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BOUNDS ON SCATTERING AMPLITUDES*

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ERRATUM

Page	Line	For	Change to
13	23	$\lambda_\gamma g_\alpha$	$\lambda_\alpha g_\alpha$
13	24	$\lambda_\gamma \geq 0$	$\lambda_\alpha \geq 0$
16	1	has already	has already been
23	7	$(P_\ell(z) - g P_\ell(w) P_\ell(w)$	$(P_\ell(z) - g P_\ell(w)) P_\ell(w)$
36	7	ℓn^2	ℓn^4
39	3	$r(s')$	$q(s')$
39	7	determined by	determined but
39	22	$L(s, s)$	$L(s', s)$
41	13	R	Q
41	13	$r(s')$	$q(s')$
42	2	R	Q
42	19	c	C
52	13	add	"Phys. Rev., to be published."

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If it is true, it can be proved. - M. L. Goldberger

I do not know. - J. L. Lagrange

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*Work supported by the U. S. Atomic Energy Commission.

ABSTRACT

The method of Lagrange multipliers, generalized to include inequality constraints, is used to derive various bounds on scattering amplitudes based on unitarity and polynomial behavior. Various physical quantities associated with two body scattering processes are treated, such as the total cross section, elastic cross section, absorptive part and real part of the scattering amplitude. Using these new mathematical techniques, several familiar problems are treated and generalized. In addition, a number of new results are presented. Both asymptotically large energy bounds and finite energy bounds with no arbitrary constants will be given.

I. INTRODUCTION

For several years, the hope has persisted that the general principles of Lorentz invariance, unitarity, and maximal analyticity of scattering amplitudes would determine most if not all the properties of the strong interactions, if, in addition, the scale of masses is given. [1] Usually, these general principles have been supplemented by additional assumptions about internal symmetries, maximal analyticity in angular momentum, and asymptotic behavior. [2] Although the phenomenological applications of this so-called S-matrix approach have been many, so far, no explicit example has been given of a model consistent with all three general principles. Consequently, in the absence of a theory, it is interesting to explore the consequences of various subsets of the general assumptions. For example, recently much attention has been given to models [3] which are Lorentz invariant and crossing symmetric but which violate unitarity. A complementary area of investigation explores the consequences of Lorentz invariance and unitarity as well as rather weak assumptions about analyticity, and it is to this subject that the current investigation is devoted.

Beginning with the classic work by Froissart, [4] it was realized that unitarity severely limited the high energy behavior of scattering amplitudes. Subsequently, a great many upper and lower bounds [5] have been derived on various quantities of physical interest, such as total cross sections, elastic cross sections, diffraction peaks, the difference between particle and antiparticle cross sections, and so forth. In general, the results obtained, however interesting, have been more useful as limitations on theoretical models than in phenomenological applications. One of the frustrations associated with this work is that very often the methods used to solve one problem are peculiar to that particular problem and not easily transferable to the next problem. A second deficiency has been that, having obtained an upper or lower bound, one seldom knows whether it is the least upper bound or greatest lower bound under the stated assumptions. To a significant degree, the methods developed in this paper relieve both of these frustrations. It is hoped that the use of these powerful mathematical techniques will allow more physical input into these types of problems.

We hope that the only prerequisite for understanding this paper is an elementary working knowledge of the method of Lagrange multipliers for finding maxima and minima.[6] To put the present discussion into a physical context, however, it would be useful to have some familiarity with the general results on this subject, as presented for example, in the book by Eden.[5] In the interest of brevity, we will omit any motivation for the introduction of Lagrange multipliers and their previous applications in physics.[7]

Our mathematical presentation will be based almost entirely upon the very elegant discussion by M. R. Hestenes.[8] However, in the interest of simplicity, some definitions and the theorems, although stated correctly, have not been given in complete generality. We highly recommend Chapters 1 and 5 of H to those physicists interested in a detailed mathematical discussion.

The outline of the paper is as follows: Section II contains the mathematical basis for the techniques used to solve problems in subsequent sections. As a pedagogical recommendation, we suggest that the reader skim over this section to try to absorb the general idea and then to proceed to the examples. He should however pay close attention to the discussions of the significance of multipliers which follow Theorems 1 and 2 and to the example following Theorem 3. He might later return to the mathematical section after seeing how the theorems are used in several examples.

Sections III through V describe applications of the method to particular problems. Sections III and IV contain applications at a fixed energy. Bounds are given for the imaginary part, the real part, and combinations thereof. Section V contains applications to averages over a finite energy range. These latter examples involve straightforward generalizations of the theorems given in Section II.

II. MATHEMATICAL PRELIMINARIES

A.

The notion of the tangent line to a curve or the tangent plane to a surface is very familiar. For the subsequent discussion, however, it is essential that these notions be generalized to the tangent cone of a set [9] S at a point $x_0 \in S$. The first step is the concept of directional convergence. In S let $\{x_i\}$ be a sequence of points converging to a point x_0 (but $x_i \neq x_0$). We will say $\{x_i\}$ converges to x_0 in the direction of the unit vector h if

$$\lim_{i \to \infty} \frac{x_i - x_0}{|x_i - x_0|} = h .$$

More geometrically, if x(t) is any curve in S terminating at x_0, the half-line tangent to the curve is generated by the unit vector h. The collection of all such half-lines originating at x_0 is called the <u>tangent cone</u> C to S at x_0. For example, if S is the surface of a sphere, the tangent cone at a point x_0 on this surface is the familiar tangent plane. The unit vectors h which generate this plane sweep out a unit disc tangent to the surface. They satisfy $(x_0, h) = 0$. If S is the surface and interior of a ball (closed ball), the generators h of the tangent cone to a point x_0 on the surface sweep out a hemisphere satisfying $(x_0, h) \leq 0$. The generators of the tangent cone at a point <u>inside</u> the ball sweep out a unit sphere. Briefly, the tangent cone C of S at x_0 is the collection of all tangent lines to all curves in S which terminated at x_0.

To discuss the maxima and minima of functions, it is necessary first to introduce some notations. Given a function f(x), we denote its gradient vector at x_0 by $f'(x_0)$. Given any vector v, the linear functional $f'(x_0, v) = (f'(x_0), v)$ will be called the first differential of f. If the vector v or point x_0 is incidental, we may suppress them and abbreviate the first differential by the notation [10] δf. Similarly, the second differential of f at x_0 is defined by the quadratic form $f''(x_0, v) = v_i \dfrac{\partial^2 f(x_0)}{\partial x_i \partial x_j} v_j$.

Let us first restate in this language and notation two familiar theorems on Lagrange multipliers. Suppose we wish to maximize [11] a function f subject to a number of (equality) constraints among the variables of the form

$$f_\alpha(x) = 0 \qquad \alpha = 1, \ldots, r$$

In the following, S will denote the subset of $\mathscr{E}^n$ satisfying these constraints. Let $x_0 \in S$ and let C denote the tangent cone of S at x_0. Then one can easily show that

$$f'_\alpha(x_0, h) = 0 \qquad \alpha = 1, \ldots, r \tag{1}$$

$(r \leq n)$ for all vectors h in C. If, conversely, every vector h satisfying these equations (1) lies in the tangent cone C, then x_0 is called a <u>regular point</u> of S. Thus, for a regular point, the tangent cone is uniquely defined by the linear homogeneous equations (1). If the gradients $f'_\alpha(x_0)$ are all linearly independent, then x_0 is called a <u>normal point</u>. One can prove that every normal point is a regular point. The fundamental theorem on the existence of Lagrange multipliers characterizes those local maxima which are regular points.

<u>THEOREM 1</u>

Suppose x_0 is a regular point of S and suppose x_0 is a local maximum of f on S.

(a) Then there exist multipliers μ_α such that the auxiliary function

$$\mathscr{L} = f + \mu_\alpha f_\alpha$$

has a vanishing gradient,

$$\mathscr{L}'(x_0) = 0$$

(b) Moreover, the second differential is nonpositive,

$$\mathscr{L}''(x_0, h) \leq 0,$$

for all h in the tangent cone C of S at x_0.

(c) If x_0 is normal, the multipliers are unique.

The tangent cone plays an essential role in the statement of the theorem in two respects: (1) Only for a regular point do the conditions $f'_\alpha(x_0, h) = 0$ uniquely determine the tangent cone, and this appears to be essential to the proof of the theorem. (2) The second differential need have a definite sign only when restricted to vectors in the tangent cone.

The usefulness of the theorem is that it almost reduces the problem of maximizing a function f on a space S defined by a number of constraints f_α to the problem of maximizing an auxiliary function[12] $\mathscr{L}$ on the whole space $\mathscr{E}^n$. The way in which this theorem is used in practice is as follows: One solves the n gradient equations, $\mathscr{L}'(x_0) = 0$, for x_0 as a function of the unknown multipliers μ_α. These solutions $x_0 = x_0(\mu_\alpha)$ are inserted into the constraint functions $f_\alpha(x_0)$ and the multipliers are chosen to satisfy the constraint conditions $f_\alpha(x_0) = 0$. The solutions are limited to those for which $\mathscr{L}''(x_0, h) \leq 0$. Thus one arrives at a number of candidates $x_0(\mu_\alpha)$ (called critical points) for local maxima. The theorem assures one that all local maxima which are regular points will be among these critical points. Unfortunately, the theorem does not say which of these critical points will be local maxima and which will only be saddle points.

However, those points for which $\mathscr{L}'(x_0, h)$ is strictly negative will in fact be local maxima. Actually, this conclusion, as given in the next theorem, holds even if x_0 is not a regular point.

THEOREM 2

Suppose there exists an auxiliary function of the form $\mathscr{L} = f + \mu_\alpha f_\alpha$ with multipliers μ_α such that its gradient vanishes at a point x_0, i.e.,

$$\mathscr{L}'(x_0) = 0$$

Suppose, furthermore, that the second differential is strictly negative

$$\mathscr{L}''(x_0, h) < 0$$

for all $h \neq 0$ in the tangent cone C of S at x_0. Then x_0 is a local maximum of f.

Before proceeding to the generalization of the above to inequality constraints, let us discuss an interesting interpretation of the multipliers. To be specific, suppose we wish to maximize function f(x) subject to a constraint of the form $f_1(x) = a - b(x) = 0$. Let x_0 be a local maximum. Then according to Theorem 1, there is a multiplier μ such that the auxiliary function

$$\mathscr{L} = f + \mu(a - b)$$

has a vanishing derivative,

$$\mathscr{L}'(x_0) = f'(x_0) - \mu b'(x_0) = 0 . \qquad (2)$$

Since $b(x_0) = a$ the point x_0 depends implicitly on a, and hence, the maximum value $f(x_0)$ also depends implicitly on a. We should like to investigate just how the maximum value changes for small change of a. By the chain rule

$$\frac{df(x_0)}{da} = \left(f'(x_0), \frac{dx_0}{da} \right) .$$

Since $f'(x_0) = \mu b'(x_0)$ by Eq. (2), it follows that

$$\frac{df(x_0)}{da} = \mu \left(b'(x_0), \frac{dx_0}{da} \right) = \mu \frac{db(x_0)}{da} \quad .$$

However, $\frac{d}{da} b(x_0) = \frac{da}{da} = 1$, so finally we arrive at

$$\mu = \frac{df(x_0)}{da} \tag{3}$$

The multiplier μ is precisely the rate of change of the maximum value of f with respect to a, which is the magnitude of the constraint function b(x). [13] This result may only be a mathematical curiousity at first sight, but in physical applications it can be very useful. Often, on intuitive grounds alone, one knows whether an increase in the magnitude of some constraint will increase or decrease the magnitude of the maximum of f and, hence, the sign of the multiplier can be immediately determined. In many circumstances one will be able to guess the order of magnitude of such changes and, therefore, estimate μ even before beginning to solve the gradient equations. In any case, Eq. (3) is the basis for the physical interpretation of the multiplier in the solution to any given problem.

