L^2(ds^2_{\mathrm{AdS}_4} + 4ds^2_{\mathbb{CP}^3}) + (d\varphi + k\ell_\varphi\, d\omega)^2 \tag{5.64}$$

where the size of the fiber

$$\ell_\varphi = \frac{2L}{k} = \left(\frac{32\pi^2 N}{k^5}\right)^{1/6} \ell_{\mathrm{p11}}, \tag{5.65}$$

$\omega$ is the Kähler form of $\mathbb{CP}^3$, and $\varphi \sim \varphi + 2\pi\ell_\varphi$ [27, 144–146].

Now we see that (5.64) is a valid description when the radius $L$ of $\mathrm{AdS}_4 \times \mathbb{CP}^3$ and the radius $\ell_\varphi$ of the $S^1$ fiber are large in Planck units $\ell_{\mathrm{p11}}$. In other words, we take the limit $\ell_{\mathrm{p11}} \to 0$ with

$$\frac{L}{\ell_{\mathrm{p11}}} = \text{fixed} \gg 1, \quad \frac{\ell_\varphi}{\ell_{\mathrm{p11}}} = \text{fixed} \gg 1. \tag{5.66}$$

We have

$$\frac{L^6}{\ell_{\mathrm{p11}}^6} = \frac{Nk}{8 \cdot 2^{2/3}\pi^{8/3}}, \quad \frac{\ell_\varphi^6}{\ell_{\mathrm{p11}}^6} = \frac{4 \cdot 2^{1/3}}{\pi^{8/3}}\frac{N}{k^5}. \tag{5.67}$$

so that the solution (5.62) is a valid description when [27]

$$N \gg 1, \quad k \ll N^{1/5}. \tag{5.68}$$

For this range of parameters, ABJM theory is holographically dual to $D = 11$ supergravity on a manifold with $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ asymptotics [27]. The limit (5.68) is a strong coupling limit of ABJM theory, because the second condition in (5.68) implies that $\lambda = \frac{N}{k} \gg 1$.

When the radius $\ell_\varphi$ of the $S^1$ fiber is small compared to the radius $L$ of $\mathrm{AdS}_4 \times \mathbb{CP}^3$, the eleven-dimensional solution can be replaced with its dimensional reduction which is a solution of type IIA supergravity in ten-dimensions as explained in the previous section. This happens when

$$N, k \gg 1, \quad N^{1/5} \ll k. \tag{5.69}$$

The reduction of the solution (5.62) is obtained by comparing (5.64) with the general reduction ansatz (5.45). We see that the resulting ten-dimensional dilaton $\phi = 0$ vanishes and that the

ten-dimensional gauge field $A_1 = k\omega$ is the Kähler form. The full ten-dimensional solution of type IIA supergravity in string frame becomes

$$ds_{10}^2 = L^2(ds_{\mathrm{AdS}_4}^2 + 4ds_{\mathbb{CP}^3}^2) \tag{5.70}$$

$$F_2 = k\ell_\varphi\, d\omega, \quad F_4 = -3L^3\ \mathrm{Vol}\,\mathrm{AdS}_4. \tag{5.71}$$

The string coupling is obtained from the formula (5.50) to be

$$g_{\mathrm{s}} = \left(\frac{32\pi^2 N}{k^5}\right)^{1/4}. \tag{5.72}$$

One can see that the $S^1$ fiber being small (5.69) means that the type IIA string is weakly coupled $g_{\mathrm{s}} \ll 1$.

The conditions (5.69) are not enough to ensure validity of the ten-dimensional solution, but we also need the curvature to be large in string units. Since we have

$$\frac{L^4}{\ell_{\mathrm{s}}^4} = \frac{2\pi^2 N}{k}, \tag{5.73}$$

we need $k \ll N$ which amounts to staying in the strongly coupled regime $\lambda \gg 1$. Hence to summarize, ABJM theory at strong coupling $\lambda \gg 1$ has two dual descriptions depending on the magnitude of $k$ as follows:

$$\text{11D supergravity:} \qquad N \gg 1, \qquad k \ll N^{1/5} \tag{5.74}$$

$$\text{10D type IIA supergravity:} \qquad N, k \gg 1, \qquad N^{1/5} \ll k \ll N. \tag{5.75}$$

Notice that we have written the solution (5.71) in conventions where the dimensional reduction ansatz (5.45) involves $\phi$ and where the $\varphi$ coordinate has period $2\pi\ell_\varphi$. However in the literature, it is usually written in conventions where the ansatz (5.45) involves $\Phi$ instead of $\phi$ and where $\varphi$ has period $2\pi\ell_{\mathrm{s}}$.[6] To obtain the solution in these conventions, we write (5.64) as

$$ds_{11}^2 = L^2(ds_{\mathrm{AdS}_4}^2 + 4ds_{\mathbb{CP}^3}^2) + \left(\frac{2L}{k\ell_{\mathrm{s}}}\right)^2 (d\varphi + k\ell_{\mathrm{s}}\, d\omega)^2 \tag{5.76}$$

where now $\varphi \sim \varphi + 2\pi\ell_{\mathrm{s}}$. Writing $L^2 = e^{-\frac{2}{3}\Phi}\ell^2$ and $\frac{2L}{k\ell_{\mathrm{s}}} = e^{\frac{2}{3}\Phi}$, the dimensional reduction formula (5.45) with $\phi$ now replaced by $\Phi$ gives

$$ds_{10}^2 = \ell^2(ds_{\mathrm{AdS}_4}^2 + 4ds_{\mathbb{CP}^3}^2) \tag{5.77}$$

$$g_{\mathrm{s}} = e^{\Phi} = \frac{2\ell}{k\ell_{\mathrm{s}}}, \quad F_2 = k\ell_{\mathrm{s}}\, d\omega, \quad F_4 = -\frac{3}{2}\,\ell^2\ell_{\mathrm{s}}\ \mathrm{Vol}\,\mathrm{AdS}_4, \quad \ell^4 = \frac{2\pi^2 N}{k}\,\ell_{\mathrm{s}}^4, \tag{5.78}$$

which agrees with [144, 147] after setting $\ell_{\mathrm{s}} = 1$.

### 5.2.3 Free energy on a sphere

We can now use the supergravity dual to compute the renormalized free energy of ABJM theory on an $S^3$ (with the standard curved Euclidean metric)

$$F_{S^3} \equiv -\log Z[S^3] \tag{5.79}$$

[6] In these conventions, $2\kappa_D^2$ and $T_{\mathrm{D}p}$ do not involve the string coupling.

in the strong coupling limit. Instead of computing (5.79) directly using the on-shell action in ten or eleven dimensions, we will use the effective action of M-theory reduced to four dimensions. This will be simpler and avoids possible problems with the need of including an additional boundary term to the higher-dimensional action as in type IIB supergravity [148, 149].

