Eur. Phys. J. C (2022) 82:51
https://doi.org/10.1140/epjc/s10052-021-09984-y

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

New Riccati equations for radiating matter

Noeleen Naidoo[a], **Sunil D. Maharaj**[b], **Keshlan S. Govinder**[c]

Astrophysics Research Centre, School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa

Received: 16 December 2021 / Accepted: 30 December 2021

Abstract The relationship between radiating stars in general relativity and Riccati equations is investigated for a general matter distribution including the electromagnetic field and the cosmological constant. A generalised transformation relating the gravitational potentials for a spherically symmetric relativistic gravitating fluid is introduced. This generates a new Riccati equation at the surface of the radiating star. Exact solutions to the boundary condition are found and the gravitational potentials are given explicitly. Some of the consistency conditions can be reduced to Bernoulli equations which admit exact solutions. We also demonstrate that the reduction of order allows us to write the boundary condition as a first order equation utilising the generalised transformation. Solutions obtained using the generalised transformation also admit a linear equation of state.

1 Introduction

The evolution of a radiating star is an interesting and long standing problem of interest in general relativity. The complete model of a radiating star was generated by Santos [1] who showed that the boundary conditions included an equation relating the radial pressure to the heat flux for an interior barotropic matter distribution. The Santos boundary conditions have been generalised to include other matter fields including the electromagnetic field, the cosmological constant, anisotropic pressure, null dust and null strings. For recent treatments of the generalised junction conditions see [2–5]. Explicit models of radiating stars are necessary to study important astrophysical processes including viscosity, thermal effects, particle production at the stellar surface, dissipative processes and gravitational collapse. Some examples of investigations in these directions are contained in [6–16]. Some other recent areas that have been studied include causal thermodynamics [17], embedding of the four dimensional spacetime containing the radiating star into higher dimensional Euclidean space [18], modified gravity theories [19] and models with minimal complexity (real radius velocity is proportional to the areal radius).

In analysing the physical features of the relativistic radiating star, including the various features mentioned above, it is necessary to solve the Santos junction condition at the surface of the star. This is a nonlinear differential equation. A systematic approach is to apply the Lie group method of infinitesimal generators [20–23] which leads to new exact models. Another approach is to write the junction condition as a Riccati equation which was first explored by Misthry et al. [24], Thirukkanesh et al. [25] and Rajah and Maharaj [26]. A third approach is to introduce a new transformation that produces a new differential equation for the Santos junction condition; the transformed junction condition is a *new* Riccati equation. The Riccati equation can be solved under certain conditions. A useful transformation, for Riccati equations, was suggested by Ivanov [27–29] called the *horizon function* as it is related to the formations of horizons. Other treatments related to the *horizon function* are contained in the works of Mahomed et al. [30–32]. A transformation related to that of Ivanov was considered by Thirukkanesh and Maharaj [33]. The intention of this paper is to find a transformation that transforms the Riccati boundary condition into a new Riccati equation. Our approach has two advantages. Firstly, it does lead to new solutions of the boundary condition which have interesting physical features. Secondly, the Riccati equation generated in our analysis regains the results of previous studies as special cases. Also it should be emphasised that new Riccati equations lead to new solutions of the Einstein or Einstein–Maxwell field equations.

In this paper we introduce a new transformation that expresses the Santos junction condition as a new Riccati equation. We show how second order derivative terms from the junction condition can be eliminated by placing restric-

[a] e-mail: 214554200@stu.ukzn.ac.za
[b] e-mail: maharaj@ukzn.ac.za (corresponding author)
[c] e-mail: govinder@ukzn.ac.za

tions on the arbitrary parameters in our transformation. We systematically show how other transformations studied are regained from our transformations. We generate new exact solutions to our new Riccati equation by placing restrictions that will transform the equation into either a simpler Riccati equation, a Bernoulli equation or a linear equation in terms of one of the dependent variables. We have included the effects of shear, electromagnetic charge, anisotropy and the cosmological constant in our comprehensive treatment.

2 The model

The interior line element of an accelerating, expanding and shearing spacetime is given by

$$\mathrm{d}s^2 = -A^2\mathrm{d}t^2 + B^2\mathrm{d}r^2 + Y^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{1}$$

where A, B and Y represent the gravitational potential functions. The potentials are functions of r and t. The kinematical quantities which describe the geometric behaviour are given by the acceleration

$$\dot{u}^a = \left(0, \frac{A_r}{AB^2}, 0, 0\right), \tag{2}$$

the expansion scalar

$$\Theta = \frac{1}{A}\left(\frac{B_t}{B} + 2\frac{Y_t}{Y}\right), \tag{3}$$

and the magnitude of the shear scalar

$$\sigma = -\frac{1}{\sqrt{3}}\frac{1}{A}\left(\frac{B_t}{B} - \frac{Y_t}{Y}\right). \tag{4}$$

Note that subscripts denote partial differentiation.

The energy momentum tensor that describes the matter field is given by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} + E_{ab}, \tag{5}$$

where ρ represents the density, p represents the isotropic pressure, $\boldsymbol{q}$ represents the heat flux, and π_{ab} represents the anisotropic stress. These quantities are measured relative to the four-velocity $\boldsymbol{u}$. The heat flux is given as

$$q^a = \left(0, \frac{1}{B}q, 0, 0\right), \tag{6}$$

and the anisotropic stress tensor π_{ab} as

$$\pi_{ab} = \left(p_\parallel - p_\perp\right)\left(n_a n_b - \frac{1}{3}h_{ab}\right), \tag{7}$$

where $p_\parallel$ represents the radial pressure and $p_\perp$ represents the tangential pressure. The relationship between the radial pressure and the tangential pressure is given by

$$p = \frac{1}{3}\left(p_\parallel + 2p_\perp\right). \tag{8}$$

Isotropic pressure is obtained when $p_\parallel = p_\perp$. The tensor $\boldsymbol{h}$ is the projection tensor and $\boldsymbol{n}$ is the unit radial vector given by

$$n^a = \frac{1}{B}\delta^a_1. \tag{9}$$

The Einstein–Maxwell equations are given by

$$R_{ab} - \frac{1}{2}R\,g_{ab} = 8\pi T_{ab} - \Lambda g_{ab}, \tag{10a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \tag{10b}$$

$$F^{ab}{}_{;b} = 4\pi J^a, \tag{10c}$$

where the tensors $\boldsymbol{R}$, $\boldsymbol{T}$, and $\boldsymbol{F}$ are the Ricci tensor, energy momentum tensor and Faraday tensor respectively, and $\boldsymbol{J}$ represents the current. We have included the cosmological constant Λ. The Faraday tensor and current can be respectively defined as

$$F_{ab} = \Phi_{b;a} - \Phi_{a;b}, \tag{11a}$$

$$J^a = \zeta u^a, \tag{11b}$$

where Φ_a and ζ respectively represent the electromagnetic potential and the proper charge density. The tensor $\boldsymbol{E}$ represents the electromagnetic field tensor which is defined by

$$E_{ab} = \frac{1}{4\pi}\left(F_a{}^c\,F_{bc} - \frac{1}{4}F^{cd}\,F_{cd}\,g_{ab}\right). \tag{12}$$

The four-potential is given by

$$\Phi_a = (\varphi(r,t), 0, 0, 0). \tag{13}$$

Using system (11) the following electromagnetic quantities can be obtained [4]

$$\varphi_{rr} - \left(\frac{A_r}{A} + \frac{B_r}{B} - 2\frac{Y_r}{Y}\right)\varphi_r = 4\pi\zeta AB^2, \tag{14a}$$

$$\varphi_{rt} - \left(\frac{A_t}{A} + \frac{B_t}{B} - 2\frac{Y_t}{Y}\right) = 0. \tag{14b}$$