There is another more general way to state this result. Let x_0 be a local maximum of f(x) subject to the constraint $f_1(x) = 0$. As before define the multiplier μ and the auxiliary function

$$\mathscr{L} = f + \mu f_1 \quad .$$

How does the maximum value $f(x_0)$ change with an infinitesimal variation δf_1 in the function f_1? Proceeding as above, one finds at x_0

$$\mu = - \frac{\delta f}{\delta f_1} \quad .$$

In the general case considered by Theorem 1, one has

$$\mu_\alpha = - \frac{\delta f}{\delta f_\alpha}$$

where it is to be understood that, during the variation of a given constraint function f_α, all other constraints are held fixed.

B.

The elegance and usefulness of the concept of tangent cone and of the interpretation of the multipliers given above will now be demonstrated by the generalization of these theorems to include inequality constraints. Consider the problem of maximizing a function f subject to a number of inequality constraints g_α which we write in the form

$$g_\alpha(x) \geq 0 \qquad \alpha = 1, 2, \ldots, q \tag{4}$$

In addition, there may also be a number of equality constraints

$$f_\alpha(x) = 0 \qquad \alpha = 1, \ldots, r \tag{5}$$

Let S denote the set of points satisfying both sets of constraints (4) and (5). Consider any point $x_0 \in S$. If $g_\gamma(x_0) > 0$ for some γ, then $g_\gamma(x) > 0$ for all x in some neighborhood $N(x_0)$ of x_0. But then, in this neighborhood, the set S is determined independently of the constraint g_γ. Hence, for studying local properties around x_0, we may ignore any constraints g_γ for which $g_\gamma(x_0) > 0$. This crucial observation essentially reduces the study of inequality constraints to equality constraints. Given a point $x_0 \in S$, it is convenient to divide up the inequality constraints into two sets

$$I_{x_0} = \{\alpha | g_\alpha(x_0) > 0\}$$

$$B_{x_0} = \{\alpha | g_\alpha(x_0) = 0\}.$$

Intuitively, one may think of I_{x_0} as those inequality constraints for which x_0 is in the Interior, whereas B_{x_0} is the collection of constraints for which x_0 is on the Boundary. One can easily show that

$$g'_\alpha(x_0, h) \geq 0 \qquad \alpha \in B_{x_0}$$

$$f'_\alpha(x_0, h) = 0 \qquad \alpha = 1, \ldots, r \tag{6}$$

for all h in the tangent cone C of S at x_0. As before, if the converse is true and every vector h satisfying these conditions is in C, then x_0 will be called a <u>regular point</u>. A point x_0 will be called <u>normal</u> if all the gradients

$$g'_\alpha \qquad \alpha \in B_{x_0}$$

$$f'_\alpha \qquad \alpha = 1, \ldots, r,$$

are linearly independent. It can be shown that a normal point is a regular point.

The analogue of Theorem 1 will now be given:

<u>THEOREM 3</u>

Let x_0 be a regular point of S and suppose x_0 is a local maximum of f on S.

(a) Then there exists multipliers $\lambda_\alpha \geq 0$, μ_α such that the auxiliary function

$$\mathscr{L} = f + \lambda_\alpha g_\alpha + \mu_\alpha f_\alpha$$

has a vanishing gradient

$$\mathscr{L}'(x_0) = 0 .$$

Furthermore, if $\alpha \in I_{x_0}$, we may choose $\lambda_\alpha = 0$. (This is simply a repetition of the statement that we may ignore any inequality constraint g_α for which $g_\alpha(x_0) > 0$.)

(b) Let Γ denote the set of indices γ for which $\lambda_\gamma > 0$, and let S_1 be the subset of S defined by

$$g_\gamma(x) = 0.$$

Then the second differential is nonpositive

$$\mathscr{L}''(x_0, h) \leq 0$$

for every h in the tangent cone C_1 of S_1 at x_0.

(c) If x_0 is normal, the multipliers are unique.

It is interesting that, in part (a) of the theorem, the existence of multipliers is supplemented by the statement that, for the inequality constraints, their sign is known ($\lambda_\gamma \geq 0$), something which cannot be said for equality constraints. One can easily show that the earlier result, which shows that the multiplier is the rate of change of the maximum with respect to the constraint, also holds for inequality constraints. Thus if we write $g_\alpha(x) = a_\alpha - b_\alpha(x) \geq 0$, we have

$$\lambda_\alpha = \frac{\partial f}{\partial a_\alpha}$$

Since increasing the upper bound a_α cannot decrease the maximum value, it follows that $\lambda_\alpha \geq 0$. Alternatively, one may vary the function g_α rather than its value to obtain

$$\lambda_\alpha = -\frac{\delta f}{\delta g_\alpha} \tag{7}$$

for <u>all</u> α. In particular, if $g_\gamma(x_0) > 0$, then clearly the maximum value $f(x_0)$ is independent of infinitesimal variations of the function g_γ and, hence, $\lambda_\gamma = 0$.

Perhaps it is worthwhile emphasizing this result by considering an example in one-dimension. Let a regular point x_0 be a local maximum of f(x) subject to the constraint $g(x) \geq 0$, and suppose x_0 is at an endpoint so that $g(x_0) = 0$. The constraint breaks up the real line into admissible subsets, e.g., $x \geq 0$ or $\sin \omega x \geq 0$. (See Fig. 1.) The tangent cone at x_0 is the half-line from x_0 to $+\infty$ (or to $-\infty$) when x_0 is approached in S from the right (or from the left). A unit vector h in C satisfies $g'(x_0, h) = hg'(x_0) \geq 0$, and $h = +1$ (-1) if x_0 is approached from the right (left). The assumption that x_0 is regular implies that $g'(x_0) \neq 0$. Since x_0 is assumed to be a local maximum of f, we have $f'(x_0, h) = hf'(x_0) \leq 0$. According to Theorem 3, there exists a multiplier λ such that the auxiliary function

$$\mathscr{L} = f + \lambda g$$

has zero derivative

$$\mathscr{L}' = f' + \lambda g' = 0.$$

Hence

$$\lambda = \frac{-f'(x_0)}{g'(x_0)} = \frac{-hf'(x_0)}{hg'(x_0)} \geq 0.$$

This simply means that,if f is rising as the boundary is approached, a positive quantity must be added to form an auxiliary function which has a zero slope at the boundary. Note the correspondence between this result and our general result (3). It is quite remarkable that the methods of differential calculus can be extended to end points [14]. Part (b) of the theorem tells us that, for those multipliers λ_γ for which $\lambda_\gamma > 0$, the corresponding inequality constraints may be treated essentially as equality constraints. In particular, if x_0 is also a regular point of S_1 (which follows if x_0 is normal), then C_1 is the set of all vectors in the tangent cone C of S at x_0 for which

$$g'_\gamma(x_0, h) = 0 \qquad (\gamma \in \Gamma)$$

Finally, there is a simple generalization of Theorem 2:

<u>THEOREM 4</u>

Suppose there exists an auxiliary function $\mathscr{L} = f + \lambda_\gamma g_\alpha + \mu_\alpha f_\alpha$ with multipliers $\lambda_\gamma \geq 0$, and μ_α such that

$$\mathscr{L}'(x_0) = 0.$$

As before, let Γ be the set of indices γ for which $\lambda_\gamma > 0$, and suppose that

$$\mathscr{L}''(x_0, h) < 0$$

for all $h \neq 0$ in the subset of tangent cone C of S at x_0 for which

$$g'_\gamma(x_0, h) = 0 \qquad (\gamma \in \Gamma)$$

also holds. Then x_0 is a local maximum.

Once again, we see in Theorem 4 that an inequality constraint for which the corresponding multiplier is positive may be treated essentially as an equality constraint.

The method by which Theorems 3 and 4 used to solve problems in practice is precisely the same procedure described after Theorem 1 for equality constraints. Perhaps the best way to illustrate this is to consider some physical examples.

III. BOUNDS ON THE ABSORPTIVE PART

A.

As a first application of the methods of Lagrange multipliers described above, we will rederive two results originally given by Singh and Roy[15]. In presenting these derivations, a notation will be established for the remainder of the paper. We will be quite explicit, perhaps even verbose, in our construction of the solution. Subsequent applications will be considerably abbreviated but these first examples are very helpful in understanding the mathematics involved. First, let us consider the problem of finding the maximum possible value of the absorptive part of an amplitude at a fixed angle which is consistent with a given value of the total cross section. The only additional constraint is that of unitarity. In particular, no assumptions of analyticity in momentum transfer will be made. It will be assumed, of course, that the partial wave expansion converges in the physical region. The usual Mandelstam variables will be used to describe the scattering of scalar, unit mass particles.

In mathematical terms, our problem is to maximize

$$A(s,t) = \sum(2\ell + 1)\, a_\ell P_\ell(z)$$

if the total cross section σ_T is given,

$$A_0 \equiv A(s,0) = \left(\frac{s-4}{16\pi}\right)\sigma_T = \sum(2\ell + 1)\, a_\ell$$

The unitarity condition is

$$u_\ell \equiv a_\ell - a_\ell^2 - r_\ell^2 \geq 0, \qquad (8)$$

where r_ℓ and a_ℓ are the real and imaginary parts of the partial wave amplitude. Since we will work at fixed s, this variable will be suppressed. Only the maximization problem will be treated here, but the minimization problem can be carried through in a similar manner by changing the signs of the multipliers.

To solve this problem, the auxiliary function $\mathscr{L}$ is introduced as

$$\mathscr{L} = A(s,t) + \alpha\left[A_0 - \sum(2\ell+1)\, a_\ell\right] + \sum(2\ell + 1)\lambda_\ell\, u_\ell$$

The multiplier α has been introduced for the equality constraint and the multipliers $\lambda_\ell \geq 0$ for the inequality constraints of unitarity. Since increasing the total cross section will undoubtedly increase the maximum value of A(s, t), the corresponding multiplier α is nonnegative. (See Eq. (3)). Since the constraint is surely nontrivial, α is strictly positive. Moreover, for a given change δA_0 in the total cross section, the maximum value of A is expected to change by a smaller amount, and one anticipates that $\alpha = \frac{\partial A}{\partial A_0} \leq 1$. By varying the r_ℓ and the a_ℓ, we obtain the equations

$$-2\,\lambda_\ell r_\ell = 0$$

and

$$P_\ell(z) - \alpha + \lambda_\ell(1 - 2\,a_\ell) = 0.$$

The second derivatives will define a nonpositive form if $\lambda_\ell \geq 0$, which has already assumed to be the case. These equations are necessary conditions on the r_ℓ and a_ℓ. Any given partial wave is either purely elastic, $u_\ell = 0$, or inelastic, i.e., it is inside the unitarity circle, $u_\ell > 0$. Thus it is natural to divide the partial waves into two classes:

$$I = \left\{ \ell | u_\ell > 0 \right\}$$

$$B = \left\{ \ell | u_\ell = 0 \right\}$$

where I(B) will be called the Interior (Boundary).

Now according to Theorem 3, for $\ell \in I$, we are instructed to set $\lambda_\ell = 0$. From Eqs. (5) and (6) it is necessary to have $P_\ell(z) = \alpha$, but no constraints are required on r_ℓ and a_ℓ except that they lie inside the unitarity circle.

For $\ell \in B$ on the other hand, we must have $\lambda_\ell \geq 0$. If $r_\ell \neq 0$ then $\lambda_\ell = 0$ and $\alpha = P_\ell(z)$, as before. If however $r_\ell = 0$, then $a_\ell = 0$ or 1 on the boundary and it is convenient to distinguish two classes in B:

$$B_0 = \left\{ \ell | r_\ell = 0, \; a_\ell = 0 \right\}$$

and

$$B_1 = \left\{ \ell | r_\ell = 0, \; a_\ell = 1 \right\}$$

Solving for λ_ℓ in each of these classes, we find that

$$\text{if } \ell \in B_0, \text{ then } \lambda_\ell = \alpha - P_\ell(z) \geq 0$$

and

$$\text{if } \ell \in B_1, \text{ then } \lambda_\ell = P_\ell(z) - \alpha \geq 0$$

In order to utilize the necessary conditions to actually construct the solution, we must try to invert them to determine their sufficiency. It is easy to see that the partial waves can be divided into three classes, so that

$$\text{if } P_\ell(z) - \alpha > 0, \text{ then } \ell \in B_1 \qquad (a_\ell = 1)$$

$$\text{if } P_\ell(z) - \alpha < 0, \text{ then } \ell \in B_0 \qquad (a_\ell = 0)$$

$$\text{if } P_\ell(z) - \alpha = 0, \text{ then no restriction on } \ell \qquad (0 \le a_\ell \le 1)$$

Thus the only ambiguity occurs when $P_\ell(z) = \alpha$.

