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Models for RG running for gravitational couplings and applications

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Declaration of Authorship

I, Lesław RACHWAŁ, declare that this thesis titled, 'Models for RG running for gravitational couplings and applications' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.

- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.

- Where I have consulted the published work of others, this is always clearly attributed.

- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.

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- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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"Eppur si corre"

almost Galileo Galilei

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Chapter 1

Introduction

The problems of quantum gravity have been in the core of research in theoretical high energy physics for last 50 years. During the course of time various approaches and methods were developed. These developments resulted in increase of understanding, not only problems in quantum gravity, but also in gauge theory and quantum field theory in general. Despite any direct experimental proof of quantum gravity prediction, research in this field have been intense and focused mainly on theoretical aspects. One of the earliest noticed problem of quantum gravity was the observation that in a simplest version it is a nonrenormalizable theory (due to the dimensionful coupling constant). Namely not all divergences in Einstein-Hilbert gravitation quantized perturbatively and covariantly around flat Minkowski spacetime can be absorbed in the redefinion of couplings present in the action. Therefore quantum gravity needs nontrivial UV completion. To find a self-consistent UV-complete theory of quantum gravity is a very difficult task. This is the reason, why in the meantime many simplified toy-models have been analyzed, which supposed to capture some of the important features of quantized gravity. However perturbative problems of quantum gravity in deep UV didn't pose an obstacle in analysis and application of low-energetic version of the theory. At low energy (much lower than the Planck scale) a marriage of quantum mechanics and general relativity was successful, especially if understood in terms of effective field theory of gravitational interactions.

In analysis of the UV behaviour of quantum gravity, one of the most useful tool revealed to be Renormalization Group methods. Understanding that couplings in quantum field theory exhibit dependence on the momentum scale of the process was one of the major achievements in field theory. Generically in any quantum field theory which is not conformal, coupling parameters are not constants and are running functions of energies. The same concerns quantum gravity, where the running of gravitational couplings was not commonly established even till recently. When speaking about RG flows we can distinguish various types, differing slightly in the underlying physical ideas. The simplest RG flow is given by perturbative analysis around gaussian fixed point, where the values of the couplings vanish. This is given by Callan-Symanzik flow in standard perturbative QFT. Another type of the RG flow we obtain by analyzing change of the average effective action with RG scale. This last effective action interpolates between bare action in UV and quantum effective action in IR. In this approach, which was termed "functional" or "exact" RG, we don't rely on perturbativity of coupling parameters. Therefore one of the advantages is that we can describe RG running of couplings around nontrivial FP of RG. This brings the connection with conformal field theories, which describe physics at FP. Brand new type of RG flow derives from holography. Although it is motivated by famous AdS/CFT correspondence, the application of holographic ideas go far beyond the original domain. It is remarkable that $d+1$-dimensional holographic spacetime can possess a knowledge about RG running of couplings in d-dimensional field theory living on boundary or a brane. On the other hand this bulk spacetime can be understood as a geometrization of RG flow. All these three types of RG flows are closely related. The evidence come from nongravitational quantum field theories as well as from field theories with dynamical gravitation.

There are basically two main ways, how the theory of quantum gravitational perturbations around flat spacetime can be completed in UV. First is that the problems of perturbatively nonrenormalizable quantum field theories are solved by inclusion of new heavy degrees of freedom. These new quanta do not appear in low-energetic spectrum and only high energetic perturbations can excite them. Moreover their interactions (with known degrees of freedom and between themselves) are tightly constrained. In the result

the theory enjoys new dynamics at high energy, which solves the renormalizability and unitarity issues. The best studied example of such version of UV completion is given by W bosons model for 4-fermion interactions and for quantum gravity by string theory. In the latter example to the low energy spectrum of quantum gravity with massless graviton, is added whole infinite tower of heavy higher spin fields. The other possibility opens up when in the UV theory flows to a nontrivial FP of RG. If additionally the critical surface, on which this FP exists, is finite dimensional, then the theory doesn't lose its predictive power. In this case we have the notion of nonperturbative renormalizability. If the parameters of the theory are chosen in such a way, that effective action lies on an RG safe trajectory, then during the RG evolution, the theory and quantum divergences are under control. There is a strong hope, that such asymptotically safe in UV theory, can heal itself from the perturbative problems present at low energy. A third hypothetical option for UV completion is one of the non-Wilsonian type. Some special theories may avoid perturbative problems of quantized versions by invoking production of special classical configurations at high energy. This is the most recent proposal and it was dubbed as classicalization. It was conjectured that Einstein gravity is self-complete and in this way classicalization is implemented there via production of black holes at trans-Planckian energy. It might be true, that there is some relation between all these three mechanisms of UV completion.

Independently of the UV completion, quantum gravity gives some unambiguous predictions at low energy. To some extent it is a unique, universal and predictive theory of massless quanta of gravitational interactions. This is the best understood in the framework of effective field theories. The low energetic action contains only terms with the smallest number of derivatives, so only the simplest Einstein-Hilbert Lagrangian is used. In this effective theory there exist observables, which do not depend on the particular way of UV completion. Although their experimental confirmation is still very far, they are genuine predictions of quantum gravity. There are different ways, by which, one can obtain quantum effective action in infrared limit. However it is without any doubt that low-energetic predictions of quantum gravity are calculable and solid, regardless of any complicated dynamics which saves the theory in UV.

The plan for this thesis is as follows. In the first part we discuss the relation between two different RG flows: functional and holographic one. The bigger emphasis is put on the novel holographic RG flow and we devote full third chapter for studying holographic RG flow geometries. We are not only interested in flows for gravitational couplings, we also consider standard RG flows from field theories with matter. The second part of this work is divided into two chapters. In the fourth chapter we study classicalization for nonlinear sigma model understood as a toy example before attacking more difficult problems of full quantum gravity. We also point there possible relations between classicalization and asymptotic safety as between two similar in some conditions mechanisms for UV completion. In the fifth chapter we consider universal 1-loop effective action in system of gravitating scalar field. We use new methods to derive its IR limit and we compute few low-energetic observables in such effective field theory of gravitational interactions. Finally in the sixth chapter we shortly collect main obtained results and conclude. The material presented in this work is partially based on two scientific articles [42] and [63], I published during my PhD studies.

Part I

Holographic vs. Exact RG Flows

Chapter 2

Planck mass and Higgs VEV in Holographic vs. Exact 4D RG

2.1 Motivation

In this chapter we describe in details the computation of the scale-dependence of the Planck mass and of the vacuum expectation value of the Higgs field using two very different renormalization group methods: a "holographic" procedure based on Einstein's equations in five dimensions with matter confined to a 3-brane, and a "functional" procedure in four dimensions based on a Wilsonian momentum cutoff. Both calculations lead to very similar results, suggesting that the coupled theory approaches a non-trivial fixed point in the ultraviolet.

One of the most remarkable recent developments in quantum field theory is the realization that the coupling of a theory to gravity in $d + 1$ dimensions can yield information about the renormalization group (RG) running of couplings in that particular theory in d dimensions. This idea is contained in the famous construction by Randall and Sundrum [1], and has been sharpened in a number of subsequent publications [3, 4, 5, 6, 7, 8]. While the notion of "holography" has come to have a rather specific meaning closely related to Quantum Gravitation and the famous AdS/CFT correspondence [9, 10], here we will generically

call "holographic RG" the flow of couplings of a d-dimensional theory, which is obtained by viewing it as living on a $(d-1)$-brane coupled to gravity in $(d+1)$ dimensions, and identifying the transverse coordinate with the RG scale.

In a different vein, there have been various significant developments in the use of "functional RG equations", *i.e.* equations which describe in a single stroke the running of infinitely many couplings [11, 12]. This method has proved particularly helpful in the study of perturbatively nonrenormalizable theories with the aim of establishing (or refuting) the existence of non-trivial UV fixed points (FPs) of Renormalization Group, that could be used for a fundamental (and not depending on perturbative scheme) definition of the theory [13], a property that has become known as "asymptotic safety" (AS) [14]. Successful attempts to "renormalize the nonrenormalizable" quantum field theories have been first reported in [15], with subsequent works using the functional RG largely focusing on gravity [16, 17, 18] and more recently also on electroweak physics [19, 20, 21]; see [22] for an overview. It must be added here, that theory of strong interactions - QCD is asymptotically safe, because asymptotic freedom is a special case of AS with vanishing FP values of the couplings. Functional RG methods have been successfully applied also to this theory in the infrared limit giving one of its nonperturbative description [24]. It still remains a challenge to solve functional RG flows equations exactly as this is equivalent to solving the full interacting quantum theory. But a particular strength of the exact RG is its flexibility allowing for a variety of systematic approximations and truncations adapted to the problem at hand, which has led to new insights [23].

To the extent that holographic and functional RG are equivalent descriptions of the same physics, they must be related in some way. There has been some work in this direction [25, 26, 27], but clearly much remains to be done. In this chapter, instead of exploring this relation from first principles, we evaluate similarities and differences of the two methods for a sample theory, which still incorporates some basic features of Nature and has some phenomenological significance in particle physics. The toy model to be considered is a $SO(N)$ non-linear sigma model coupled to gravity with an Euclidean action of the form $S = S_g + S_m$, where the gravitational action is in the form of Einstein-Hilbert:

$$S_g = m_P^2 \int d^4x \sqrt{g}\, R \tag{2.1}$$

with $m_P^2 = 1/(16\pi G)$ and S_m is the matter action minimally coupled to gravitational background. The matter action, for the $SO(N)$ non-linear sigma model can be obtained by a limiting procedure from the corresponding linear theory, which contains a multiplet of N real scalars ϕ^a with an action

$$S_m = \int d^4x \sqrt{g}\, \left(\frac{1}{2} \sum_{a=1}^{N} g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a + V(\rho^2) \right), \tag{2.2}$$

where the square of the radius equals $\rho^2 = \sum_{a=1}^{N} \phi^a \phi^a$, and the potential is in the form of Higgs potential $V = \lambda(\rho^2 - v^2)^2$ with $v^2 = \langle \rho^2 \rangle$. The action (2.2) represents Higgs model in linear representation, which is invariant under global special orthogonal transformations from $SO(N)$ group. In a phase with spontaneous symmetry breaking, we have $v^2 > 0$. Without loss of generality we can assume that the background field is $\phi^\alpha = 0$ for $\alpha = 1 \ldots N-1$ and $\phi^N = v$. (Therefore we choose Higgs vev pointing exactly in the last N-th direction in the field space and with such pattern the model possesses only $SO(N-1)$ as the remaining symmetry group.) Then the $N-1$ fields ϕ^α are the Goldstone bosons, while the radial mode $\delta\rho = \phi^N - v$ corresponds to the physical Higgs field. The square of the mass of the radial mode is given by $m^2 = 8\lambda v^2$, whereas $N-1$ Goldstone modes remain massless. Note that the potential is always zero at the minimum; here we will not discuss the running of the cosmological constant. The non-linear sigma model is achieved in the limit $\lambda \to \infty$ with v kept constant. Then the potential becomes a constraint for ρ: $\rho^2 = v^2$, which can be solved to eliminate one scalar field and describe the theory in terms of the remaining dynamical $N-1$ fields φ^α transforming non-linearly under $SO(N)$ - the coordinates on the sphere S^{N-1}. (In particular there exist coordinate choices for which one can identify $\varphi^\alpha = \phi^\alpha$.) In this limit physical Higgs field becomes infinitely heavy, so decouples from the system of interacting Goldstone bosons and the theory is perturbatively non-unitary. Later we will see, how this can be healed. In an arbitrary coordinate system, the action becomes

$$S_m = \frac{1}{2} \upsilon^2 \int d^4x \, \sqrt{g} \, g^{\mu\nu} \partial_\mu \varphi^\alpha \partial_\nu \varphi^\beta h_{\alpha\beta}(\varphi) \, , \qquad (2.3)$$

where $h_{\alpha\beta}$ denotes the general, positive definite metric on the target space of nonlinear sigma model. Our toy model contains two dimensionful couplings m_P^2 and υ^2, which we identify with the square of the Planck mass and of the Higgs VEV. They appear in a very similar manner as prefactors of the respective terms in the Lagrangian. From here on we will consider their inverses as the couplings in our model.