The four-dimensional effective action of M-theory is given by

$$I_{4\mathrm{D}} = -\frac{1}{2\kappa_4^2}\int d^4x\sqrt{g_4}\,(R_4 - 2\Lambda + \ldots) \tag{5.80}$$

where $g_4$ is the four-dimensional Euclidean metric, $R_4$ is its Ricci scalar, $\Lambda$ is the cosmological constant and dots denote additional terms that depend on fields beyond the metric. We will reduce the M-theory solution (5.62) along the internal space which gives the 4-dimensional Newton's constant

$$\frac{1}{2\kappa_4^2} = \frac{(2L)^7}{2\kappa_{11}^2}\,\mathrm{Vol}\,(S^7/\mathbb{Z}_k). \tag{5.81}$$

where the volume factor arises from integrating over $S^7/\mathbb{Z}_k$. Since (5.62) has only the $F_4$-flux turned on, the resulting four-dimensional on-shell action is given completely by the Einstein–Hilbert part of (5.80) as [150]

$$I_{4\mathrm{D}}^{\text{on-shell}} = \frac{(2L)^7\,\mathrm{Vol}\,(S^7/\mathbb{Z}_k)}{2\kappa_{11}^2}\int_{\mathrm{AdS}_4}\sqrt{g_4}\left(R_4 + \frac{3\cdot 2}{L^2}\right) \tag{5.82}$$

where now $R_4 = -12/L^2$ and $\sqrt{g_4} = L^4$. The renormalized Euclidean on-shell action becomes

$$I_{4\mathrm{D}}^{\mathrm{ren}} = -\frac{3\,\mathrm{Vol}\,(S^7/\mathbb{Z}_k)}{\pi^8}\,V_{\mathrm{AdS}_4}^{\mathrm{ren}}\left(\frac{L}{\ell_{\mathrm{p}11}}\right)^9. \tag{5.83}$$

where $V_{\mathrm{AdS}_4}^{\mathrm{ren}}$ denotes the renormalized volume unit Euclidean $\mathrm{AdS}_4$. We have $V_{\mathrm{AdS}_4}^{\mathrm{ren}} = \frac{(2\pi)^2}{3}$ [151] so that the free energy (5.79) becomes (see also [147, 150])

$$F_{S^3} = \frac{\pi\sqrt{2}}{3}N^{3/2}k^{1/2} = \frac{\pi\sqrt{2}}{3}\frac{N^2}{\sqrt{\lambda}} \tag{5.84}$$

where we used (5.59). At fixed 't Hooft coupling $\lambda$, the free energy exhibits the familiar $\sim N^2$ behaviour as predicted by planar diagrams, and at fixed $k$, it exhibits the characteristic behaviour $\sim N^{3/2}$ of M2-branes.

## 5.3 Holographic dual of flavor in ABJM theory

Holographic duals of string theories do not contain matter transforming in the fundamental representation of the gauge group as is the case for example with $\mathcal{N} = 4$ SYM whose scalar fields (corresponding to transverse fluctuations of the brane) transform in the adjoint of $U(N)$. The reason for this limitation is the fact that open strings have to start and end on identical branes. If there were at least two types of branes present to which open strings can attach fields transforming in the fundamental representation could arise [28].

Consider $N$ D3-branes inside $N_f$ D7-branes gives a low energy $U(N)\times U(N_f)$ gauge theory on the worldvolume of the D3-brane with adjoint scalar fields coming from D3-D3 and D7-D7 open strings. Now the system also includes D3-D7 strings that have one end-point coupled to

transverse fluctuations of the D3-brane while the other end-point is coupled to fluctuations of the D7-brane. From the point of view of the gauge theory, these modes are encoded in scalar fields that transform in the bifundamental representation of $U(N) \times U(N_f)$. In the decoupling limit, D7-D7 strings decouple so that the gauge group reduces to $U(N)$ and $U(N_f)$ becomes a global symmetry [152]. As a result, the remaining D3-D7 scalars appear as quarks (flavor) that transform in the fundamental representation of $U(N)$. This is the original proposal of [28] for adding flavor to $\mathcal{N} = 4$ SYM using the D3-D7 system.

The addition of flavor can be done in the double scaling limit

$$N, N_f \to \infty, \quad \frac{N_f}{N} = \text{fixed}, \tag{5.85}$$

which is known as the Veneziano limit [153]. In this limit, one has to take into account backreaction of the D7-branes to the geometry sourced by the D3-branes which is a difficult problem in general so that various approximations such as smearing have been proposed to simplify the problem (see [153] for a review). The problem simplifies if we also assume that $N_f/N \ll 1$ is small and compute to leading order in an expansion in $N_f/N$. In this case, the D7-brane is a probe brane whose backreaction is neglected (on the field theory side it corresponds to the quenched approximation).

A similar story exists for ABJM theory [145, 154–156] which admits an extension with flavor that transform in the fundamental representations $(N, 1)$ and $(1, N)$ of $U(N) \times U(N)$. These flavor fields arise by the inclusion of D5-branes into the original type IIB brane system that engineers pure ABJM theory as its low energy limit. The flavored brane system can then be uplifted to M-theory using the same steps as in the flavorless case. After taking the near horizon limit and reducing to type IIA supergravity, the addition of flavor corresponds to the addition of D6-branes that extend in all $\text{AdS}_4$ directions and wrap an $S^3/\mathbb{Z}_2 = \mathbb{RP}^3 \subset \mathbb{CP}^3$ in the internal space [145, 154–156]. This is similar to the D7-branes wrapping an $S^3 \subset S^5$ in the near horizon limit of the above D3-D7 system.

In this section, we compute flavor corrections to the sphere free energy (5.84) of ABJM theory using its holographic dual. This can be done using both M-theory and type IIA supergravity descriptions. In M-theory we can work with any $N_f$, while in ten-dimensions it is simplest to take the probe limit $N_f/N \ll 1$ to neglect backreaction of the D6-branes. The power of the probe limit is that it also allows to work with non-zero flavor mass [A2].

### 5.3.1 Adding flavor in M-theory

Let us consider ABJM theory with $N_f$ flavor fields such that the parameters are in the regime (5.74) where the dual description in terms of $D = 11$ supergravity is valid. We want to compute the sphere free energy $F_{S^3}$ of this flavored theory. The presence of flavor modifies the near horizon solution (5.62) by deforming the internal space from $S^7/\mathbb{Z}_k$ to an Einstein–Sasaki manifold $X_7$ whose metric depends on the number of flavors $N_f$. More precisely, $X_7$ is a tri-Sasakian parametrized by the three integers $(N_f, N_f, k)$ [157, 158]. The new solution is explicitly

$$ds_{11}^2 = L^2(ds_{\text{AdS}_4}^2 + 4ds_{X_7}^2) \tag{5.86}$$

$$F_4 = -3L^3 \,\text{Vol}\,\text{AdS}_4. \tag{5.87}$$

The charge of this solution is the same as (5.58), but with $S^7/\mathbb{Z}_k \to X_7$ so that the radius of curvature is modified to

$$L = \pi \left( \frac{N}{6 \operatorname{Vol}(X_7)} \right)^{1/6} \ell_{\text{p11}}. \tag{5.88}$$

The four-dimensional renormalized on-shell action has the same form (5.83), but with $L$ given by (5.88) and the volume being that of $X_7$. The result is

$$I_{\text{4D}}^{\text{ren}} = -\frac{\pi^3 \sqrt{2}}{3\sqrt{3}} \frac{N^{3/2}}{(\operatorname{Vol} X_7)^{1/2}}. \tag{5.89}$$

The volume of $X_7$ is given by [157]

$$\operatorname{Vol} X_7 = \frac{\operatorname{Vol}(S^7/\mathbb{Z}_k)}{f(N_f/k)^2}, \quad f(s) = \frac{1+s}{\sqrt{1+s/2}} \tag{5.90}$$

so that the sphere free energy becomes [158]

$$F_{S^3} = \frac{\pi\sqrt{2}}{3} N^{3/2} k^{1/2} \frac{1 + N_f/k}{\sqrt{1 + N_f/(2k)}}. \tag{5.91}$$

We will now consider the probe limit. Since $N_f/k = \lambda\,(N_f/N)$ with the 't Hooft coupling fixed, the probe limit $N_f/N \ll 1$ in ABJM theory amounts to $N_f/k \ll 1$ as long as $N_f/N \ll \lambda^{-1}$. Expanding (5.91) in powers of $N_f/k$ gives

$$F_{S^3} = \frac{\pi\sqrt{2}}{3} N^{3/2} k^{1/2} + \frac{\pi}{4} N_f N \sqrt{2\lambda} + \mathcal{O}(N_f^2/N^2). \tag{5.92}$$

The leading $N_f/N$-correction can be derived directly from the field theory side using matrix theory formulation of ABJM theory [158]. In the next section, we reproduce the same correction by looking at probe D6-branes in ten dimensions.

