



Noether symmetries of test charges in the magnetic monopole field

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Abstract

We consider the motion of charged test particles in the presence of a Dirac magnetic monopole. We use an extension of Noether's theorem for systems with magnetic forces and integrate explicitly the corresponding equations of motion.

Keywords Magnetic curves · Noether symmetries · Dirac monopole · Analytic solutions

1 Introduction

The exploration of the properties of magnetic curves – the trajectories followed by charged test particles in the presence of a magnetic field – is a timely problem in mathematical physics and geometry. In particular, it can be shown that (see Proposition 2.1 in [1]) magnetic curves cannot be the geodesics of any affine connection on the Riemannian manifold they propagate. In this sense, the natural setting for studying such systems from a variational point of view is not that of Riemannian but Finslerian geometry. Nevertheless, the problem of integrating the non-geodesic problem of magnetic curves can be tackled using standard Riemannian techniques together with a generalization of Noether's theorem [2].

The simplest non-trivial static magnetic field is that of a monopole. Albeit “physically prohibited” according to classical electromagnetism formulated on a topologically trivial region (e.g. Minkowski spacetime), they can be easily realised as topological defects. That is, while the local dynamics of the electromagnetic field is determined by the differential form of Maxwell's equations, its global properties

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depend, in a crucial manner, on the topology of the domain where the fields are defined [3]. Moreover, it turns out that the condition for electric charge quantization in vacuum is equivalent to a non-trivial topology for spacetime itself (see [4–7]).

Interestingly, long before the modern tools of symmetry methods for integrating differential equations, Poincaré presented the general properties of the magnetic curves of the magnetic monopole in a rather brief note published in 1896 [8] (cf. McDonald’s notes [9] for a more detailed explanation), arguing that the trajectories should lie on the surface of a cone whose vertex is located at the monopole. Then, in 1960 Lapidus and Pietenpol extended Poincaré’s result and obtained the solution in a coordinate system adapted to the cone on which the motion takes place [10]. More recently, Sivardière [11] revisited the problem and used a rotating frame to reach the solution. Then, Mayrand [12] generalised the study of the integrability of the particle dynamics on the cones to higher dimensions. In this work, we frame the general problem of magnetic curves in the language of differential geometry and then use an extension of Noether’s theorem for the non-geodesic motion of test charged particles in the presence of a magnetic monopole field to obtain an exact analytical expression for the corresponding trajectories. Although previous studies have employed symmetry-based methods to address this problem [13–18], to the best of our knowledge, this particular methodology has not been used in this context.

The manuscript is structured as follows. In section 2 we present the magnetic curves problem from a differential geometric Newtonian point of view. In section 3, we recall Sternberg’s lemma “*the electromagnetic field determines the symplectic structure of spacetime*” and lift the field 2-form to phase space to assemble a non-canonical symplectic structure such that the resulting projected Hamiltonian flow of free particles corresponds precisely to the magnetic curves on spacetime. Then, recalling the result by Ikawa [2], we present the first integrals to be obtained in section 4. Then, providing a set of initial conditions on the “equatorial plane” we obtain explicitly the exact trajectories of charged particles moving in the monopole field. We present a graphic analysis showing emergent features such as “velocity ripples” associated with the spring-like trajectory of each individual particle. Thus, the method itself is a novel combination of differential geometry and magnetic integrable systems exhibiting interesting emergent properties.

2 The magnetic curves problem

Consider an n -dimensional ($n \geq 3$) spacetime manifold (M, g) . The path followed by a point particle of mass m is a curve $\gamma : I \subset \mathbb{R} \longrightarrow M$ such that Newton’s second law

$$f^\sharp = ma \quad \text{with} \quad f^\sharp = g^\sharp(f), \quad (1)$$

is satisfied. Here $f \in T^*M$ is a 1-form expressing the external forces responsible for the non-conservation of the particle's momentum, i.e.

$$f = \frac{d}{d\tau} p \quad \text{where} \quad p = mg^b(\dot{\gamma}) \tag{2}$$

$a \in TM$ is the acceleration vector field associated with the Levi-Civita connection ∇ given by

$$a = \frac{D}{d\tau} \frac{d}{d\tau} \gamma = \nabla_{\dot{\gamma}} \dot{\gamma} \quad \text{with} \quad \dot{\gamma} \equiv \frac{d}{d\tau} \gamma \in TM, \tag{3}$$

and τ is an arc-length affine parameter for the curve representing its *proper time*. Derivatives with respect to such parameter are computed by

$$\frac{d}{d\tau} \equiv \mathfrak{L}_{\dot{\gamma}}, \tag{4}$$

where $\mathfrak{L}_{\dot{\gamma}}$ is the Lie derivative along the particle velocity vector field.