The system (14) can be solved to yield

$$\varphi_r = \frac{AB}{Y^2}Q, \tag{15}$$

$$Q = 4\pi\int^r \zeta B(w,t)Y^2(w,t)\mathrm{d}w, \tag{16}$$

where Q is a function of r and represents the total charge contained in the star. Throughout this paper w represents the dummy variable of integration. The Einstein–Maxwell field equations (10) with shear are given by

$$8\pi\rho = \frac{2}{A^2}\frac{B_t Y_t}{BY} + \frac{1}{Y^2} + \frac{Y_t^2}{A^2Y^2} - \frac{Q^2}{Y^4} - \frac{1}{B^2}\left(2\frac{Y_{rr}}{Y} + \frac{Y_r^2}{Y^2} - 2\frac{B_r Y_r}{BY}\right) - \Lambda, \tag{17a}$$

$$8\pi\left(p + \frac{2}{3}\Delta\right) = \frac{1}{A^2}\left(-2\frac{Y_{tt}}{Y} - \frac{Y_t}{Y^2} + 2\frac{A_t Y_t}{AY}\right) + \frac{Q^2}{Y^4} + \frac{1}{B^2}\left(\frac{Y_r^2}{Y^2} + 2\frac{A_r Y_r}{AY}\right) - \frac{1}{Y^2} + \Lambda, \tag{17b}$$

$$8\pi\left(p - \frac{1}{3}\Delta\right) = -\frac{1}{A^2}\left(\frac{B_{tt}}{B} - \frac{A_t B_t}{AB} + \frac{B_t Y_t}{BY} - \frac{A_t Y_t}{AY} + \frac{Y_{tt}}{Y}\right) - \frac{Q^2}{Y^4} + \frac{1}{B^2}\left(\frac{A_{rr}}{A} - \frac{A_r B_r}{AB} + \frac{A_r Y_r}{AY} - \frac{B_r Y_r}{BY} + \frac{Y_{rr}}{Y}\right) + \Lambda, \tag{17c}$$

$$8\pi q = -\frac{2}{AB}\left(\frac{B_t Y_r}{BY} + \frac{A_r Y_t}{AY} - \frac{Y_{rt}}{Y}\right), \tag{17d}$$

$$\zeta = \frac{Q_r}{4\pi BY^2}, \tag{17e}$$

where Δ represents the degree of anisotropy, and is given by

$$\Delta = p_\parallel - p_\perp. \tag{18}$$

The line element that describes the exterior spacetime, represented by the generalised Vaidya metric [34,35], is given by

$$\mathrm{d}s^2 = -\left(1 - 2\frac{m(v,\bar{r})}{\bar{r}}\right)\mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}\bar{r} + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{19}$$

where $m(v,\bar{r})$ represents the mass function. In the exterior v and $\bar{r}$ are the retarded time and the radial coordinates respectively. In our case the mass function becomes

$$m(v,\bar{r}) = m(v) + \frac{1}{2}\frac{Q^2}{\bar{r}} - \frac{1}{6}\Lambda\bar{r}^3. \tag{20}$$

The matching of the interior spacetime and extrinsic curvature to the exterior spacetime was completed by Santos [1] for shear-free, uncharged matter. For the extension to charged matter and shear, see the treatments of Maharaj and Govender [36] and De Oliveira and Santos [37]. The cosmological constant was included in the investigations of Thirukkanesh et al [38] and Bhatti [39]. For a composite fluid the matching was completed in four dimensions by Maharaj and Brassel [4]. The mass function (20) includes both charge and the cosmological constant in our unified treatment for a shearing spherically symmetric interior. The junction conditions at the stellar surface Σ then gives

$$(p)_\Sigma = (q)_\Sigma\,. \tag{21}$$

We substitute (17b) and (17d) into (21) to obtain

$$2AB^2Y^3Y_{tt} + AB^2Y^2Y_t^2 - 2B^2Y^3A_tY_t - 2ABY^3Y_tA_r + 2A^2BY^3Y_{rt} - 2A^2Y^3A_rY_r - 2A^2Y^3B_tY_r - A^3Y^2Y_r^2 + A^3B^2Y^2 - A^3B^2Q^2 - \Lambda A^3B^2Y^4 = 0. \tag{22}$$

The partial differential equation (22) represents the junction condition at Σ for a spherically symmetric radiating star inclusive of shear, charge and the cosmological constant. We can recover earlier results by placing restrictions on A, B, Y, Q and Λ in (22). The general solution to (22) is currently unknown.

We write (22) in the equivalent form

$$B_t + \mathcal{L}_1B^2 + \mathcal{L}_2B + \mathcal{L}_3 = 0, \tag{23}$$

where

$$\mathcal{L}_1 = \frac{A_tY_t}{A^2Y_r} + \frac{Q^2A}{2Y^3Y_r} + \frac{\Lambda AY}{2Y_r} - \frac{Y_t^2}{2AYY_r} - \frac{Y_{tt}}{AY_r} - \frac{A}{2YY_r}, \tag{24a}$$

$$\mathcal{L}_2 = -\frac{Y_{rt}}{Y_r} + \frac{A_rY_t}{AY_r}, \tag{24b}$$

$$\mathcal{L}_3 = A_r + \frac{1}{2}\frac{AY_r}{Y}. \tag{24c}$$

We observe that (23) is a Riccati equation in the potential B. Riccati equations are useful as Bernoulli equations, linear equations and other integrable forms can be obtained as special cases by placing appropriate restrictions on the coefficients in the Riccati equation. This feature was explored in several investigations including that of Ivanov [29].

The boundary condition may be supplemented with a barotropic equation of state

$$p_\parallel = p_\parallel(\rho), \tag{25}$$

based on physical grounds. The linear stiff equation of state is a special case and can be written as

$$p_\parallel = \rho. \tag{26}$$

The geodesic junction condition is obtained by assuming the particles travel in geodesic motion and this can be achieved by setting $A = 1$ in (23). The charge can be removed by setting $Q = 0$ and the cosmological constant can be excluded by setting $\Lambda = 0$. The presence of charge Q is an important physical quantity, particularly in the early stages of stellar evolution. The cosmological constant Λ represents the background energy density of spacetime. In 1998 the Supernova Cosmology Project [40] and High-Z Supernova Search Team [41] independently produced results that suggested that the universe could be expanding at an accelerating rate. This implies that the cosmological constant on a cosmological scale could be a strictly positive number [42]. There are cases on an astrophysical scale where the cosmological constant could be a negative number. An example of this is the anti-de Sitter spacetime. The anti-de Sitter spacetime can be used to model static black holes. The accelerating expansion of the universe is related to the concept of dark energy, which is a repulsive force. It is therefore important to consider both cases for the cosmological constant: $\Lambda = 0$ and $\Lambda \neq 0$.

There are fewer charged models of radiating stars as adding an electromagnetic field results in the inclusion of the Maxwell equations in the modelling process. Radiating stars for different compositions of matter with the inclusion of electromagnetic fields were studied by Maharaj and Brassel [4]. The presence of charge in the gravitational potential functions affects the rate of gravitational collapse and other physical features. We will show this later. Other interesting features such as how an electromagnetic field reduces the instability of an expansion free radiating fluid during dissipative collapse were explored by Sharif and Azam [43]. There is evidence to suggest that massive stars also have a prevalent magnetic field [44].

3 An invariant transformation

Transformations are useful as they allow us to transform coordinate systems of equations which might allow us to express the equation in a form that is simpler to solve, or could allow us to obtain solutions that are group invariant with regard to the transformation. In this paper we investigate transformations that remove second order derivative terms from (23), that still allow us to express the equation as a Riccati equation in one of the dependent variables.