There are two main motivations for choosing this model as opposed to gravitation coupled to linearly transforming scalars. Firstly, in the absence of gravity and in four dimensions, the linear scalar theory displays a unique Gaussian FP, and it is perturbatively renormalizable and trivial. On the other hand the non-linear model has a coupling constant with inverse mass dimension and is power-counting nonrenormalizable, similar to gravity itself. It also suffers from violation of unitarity at high energy. Recent studies showed that it displays an UV FP [19, 28], with, incidentally, identical critical exponents as found within pure Einstein gravity [17]. It has therefore been suggested that, quite independently of gravity, a strongly interacting Goldstone boson sector may exist, able to overcome its perturbative issues in a dynamical way [19, 20, 21].

Secondly, given the existing evidences for asymptotic safety of the non-linear scalar theory and gravity separately, one may expect to find a non-trivially interacting FP also for the coupled theory. This would provide an alternative to the scenario discussed in [29, 30], where a "Gaussian matter FP" was found, with asymptotically free scalar matter but non-trivial gravitational couplings. This scenario has been used to put new bounds on the mass of the Higgs particle [31], which agree remarkably well with the experimental measurements of recently discovered particle. Although now collected evidences show undeniably the existence of Higgs particle in the Standard Model, still usage of the non-linear theory may be considered for explanation, how in a gauge-invariant way the masses are provided for the W and Z bosons. We can also treat it as a simple toy model.

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2.2   Holographic RG in pure AdS

In this section we evaluate the running of the two dimensionful couplings m_P^2 and v^2 of the four-dimensional toy model using a holographic technique. Following [1], we consider a 5-dimensional spacetime with coordinates $y^m = (x^\mu, t)$, $\mu = 1, 2, 3, 4$ and metric G_{mn}. Set of coordinates x^μ will describe 4-dimensional leaves with usual Minkowskian metric's signature $(+ - --)$, while the t coordinate is a transverse direction to this foliation. The gravitational part of the action is

$$S_{\text{grav}} = \int d^5y \sqrt{-G}(2M^3 R - \Lambda) , \tag{2.4}$$

where M is the 5-dimensional Planck mass and $\Lambda < 0$ is the bulk cosmological constant. These parameters of the 5-dimensional theory are not dynamical and they do not undergo RG evolution. We make a particular ansatz for the metric of the form

$$ds^2 = e^{2t}\bar{g}_{\mu\nu}(x)dx^\mu dx^\nu + r_c^2 dt^2 . \tag{2.5}$$

Using the 5-dimensional Einstein equations we get the AdS solution with $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, where we have identified the arbitrary length scale r_c with the AdS radius $\sqrt{24M^3/|\Lambda|}$. We can make the coordinate transformation $t = -\log(z/r_c)$, which brings the metric to the form

$$ds^2 = \frac{r_c^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) . \tag{2.6}$$

This is the AdS metric in the so called Poincarè patch, as mostly used in cosmology. From its form we easily read out information about conformal structure of the AdS spacetime. We note that the hypersurface $z = 0$ corresponds to a conformal boundary at $t = \infty$. In the holographic interpretation of the 5-dimensional metric such as the RS model, the 5-th dimension is identified with the (logarithm of the) RG scale k [25] of the quantum 4-dimensional theory living on a 3-brane. Following [6, 5, 33], we make the identification $z = 1/k$, which implies $t = \log(kr_c)$, independently of the number of dimension of AdS spacetime. This provides a precise mapping between 5d calculations and 4d interpretations

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in terms of RG flow. We choose the origin of t coordinate to correspond to the electroweak scale $k_0 = v_0 = 246\text{GeV}$, which implies $r_c = 1/v_0$ for the AdS radius. It is convenient, for future purposes, to introduce also dimensionful radial coordinate $r = r_c t$.

To read off the β-functions of matter couplings we imagine putting a test brane at a given value of t. As noticed in [4], the use of a brane provides information on the quantum behaviour of the matter couplings themselves, as well as on gravity coupled to matter. Dimensionless couplings in general run logarithmically. All masses in the 4-dimensional matter theory are proportional to v, whose running is governed by the formula

$$v(t) = v_0 \, e^t \ . \tag{2.7}$$

In other words the scale-dependence is given by the exponential warping factor e^t, which was present in the AdS metric (2.5). This result is completely general: any mass parameter on the 3-brane, in the fundamental higher-dimensional theory will correspond to a RG rescaled mass according to the formula above, when measured with the metric $\bar{g}_{\mu\nu}$ [2]. This is the metric that appears in the effective Einstein action. All operators on the boundary get rescaled according to their four-dimensional energy dimensions. Note that, there is a freedom in choosing normalization of t. The choice, we made in (2.5) for the AdS metric, is such that all dimensionful couplings (except gravitational ones) scale like at FP of RG. This choice doesn't depend on the dimension of spacetime. In this way we set the normalization of distances in transverse direction to the brane. The AdS solution thus corresponds to linear running of v with RG momentum scale k, which is a manifestation of the quadratic divergences in the running $(\text{mass})^2$ in the underlying field theory.

Next we can obtain holographic RG running of the gravitational coupling constant m_P. To do this we consider small metric perturbations $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$ on the 3-brane and couple them to energy-momentum tensor of the matter living there. These are the massless gravitational fluctuations about our classical AdS solution and they will provide the gravitational fields for our effective theory. They are the zero-modes of our classical solution, and take the form

$$ds^2 = e^{2t} \left(\eta_{\mu\nu} + \bar{h}_{\mu\nu} \right) dx^\mu dx^\nu + r_c^2 dt^2 \ . \tag{2.8}$$

The four-dimensional effective theory now follows by inserting the ansatz (2.5) in the action (2.4) and we find that the action for the metric $\bar{g}_{\mu\nu}(x)$ is equal to

$$S_{\text{grav}} = 2M^3 r_c \int^t dt' e^{2t'} \int d^4x \sqrt{-\bar{g}} \bar{R} \ . \tag{2.9}$$

Here the warping factor enters in the form as it is originally in the metric ansatz (2.5), regardless of the dimensionality of the effective gravitational coupling constant. We denote by $\bar{R}$ curvature scalar of the metric $\bar{g}_{\mu\nu}$ in contrast to the five-dimensional Ricci scalar, R, made out of metric G_{mn}. The relation connecting the 4-dimensional Planck mass m_P and the 5-dimensional parameter M is obtained by performing in (2.9) the integral over t' explicitly and then comparing with the effective 4-dimensional action in the form (2.1). This leads to

$$m_P^2(t) = m_P^2(0) + \frac{M^3 r_c}{2} \left[e^{2t} - 1 \right] \ , \tag{2.10}$$

independent of the lower end of integration in (2.9).

The requirement that $m_P^2(t)$ be positive for all t implies $m_P^2(0) > M^3 r_c / 2$. Equation (2.10) contains the unobservable five-dimensional Planck mass. We can rewrite it in terms of four-dimensional measurable quantities as follows. We assume that the Planck mass at the TeV scale is not so different from the measured value at macroscopic scales (deep IR limit) $m_P(0) \approx m_P(-\infty)$. Then, knowing the empirical values of v_0 and $m_P(0)$ we have $t_P = t(k = m_P(0)) \approx 38$. Furthermore we define the coefficient $c_P = \left(\frac{m_P(t_P)}{m_P(0)} \right)^2 - 1$, which measures the relative change of the effective Planck mass between the TeV and (the initial) Planck scale $m_P(0)$. We expect the value of c_P to be of order one. Since $M^3 r_c > 0$ we must have $c_P > 0$. We note, that since the AdS curvature v_0^2 is much smaller than the five-dimensional Planck scale M^2, it is justified to treat the five-dimensional gravitational field classically. From the definition of c_P and the assumption that $m_P(0) \gg v_0$ we get the relation $M^3 r_c = 2 c_P v_0^2$ with the help of which we can rewrite formula (2.10) as

$$m_P^2(t) = m_P^2(0) + c_P \, v_0^2 \left[e^{2t} - 1 \right] \ , \tag{2.11}$$

where we have replaced the 5-dimensional parameters M and r_c by the Higgs VEV v and the parameter c_P.

We observe that equation (2.7) describes a mass that scales with the cutoff exactly as dictated by dimensional analysis $m \sim k$. Therefore, when the mass is measured in units of the cutoff k, it is constant. If we regard this mass as the (inverse) coupling constant of the non-linear sigma model (2.3), we are at a FP. Likewise, when $t \to \infty$, also the Planck mass scales asymptotically in the same way, so if we regard it as the (inverse) gravitational coupling, (2.11) describes an RG trajectory where gravity (coupled to matter) approaches a non-trivial FP. Interestingly, in this limit the decoupling of gravity $G \to 0$ can be viewed as a consequence of a non-trivial FP. It is worthy to emphasize that only in four spacetime dimension equation for RG running of a power of the effective Planck scale (2.11) describes in the UV limit FP for this dimensionful coupling. In complete generality in equation (2.10) we have always e^{2t} factor appearing from our AdS metric ansatz, however the power of the effective Planck mass on the left hand side of this equation is given by $d-2$, where d is the dimensionality of the brane. In higher dimensions the action for gravitation contains higher powers of the Planck mass in contrast to the second power of cutoff momenta originated from the warping factor, so in such circumstances holographic method doesn't confirm the existence of nontrivial FP of RG for dimensionful gravitational constant. This finishes the discussion of RG running of dimensionful couplings (of matter and gravitational character) from the holographic perspective in pure AdS spacetime.

2.3 Functional RG

In this section we evaluate the scale-dependence of m_P^2 and v^2 directly in the four-dimensional theory. To do this we will use techniques of functional (also known as exact) Renormalization Group. Our starting point is the "average effective action" Γ_k, a coarse-

grained version of the effective action, which interpolates between some microscopic action at $k = k_0$ and the full quantum effective action at $k = 0$. The RG momentum scale k is introduced at the level of the functional path integral by adding suitable momentum-dependent kernels $R_k(q^2)$ to the inverse propagators of all propagating fields, which for bosonic fields take the standard q^2 form at high energy. These kernels must decrease monotonically with k^2, tend to 0 for $k^2/q^2 \to 0$ (in order to leave the propagation of large momentum modes intact), and tend to k^2 for $q^2/k^2 \to 0$ (in order to suppress the low momentum modes). The change of Γ_k with logarithmic RG "time" $t = \log(k/k_0)$ is given by a functional differential equation [12]

$$\partial_t \Gamma_k = \frac{1}{2} \text{STr} \left(\Gamma_k^{(2)} + R_k \right)^{-1} \partial_t R_k . \tag{2.12}$$

Here, $\Gamma_k^{(2)}$ denotes the matrix of second functional derivatives with respect to all propagating fields, and the supertrace stands for a sum over all modes including a minus sign for fields of fermionic type. The RG flow (2.12) is an exact functional identity, which derives from the path-integral representation of the theory in the vicinity of a gaussian fixed point. The flow reduces to the Callan-Symanzik equation in the special limit where R_k becomes a simple mass term k^2, and is related to the Wilson-Polchinski RG [11] by a Legendre transform. Most importantly, the functional flow is finite and well-defined for all fields including the UV and IR ends of integration, which makes it a useful tool for our purposes. The requirements of diffeomorphism or gauge invariance of the average effective action are implemented with the help of the background field technique [34]. For optimized choices of the momentum cutoff all the operator traces can be performed analytically [35], also using the heat kernel methods.

