

Single particle entanglement of a massive relativistic particle: Dirac bispinors and spin 1/2 states

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Abstract. In this paper we study the single particle entanglement induced by a Lorentz boost. We described the particle as a solution of the free Dirac equation, a Dirac bispinor, and compare the induced momentum-spin entanglement with the result obtained in the widely considered framework of relativistic spin 1/2 states. The spin linear entropy for both approaches agree in the ultra relativistic limit. We also verify that the spin-momentum entanglement differs from the spin entropy for the bispinorial case, indicating a true multipartite entanglement involving the degrees of freedom of a Dirac bispinor state: momentum, spin and intrinsic parity. The fact that Dirac bispinors belongs to an irreducible representation of the complete Lorentz group, which also included parity as a symmetry, is the ultimate reason behind such non-trivial structure.

1. Introduction

Quantum entanglement is a resource for performing computing tasks, engineering safe communication protocols and devising metrological procedures [1, 2]. The last two applications have a special intersection with relativistic aspects of quantum information theory, in particular regarding frame transformations and their effects on quantum correlations [3]. This is important, for instance, for the quantum GPS protocol [5] and clock synchronization [4], in which frame transformations are an important issue. Other fundamental questions, such as a proper definition of spin density matrix [6], have also been addressed in the scope of relativistic quantum information.

In the general context of relativistic quantum information, the correlation and information carriers are particle states, described via representations of the Lorentz or the Poincaré groups¹. The specific representation associated to a type of particle determines how its state transforms under the symmetries included in the group, with the most iconic example being the Lorentz

¹ Here and in the following what we refer to as Lorentz group is actually the restricted Lorentz group, not to be confused with the complete Lorentz group (see below).



Boosts, connecting two inertial frames moving with relative constant speed. With such transformation laws, one can address how quantum correlations, such as entanglement, encoded in pairs of particles (or in a single particle) transform under frame transformations, which is the main study ground of several works since the early 2000s [7, 8, 9, 10, 11, 12, 13, 14, 15].

A particular interest lies on spin $1/2$ particles, since those are prototypical carriers of a quantum bit encoded in the spin degree of freedom. The relevant irreducible representation (irrep) of the Poincaré group (or the Lorentz group if translations are not considered) usually adopted in many relativistic quantum information studies is obtained via the induced representation technique and it naturally includes spin as a degree of freedom associated with one the Casimir invariants of the group [16, 17]. This framework gave important insights to several scenarios, such as the aforementioned transformation properties of quantum correlations under boosts [7, 8] and the discussion of Bell's inequality [9, 10].

Despite the effectiveness of the description via the irrep of the Poincaré group, in the covariant formulation of relativistic quantum mechanics, spin $1/2$ particles are described via the solutions of the Dirac equation, the so-called Dirac bispinors. These are 4-component objects which, besides spin, also carry an additional discrete degree of freedom, the intrinsic parity, related with the inclusion of spatial parity as a symmetry [18, 19, 20]. Dirac bispinors were investigated in different relativistic quantum information scenarios. The transformation properties of bispinors under Lorentz boosts were described via Wigner rotations [21] and in the context of the Fouldy-Wouthuysen (FW) spin operator [22]. As in the case of the two-component irreps of the Poincaré group, bispinors were studied in the context Bell's inequalities [23] and, in particular, they play an important role in the discussion of a proper definition of spin and position operators in the context of relativistic quantum mechanics [24]. Still in the context of relativistic quantum information, bispinor structures were recently used as a basis to study relativistic chiral qubits [25]. In a different note, Dirac bispinors also effectively describe low energy excitation in some materials, which the most infamous example the particular graphene [26], which furthers motivates the study of such structures in an informational context [27].