We present the new transformation in the form

$$H = \left(\alpha \frac{Y_r}{B} + \beta \frac{Y_t}{A}\right) \mathcal{F} + \mathcal{G}, \tag{27}$$

where α and β are arbitrary constants, and $\mathcal{F}$ and $\mathcal{G}$ are arbitrary functions of r, t, A and Y. In general

$$\mathcal{F} = \mathcal{F}\left(r, t, A, Y\right), \tag{28a}$$

$$\mathcal{G} = \mathcal{G}\left(r, t, A, Y\right), \tag{28b}$$

which we will show leads to new solutions. Note, even though A and Y are functions of r and t, they have to appear explicitly in $\mathcal{F}$ and $\mathcal{G}$. Also note that $\mathcal{F}$ and $\mathcal{G}$ are independent of B, which implies that we can substitute H for B. Indeed, requiring this relationship between H and B guarantees that the resulting differential equation will remain a Riccati equation as Riccati equations are form invariant under reciprocal transformations.

The four transformations used in [27,28,32,33] are contained in the generalised transformation (27). We now consider some special cases for $\mathcal{F}$ and $\mathcal{G}$.

3.1 $\mathcal{F} = 1$ and $\mathcal{G} = 0$

We set

$$\mathcal{F} = 1, \tag{29a}$$

$$\mathcal{G} = 0, \tag{29b}$$

$$\alpha = 1, \tag{29c}$$

$$\beta = 0, \tag{29d}$$

$$Z = \frac{1}{H}, \tag{29e}$$

in (27) to obtain

$$Z = \frac{B}{Y_r}. \tag{30}$$

Equation (30) is the Thirukkanesh and Maharaj [33] transformation.

3.2 $\mathcal{F} = 1$ and $\mathcal{G} = \gamma$

We set

$$\mathcal{F} = 1, \tag{31a}$$

$$\mathcal{G} = \gamma, \tag{31b}$$

in (27) to obtain

$$H = \alpha \frac{Y_r}{B} + \beta \frac{Y_t}{A} + \gamma, \tag{32}$$

where γ is an arbitrary constant. Equation (32) is the *generalised horizon function* which was used by Mahomed et al. [32].

We observe that the γ term in (32) is redundant. We can see this by defining

$$\mathcal{H} = H - \gamma. \tag{33}$$

Substituting (33) into (32) results in

$$\mathcal{H} = \alpha \frac{Y_r}{B} + \beta \frac{Y_t}{A}. \tag{34}$$

We conclude from (32), that the γ term in (32) may be eliminated.

In the special case when $\alpha = \beta$ we can define

$$\mathcal{H} = \frac{H - \gamma}{\beta}. \tag{35}$$

Substituting (35) into (32) results in

$$\mathcal{H} = \frac{Y_r}{B} + \frac{Y_t}{A}. \tag{36}$$

Hence (32) reduces to (36) when $\alpha = \beta$, which is the *horizon function* first introduced by Ivanov [28].

3.3 $\mathcal{F} = \mathcal{F}(r, t)$ and $\mathcal{G} = \mathcal{G}(r, t)$

We set

$$\mathcal{F} = \mathcal{F}(r, t), \tag{37a}$$
$$\mathcal{G} = \mathcal{G}(r, t), \tag{37b}$$

in (27) to obtain

$$H = \left(\alpha \frac{Y_r}{B} + \beta \frac{Y_t}{A}\right) \mathcal{F}(r, t) + \mathcal{G}(r, t). \tag{38}$$

This means that A and Y do not explicitly appear in $\mathcal{F}$ and $\mathcal{G}$. The expression (38) can be transformed to the simpler form if we can introduce a new function $\mathcal{H}$ by setting

$$\mathcal{H} = \frac{H - \mathcal{G}(r, t)}{\mathcal{F}(r, t)}. \tag{39}$$

Substituting (39) into (38) results in

$$\mathcal{H} = \alpha \frac{Y_r}{B} + \beta \frac{Y_t}{A}, \tag{40}$$

which is the *generalised horizon function* obtained in [32]. The *horizon function* (36) is a special case when $\alpha = \beta$.

3.4 Dependence on potential functions

When either or both functions A and Y appear explicitly in $\mathcal{F}$ and/or $\mathcal{G}$ in the transformation (27), a *new* transformation is produced. It should be emphasised that the dependence of the potentials A and Y on the functions $\mathcal{F}$ and $\mathcal{G}$ have not been considered previously. Consequently these cases will lead to new solutions via the generalised transformation (27). The explicit forms of the potentials A, B and Y will then lead to new solutions of the Einstein–Maxwell system (10).

We solve (27) for B to obtain

$$B = -\frac{\alpha A Y_r \mathcal{F}}{\beta Y_t \mathcal{F} A \mathcal{G} - A H}. \tag{41}$$

We substitute (41) into (23) to obtain the master equation

$$H_t + \mathcal{L}_4 H^2 + \mathcal{L}_5 H + \mathcal{L}_6 = 0, \tag{42}$$

where

$$\mathcal{L}_4 = -\frac{1}{2\alpha} \frac{1}{Y Y_r \mathcal{F}} (2A_r Y + A Y_r), \tag{43a}$$

$$\mathcal{L}_5 = -\frac{1}{\alpha} \frac{1}{A Y Y_r \mathcal{F}} \big(A \left(Y_r \left(\alpha Y \left(A_t \mathcal{F}_A + Y_t \mathcal{F}_Y + \mathcal{F}_t\right) - \beta Y_t \mathcal{F}\right) - 2A_r Y \mathcal{G}\right) + (\alpha - 2\beta) A_r Y Y_t \mathcal{F} - A^2 Y_r \mathcal{G}\big), \tag{43b}$$

$$\begin{aligned}\mathcal{L}_6 = -\frac{1}{2\alpha A^2 Y^3 Y_r \mathcal{F}} \big(&-2A^2 Y^2 Y_r \mathcal{G} \left(\alpha Y \left(A_t \mathcal{F}_A + Y_t \mathcal{F}_Y + \mathcal{F}_t\right) - \beta Y_t \mathcal{F}\right) + 2\alpha A^2 Y^3 Y_r \mathcal{F} \left(A_t \mathcal{G}_A + Y_r \mathcal{G}_Y + \mathcal{G}_t\right) \\ &- 2(\alpha - 2\beta) A_r A Y^3 Y_t \mathcal{F} \mathcal{G} \\ &+ 2(\alpha - \beta) Y^3 Y_t \mathcal{F}^2 \left(\alpha A_t Y_r - \beta A_r Y_t\right) + 2A_r A^2 Y^3 \mathcal{G}^2 \\ &+ \alpha^2 A^3 Y_r \mathcal{F}^2 \left(\Lambda Y^4 - Y^2 + Q^2\right) - (\alpha - \beta) A Y^2 Y_r \mathcal{F}^2 \\ &\times \left((\alpha + \beta) Y_t^2 + 2\alpha Y Y_{tt}\right) + A^3 Y^2 Y_r \mathcal{G}^2\big).\end{aligned} \tag{43c}$$

A comparison between (23) and (42) reveals that both are Riccati equations. There are two distinguishing features of the master equation (42). Firstly (42) contains only one second order term Y_{tt}. Our transformation (27) simplified (23) by removing the second order term Y_{rt}. Secondly new terms containing $\mathcal{F}_t, \mathcal{F}_A, \mathcal{F}_Y, \mathcal{G}_t, \mathcal{G}_A$ and $\mathcal{G}_Y$ arise which provide us with additional options to solve the Riccati equation.