We want to calculate the RG flow of Γ_k for the system described by the classical action, whose two pieces were given in (2.1) and (2.3). This type of calculation for pure gravity was first described in [36, 37, 17] and in [19] for the non-linear sigma model. Here we apply the same technique to the coupled system starting with $\Gamma_k = S_g + S_m + S_{gf} + S_{gh}$, where it is understood that the couplings in the RHS are replaced by running couplings

(k-dependent), evolving under the RG flow (2.12). Since the classical action is invariant under diffeomorphisms, we have introduced a gauge-fixing term S_{gf} and a ghost term S_{gh} in addition to the gravitational action (2.1) (for vanishing cosmological constant) and the matter action (2.2). Using the split of the metric and the scalar fields into background fields $g_{\mu\nu}$, ϕ^a and quantum fields $h_{\mu\nu}$, η^a, the gauge fixing term reads

$$S_{gf} = \frac{m_P^2}{2\alpha} \int d^4x \, \sqrt{g} \chi_\mu g^{\mu\nu} \chi_\nu \qquad (2.13)$$

with $\chi_\mu = \nabla^\nu h_{\nu\mu} + \frac{1}{2}\nabla_\mu h$. The trace of the metric perturbations computed using background value of metric $g^{\mu\nu} h_{\mu\nu}$ we denoted by h. The corresponding Faddeev-Popov ghost action is

$$S_{gh} = \int d^4x \, \sqrt{g} \bar{C}^\mu (-\nabla^2 \delta_\mu^\nu - R_\mu^\nu) C_\nu \ . \qquad (2.14)$$

Below we work in Feynman gauge ($\alpha = 1$) for simplicity, but this is not essential. In order to find terms in (2.12) we have to invert the matrix $(\Gamma_k^{(2)} + R_k)$ in field space. For the Einstein-Hilbert action we can follow the procedure of [38], Section IV B. Expanding the matter action up to quadratic order in the fluctuation fields $\delta\phi^a = \eta^a$ and $h_{\mu\nu}$, the second variation $S_m^{(2)}$ reads

$$\frac{1}{2} \int d^4x \sqrt{g} \left[V \left(\frac{1}{4}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu} \right) + 2V'\phi^a \delta\phi^a h + \delta\phi^a \left(-\nabla^2 \delta^{ab} + 2V'\delta^{ab} + 4V''\phi^a\phi^b \right) \delta\phi^b \right] .$$
$$(2.15)$$

Separating the radial mode ρ from the Goldstone modes, and splitting the graviton field into traceless, transverse part and other fields ξ and σ as $h_{\mu\nu} = h_{\mu\nu}^{TT} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \nabla_\mu \nabla_\nu \sigma - \frac{1}{4} g_{\mu\nu} \nabla^2 \sigma + \frac{1}{4} g_{\mu\nu} h$, where $\nabla^\mu h_{\mu\nu}^{TT} = 0$, $\nabla^\mu \xi_\mu = 0$, the expansion of the average effective action Γ_k to quadratic order in the fluctuations becomes

$$
\begin{aligned}
\Gamma_k|_{\text{quad}} \ = \ & \frac{1}{2} \int d^4x \sqrt{g} \Big[\frac{1}{2}m_P^2 h^{TT\mu\nu} \left(-\nabla^2 + \frac{2}{3}R - \frac{V}{m_P^2} \right) h_{\mu\nu}^{TT} + m_P^2 \hat{\xi} \left(-\nabla^2 + \frac{1}{4}R - \frac{V}{m_P^2} \right) \hat{\xi} \\
& + \frac{3}{8}m_P^2 \hat{\sigma} \left(-\nabla^2 - \frac{V}{m_P^2} \right) \hat{\sigma} - \frac{1}{8}m_P^2 h \left(-\nabla^2 - \frac{V}{m_P^2} \right) h + \delta\rho \left(-\nabla^2 + 2V' + 4v^2 V'' \right) \delta\rho
\end{aligned}
$$

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$$+2V'\upsilon\, h\delta\rho + \delta\varphi^{\alpha}\left(-\nabla^2 + 2V'\right)\delta\varphi^{\alpha}\Big] + S_{gh}|_{\text{quad}} \ , \tag{2.16}$$

where we have defined hatted variables by field redefinitions according to the formulas $\hat{\xi}_{\mu} = \sqrt{-\nabla^2 - \frac{R}{4}}\,\xi_{\mu}, \hat{\sigma} = \sqrt{-\nabla^2}\sqrt{-\nabla^2 - \frac{R}{3}}\sigma$. We observe that the radial mode $\delta\rho = \rho - \upsilon$ mixes with the trace h, whereas the Goldstone bosons do not. However, it is easy to see that this mixing is absent once the background scalar is at the minimum of its potential. Then (2.16) is already diagonal in field space and the inversion of the matrix $(\Gamma_k^{(2)} + R_k)$ becomes straightforward. Defining the graviton "anomalous dimension" $\eta = \partial_t m_P^2/m_P^2$, the flow equation (2.12) reads

$$
\begin{aligned}
\partial_t\Gamma_k &= \frac{1}{2}\text{Tr}_{(2)}\frac{\partial_t R_k + \eta R_k}{P_k + \frac{2}{3}R} + \frac{1}{2}\text{Tr}'_{(1)}\frac{\partial_t R_k + \eta R_k}{P_k + \frac{1}{4}R} + \frac{1}{2}\text{Tr}_{(0)}\frac{\partial_t R_k + \eta R_k}{P_k} \\
&\quad + \frac{1}{2}\text{Tr}''_{(0)}\frac{\partial_t R_k + \eta R_k}{P_k} - \text{Tr}_{(1)}\frac{\partial_t R_k}{P_k - \frac{1}{4}R} - \text{Tr}'_{(0)}\frac{\partial_t R_k}{P_k - \frac{1}{2}R} \\
&\quad + \frac{N-1}{2}\text{Tr}_{(0)}\frac{\partial_t R_k}{P_k} + \frac{1}{2}\text{Tr}_{(0)}\frac{\partial_t R_k}{P_k + 8\lambda\upsilon^2} \ , \tag{2.17}
\end{aligned}
$$

where $P_k \equiv -\nabla^2 + R_k(-\nabla^2)$. For a definition of the remaining (primed and unprimed) traces over the various tensor, vector and scalar modes, we refer to [38]. The first six terms originate from the gravitational sector and the ghosts while the last two terms come from the Goldstone bosons and the radial mode, respectively.

We make an ansatz for Γ_k of the form $S_g + S_m + S_{gf} + S_{gh}$, where G_k, λ_k and υ_k are k-dependent coupling constants in our model. The β-functions for the couplings are obtained from (2.17) by projection onto the truncation ansatz for the action as given in Γ_k. To that end we polynomially expand the functional flow on both sides about $R = 0$ and $\rho^2 = \upsilon^2$. The flow for the inverse gravitational coupling m_P^2, the quartic coupling λ, and for the vacuum expectation value υ^2 are then given by $\frac{d}{dR}(\partial_t\Gamma_k)$, $\frac{1}{2}(\frac{d}{d\rho^2})^2\partial_t\Gamma_k$ and $-\frac{d}{d\rho^2}\partial_t\Gamma_k/(2\lambda)$ at $R = 0$ and $\rho^2 = \upsilon^2$, respectively. For completeness we have listed here also the RG flow for coupling λ, although it doesn't appear in the final formulation of nonlinear sigma model. In the following we will neglect the terms linear in η on the RHS of (2.17). Moreover we work on one-loop level (if we were to refer to perturbative

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computations in QFT), therefore as a first approximation we forget about the effects driven by the graviton anomalous dimension. Using the heat kernel expansion together with an optimized cutoff function [35] $R_k(z) = (k^2 - z)\theta(k^2 - z)$ with Heaviside θ step function, the β-function for λ reads

$$\partial_t \lambda \;=\; \frac{\lambda^2}{2\pi^2}\left(N - 1 + \frac{9}{(1+\tilde{m}^2)^3}\right) + \tilde{G}\lambda\,\frac{5 + 6\tilde{m}^2 + 3\tilde{m}^4}{(1+\tilde{m}^2)^2}\,, \tag{2.18}$$

where we have introduced the square of the Higgs mass in units of the RG scale, $\tilde{m}^2 = 8\lambda v^2/k^2$ and $\tilde{G} = G\,k^2$. The terms proportional to λ^2 contains the contributions of the $N - 1$ Goldstone modes and the Higgs field. Notice the threshold behaviour of the Higgs contribution at the Higgs mass $m^2 \approx k^2$. The last term is the leading gravitational correction. The β-function of v^2 is

$$\partial_t v^2 = \frac{k^2}{16\pi^2}\left(N - 1 + \frac{3}{(1+\tilde{m}^2)^2}\right). \tag{2.19}$$

It receives contributions from the Higgs and the Goldstone bosons, but, remarkably, not from the fluctuations of the metric field. Now we take the non-linear limit $\lambda \to \infty$ (or $\tilde{m}^2 \to \infty$) with v^2 held constant. In this limit (2.18) becomes useless, the Higgs field becomes infinitely massive and the radial mode contribution to (2.19) drops out. The Goldstone bosons remain fully dynamical, in fact their action is completely unaffected by the limit. We end up with

$$\partial_t v^2 \;=\; B_H k^2\;; \qquad B_H = \frac{N - 1}{16\pi^2}\,, \tag{2.20}$$

$$\partial_t m_P^2 \;=\; B_P k^2\;; \qquad B_P = \frac{N_c - N}{96\pi^2}\,, \tag{2.21}$$

where we have just defined the critical number of fields in $SO(N)$ model equal to $N_c = 109/4$. The dependence of the result on the number of Goldstone modes is simple to understand. In (2.20), only the Goldstone modes contribute to the running of the VEV. In the running for effective 4-dimensional Planck mass (equation (2.21)), the contribution from

the Goldstone modes compete with this originating from the graviton self-interaction. For $N < N_c$, the gravitons keep the lead and the combined effect is to increase m_P ($B_P > 0$) with increasing RG time t. In the opposite regime the Goldstone modes take over and change the sign of the coefficient B_P. More generally, matter field can contribute to (2.21) with either sign and hence the global sign of RG running will depend on the number of scalars, spinor, or vector fields coupled to gravity [29]. This pattern is similar to scale-dependence of strong coupling in QCD (where we have asymptotic freedom for sufficiently small number of fermions) and its dependence on the number of fermionic quark species. We will later come back to the issue of results' dependence on the number of fields N.

For a better understanding of the system of our nonlinear sigma model coupled to Einstein-Hilbert gravitation it is convenient to use the inverses $G = 1/(16\pi m_P^2)$, $f^2 = 1/v^2$, and to introduce dimensionless couplings $\tilde{v}^2 = v^2/k^2$, $\tilde{f}^2 = f^2 k^2$, $\tilde{m}_P^2 = m_P^2/k^2$, $\tilde{G} = G k^2$. This is because the perturbative analysis of the sigma model and gravitational theory is an expansion in the couplings $\tilde{f}^2$ and $\tilde{G}$, respectively. Their β-functions are given by

$$\partial_t \tilde{G} \;=\; 2\tilde{G} - B_P \, \tilde{G}^2 \tag{2.22}$$

$$\partial_t \tilde{f}^2 \;=\; 2\tilde{f}^2 - B_H \, \tilde{f}^4 \; . \tag{2.23}$$

Also on this level we observe a lot of similarities. Each one of these β-functions admits two FPs: an IR FP at zero coupling and an UV FP at finite coupling $\tilde{f}^2 = 2/B_H$ and $\tilde{G} = 2/B_P$ respectively. The gravitational FP is in the physical domain provided the number of Goldstone modes is small enough, or else the FP turns negative and cannot be reached, because in the RG evolution we cannot cross zero value of the coupling.

The two couplings have completely independent but very similar behaviour. For $k \ll v$ (so in the deep infrared limit of energies), $\tilde{v}$ is close to the Gaussian FP. This is the domain, where the dimensionful coupling v is nearly constant, the dimensionless $\tilde{v}$ has an inversely linear "classical" running with energy (derived from the canonical energy dimension of v), and perturbation theory is rigorously applicable. Then for higher energies there is a regime where $\tilde{v}$ is nearly constant and close to the non-trivial FP, while the dimensionful v scales

linearly with energy. Note that on such trajectories it never happens that $k \gg v$. The transition between the two regimes alluded before is near the scale determined by v, so this is way below in energies than the Planck scale. These considerations can be repeated verbatim for m_P, the sole difference being that the RG scale, where the transition from "classical running" to non-classical behaviour driven by quantum effects occurs, will be near the Planck scale. Thus, there are three regimes: the low energy regime $k \ll v \ll m_P$, where both G and f are constant, the intermediate regime where $\tilde{f}$ has reached its FP value but G is still constant and the FP (high energy) regime, where both dimensionless couplings have reached the FP.