In this brief contribution, we further discuss the differences and connection between the bispinor approach and the spin $1/2$ states framework for describing frame transformation effects on quantum correlations encoded in pairs of spin $1/2$ particles. First, we present a short review of the spin $1/2$ representation of the Lorentz and Poincaré groups and introduce the objects used in the study of frame transformation effects on correlations. We then compare both frameworks in a simple and illustrative example: we consider the action of a Lorentz boost in the state of a particle in a superposition of momenta, which induced spin-momentum entanglement [8, 12, 28]. We verify that both approaches lead to the same qualitative result for particles with small mass, which corroborates previous results obtained in the scenario of particle-particle entanglement [29, 30]. Moreover, we observe a difference between the spin linear entropy and the spin-momentum entanglement for the bispinor state, indicating a true multipartite entanglement among spin, momentum and intrinsic parity.

2. Spin $1/2$ representations of the Lorentz and Poincaré groups and the Dirac bispinors.

In this section we present a short review of the representations of the Lorentz and the Poincaré groups. For a detailed description of the subject, we suggest the textbooks on group theory by W. K. Tung [17] and by Fonda and Ghirardi [16].

In its seminal paper, Wigner stated that the physical states of a particles are represented by unitary representations of the Poincaré group [31], which includes the homogeneous Lorentz transformations and space-time translations. Before we talk about the representations of the Poincaré group, let's review the representations of the Lorentz group. Those are obtained by studying the generators of the Lorentz transformations in its defining representation, the

coordinate representation. The generators of Lorentz transformations can be divided into two types: boosts (connecting two inertial frames moving with relative constant speed), and rotations. Although the rotations by themselves form a closed group, the boosts do not, and in order to obtain a closed algebra for the generators, rotations and boosts need to be combined. The algebra thus obtained is identical to the Lie algebra associated to the group $SU(2) \otimes SU(2)$: the representations of the Lorentz group can be labeled by two numbers $j_{\pm} = 0, 1/2, 1, 3/2, \dots$ as (j_+, j_-) .

The first non-trivial irreps of the Lorentz group obtained as aforementioned are $(1/2, 0)$ and $(0, 1/2)$, the so called *spinor representations*. The objects transforming according with those representations are called *Weyl spinors*, that are well known for describing massless fermions. Although being irreps, the spinor representations are not invariant under parity transformation, which transforms one representation into the other. This is relevant when considering massive particles, and for such one needs to combine these representations to form a new one. The reducible representation $(1/2, 0) \oplus (0, 1/2)$ is then invariant also under parity, and the objects transforming according to it are the 4-component spinors, or bispinors. The solutions of the Dirac equation belong to this representation. Although the $(1/2, 0) \oplus (0, 1/2)$ is a reducible representation of the restricted Lorentz group, it is an irreducible representation of the *complete* Lorentz group, which also includes parity as a symmetry.

The Poincaré group includes the homogeneous Lorentz transformations plus space-time translations. Its unitary irreducible representations are obtained via the induced representation technique, which relies on using the Casimirs of the group to construct the irreps. For the Poincaré group, the first step is to choose the basis vector of the representations as eigenvectors of the generator of translations, the 4-momentum operator. The square of this operator is a Casimir of the Poincaré group and its eigenvalue, the mass, gives the first classification of the different types of representations. Then, on the subspace corresponding to a given 4-momentum, the components of the other Casimir of the group, the Pauli-Lubanski vector, form the (Lie) algebra of the so called little group of such momentum. Finally, a different irrep of the full Poincaré group can be derived for every irrep of the little group by successive application of homogeneous Lorentz transformation. The irreps obtained by this induced representation technique are classified by the eigenvalue of the two Casimir of the groups: the mass (related to the translations) and the eigenvalue of the Pauli-Lubanski operator, which is directly related to the spin of the particle.

The characteristics of a given irrep of the Poincaré group depends strongly on the specific type of the 4-momentum associated to it: time-like, space-like or light-like. For massive particles, the associated little group is the rotation group $SO(3)$, which in turns is isomorphic to the $SU(2)$, through which one can then construct the irrep describing massive particles with spin $1/2$. For massless particles it can be shown that when the mass tends to zero, the original little group $SO(3)$ selects single and double valued representations of a restricted little group, the $SO(2)$, which is associated with the fact the massless particles have definite helicity.