We were able to generate several previous transformations from (27) that were used to simplify the different versions of junction conditions that are obtained from (23). These results are sumarised in Table 1. The first transformation we looked at was that obtained by Thirukkanesh and Maharaj [33]. Their transformation was used to simplify the geodesic junction condition exclusive of both charge and a cosmological constant. This junction condition can be obtained by setting $A = 1$ and $Q = \Lambda = 0$ in (23). We note that Thirukkanesh and Maharaj's [33] transformation removed one of the second order terms, the Y_{rt} term, from this junction condition. The next transformation we explored was the *horizon function* introduced by Ivanov [27], which simplified the geodesic junction condition exclusive of both charge and a cosmological constant. This transformation removed both of the second order terms, Y_{rt} and Y_{tt}, from the junction condition. The *horizon function* [27] also transformed the junction condition from a second order equation into a first order equation. In 2016 Ivanov [28] modified the *horizon function* to simplify the non-geodesic junction condition exclusive of both charge and cosmological constant. This junction condition can be obtained by setting $Q = \Lambda = 0$ in (23). This transformation removed both second order terms, Y_{rt} and Y_{tt}, from the junction condition. The final transformation we considered was the Mahomed et al. [32] *generalised horizon function*, which simplified (23) by removing the second order term Y_{rt}.

4 Linear equation: $\mathcal{L}_4 = 0$

We impose the restriction

$$2A_r Y + A Y_r = 0, \tag{44}$$

on (42) to obtain a linear equation in H. In order to avoid having implicit solutions we make the assumption that $\mathcal{F}$ and $\mathcal{G}$ are independent of A so that

$$\mathcal{F} = \mathcal{F}(r, t, Y), \tag{45a}$$
$$\mathcal{G} = \mathcal{G}(r, t, Y). \tag{45b}$$

Table 1 Particular transformations that can be generated from (27)

Reference	Restrictions	Transformation	Junction condition	Spacetime
[33]	$\beta=\mathcal{G}=0$ $\alpha=\mathcal{F}=1$ $H=\frac{1}{Z}$ $A=1$	$Z=\frac{B}{Y_r}$	$B_t=\frac{1}{2}\frac{B^2\left(Y_t^2+2YY_{tt}+1\right)}{YY_r}+\frac{BY_{rt}}{Y_r}-\frac{1}{2}\frac{Y_r}{Y}$	Geodesic and exclusive of both charge and cosmological constant
[27]	$\alpha=\beta=\mathcal{F}=1$ $\mathcal{G}=0$ $A=1$	$H=\frac{Y_r}{B}+Y_t$	$B_t=\frac{1}{2}\frac{B^2\left(Y_t^2+2YY_{tt}+1\right)}{YY_r}+\frac{BY_{rt}}{Y_r}-\frac{1}{2}\frac{Y_r}{Y}$	Geodesic and exclusive of both charge and cosmological constant
[28]	$\alpha=\beta=\mathcal{F}=1$ $\mathcal{G}=0$	$H=\frac{Y_r}{B}+\frac{Y_t}{A}$	$B_t=\frac{1}{2}\frac{B^2\left(-2A_tYY_t+A\left(Y_t^2+2YY_{tt}\right)+A^3\right)}{A^2YY_r}$ $+\frac{1}{2}\frac{B\left(2A^2YY_{rt}-2AA_rYY_t\right)}{A^2YY_r}$ $+\frac{1}{2}\frac{A^3\left(-Y_r^2\right)-2A_rA^2YY_r}{A^2YY_r}$	Nongeodesic and exclusive of both charge and cosmological constant
[32]	$\mathcal{F}=1$ $\mathcal{G}=\gamma$ where α, β and γ are arbitrary constants	$H=\alpha\frac{Y_r}{B}+\beta\frac{Y_t}{A}+\gamma$	$B_t=\mathcal{L}_1B^2+\mathcal{L}_2B+\mathcal{L}_3$	Nongeodesic and exclusive of both charge and cosmological constant

We solve (44) to obtain the restriction

$$A=\frac{\mathcal{T}_0}{\sqrt{Y}}, \tag{46}$$

where $\mathcal{T}_0$ is an arbitrary function of t. We substitute (46) into (42) to obtain the linear equation

$$H_t+\mathcal{L}_5H+\mathcal{L}_6=0, \tag{47}$$

where

$$\mathcal{L}_5=-\left(\frac{2}{\mathcal{F}}\left(\mathcal{F}_t+Y_t\mathcal{F}_Y\right)-\frac{Y_t}{Y}\right) \tag{48a}$$

$$\begin{aligned}\mathcal{L}_6=&-\frac{1}{2}\frac{1}{Y^{7/2}}\left(\frac{Y^2\mathcal{G}}{\mathcal{F}}\left(-2Y^{3/2}\left(Y_t\mathcal{F}_Y+\mathcal{F}_t\right)\right)\right.\\&+\frac{\mathcal{F}}{\mathcal{T}_0^2}\left(2(\alpha-\beta)\mathcal{T}_0'Y_tY^4\right.\\&+\alpha\mathcal{T}_0^3\left(\Lambda Y^4-Y^2+Q^2\right)\\&\left.-2(\alpha-\beta)\mathcal{T}_0\left(Y_t^2+YY_{tt}\right)Y^3\right)\\&\left.+Y_tY^{5/2}\left(2Y\mathcal{G}_Y+\mathcal{G}\right)\right)+\mathcal{G}_t.\end{aligned} \tag{48b}$$

We solve (47) to obtain

$$H=\frac{\mathcal{F}\left(\int_1^t\wp_1\mathrm{d}w+\mathcal{R}_0\right)}{\sqrt{Y}}, \tag{49}$$

where $\mathcal{R}_0$ is an arbitary function of r, and

$$\begin{aligned}\wp_1=&\frac{1}{2\mathcal{T}_0^2Y^3\mathcal{F}^2}\left(-2\mathcal{T}_0^2Y^{7/2}\left(Y_w\mathcal{F}_Y+\mathcal{F}_w\right)\mathcal{G}\right.\\&+\mathcal{T}_0^2Y^{5/2}\mathcal{F}\left(Y_w\left(2Y\mathcal{G}_Y+\mathcal{G}\right)+2Y\mathcal{G}_w\right)\\&+\mathcal{F}^2\left(Y^2\left(2(\alpha-\beta)\mathcal{T}_0'Y^2Y_w-2(\alpha-\beta)\mathcal{T}_0Y\right.\right.\\&\times\left.\left.\left.\left(Y_w^2+YY_{ww}\right)+\alpha\mathcal{T}_0^3\left(\Lambda Y^2-1\right)\right)+\alpha\, Q^2\mathcal{T}_0^3\right)\right).\end{aligned} \tag{50}$$

In the integrand $\wp_1$ in (50) the functions $\mathcal{T}_0$, Y, $\mathcal{F}$ and $\mathcal{G}$ respectively represent $\mathcal{T}_0(w)$, $Y(r,w)$, $\mathcal{F}(r,w,Y(r,w))$ and $\mathcal{G}(r,w,Y(r,w))$. Substituting (45), (46) and (49) into (41), we obtain

$$B=\frac{\alpha\mathcal{T}_0\sqrt{Y}Y_r\mathcal{F}}{-\mathcal{T}_0\sqrt{Y}\mathcal{G}+\mathcal{F}\left(\mathcal{T}_0(\int_1^t\wp_1\mathrm{d}w+\mathcal{R}_0)-\beta YY_t\right)}. \tag{51}$$

Restriction (44) was also solved by Mahomed et al [32] to obtain the potential A given in (46). However, the potential B in (51) differs from that obtained in [32] as different transformations were used. We can regain the same form of the gravitational potential function B in [32] (see their Eq. (24)) when we set $\mathcal{F}=1$ and $\mathcal{G}=0$ in (51). An alternative approach is to solve (44) for Y as was done in [25]. As expected, the gravitational potential B obtained there is different from (51).