We have also introduced the metric isomorphisms $g^b : TM \rightarrow T^*M$ together with $g^\sharp : T^*M \rightarrow TM$ where $g^\sharp \circ g^b = \text{id}_{TM}$ and $g^b \circ g^\sharp = \text{id}_{T^*M}$ for every $x \in M$ to establish (1) as a relation in TM .

Expression (1) is a mere statement of the compatibility between the Lie derivative and the Levi-Civita connection (cf. [19]),

$$\mathfrak{L}_v v^b = (\nabla_v v)^b + \frac{1}{2} dg(v, v), \tag{5}$$

for the velocity vector field of an affinely parametrised curve γ . As it is well known, a free particle is determined by $f = 0$ and thus satisfies the geodesic equation $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.

An electromagnetic field F on (M, g) is a solution to Maxwell's equations

$$dF = 0 \quad \text{and} \quad \star d\star F = j, \tag{6}$$

where F is a closed 2-form associated with the conservation of the electromagnetic flux while j is a closed $(n - 1)$ -form expressing the *global* conservation of electric charge, namely

$$\oint_{\partial\Omega^3} F \stackrel{!}{=} 0 \quad \text{and} \quad \oint_{\partial\Omega^n} j \stackrel{!}{=} 0, \tag{7}$$

where \star is the Hodge dual operator while $\partial\Omega^3$ and $\partial\Omega^n$ denote the *boundaries* of 3 and n dimensional regions of M , respectively; and we use $\stackrel{!}{=}$ to denote an equality that is physically required.

In this setting, the Lorentz force 1-form is postulated to be [20]

$$f \stackrel{!}{=} q i_{\dot{\gamma}} F, \tag{8}$$

or, equivalently,

$$f^\sharp \equiv q \phi(\dot{\gamma}) = q g^\sharp(i_{\dot{\gamma}}F), \quad (9)$$

where $\phi : TM \rightarrow TM$ is a tangent bundle automorphism which, by construction, satisfies the compatibility condition

$$g(\phi(v), u) = F(v, u), \quad (10)$$

for every pair of vector fields v and u . Here i denotes the interior product of vector fields and differential forms (we use the same convention as in [21]). In a local coordinate chart, ϕ is nothing but the electromagnetic field with an index raised by the metric, namely $\phi_a^b = g^{cb}F_{ac}$.

Therefore, the motion of a particle with mass m and charge q in the presence of the field F is given by

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{q}{m} \phi(\dot{\gamma}), \quad (11)$$

This is the geometric form of the familiar coordinate expression for the Lorentz force $f^{\sharp a} = g^{ab}F_{bc}(q\dot{\gamma}^c)$.

It follows directly from Eq. (10) and (11) that the force is orthogonal to the particle velocity

$$g(\phi(\dot{\gamma}), \dot{\gamma}) = F(\dot{\gamma}, \dot{\gamma}) = 0, \quad (12)$$

and that the norm of the particle velocity is preserved along its own motion

$$\frac{d}{d\tau}|\dot{\gamma}|^2 = \mathfrak{L}_{\dot{\gamma}}g(\dot{\gamma}, \dot{\gamma}) = 2g(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}) = 0, \quad (13)$$

where we have used the fact that for a metric compatible connection

$$\mathfrak{L}_v g(u, w) = g(u, \nabla_v w) + g(\nabla_v u, w). \quad (14)$$

The magnetic curves problem is that of integrating the system of n second order, non-linear, coupled, ordinary differential equations given by (11) for a force depending linearly in the particle velocity satisfying (12) and (13). The linear dependence of the Lorentz force in the velocity, makes the integral curves of (11) non-reversible under the change $q \rightarrow -q$ or $\tau \rightarrow -\tau$ and is analogous to a particular form of the Zermelo navigation problem studied in the context of geodesic motion on Finsler manifolds [22, 23]

3 First integrals in the Hamiltonian formulation

The equations of motion (11) are in general hard to integrate. However, as it is usually the case, a significant simplification arises when there are symmetries present in the formulation of the problem, namely symmetries of g and ϕ . For the geodesic problem, Noether's theorem provides us with the standard tool for obtaining first integrals of the geodesic equation. However, it is not directly applicable to the non-geodesic motion of the magnetic curves problem. Here, we will *lift* the problem to phase space and use Sternberg's construction of a non-canonical symplectic structure $\omega_F \in \Lambda^2(TM)$ generated by the electromagnetic field F [24, 25] together with the extension of Noether's theorem developed by Ikawa [2, 26, 27] to find first integrals for the *non-geodesic* dynamics of the *free* particle hamiltonian flow associated with ω_F . Here, $\Lambda^2(TM)$ denotes the space of 2-forms on TM .