2.4 Comparison between holographic and functional RG results

In the previous two sections we obtained results for the RG running for coupling parameters in the nonlinear sigma model and in the gravitation computed using two conceptually completely different methods. In this section we try to draw a comparison between these results. To find a relation between them is the main goal of this chapter. For the sake of comparison with the results of the holographic procedure, we can write the general solutions of equations (2.20), (2.21) from the previous section as:

$$v^2(t) = v_0^2 + \frac{1}{2}B_H(k^2 - k_0^2) = v_0^2\left[1 + \frac{1}{2}B_H(e^{2t} - 1)\right] , \qquad (2.24)$$

$$m_P^2(t) = m_{P0}^2 + \frac{1}{2}B_P(k^2 - k_0^2) = m_{P0}^2 + \frac{1}{2}B_P v_0^2(e^{2t} - 1) , \qquad (2.25)$$

where we have defined, in accordance with the definitions in Section 2.3, $k(t) = v_0 e^t$, $k_0 = k(0) = v_0$, and v_0, m_{P0} are the values of the respective couplings at k_0. Strictly speaking, when all dimensionful parameters of the theory undergo RG running, the only physical and measurable parameter of the theory is the ratio of the mass scales, which are present,

$$\alpha(t) \equiv \frac{m_P(t)}{\upsilon(t)} . \tag{2.26}$$

The plot of its natural logarithm $\log \alpha(t)$ is shown on Fig. 2.1 and illustrates the three regimes of the theory alluded to in the end of the preceding section. We may already analyze the behaviour of this ratio computed on the basis of exact (functional) RG. Namely for $t \to \infty$ the square of the ratio tends, for all trajectories, to the constant value B_P/B_H, while for $t \to -\infty$ it tends to a number that depends on the initial conditions and is equal to

$$\lim_{t \to -\infty} \alpha(t) = \frac{m_{P0}^2 - \frac{1}{2}B_P\upsilon_0^2}{\upsilon_0^2 \left(1 - \frac{1}{2}B_H\right)} . \tag{2.27}$$

After neglecting $B_P\upsilon_0^2$ with respect to m_{P0}^2 we can conclude that this number is of order m_{P0}^2/υ_0^2, so roughly of 10^{38} magnitude. We must be however careful here, because the precise value of the limit depends strongly on the value of the coefficient B_H and is singular for it equal 2.

Returning to equations (2.24) and (2.25), we see that if we could set $B_H = 2$ and $B_P = 2c_P$, they would agree with the flow obtained by the holographic method as encoded in formulas (2.7) and (2.11). There is a difference here between the holographic RG flows of υ and m_P: whereas c_P is a free parameter in the holographic model for the running of the gravitational coupling, which can be adjusted to match the result of the functional RG, there is no corresponding free parameter for υ. One is thus left with a prediction for the parameter B_H, that does not seem to match the result of the functional RG, which shows explicitly dependence on N. One could try to exploit the fact that the parameter B_H is scheme-dependent, to try and force a match, however this could not hide the important difference that whereas in the functional RG there are infinitely many trajectories for both υ and m_P, parametrized by their values at k_0, in the holographic RG there is a single trajectory for υ characterized by the initial value υ_0 and the fixed value of the coefficient $B_H = 2$.

To clarify this difference further, we observe that if we set $B_H = 2$, as the pure AdS
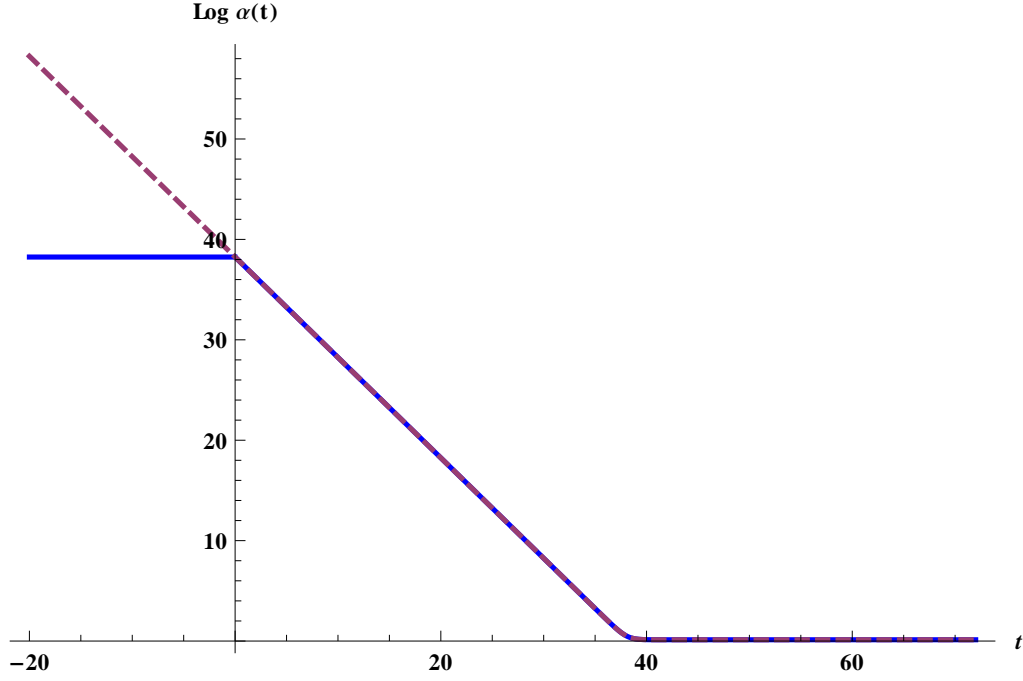
Figure 2.1: The running of the mass ratio $\alpha(t)$ defined in (2.26), for $N = 4$, on a logarithmic scale as a function of t. Solid curve: solution of the functional RG; dashed curve: solution of the holographic RG. For large t the curves tend to the value 0.13.

holographic RG seems to demand, v tends not to finite value $v_0\sqrt{1 - B_H/2}$, but to zero in the IR and therefore α diverges. The ratio α in the far IR grows linearly to $+\infty$. This is shown by the dashed line in Fig. 2.1. Thus, the holographic description of the preceding section agrees well with functional RG in the second and third regime, but fails to reproduce even at a qualitative level the generic low-energy regime of the theory. This is due to the fact, that the holographic RG trajectory is such that v tends to zero in the IR, which is just one amongst infinitely many RG trajectories allowed in (2.24), that would tend to different finite limits in the IR. In contrast, m_P^2 can have an arbitrary limit in the IR $m_{P0}^2 - c_P v_0^2$: this is due to the freedom of choosing the parameter c_P. The difference in the behaviour of the two couplings can be traced back to the fact, that in the five-dimensional description, gravity is free to propagate in all directions, whereas all the other matter fields are confined to the 3-brane.

Here we also want to touch on the issue, which is of great importance for the question of UV completion of quantum gravity. Namely we will shortly discuss the running of the Planck scale. We know, from macroscopic physics, that IR (classical) value of parameter setting the strength of gravitational interactions is finite. We call it here Planck constant and denote by $m_{P\,\mathrm{IR}}$. Due to quantum effects this parameter, as any other in the Lagrangian, exhibits scale-dependence. From the energy range explored so far in particle physics (up to TeV scale), we conclude that this running Planck scale runs very weakly and that it is justified there to neglect gravitational effects. However still we can consider function $m_P(t)$ giving us the RG running of Planck scale at different energies $k = v_0 \exp t$. Let us call the energy scale, at which gravitational interactions are becoming important for quantum physics of elementary particles, a proper Planck scale. This is a scale, which can be determined quite uniquely. It is very important that the proper Planck scale m_{P*} doesn't run under RG transformation of scales. It is an RG-invariant in the same way as masses of the physical particles (the latter are determined from poles of the exact quantum propagators). The fact, that the Planck scale, describing the strength of gravitation, measured by a dimensionless product $k^2 m_P^{-2}$, undergoes RG running in an effective field theory approach, means in this context only that an IR estimation for the proper Planck scale ($m_{P*} \approx m_{P\,\mathrm{IR}}$) must be corrected. Quantum effects in gravity make it antiscreening, so the correction is positive. This means that at $k = m_{P\,\mathrm{IR}}$ we don't meet an outset of quantum gravity for elementary particles. The energy scale must be raised. At new increased energy k we check for the running value of Planck scale and compare it with k - by this way we decide, whether the energy must be raised again and so on. The non-iterative solution to this problem comes after noticing that the proper Planck scale is a mathematical fixed point of the function $m_P(k)$, this is

$$m_{P*} = m_P(k = m_{P*}). \qquad (2.28)$$

If we know, the running of Planck scale (analytically or numerically $m_P(k)$) over some energy scale range, then it is easy to find such fixed point. It is at the section of a curve

$m_P(k)$ with a line showing relation $m_P = k$. For monotonic RG runnings such fixed point is determined uniquely. All this considerations are in perfect analogy to the issue of determining masses of W and Z bosons. Their masses can be traced back from the low-energetic interactions of weak and neutral currents. However their IR value must be renormalized at higher energy scale to give the proper values at physical poles in the corresponding propagators. Here we may interpret the proper Planck scale as a mass present in the denominator of some exact quantum 2-point function. This is really like that, because in some theories physical masses of heavy particles, which appear in UV completions, are proportional to the proper Planck scale. Parameters determining the strengths of interactions do run, masses of related particles do not, however these two facts do not contradict for the existence of connecting the two relation.

In our case from equation (2.25) we see that the unique answer for the square of the proper Planck scale is given by

$$m_{P*}^2 = \frac{m_{P0}^2}{1 - \frac{1}{2}B_P} = \frac{m_{P0}^2}{1 - c_P}, \tag{2.29}$$

when we neglected a constant term $\frac{1}{2}B_P v_0^2$ as smaller compared to m_{P0}^2. We easily see that the sensible solution exists only for $c_P < 1$. The best picture we get, when we plot m_P^2 versus k^2. Then the RG running of Planck scale and $m_P = k$ are given by straight lines. For $c_P > 1$ two straight lines cross for negative value of energy scale k. This result is nonsensical on physical grounds and means that simply m_{P*} doesn't exist. For higher and higher energies two lines diverge and this is means that iterative correction to the proper Planck scale are bigger and bigger, when the energy is raised. In that case quantum gravity never becomes significant in the interaction of elementary particles and there is no new characteristic quantum gravity scale. When $c_P = 1$ exactly, then the lines meet at infinity (are parallel) and indeed $m_{P*} = +\infty$ is a limiting solution for the mathematical fixed point of the function $m_P(k)$. In the remaining case $c_P < 1$ lines cross for finite positive k and there exists a finite solution for the proper Planck scale (2.29). Summarizing, when $c_P < 1$, proper scale of quantum gravity exists and in some theories of quantum gravity

one can expect there appearing of new physics (i.e. new degrees of freedom). However for $c_P \geq 1$ UV completion can be achieved only by requiring existence of nontrivial FP of RG. Moreover in the RG flow of couplings we demand to remain on the RG safe trajectory towards this FP. With other small technical details this is the idea of asymptotic safety for quantum gravitation. We remind here that value of the coefficient c_P is equal to the inverse UV FP value of the dimensionless gravitational constant $\tilde{G}_*$. This means that the above given conditions can be phrased equivalently in terms of the details of the FP in UV and in this way we can avoid studying the RG flow at intermediate energies (around $k = m_{P0}$), where we are not sure about full dynamics of the gravitational system.

Whole asymptotic safety program concerns UV limit of energies and as such is not sensitive to the particular behaviour of couplings in a range between v_0 and m_{P0} scales (as in original definition of c_P). We give as a partial evidence for asymptotic safety the fact, that for Einstein-Hilbert system the values of $\tilde{G}_*$ at UV FP are smaller than unity (in type II and III cutoff scheme). When running of cosmological constant is also included, then $\tilde{G}_* < 1$ in all types of cutoff (beyond 1-loop approximations) [38]. This reinforces asymptotic safety conjecture, because running of Planck scale in the vicinity of hypothetical UV FP is consistent with the absence of any new energy scale for quantum gravity. When extrapolating RG running equations from UV to intermediate energy scale, we don't meet any new scale, so the existence of nontrivial FP of RG in UV is inevitable, if our theory is to be fully consistent. When in asymptotic safety scenario, RG safe trajectory reaches FP in UV, then there exists an energy scale, at which RG flows enters into FP regime. From equation (2.25) we see that this scale is set by $\frac{m_{P0}}{c_P}$, which is smaller than m_{P0} for $c_P > 1$. However this is not a new mass scale in the sense explained above. We must add here, that asymptotic safety scenario as a possible UV completion for quantum gravity works also for $c_P < 1$, however the evidences for it are not so strong in this case.

The behaviour of the proper Planck scale with changes of the coefficient c_P can be understood very intuitively. We recall that this coefficient measure the changes in the running Planck scale between v_0 and m_{P0} scales. When this change is small $c_P < 1$, then we expect correction to m_{P0} to be small and soon we should converge with the finite value

of m_{P*}. In opposite case, when the change is bigger than the initial estimate for proper Planck scale ($m_{P*} \approx m_{P0}$), then our procedure gives a divergent result and there is no a crossing point. We want also to remark here, that the above presented analysis for the proper Planck scale is insensitive to any IR modification of the flows, because our iterative procedure starts at $k = m_{P0}$, which is very high energy compared to $k = v_0$. If $c_P > 1$, then above the scale $k = m_{P0}$, we are with big confidence in a fixed point regime of the flow, when our flow equation (2.24), (2.25) hold true. When $c_P < 1$, then probably more detailed analysis is required especially in the intermediate region of energies.