Since B_1 must be a finite class in order that the sum over ℓ converge, α must be positive. If B_1 and I are to have any members, then $\alpha < 1$ as expected. Now let us turn to the determination of the multiplier α from the equality constraint (2):

$$A_0 = \sum_{B_1} (2\ell+1) + \sum_{I} (2\ell+1) a_\ell \qquad (9)$$

At this juncture it is useful to recall that $P_\ell(z)$ for fixed z oscillates as a function of ℓ with decreasing amplitudes $\left(\text{e.g., } P_\ell(z) \sim J_0\left[2\ell \sin \frac{\theta}{2}\right] \text{ for } \ell \to \infty \text{ and small } \theta\right)$. Therefore one can choose α to be as small as possible consistent with the sum over B_1 being still less than or equal to A_0. This is always possible since as α decreases from one to zero, the sum over B_1 increases monotonically (although not continuously) from zero to infinity. Then it is easy to show that one can always choose the second sum to fit the given value of A_0. If there is only one integer with $P_\ell(z) = \alpha$, then the corresponding $0 \le a_\ell \le 1$ is uniquely determined. If there are more than one such integer, then only their sum is determined. However, since they all correspond to the same value of $P_\ell(z) = \alpha$, this indeterminancy does not affect the maximum value of A; i.e.,

$$A \le A_{max} = \sum_{B_1} (2\ell+1) P_\ell(z) + \alpha \sum_{I} (2\ell+1) a_\ell \qquad (10)$$

Because of the presence of I, the second differential is not strictly negative, so it is impossible to apply Theorem 4 to establish that A_{max} is truly a maximum. This kind of local ambiguity arises occasionally, especially when maximizing linear functions or when dealing with linear constraints. There is an almost obvious theorem (unstated in H or in L) which may be useful.

THEOREM 0

Consider a function f(x) on a domain S and let $x_0 \epsilon S$. Suppose $f'(x_0, h) < 0$ for all $h \neq 0$ in the tangent cone C of S at x_0. Then x_0 is a local maximum of f.

The realm of applicability of the hypothesis is rather limited since, for a regular point, a necessary and sufficient condition that there be a vector $h_1 \neq 0$ such that $f'(x_0, h_1) = 0$ is that the set C_1 (defined in Theorem 3) be nonempty. However, the theorem is relevant to the above example, since usually the set I will have at most one element $\ell = L$. From the inequality constraint $\mu_\ell \geq 0$ we determine that, for $\ell \epsilon B$,

$$(1 - 2a_\ell)h_\ell \geq 0$$

for all h in the tangent cone C. From the equality constraint, we have

$$(2L + 1)h_L + \sum_B (2\ell + 1)h_\ell = 0.$$

However,

$$A'(x_0, h) = - \sum_B (2\ell + 1)\lambda_\ell(1 - 2a_\ell)h_\ell \leq 0$$

Every term in the sum is nonnegative and, hence, the sum connot vanish unless $h_\ell = 0$ for all $\ell \epsilon B$. But then $h_L = 0$ and so $h = 0$. Thus $A'(x_0, h) \neq 0$ for all $h \neq 0$ in C. By Theorem 0, x_0 is a local maximum.

The argument fails when there is more than one element in I, for the problem is invariant under variations in the subspace of S determined by I. However, since A(s, t) is independent of such a degeneracy, the redundant variables can be simply eliminated by defining a new variable ξ by

$$\xi \sum_I (2\ell + 1) = \sum_I (2\ell + 1) a_\ell .$$

Then, in spite of the ambiguity among the elements of I, both ξ and the maximum of A are unique. (See Singh and Roy [15].)

B.

The above derivation contains the essence of all our subsequent applications. As a second example of this technique, let us add an additional constraint to the problem just solved. Again following Singh and Roy [15], it will be assumed in addition that the total elastic cross section is fixed,

$$\frac{k^2}{4\pi} \sigma_{el} \equiv \Sigma_{el} = \sum (2\ell + 1) \left(a_\ell^2 + r_\ell^2\right) .$$

Therefore consider the auxiliary function

$$\mathscr{L} = A + \alpha\left[A_0 - \Sigma(2\ell + 1)a_\ell\right] + \frac{1}{2a}\left[\Sigma_{el} - \Sigma (2\ell + 1) \left(a_\ell^2 + r_\ell^2\right)\right] + \Sigma(2\ell + 1)\lambda_\ell u_\ell .$$

We assume of course, that $\sigma_{el} < \sigma_T$, so the new constraint will be nontrivial. Increasing A_0 for fixed Σ_{el} will have the same effect on the maximum value of A as in the preceding example, so we anticipate $0 < \alpha \leq 1$. Increasing Σ_{el} for fixed A_0 will surely increase A_{max} so it is expected that $a > 0$.

The variational equations are easily obtained in the usual manner. For $\ell \in I$, one readily obtains

$$r_\ell = 0 \qquad a_\ell = a(P_\ell(z) - \alpha)$$

where one must have $0 < a_\ell < 1$ by unitarity. For $\ell \in B$, introducing B_1 and B_0 as before, one finds that $r_\ell = 0$, and

$$\text{if } \ell \in B_1, \text{ then } \lambda_\ell = P_\ell - \alpha - \frac{1}{a} \geq 0$$

whereas

$$\text{if } \ell \in B_0, \text{ then } \lambda_\ell = \alpha - P_\ell \geq 0$$

The second derivatives are

$$\frac{\partial^2 \mathscr{L}}{\partial a_\ell \partial a_{\ell'}} = \frac{\partial^2 \mathscr{L}}{\partial r_\ell \partial r_{\ell'}} = -\delta_{\ell\ell'} 2(2\ell + 1) \left(\frac{1}{2a} + \lambda_\ell\right)$$

and

$$\frac{\partial^2 \mathscr{L}}{\partial a_\ell \partial r_{\ell'}} = 0.$$

Assuming that I is not empty, this will define a nonpositive form if and only if $a \geq 0$. Assuming that the total and elastic cross sections are not equal, the additional constraint will be nontrivial, i.e., $a \neq 0$. Hence, we must require $a > 0$, which in turn implies that the second variation is negative definite. Thus by Theorem 2, we are assured that any solution we construct <u>will</u> be a local maximum. In this sense, this problem is simpler than the previous example. This problem is also simpler in a second respect, viz., the necessary conditions displayed above are also completely sufficient to define uniquely the solution.

The multipliers a and α are determined from the equations

$$A_0 = \sum_{B_1} (2\ell + 1) + \sum_{I} (2\ell + 1)\, a(P_\ell(z) - \alpha)$$

$$\Sigma_{el} = \sum_{B_1} (2\ell + 1) + \sum_{I} (2\ell + 1)\, a^2 (P_\ell(z) - \alpha)^2 \qquad (11)$$

Having determined a and α, one then computes the maximum value of A. Since both of these problems have been discussed by Singh and Roy [15] an explicit evaluation will be omitted. Our main purpose here was to present illustrations of the mathematical method in a familiar context. We have already established that $a > 0$ and it can be shown that $0 \leq \alpha < 1$ as was conjectured.

It is straightforward to generalize the problem to include more equality constraints. Such generalizations have been considered by Jacobs et al., [16], although these authors obtained the correct form of the maximum only in the case that the set B_1 is empty.

It is interesting to note that the two examples quoted above required no assumptions of analyticity in momentum transfer, but required only the convergence of the partial wave sum in the physical region. Mathematically, the reason that bounds exist in such cases is that the imaginary parts of the partial wave amplitude have a definite sign. If similar problems are posed for the real parts of the scattering amplitude, no solution can be found because the r_ℓ's can oscillate in sign. One way to get finite bounds is to force the partial waves to fall off rapidly for large ℓ, and this can be accomplished by the requirements of analyticity in momentum transfer. Another way would be to require all r_ℓ for sufficiently large ℓ to have a definite sign. Let us now turn to these considerations.

C.

In this section, the Lagrange multiplier method will be used to improve some of the bounds for problems which are well known. For the problem to be discussed here, the new physical requirement to be imposed is that of Jin and Martin [17].

$$G \equiv A(s,t_1) = \sum (2\ell + 1)\, a_\ell(s)\, P_\ell(w) \leq \frac{k}{\sqrt{s}} \left(\frac{s}{s_0}\right)^2 \qquad (12)$$

where

$$w = 1 + \frac{2t_1}{s-4}$$

This is assumed to be true for t_1 below the nearest continuum singularity in the t-channel, which occurs at $t_1 = 4$ or $4M^2$ if the mass is explicitly restored.

Although the scale s_0 is unknown, what is significant for asymptotic bounds is the power of s. For purposes of discussion, it will be assumed that for any given value of s and t_1, s_0 is chosen sufficiently small so that the inequality (12) holds. Also, s_0 approaches a constant as $s \to \infty$.

The first natural problem to consider is the problem of maximizing the absorptive part A(s, t) given the Jin-Martin bound. This is essentially the problem of Froissart[4] and leads in the forward direction, to the well-known bound,

$$\sigma_T \leq \frac{\pi}{m^2} \ln^2\left(\frac{s}{s_0}\right).$$

We will consider explicitly the slightly more difficult problem of maximizing A(s, t) if we are given the total elastic cross section in addition to the Jin-Martin bound. Thus consider the auxiliary function

$$\mathscr{L} = A + \frac{1}{2a}\left[\Sigma_{el} - \sum (2\ell + 1)\left(a_\ell^2 + r_\ell^2\right)\right] + g\left[\frac{k}{\sqrt{s}}\left(\frac{s}{s_0}\right)^2 - G\right] + \sum(2\ell + 1)\lambda_\ell u_\ell$$

where the inequality multipliers g and λ_ℓ are nonnegative. Since A_{max} will increase with increasing $\Sigma_{e\ell}$, we expect that $a > 0$. The variations with respect to r_ℓ and a_ℓ yield

$$-2\, r_\ell\left(\lambda_\ell + \frac{1}{2a}\right) = 0$$

and

$$P_\ell(z) - g\, P_\ell(w) - \frac{1}{a}\, a_\ell + \lambda_\ell(1 - 2a_\ell) = 0$$

The second derivative constraints require that $(\lambda_\ell + 1/2a)$ be positive. As before one concludes that $r_\ell = 0$ and that $a > 0$. The second variation is strictly negative, so our construction will lead to a local maximum. The ℓ's split into the usual three classes according to the conditions

$$\text{if } 0 < a_\ell = a(P_\ell(z) - g\, P_\ell(w)) < 1, \quad \text{then } \ell \in I$$

$$\text{if } \lambda_\ell = P_\ell(z) - g\, P_\ell(w) - \frac{1}{a} \geq 0, \quad \text{then } \ell \in B_1$$

and if

$$\lambda_\ell = g P_\ell(w) - P_\ell(z) \geq 0, \quad \text{then } \ell \in B_0.$$

Since all sufficiently large ℓ must lie in B_0, g is positive definite and hence the maximization requires that $G = \frac{k}{\sqrt{s}} (s/s_0)^2$. In order to have B_1 or I nonempty, one must have $g \leq 1$. Having determined a_ℓ and λ_ℓ in terms of a and g, these latter parameters are determined from the conditions

$$\frac{k}{\sqrt{s}} \left(\frac{s}{s_0}\right)^2 = \sum_{B_1} (2\ell + 1)\, P_\ell(w) + a \sum_I (2\ell + 1)\, (P_\ell(z) - g P_\ell(w)\, P_\ell(w)$$

and

$$\Sigma_{el} = \sum_{B_1} (2\ell + 1) + a^2 \sum_I (2\ell + 1)\, (P_\ell(z) - g P_\ell(w))^2$$

If suffices for our purposes to consider the case when B_1 is empty. After solving the problem, one can then find out what are the permissible values of Σ_{el} and s_0 for this to pertain.