### 5.3.2 Adding flavor in type IIA supergravity

We will now consider flavored ABJM theory in the regime (5.75) where the description in terms of type IIA supergravity is valid. This regime involves taking $k \to \infty$ so we can work in the Veneziano limit (5.85) with $N_f/k = \lambda^{-1}\,(N_f/N)$ also being fixed. In ten dimensions, massless flavor is described by $N_f$ D6-branes that wrap the three-cycle $\mathbb{RP}^3 \subset \mathbb{CP}^3$ in the internal space [145, 154–156]. The total bulk action is thus

$$I = I_{\text{IIA}} + N_f I_{\text{D6}}, \tag{5.93}$$

where $I_{\text{IIA}}$ is the type IIA supergravity action (5.46) and the action of a single D6-brane in the string frame is

$$I_{\text{D6}} = T_{\text{D6}} \int e^{-\phi} \sqrt{h} - T_{\text{D6}} \int \widehat{C}_7, \tag{5.94}$$

where in addition to the DBI action[7] (5.14) there is a WZ term that couples the brane to the background $F_8 = dC_7 = *F_2$ with $\widehat{C}_7$ denoting the pull-back of $C_7$.

[7] The overall minus sign compared to (5.14) is due to being in Euclidean signature

In principle, the embeddings of the D6-branes and their backreaction to the solution (5.71) can be computed at finite $N_f/k$ by solving the equations of motion of the action (5.93) and the resulting Euclidean on-shell action should match with (5.91). However, this is a difficult task due to the delta-function nature of the D6-brane stress tensor, but progress can be made by smearing the branes in transverse directions. For backreacting smeared D6-branes, the on-shell action (5.93) was computed in [147] and the result looks very similar to the exact result (5.91), but does not agree with it perfectly.

Instead of attempting the full backreacted analysis, we will consider the probe limit $N_f/k \ll 1$ (equivalently $N_f/N \ll 1$). In this case, the zeroth order term in $N_f/N$ is simply the supergravity action evaluated on the solution (5.71) which gives (5.91). The leading $N_f/N$-correction then comes only from the on-shell D6-brane action (5.94) evaluated at the brane embedding that minimizes its action on the non-backreacted background (5.71) [A2]:

$$F_{S^3} = \frac{\pi\sqrt{2}}{3} N^{3/2} k^{1/2} + N_f I_{\text{D6}}^{\text{on-shell}} + \mathcal{O}(N_f^2/N^2). \tag{5.95}$$

The result for the renormalized Euclidean on-shell D6-brane action is [A2]

$$I_{\text{D6}}^{\text{on-shell}} = \frac{\pi}{4} N \sqrt{2\lambda} \tag{5.96}$$

so that the leading $N_f/N$-correction in (5.95) matches with (5.92). The same correction can also be found in the smeared D6-brane analysis [147]. The power of the D6-brane approach in ten dimensions is that it can be extended to massive flavor (as done in [A2]) which would be intractable in M-theory.

## 5.4 Relation to the articles

The top-down holographic duality involving ABJM theory presented in this chapter was utilized in the article [A2]. We used the ABJM duality, because we wanted to apply holography to study renormalization group monotones in a non-trivial interacting three-dimensional QFT of which ABJM theory is an example. ABJM theory is also suitable for the problem, because its extension with flavor admits a relevant mass deformation that triggers a renormalization group (RG) flow from the flavored theory in the UV to pure ABJM theory in the IR. This flow on the three-sphere was the main focus of our article [A2]. The recipe for describing flavor using D6-branes on the supergravity side allowed us to compute free energies holographically and to study the F-theorem. Approaching the RG flow directly in field theory would have been impossible.

# Chapter 6

# Summary and conclusion

Quantum field theories without gravity are related to theories of (quantum) gravity via the holographic duality. The duality provides a two-way street to approach difficult problems that appear in both theories. In particular, strongly coupled QFT phenonema such as confinement are equivalently described by weakly coupled gravitational physics in the gravity dual that are easier to solve. On the other hand, semi-classical quantum gravity contains unsolved problems such as the black hole information paradox for which progress can be made using well understood quantum mechanical tools of the QFT. The duality itself is also interesting on its own and giving constraints for what types of QFTs are holographic plays a central role in mapping the landscape of quantum gravity.

In this thesis, we gave an introduction to the holographic duality and related topics with the above perspectives in mind. The first half of the thesis was reserved for reviewing necessary concepts in conformal field theory with emphasis on two-dimensional CFTs. Our description of conformal symmetry was general and included CFTs living on curved manifolds for which the definition of the conformal group as a subgroup of the Diff $\ltimes$ Weyl group was given. We studied quantum mechanical properties of CFTs using both path integral and operator formulations (canonical quantization) of the theory with latter described in a less conventional Lorentzian signature. The quantum mechanical breaking of Diff $\ltimes$ Weyl symmetry is realized in the two formulations in different, but equivalent ways: the Weyl non-invariance of the path integration measure is consistent with the Schwinger term in the algebra of generators of conformal symmetry. We also considered two-dimensional CFTs with a boundary where open closed string duality leads to strong constraints on the spectrum of the theory.

In the second half of the thesis, we reviewed both bottom-up and top-down approaches to the holographic duality. In the bottom-up approach, theories of classical gravity play a central role as toy-models of string theory. We reviewed properties of Lovelock gravity which is a natural generalizations of Einstein's general relativity to higher-dimensions. In particular, we explained how boundaries and corners can be treated in these theories. Our main focus was on the bottom-up description of the Weyl anomaly of the dual CFT. To this end, we considered the description of the Diff $\ltimes$ Weyl symmetry on the gravity side as the asymptotic symmetry group of asymptotically locally AdS spaces. In $\mathrm{AdS}_3/\mathrm{CFT}_2$, the action of the asymptotic symmetry group produced the Schwarzian transformation of the stress tensor of the dual CFT. For holographic description of boundaries, we considered models with end-of-the-world branes that can be used to accommodate primary or boundary condition changing operators of the BCFT.

In our top-down introduction to holography, we described the low-energy effective theories of closed and open strings. In particular, we gave a detailed account of Dirichlet–Neumann open strings whose effective field theory is the DBI action of a D-brane in the target space. We also explained that membrane theory in eleven dimensions produces superstrings in ten dimensions and how the effective actions are related. In the process, we derived relationships between M-theory and stringy theory parameters. Based on the effective actions, we derived the holographic duality starting with a stack of D3-branes which lead to the duality between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ SYM in four dimensions. We also presented another example of the duality involving M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ and type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ that are dual to different regimes of ABJM theory in three dimensions. These CFTs can be supplemented with matter transforming in the fundamental representation of the gauge group by adding more branes in the dual gravity description.

Since its discovery, the holographic duality has been extremely successful in shaping our understanding of the landscape of quantum field theories and string theories. Since string theory is by current knowledge a complete theory of quantum gravity, this development has naturally lead to better understanding to quantum properties of gravity. For example, applying ideas from quantum information theory has been fruitful in this regard. It will be interesting to see what the future holds.

# Articles by the author


[A1] Shovon Biswas, Jani Kastikainen, Sanjit Shashi, and James Sully. "Holographic BCFT Spectra from Brane Mergers". In: *Journal of High Energy Physics* 2022.11 (Nov. 2022), p. 158. ISSN: 1029-8479. arXiv: 2209.11227.

[A2] Niko Jokela, Jani Kastikainen, Elias Kiritsis, and Francesco Nitti. "Flavored ABJM Theory on the Sphere and Holographic F-functions". In: *Journal of High Energy Physics* 2022.3 (Mar. 2022), p. 91. ISSN: 1029-8479. arXiv: 2112.08715.

[A3] Jani Kastikainen and Sanjit Shashi. "Structure of Holographic BCFT Correlators from Geodesics". In: *Physical Review D* 105.4 (Feb. 2022), p. 046007. arXiv: 2109.00079.