In the description of Fig. 2.1 we said, that it was prepared for value $N = 4$ for the solution of the functional as well as holographic RG. Solid curve (from exact RG) was indeed obtained for such input data, however the dashed one symbolizing the results of holography was obtained for $c_P \approx 1.292$. There is no any N-dependence in holographic running of v or m_P, the only parameter governing RG flow of m_P is c_P and we can adjust only its value. Later we considered the possibility of having the agreement of two RG flows, from which one of the first implications is that the coefficient c_P as a function of N is given by the formula $c_P = \frac{N_c - N}{6(N-1)}$ and this evaluated for $N = 4$ gives mentioned above numerical value of it. To get this conclusion we must have used formulas (2.20) and (2.21). For these values of the parameters (N and c_P), describing the two curves, we get that both at $+\infty$ tend to the same value $\frac{1}{2} \log c_P$, which is numerically, what we found as 0.13.

Lastly, the second implication of our matching holography with functional RG methods is that we are forced to admit, that $B_H = 2$. With this, from formula (2.20) we can find that $N \approx 316.8$. This is the value for which problems described above arise ($N > N_c$, so c_P is negative!) and this is a determined finite number of matter flavors, which could be present in our theory. We wouldn't expect, that by forcing matching of two RG flows, we could find the unique value of N (which is by the way wrong, because for it $c_P < 0$). We share the opinion, that it shall remain free, not determined, parameter of our model. Formulas (2.20) and (2.21) were used to determine c_P, for some value of N, knowing that it must be equal from the matching to the ratio $\frac{B_P}{B_H}$. But in the same moment we were able in principle to use the full conditions of matching ($B_H = 2$ and $B_P = 2c_P$) to determine

the value of N from (2.20). The only cure for all these issues are the modifications for both RG flows, which we will describe further in this and next chapter.

We can modify the holographic RG to resemble more closely the functional one by stopping the flow of v at $k = v_0$. From Fig. 2.1 we notice, that without such modification, holographic RG flow is good only for energy scales much bigger than masses. This can be achieved by putting a source brane at $t = 0$ with action

$$\sqrt{6\,M^3\,|\Lambda|}\,\int d^5y\,\delta(t)\,. \tag{2.30}$$

Therefore we generalise the ansatz (2.5) by replacing e^{2t} with more general warping factor $e^{2A(t)}$. Then solving the five-dimensional Einstein equations with this source gives a second order differential equation for the warp factor $A'' = -\frac{Vr_c}{12M^3}\delta(t)$. Since we want to have $A(t) = t$ for $t > 0$, we get from the equation above that $A(t) = 0$ for $t < 0$. Thus, we have a solution where the brane at the origin joins continuously a flat spacetime for $k\,r_c < 1$ with AdS spacetime for $k\,r_c > 1$, where we recall that $t = \log(k\,r_c)$. By doing this we have modified significantly only the IR part of holographic geometry, so we changed the RG running for couplings only in this regime. Since the Higgs VEV scales in general as $v_0 e^{A(t)}$, we find that it becomes constant for $t < 0$. For the Planck mass the above construction implies a weak, logarithmic running for $t < 0$, which would reduce it to zero once $t_{\mathrm{IR}} \sim -10^{32}$. This is so far in the infrared that we can disregard this effect for all practical purposes.

The behaviour of the couplings for $t < 0$ is not exactly the same as the solution that we found from the functional RG, but it is qualitatively very similar. The comparison could be improved further by making the model more realistic. Equations (2.24) and (2.25) show that the running of the couplings continues all the way down to $k = 0$ without thresholds. This is due to the fact that all degrees of freedom of the theory (gravitons and Goldstone bosons) are massless. In the real world, the Goldstone bosons are coupled to gauge fields and are not physical degrees of freedom. Instead, they become the longitudinal components of the W and Z bosons. These gauge fields are massive and their contributions to the β-

functions will exhibit threshold phenomena, whose effect is to switch off the running of v below $k_0 = v_0$ [20]. It appears therefore that branes can be naturally associated to the presence of thresholds.

We conclude that with the addition of the source brane at $t = 0$ the five-dimensional space has become very similar to the Randall–Sundrum one [1]. This can be generalised: one can modify the holographic flow by introducing branes at specific locations and with specific cosmological constants, or more generally a continuous distribution of branes with a given density. With placing a source brane at $t = 0$ and stopping the functional RG flow due to threshold phenomena, we can obtain a situation in which both flows are qualitatively very similar, but in the same time N-dependence in formulas (2.20) and (2.21) is not rigorously correct. The matching conditions and UV behaviour of RG runnings for particular couplings are fortunately insensitive to these changes. Moreover in the next chapter we will concentrate on the other distinct possible deformation of the RG flow geometries, with which most of the problems, we mentioned here, will find its field-theoretical solutions.

2.5 Discussion

In this section we would like to discuss some aspects of the considered model in the light of found relation between holographic and functional Renormalization Group Flows. The first issue concerns the physical meaning of a non-trivial FP for gravitation coupled to a non-linear sigma model.

We have shown that in the simplest approximation, retaining only terms with two derivatives of the fields, the non-linear sigma model minimally coupled to gravity exhibits a non-trivial, UV attractive FP, which could be used to define this theory nonperturbatively according to Asymptotic Safety proposal. Therefore we can hope that its perturbative problems (like apparent violation of unitarity at high energy) can be solved if the theory is on the RG safe trajectory. The functional RG calculation presented here can be easily extended beyond the one-loop level by keeping the back-coupling of the graviton "anomalous dimension" η, which we neglected, and its analog for the non-linear sigma model. Similarly,

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the inclusion of a cosmological constant term in this framework is straightforward. These extensions bring only relatively minor changes to the picture we have found here. Inclusion of higher derivative terms would require a more significant calculational effort but the existing results for gravity and the sigma model separately suggest that the non-trivial fixed point should persist.

The physical application of our results is in the construction of an asymptotically safe quantum field theory of all matter and gravitational interactions. Much work has gone into trying to prove that gravity is asymptotically safe, but in order to be applicable to the real world one would have to extend this result also to the other interactions. Strong interactions are already asymptotically safe (as a particular case - they are asymptotically free in UV) on their own, so presumably they pose the least problem. The main issues seem to be in the electroweak sector, and in particular in the abelian and scalar subsectors. There are mainly two ways in which these issues could be overcome. In the first, asymptotic safety would be an essentially gravitational phenomenon: the standard model (or a grand unified extension thereof) coupled to gravity would not be UV complete and gravity would fix the UV behaviour of all couplings, including the matter ones. In this case the matter theory would be an effective field theory that need only hold up to the Planck scale; thereafter all couplings would approach a FP together. This is probably the most preferred scenario due to the lack of experimental hints beyond Standard Model of elementary partice physics. This is the point of view that is implicit in [29, 39, 40]. Recent discovery of Higgs-like particle in LHC at CERN reinforces the claim that sector of electroweak interactions is perturbatively renormalizable, however really important issue become UV behaviours of running electric charge and quartic coupling in the Higgs potential. One of the possibility of securing UV limit of such theory is to require that these two couplings reach FP. This is the second case, when each interaction would be asymptotically safe by itself, and each coupling would reach the FP at a different energy scale: the TeV scale for electroweak interactions and the Planck scale for the gravitational interactions. This is the point of view that we tried to propose by consideration of our model.

Taking this seriously, one is led to a non-standard picture of all interactions, where both

electroweak and gravitational interactions would be in their respective "broken" phases, characterized by non-vanishing VEVs, and carrying non-linear realizations of the respective local symmetries. Gravitation is in broken phase due to nonvanishing value of spacetime metric, when we are way from the topological phase. The theory as formulated does not admit the possibility of symmetry restoration at high energy. In fact, rather than going to zero, the Higgs VEV goes to infinity asymptotically for $t \to \infty$. At high energy, when the FP of RG is reached, symmetry of the theory is enhanced, because we have scale-invariance, which can be enhanced even more to the full conformal symmetry. We can see it from the behaviour of the ratio α, illustrated in Fig. 2.1, which characterizes the three regimes of the theory, with the electroweak and gravitational interactions becoming scale-invariant above their characteristic mass scales. The approach to the FP would fix the behaviour of the electroweak Goldstone sector, in a way that is still to be understood in detail, but has nothing to do with gravity. For the abelian gauge interaction one would have to invoke unification into a simple group, or gravity, as in [40].

We now come to the striking correspondence between the RG flows computed by holographic and functional methods. Working examples of holography are hard to come by outside the original domain of superstring theory, but in spite of this there seems to be a trend towards viewing holography as a field-theoretic phenomenon [27]. In the famous gauge/gravity duality the correspondence is conjectured between any quantum gravity theory in the bulk and the boundary theory with some local symmetries. In some sense the correspondence is surprising, because it is not a priori clear why the dynamics of gravity in five dimensions should have anything to do with the RG in four dimensions understood on the level of field theories. The idea of holography is often thought to be a fundamental ingredient of the construction of consistent quantum gravity theory and it has a strong support from open/closed string modes duality in string theory. On the other hand, our understanding of holographic RG is based to a large extent on the AdS$_5$ solution interpreted in the framework of Randall-Sundrum model. Given that the isometry group of AdS$_5$ is the group $SO(3,2)$, which can be interpreted as the conformal group in four dimensions with standard Minkowskian signature, it is not so surprising that this space can be used

to describe in geometric terms a theory at a FP. Our view here is therefore to interpret the five-dimensional metric as a geometrization of the four-dimensional RG flow at or near FP. In opposite direction we read out here, from spacetime geometry, RG runnings for couplings of four-dimensional theory following RS prescription, which is very similar to general AdS/CFT recipes. In RS prescription running of matter dimensionful coupling is derived from warping factor of the spacetime metric. Gravitational coupling on a 3-brane is of different nature and we obtain its scaling with energy by doing an integral over some interval of radial coordinate in AdS-like spacetime. The brane introduced in section §2.2, devoted for holographic RG, can be regarded as a true boundary of AdS located at some small but finite positive z. [1]

In this chapter we have neglected completely holographic RG running for the cosmological constant on the brane. The reason for this is quite technical. Despite the presence of the bulk cosmological constant (giving the background AdS spacetime), on our flat probe brane observer doesn't see any 4d-gravitational effect originating from vacuum energy. We chose to foliate 5-dimensional spacetime using flat Minkowski slices. That's why the bulk cannot induce any effect on the brane vacuum energy. Possible solutions would be to foliate 5-dimensional AdS using curved slices with maximal symmetry (dS_4 and AdS_4 for positive and negative 4-dimensional cosmological constant respectively), however this is not always an option. Moreover another additional difficulty appears in such setup, because then the value and the impact of the cosmological constant on the physics on the brane is nonvanishing and finite. In the true gravitational interactions (mediated by gravitons) the strength of interactions can be tuned to be infinitesimal, even for finite value of the coupling m_P^2, if only the energy excitations on the brane carry infinitesimal energy. This is because the product $m_P^2 k^2$, where k is the characteristic energy scale for matter perturbations, measures that strength. This means that, the impact of such perturbations on the background geometry of the brane can be safely neglected. Without back-reaction in

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[1]For AdS to be a solution in the presence of such a boundary one has to add to the action the Gibbons-Hawking boundary term [41], which in the present case just reduces to a cosmological constant on the brane.

this case we can study the linearized theory of gravitons (metric perturbations) and derive their scaling with the radial dimension of AdS spacetime $h_{\mu\nu} = e^{2t}\bar{h}_{\mu\nu}$. This was in the core of our derivation of the holographic RG flow for Planck mass. In the case of interaction with the brane cosmological constant, we don't have the possibility to turn off this coupling smoothly to zero and its effect on the background brane geometry is non negligible. It is incorrect to consider here the linearization around flat Minkowski background of the 3-brane. 4-dimensional gravitation here must be treated nonperturbatively in order to determine the correct background and the response for brane vacuum energy. Additionally on the field theory side we would have to work in the quantum field theory on the curved background. Nontrivial scaling with t of brane cosmological constant would correspond to a foliation of bulk spacetime by leaves with changing internal curvature. This is much more complicated setup for analyzing holographic RG flows. Fortunately this problem doesn't arise for holographic RG flow of couplings in front of gravitational higher derivative terms.