To sum up, we have two distinct objects that potentially describe massive spin $1/2$ particles. On one hand, the Dirac bispinors, belonging to the irreducible representation of the complete Lorentz group, which are physically related to the solutions of a covariant wave equation, the Dirac equation. On the other hand, the irreducible representations of the Poincaré group, obtained via the irreducible representation technique and described via a little group, which is isomorphic to the usual $SU(2)$, the group widely used to describe quantum bits [2], for example via two-level systems.

3. Quantum entanglement of a single particle under a boost - Dirac bispinors and the irreps of the Poincaré group

We discuss now the implications of using the different representations described in the previous section for a widely considered quantum information scenario, the transformation of quantum superpositions under Lorentz boosts and the induced change on the quantum correlations encoded on them. Specifically, we consider a simplified and illustrative example: a single spin 1/2 particle, with spin polarized along the \mathbf{e}_z direction is in a superposition of opposite momenta with respect to a reference frame \mathcal{S} . A Lorentz boost with rapidity ω is then performed in the \mathbf{e}_x direction, describing the frame transformation to another observer \mathcal{S}' , as depicted in Fig. 1. In this section we use units $\hbar = c = 1$.

In our considered scenario, the state of the particle with respect to \mathcal{S} is described in the framework of spin 1/2 states as

$$|\phi\rangle = \frac{|\mathbf{p}, \uparrow\rangle + |-\mathbf{p}, \uparrow\rangle}{\sqrt{2}}, \quad (1)$$

where $|\mathbf{p}, \uparrow\rangle \equiv |\mathbf{p}\rangle \otimes |\uparrow\rangle$ and we designate the eigenstates of the operator $\hat{\sigma}_z$ by $|\uparrow\rangle$ ($|\downarrow\rangle$). We consider a simple toy model, adopted for example in [12, 13, 15, 29, 30], in which the momentum has sharply peaked distribution such that we can assume $\langle \mathbf{p} | -\mathbf{p} \rangle = 0$ and $\langle \mathbf{p} | \mathbf{p} \rangle = \langle -\mathbf{p} | -\mathbf{p} \rangle = 1$. At the bispinor level, the state of the particle is described as

$$|\psi\rangle = \frac{|\mathbf{p}, u_\uparrow(\mathbf{p})\rangle + |-\mathbf{p}, u_\uparrow(-\mathbf{p})\rangle}{\sqrt{2}}, \quad (2)$$

where $|\mathbf{p}, u_\uparrow(\mathbf{p})\rangle \equiv |\mathbf{p}\rangle \otimes |u_\uparrow(\mathbf{p})\rangle$. Here $|u_\uparrow(\mathbf{p})\rangle$ is the positive energy solution of the free Dirac equation (in the momentum space) with spin polarized along the \mathbf{e}_z direction [32]. Since, with respect to \mathcal{S} we are assuming $\mathbf{p} = p\mathbf{e}_z$, we can write the bispinor $|u_\uparrow(\mathbf{p})\rangle$ [20, 29, 30, 32] as

$$|u_\uparrow(p)\rangle = \sqrt{\frac{E_p + m}{2E_p}} \left[|+\rangle_P + \frac{p}{E_p + m} |-\rangle_P \right] \otimes |\uparrow\rangle_S, \quad (3)$$

with $E_p = \sqrt{p^2 + m^2}$, m the particle mass and $p = \sqrt{\mathbf{p} \cdot \mathbf{p}}$. In this form of writing the bispinor we make explicitly the two discrete degrees of freedom carried by a Dirac particle as a consequence of it being described as an irrep of the complete Lorentz group, as discussed in the previous section. The state in Eq. (3) belongs to a composite Hilbert space $\mathcal{H}_P \otimes \mathcal{H}_S$ with $\dim[\mathcal{H}_P] = \dim[\mathcal{H}_S] = 2$. While \mathcal{H}_S is related to the spin, the other Hilbert space \mathcal{H}_P describes the intrinsic parity degree of freedom arising from the invariance under parity of the Dirac equation. A Dirac particle can then be understood as carrying two quantum bits: spin and intrinsic parity [18, 19, 33]. Here we are using the notation $|\cdot\rangle_{S(P)} \in \mathcal{H}_{S(P)}$ to indicate states belonging to the spin(intrinsic parity) spaces, $\hat{O}^{(X)}$ indicates an operator acting on the $X = S, P$ Hilbert space and $\hat{\sigma}_z^{(P)} |\pm\rangle_P = \pm |\pm\rangle_P$.