We summarise our results in the following theorem:

Theorem 1 *The boundary condition*

$$H_t + \mathcal{L}_5 H + L_6 = 0, \tag{52}$$

which is a linear equation in H can be solved in general with $\mathcal{F} = \mathcal{F}(r, t, Y)$ and $\mathcal{G} = \mathcal{G}(r, t, Y)$ for a general relativistic radiating star with explicit forms for the potentials A and B obtained.

Corollary 1.1 *Known potentials are regained if $\mathcal{F} = \mathcal{F}(r, t)$ and $\mathcal{G} = \mathcal{G}(r, t)$ including models with accelerating particles. Physical solutions that include particles that travel in geodesic trajectories are not contained in this class of models.*

5 Bernoulli equation: $\mathcal{L}_6 = 0$

We can rewrite (42) as the Bernoulli equation

$$H_t + \mathcal{L}_4 H^2 + \mathcal{L}_5 H = 0, \tag{53}$$

subject to the restriction

$$\begin{aligned}
&-\frac{1}{2\alpha A^2 Y^3 Y_r \mathcal{F}}\Big(-2A^2Y^2Y_r\mathcal{G}\left(\alpha Y\left(A_t\mathcal{F}_A + Y_t\mathcal{F}_Y + \mathcal{F}_t\right)\right.\\
&\left.-\beta Y_t\mathcal{F}\right) + 2\alpha A^2Y^3Y_r\mathcal{F}\left(A_t\mathcal{G}_A + Y_r\mathcal{G}_Y + \mathcal{G}_t\right)\\
&- 2(\alpha-2\beta)A_rAY^3Y_t\mathcal{F}\mathcal{G}\\
&+ 2(\alpha-\beta)Y^3Y_t\mathcal{F}^2\left(\alpha A_tY_r - \beta A_rY_t\right) + 2A_rA^2Y^3\mathcal{G}^2\\
&+ \alpha^2A^3Y_r\mathcal{F}^2\left(\Lambda Y^4 - Y^2 + Q^2\right)\\
&- (\alpha-\beta)AY^2Y_r\mathcal{F}^2\Big((\alpha+\beta)Y_t^2\\
&+2\alpha YY_{tt}\Big) + A^3Y^2Y_r\mathcal{G}^2\Big) = 0.
\end{aligned} \tag{54}$$

It is difficult to obtain the general solution to the partial differential equation (54). Particular exact solutions to (54) can be obtained by imposing appropriate restrictions. We will show later that such solutions to (54) do exist. We solve (53) to obtain

$$H = \frac{e^{\int_1^t \wp_2 d\bar{w}}}{\mathcal{R}_1 - \int_1^t \wp_3 e^{\int_1^w \wp_2 d\bar{w}} dw}, \tag{55}$$

where $\mathcal{R}_1$ is an arbitrary function of r and

$$\begin{aligned}
\wp_2 =& \frac{1}{\alpha AYY_r\mathcal{F}}\left(A\left(Y_r\left(\alpha Y\left(A_{\bar{w}}\mathcal{F}_A + Y_{\bar{w}}\mathcal{F}_Y + F_{\bar{w}}\right) - \beta Y_{\bar{w}}\mathcal{F}\right)\right.\right.\\
&\left.\left.-2A_rY\mathcal{G}\right) + (\alpha-2\beta)A_rYY_{\bar{w}}\mathcal{F} - A^2Y_r\mathcal{G}\right),
\end{aligned} \tag{56a}$$

$$\wp_3 = \frac{1}{2\alpha}\frac{1}{YY_r\mathcal{F}}\left(2A_rY + AY_r\right). \tag{56b}$$

Note that A and Y are functions of r and $\bar{w}$, and $\mathcal{F}$ and $\mathcal{G}$ are functions of r, $\bar{w}$, $A(r, \bar{w})$ and $Y(r, \bar{w})$ in (56a), while A and Y are functions of r and w, and $\mathcal{F}$ is a function of r, w, $A(r, w)$ and $Y(r, w)$ in (56b). We substitute (55) into (41) to obtain the potential function B given by

$$B = -\frac{\alpha AY_r\mathcal{F}}{\beta Y_t\mathcal{F} + A\left(\mathcal{G} - \frac{\exp\left(\int_1^t \wp_2 d\bar{w}\right)}{\mathcal{R}_1 - \int_1^t \exp\left(\int_1^w \wp_2 d\bar{w}\right)\wp_3 dw}\right)}. \tag{57}$$

We now consider particular cases where we can show that the potential A can be found explicitly by solving (54).

5.1 Solution I

We set

$$\beta = 0, \tag{58a}$$
$$\mathcal{F} = \mathcal{F}(r, t), \tag{58b}$$
$$\mathcal{G} = 0, \tag{58c}$$
$$Y = \left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)^{n_3}, \tag{58d}$$

where $\mathcal{R}_2$ and $\mathcal{R}_3$ are arbitrary functions of r, $\mathcal{T}_1$ is an arbitrary function of t and n_1 to n_3 are arbitrary constants. We substitute (58) into (54) to obtain the restriction

$$\begin{aligned}
&2n_1n_3\wp_4^{4n_3-1}\mathcal{R}_2\mathcal{T}_1'A_t + A^3\left(\Lambda\wp_4^{4n_3} - \wp_4^{2n_3} + Q^2\right)\\
&+ n_1n_3\wp_4^{4n_3-2}\mathcal{R}_2A\left(n_1(2-3n_3)\mathcal{R}_2\mathcal{T}_1'^2 - 2\wp_4\mathcal{T}_1''\right) = 0,
\end{aligned} \tag{59}$$

where

$$\wp_4 = n_1\mathcal{T}_1\mathcal{R}_2 + n_2\mathcal{R}_3 \tag{60}$$

We note that (59) is a Bernoulli equation in A which can be solved to yield

$$A = \mp\frac{\sqrt{3}n_1n_3\mathcal{R}_2\mathcal{T}_1'\wp_4^{2n_3}}{\sqrt{\wp_4^2\left(-3Q^2 + \wp_4^{n_3}\left(3n_1^2n_3^2\mathcal{R}_2^2\mathcal{R}_4 + \Lambda\wp_4^{3n_3} - 3\wp_4^{n_3}\right)\right)}}, \tag{61}$$

where $\mathcal{R}_4$ is an arbitrary function of r.