The multiplier g is determined from the equation

$$\frac{1}{k} \sqrt{s}\, \Sigma_{e\ell} \left(\frac{s_0}{s}\right)^2 = \frac{\left[\sum_I (2\ell + 1)\left(P_\ell(z) - g P_\ell(w)\right)^2\right]^{1/2}}{\left[\sum_I (2\ell + 1)\left(P_\ell(z) - g P_\ell(w)\right) P_\ell(w)\right]}.$$

In the appendix we show that g is uniquely determined whenever the left-hand side is between zero and one. This will certainly be true for large enough s. Then a is found to be

$$a = \left[\Sigma_{el} \;/\; \sum_I (2\ell + 1)(P_\ell(z) - g P_\ell(w))^2\right]^{1/2}.$$

These parameters must satisfy the conditions that all a_ℓ must lie between zero and one. Finally, the maximum value of A will be given by

$$A_{max}(s, t) = a \sum_I (2\ell + 1) \left(P_\ell(z) - g P_\ell(w)\right) P_\ell(z).$$

For large s, small t, the results of the appendix lead to

$$A(s,t) \leq A_{max} \approx \frac{s}{16\,m^2}\left[\frac{m^2\sigma_{el}}{\pi}\left(J_0^2(x) + J_1(x)^2\right)\right]^{1/2} \ln C(s) \qquad (13)$$

where

$$x = \left(\frac{-t}{4m^2}\right)^{1/2} \ln C(s)$$

and

$$C(s) \approx 4ms\left(s_0^4\,\sigma_{el}\right)^{-1/2}$$

where m is the exchanged particle mass which has been restored for dimensional reasons. As discussed in the appendix, this formula is valid for $x < x_0 \approx 2.4$, the first zero of the Bessel function $J_0(x)$, and only if

$$\sigma_{el} \leq \left[J_0^2(x) + J_1(x)^2\right] \frac{\pi}{m^2} \ln^2 C(s)$$

If σ_{el} violates this condition, then the set B_1 must be included.

From (13), a bound on the total cross section is obtained by setting t = 0; one finds

$$\sigma_T \leq \left(\frac{\pi\,\sigma_{el}}{m^2}\right)^{1/2} \ln C(s) \qquad (14)$$

If σ_{el} goes to zero as $s \to \infty$, then this bound is an improvement over the result of Singh and Roy [18] who have $1/\sigma_{el}$ in the log rather than its square root.

IV. BOUNDS ON THE REAL PART

A.

Let us now turn to bounds involving the real part of the scattering amplitude. As noted previously, because the sign of the real part is not fixed, analyticity in momentum transfer outside the physical region (in the form of the Jin-Martin bound) is essential for obtaining finite results. One might imagine maximizing the real part in the forward direction given only the Jin-Martin bound and unitarity. This problem is essentially the same as deriving the Froissart bound on A_0, except that unitarity requires $|r_\ell| \leq 1/2$. Thus one expects that the maximum real part is only one-half the Froissart bound, and this may be verified by explicit calculation. This problem will be returned to later.

We will consider in this section the problem of maximizing the real part of the amplitude in a definite direction,

$$R(s,t) = \sum (2\ell + 1)\, r_\ell\, P_\ell(z),$$

for a fixed value of the elastic cross section, the Jin-Martin bound, and unitarity. The auxiliary function is written as

$$\mathscr{L} = R + \beta\left[\Sigma_{el} - \sum(2\ell+1)\left(a_\ell^2 + r_\ell^2\right)\right] + g\left[\frac{k}{s}\left(\frac{s}{s_0}\right)^2 - G\right] + \sum(2\ell+1)\lambda_\ell u_\ell \, .$$

Increasing Σ_{el} will surely increase R_{max}, so $\beta \geq 0$. Proceeding in the by-now familiar manner, one finds that the interior I is empty. On the boundary, which corresponds to purely elastic scattering, the solution is

$$r_\ell = P_\ell(z)/2(\lambda_\ell + \beta)$$

$$a_\ell = P_\ell^2(z)/2(\lambda_\ell + \beta)(\lambda_\ell + 2\beta + g\, P_\ell(w))$$

where

$$\lambda_\ell + \beta = \left[P_\ell^2(z) + \left(\beta + g\, P_\ell(w)\right)^2\right]^{1/2}$$

Notice that the sign of r_ℓ oscillates as does that of $P_\ell(z)$. Using the methods developed in the appendix the values of β and g are directly evaluated in terms of Σ_{el} and s_0. Assuming that $\beta + g >> 1$, which is the physically interesting case, the limit becomes

$$R \leq R_{max} = \frac{s}{16}\left(\sigma_{el}/\pi m^2\right)^{1/2} \ln\left(s/s_0^2\, \sigma_{el}\right)\left[J_0^2(x) + J_1^2(x)\right]^{1/2} \qquad (15)$$

where

$$x = (-t/4m^2)^{1/2} \ln\left(s/s_0^2 \sigma\right)$$

The approximations made in deriving this form for R_{max} hold if

$$4m^2 \sigma / \pi << \ln^2\left(s/s_0^2 \sigma\right) \left[J_0^2(x) + J_1^2(x)\right] .$$

In the forward direction, $(x = 0)$ this condition means that the above form of the solution is invalid if σ_{el} approaches the Froissart limit. It is reasonable to expect the condition to fail under such circumstances, since the Froissart bound on σ_{el} is already implied by the Jin-Martin bound. In mathematical terms, this means that the constraint imposed by σ_{el} is really not an independent constraint and, hence we expect the corresponding multiplier β to vanish.

Noting that the constraints in $\mathscr{L}$ are invariant under the transformation $r_\ell \longrightarrow -r_\ell$, one can easily show that $-R_{max}$ is the minimum of R. Hence, one has

$$|R| \leq R_{max}.$$

This result can be written in a number of interesting ways. For example, in the forward direction[19],

$$|R_0|/A_0 \ln(s/s_0) \leq \left[\pi \sigma_{el}/m^2 \sigma_T^2\right]^{1/2} \qquad (16)$$

Away from the forward direction, this can be written

$$R_{max}(s, t)/R_{max}(s, 0) = \left[J_0^2(x) + J_1^2(x)\right]^{1/2} .$$

and for $s \longrightarrow \infty$, this ratio goes to zero as

$$\approx [2/\pi x]^{1/2} \sim [1/\ln s]^{1/2}$$

The vanishing of the ratio for $s \longrightarrow \infty$ at fixed nonzero t is the effect of the shrinkage of the diffraction peak on the real part of the amplitude.

B.

In this section a new type of problem will be treated. An upper bound on a two body reaction amplitude F will be derived by using unitarity, a fixed total reaction cross section σ_r, and the Jin-Martin bound on the associated elastic scattering channels. We have in mind here charge exchange, or other, more general, two body inelastic reactions. The amplitude will be written as

$$|F| \leq \hat{F} = \sum (2\ell + 1)\, f_\ell |P_\ell(z)|,$$

where $|F_\ell| = f_\ell$. The corresponding coupled elastic scattering amplitude will be described in terms of a_ℓ and r_ℓ. Unitarity in this latter channel demands that

$$a_\ell - a_\ell^2 - r_\ell^2 - f_\ell^2 \geq 0.$$

The auxiliary function is chosen to be

$$\mathscr{L} = \hat{F} + g\left[\frac{k}{\sqrt{s}}\left(\frac{s}{s_0}\right)^2 - G\right] + \eta\left[\Sigma_r - \sum(2\ell+1)f_\ell^2\right] + \sum(2\ell+1)\,\nu_\ell(u_\ell - f_\ell^2)\ ,$$

where Σ_r is proportional to the total reaction cross section. Intuitively we expect $\hat{F}_{max}$ to increase with increasing Σ_r, so the multiplier η should be positive. The variational equations yield the amplitudes

$$2a_\ell = 1 - g\,P_\ell(w)/\nu_\ell$$

$$r_\ell = 0$$

$$2f_\ell = P_\ell(z)/(\nu_\ell + \eta),$$

and the second derivative conditions demand that $\nu_\ell > 0$, and $(\nu_\ell + \ \ )\ \ 0$. Since ν_ℓ cannot vanish, the solutions must be on the boundary, that is, $u_\ell = f_\ell^2$, and the resulting equation for ν_ℓ is

$$1 = P_\ell^2/(\nu_\ell + \eta)^2 + g^2\, P_\ell^2(w)/\nu_\ell^2.$$

For large ℓ, ν_ℓ must grow, and one finds

$$\nu_\ell^2 = P_\ell^2(z) + g^2 P_\ell^2(w) + O\left(1/\nu_\ell^2\right)$$

Using the techniques of integration and summation discussed in the appendix, it is straightforward to show that

$$\hat{F} \leq \frac{s}{16}\left(\sigma_r/\pi m^2\right)^{1/2} \ln\left(s/s_0^2\sigma_r\right)\left[J_0^2(x) + J_1^2(x)\right]^{1/2} \tag{17}$$

and x has been defined before, Eq. (15). The close resemblance between this bound and the bound on R should be no surprise due to the similar nature of the unitarity conditions in the two cases. In the forward direction, this bound should be compared with the result of Roy and Singh[20].

However, it is interesting to note that the Jin-Martin bound on G was applied to the imaginary part of the associated elastic scattering amplitude. The lowest threshold in t is at $4m^2$, which is reflected in the bound [17]. In some of the Roy and Singh [20] bounds, such as nucleon-nucleon scattering, one pion exchange is used as the lowest threshold. However, two pion exchange is equally correct and yields a better bound by a factor of two. In view of their comparisons with experiments in the pion-nucleon case, a factor of two improvements may be very important.

C.

It is interesting to add an additional constraint to the problem in Section IV.A, viz., suppose the total cross section σ_T is fixed as well. Assuming that $\sigma_{el} < \sigma_T$, the new constraint will be nontrivial and so, at least some of the partial waves must be inelastic, i.e., I cannot be empty. Let us also recall the upper bound on σ_T determined by σ_{el} given in Section III.C.,

$$\sigma_T \leq \left(\frac{\pi\sigma_{el}}{m^2}\right)^{1/2} \ln C(s) \tag{14}$$

This bound was derived by maximizing the imaginary part, given σ_{el}. Thus, if the bound is almost saturated, inelastic partial waves will be important and the real part must be very small. In the other extreme, if $\sigma_{el} \approx \sigma_T$, most partial waves will be elastic and the bound on the real part will be the same as that derived in Section IV.A.

The new auxiliary function is

$$\mathscr{L} = R - \alpha\left[A_0 - \sum(2\ell+1)a_\ell\right] + \beta\left[\Sigma_{e\ell} - \sum(2\ell+1)\left(a_\ell^2 + r_\ell^2\right)\right] + g\left[\frac{k}{\sqrt{s}}\left(\frac{s}{s_0}\right)^2 - G\right] + \sum(2\ell+1)\lambda_\ell u_\ell$$

We expect that $\beta > 0$, since an increase in σ_{el} for fixed σ_T will increase the maximum value of the real part R_{max}. On the other hand, according to the inequality given above, Eq. (14), increasing σ_T for fixed σ_{el} must reduce R_{max}, and hence, $\alpha > 0$ also.