Under a Lorentz Boost in the \mathbf{e}_x direction, the spin 1/2 state (1) transforms as [15, 34]

$$|\phi\rangle \rightarrow |\phi'\rangle = \frac{1}{\sqrt{2}} \left[|\mathbf{p}'\rangle \otimes \left(\hat{D}[\omega, \mathbf{p}] |\uparrow\rangle \right) + |-\mathbf{p}'\rangle \otimes \left(\hat{D}[\omega, -\mathbf{p}] |\uparrow\rangle \right) \right], \quad (4)$$

where \mathbf{p}' indicates the momentum with respect to \mathcal{S}' , obtained via a usual Boost transformation. Again we assume that $\langle \mathbf{p}' | \mathbf{p}' \rangle = \langle -\mathbf{p}' | -\mathbf{p}' \rangle = 1$ and $\langle \mathbf{p}' | -\mathbf{p}' \rangle = 0$. Within this framework momentum will be described as a dichotomic variable, and its associated Hilbert space as two dimensional. For the considered geometry, the operator $\hat{D}[\omega, \mathbf{p}]$ describing the spin rotation induced by the boost, the so called Wigner rotation, is given by [15]

$$\hat{D}[\omega, \mathbf{p}] = \cos\left(\frac{\Omega_p}{2}\right) + i\mathbf{e}_W \cdot \hat{\boldsymbol{\sigma}} \sin\left(\frac{\Omega_p}{2}\right),$$

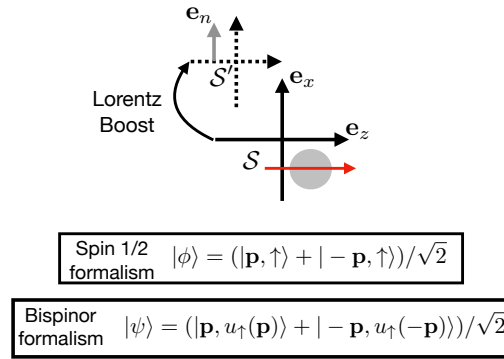


Figure 1. The framework considered. A single spin 1/2 particle polarized along the e_z direction is in a superposition of opposite momenta. A Lorentz boost along the e_x direction will induce a change on the superposition, generating quantum correlations between the different degrees of freedom carried by the particle. We compare the scenarios for the spin 1/2 states (the state belonging to a irrep of the Poincaré group) and for the bispinor frameworks (the discrete degrees of freedom described via free particle solutions of the Dirac equation).

$$\cos\left(\frac{\Omega_p}{2}\right) = \sqrt{2} \frac{\cosh\left(\frac{\omega}{2}\right) \cosh\left(\frac{\omega_0}{2}\right)}{\sqrt{1 + \cosh(\omega) \cosh(\omega_0)}}, \quad (5)$$

$$e_W \sin\left(\frac{\Omega_p}{2}\right) = \sqrt{2} \frac{\sinh\left(\frac{\omega}{2}\right) \sinh\left(\frac{\omega_0}{2}\right) \mathbf{e}_x \times \mathbf{p}/p}{\sqrt{1 + \cosh(\omega) \cosh(\omega_0)}},$$

with ω the boost rapidity and $\omega_0 = \text{arccosh}\left[\frac{E_p}{m}\right]$ the particle rapidity with respect to the unboosted frame \mathcal{S} . Since the spin rotation depends on the direction of the unboosted momentum, one concludes that $|\phi'\rangle$ will not be separable, although $|\phi\rangle$ is. Therefore, a boost induces spin-momentum entanglement, a well known result in relativistic quantum information, see for example [8, 12, 28].