Solving (11) is equivalent to obtaining the integral curves of the *canonical lift* of $\dot{\gamma}$ given by its prolongation to the tangent bundle [28]

$$X_h = \text{Pr}(\dot{\gamma}) \in T(TM), \quad (15)$$

satisfying Hamilton's equations

$$i_{X_h}\omega_F = dh. \quad (16)$$

Here, ω_F is the 2-form given by

$$\omega_F = \omega - q\pi^*(F), \quad (17)$$

where the standard symplectic form ω is given by

$$\omega = -d\pi^*(p) = -d\pi^*[mg^b(\dot{\gamma})], \quad (18)$$

where $\pi^* : T^*M \rightarrow T^*(TM)$ is the induced map associated with the bundle projection $\pi : TM \rightarrow M$ and $h : TM \rightarrow \mathbb{R}$ is the *free* particle hamiltonian

$$h = \frac{1}{2}m g(\dot{\gamma}, \dot{\gamma}). \quad (19)$$

It follows from (6) and (18) that ω is closed and non-degenerate. Indeed, ω_F is a symplectic form on TM .

The evolution of a function $\lambda : TM \rightarrow \mathbb{R}$ along the hamiltonian flow X_h is given by the Poisson bracket $\{ \}_F$ associated with ω_F , that is

$$\frac{d}{d\tau}\lambda = \mathfrak{L}_{X_h}\lambda = \{\lambda, h\}_F \equiv \omega_F(X_\lambda, X_h). \quad (20)$$

Note that derivations with respect to the proper time of γ are given by the corresponding Lie derivative with respect to the lift of $\dot{\gamma}$. This is the extension of the correspondence between the hamiltonian and geodesic flows to the case of magnetic curves.

Let us assume that the set of symmetry generators of the metric *and* the electromagnetic field

$$K = \{ \xi \in TM \mid \mathfrak{L}_\xi g = 0 \text{ and } \mathfrak{L}_\xi F = 0 \}, \quad (21)$$

is non-empty. In such case $\#K = s \leq n(n+1)/2$. Then, for each $\xi \in K$ we have

$$\mathfrak{L}_\xi F = i_\xi dF + d i_\xi F = 0. \quad (22)$$

Since F is closed, it follows that, at least locally, there is a function $\psi_\xi : U \subset M \rightarrow \mathbb{R}$ such that

$$i_\xi F = d\psi_\xi. \quad (23)$$

Furthermore, the function

$$\lambda_\xi = i_{\dot{\gamma}} p_\xi - \psi_\xi \quad \text{where} \quad p_\xi = g^b(\xi), \quad (24)$$

is constant along the free particle Hamiltonian flow (20), namely

$$\frac{d}{d\tau} \lambda_\xi = \frac{d}{d\tau} (i_{\dot{\gamma}} p_\xi - \psi_\xi) = 0. \quad (25)$$

That is, Eq. (24) is a first integral of (11) associated with the symmetry generator ξ . Additionally, since the norm of $\dot{\gamma}$ is also a constant of motion, we have a set of $s+1$ first integrals for the system. In this sense, we can consider Eq. (25) as an extension of Noether's theorem for the problem of magnetic curves.

4 The magnetic curves of the Dirac monopole

Let us consider (M, g) to be Minkowski spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ with the section $r = 0$ removed, $M = \mathbb{R}^4 - \{r = 0\}$, with

$$g = -dt \otimes dt + dr \otimes dr + r^2 g_{S^2}, \quad (26)$$

where

$$g_{S^2} = d\vartheta \otimes d\vartheta + \sin(\vartheta)^2 d\varphi \otimes d\varphi, \quad (27)$$

is the standard metric tensor on the 2-sphere S^2 .