Proceeding in the usual way the solutions in the interior I are

$$r_\ell = r\, P_\ell(z)$$

$$a_\ell = a - \frac{\gamma}{2} P_\ell(w) ,$$

where we have defined more convenient multipliers

$$r = \frac{1}{2\beta}, \qquad a = \frac{\alpha}{2\beta}, \qquad \gamma = \left(\frac{g}{2\beta}\right) .$$

In B, the solutions are

$$r_\ell = \frac{P_\ell(z)}{2\lambda_\ell'}$$

$$a_\ell = \frac{P_\ell(z)^2}{2\lambda_\ell'\left(\lambda_\ell' + \frac{1}{2r}\left(1 - 2a + \gamma P_\ell(w)\right)\right)}$$

where

$$\lambda_\ell' = \lambda_\ell + \beta = \left[P_\ell(z)^2 + \left(\frac{1}{2r}\right)^2 (1 - 2a + \gamma P_\ell(w))^2 \right]^{1/2}$$

The multipliers r, a, and γ, must be determined by the constraints

$$\frac{k}{\sqrt{s}} \left(\frac{s}{s_0}\right)^2 = \sum_I (2\ell + 1) a_\ell P_\ell (w) + \sum_B (2\ell + 1) a_\ell P_\ell(w)$$

$$\Sigma_{e\ell} = \sum_I (2\ell + 1) \left(a_\ell^2 + r_\ell^2\right) + \sum_B (2\ell + 1) a_\ell$$

$$A_0 = \sum_I (2\ell + 1) a_\ell + \sum_B (2\ell + 1) a_\ell$$

Motivated by our introductory observations, we expect the relative importance of the sum over the inelastic partial waves (I) with respect to the elastic partial waves (B) will be sensitive to the inelastic cross section, $\sigma_{in} \equiv \sigma_T - \sigma_{e\ell}$. Based on the previous examples, one anticipates that the cutoff $1/\gamma$ will be of order s, and that the contribution to any one partial wave will be small, $a^2 \ll a \ll 1$. Unitarity requires that $r^2 < a$; however,

if $\sigma_{el} \approx \sigma_T$, then $r^2 \approx a$

but if $\sigma_{el} \ll \sigma_T$, then $r^2 \ll a$.

This is the mathematical transcription of the introductory remarks that the real part is a measure of the inelasticity. Having established these orders of magnitude, we may approximate

$$\lambda_\ell' \approx (1 + \gamma P_\ell(w))/2r$$

so that, in B,

$$r_\ell \approx \frac{r P_\ell(z)}{(1 + \gamma P_\ell(w))}$$

$$a_\ell \approx r_\ell^2 .$$

The condition that any given partial wave ℓ lie in I is approximately

$$r^2 P_\ell(z)^2 < a - \frac{\gamma}{2} P_\ell(w) .$$

There are solutions of this inequality only if $\gamma/2 < a - r^2$. We will assume that $\sigma_T - \sigma_{e\ell}$ is sufficiently large so this is indeed satisfied. In fact, it will be assumed that $\gamma/2 \ll a - r^2$, i.e., there are many inelastic partial waves. Then all partial waves are in I from $\ell = 0$ to $\ell = L$, defined by $r^2 P_L(z)^2 = a - \frac{\gamma}{2} P_L(w)$. (For simplicity, we will assume there is a unique solution of this equation. This is not an essential assumption.) All partial waves for $\ell > L$ will be purely elastic and hence in B. This situation is depicted in Fig. 2.

One may now proceed to approximate the sums using the techniques and results of the appendix. The mathematical methods are essentially the same as in the preceding problems, except that, in general, the contributions from the interior I and the boundary B are of the same order and so must be treated quite carefully. It is informative to study the solution first in the forward direction. One finds

$$\frac{\zeta^2}{4}\left(\frac{s}{s_0}\right)^2 \approx \frac{1}{\gamma}\,\ell n\left(\frac{1}{\gamma}\right)\left\{a(a - r^2) + r^2\right\}$$

$$\zeta^2 \Sigma_{el} \approx (a^2 + r^2)\,\ell n^2\left(\frac{1}{\gamma}\right)$$

$$\zeta^2 A_0 \approx a\,\ell n^2\left(\frac{1}{\gamma}\right) \qquad (18)$$

where

$$\zeta^2 = 4t_1/s .$$

These equations determine the multipliers. Once they are known, the maximum value of the real part can be evaluated

$$\zeta^2 R_{0\,max} \approx r\,\ell n^2\left(\frac{1}{\gamma}\right)$$

Solving for the multipliers, one finds

$$\zeta^2 R_{0\,max} \approx \left[\zeta^2 \Sigma_{el} - \frac{\left(\zeta^2 A_0\right)^2}{\ell n^2 \left(\frac{1}{\gamma}\right)}\right]^{1/2} \ell n \left(\frac{1}{\gamma}\right)$$

where $1/\gamma$ is to be determined from

$$\frac{1}{\gamma} \frac{1}{\ell n^3 \left(\frac{1}{\gamma}\right)} \approx \frac{\zeta^2}{4} \left(\frac{s}{s_0}\right)^2 \Big/ \left[\left(\zeta^2 A_0\right)\left(\zeta^2 A_0 - \zeta^2 \Sigma_{el}\right) + \left(\zeta^2 \Sigma_{el} \ell n^2 \left(\frac{1}{\gamma}\right) - \left(\zeta^2 A_0\right)^2\right)\right]$$

The regimes for which these approximate formulas apply are $a^2 \ll a \ll 1$, $r^2 < a$, $\gamma/2 \ll a - r^2$.

These equations become more transparent by setting $t_1 = 4$. $\zeta^2 A_0 = \sigma_T/\pi$; $\zeta^2 \Sigma_{el} = \sigma_{el}/\pi$. It can be easily checked that this result has the expected behavior in various limiting cases. For example, if $\sigma_{el} \approx \sigma_T$, the result of the preceding section is recovered.

If $\sigma_{e\ell}$ is of the order of $\sigma_T^2/\pi\left(\ell n^2\left(s/s_0^2\sigma_T^2\right)\right) \ll \sigma_T$, then

$$\zeta^2 R_{0\,max} = \left[\frac{\zeta}{\pi} \sigma_{el} - \frac{\left(\frac{1}{\pi}\sigma_T\right)}{\ell n^2 \left(\frac{1}{\gamma}\right)}\right]^{1/2} \ell n \left(\frac{1}{\gamma}\right), \qquad (19)$$

where

$$\ell n \left(\frac{1}{\gamma}\right) \approx \ell n \left(\frac{4\pi^2 s}{s_0^2 \sigma_T^2}\right) .$$

In this case, the bound depends on the precise relationship between σ_{el} and $\sigma_T/\ell n(s/s_0)$ and the approximations made leading to the Eq. (18) for the multipliers must be revised. The physics however is quite clear. If σ_{el} falls as rapidly as possible consistent with σ_T, the amplitude becomes purely imaginary and a very strict bound is imposed upon the real part. One might argue that all this concern over logarithms, whose scale (s_0) is unknown, is unwarranted. However, by the methods of Section V, these results could be converted to finite s bounds and the scale of the logarithm determined by low energy data such as scattering lengths or the elastic cross section at low and intermediate energies. These bounds then impose limitations on the ratio of the real to imaginary part which can be compared with experiment or with theoretical models.

The determination of the bound at a fixed nonforward angle involves some rather complicated integrals, probably most easily done with the help of a computer. However, as in the preceding section, it is not difficult to obtain the form in the near-forward direction. The equations for the multipliers become

$$\frac{\zeta^2}{4}\left(\frac{s}{s_0}\right)^2 = \frac{1}{\gamma}\,\ln\left(\frac{1}{\gamma}\right)\left\{a\left(a - r^2 J_0^2(x)\right) + r^2 J_0^2(x)\right\}$$

$$\zeta^2 \Sigma_{e\ell} \approx \ln^2\left(\frac{1}{\gamma}\right)\ (a^2 + r^2)\ \left\{J_0^2(x) + J_1^2(x)\right\}$$

and

$$\zeta^2 A_0 \approx \ln^2\left(\frac{1}{\gamma}\right) a \left\{J_0^2 + J_1^2\right\}$$

where the argument of the Bessel functions is

$$x = \sqrt{\frac{-t}{4m^2}}\,\ln\left(\frac{1}{\gamma}\right) .$$

Finally, R_{max} is given by

$$R_{max} \approx \ln^2\left(\frac{1}{\gamma}\right) r \left\{J_0^2 + J_1^2\right\}$$

As in the preceding section, one sees the effect of shrinkage but it is considerably more complicated in this solution.

D.

Next let us consider the problem of maximizing $\left.\frac{d\sigma}{dt}\right|_{t=0} \propto A_0^2 + R_0^2$, given the Jin-Martin bound. First, let us anticipate the result. As noted in the introduction to Section IV.A, the maximum value of R_0, the real part in the forward direction, is one-half the Froissart bound. This limit is achieved by having all the partial waves purely elastic. If the imaginary part to which this solution corresponds is then calculated, not surprisingly, one finds A_0 equal to one-half the Froissart bound. Thus the corresponding value of $A_0^2 + R_0^2$ is also one-half the Froissart limit. To increase A_0, one must decrease R_0; however, one can clearly do better by having a vanishing real part and setting A_0 equal to the Froissart limit. Thus, we expect that the maximum value of $A_0^2 + R_0^2$ will correspond to an asymptotically vanishing real part and a maximum imaginary part. This expectation will be born out by the calculation below.

Using a notation which should be familiar by now, let us define the auxiliary function

$$\mathscr{L} = A_0^2 + R_0^2 + 2g\left[\frac{k}{\sqrt{s}}\left(\frac{s}{s_0}\right)^2 - G\right] + \sum(2\ell+1)\lambda_\ell u_\ell$$

Assuming $R_0 \neq 0$, we find that I must be empty. In B, the solutions are

$$\lambda_\ell r_\ell = R_0$$

$$\lambda_\ell(a_\ell - 1/2) = A_0 - g P_\ell$$

Solving for λ_ℓ, we obtain

$$\lambda_\ell = 2\left[R_0^2 + (A_0 - g P_\ell)^2\right]^{1/2}$$

These solutions involve R_0 and A_0 which we hoped to compute, so in what sense are these "solutions" at all? In this problem, because of its nonlinearity, one cannot obtain an explicit formula for the r_ℓ and a_ℓ in terms of the multipliers alone. However, what we have here is an implicit representation of the solution, very much analogous to implicit differentiation in ordinary calculus. The implicit solutions are made explicit by returning to the defining equations for R_0 and A_0. First, let us modify the notation slightly. Since the problem is symmetric under $r_\ell \to -r_\ell$, we suppose, without loss of generality, that $R_0 > 0$. Let us define $\rho = R_0/A_0$, and $g = A_0\gamma$. Then our implicit solutions become

$$r_\ell = \frac{\rho}{2\left[\rho^2 + (1-\gamma P_\ell)^2\right]^{1/2}}$$

$$a_\ell = \frac{1}{2}\left[1 + \frac{(1-\gamma P_\ell)}{\left[\rho^2 + (1-\gamma P_\ell)^2\right]^{1/2}}\right]$$

By the definition of ρ, we have

$$A_0 - \frac{1}{\rho} R_0 = 0$$

Using the solutions given above, and introducing

$$\xi_\ell \equiv a_\ell - \frac{1}{\rho} r_\ell = \frac{1}{2}\left[1 - \frac{\gamma P_\ell}{\left[\rho^2 + (\gamma P_\ell - 1)^2\right]^{1/2}}\right],$$

we may write the requirement as

$$I(\rho, \gamma) \equiv \sum (2\ell+1) \; \xi_\ell(\rho, \gamma) = 0.$$

This implicitly defines the ratio ρ as a function of the multiplier γ. Having solved for $\rho = \rho(\gamma)$, the multiplier γ is then determined in the usual way from the constraint

$$\frac{k}{\sqrt{s}}\left(\frac{s}{s_0}\right)^2 = \sum(2\ell+1)a_\ell P_\ell = \frac{1}{2}\sum(2\ell+1)\left[1 + \frac{(1-\gamma P_\ell)}{\left[\rho^2 + (\gamma P_\ell - 1)^2\right]^{1/2}}\right]P_\ell$$

This completes the statement of the formal solution.