The state at the bispinor level transforms as [32, 34]

$$|\psi\rangle \rightarrow |\psi'\rangle = \frac{1}{\sqrt{2}} \left[|\mathbf{p}'\rangle \otimes \left(\hat{S}[\omega] |u_{\uparrow}(\Lambda^{-1}\mathbf{p})\rangle \right) + |-\mathbf{p}'\rangle \otimes \left(\hat{S}[\omega] |u_{\uparrow}(-\Lambda^{-1}\mathbf{p})\rangle \right) \right], \quad (6)$$

where the boost operator $\hat{S}[\omega]$ is given by [20, 29, 30]

$$\hat{S}[\omega] = \frac{1}{\sqrt{\cosh(\omega)}} \left[\cosh\left(\frac{\omega}{2}\right) \hat{I}^{(P)} \otimes \hat{I}^{(S)} + \sinh\left(\frac{\omega}{2}\right) \hat{\sigma}_x^{(P)} \otimes \hat{\sigma}_x^{(S)} \right]. \quad (7)$$

As in the spin 1/2 case, the boost will change the superposition, inducing non-trivial correlations. Since the bispinor state has an additional degree of freedom, the intrinsic parity, the obtained spin-momentum correlation in \mathcal{S}' will, in general, be different from the one obtained in the spin 1/2 case.

With the transformed states $|\phi'\rangle$ and $|\psi'\rangle$ we can construct the density matrices $\rho_{\phi'} = |\phi'\rangle\langle\phi'|$ and $\rho_{\psi'} = |\psi'\rangle\langle\psi'|$ through which we can study the correlation encoded between the different degrees of freedom of the particle. For the first case, $\rho_{\phi'}$, the particle has only the momentum and the spin degrees of freedom. Since $\rho_{\phi'}$ is always pure and describes a bipartite system, the amount of spin-momentum entanglement can be calculated via the linear spin entropy [1]

$$E_{\text{Spin}}[\rho_{\phi'}] = 2 \left(1 - \text{Tr} \left[\left(\rho_{\phi'}^{(S)} \right)^2 \right] \right), \quad (8)$$

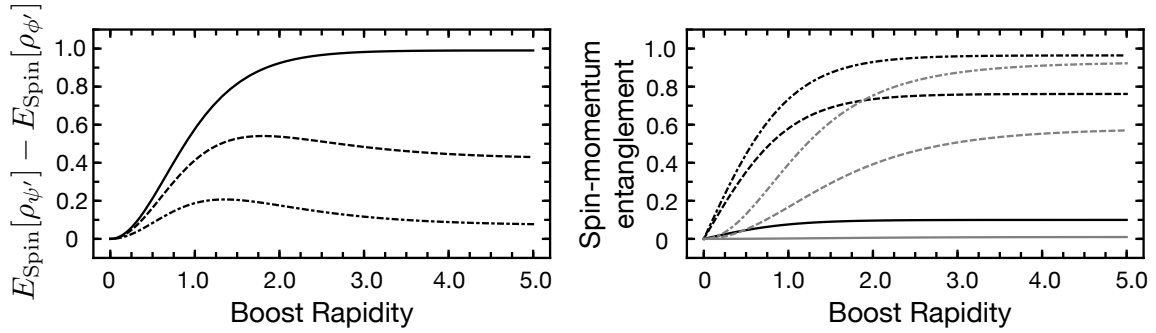


Figure 2. Left: difference between the spin linear entropies $E_{\text{Spin}}[\rho_{\psi'}] - E_{\text{Spin}}[\rho_{\phi'}]$ between the boosted bispinor state (6) and the boosted spin 1/2 state (4). Right: Spin-momentum entanglement for the boosted bispinor state (black curves) and for the boosted spin 1/2 state (grey curves). Both plots are as function of the boost rapidity ω and for different values of the unboosted rapidity $\omega_0 = \text{arccosh}(E_p/m)$: 0.1 (solid), 1.0 (dashed) and 2.0 (dot-dashed).

where $\rho_{\phi'}^{(S)}$ is the density matrix reduced to the spin degree of freedom: $\rho_{\phi'}^{(S)} = \text{Tr}_{\text{momentum}}[\rho_{\phi'}]$.