The Dirac monopole electromagnetic field is given by

$$F = k \sin(\vartheta) d\vartheta \wedge d\varphi. \quad (28)$$

It is straightforward to verify that F satisfies the sourceless Maxwell's equations. Indeed

$$dF = 0 \quad \text{and} \quad d\star F = d\left[\frac{k}{r^2} dr \wedge dt\right] = 0, \quad (29)$$

are closed 2-forms on M . However, integrating F over a space-like 2-surface homeomorphic to S^2 enclosing the origin at any given coordinate time t yields

$$\oint_{S^2} F = 4\pi k, \quad (30)$$

which follows from the non-trivial topological nature of M . That is, the non-trivial homology class $[S^2] \in H^2(M)$ is associated with the magnetic charge producing the field F , albeit in the domain $M \subset \mathbb{R}^4$ the field itself is indeed sourceless. Moreover, the 'spatial' vector fields measured by a static observer are

$$\vec{E}_{\text{static}} = g^\sharp \left[i_{\frac{\partial}{\partial t}} F \right] = 0, \quad (31)$$

and

$$\vec{B}_{\text{static}} = g^\sharp \left[i_{\frac{\partial}{\partial t}} \star F \right] = -\frac{k}{r^2} \hat{e}_{(r)}, \quad (32)$$

where $\hat{e}_{(r)}$ is the normalised, outgoing, space-like, radial vector field.

The field F appears to be regular everywhere except at the poles of the 2-sphere, $\vartheta = 0, \pi$ where it becomes singular. However this is merely a coordinate issue which is solved by considering the two overlapping charts covering the entire surface of the sphere except for the north and south poles, respectively. This path leads to the well known Dirac's charge quantization which we will not use in this work [6, 7].

Let us write the velocity of a charged particle in M as

$$\dot{\gamma} = \dot{t} \frac{\partial}{\partial t} + \dot{r} \frac{\partial}{\partial r} + \dot{\vartheta} \frac{\partial}{\partial \vartheta} + \dot{\varphi} \frac{\partial}{\partial \varphi}. \quad (33)$$

Then, the Lorentz force per charge q is given by

$$\phi(\dot{\gamma}) = \frac{k}{r^2} \left[\dot{\varphi} \sin(\vartheta) \frac{\partial}{\partial \vartheta} - \frac{\dot{\vartheta}}{\sin(\vartheta)} \frac{\partial}{\partial \varphi} \right], \quad (34)$$

and the equations of motion (11) are

$$\ddot{i} = 0, \quad (35)$$

$$\ddot{r} = r \left[\sin(\vartheta)^2 \dot{\varphi}^2 + \dot{\vartheta}^2 \right], \quad (36)$$

$$\ddot{\vartheta} = \sin(\vartheta) \cos(\vartheta) \dot{\varphi}^2 - \frac{2}{r} \dot{r} \dot{\varphi} + \frac{q}{m} \frac{k}{r^2} \sin(\vartheta) \dot{\varphi}, \quad (37)$$

$$\ddot{\varphi} = -2 \cot(\vartheta) \dot{\vartheta} \dot{\varphi} - \frac{2}{r} \dot{r} \dot{\varphi} - \frac{q}{m} \frac{k}{r^2} \frac{\dot{\vartheta}}{\sin(\vartheta)}. \quad (38)$$

The lift of $\dot{\gamma}$ to $T(TM)$ is

$$\begin{aligned} \text{Pr}(\dot{\gamma}) &= \dot{i} \frac{\partial}{\partial t} + \dot{r} \frac{\partial}{\partial r} + \dot{\vartheta} \frac{\partial}{\partial \vartheta} + \dot{\varphi} \frac{\partial}{\partial \varphi} \\ &\quad + \ddot{i} \frac{\partial}{\partial \dot{i}} + \ddot{r} \frac{\partial}{\partial \dot{r}} + \ddot{\vartheta} \frac{\partial}{\partial \dot{\vartheta}} + \ddot{\varphi} \frac{\partial}{\partial \dot{\varphi}}, \end{aligned} \quad (39)$$

while the symplectic two form ω_F , Eq. (17), is given by

$$\begin{aligned} \omega_F &= -m \, dt \wedge \dot{d}i + m \, dr \wedge \dot{d}r \\ &\quad + mr^2 \left[d\vartheta \wedge \dot{d}\vartheta + \sin^2(\vartheta) \, d\varphi \wedge \dot{d}\varphi \right] \\ &\quad - 2r \left[m \dot{\vartheta} \, dr \wedge d\vartheta + m \sin^2(\vartheta) \, \dot{\varphi} dr \wedge d\varphi \right. \\ &\quad \left. + \sin(\vartheta) (mr \dot{\varphi} \cos(\vartheta) + qk) \, d\vartheta \wedge d\varphi \right], \end{aligned} \quad (40)$$

and the free particle hamiltonian is simply

$$h = \frac{1}{2} m \left[-\dot{i}^2 + \dot{r}^2 + r^2 \dot{\vartheta}^2 + r^2 \sin^2(\vartheta) \dot{\varphi}^2 \right]. \quad (41)$$