From the four-dimensional perspective, the corresponding large but finite value of t defines a UV cutoff. Due to this boundary, five-dimensional graviton modes are normalizable in the cut out region of AdS, and this setup describes gravity coupled to a conformal field theory with a UV cutoff [7, 4]. This is exactly the construction as presented in [1]. In this connection, it is important to clarify the following point, which could be cause of misunderstanding. In the limit $z \to 0$ (conformal boundary of AdS) we have seen that $G \to 0$, and for this reason it is usually said that gravity decouples. However, the strength of gravitation in a certain process is measured by the dimensionless product Gp^2, where p^2 is the characteristic momentum. In the vicinity of UV FP the following quantity If we identify the cutoff k with the momentum p, the strength of gravity is given by $\tilde{G} = Gk^2$, which in the limit $z \to 0$ tends to a finite constant (nb. this is the FP value of dimensionless gravitational coupling). It is in this sense that the decoupling of gravity can be seen as the consequence of a nontrivial FP for gravity.

It is not obvious at all that this five-dimensional theory has a dual CFT description. If it exists, it must correspond to the putative nontrivial fixed point of the $O(N)$ nonlinear sigma model coupled to gravitation. Note that the non-linear sigma model has a

dimensionful coupling and therefore, for fixed coupling it is certainly far from conformal. It is the quantum running of the coupling that would make it scale-invariant at the nontrivial fixed point. It should be possible to describe this fixed point also in terms of an effective Lagrangian containing only dimensionless couplings and also in terms of fields suitable for UV degrees of freedom. We will comment on this issue also in the next chapter.

Chapter 3

Holographic RG flow geometries for gravitational coupling

3.1   Holographic setup

In this chapter we focus on the precise realisation of RG flow geometries, which are different from pure AdS spacetime. Mainly we will consider flow for the gravitational coupling (which stands in front of the curvature scalar in Einstein-Hilbert action), but we will say few words about RG flow for ordinary matter couplings too. We will read RG flows from geometry following Randall-Sundrum prescription [1, 2], but in general geometries, which admit foliation by 4-dimensional flat Poincarè slices. Moreover we demand that these spacetimes asymptotically tend to AdS spacetime, so we will work with asymptotically Anti-de Sitter (AAdS).

The main problem, we would like to address in this chapter is how to find holographic geometries describing RG flows of gravitational coupling, which we found primarily using other methods (like functional RG).

For a scalar field with standard kinetic term and minimally coupled to gravitation, the only place, where the difference between different RG flows originates, is a scalar potential. Its shape $V(\Phi)$ determines the flow completely. However the opposite is not

always true, because the same configurations (scalar profiles) can be solutions of various potentials. We have only the equivalence between RG flows and scalar profiles. Giving the potential completes the ingredients necessary for building the holographic description of flows. Hence the task of this chapter will be to find explicit 5-dimensional potentials for explicit gravitational RG flows. We will also describe quickly its impact on matter RG flows.

We will describe holographic geometries corresponding to RG flows as solutions of Einstein-Hilbert system with minimally coupled scalar field with a potential. For generality we will study $(d+1)$-dimensional theory, of which our 5-dimensional description is a special case, with the action of the form

$$S = \int d^{d+1}x \sqrt{g} \, M^{d-1} \left(R + \mathcal{L} \right) = \int d^{d+1}x \sqrt{g} \, M^{d-1} \left(R \mp \frac{1}{2}(\partial \Phi)^2 - V(\Phi) \right), \qquad (3.1)$$

where M is the $(d+1)$-dimensional Planck mass (constant gravitational coupling in the bulk). We emphasise that the scalar field Φ is chosen to be dimensionless and also the scalar potential $V(\Phi)$ has energy dimension equal to two, independent of d. Such choice enables us for simultaneous studying of holo-duals in any bulk spacetime dimension $d+1$. In Lagrangian $\mathcal{L}$ for scalar field we admit also the possibility, that scalar field has negative sign of the kinetic term. Later we will decide and comment about it. We use the following ansatz for the metric, which preserves full d-dimensional Poincarè symmetry of constant r, flat Minkowski slices:

$$ds^2 = e^{2A(r)} \, \eta_{ij} \, dx^i dx^j + dr^2 \quad i,j = 1,\ldots,d \,. \qquad (3.2)$$

It means that spacetimes of our interest are warped (conformally scaled) d-dimensional flat Minkowski slices with original metric η_{ij}. We set all coordinates to have inverse energy dimension, this establishes the proportionality relation between radial r coordinate in the bulk of AdS and logarithmic RG time t. For definiteness we assume that $r = r_c t$, where $r_c = {}^1/v = (246\,\text{GeV})^{-1}$ is the length associated to the electroweak scale. It is also equal to the AdS radius in far UV as chosen in [42]. The requirement that our holographic

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spacetimes are asymptotically AdS, when the radius r tends to plus or minus infinity, is satisfied in situations, in which we have that $A(r) \to {}^r/r_{c\pm}$ in respective asymptotics in r. $r_{c\pm}$ denotes here the radii of AdS in asymptotic region of spacetime, they must be different from each other, when we ask for nontrivial RG flow. The direction of r coordinate, taking values on whole real line, is such that decreasing its value corresponds to following RG flow from UV to IR FP's.

Now we are going to present the method, which allows us to find a scalar potential $V(\Phi)$ for a given RG flow. From our coupled system of classical equations of motion, we first derive the scalar EOM in a fully generally covariant form $\pm \nabla^2 \Phi - \frac{\delta V}{\delta \Phi} = 0$, where $\nabla^2 \Phi = \sqrt{g}^{-1} \partial_\mu \left(\sqrt{g} \, \partial^\mu \Phi \right)$. For our spacetime metric ansatz (3.2) we have that $\sqrt{g} = \exp(d\,A)$. We assume that our scalar profiles vary only along the radial direction, being constant on Minkowski sections. With this simplification we have the scalar equation of motion given by

$$\pm \left(\Phi'' + d\,A'\,\Phi' \right) - \frac{\delta V}{\delta \Phi} = 0, \tag{3.3}$$

where by prime we denote radial derivatives $\frac{\partial}{\partial r} = \frac{1}{r_c} \frac{\partial}{\partial t}$. Our equations of motion take the form of ordinary differential equation, where r is the independent variable. Warp function $A(r)$ contains all RG flow data. However we don't want to solve (3.3) for scalar profile $\Phi(r)$, but rather to find a potential $V(\Phi_{\text{sol}})$ on a consistent solution $\Phi_{\text{sol}}(r)$ of the full gravitating system, provided some boundary conditions are satisfied in UV and IR limit of r coordinate. To do this we need at least one equation of gravitational character, because we don't know explicitly the scalar solution $\Phi_{\text{sol}}(r)$. If we knew this explicitly we could invert the relation and plug $r(\Phi_{\text{sol}})$ in the right side of the formula for the first derivative of the scalar potential $\frac{\delta V}{\delta \Phi} = \pm \left(\Phi''(r) + d\,A'(r)\,\Phi'(r) \right)$. Now integrating the RHS (understood as a function of independent variable Φ_{sol}) over Φ_{sol} in some limits, we would get searched potential $V(\Phi_{\text{sol}})$. We will use a different method, which doesn't use the explicit scalar profile solution. Gravitational equations of motion must be exploited to reach this goal.

The Einstein tensor, of satisfying our metric ansatz spacetimes, contains only two

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interesting components $G^r{}_r$ and $G^i{}_j$. We compute a mixed covariant-contravariant form (to get rid of warping factor). We have explicitly interesting tensors: $R_{rr} = -d(A'' + A'^2)$, $R_{ij} = -e^{2A} \eta_{ij}(A'' + d\, A'^2)$ and $R = -d\,(2A'' + (d+1)A'^2)$. With this we get two diagonal components of Einstein tensor as

$$G^r{}_r = \frac{d(d-1)}{2} A'^2 \quad \text{and} \quad G^i{}_j = \frac{d-1}{2} \delta^i{}_j \left(2A'' + d\, A'^2\right). \tag{3.4}$$

For a pair of equal transverse spacetime indices (i.e. i, $j = t_M$ d-dimensional Minkowski time), we see that it holds

$$G^{t_M}{}_{t_M} - G^r{}_r = (d-1)A'' \tag{3.5}$$

nicely relating geometrical structure on the manifold with the second derivative of the warp factor. On the other side of Einstein equations of motion $G_{\mu\nu} = M^{-(d-1)} \tilde{T}_{\mu\nu}$, we have energy-momentum tensor $\tilde{T}_{\mu\nu}$ of matter, which here is only in a form of scalar field Φ. We are going to simplify the notation by rescaling energy-momentum tensor by $M^{-(d-1)}$, which now has energy dimension equal to two and this is the most convenient choice for dimensionless scalars. We denote it by $T_{\mu\nu}$. Then we have gravitational equations in the simple form

$$G^\mu{}_\nu = T^\mu{}_\nu. \tag{3.6}$$

From the action functional (3.1) we derive the energy-momentum tensor for the scalar field Φ:

$$T^\mu{}_\nu = \pm(\partial^\mu \Phi)(\partial_\nu \Phi) - \delta^\mu{}_\nu \mathcal{L}, \tag{3.7}$$

where the structure of the scalar Lagrangian $\mathcal{L}$ is not as crucial as the sign in front of the kinetic term. Combining all three last numbered equations we arrive at the one, we really need for our method, namely

$$(d-1)A'' = \mp \Phi'^2. \tag{3.8}$$

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This last equation tells an interesting thing, that the convexity of warp factor is entirely determined by the sign of the kinetic term for the scalar field. For RG flows, which corresponds to concave warp factor, we have standard positively defined kinetic term. Those, for which $A(r)$ is convex, may be described holographically by phantom scalar field.

Now we parallel the derivation of a scalar potential $V(\Phi)$ based on the method of fake superpotential. We must remark that the presence of supersymmetry in the bulk is by no means a necessary assumption. We only borrow the method for finding special scalar potential from supergravity written in terms of superpotential. This is not completely general potential, but one, which is a representative in a wider class of potentials solving our issue. In this derivation we try to follow [46, 48, 47], where such potential was derived in general Einstein-Hilbert system with standard sign of the kinetic term for the scalar field, without at all invoking supersymmetry and for arbitrary dimension $d+1$. However we must also modify the form of this derivation for case of the scalar field with negative kinetic term. In the case of a single scalar field, when the target space metric is diffeomorphic to a constant and V becomes a function of a single real variable we can write

$$V = -2(d-1)^2 \left(\frac{\delta W}{\delta \Phi}\right)^2 - 2d(d-1)W^2. \tag{3.9}$$

The corresponding form of the scalar potential for standard (not phantomic) scalar field would satisfy the requirement of nonperturbative gravitational stability of AdS vacua (as found in [56, 57]). This would easily translate itself into a condition of positive energy solutions in the gravitational framework and preservation of null energy condition for matter content of the bulk theory. The situation with phantomic scalar field is however different and this is why the sign is flipped of the first term in (3.9). Next we note that

$$\frac{\delta V}{\delta \Phi} = 4(d-1)\left[-(d-1)\frac{\delta^2 W}{\delta \Phi^2} - dW\right]\frac{\delta W}{\delta \Phi}, \tag{3.10}$$

which tells us that AdS vacua are at points, where $\frac{\delta^2 W}{\delta \Phi^2} = -\frac{d}{d-1}W$. In addition to such would-be supersymmetric vacua in our theory we have also domain wall solutions. We

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will mainly focus on would-be BPS domain walls, which interpolate between two would-be supersymmetric AdS vacua. The reason is that for them we are able to easily solve resulting equations of motion.