Substituting A, using (61), in (57) yields

$$\begin{aligned}
B =& \alpha n_3\wp_4^{n_3-1}\left(n_1\mathcal{T}_1\mathcal{R}_2' + n_2\mathcal{R}_3'\right)\exp\left(-\int_1^t \wp_5 d\bar{w}\right)\\
&\times\left(\mathcal{R}_1 - \int_1^t \exp\left(\int_1^w \wp_5 d\bar{w}\right)\wp_6 dw\right)\mathcal{F},
\end{aligned} \tag{62}$$

where

$$
\begin{aligned}
\wp_5 &= \frac{\mathcal{F}_{\bar{w}}}{\mathcal{F}} + \left(2\alpha\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)\left((n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3)^{n_3}\right.\right.\\
&\times \left(3n_1^2n_3^2\mathcal{R}_2^2\mathcal{R}_4 + \Lambda\left(n_1\mathcal{R}_2\mathcal{T}_1\right.\right.\\
&+ \left.\left. n_2\mathcal{R}_3\right)^{3n_3} - 3\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)^{n_3}\right) - 3Q^2\Big)\\
&\times \left(n_1\mathcal{R}_2'\mathcal{T}_1 + n_2\mathcal{R}_3'\right)\Big)^{-1}\left(n_1(\alpha - 2\beta)\mathcal{T}_1'\right.\\
&\times \left((n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3)^{n_3}\left(3n_1^2n_3^2\mathcal{R}_2^3\left((3n_3 - 2)\right.\right.\right.\\
&\times \ \mathcal{R}_4\left(n_1\mathcal{R}_2'\mathcal{T}_1 + n_2\mathcal{R}_3'\right)\\
&- \ \left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)\mathcal{R}_4'\Big) + 2\Lambda n_2\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)^{3n_3}\\
&\times \left(\mathcal{R}_3\mathcal{R}_2' - \mathcal{R}_2\mathcal{R}_3'\right)\\
&- \ 6\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)^{n_3}\left(n_2(n_3 - 1)\mathcal{R}_2\mathcal{R}_3'\right.\\
&+ \ \left.\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)\mathcal{R}_2'\right)\Big)\\
&+ 6Q\left(\mathcal{R}_2\left(Q'(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3) + n_2(1 - 2n_3)Q\mathcal{R}_3'\right)\right.\\
&- \ \left.\left.\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)Q\mathcal{R}_2'\right)\right)\Big), \qquad (63a)\\
\wp_6 &= \sqrt{3}n_1\mathcal{T}_1'\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)^{n_3+2}\\
&\times \left(2\alpha\mathcal{F}\left((n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3)^2\left((n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3)^{n_3}\right.\right.\right.\\
&\times \ \left(3n_1^2n_3^2\mathcal{R}_2^2\mathcal{R}_4 + \Lambda\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)^{3n_3}\right.\\
&- \ \left.\left.\left.3\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)^{n_3}\right) - 3Q^2\right)\right)^{3/2}\\
&\times \left(n_1\mathcal{R}_2'\mathcal{T}_1 + n_2\mathcal{R}_3'\right)\Big)^{-1}\left(\left(\pm\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)^{n_3}\right.\right.\\
&\times \left(3n_1^2n_3^2\mathcal{R}_2^3\left(2(2n_3 - 1)\mathcal{R}_4\left(n_1\mathcal{R}_2'\mathcal{T}_1 + n_2\mathcal{R}_3'\right)\right.\right.\\
&- \ \left.\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)\mathcal{R}_4'\right) - 3\left(n_1\mathcal{R}_2\mathcal{T}_1\right.\\
&+ \ \left.n_2\mathcal{R}_3\right)^{n_3}\left(\mathcal{R}_2'(3n_1n_3\mathcal{R}_2\mathcal{T}_1 + 2n_2\mathcal{R}_3)\right.\\
&+n_2(3n_3 - 2)\mathcal{R}_2\mathcal{R}_3'\Big) + \Lambda\left(n_1\mathcal{R}_2\mathcal{T}_1\right.\\
&+ \ \left.n_2\mathcal{R}_3\right)^{3n_3}\left(n_2(n_3 - 2)\mathcal{R}_2\mathcal{R}_3' + (n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3)\mathcal{R}_2'\right)\Big)\\
&+ 3Q^2\left(n_2(2 - 5n_3)\mathcal{R}_2\mathcal{R}_3' - \left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)\mathcal{R}_2'\right)\\
&+ 6\left(n_1\mathcal{R}_2\mathcal{T}_1 + n_2\mathcal{R}_3\right)Q\mathcal{R}_2Q'\Big)\Big), \qquad (63b)
\end{aligned}
$$

$\mathcal{F}$ is a function of r and $\bar{w}$, and $\mathcal{T}_1$ is a function of $\bar{w}$ in (78) while $\mathcal{F}$ is a function of r and w, and $\mathcal{T}_1$ is a function of w in (63b). This is a new solution of the master equation (42)

We can regain the gravitational potential Y of Ivanov [28] if we set

$$n_1 = n_2 = 1, \qquad (64a)$$
$$\mathcal{T}_1 = t, \qquad (64b)$$
$$n_3 = \frac{2}{3}, \qquad (64c)$$

in (58d). However, the potential function Y obtained in [28] corresponds to a geodesic model of the junction condition excluding charge and a cosmological constant. Therefore the gravitational potential B in [28] is different from (62) which corresponds to an accelerating model. We have thus shown that the Ivanov [28] geodesic model with $A = 1$ can be extended to a larger class of nongeodesic models with $A \neq 1$.

5.2 Solution II

We set

$$\beta = 0, \qquad (65a)$$
$$\mathcal{F} = \mathcal{F}(r, t, Y), \qquad (65b)$$
$$\mathcal{G} = 0, \qquad (65c)$$

in (54) to obtain the restriction

$$
\begin{aligned}
&2A_tY^3Y_t + A^3\left(\Lambda Y^4 - Y^2 + Q^2\right)\\
&-AY^2\left(Y_t^2 + 2YY_{tt}\right) = 0, \qquad (66)
\end{aligned}
$$

which is again a Bernoulli equation in A and can be solved to obtain

$$A = \mp\frac{YY_t}{\sqrt{\mathcal{R}_2Y + \frac{1}{3}\Lambda Y^4 - Y^2 - Q^2}}, \qquad (67)$$

where $\mathcal{R}_2$ is an arbitrary function of r. Note the non-appearance of $\mathcal{F}$ in (66).

Using (57), (65) and (67) we express B as

$$
\begin{aligned}
B &= \alpha Y_r\mathcal{F}\exp\left(-\int_1^t \wp_7 \mathrm{d}\bar{w}\right)\\
&\times\left(-\int_1^t \exp\left(\int_1^w \wp_7\mathrm{d}\bar{w}\right)\wp_8\mathrm{d}w + \mathcal{R}_1\right), \qquad (68)
\end{aligned}
$$

where

$$
\begin{aligned}
\wp_7 &= \frac{1}{\mathcal{F}}\left(Y_{\bar{w}}\mathcal{F}_Y + \mathcal{F}_{\bar{w}}\right) + \frac{1}{2}\frac{1}{Y_r}\left(2Y_{r\bar{w}}\right.\\
&+ Y_{\bar{w}}\left(Y\left(-3Q^2 + Y\left(\Lambda Y^3 - 3Y + 3\mathcal{R}_2\right)\right)\right)^{-1}\\
&\times\left(Y_r\left(3\mathcal{R}_2Y - 2\Lambda Y^4 - 6Q^2\right)\right.\\
&- \left.\left. 3Y\left(-2QQ' + \mathcal{R}_2'Y\right)\right)\right), \qquad (69a)\\
\wp_8 &= \mp\sqrt{3}\left(2\alpha Y_r\mathcal{F}\left(-3Q^2 + Y\left(\Lambda Y^3 - 3Y + 3\mathcal{R}_2\right)\right)^{3/2}\right)^{-1}\\
&\times\left(2YY_{rw}\left(-3Q^2 + Y\left(\Lambda Y^3 - 3Y + 3\mathcal{R}_2\right)\right)\right.\\
&-Y_w\left(3Y\left(-2QQ' + Y\mathcal{R}_2'\right)\right.\\
&+ \left.\left. Y_r\left(9Q^2 + Y\left(\Lambda Y^3 + 3Y - 6\mathcal{R}_2\right)\right)\right)\right), \qquad (69b)
\end{aligned}
$$

$\mathcal{F}$ is a function of r $\bar{w}$, and Y, and Y is a function of r and $\bar{w}$ in (69a) while $\mathcal{F}$ is a function of r, w and Y, and Y is a function of r and w in (69b).