The reader can easily verify that the hamiltonian vector field X_h satisfying Hamilton's equations (16) corresponds precisely to substituting (35) - (38) into (39). Indeed, Hamilton's equations of motion (16) are equivalent to (11), i.e.

$$i_{X_h} \omega_F = dh \iff X_h = \text{Pr}(\dot{\gamma}) \Big|_{\nabla_{\dot{\gamma}} \dot{\gamma} = \frac{q}{m} \phi(\dot{\gamma})}. \quad (42)$$

To integrate the system (35) - (38), we observe that the elements of K [cf. Eq. (21)] correspond to the symmetry generators of S^2 together with that of time translations,

that is

$$\xi_1 = \frac{\partial}{\partial t}, \quad (43)$$

$$\xi_2 = \frac{\partial}{\partial \varphi}, \quad (44)$$

$$\xi_3 = \cot(\vartheta) \sin(\varphi) \frac{\partial}{\partial \varphi} - \cos(\varphi) \frac{\partial}{\partial \vartheta}, \quad (45)$$

$$\xi_4 = \cot(\vartheta) \cos(\varphi) \frac{\partial}{\partial \varphi} + \sin(\varphi) \frac{\partial}{\partial \vartheta}. \quad (46)$$

Then, integrating (23) for each symmetry generator yields

$$\psi_1 = c_1, \quad (47)$$

$$\psi_2 = \frac{q}{m} k \cos(\vartheta) + c_2, \quad (48)$$

$$\psi_3 = -\frac{q}{m} k \sin(\vartheta) \sin(\varphi) + c_3, \quad (49)$$

$$\psi_4 = -\frac{q}{m} k \sin(\vartheta) \cos(\varphi) + c_4, \quad (50)$$

where the c_i 's are integration constants. Finally, the first integrals of the system are [cf. Eq. (24)]

$$\dot{t} = \lambda_1, \quad (51)$$

$$r^2 \dot{\varphi} \sin^2(\vartheta) - \frac{q}{m} k \cos(\vartheta) = \lambda_2, \quad (52)$$

$$r^2 [\sin(\vartheta) \cos(\vartheta) \sin(\varphi) \dot{\varphi} - \cos(\varphi) \dot{\vartheta}] + \frac{q}{m} k \sin(\vartheta) \sin(\varphi) = \lambda_3, \quad (53)$$

$$r^2 [\sin(\vartheta) \cos(\vartheta) \cos(\varphi) \dot{\varphi} + \sin(\varphi) \dot{\vartheta}] + \frac{q}{m} k \sin(\vartheta) \cos(\varphi) = \lambda_4, \quad (54)$$

together with the norm of the velocity [which is equivalent to the conservation of hamiltonian (41)]

$$-\dot{t}^2 + \dot{r}^2 + r^2 \dot{\vartheta}^2 + r^2 \sin^2(\vartheta) = -\lambda_5^2. \quad (55)$$

The fact that we have five constants of motion when we have four degrees of freedom yields the system superintegrable.

A long but straightforward algebraic manipulation allows us to express φ as a function of ϑ , namely

$$\tan(\varphi) = \frac{\zeta_1(\vartheta)\lambda_3 - \zeta_2(\vartheta)\lambda_4}{\zeta_1(\vartheta)\lambda_4 + \zeta_2(\vartheta)\lambda_3}, \quad (56)$$

where

$$\zeta_1(\vartheta) = qk - m\lambda_2 \cos(\vartheta), \quad (57)$$

and

$$\zeta_2(\vartheta) = \left[\left(\lambda_3^2 + \lambda_4^2 \right) m^2 + 2kmq\lambda_2 \cos(\vartheta) - k^2 q^2 - m^2 \left(\lambda_2^2 + \lambda_3^2 + \lambda_4^2 \right) \cos^2(\vartheta) \right]^{\frac{1}{2}}. \quad (58)$$

Thus, the corresponding angular momenta are given by

$$L_\varphi = mr^2 \dot{\varphi} = \frac{kq \cos(\vartheta) - m\lambda_2}{\sin^2(\vartheta)}, \quad (59)$$

and

$$L_\vartheta = mr^2 \dot{\vartheta} = -\frac{\zeta_2(\vartheta)}{\sin(\vartheta)}. \quad (60)$$