Before making any approximations, it is useful to understand qualitatively the mathematical problems encountered. One can show that $I(\rho, \gamma)$, for fixed γ, is continuous and monotonic in ρ and will always change sign once as ρ varies from 0 to $+\infty$. Hence $I(\rho, \gamma) = 0$ has a unique solution $\rho = \rho(\gamma) > 0$. Solving, one finds to a good approximation,

$$\frac{1}{\rho} = \left(\frac{1}{\gamma}\right)^{\frac{1}{2\sqrt{2}}}$$

so that as $\gamma \to 0$, $\rho \to 0$ although more slowly than γ. Consequently, one is not surprised to find that γ has its canonical value

$$\ln\left(\frac{1}{\gamma}\right) \approx \ln\left(\frac{\zeta^2}{4}\left(\frac{s}{s_0}\right)^2\right) \cong \ln(s/s_0),$$

and that A_0 achieves the Froissart limit

$$\frac{s}{16\,m^2}\,\ln^2(s/s_0)$$

which was the result anticipated. Thus, the final result is

$$\left.\frac{d\sigma}{dt}\right|_{t=0} \leq \frac{16\pi A_0^2}{s^2}\,(1+\rho^2) \approx \frac{\pi}{16m^4}\,\ln^2(s/s_0) \tag{20}$$

Although the mathematics is complicated and the result not surprising, we believe this problem is interesting in two respects. First, it illustrates the power of using Lagrange multipliers for inequality constraints. Secondly, using the trick of implicitly representing the solution for the variables (in this case, the partial wave amplitudes), one sees the method may be used to treat complicated nonlinear functions. Bounds on $d\sigma/dt$ will be much more useful experimentally than on either the real or imaginary parts separately and will eliminate the necessity for having to suppose at high energy that the real part is negligible.

V. INTEGRAL BOUNDS

Reviewing the problems considered so far we see that all the bounds have been derived for a fixed energy and fixed angle or momentum transfer. Except for the first two examples, analyticity in the form of the Jin-Martin bound has been essential to obtain finite results. Unfortunately, the actual magnitude of these results remains unknown because the scale s_0 is unknown. However, as pointed out by Jin-Martin [17] one consequence of their bound is the convergence of the Froissart-Gribov dispersion formulas for the partial wave scattering lengths d_ℓ for $\ell \geq 2$,

$$d_\ell = \frac{1}{2} \frac{\Gamma(\ell+1)}{\Gamma(\ell+3/2)\,\Gamma(1/2)} \int_4^\infty \frac{\sqrt{s}}{k} \frac{A(s,4)ds}{s^{\ell+1}} \tag{21}$$

Conversely, knowing d_ℓ, the imaginary part A cannot grow as fast as s^ℓ. Clearly, the most stringent requirement on the growth is given by requiring the D-wave scattering length d_2 to be finite. However, knowing the value of d_2, we have more information, since then the scale of the bound is determined. This observation is the basis of the bound with no arbitrary constraints, discussed recently by Yndurain [21] and extended by Common [22]. Since the new input is in the form of an integral, the quantity to be maximized must be compatible with this form of the constraint, i.e., it must involve a range of energies.

Since the D-wave scattering length may be difficult to determine experimentally, it may be more convenient in certain cases to use, say, the P-wave effective range. If there is a low energy resonance in the P-wave system, then the effective range may be more accessible to measurement. This latter problem is very similar to the one worked below, and details will be dealt with elsewhere.

From a mathematical standpoint, this involves a generalization of the theorems presented in Section I, since we now deal with a continuum of variables (i.e., functions) rather than a countable number. However such natural generalizations are familiar from other physical problems (notably, in field theory) and, consequently, we will not pause to state these explicitly. The rigorous justification for the following discussion may be found in Chapter 5 of H.

A.

The first problem we choose to consider is to maximize the average total cross section $\bar{\sigma}_T$, defined by

$$\bar{\sigma}_T = \frac{1}{Q(s)} \int_b^s ds'(s' - 4)\, q(s')\, \sigma_T(s')$$

$$= \frac{16\pi}{Q(s)} \int_b^s ds'\, q(s')\, \sum(2\ell + 1)\, a_\ell(s') \qquad (22)$$

given the D-wave scattering length $d_2 = (8/15\pi)d$, where

$$d = \int_4^\infty ds\, K(s) \sum_\ell (2\ell + 1)\, a_\ell(s)\, P_\ell(w),$$

$$w = (s + 4)/(s - 4),$$

and

$$K(s) = \sqrt{s}/ks^3 .$$

The function q(s) is an arbitrary weight function and

$$Q(s) = \int_b^s ds'(s' - 4)\, q(s').$$

The auxiliary function is written as

$$\mathscr{L} = Q\bar{\sigma}_T/16\pi + D(s)\left[d - \int_4^\infty ds\, K \sum(2\ell + 1)\, a_\ell P_\ell(w)\right] + \sum(2\ell + 1) \int_4^\infty ds'\, K \lambda_\ell u_\ell(s').$$

Notice that the bound is required for a fixed value of s, and hence the multiplier D depends only on s. On the other hand, since unitarity must hold for all values of s', the unitarity multiplier $\lambda_\ell(s')$ must be a function of s'. As usual, $\lambda_\ell(s') \geq 0$, and since increasing d will increase the maximum value of $\bar{\sigma}_T$, one expects $D(s) > 0$.

The variational equations are

$$-2\lambda_\ell r_\ell = 0,$$

and

$$\theta(s - s')\,\theta(s' - b)\,h(s') - DP_\ell(w') + \lambda_\ell(1 - 2a_\ell) = 0,$$

where $h(s') = r(s')/K(s')$. As in our earlier discussions two classes of ℓ's are introduced, namely I and B. Since all sufficiently large ℓ must belong to B_0, the multiplier D must be positive. For s' outside the range b to s, it is easy to see that all $\ell \in B_0$. Therefore let us examine only this nontrivial interval. For $\ell \in I$, since $\lambda_\ell = 0$, r_ℓ and a_ℓ are not determined by the integer ℓ must be such that

$$h = DP_\ell(w').$$

The set B_1, which has $a_\ell = 1$, must have

$$\lambda_\ell(s', s) = h(s') - D(s)\,P_\ell(w') \geq 0.$$

If h(s') is sufficiently small for certain ranges of s', then the inequality fails, which means that the class B_1 will be empty for these values of s'. To describe this possibility, it is convenient to define a function $s_1 = s_1(s)$ by the equation $D(s) = h(s_1) = h_1$. Then the condition to be in B_1 can be written as

$$P_\ell(w') \leq h'/h_1 .$$

The cutoff on the ℓ sum, L(s', s) is then defined by

$$P_L(w') = h'/h_1 .$$

We will content ourselves to solve these equations in the large s limit, thereby throwing away part of one of the important features of this type of problem, namely completely determined finite energy bounds. These will be examined in a later paper. Using the results in the appendix, one finds that

$$L(s, s) \simeq \frac{k'}{2m}\ \ell n(h'/h_1)$$

and

$$\sum(2\ell + 1)\, a_\ell(s')\, P_\ell(w') \simeq \frac{k'}{M}\, L(s', s)\,(h'/h_1)$$

The condition due to the scattering length which determines the multiplier h_1 can be written as

$$h_1 = \frac{1}{8d}\int_b^s ds'(s' - 4)\, r(s')\, \theta(s' - s_1)\, \ln(h'/h_1)$$

Finally, the result for $\bar{\sigma}_T$ becomes

$$\bar{\sigma}_T \leq \frac{\pi}{Q} \int_b^s ds'(s' - 4)\, r(s')\, \theta(s' - s_1)\, \ln^2(h'/h_1)\ . \tag{23}$$

Let us consider the case treated by Common [21] in which $r(s\ ) \sim s^{N-1}$ for large s. One then finds

$$h(s)/Q(s) \simeq \frac{1}{2}\, s(N + 1),$$

and

$$(s/s_1)^{N+2}\, \ln(s/s_1) \simeq 8 s\, d\, (N + 1)/(N + 2).$$

This latter equation implies that $4 \ll s_1 \ll s$ for large s. Therefore in this limit the leading term of the right-hand side of the bound becomes

$$\bar{\sigma}_T \quad \leq \frac{\pi}{M^2}\, \ln^2\left[4M^2\, sd(N + 1)\right] + \ldots \tag{24}$$

It should be stressed that it is not at all necessary to take the limit of large s; the exact solution given above exists for any energy value. The asymptotic limit was taken only to show the form of the result.

This bound can be improved in a simple manner by introducing more physical information in the following way. It will be assumed that one of the amplitudes, say a_J, is known in the energy region $4 \leq s \leq c$, where $c \leq b$. Then the scattering length condition can be written as

$$d_J = d - (2J + 1)\int_4^c ds'\, \sqrt{s'}\, a'_J\, P_J(w')/h's'^3$$

$$= \int_4^{\infty} ds(\sqrt{s'}/a's'^3)\, \Sigma'(2\ell + 1)\, a_\ell P_\ell(w) \tag{25}$$

where the prime on the sum means to omit a_J from the sum and integral in the region $s' < c$. The previous discussion goes through as before with d_J occurring in the logarithm in place of d. If more than one value of J is known, then the bound can be improved by subtracting their contributions to d as above. This subtraction process may be useful in the physically interesting cases of pion-pion scattering in which the contribution due to rho and f exchange can be subtracted from the input information which is either the D-wave scattering length or the P-wave effective range.

B.

As a second example of these techniques, let us treat a problem that involves both an integral constraint and a constraint that is local in the energy. Defining A(s, t) as before, the problem is to find the maximum value of $\overline{A/(s-4)}$,

$$\overline{A/(s-4)} = \frac{1}{R} \int_b^s ds' \, r(s') \sum (2\ell+1) \, a_\ell(s') \, P_\ell(z),$$

where

$$Q = \int_b^s ds' \, q(s') \, (s'-4)$$

$$z = 1 + 2t/(s'-4),$$

and $r(s')$ is an arbitrary weight function. The integral constraint will be the same as before, and the local constraint will be chosen to be a given value of the total elastic cross section σ_{el} at each value of s',

$$\Sigma_{el}(s') = (s'-4) \, \sigma_{el}(s')/16\pi = \sum (2\ell+1) \left(a_\ell^2 + r_\ell^2\right) .$$

The auxiliary function $\mathscr{L}$ is written as

$$\mathscr{L} = R\,\overline{A/(s-4)} + D(s)\left[d - \int_4^\infty ds'\, K' \Sigma(2\ell+1)\, a_\ell P_\ell(w)\right]$$

$$+ \Sigma(2\ell+1)\int_4^\infty ds'\, K'\lambda_\ell(s')\, u_\ell(s')$$

$$+ \int_4^\infty ds'\, K'C(s')\left[\Sigma_{e\ell}(s') - \Sigma(2\ell+1)\left(a_\ell^2 + r_\ell^2\right)\right]$$

In order to find the maximum it is convenient to divide the s' interval into two regions. Region O will be the ranges $4 \leq s' \leq b$ and $s < s' < \infty$. Region R will be the range $b < s' < s$. It has been assumed that σ_{el} is given for all s' whereas the maximization involves only the region R. Increasing σ_{el} inside R will increase the maximum, hence $C(s') \geq 0$ for $s' \epsilon$ R. On the other hand, increasing σ_{el} outside this range will decrease the maximum, thus we expect $C(s') \leq 0$ for $s' \epsilon$ O. The variational equations are

$$h(s')\, P_\ell(z)\, \theta(s - s')\, \theta(s' - b) - DP_\ell(w) - C\, a_\ell + \lambda_\ell(1 - 2a_\ell) = 0 \qquad (26)$$

and

$$-\, r_\ell(C + 2\lambda_\ell) = 0.$$

The requirement that the second derivatives are nonpositive is

$$(C + 2\lambda_\ell) \geq 0.$$

Let us first examine region O. In this region the term involving h(s') is not present in the equation for a_ℓ. It is easy to see that there are no $\ell \epsilon$ I since if $\lambda_\ell = 0$, $c \geq 0$, and there is no positive solution for a_ℓ. Therefore all the ℓ's are on the boundary. In the class B_1, the multiplier is

$$\lambda_\ell = -C - DP_\ell(w) \geq 0$$

and in B_0,

$$\lambda_\ell = DP_\ell(w) \geq 0.$$

However, if $C + 2\lambda_\ell$ vanishes, which demands that $\lambda_\ell = DP_\ell$, then the equations do not determine a_ℓ and r_ℓ. Values of ℓ in this class will be denoted by B. Since all large ℓ's must be in B_0, $D > 0$, and in order to have any nonzero values of a_ℓ, the equality multiplier $C(s')$ must be negative in region O. The solution is that all $\ell \in B_1$ for $\ell < L_1$, $\ell \in B$ for $\ell = L_1$, and $\ell \in B_0$ for $\ell > L_1$, where the integer L_1 is determined from

$$\Sigma_{e\ell}(s') = \sum_0^{L_1-1} (2\ell + 1) + (2L_1 + 1)\left(a_{L_1}^2 + r_{L_1}^2\right)$$

$$= L_1^2 + (2L_1 + 1)\, a_{L_1} .$$

Thus the existence of the class B allows a fit to any value of Σ_{el} for an integer value of $L_1(s')$. As in Section II. A, this equation uniquely determines both L_1 and $0 \le a_{L_1} \le 1$.