[A4] Jani Kastikainen. "Conical Defects and Holography in Topological AdS Gravity". In: *Classical and Quantum Gravity* 37.19 (Oct. 2020), p. 195010. ISSN: 0264-9381, 1361-6382. arXiv: 2006.02803.

[A5] Jan de Boer, Victor Godet, Jani Kastikainen, and Esko Keski-Vakkuri. "Quantum Hypothesis Testing in Many-Body Systems". In: *SciPost Physics Core* 4.2 (June 2021), p. 019. ISSN: 2666-9366. arXiv: 2007.11711.

[A6] Jani Kastikainen. "Quasi-Local Energy and ADM Mass in Pure Lovelock Gravity". In: *Classical and Quantum Gravity* 37.2 (Jan. 2020), p. 025001. ISSN: 0264-9381, 1361-6382. arXiv: 1908.05522.

# References


[1] Joseph Polchinski. "Scale and Conformal Invariance in Quantum Field Theory". In: *Nuclear Physics B* 303.2 (June 1988), pp. 226–236. ISSN: 0550-3213.

[2] Yu Nakayama. "Scale Invariance vs Conformal Invariance". In: *arXiv:1302.0884 [cond-mat, physics:hep-th]* (Feb. 2014). arXiv: 1302.0884 [cond-mat, physics:hep-th].

[3] H. B. G. Casimir. "On the Attraction Between Two Perfectly Conducting Plates". In: *Indag. Math.* 10 (1948), pp. 261–263.

[4] D. Deutsch and P. Candelas. "Boundary Effects in Quantum Field Theory". In: *Physical Review D* 20.12 (Dec. 1979), pp. 3063–3080.

[5] John L. Cardy. "Conformal Invariance and Surface Critical Behavior". In: *Nuclear Physics B* 240.4 (Nov. 1984), pp. 514–532. ISSN: 0550-3213.

[6] John L. Cardy. "Operator Content of Two-Dimensional Conformally Invariant Theories". In: *Nuclear Physics B* 270 (Jan. 1986), pp. 186–204. ISSN: 0550-3213.

[7] John L. Cardy. "Boundary Conditions, Fusion Rules and the Verlinde Formula". In: *Nuclear Physics B* 324.3 (Oct. 1989), pp. 581–596. ISSN: 0550-3213.

[8] Jin Dai, R.g. Leigh, and Joseph Polchinski. "New Connections between String Theories". In: *Modern Physics Letters A* 04.21 (Oct. 1989), pp. 2073–2083. ISSN: 0217-7323.

[9] Petr Hořava. "Background Duality of Open-String Models". In: *Physics Letters B* 231.3 (Nov. 1989), pp. 251–257. ISSN: 0370-2693.

[10] R.g. Leigh. "DIRAC-BORN-INFELD ACTION FROM DIRICHLET $\sigma$-MODEL". In: *Modern Physics Letters A* 04.28 (Dec. 1989), pp. 2767–2772. ISSN: 0217-7323.

[11] C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry. "Strings in Background Fields". In: *Nuclear Physics B* 262.4 (Dec. 1985), pp. 593–609. ISSN: 0550-3213.

[12] C. G. Callan, I. R. Klebanov, and M. J. Perry. "String Theory Effective Actions". In: *Nuclear Physics B* 278.1 (Dec. 1986), pp. 78–90. ISSN: 0550-3213.

[13] Gary T. Horowitz and Andrew Strominger. "Black Strings and P-Branes". In: *Nuclear Physics B* 360.1 (Aug. 1991), pp. 197–209. ISSN: 0550-3213.

[14] Joseph Polchinski. "Dirichlet-Branes and Ramond-Ramond Charges". In: *Physical Review Letters* 75.26 (Dec. 1995), pp. 4724–4727. ISSN: 0031-9007, 1079-7114. arXiv: hep-th/9510017.

[15] H. A. Kramers and G. H. Wannier. "Statistics of the Two-Dimensional Ferromagnet. Part I". In: *Physical Review* 60.3 (Aug. 1941), pp. 252–262.

[16] H. A. Kramers and G. H. Wannier. "Statistics of the Two-Dimensional Ferromagnet. Part II". In: *Physical Review* 60.3 (Aug. 1941), pp. 263–276.

[17] Edward Witten. "String Theory Dynamics In Various Dimensions". In: *Nuclear Physics B* 443.1-2 (June 1995), pp. 85–126. ISSN: 05503213. arXiv: hep-th/9503124.

[18] B. Sathia p palan. "Duality in Statistical Mechanics and String Theory". In: *Physical Review Letters* 58.16 (Apr. 1987), pp. 1597–1599.

[19] T. H. Buscher. "A Symmetry of the String Background Field Equations". In: *Physics Letters B* 194.1 (July 1987), pp. 59–62. ISSN: 0370-2693.

[20] T. H. Buscher. "Path-Integral Derivation of Quantum Duality in Nonlinear Sigma-Models". In: *Physics Letters B* 201.4 (Feb. 1988), pp. 466–472. ISSN: 0370-2693.

[21] C. G. Callan, C. Lovelace, C. R. Nappi, and S. A. Yost. "Adding Holes and Crosscaps to the Superstring". In: *Nuclear Physics B* 293 (Jan. 1987), pp. 83–113. ISSN: 0550-3213.

[22] Tetsuya Onogi and Nobuyuki Ishibashi. "Conformal Field Theories on Surfaces with Boundaries and Crosscaps". In: *Modern Physics Letters A* 04.02 (Jan. 1989), pp. 161–168. ISSN: 0217-7323.

[23] Nobuyuki Ishibashi. "The Boundary and Crosscap States in Conformal Field Theories". In: *Modern Physics Letters A* 04.03 (Feb. 1989), pp. 251–264. ISSN: 0217-7323.

[24] Juan M. Maldacena. "The Large N Limit of Superconformal Field Theories and Supergravity". In: (Jan. 1998). arXiv: hep-th/9711200.

[25] Edward Witten. "Anti De Sitter Space And Holography". In: *arXiv:hep-th/9802150* (Feb. 1998). arXiv: hep-th/9802150.

[26] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov. "Gauge Theory Correlators from Non-Critical String Theory". In: *Physics Letters B* 428.1-2 (May 1998), pp. 105–114. ISSN: 03702693. arXiv: hep-th/9802109.

[27] Ofer Aharony, Oren Bergman, Daniel Louis Jafferis, and Juan Maldacena. "N=6 Superconformal Chern-Simons-Matter Theories, M2-Branes and Their Gravity Duals". In: *Journal of High Energy Physics* 2008.10 (Oct. 2008), pp. 091–091. ISSN: 1029-8479. arXiv: 0806.1218.

[28] Andreas Karch and Emanuel Katz. "Adding Flavor to AdS/CFT". In: *Journal of High Energy Physics* 2002.06 (June 2002), pp. 043–043. ISSN: 1029-8479. arXiv: hep-th/0205236.

[29] Joseph Polchinski and Matthew J. Strassler. "The String Dual of a Confining Four-Dimensional Gauge Theory". In: arXiv:hep-th/0003136 (Mar. 2000). arXiv: hep-th/0003136.

[30] Edward Witten. "Anti-de Sitter Space, Thermal Phase Transition, And Confinement In Gauge Theories". In: arXiv:hep-th/9803131 (Apr. 1998). arXiv: hep-th/9803131.

[31] Anton Rebhan. "The Witten-Sakai-Sugimoto Model: A Brief Review and Some Recent Results". In: arXiv:1410.8858 (Nov. 2014). arXiv: 1410.8858 [hep-ph, physics:hep-th, physics:nucl-th].

[32] Matti Jarvinen. "Holographic Dense QCD in the Veneziano Limit". In: *EPJ Web of Conferences* 274 (2022), p. 08006. ISSN: 2100-014X. arXiv: 2211.10005 [hep-ph, physics:hep-th].