It is easy to see that these are not conserved [cf. Eq. (20), above], that is

$$\frac{d}{d\tau} L_\varphi = -\frac{qk}{\sin^3(\vartheta)} \left[1 + \cos^2(\vartheta) - \frac{2m\lambda_2 \cos(\vartheta)}{kq} \right] \dot{\vartheta}, \quad (61)$$

and

$$\frac{d}{d\tau} L_\vartheta = -\frac{[qk - m\lambda_2 \cos(\vartheta)][kq \cos(\vartheta) - m\lambda_2]}{\sin^2(\vartheta) \zeta_2(\vartheta)} \dot{\vartheta}. \quad (62)$$

Indeed, the effect of the magnetic field on the non-radial motion of charged particles is a torque. This also illustrates that the first integrals associated with the extension of Noether's theorem we are using, cf. Eq. (24), and in particular Eqs. (52)–(54), do not correspond to the usual conserved momenta of the corresponding geodesic problem. Indeed, magnetic curves are not geodesics.

Finally, substituting (51), (52) and (60) into (55) yields the radial equation

$$\dot{r}^2 = \frac{1}{r^2} \left[\frac{k^2 q^2}{m^2} - \left(\lambda_2^2 + \lambda_3^2 + \lambda_4^2 \right) \right] + \lambda_1^2 - \lambda_5^2. \quad (63)$$

Therefore, since φ is determined solely in terms of ϑ , the magnetic curves in a Dirac monopole are completely determined by solving the radial Eq. (63) and then integrating the angular momentum (60) while the coordinate time evolves as the solution of (51).

4.1 Equatorial initial conditions

From symmetry considerations any given initial condition can be placed in the equatorial plane. Thus, let us consider the set of initial conditions

$$\dot{r}(0) = v_r, \quad r(0) = r_0, \quad (64)$$

$$\dot{\phi}(0) = \omega_\phi, \quad \phi(0) = 0, \quad (65)$$

$$\dot{\vartheta}(0) = 0, \quad \vartheta(0) = \pi/2. \quad (66)$$

The constants of motion can be expressed in terms of the initial data as

$$\lambda_1 = \sqrt{1 + v_\phi^2 + v_r^2}, \quad (67)$$

$$\lambda_2 = -r_0^2 \omega_\phi, \quad (68)$$

$$\lambda_3 = 0, \quad (69)$$

$$\lambda_4 = -qk/m, \quad (70)$$

$$\lambda_5 = 1. \quad (71)$$

Here, the last constant represents the speed of light normalisation for a timelike trajectory and the equatorial initial velocity is defined as

$$v_\phi = r_0 \omega_\phi. \quad (72)$$

Using (67) - (71) to integrate (63) and (60) we have

$$r^2 = (v_\phi^2 + v_r^2)\tau^2 - 2r_0 v_r \tau + r_0^2, \quad (73)$$

and

$$\cos(\vartheta) = -kq \frac{L_0}{\ell^2} [1 - \cos(\alpha(\tau))], \quad (74)$$

respectively. Here the function $\alpha(\tau)$ is given by

$$\alpha(\tau) = \frac{\ell}{L_0} \left[\arctan \left(\frac{[v_\phi^2 + v_r^2]\tau + r_0 v_r}{r_0 v_\phi} \right) - \arctan \left(\frac{v_r}{v_\phi} \right) \right], \quad (75)$$

with the angular momentum constants defined as

$$\ell^2 = k^2 q^2 + L_0^2 \quad \text{and} \quad L_0 = m r_0 v_\phi. \quad (76)$$

Note that Eq. (74) reveals that the constant k , quantifying the monopole strength [cf. Eq. (30)], should have units of action per charge.