In region R, the presence of the h(s') term in Eq. (26) allows some of the a_ℓ and r_ℓ to lie in the interior I of the unitarity circle. For this class one has $r_\ell = 0$ and

$$a_\ell = \frac{1}{C(s')}\left[h(s')\, P_\ell(z) - D(s)\, P_\ell(w)\right] .$$

The multiplier C(s') must evidently be positive in region R as expected. For values of ℓ for which this expression yields an a_ℓ which is greater than one, the correct value lies in B_1 and λ_ℓ is easily seen to be positive. If this expression is negative then it is easily seen that the correct ℓ's lie in B_0. Therefore, one has

$$\Sigma_{el} = \sum_{B_1} (2\ell + 1) + \sum_I (2\ell + 1)\left[hP_\ell(z) - DP_\ell(w)\right]^2 / C^2(s')$$

which determines the parameter C(s').

The multiplier D(s) is determined by requiring that the scattering length d be given correctly. Finally, now that all the parameters are determined, the value of the maximum possible value of $\overline{A/(s - 4)}$ can be evaluated in terms of σ (s') and d. In order to illustrate the general form of the result, the various quantities will be evaluated in the limit of large s and large Σ_{el}. It will be assumed that d is such that the class B_1 is empty.

For large values of Σ_{el}, one has

$$L_1^2 \cong \Sigma_{el}$$

and hence using the results of the appendix,

$$\sum_0^{L_1} (2\ell + 1)\, P_\ell(w) \cong \frac{(s' - 4)}{4m^2} \left[\frac{\sigma_{el}\, m^2}{4\pi}\right]^{1/2} I_1\left(\left[\frac{\sigma_{el}\, m^2}{\pi}\right]^{1/2}\right)$$

The contribution of region O to d will be denoted by d_O, and it is given by

$$d_O = \left[\int_4^b + \int_s^\infty\right] ds'\, K(s')\, \frac{(s' - 4)}{4m^2} \left[\frac{\sigma_{el}\, m^2}{4\pi}\right]^{1/2} I_1\left(\left[\frac{\sigma_{el}\, m^2}{\pi}\right]^{1/2}\right)$$

In evaluating the contribution of region R to d, it is necessary to know the sums (see appendix)

$$\sum_0^{L_1} (2\ell + 1) \left[P_\ell(z) - g\, P_\ell(w)\right] P_\ell(w) \equiv L^2\, P_L(w)\, G_L$$

and

$$\sum_0^{L_1} (2\ell + 1) \left[P_\ell(z) - g\, P_\ell(w)\right]^2 \equiv L^2 E_L \quad ,$$

where

$$g = D(s)/h(s') = K(s')\, D(s)/q(s') = P_L(z)/P_L(w)$$

Therefore,

$$d - d_O = \int_b^s ds'\, K(s') \left[\Sigma_{el}/E_L\right]^{1/2} L\, P_L(w)\, G_L$$

or

$$D(s)\,(d - d_O) = \frac{1}{8m} \int_b^s ds'\, q(s')\,(s' - 4) \left[\sigma_{el}(s')/4\pi E_L\right]^{1/2} P_L(z) \left[\, yG_L \,\right]$$

where $\left[\, yG_L \,\right]$ is given in the appendix. This equation determined D(s). The maximum value of $\overline{A/(s-4)}$ is therefore

$$\overline{A/(s-4)} \le \frac{1}{8mQ} \int_b^s ds'\, q(s')\,(s' - 4) \left[\sigma_{el}(s')\, E_L/4\pi\right]^{1/2} \cdot y(s')$$

where y is determined from

$$I_0(y) = J_0(x)\; r(s')/k(s')\; D(s)\;,$$

and D is given in terms of $(d - d_O)$. For $r \sim s'_n$, the limit becomes

$$\overline{A/(s-4)} \lesssim \frac{1}{8m} \left[\sigma_{el}/4\pi\right]^{1/2} \left[J_0^2(x) + J_1^2(x)\right]^{1/2} \cdot y(s)$$

where

$$y(s) \simeq \ln\left[4ms(d - d_O)(n+2)/(\sigma_{el}/4\pi)^{1/2}\right]\,,$$

and terms of order unity have been dropped in the argument of the logarithm.

In the forward direction, this expression yields an asymptotic bound on the total cross section

$$\bar{\sigma}_T \lesssim \left[\pi\sigma_{el}/m^2\right]^{1/2} \cdot y(s)$$

Thus it is seen that d and σ_{el} set the energy scale in the log. Also, since the amplitudes have been constrained to yield a given total elastic cross section at all energies, this improves the bound given in the previous section by decreasing d by the minimum amount d_O that can come from region O. The presence of σ_{el} under the square root is also an improvement on the previous result.

VI. CONCLUSION

In conclusion, let us reflect on the mathematical machinery used here and its potential usefulness. First, one should note that the mathematical theory presented in Section II gave only necessary conditions on local extrema. One may then wonder why, in the subsequent applications, the solution to each problem was almost always unique. The answer [23] is that we were dealing with convex functionals with convex constraints. This choice of problems was due less to design than to luck.

Secondly, we should stress our hope that the mathematical tools developed here, in addition to unifying the treatment of these diverse types of bounds, will allow more physical input to be injected into similar problems. If it should be the case that scattering amplitudes grow logarithmically with energy, then the asymptotic bounds may be of more than just academic interest and may have phenomenological applications.

Roy and Singh 20 have used bounds on reaction amplitudes to discuss the Pomeranchuk theorem and to bound the difference between particle and anti-particle total cross sections at large energies. Unfortunately, their very important results hold only at infinite energies. It is possible to derive an integral form of the very important bound on reaction amplitudes. These integral forms will hold at finite energies and may prove to be very interesting. This problem will be discussed elsewhere.

No paper on bounds would be complete without mentioning the tremendous impetus given to the subject by A. Martin [24] and collaborators. The recent revival of interest in the subject is due, we believe, to the Serpukhov data [25] suggesting a violation of the Pomeranchuk theorem [26] and to the very interesting results by S. M. Roy and V. Singh [15, 18, 20].

APPENDIX

Reluctantly, we must now turn to some rather tomentose calculations. The basic formula used to perform the sums in the text is

$$\sum_0^L (2\ell + 1)\, P_\ell(z)\, P_\ell(w) = (L+1)\left[P_L(z)\, P_{L+1}(w) - P_L(w)\, P_{L+1}(z)\right]/(w-z)$$

and its limiting forms when w = 1, and w = z. After performing the sums, approximations have been used to simplify the final formulas. The most useful approximations are

$$P_L(z) \approx J_0(x)$$

$$P_L(w) \approx I_0(y)$$

where

$$x = L\left[\frac{-4t}{s-4}\right]^{1/2}$$

and

$$y = L\left[\frac{16}{s-4}\right]^{1/2}$$

An approximate formula used many times in the text which follows from the above relations is

$$\sum_0^L (2\ell + 1)\, P_\ell(z)^2 \approx (L+1)^2\left[J_0^2(x) + J_1^2(x)\right]$$

In many applications it is necessary to perform sums of the form

$$S = \Sigma a_\ell b_\ell$$

where a_ℓ is a slowly varying function of ℓ for $\ell < L$ and for $\ell > L$ and rapidly varying for $\ell \sim L$. It is convenient in such cases to perform a summation by parts by introducing

$$B_\ell = \sum_0^\ell b_j$$

so that

$$b_\ell = B_\ell - B_{\ell-1}$$

Inserting this into S and redefining summation variables achieves the form

$$S = \sum (a_\ell - a_{\ell+1}) \, B_\ell$$

Since the a_ℓ change most rapidly for $\ell \sim L$, the expansion of B_ℓ about this value yields

$$S = a_0 \, B_L + \sum (a_\ell - a_{\ell+1}) \, (B_\ell - B_L)$$

This will be a useful procedure if the second term can be shown to be small in comparison with the first. This will be the case under the conditions stated.

In Section III. C, it is necessary to perform sums over positive functions of the form $(P_\ell(z) - g\,P_\ell(w))$ and its square. Since $P_\ell(z)$ oscillates, for simplicity, it will be assumed that the momentum transfer is small enough so that only the first cycle of $P_\ell(z)$ need be considered. The second term $P_\ell(w)$ grows monotonically with ℓ and the sum must terminate. Define L to be the last value of ℓ to contribute, then it follows that

$$\frac{P_L(z)}{P_L(w)} \geq g \geq \frac{P_{L+1}(z)}{P_{L+1}(w)}$$

In the equation that determines g it is clear that for $g \sim 0$, or $L \sim \infty$, the ratio of sums on the right-hand side vanishes. As $g \sim 1$, this ratio also goes to one. It is straightforward to prove using Schwartz's inequality that the derivation of the ratio with respect to g is positive definite for $0 < g < 1$. Hence the ratio is monotonic and g is unique. Using the summation formulas and the asymptotic behavior of $I_0(y)$, the solution for A_{max}, Eq. (15), is easily derived. The condition that only the first cycle of $P_\ell(z)$ contribute is the requirement that x be less than the first zero of $J_0(x)$.

The summations required for R_{max} in Section IV.A are easily performed by using summation by parts and the usual summation formulas.

In Section IV. D, some care must be taken in evaluating the sum over ξ_ℓ which yields the connection between γ and ρ. It is convenient to divide the sum over ℓ into three regions. Assuming that γ and ρ are small, the regions are defined by

1. $\gamma P_\ell < 1 - \rho$.
2. $1 - \rho < \gamma P_\ell < 1 + \rho$
3. $1 + \rho < \gamma P_\ell$.

Expansions in ρ can be made in each of these regions, and there are important cancellations between region 1 and 3 which must be treated carefully.

In Section V, the summations are of the canonical form and one finds

$$E_L \approx J_0^2(x) + J_1^2(x)$$

$$G_L \approx \frac{2}{y^2+x^2}\left[\frac{y\, I_1(y)}{I_0(y)} J_0(x) + x\, J_1(x)\right] - J_0(x)\left[1-\frac{I_1^2(y)}{I_0^2(y)}\right] .$$

Assuming that $y \gg 1$, G_L becomes

$$y\, G_L \approx \left[(y^2 - x^2)\, J_0(x) + 2xy\, J_1(x)\right]/(y^2 + x^2) .$$

REFERENCES AND FOOTNOTES

1. The classic source here is G. F. Chew, S-Matrix Theory of Strong Interactions, (W. A. Benjamin, Inc., New York, 1962).
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3. For a review of work based on Veneziano's Model and its generalizations, see D. Silvers and J. Yellin, "Review of Recent Work on Narrow Resonance Models," UCRL-19418, preprint, (1969), submitted to Rev. Mod. Phys.
4. M. Froissart, Phys. Rev. 123, 1053 (1961).
5. An excellent review of these results is contained in Chapter 6 of R. J. Eden, High Energy Collisions of Elementary Particles, (Cambridge University Press, New York, 1967). This has been up-dated in a recent talk by R. J. Eden, "Theorems on high energy collisions," invited paper for presentation at the Wisconsin Conf. on Expectations for Particle Reactions at the New Accelerators, March 30-April 1 (1970), unpublished.
6. See for example, the discussion in G. B. Arfken, Mathematical Methods for Physicists, (Academic Press, Inc., New York/London, 1966);

   M. L. Boas, Mathematical Methods in the Physical Sciences, (J. Wiley and Sons, Inc., New York, 1966);

   J. Mathews and R. L. Walker, Mathematical Methods of Physics, (W. A. Benjamin, Inc., New York, 1964).
7. In addition to the references in footnote [6], see H. Goldstein, Classical Mechanics, (Addison-Wesley Publishing Co., Inc., Reading, 1950).