We regain the result of Mahomed et al. [32] if we set $\mathcal{F} = 1$ in (65b). Since they also solved (66), the potential A is the same. However, our potential B differs from their result. We can regain the potential B in [32] by setting $\mathcal{F} = 1$ in (68). We summarise our results in the following theorem:

Theorem 2 *The boundary condition*

$$H_t + \mathcal{L}_4H^2 + \mathcal{L}_5H = 0, \qquad (70)$$

which is a Bernoulli equation in H can be solved in general with $\mathcal{F}=\mathcal{F}(r,t,A,Y)$ and $\mathcal{G}=\mathcal{G}(r,t,A,Y)$ for a general relativistic radiating star. The potential B is found explicitly and the potentials A and Y satisfy a constraint equation which can be solved.

Corollary 2.1 *Known solutions are regained if $\mathcal{F}=\mathcal{F}(r,t)$ and $\mathcal{G}=\mathcal{G}(r,t)$ in models with accelerating particles.*

6 Reduction of order

We now place restrictions on the arbitrary constants that will allow us to eliminate all second order derivatives from (23), thereby allowing us to reduce the order of (23) from a second order equation to a first order equation. This reduction will also assist with simplifying the Bernoulli equation (53), thereby allowing us to obtain a simpler version of the restriction (54) to solve. We consider the special case of the transformation (27) that transforms (23) into a first order equation in all of the dependent variables. We obtain this special case by placing restrictions on the arbitrary parameters in (27). We set $\alpha=\beta$ to obtain the special case

$$H=\left(\frac{Y_r}{B}+\frac{Y_t}{A}\right)\mathcal{F}+\mathcal{G}, \tag{71}$$

which we solve for B to obtain

$$B=-\frac{AY_r\mathcal{F}}{Y_t\mathcal{F}+A\mathcal{G}-AH}. \tag{72}$$

We substitute (72) into (23) to obtain

$$H_t+\mathcal{L}_4H^2+\mathcal{L}_5H+\mathcal{L}_6=0, \tag{73}$$

where

$$\mathcal{L}_4=-\frac{1}{2}\frac{1}{YY_r\mathcal{F}}\left(2A_rY+AY_r\right), \tag{74a}$$

$$\mathcal{L}_5=-\frac{1}{AYY_r\mathcal{F}}\Big(A\left(Y_r\left(Y\left(A_t\mathcal{F}_A+Y_t\mathcal{F}_Y+\mathcal{F}_t\right)-Y_t\mathcal{F}\right)\right.$$
$$\left.-2A_rY\mathcal{G}\right)-A_rYY_t\mathcal{F}-A^2Y_r\mathcal{G}\Big), \tag{74b}$$

$$\mathcal{L}_6=\frac{\mathcal{G}}{\mathcal{F}}\left(A_t\mathcal{F}_A+Y_t\mathcal{F}_Y+\mathcal{F}_t\right)-\frac{A_r\mathcal{G}^2}{Y_r\mathcal{F}}-A_t\mathcal{G}_A-\frac{A_rY_t\mathcal{G}}{AY_r}$$
$$-\frac{1}{2}\frac{1}{Y^3\mathcal{F}}A\left(\mathcal{F}^2\left(\Lambda Y^4-Y^2+Q^2\right)+Y^2\mathcal{G}^2\right)$$
$$-Y_t\mathcal{G}_Y-\mathcal{G}_t-\frac{Y_t\mathcal{G}}{Y}. \tag{74c}$$

A comparison between (23) and (73) reveals that both are Riccati equations, however both of the second order terms Y_{tt} and Y_{rt} that are present in (23) are absent in (73). Our transformation (71) simplified (23) by transforming the equation from a second order equation with two second order terms into a *first order equation*.

Equation (73) admits exact solutions. We continue by expressing (73) as a Bernoulli equation in H. We do not consider expressing (73) as a linear equation in H, as the restriction is the same as (44); we would not gain any new solutions.

6.1 Bernoulli equation: $\mathcal{L}_6=0$

We can write (73) as the Bernoulli equation

$$H_t+\mathcal{L}_4H^2+\mathcal{L}_5H=0, \tag{75}$$

subject to the restriction

$$\frac{\mathcal{G}}{\mathcal{F}}\left(A_t\mathcal{F}_A+Y_t\mathcal{F}_Y+\mathcal{F}_t\right)-\frac{A_r\mathcal{G}^2}{Y_r\mathcal{F}}-A_t\mathcal{G}_A-\frac{A_rY_t\mathcal{G}}{AY_r}$$
$$-\frac{1}{2}\frac{1}{Y^3\mathcal{F}}A\left(\mathcal{F}^2\left(\Lambda Y^4-Y^2+Q^2\right)+Y^2\mathcal{G}^2\right)-Y_t\mathcal{G}_Y$$
$$-\mathcal{G}_t-\frac{Y_t\mathcal{G}}{Y}=0. \tag{76}$$

The restriction (76) is simpler to solve than (54), as there are no second order derivative terms present. We first solve (75) to obtain

$$H=\frac{\exp\left(\int_1^t\wp_9\mathrm{d}\bar{w}\right)}{\mathcal{R}_1-\int_1^t\exp\left(\int_1^w\wp_9\mathrm{d}\bar{w}\right)\wp_{10}\mathrm{d}w}, \tag{77}$$

where $\mathcal{R}_1$ is an arbitrary function of r and

$$\wp_9=\frac{1}{AYY_r\mathcal{F}}\Big(A\left(Y_r\left(Y\left(A_{\bar{w}}\mathcal{F}_A+Y_{\bar{w}}\mathcal{F}_Y+\mathcal{F}_{\bar{w}}\right)-Y_{\bar{w}}\mathcal{F}\right)\right.$$
$$\left.-2A_rY\mathcal{G}\right)-A_rYY_{\bar{w}}\mathcal{F}-A^2Y_r\mathcal{G}\Big), \tag{78a}$$

$$\wp_{10}=\frac{1}{2YY_r\mathcal{F}}\left(2A_rY+AY_r\right). \tag{78b}$$

In the above A and Y are functions of r and $\bar{w}$, and $\mathcal{F}$ and $\mathcal{G}$ functions of r, $\bar{w}$ $A(r,\bar{w})$, and $Y(r,\bar{w})$ in (78a) while A and Y are functions of r and w, and $\mathcal{F}$ and $\mathcal{G}$ are functions of r, w $A(r,w)$ and $Y(r,w)$ in (78b). We substitute (77) into (72) to obtain

$$B=-\frac{AY_r\mathcal{F}}{Y_t\mathcal{F}+A\left(\mathcal{G}-\frac{\exp\left(\int_1^t\wp_9\mathrm{d}\bar{w}\right)}{\mathcal{R}_1-\int_1^t\exp\left(\int_1^w\wp_9\mathrm{d}\bar{w}\right)\wp_{10}\mathrm{d}w}\right)}. \tag{79}$$

We now demonstrate that exact solutions to the restriction (76) exist.

We set

$$\mathcal{F}=\mathcal{F}(r,t,A), \tag{80a}$$
$$\mathcal{G}=0, \tag{80b}$$

in (76) to obtain

$$\Lambda Y^4-Y^2+Q^2=0. \tag{81}$$

We solve the algebraic Eq. (81) to obtain the two cases

$$Y^2 = \begin{cases} Q^2, & \Lambda = 0 \\ \frac{1}{2\Lambda} \pm \frac{\Lambda}{2}\sqrt{1-4\Lambda Q^2}, & \Lambda \neq 0 \end{cases} \tag{82}$$

The solution (82) gives an explicit functional form for Y which is a function of r only. Observe that (82) allows for positive and negative Λ. It is important to note that in this example the potential A is arbitrary. The model obtained by Mahomed et al. [32] is contained in our results.