[33] Juan M. Maldacena. "Eternal Black Holes in AdS". In: *Journal of High Energy Physics* 2003.04 (Apr. 2003), pp. 021–021. ISSN: 1029-8479. arXiv: hep-th/0106112.

[34] Mark Van Raamsdonk. "Comments on Quantum Gravity and Entanglement". In: (July 2009). arXiv: 0907.2939 [gr-qc, physics:hep-th, physics:quant-ph].

[35] Mark Van Raamsdonk. "Building up Spacetime with Quantum Entanglement". In: *International Journal of Modern Physics D* 19.14 (Dec. 2010), pp. 2429–2435. ISSN: 0218-2718, 1793-6594. arXiv: 1005.3035.

[36] Shinsei Ryu and Tadashi Takayanagi. "Holographic Derivation of Entanglement Entropy from AdS/CFT". In: *Physical Review Letters* 96.18 (May 2006). ISSN: 0031-9007, 1079-7114. arXiv: hep-th/0603001.

[37] Shinsei Ryu and Tadashi Takayanagi. "Aspects of Holographic Entanglement Entropy". In: *Journal of High Energy Physics* 2006.08 (Aug. 2006), pp. 045–045. ISSN: 1029-8479. arXiv: hep-th/0605073.

[38] Veronika E. Hubeny, Mukund Rangamani, and Tadashi Takayanagi. "A Covariant Holographic Entanglement Entropy Proposal". In: *Journal of High Energy Physics* 2007.07 (July 2007), pp. 062–062. ISSN: 1029-8479. arXiv: 0705.0016.

[39] Ahmed Almheiri, Raghu Mahajan, Juan Maldacena, and Ying Zhao. "The Page Curve of Hawking Radiation from Semiclassical Geometry". In: *arXiv:1908.10996 [gr-qc, physics:hep-th]* (Aug. 2019). arXiv: 1908.10996 [gr-qc, physics:hep-th].

[40] Ahmed Almheiri, Netta Engelhardt, Donald Marolf, and Henry Maxfield. "The Entropy of Bulk Quantum Fields and the Entanglement Wedge of an Evaporating Black Hole". In: *Journal of High Energy Physics* 2019.12 (Dec. 2019), p. 63. ISSN: 1029-8479. arXiv: 1905.08762 [gr-qc, physics:hep-th].

[41] Geoff Penington, Stephen H. Shenker, Douglas Stanford, and Zhenbin Yang. "Replica Wormholes and the Black Hole Interior". In: (Apr. 2020). arXiv: 1911.11977.

[42] Ahmed Almheiri, Thomas Hartman, Juan Maldacena, Edgar Shaghoulian, and Amirhossein Tajdini. "Replica Wormholes and the Entropy of Hawking Radiation". In: *Journal of High Energy Physics* 2020.5 (May 2020), p. 13. ISSN: 1029-8479. arXiv: 1911.12333.

[43] J. D. Brown and Marc Henneaux. "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity". In: *Communications in Mathematical Physics* 104.2 (1986), pp. 207–226. ISSN: 0010-3616, 1432-0916.

[44] Edward Witten. "Three-Dimensional Gravity Revisited". In: *arXiv:0706.3359 [hep-th]* (June 2007). arXiv: 0706.3359 [hep-th].

[45] Thomas Hartman, Christoph A. Keller, and Bogdan Stoica. "Universal Spectrum of 2d Conformal Field Theory in the Large c Limit". In: *Journal of High Energy Physics* 2014.9 (Sept. 2014), p. 118. ISSN: 1029-8479. arXiv: 1405.5137.

[46] Jeevan Chandra, Scott Collier, Thomas Hartman, and Alexander Maloney. "Semiclassical 3D Gravity as an Average of Large-c CFTs". In: *arXiv:2203.06511 [hep-th]* (Mar. 2022). arXiv: 2203.06511 [hep-th].

[47] Tadashi Takayanagi. "Holographic Dual of BCFT". In: *Physical Review Letters* 107.10 (Aug. 2011), p. 101602. ISSN: 0031-9007, 1079-7114. arXiv: 1105.5165.

[48] Mitsutoshi Fujita, Tadashi Takayanagi, and Erik Tonni. "Aspects of AdS/BCFT". In: *Journal of High Energy Physics* 2011.11 (Nov. 2011), p. 43. ISSN: 1029-8479. arXiv: 1108.5152.

[49] Carlo Rovelli. *Quantum Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2004. DOI: 10.1017/CBO9780511755804.

[50] Paul Ginsparg. "Applied Conformal Field Theory". In: *arXiv:hep-th/9108028* (Nov. 1988). arXiv: hep-th/9108028.

[51] David Simmons-Duffin. "TASI Lectures on the Conformal Bootstrap". In: (Feb. 2016). arXiv: 1602.07982 [cond-mat, physics:hep-th].

[52] Slava Rychkov. "EPFL Lectures on Conformal Field Theory in D>= 3 Dimensions". In: *arXiv:1601.05000 [cond-mat, physics:hep-ph, physics:hep-th]* (2017). arXiv: 1601.05000 [cond-mat, physics:hep-ph, physics:hep-th].

[53] Martin Schottenloher, ed. *A mathematical introduction to conformal field theory*. Vol. 759. 2008. DOI: 10.1007/978-3-540-68628-6.

[54] Liang Kong and Ingo Runkel. "Algebraic Structures in Euclidean and Minkowskian Two-Dimensional Conformal Field Theory". In: arXiv:0902.3829 (Feb. 2009). arXiv: 0902.3829 [hep-th, physics:math-ph].

[55] Sam Crawford, Kasia Rejzner, and Benoît Vicedo. "Lorentzian 2D CFT from the pAQFT Perspective". In: *Annales Henri Poincaré* 23.10 (Oct. 2022), pp. 3525–3585. ISSN: 1424-0661.

[56] Blagoje Oblak. "BMS Particles in Three Dimensions". In: *arXiv:1610.08526 [gr-qc, physics:hep-th, physics:math-ph]* (2017). arXiv: 1610.08526 [gr-qc, physics:hep-th, physics:math-ph].

[57] Mert Besken, Jan de Boer, and Grégoire Mathys. "On Local and Integrated Stress-Tensor Commutators". In: *arXiv:2012.15724 [hep-th]* (Dec. 2020). arXiv: 2012.15724 [hep-th].

[58] Andrea Cappelli and Antoine Coste. "Properties of the Stress Tensor in More than Two Dimensions". In: *Nuclear Physics B - Proceedings Supplements* 5.2 (Dec. 1988), pp. 57–64. ISSN: 0920-5632.

[59] Andrea Cappelli and Antoine Coste. "On the Stress Tensor of Conformal Field Theories in Higher Dimensions". In: *Nuclear Physics B* 314.3 (Mar. 1989), pp. 707–740. ISSN: 0550-3213.

[60] G Mack and Abdus Salam. "Finite-Component Field Representations of the Conformal Group". In: *Annals of Physics* 53.1 (June 1969), pp. 174–202. ISSN: 0003-4916.

[61] H. Osborn and A. Petkos. "Implications of Conformal Invariance in Field Theories for General Dimensions". In: *Annals of Physics* 231.2 (May 1994), pp. 311–362. ISSN: 00034916. arXiv: hep-th/9307010.

[62] J. Erdmenger and H. Osborn. "Conserved Currents and the Energy Momentum Tensor in Conformally Invariant Theories for General Dimensions". In: *Nuclear Physics B* 483.1-2 (Jan. 1997), pp. 431–474. ISSN: 05503213. arXiv: hep-th/9605009.

[63] Andreas Stergiou, Gian Paolo Vacca, and Omar Zanusso. "Weyl Covariance and the Energy Momentum Tensors of Higher-Derivative Free Conformal Field Theories". In: *arXiv:2202.04701 [hep-th]* (Feb. 2022). arXiv: 2202.04701 [hep-th].