The linear spin entropy for the bispinor case is $E_{\text{Spin}}[\rho_{\psi'}] = 2 \left(1 - \text{Tr} \left[\left(\rho_{\psi'}^{(S)} \right)^2 \right] \right)$, where now $\rho_{\psi'}^{(S)}$ is obtained by tracing $\rho_{\psi'}$ over both momentum and intrinsic parity. In the bispinor case $E_{\text{Spin}}[\rho_{\psi'}]$ quantifies the entanglement between the spin and the joint momentum-intrinsic parity degree of freedom. To quantify the amount of spin-momentum entanglement in the boosted bispinor state, we need to consider the density matrix obtained by tracing out the intrinsic parity degree of freedom $\rho_{\psi'}^{(\text{mom.},S)} = \text{Tr}_{\text{Parity}}[\rho_{\psi'}]$. Since all the DoFs of $\rho_{\psi'}$ are in general entangled, $\rho_{\psi'}^{(\text{mom.},S)}$ is a mixed state and to quantify the entanglement between spin and momentum we use the so called negativity [1, 35, 36], defined via the Peres separability criterion [35] and which involves partially transposing the density matrix and evaluating its eigenvalues [36].

In Figure 2 we show in the left plot the difference between spins linear entropies $E_{\text{Spin}}[\rho_{\psi'}] - E_{\text{Spin}}[\rho_{\phi'}]$ and in the right plot the spin-momentum entanglement encoded in $\rho_{\psi'}$ and in $\rho_{\phi'}$ as a function of the boost rapidity ω for several values of the unboosted rapidity ω_0 . The spin entropies tend to the same value irrespective of the boost rapidity for states with large ω_0 , that is states with $m \ll E_p$. This is due to the fact that in the massless limit, Dirac bispinors have definite chirality which is associated to the factorization of the two irreps of the Lorentz group that compose a bispinor [16, 17]. This is also observed in the spin-momentum entanglement shown in plot (b). The difference on both quantities is due to the intrinsic parity degree of freedom. Interestingly, spin and intrinsic parity are always separable for $\rho_{\psi'}$, so the difference between the spin linear entropy and the spin-momentum entanglement depicted on the plots of Fig. 2 is due to some true multipartite entanglement among spin, momentum and intrinsic parity.

The result obtained in this simple scenario is in agreement with our previous discussions relating both frameworks. In a previous work, we considered a pair of particles in an entangled state and we have shown that the different approaches give the same results for the global amount of entanglement encoded in the states provided that the particles have small mass [30]. Also in this limit and for the same two particles scenarios, the spin-momentum entanglement obtained with the bispinor approach has the same qualitative behaviour under boosts as the one obtained in the context of spin 1/2 states [29]. Finally, the action of the boost operator (7) can

also be given in terms of a rotation. Entangled bispinor states can be generated via quantum electrodynamical processes [37].

4. Conclusions

In conclusion, in this paper we discussed a simple scenario to further emphasize the differences in two approaches for describing spin 1/2 particles in relativistic quantum information scenarios: one based on the irreducible representations of the Poincaré group and the other in terms of Dirac bispinors. We considered a spin 1/2 particle in a momentum superposition and studied the effects of a Lorentz boost on the superposition and the amount of entanglement generated by it. We verified that spin linear entropy, quantifying the amount of entanglement between the spin degree of freedom and all the other degrees of freedom of the state, obtained in both approaches is the same for states with small mass. This is related to the fact, discussed also in previous works, that in the massless limit bispinors have definite chirality and can be described via two component objects. Additionally, we notice a difference between the spin linear entropy and the spin-momentum entanglement which, since intrinsic parity is always separable from spin for the considered state, can be attributed to a true multipartite entanglement among spin, momentum and intrinsic parity. Such multipartite entanglement can be quantified, for example, with a geometric measurement as proposed in [38]. Although we worked in a simplified framework, treating momentum as a discrete degree of freedom which corresponds to sharp momentum distributions, it is interesting to also compare the frameworks when the particle has a gaussian momentum distribution [8]. More interesting, given that bispinors are the natural language to describe scattering and creation/annihilation processes in quantum electrodynamics [39], thus the formalism is the basis to future studies of quantum information generation and transformation in such processes [40].

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