We summarise our results in the following theorem.

Theorem 3 *The boundary condition*

$$H_t + \mathcal{L}_4 H^2 + \mathcal{L}_5 H + \mathcal{L}_6 = 0, \tag{83}$$

can be transformed to a first order equation in Y when $\alpha = \beta$, $\mathcal{F} = \mathcal{F}(r, t\, A, Y)$ and $\mathcal{G} = \mathcal{G}(r, t, A, Y)$ in a general relativistic star. The potential B is given explicitly, and A and Y satisfy a constraint equation.

Corollary 3.1 *Solutions found previously related to the horizon function*

$$H = \frac{Y_r}{B} + \frac{Y_t}{A}, \tag{84}$$

are special cases with $\mathcal{F} = \mathcal{F}(r, t)$ and $\mathcal{G} = \mathcal{G}(r, t)$ in (71).

7 Equation of state

A physical analysis of the results generated in this paper should yield new physical insights. As an example, Paliathanasis et al [23] studied the dissipative effects and temporal solution of radiating stars during the collapse phase. One could also analyse energy conditions which are important quantities in a physical treatment. This will be the object of future research for solutions of the generalised Riccati equations arising in this paper. For now, we concentrate on an equation of state.

We can express the equation of state as a partial differential equation by substituting (17a) and (17b) into (26) to obtain

$$\begin{aligned} &\frac{A_t B^2 Y^2 Y_t}{A} + A_r A Y^2 Y_r \\ &\quad + \frac{A^2}{BY}\left(-B_r Y^3 Y_r + B^3\left(\Lambda Y^4 - Y^2 + Q^2\right)\right. \\ &\quad \left. + BY^2\left(Y_r^2 + Y Y_{rr}\right)\right) \\ &-BY\left(Y\left(B_t Y_t + B Y_{tt}\right) + B Y_t^2\right) = 0. \end{aligned} \tag{85}$$

Note that although (85) represents a linear stiff equation of state, it is a nonlinear partial differential equation.

In the same manner in which we utilised the generalised transformation discussed in this paper, we use special cases of the transformation (27) to solve the system of equations consisting of (85) and the junction condition (23). We obtain three exact models in which solutions of the junction condition admit a linear stiff equation of state. The three models are listed in Table 2. Note that n in Table 2 is an arbitrary constant. The Restrictions column represents the conditions we placed on (27). The Transformation column represents the resulting new generalised transformation (27). The Spacetime column gives the explicit line element with forms of the potential functions that satisfy the junction condition and admit a linear equation of state. The existence of these exact solutions with an equation of state indicates that the generalised transformation (27) leads to physically acceptable models for radiating stars.

8 Discussion

We have introduced a generalised transformation relating the gravitational potential A, B and Y for a spherically symmetric relativistic fluid. In this transformation new general functions $\mathcal{F} = \mathcal{F}(r, t, A, Y)$ and $\mathcal{G} = \mathcal{G}(r, t, A, Y)$ appear. This *new* transformation, which has not been considered in previous investigations, reduces to the case of Ivanov [32], called the *horizon function* and Mahomed et al [28], called the *generalised horizon function*, in the relevant limits. The generalised transformation leads to a new form of the boundary condition at the surface of the relativistic radiating star. The boundary condition now also depends on the functions $\mathcal{F}$ and $\mathcal{G}$. This dependence allows us to obtain new solutions under certain conditions. We show that earlier results are contained in our treatment by placing appropriate restrictions on the parameters and choosing particular forms for $\mathcal{F}$ and $\mathcal{G}$. Solutions to the new generalised Riccati equation are also found by writing the new generalised equation as Bernoulli and linear equations. It should be pointed out that an advantage of our approach is that the gravitational potential B can be given explicitly in all models while the potentials A and Y satisfy constraint equations for which exact solutions exist. We also demonstrated that our approach permits reduction of order: the boundary condition can be written as a *first order equation* where all second order terms have been eliminated. Finally we showed that particular forms of our new generalised transformation simplifies the system of nonlinear partial differential equations consisting of an equation of state and boundary condition. We comment that reduction of order calculations for ordinary differential equations can be performed using Lie symmetries [45]. Lie symmetries also generate group invariant solutions under Lie groups [45]. Lie symmetries have been previously used to great effect in studying radiating relativistic stars [46–50]. We will study the relationship between our new generalised transformation and Lie group analysis in future work.

Table 2 Models satisfying the boundary condition that admit an equation of state

Model	Restrictions	Transformation	Spacetime
I	$\mathcal{F}=Y$ $\mathcal{G}=\beta=0$ $Y=\frac{1}{\sqrt{2}}\sqrt{\frac{\sqrt{1-4\Lambda Q^2}+1}{\Lambda}}$	$H=-\frac{\alpha Q Q'}{\sqrt{1-4\Lambda Q^2}B}$	$\mathrm{d}s^2=-\left(\frac{\left(t-\frac{\alpha\sqrt{1-4\Lambda Q^2}}{4\Lambda}\right)^{-n}}{\sqrt[4]{\sqrt{1-4\Lambda Q^2}+1}}\right)^2\mathrm{d}t^2+\left(\frac{\alpha Q Q'\left(t-\frac{\alpha\sqrt{1-4\Lambda Q^2}}{4\Lambda}\right)^{-n}}{\sqrt{1-4\Lambda Q^2}\sqrt[4]{\sqrt{1-4\Lambda Q^2}+1}}\right)^2\mathrm{d}r^2$ $+\left(\sqrt{\frac{\sqrt{1-4\Lambda Q^2}+1}{2\Lambda}}\right)^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)$
II	$\mathcal{F}=Y$ $\mathcal{G}=\beta=0$ $Y=\frac{\sqrt{\frac{1-\sqrt{1-4\Lambda Q^2}}{\Lambda}}}{\sqrt{2}}$	$H=\frac{\alpha Q Q'}{\sqrt{1-4\Lambda Q^2}B}$	$\mathrm{d}s^2=-\left(\frac{\sqrt[4]{\sqrt{1-4\Lambda Q^2}+1}\left(\frac{\alpha\sqrt{1-4\Lambda Q^2}}{4\Lambda}+t\right)^{-n}}{\sqrt{Q}}\right)^2\mathrm{d}t^2$ $+\left(\frac{\alpha\sqrt{Q}\sqrt[4]{\sqrt{1-4\Lambda Q^2}+1}Q'\left(\frac{\alpha\sqrt{1-4\Lambda Q^2}}{4\Lambda}+t\right)^{-n}}{\sqrt{1-4\Lambda Q^2}}\right)^2\mathrm{d}r^2$ $+\left(\frac{\sqrt{\frac{1-\sqrt{1-4\Lambda Q^2}}{\Lambda}}}{\sqrt{2}}\right)^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)$
III	$\mathcal{F}=Y$ $\mathcal{G}=\beta=\Lambda=0$ $Y=Q$	$H=\alpha\frac{QQ'}{B}$	$\mathrm{d}s^2=-\left(\frac{\left(t-\frac{1}{2}\alpha Q^2\right)^{-n}}{\sqrt{Q}}\right)^2\mathrm{d}t^2+\left(\alpha\sqrt{Q}Q'\left(t-\frac{1}{2}\alpha Q^2\right)^{-n}\right)^2\mathrm{d}r^2$ $+Q^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)$

Acknowledgements NN and KSG thank the National Research foundation for financial support. SDM acknowledges that this work is based upon research supported by the South African Research chair Initiative of the Department of Science and Technology and the National Research Council.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and the results can be verified from the information available.]

Funded by SCOAP^3.

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