[64] Georgios K. Karananas and Alexander Monin. "Weyl vs. Conformal". In: *Physics Letters B* 757 (June 2016), pp. 257–260. ISSN: 03702693. arXiv: 1510.08042 [hep-th].

[65] Yu Nakayama. "Hidden Global Conformal Symmetry without Virasoro Extension in Theory of Elasticity". In: *Annals of Physics* 372 (Sept. 2016), pp. 392–396. ISSN: 00034916. arXiv: 1604.00810 [cond-mat, physics:hep-th, physics:math-ph].

[66] Joseph Polchinski. "Vertex Operators in the Polyakov Path Integral". In: *Nuclear Physics B* 289 (tammikuu 1, 1987), pp. 465–483. ISSN: 0550-3213.

[67] P. Ginsparg and Gregory Moore. "Lectures on 2D Gravity and 2D String Theory (TASI 1992)". In: arXiv:hep-th/9304011 (Apr. 1993). arXiv: hep-th/9304011.

[68] S. Deser and A. Schwimmer. "Geometric Classification of Conformal Anomalies in Arbitrary Dimensions". In: *Physics Letters B* 309.3-4 (July 1993), pp. 279–284. ISSN: 03702693. arXiv: hep-th/9302047.

[69] M. Knecht, S. Lazzarini, and F. Thuillier. "Shifting the Weyl Anomaly to the Chirally Split Diffeomorphism Anomaly in Two Dimensions". In: *Physics Letters B* 251.2 (Nov. 1990), pp. 279–283. ISSN: 0370-2693.

[70] Philippe Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. Springer Science & Business Media, 2012.

[71] Elliott Lieb, Theodore Schultz, and Daniel Mattis. "Two Soluble Models of an Antiferromagnetic Chain". In: *Annals of Physics* 16.3 (Dec. 1961), pp. 407–466. ISSN: 0003-4916.

[72] Ian Affleck. "FIELD THEORY METHODS AND QUANTUM CRITICAL PHENOMENA". In: *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena*. June 1988.

[73] Per Moosavi. "Inhomogeneous Conformal Field Theory out of Equilibrium". In: (Dec. 2019). arXiv: 1912.04821 [cond-mat, physics:math-ph].

[74] David G. Boulware and S. Deser. "Stress-Tensor Commutators and Schwinger Terms". In: *Journal of Mathematical Physics* 8.7 (July 1967), pp. 1468–1477. ISSN: 0022-2488.

[75] M. Katanaev. *Global Conformal Gauge in String Theory*. Mar. 2021. arXiv: 2106.05839 [physics].

[76] M. O. Katanaev. "On the Existence of the Global Conformal Gauge in String Theory". In: *The European Physical Journal C* 81.7 (July 2021), p. 581. ISSN: 1434-6044, 1434-6052. arXiv: 1912.08052 [hep-th, physics:physics].

[77] M. Henneaux. "On the Inverse Problem of the Calculus of Variations in Field Theory". In: *Journal of Physics A: Mathematical and General* 17.1 (Jan. 1984), p. 75. ISSN: 0305-4470.

[78] A M Polyakov. "QUANTUM GEOMETRY OF BOSONIC STRINGS". In: *PHYSICS LETTERS* 103.3 (1981), p. 4.

[79] R. Zucchini. "A Polyakov Action on Riemann Surfaces". In: *Physics Letters B* 260.3 (May 1991), pp. 296–302. ISSN: 0370-2693.

[80] Roberto Zucchini. "A Polyakov Action on Riemann Surfaces (II)". In: *Communications in Mathematical Physics* 152.2 (Mar. 1993), pp. 269–297. ISSN: 1432-0916.

[81] Kevin Nguyen. "Holographic Boundary Actions in AdS$_3$/CFT$_2$ Revisited". In: *Journal of High Energy Physics* 2021.10 (Oct. 2021), p. 218. ISSN: 1029-8479. arXiv: 2108.01095.

[82] Roberto Longo and Karl-Henning Rehren. "Local Fields in Boundary Conformal QFT". In: *Reviews in Mathematical Physics* 16.07 (Aug. 2004), pp. 909–960. ISSN: 0129-055X, 1793-6659. arXiv: math-ph/0405067.

[83] John Cardy. "Boundary Conformal Field Theory". In: *arXiv:hep-th/0411189* (Nov. 2004). arXiv: hep-th/0411189.

[84] James W. York. "Role of Conformal Three-Geometry in the Dynamics of Gravitation". In: *Physical Review Letters* 28.16 (Apr. 1972), pp. 1082–1085.

[85] S. W. Hawking and Gary T. Horowitz. "The Gravitational Hamiltonian, Action, Entropy, and Surface Terms". In: *Classical and Quantum Gravity* 13.6 (June 1996), pp. 1487–1498. ISSN: 0264-9381, 1361-6382. arXiv: gr-qc/9501014.

[86] Geoff Hayward. "Gravitational Action for Spacetimes with Nonsmooth Boundaries". In: *Physical Review D* 47.8 (Apr. 1993), pp. 3275–3280.

[87] T. Padmanabhan and Dawood Kothawala. "Lanczos-Lovelock Models of Gravity". In: *Physics Reports* 531.3 (Oct. 2013), pp. 115–171. ISSN: 03701573. arXiv: 1302.2151.

[88] David Lovelock. "The Einstein Tensor and Its Generalizations". In: *Journal of Mathematical Physics* 12.3 (Mar. 1971), pp. 498–501. ISSN: 0022-2488.

[89] A. H. Chamseddine. "Topological Gravity and Supergravity in Various Dimensions". en. In: *Nuclear Physics B* 346.1 (Dec. 1990), pp. 213–234. ISSN: 0550-3213.

[90] Branislav Cvetković, Olivera Miskovic, and Dejan Simić. "Holography in Lovelock Chern-Simons AdS Gravity". In: *Physical Review D* 96.4 (Aug. 2017), p. 044027. ISSN: 2470-0010, 2470-0029. arXiv: 1705.04522.

[91] Juan Crisostomo, Ricardo Troncoso, and Jorge Zanelli. "Black Hole Scan". In: *Physical Review D* 62.8 (Sept. 2000). ISSN: 0556-2821, 1089-4918. arXiv: hep-th/0003271.

[92] Pablo Mora, Rodrigo Olea, Ricardo Troncoso, and Jorge Zanelli. "Transgression Forms and Extensions of Chern-Simons Gauge Theories". In: *Journal of High Energy Physics* 2006.02 (Feb. 2006), pp. 067–067. ISSN: 1029-8479. arXiv: hep-th/0601081.

[93] Jie Jiang and Hongbao Zhang. "Surface Term, Corner Term, and Action Growth in F(Riemann) Gravity Theory". In: *Physical Review D* 99.8 (Apr. 2019), p. 086005. ISSN: 2470-0010, 2470-0029. arXiv: 1806.10312.

[94] Robert C. Myers. "Higher-Derivative Gravity, Surface Terms, and String Theory". In: *Physical Review D* 36.2 (July 1987), pp. 392–396.

[95] C Teitelboim and J Zanelli. "Dimensionally Continued Topological Gravitation Theory in Hamiltonian Form". en. In: *Classical and Quantum Gravity* 4.4 (July 1987), pp. L125–L129. ISSN: 0264-9381, 1361-6382.

[96] Pablo A. Cano. "Lovelock Action with Nonsmooth Boundaries". In: *Physical Review D* 97.10 (May 2018), p. 104048.

[97] Nathalie Deruelle, Misao Sasaki, Yuuiti Sendouda, and Daisuke Yamauchi. "Hamiltonian Formulation of f(Riemann) Theories of Gravity". In: *Progress of Theoretical Physics* 123.1 (Jan. 2010), pp. 169–185. ISSN: 0033-068X, 1347-4081. arXiv: 0908.0679.