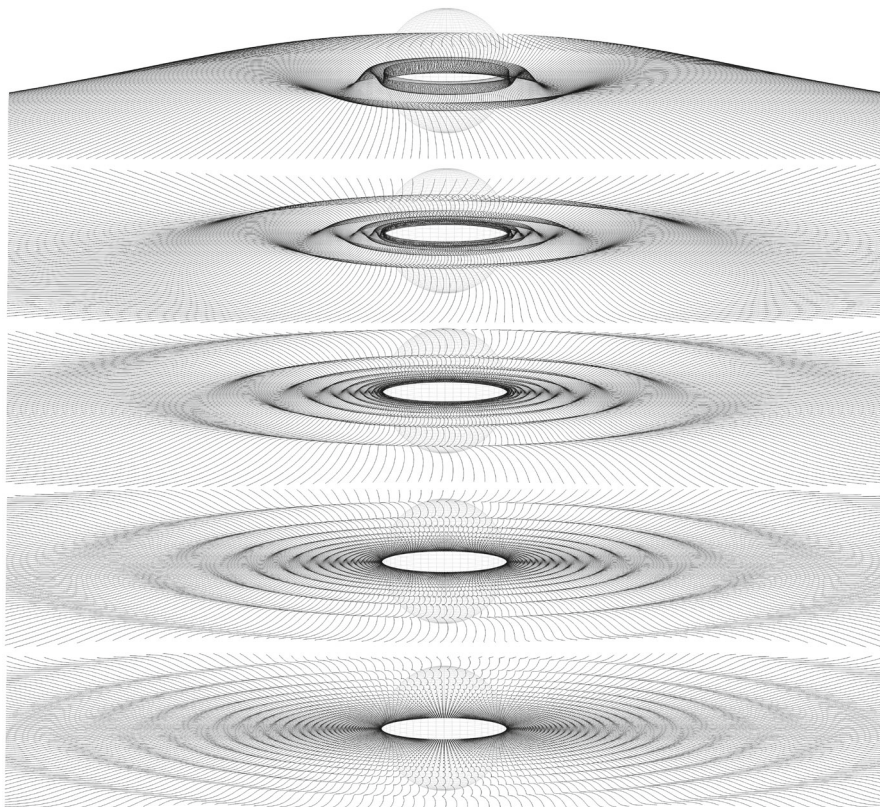


Fig. 1 The magnetic curves of the Dirac monopole. From top to bottom, we show the curves corresponding to initial velocity $v_\varphi = 1/10, 1/25, 1/50, 1/100$ and $1/200$, tangent to the equatorial plane. Here, $r_0 = 1$

Substituting (74) together with the constants (68)–(70) into (56) we obtain

$$\tan(\varphi) = -\frac{L_0 \ell \sin(\alpha(\tau))}{L_0^2 \cos(\alpha(\tau)) + k^2 q^2}. \quad (77)$$

Finally, the time coordinate is affinely related to proper time as

$$t = \sqrt{1 + v_\varphi^2 + v_r^2} \tau. \quad (78)$$

It can be directly verified that, indeed, (73), (74), (77) and (78) solve the equations of motion (35) - (38). Therefore, we have obtained the explicit solution to the magnetic curves problem for the Dirac monopole.

4.2 Some features of the magnetic curves

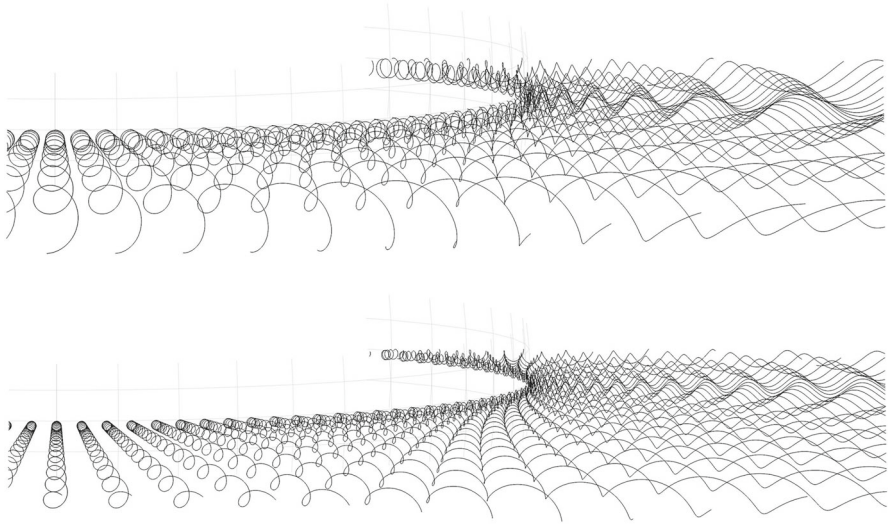


Fig. 2 Detail of the magnetic curves shown in Fig. 1 corresponding to $v_\varphi = 1/50$ (top) and $v_\varphi = 1/100$ (bottom)