Let us turn back to the integral (3.1) giving us the classical action of the whole system. We plug there our ansatz for the metric (3.2) and the form of the gravitational Lagrangian given by the curvature scalar. We already properly integrated it by parts from the initial form $R \cong -d(d-1)A'^2$ with total derivative term $\left(2e^{dA} A'\right)'$ abandoned. After neglecting the integration over transverse d-dimensional space we can rewrite the action integral as the following energy functional in only one integration variable r as

$$E[A, \Phi] = -\int_{-\infty}^{\infty} dr \, e^{dA} \left[-\frac{1}{2}\Phi'^2 - d(d-1)A'^2 + V \right]. \tag{3.11}$$

With the use of (3.9) and Bogomol'nyi method this functional can be presented in the following form

$$E = -\int_{-\infty}^{\infty} dr \, e^{dA} \left[-\frac{1}{2}\left(\Phi' \mp 2(d-1)\frac{\delta W}{\delta \Phi} \right)^2 - d(d-1)\left(A' \mp 2W \right)^2 \right] \pm 2(d-1)\left[e^{dA} W \right]_{-\infty}^{\infty}. \tag{3.12}$$

We obtain so called BPS equations by requiring for extremisation of this expression with respect to all A and Φ. In the result a pair of first-order differential equations is derived:

$$A' = \pm 2W \tag{3.13}$$

$$\Phi' = \pm 2(d-1)\frac{\delta W}{\delta \Phi}. \tag{3.14}$$

A posteriori we check that solutions of these BPS equations indeed solve the full system of equations of motion, given explicitly by:

$$d(d-1)A'^2 + \Phi'^2 + 2V = 0 \tag{3.15}$$

$$2(d-1)A'' + d(d-1)A'^2 - \Phi'^2 + 2V = 0 \tag{3.16}$$

$$-\Phi'' - d\,A'\,\Phi' - \frac{\delta V}{\delta \Phi} = 0. \tag{3.17}$$

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Although in the BPS equations (3.14) we have two signs allowed, this ambiguity cancels, when we go to the formula for the scalar potential representative written entirely using derivatives of the warping factor:

$$V = -\frac{1}{2}d(d-1)A'^2 - \frac{1}{2}(d-1)A'',\tag{3.18}$$

where we also used the relation $A'' = 4(d-1)\left(\frac{\delta W}{\delta \Phi}\right)^2$. This is valid for both signs in (3.14) and was derived from (3.8). The formula (3.18) is a crucial step in our method for finding a scalar potential valid for given scalar configurations. By knowing spacetime dependence of warping factor $A(r)$ in this way we can find exact radial dependence of the scalar potential $V(r)$ understood as evaluated on particular solution $\Phi_{\text{sol}}(r)$, though we don't know it yet.

To find a unique function $V(\Phi)$ we must determine this scalar configuration solution and invert it:

$$V(\Phi) = V(r(\Phi)) \quad \text{for } r(\Phi) = (\Phi_{\text{sol}}(r))^{-1}.\tag{3.19}$$

In this method one integration (over Φ variable) is avoided compared to the method previously suggested. We must note however, that these two methods are equivalent, because they give the same answer for the potential. We must still find a solution $\Phi_{\text{sol}}(r)$. This can be achieved by integration of equation (3.8) over radial coordinate. With obvious notation we get that

$$\Phi(r) = \Phi_{\text{UV}} - \int_r^\infty d\tilde{r}\, \Phi'(\tilde{r}) = \Phi_{\text{UV}} - \int_r^\infty d\tilde{r}\, \sqrt{(d-1)A''(\tilde{r})}.\tag{3.20}$$

If this integral can be done analytically and resulting function inverted, then equation (3.19) will yield an analytic expression for the desired potential $V(\Phi)$.

3.2 Gravitational RG flows

Main part of this section we will devote for the description of gravitational RG flow geometries in the holographic perspective. But before this, let us describe briefly the

RG runnning of ordinary matter couplings from the boundary theory. As it is common in AdS/CFT we will describe the RG running of a scalar operator $\mathcal{O}$, which triggers nontrivial flow and hence explicitly breaks the conformal invariance. According to the dictionary it is dual to some scalar field in the bulk ϕ. Nontrivial radial dependence of this bulk field $\phi(r)$ means that we have nonconformal RG flow for our deformation $\mathcal{O}$ in the boundary theory. Of course the flow of such operator in the boundary theory can be interpreted as the RG flow of a coupling parameter g, which is used to couple it. This coupling possesses such an energy dimension that the product $g\mathcal{O}$ has dimension d proper for the Lagrangian in the boundary theory. Although to different operators we have corresponding different bulk scalars, the RG running of those is read not from their corresponding profiles, but from the universal warp factor. Dynamics of all bulk fields have the impact on the actual form of the warping factor $A(r)$ due to the gravitational sourcing. Following [58] and [42] we accept the following identification

$$k(r) = k_0 \, e^{A(r)} \tag{3.21}$$

between radial bulk direction and the momentum scale in the boundary theory. This is the generalisation of the relation $k(r) = k_0 \, e^{\frac{r}{r_c}}$ to bulk spacetimes different from AdS, but still having the appropriate properties in the transverse directions. We may write the expression for the beta function of the coupling g:

$$\beta_g = k\frac{dg}{dk} \leftrightarrow \frac{1}{A'}\phi'(r), \tag{3.22}$$

where the first equality gives the field theory definition of such object, while the second relation gives a holographic interpretation in the bulk spacetime. The most RHS of the above equation can be rewritten further using equations of motion in the bulk and put in a form, where there is only dependence on the bulk scalar ϕ. This correspondence can be viewed as another fundamental formula in AdS/CFT duality relating boundary to bulk quantities.

Now we come to discuss the scaling properties of dimensionful couplings in boundary

theory. If in boundary field theory we are in FP regime, then this scaling is in the form of a power law

$$g_{\text{IR,UV}}(k) = g_0 \left(\frac{k}{k_0} \right)^{\alpha_{\text{UV,IR}}}. \tag{3.23}$$

Conformal scaling dimensions α_{IR} and α_{UV} need not be identical, but they reach fixed values at CFT FP's in IR and UV respectively. There we have valid the simple expression for the beta function $\beta_g = \alpha g$. This agrees precisely with the way, how we have read the scale-dependence in §2.2, when we assumed that our considered couplings come with some defined scaling dimension α. Because in our model in infrared limit gaussian FP exists, then the scaling dimensions α_{IR} are given by classical energy dimension of couplings. To leading order we continued with the assumption that they are not changed significantly in UV, in other words we neglected anomalous dimensions of these couplings. Namely we stuck with UV dimensions 1 for Higgs vev and 2 for the square of the 4-dimensional Planck mass. With this we were able to read correctly in §2.2 the RG running in holographic method. In full generality we have running dimension $\alpha = \alpha(k)$ interpolating between two scaling dimensions of the same operator $\mathcal{O}$ in two CFTs. They do not have to correspond to classical dimension of this operator. We use the following definition for varying α, $g(k) = g_0 \left(\frac{k}{k_0} \right)^{\alpha}$. In the intermediate region between two FPs of RG, we have an expression for the beta function $\beta_g = g \left(\alpha + \beta_\alpha \log \frac{k}{k_0} \right)$. We see that it was corrected by the beta function of α itself multiplied by a logarithm of the energy scale. It often appears, when we rescale a dimensionful coupling by power of energy scale with the classical dimension in the exponent $\tilde{g} = g \, k^{-\alpha_{\text{cl}}}$. This removes powers of momentum from RG running and the corresponding beta function equals $\beta_{\tilde{g}} = \tilde{g} \left(\alpha - \alpha_{\text{cl}} + \beta_\alpha \log \frac{k}{k_0} \right)$.

Now we are in position to apply method, described in the previous section, for finding scalar potential for given RG flows. We will consider a flow of the gravitational coupling in 4-dimensional boundary theory, whose tree level action (and also our truncation) is Einstein-Hilbert for gravitation. We will consider an RG flow of m_P caused by quantum effects. From here on we work explicitly in $d = 4$. The reason is that only in this dimension

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we have the correct description from holographic RG flow as it was elucidated in the previous chapter. We adopt the following conventions for dimensionless gravitational couplings: $\tilde{G} = G\,k^2 = k^2(m_P^2)^{-1}$, where k is the RG energy scale related to radial coordinate by (3.21) [52]. Let us pay attention to the fact that dimensionful Newton's constant is given by $G_N = (8\pi m_P^2)^{-1}$, so it is off by numerical factors from the coupling G. In [45] was derived a one-loop equation governing the RG flow of $\tilde{G}$. This is in the form

$$\frac{d\tilde{G}}{dt} = \dot{\tilde{G}}(t) = (d-2)\tilde{G} + B_1\,\tilde{G}^2. \tag{3.24}$$

The solution of this equation is given by

$$\tilde{G}(t) = \frac{2\tilde{G}(0)e^{2t}}{2 + B_1\,\tilde{G}(0)(1 - e^{2t})}. \tag{3.25}$$

It is convenient (in four dimension) to analyse the RG running of the square of the effective Planck mass. From (3.25) it is given by

$$m_P^2(t) = m_P^2(0) + \frac{B_1}{2}r_c^{-2}\left(1 - e^{2t}\right). \tag{3.26}$$

It is very important that all investigated types of cutoffs give negative values of the coefficient B_1. This signifies that the effective 4-dimensional Planck mass grows, when the energy scale increases. This means that quantum gravity perturbed around flat Minkowski spacetime shows its antiscreening nature.

To find a holographic geometry, which gives rise to a valid description of such a flow, we recall how the running of effective couplings in gravitational theory living on hypothetical brane of codimension 1 in the bulk, is seen from 5-dimensional perspective. This is basically the argument presented by Randall and Sundrum in [1], which we showed already in §2.2. Here it is generalised to a t-dependent conformal factor $A(t)$. We have the action on a probe brane, located at some position given by the RG time t,

$$S_{\text{grav}} = m_P^2(t)\int d^4x\,\sqrt{-\bar{g}}\,\bar{R}, \tag{3.27}$$

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where barred geometric quantities are induced on a brane from the bulk. An observer in the bulk sees this action as resulting from integration of the gravitational action in the bulk over some interval of radial coordinate (equivalently RG time t) according to the formula

$$S_{\text{grav}} = M^3 \, r_c \int_0^t dt' \, e^{2A(t')} \int d^4x \, \sqrt{-\bar{g}} \, \bar{R}. \tag{3.28}$$

From two above formula we derive that the holographic running of 4-dimensional m_P^2 is expressed by

$$m_P^2(t) = m_P^2(0) + M^3 \, r_c \int_0^t dt' \, e^{2A(t')}. \tag{3.29}$$

It means that roughly, when going in direction from IR to UV, the square of the Planck mass gets increased by integrating always positive warping function $e^{A(t)}$. This is another, holographic proof, of the character of running of this gravitational coupling parameter - it is bigger at higher energy scales. By differentiating (3.29) we get

$$\frac{d}{dt} m_P^2(t) = M^3 \, r_c \, e^{2A(t)} \tag{3.30}$$

and an explicit expression for the warping factor

$$A(t = \frac{r}{r_c}) = \frac{1}{2} \log \left(\frac{\frac{d}{dt} m_P^2(t)}{M^3 \, r_c} \right). \tag{3.31}$$

We see, that the whole construction of the holographic RG geometry is derived not from the scale dependence of coupling itself, but from the beta function of the gravitational coupling. This means, that holography is insensitive to any additive constant, which might be present in explicit running $m_P^2(t)$. In order to read local curvature of AdS part of spacetime we have to compute $A'(r)$ with the help of

$$A' = \frac{\ddot{m}_P^2}{2 r_c \, \dot{m}_P^2}. \tag{3.32}$$

Constancy of the above quantity over some range of radial coordinate values means that this region of spacetime is exactly a piece of AdS with given radius. However for a typical

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RG flow such good situation does not happen and only in asymptotic limits $r \to \pm\infty$ we obtain constant value of local radius $(A')^{-1}$. Therefore we work only with asymptotically AdS spacetimes.