8. M. R. Hestenes, Calculus of Variations and Optimal Control Theory, (John Wiley and Sons, Inc., New York, 1966), hereafter referred to as H. The theorems presented here have been proved by Hestenes only for finite dimensional spaces with a finite number of constraints, whereas the subsequent applications in this paper are to infinite-dimensional spaces. Many of the results in H can be generalized to arbitrary normed vector spaces. See D. G. Luenberger, Optimization by Vector Space Methods, (John Wiley and Sons, Inc., New York 1969), hereafter referred to as L.

9. We will work with subsets S of n-dimensional Euclideon space $\mathscr{E}^n$. If $x, y \in S$, the usual dot product is $(x, y) = x_i y_i$ where the summation convention has been used. Also the usual distance between two points x, y is $|x - y| = \sqrt{(x-y, x-y)}$.

10. δf is also often called the first variation of f. It is important not to confuse the scalar δf with the vector $f'(x_0)$.

11. The following discussion is concerned with the maxima of functions subject to constraints. Since a maximum of (-f) is a minimum of f, it is a simple matter to convert them to theorems on minima. Similarly, our applications concentrate on upper bounds, and omit correspondent results on lower bounds.

12. $\mathscr{L}$ seems to be called the Lagrangian in much of the mathematical literature (and for good reason!), however, we have chosen to reserve this term for dynamical problems.

13. This interpretation is not given in texts on the calçulus of variations. Moreover, even though it is used often to give physical significance to the multipliers introduced in the derivation of thermodynamics, it goes unstated in statistical mechanics texts. Indeed, given the hypothesis of equal a priori probability, all of statistical mechanics involves finding the most probable distribution

of states subject to various kinds of constraints. Although a geometrical interpretation of multipliers appears in H, this simple result does not appear. We found that the result is well-known in that area of control theory concerned with the sensitivity of an optimum to small changes in the constraints, and, in fact, the multipliers are sometimes referred to as sensitivity coefficients. See e.g., D. J. Wilde and C. S. Beightler, Foundations of Optimization, (Prentice Hall, Inc., Englewood Cliffs, 1967); p. 38. See also p. 222 of L.

14. Cf., M. L. Boas, op. cit., p. 150.
15. V. Singh and S. M. Roy, "Unitarity upper and lower bounds on the absorptive parts of elastic scattering amplitudes," preprint, Tata Institute, (1970).
16. M. A. Jacobs et al., "An improved upper bound on the imaginary part of elastic scattering amplitudes," preprint, Tel-Aviv University, (1970).
17. Y. S. Jin and A. Martin, Phys. Rev. 135, B1375 (1964).
18. V. Singh and S. M. Roy, Ann. Phys. 57, 461 (1970).
19. See T. Kinoshita, "The Pomeranchuk Theorem," in Perspectives in Modern Physics, (R. E. Marshak, ed.), (Interscience Publishers, New York, 1966). See also R. J. Eden, Phys. Rev. Letters 16 (1966).
20. S. M. Roy and V. Singh, Phys. Letters 32B, 50 (1970).
21. F. J. Yndurain, Phys. Letters 31B, 368 (1970).
22. A. K. Common, "Froissart bounds with no arbitrary constants," preprint, CERN-TH-1145, April (1970).
23. See Chapter 7 of L. Strictly speaking, we have used the multiplier theorems beyond their realm of applicability. Although the theorems in L make weaker demands on the underlying vector spaces than in H, they usually require far more stringent properties of the functional f and the constraint functionals

f_α and g_α. Unfortunately for our applications, the results in L stop short of any theorems which combine equality and inequality constraints. Incidently, it may be somewhat surprising that inequality constraints appear to be far easier to treat mathematically than equality constraints.

24. For a review, see A. Martin, "Rigorous results from field theory and unitarity," Lectures given at the International School of Subnuclear Physics, Erice (1970), CERN Report TH.1181.

25. J. V. Allaby et al., Phys. Letters 30B, 500 (1969).

26. See, e.g., Martin's review, Ref. [24].

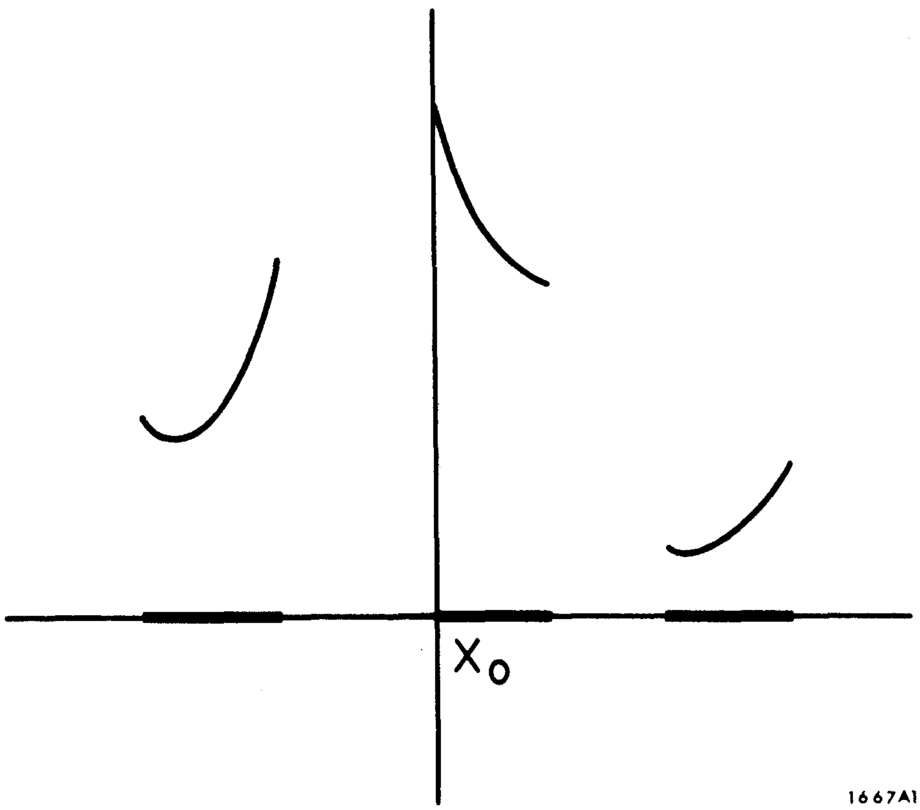

Fig. 1

Maximizing f(x) on the set $g(x) \geq 0$.

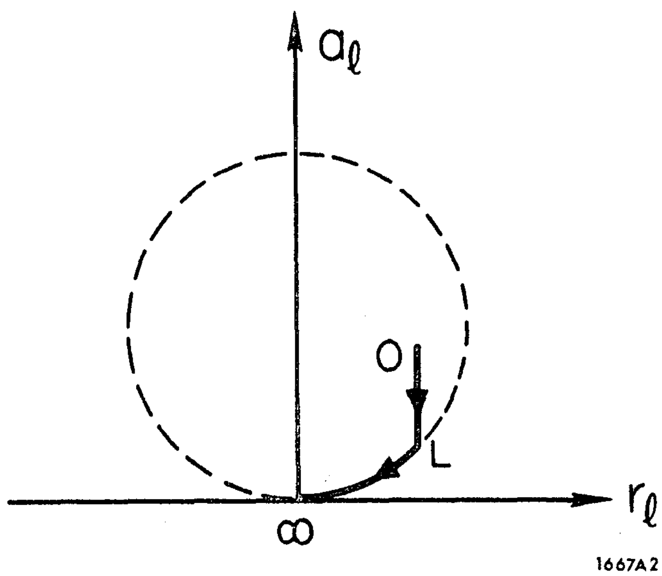

Fig. 2

Argand diagram. The unitarity circle is $a_\ell^2 + r_\ell^2 = a_\ell$. The heavy line indicates the values of the partial wave amplitudes; inelastic for $\ell < L$, elastic for $\ell > L$.

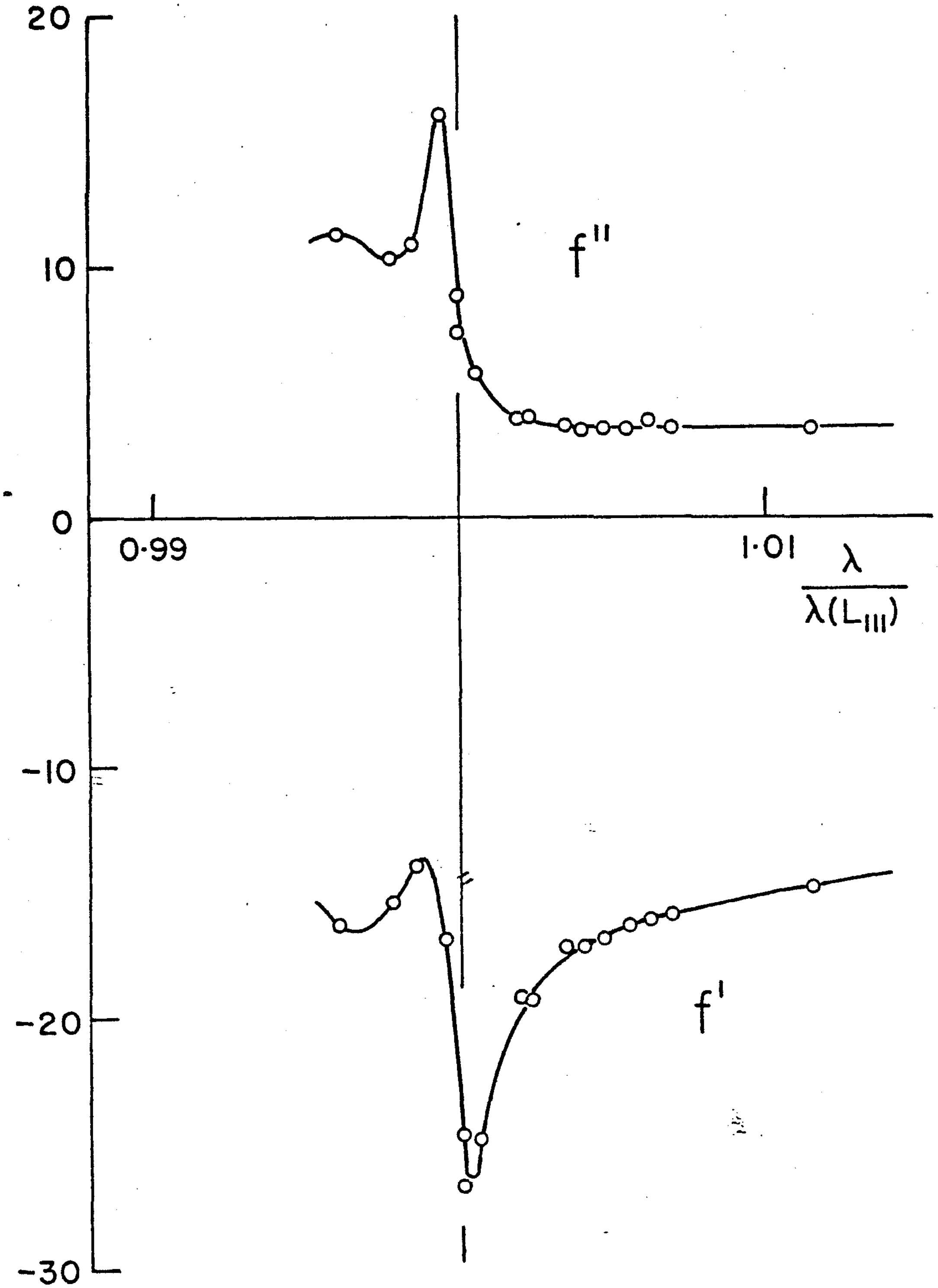

Figure (3) Expansion of figure (2) for the region near the L_{III} edge, showing the oscillations in f'' and f' through the edge.