[98] Daniel Grumiller, Robert Mann, and Robert McNees. "Dirichlet Boundary-Value Problem for Chern-Simons Modified Gravity". In: *Physical Review D* 78.8 (Oct. 2008), p. 081502.

[99] Charles Fefferman and C. Robin Graham. "The Ambient Metric". In: (Oct. 2007). arXiv: 0710.0919 [math].

[100] C. Imbimbo, A. Schwimmer, S. Theisen, and S. Yankielowicz. "Diffeomorphisms and Holographic Anomalies". In: *Classical and Quantum Gravity* 17.5 (Mar. 2000), pp. 1129–1138. ISSN: 0264-9381, 1361-6382. arXiv: hep-th/9910267.

[101] Kostas Skenderis and Sergey N. Solodukhin. "Quantum Effective Action from the AdS/CFT Correspondence". In: *Physics Letters B* 472.3-4 (Jan. 2000), pp. 316–322. ISSN: 03702693. arXiv: hep-th/9910023.

[102] Maximo Banados, Olivera Miskovic, and Stefan Theisen. "Holographic Currents in First Order Gravity and Finite Fefferman-Graham Expansions". In: *Journal of High Energy Physics* 2006.06 (June 2006), pp. 025–025. ISSN: 1029-8479. arXiv: hep-th/0604148.

[103] Maximo Banados. "Three-Dimensional Quantum Geometry and Black Holes". In: *arXiv:hep-th/9901148* (1999), pp. 147–169. arXiv: hep-th/9901148.

[104] G. Compère, P. Mao, A. Seraj, and M. M. Sheikh-Jabbari. "Symplectic and Killing symmetries of $AdS_3$ gravity: holographic vs boundary gravitons". In: *Journal of High Energy Physics* 2016.1 (Jan. 2016), p. 80. ISSN: 1029-8479. arXiv: 1511.06079.

[105] Sebastian de Haro, Kostas Skenderis, and Sergey N. Solodukhin. "Holographic Reconstruction of Spacetime and Renormalization in the AdS/CFT Correspondence". In: *Communications in Mathematical Physics* 217.3 (Mar. 2001), pp. 595–622. ISSN: 0010-3616, 1432-0916. arXiv: hep-th/0002230.

[106] M. Banados, O. Chandia, and A. Ritz. "Holography and the Polyakov Action". In: *Physical Review D* 65.12 (June 2002), p. 126008. ISSN: 0556-2821, 1089-4918. arXiv: hep-th/0203021.

[107] Vijay Balasubramanian and Per Kraus. "A Stress Tensor for Anti-de Sitter Gravity". In: *Communications in Mathematical Physics* 208.2 (Dec. 1999), pp. 413–428. ISSN: 0010-3616, 1432-0916. arXiv: hep-th/9902121.

[108] M. Banados, A. Schwimmer, and S. Theisen. "Chern-Simons Gravity and Holographic Anomalies". In: *Journal of High Energy Physics* 2004.05 (May 2004), pp. 039–039. ISSN: 1029-8479. arXiv: hep-th/0404245.

[109] Maximo Banados, Rodrigo Olea, and Stefan Theisen. "Counterterms and Dual Holographic Anomalies in CS Gravity". In: *Journal of High Energy Physics* 2005.10 (Oct. 2005), pp. 067–067. ISSN: 1029-8479. arXiv: hep-th/0509179.

[110] Masamichi Miyaji and Chitraang Murdia. "Holographic BCFT with a Defect on the End-of-the-World Brane". In: arXiv:2208.13783 (Aug. 2022). arXiv: 2208.13783 [cond-mat, physics:hep-th, physics:quant-ph].

[111] Masamichi Miyaji, Tadashi Takayanagi, and Tomonori Ugajin. "Spectrum of End of the World Branes in Holographic BCFTs". In: *arXiv:2103.06893 [cond-mat, physics:hep-th, physics:quant-ph]* (Mar. 2021). arXiv: 2103.06893 [cond-mat, physics:hep-th, physics:quant-ph].

[112] Wyatt Reeves, Moshe Rozali, Petar Simidzija, James Sully, Christopher Waddell, and David Wakeham. "Looking for (and Not Finding) a Bulk Brane". In: *arXiv:2108.10345 [hep-th]* (Aug. 2021). arXiv: 2108.10345 [hep-th].

[113] Yuya Kusuki. "Analytic Bootstrap in 2D Boundary Conformal Field Theory: Towards Braneworld Holography". In: *Journal of High Energy Physics* 2022.3 (Mar. 2022), p. 161. ISSN: 1029-8479. arXiv: 2112.10984 [hep-th].

[114] Yuya Kusuki. "Semiclassical Gravity from Averaged Boundaries in Two-Dimensional BCFTs". In: (Sept. 2022). arXiv: 2206.03035 [gr-qc, physics:hep-th].

[115] D. V. Fursaev and S. N. Solodukhin. "On the Description of the Riemannian Geometry in the Presence of Conical Defects". In: *Physical Review D* 52.4 (Aug. 1995), pp. 2133–2143. ISSN: 0556-2821. arXiv: hep-th/9501127.

[116] Xi Dong. "Holographic Entanglement Entropy for General Higher Derivative Gravity". In: *Journal of High Energy Physics* 2014.1 (Jan. 2014). ISSN: 1029-8479. arXiv: 1310.5713.

[117] Joseph Polchinski. *String Theory*. Vol. 2. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1998. DOI: 10.1017/CBO9780511618123.

[118] C. G. Callan, C. Lovelace, C. R. Nappi, and S. A. Yost. "String Loop Corrections to Beta Functions". In: *Nuclear Physics B* 288 (Jan. 1987), pp. 525–550. ISSN: 0550-3213.

[119] E. S. Fradkin and A. A. Tseytlin. "Effective Field Theory from Quantized Strings". In: *Physics Letters B* 158.4 (Aug. 1985), pp. 316–322. ISSN: 0370-2693.

[120] H. Dorn and H. J. Otto. "Open Bosonic Strings in General Background Fields". In: *Zeitschrift für Physik C Particles and Fields* 32.4 (Dec. 1986), pp. 599–607. ISSN: 1431-5858.

[121] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda, and R. Russo. "Classical P-Branes from Boundary State". In: *Nuclear Physics B* 507.1-2 (Dec. 1997), pp. 259–276. ISSN: 05503213. arXiv: hep-th/9707068.

[122] Martin Ammon and Johanna Erdmenger. *Gauge/Gravity Duality: Foundations and Applications*. Cambridge University Press, 2015. DOI: 10.1017/CBO9780511846373.

[123] Edward Witten. "Bound States Of Strings And $p$-Branes". In: *Nuclear Physics B* 460.2 (Feb. 1996), pp. 335–350. ISSN: 05503213. arXiv: hep-th/9510135.

[124] R. C. Myers. "Dielectric-Branes". In: *Journal of High Energy Physics* 1999.12 (Dec. 1999), pp. 022–022. ISSN: 1029-8479. arXiv: hep-th/9910053.

[125] Clifford V. Johnson. "D-Brane Primer". In: *Strings, Branes and Gravity*. Oct. 2001, pp. 129–350. arXiv: hep-th/0007170.

[126] G. 't Hooft. "A Planar Diagram Theory for Strong Interactions". In: *Nuclear Physics B* 72.3 (Apr. 1974), pp. 461–473. ISSN: 0550-3213.

[127] Clifford P. Burgess, Neil R. Constable, and Robert C. Myers. "The Free Energy of N = 4 Super-Yang-Mills and the AdS/CFT Correspondence". In: *Journal of High Energy Physics* 1999.08 (Sept. 1999), p. 017. ISSN: 1126-6708.

[128] Rafael I. Nepomechie. "Magnetic Monopoles from Antisymmetric Tensor Gauge Fields". In: *Physical Review D* 31.8 (Apr. 1985), pp. 1921–1924.