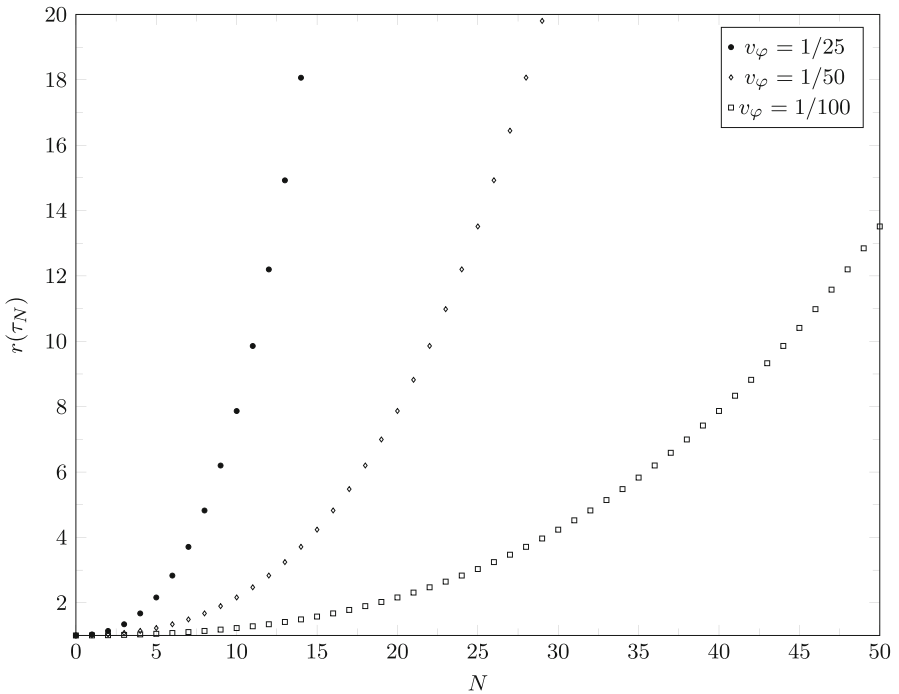


Fig. 3 Radial spreading of the apparent oscillations shown in Fig. 1. The dots represent the radial distance for each cycle where $\vartheta(\tau_N) = \pi/2$ for various initial tangential velocities

In Fig. 1 we show the magnetic curves of the Dirac monopole for various initial data. Discarding dissipative effects due to radiation, which is easily justified for non-relativistic velocities [29, 30] and as noted by Poincaré [8], the trajectories of charged particles in the monopole field are constrained to lie on a cone with apex at the location of the monopole. This can be directly verified by noting that the aperture is easily determined to be the constant

$$\theta_{\text{aperture}}^{\pm} = \frac{\pi}{2} \pm \arctan\left(\frac{\ell^2 - 2L_0^2}{2kqL_0}\right), \quad (79)$$

where \pm denotes the direction of the initial velocity v_{φ} (cf. Figure 2). This result is in well agreement with that of Zia [31].

Moreover, there is a minimum for the radial distance reached by a charged particle given by

$$r_{\min}^2 = \left(\frac{v_{\varphi}^2}{v_r^2 + v_{\varphi}^2}\right) r_0^2. \quad (80)$$

Finally, note that the emergent wave pattern shown in Fig. 1 is a consequence of the elongation of the spring-like trajectories and the ever decreasing angular velocities. Indeed,

$$\lim_{\tau \rightarrow \infty} \dot{\vartheta} = 0 \quad \text{and} \quad \lim_{\tau \rightarrow \infty} \dot{\varphi} = 0, \quad (81)$$

while each cycle, considering a purely tangential initial velocity, occurs every time that $\vartheta(\tau_N) = \pi/2$ (cf. Eqs. (21)–(23) in [9]), that is

$$\tau_N = \frac{r_0}{v_{\varphi}} \tan\left(2\pi N \frac{L_0}{\ell}\right). \quad (82)$$

It is worth recalling that the tangential velocity is constrained by the speed of light. Furthermore, since we are neglecting radiation effects due to the acceleration of the charged particles, we are in the non-relativistic limit, that is $v_{\varphi} \ll 1$. In such case, we can approximate (82) as

$$\tau_N \approx \left[\frac{2\pi r_0^2 m}{kq} - \left(\frac{m}{kq}\right)^3 \pi r_0^4 v_{\varphi}^2\right] N + \left[\frac{8}{3} \left(\frac{\pi m}{kq}\right)^3 r_0^4 v_{\varphi}^2\right] N^3. \quad (83)$$

In Fig. 3 we can see the way in which higher tangential velocities yield a very rapid radial spreading, while in the low velocity limit more “oscillations” are observed.

5 Closing remarks

In this work, we have obtained in a detailed and explicit manner the solution to the magnetic curves problem in the case of a magnetic monopole. The geometric formulation allows for a direct extension to arbitrary Riemannian manifolds and magnetic fields. In particular, we present the exact analytical expressions in spherical coordinates for the curves together with a visualization of the velocity field, exhibiting an emerging ripple pattern. Moreover, this geometric perspective can be directly extended to include dissipative effects due to radiation by means of contact geometric techniques, which will be the subject of a forthcoming work.

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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