[129] Claudio Teitelboim. "Monopoles of Higher Rank". In: *Physics Letters B* 167.1 (Jan. 1986), pp. 69–72. ISSN: 0370-2693.

[130] Per Kraus, Finn Larsen, and Sandip P. Trivedi. "The Coulomb Branch of Gauge Theory from Rotating Branes". In: *Journal of High Energy Physics* 1999.03 (May 1999), p. 003. ISSN: 1126-6708.

[131] Douglas J. Smith. "Intersecting Brane Solutions in String and M-theory". In: *Classical and Quantum Gravity* 20.9 (May 2003), R233–R300. ISSN: 0264-9381, 1361-6382. arXiv: hep-th/0210157.

[132] Igor R. Klebanov and Giuseppe Torri. "M2-Branes and AdS/CFT". In: *International Journal of Modern Physics A* 25.02n03 (Jan. 2010), pp. 332–350. ISSN: 0217-751X, 1793-656X. arXiv: 0909.1580.

[133] Jonathan Bagger, Neil Lambert, Sunil Mukhi, and Constantinos Papageorgakis. "Multiple Membranes in M-theory". In: *Physics Reports* 527.1 (June 2013), pp. 1–100. ISSN: 03701573. arXiv: 1203.3546 [hep-th].

[134] E. Bergshoeff, E. Sezgin, and P. K. Townsend. "Supermembranes and Eleven-Dimensional Supergravity". In: *Physics Letters B* 189.1 (Apr. 1987), pp. 75–78. ISSN: 0370-2693.

[135] U. Marquard, R. Kaiser, and M. Scholl. "Lorentz Algebra and Critical Dimension for the Supermembrane". In: *Physics Letters B* 227.2 (Aug. 1989), pp. 234–238. ISSN: 0370-2693.

[136] Krzysztof A. Meissner and Hermann Nicolai. "Fundamental Membranes and the String Dilaton". In: *Journal of High Energy Physics* 2022.9 (Sept. 2022), p. 219. ISSN: 1029-8479. arXiv: 2208.05822 [hep-th].

[137] E Bergshoeff, E Sezgin, and P. K Townsend. "Properties of the Eleven-Dimensional Supermembrane Theory". In: *Annals of Physics* 185.2 (Aug. 1988), pp. 330–368. ISSN: 0003-4916.

[138] E. Cremmer, B. Julia, and J. Scherk. "Supergravity in Theory in 11 Dimensions". In: *Physics Letters B* 76.4 (June 1978), pp. 409–412. ISSN: 0370-2693.

[139] Edward Witten. "On Flux Quantization in M-theory and the Effective Action". In: *Journal of Geometry and Physics* 22.1 (Apr. 1997), pp. 1–13. ISSN: 0393-0440.

[140] M. J. Duff, James T. Liu, and R. Minasian. "Eleven Dimensional Origin of String/String Duality: A One Loop Test". In: *Nuclear Physics B* 452.1-2 (Oct. 1995), pp. 261–282. ISSN: 05503213. arXiv: hep-th/9506126.

[141] Pablo Mora. *Chern-Simons Branes with Enhanced Gauge Invariance*. May 2023. eprint: 2305.06186 (hep-th).

[142] M. J. Duff, P. S. Howe, T. Inami, and K. S. Stelle. "Superstrings in D=10 from Supermembranes in D=11". In: *Physics Letters B* 191.1 (June 1987), pp. 70–74. ISSN: 0370-2693.

[143] M. J. Duff and K. S. Stelle. "Multi-Membrane Solutions of D = 11 Supergravity". In: *Physics Letters B* 253.1 (Jan. 1991), pp. 113–118. ISSN: 0370-2693.

[144] Tatsuma Nishioka and Tadashi Takayanagi. "On Type IIA Penrose Limit and N=6 Chern-Simons Theories". In: *Journal of High Energy Physics* 2008.08 (Aug. 2008), pp. 001–001. ISSN: 1029-8479. arXiv: 0806.3391.

[145] Yasuaki Hikida, Wei Li, and Tadashi Takayanagi. "ABJM with Flavors and FQHE". In: *Journal of High Energy Physics* 2009.07 (July 2009), pp. 065–065. ISSN: 1029-8479. arXiv: 0903.2194.

[146] Gabi Zafrir. "Embedding Massive Flavor in ABJM". In: *Journal of High Energy Physics* 2012.10 (Oct. 2012), p. 56. ISSN: 1029-8479. arXiv: 1202.4295.

[147] Eduardo Conde and Alfonso V. Ramallo. "On the Gravity Dual of Chern-Simons-Matter Theories with Unquenched Flavor". In: *Journal of High Energy Physics* 2011.7 (July 2011), p. 99. ISSN: 1029-8479. arXiv: 1105.6045.

[148] S. A. Kurlyand and A. A. Tseytlin. "Type IIB supergravity action on M5×X5 solutions". In: *Physical Review D* 106.8 (Oct. 2022), p. 086017. ISSN: 2470-0010, 2470-0029. arXiv: 2206.14522 [hep-th].

[149] Subhroneel Chakrabarti, Divyanshu Gupta, and Arkajyoti Manna. "On-shell action for type IIB supergravity and superstrings on AdS5×S5". In: *Physics Letters B* 835 (Dec. 2022), p. 137578. ISSN: 03702693. arXiv: 2211.02345 [hep-th].

[150] Nadav Drukker, Marcos Marino, and Pavel Putrov. "From Weak to Strong Coupling in ABJM Theory". In: *Communications in Mathematical Physics* 306.2 (Sept. 2011), pp. 511–563. ISSN: 0010-3616, 1432-0916. arXiv: 1007.3837.

[151] C. Robin Graham. "Volume and Area Renormalizations for Conformally Compact Einstein Metrics". In: *arXiv:math/9909042* (Sept. 1999). arXiv: math/9909042.

[152] Johanna Erdmenger, Nick Evans, Ingo Kirsch, and Ed Threlfall. "Mesons in Gauge/Gravity Duals - A Review". In: *The European Physical Journal A* 35.1 (Jan. 2008), pp. 81–133. ISSN: 1434-6001, 1434-601X. arXiv: 0711.4467.

[153] Carlos Nunez, Angel Paredes, and Alfonso V. Ramallo. "Unquenched Flavor in the Gauge/Gravity Correspondence". In: *arXiv:1002.1088 [hep-th]* (Feb. 2010). arXiv: 1002.1088 [hep-th].

[154] Stefan Hohenegger and Ingo Kirsch. "A Note on the Holography of Chern-Simons Matter Theories with Flavour". In: *Journal of High Energy Physics* 2009.04 (Apr. 2009), pp. 129–129. ISSN: 1029-8479. arXiv: 0903.1730.

[155] Davide Gaiotto and Daniel Louis Jafferis. "Notes on Adding D6 Branes Wrapping RP3 in AdS4 x CP3". In: *Journal of High Energy Physics* 2012.11 (Nov. 2012), p. 15. ISSN: 1029-8479. arXiv: 0903.2175.

[156] Martin Ammon, Johanna Erdmenger, Rene Meyer, Andy O'Bannon, and Timm Wrase. "Adding Flavor to AdS4/CFT3". In: *Journal of High Energy Physics* 2009.11 (Nov. 2009), pp. 125–125. ISSN: 1029-8479. arXiv: 0909.3845.

[157] Ki-Myeong Lee and Ho-Ung Yee. "NewAdS4×X7geometries with Script N = 6 in M Theory". en. In: *Journal of High Energy Physics* 2007.03 (Mar. 2007), pp. 012–012. ISSN: 1126-6708.

[158] Ricardo Couso Santamaria, Marcos Marino, and Pavel Putrov. "Unquenched Flavor and Tropical Geometry in Strongly Coupled Chern-Simons-Matter Theories". In: *Journal of High Energy Physics* 2011.10 (Oct. 2011), p. 139. ISSN: 1029-8479. arXiv: 1011.6281.