It is easy to convince ourselves that the RG flow as given by (3.26) is described by an exactly affine function of the radial coordinate r: $2A(r) = 2\frac{r}{r_c} + \log\left(\frac{-B_1}{(M\,r_c)^3}\right)$. However recalling from [42] that $-B_1 = 2c_P = (M\,r_c)^3$ we obtain linear radial dependence of the warp factor $A(r) = \frac{r}{r_c}$, exactly like in AdS spacetime all the way along this RG flow. The AdS radius equals always to r_c. We state this fact as that the one-loop perturbative Einstein-Hilbert flow is described by the pure AdS holographic spacetime. We are not already at a conformal fixed point, because only at high energy we can neglect any additive constant in the solution (3.26). In FP regime dimensionful Planck constant scales according to $m_P^2(t) = G_*^{-1} r_c^{-2} e^{2t}$. That we are in pure AdS spacetime is not a surprise, because in holographic construction in §2.2 this was exactly our initial assumption about the bulk spacetime. We must consider this kind of flow deeper. We have for it, that $\Delta A' = 0$, because $A' = r_c^{-1}$. This flow (3.26) is valid for small (perturbative) values of the dimensionless gravitational coupling $\tilde{G}$ and significant corrections appear only, when running $\tilde{G}(t)$ is of order 1. So this happens for t around $t_P = -\frac{1}{2}\log\tilde{G}(0)$. (Knowing the approximate experimental value of G at electroweak scale $t = 0$, we find that $\tilde{G}(0) \approx 10^{-33}$ and $-\frac{1}{2}\log\tilde{G}(0) \approx 38$) As it stands this flow of coupling possesses a nontrivial UV FP with the fixed value of the coupling $\tilde{G}_* = -\frac{2}{B_1}$. Nevertheless we expect some changes to details of this picture due to higher loops and nonperturbative corrections. This is because close to this FP we are away from the regime of validity, where this flow was derived. It is important to analyse the limiting behaviours of this flow in the UV and IR. Namely in the UV regime we have that the flow is approximated by

$$\tilde{G}(t) \approx -\frac{2}{B_1}\left(1 + \frac{2e^{-2t}}{B_1\,\tilde{G}(0)}\right), \tag{3.33}$$

so we conclude, that the fixed limiting value of the coupling $\tilde{G}_*$ is reached exponentially fast for $t > t_P$. In the IR regime the flow asymptotically coincides with the flow from a

gaussian (trivial) FP $\tilde{G}(t) = \tilde{G}(0)e^{2t}$ and is given by an approximate formula

$$\tilde{G}(t) \approx \frac{2\tilde{G}(0)}{2 + B_1\,\tilde{G}(0)}e^{2t}\left(1 + \frac{B_1\,\tilde{G}(0)}{2}e^{2t}\right) \tag{3.34}$$

and again we have the conclusion of exponentially fast reaching of IR FP regime in negative t variable. Of course in infrared the dimensionless constant has vanishing limit. In the very far IR regime this flow as well as gaussian one is a solution of a simplified differential equation $\dot{\tilde{G}} = 2\tilde{G}$. This equation just governs the behaviour at trivial FP, where the dimensionless values of the couplings vanish. At this FP the dimensionful Planck mass is exactly constant and this naively means the breakdown of holographic description, because the warping factor blows up ($A = -\infty$) and we cannot define the radius of curvature. Indeed strong-weak duality arguments suggest that to infinitesimally weakly coupled boundary theory (as when originating from gaussian FP) corresponds infinitely high curved holographically dual spacetime. And such spacetime without including quantum 5d gravitational corrections to bulk theory doesn't make much sense. This is one of the problems we want to address by later modification of the flow given in (3.26).

It is not necessary to modify the RG flow in the UV, because for general flows ending at nontrivial UV FP, we have the scale invariance of the gravitational coupling, which means that $m_P^2(k) \sim k^2$ and hence $A' = r_c^{-1}$. The exponent 2 in the formula (3.33) comes because of such dimensionality of Newton's constant in four dimensions (negative to classical energy dimension). We remind that here we have completely neglected the anomalous graviton dimension. That the local radius of curvature equals to parameter r_c, is a very robust feature for all approximations to gravitational flows near UV FP. For example for a flow obtained from the exact RG differential equation

$$\frac{d\tilde{G}}{dt} = 2\tilde{G} + \frac{B_1\,\tilde{G}^2}{1 + B_2\,\tilde{G}} \tag{3.35}$$

with constant coefficients B_1 and B_2, we find the following high energy behaviour of the warping factor

$$A(r) = \frac{r}{r_c} + \frac{1}{2}\log\left(\frac{-B_1}{2c_P}\left(1 + B_2\,\tilde{G}\right)\right). \tag{3.36}$$

This expression near UV FP reduces to $A(r) = \frac{r}{r_c} - \frac{1}{2}\log\left(c_P\,\tilde{G}_*\right)$, because the fixed point value of the coupling is given by the relation $\tilde{G}_* = -\frac{2}{B_1+2B_2}$. This exact flow, although precisely is not as the flow governed by the FP in formula (3.25), returns back to it in the deep IR and UV. In IR it coincides precisely, because the effective value of $\tilde{G}$ is small and we can neglect the denominator in (3.35). Hence we arrive at flow equation (3.24). In deep UV we have however only a quantitative difference between the two flows showing itself up in the presence of free term in the conformal factor for the latter flow in formula (3.36). Additionally there is a difference between values of the coupling at fixed points. For our purposes improvement given by functional RG doesn't change the qualitative characteristics of running of gravitational coupling with an energy scale. Nevertheless we must note that the flow, which is a solution of (3.35) (not existing in a closed form) is truly an interpolating flow between two almost identical CFT's. The only difference between them is in the fixed values of dimensionless couplings (between $\tilde{G}_* = 0$ and $\tilde{G}_* = -\frac{2}{B_1+2B_2}$ in UV). This is because in the limit $r \to \pm\infty$ the inverse AdS radius tends to the same value $A' \to r_c^{-1}$. Moreover it seems, that their corresponding central charges are the same! In global sense we have precisely that $\Delta A' = 0$ for the whole flow. But this and the continuity of the flow implies that the sign of the second derivative A'' is undetermined. So we arrive at the conclusion, that the local version of the c-theorem doesn't hold here. And moreover in the holographic description of this flow the sign of the kinetic term for interpolating scalar field is undetermined too. It is fair to say that from holographic perspective flows given by (3.25) and solving (3.35) are closer to being at UV FP (where $A' = r_c^{-1}$) than at IR gaussian FP (with $A = -\infty$). The conclusion might be opposite, when looking naively for running of dimensionful gravitational coupling $m_P^2(t)$.

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3.3 Holographic description of the interpolating gravitational flow

To solve above mentioned problems with holographic interpretations, we may try to modify the RG flow only in the IR, not spoiling therefore nice properties of asymptotic safety scenario holding in UV. We cannot allow the warping factor to be or to tend to minus infinity limit in the far IR region. We must include threshold effects and stop or better neutralise the running in this low energetic regime. We obtain the most harmless running derived from (3.29), when we put to zero the warping factor asymptotically for $t \rightarrow -\infty$. If we put it to a negative value, then the running would be even smaller, but then we would enter another nontrivial inverse FP regime. Namely in the deep IR Planck mass would tend to zero value exponentially fast. And there wouldn't be a remnant non-zero value for the effective Planck mass at IR limit - we want however opposite. Both these choices of flow's modification lead to different geometries in IR region of holographic spacetime. But regardless of them we are forced to accept the global change of the inverse local AdS radius to be positive $\Delta A' > 0$. This is a necessary consequence, when we want to soften the flow in the IR. The problem with the c-theorem can not be solved by this method. We can achieve such a smooth change of warp factor that always $A'' > 0$ and asymptotically $A'' \rightarrow 0$. This means that in the holographic 5-dimensional description in the middle of our interpolating geometry we excite a phantomic scalar field Φ (from formula (3.8)). With the choice that $A = A' = 0$ for deep IR we get the following flow of the gravitational coupling $m_P^2(t) = m_P^2(0) + 2c_P \, r_c^{-2} \, t$. This is also a solution of an unautonomous flow equation $\frac{d\tilde{G}}{dt} = 2\tilde{G}\left(1 - c_P \, \tilde{G} \, e^{-2t}\right)$. We obtain the interpolating flow between IR and UV by simply adding and adjusting two limiting behaviours in the IR and UV. This is possible, because in opposite limits each of these flows is negligible with respect to the other one. In UV limiting behaviour of the flow is $m_P^2(t) = m_P^2(0) + c_P \, r_c^{-2} \, (e^{2t} - 1)$ (compare with (3.26)). The interpolating flow has the explicit form

$$m_P^2(t) = c_P \, r_c^{-2} \left(e^{2t} + 2t - 1 \right) + m_P^2(0). \tag{3.37}$$

This is the flow, which we are going to analyse in this section looking for its fully-fledged holographic description. We treat this particular flow as an example, for which we are able to bring the explicit computation of scalar potential to the very end. Our method for finding holographic RG geometries is however general and works also for other examples of gravitational RG flows. The necessary ingredient is the explicit form of the function $m_P(t)$.

We now come to the construction of a holographic RG flow spacetime, which may be understood as a geometrization of the above flow. Equation (3.37) governs the running of the dimensionful gravitational coupling: Planck mass square. In terms of dimensionless coupling $\tilde{G}$ (convenient for describing physics from nontrivial UV FP perspective) the flow is expressed as

$$\tilde{G}(t) = \frac{\tilde{G}(0) \, e^{2t}}{\tilde{G}(0) \, c_P \left(e^{2t} + 2t - 1 \right) + 1}. \tag{3.38}$$

For very small values of $\tilde{G}(0)$ this flow has a big variability around $t = t_P$ and this lasts for around 5 units in logarithmic RG time t. Before this transition region the value of $\tilde{G}(t)$ almost vanishes and after it attains c_P^{-1} equal to the fixed point value. Warping factor function, because it gives the holographic description, is quite universal and doesn't depend on specific parameters of the flow expressed by its initial value $\tilde{G}(0)$ and the rescaling parameter c_P. Hence, regardless of these parameters, for this type of flow warping factor equals to

$$A(t) = \frac{1}{2} \log \left(1 + e^{2t} \right). \tag{3.39}$$

It has required properties, which we described in the previous section, and this is easily visible from the plot of $A(t)$ shown on the left panel of Fig. 3.1. The value at the origin is given by $A(0) = \frac{1}{2} \log 2 \approx 0.35$. The origin is also a point, where the two asymptotic limit of $A(t)$ are joined smoothly. The first derivative of $A(t)$ with respect to radial coordinate
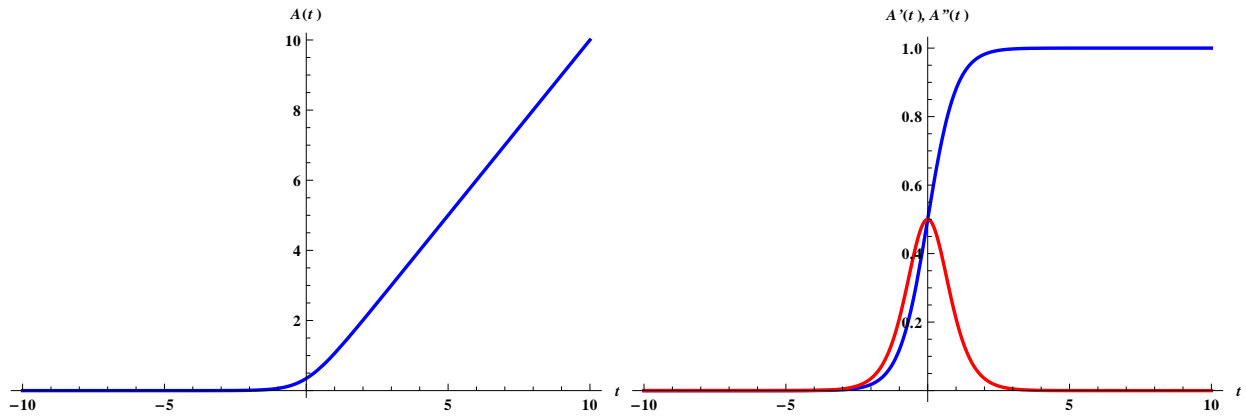
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Figure 3.1: On the left: Warping factor $A(t)$ as a function of t. On the right: First (in blue) and second (in red) derivative with respect to radial coordinate t of the warping factor $A(t)$.

r is a symmetric function interpolating between values of zero (in IR) and two (in UV). The second derivative given by

$$A''(t) = \frac{2e^{2t}}{r_c^2 \left(1 + e^{2t}\right)^2} \tag{3.40}$$

is an even and positive function (as we demanded) and its maximal value $\frac{1}{2}r_c^{-2}$ is reached at the origin. In infinity this function attains vanishing limits. Plots of the rescaled to be dimensionless, first and second derivative are shown on the right panel of Fig. 3.1.

Now using formula (3.18) we have the explicit radial dependence of the scalar potential

$$V(t) = -\frac{3e^{2t} \left(1 + 2e^{2t}\right)}{r_c^2 \left(1 + e^{2t}\right)^2} \, . \tag{3.41}$$

This potential has vanishing IR limit $V_{\text{IR}} = 0$. From more closer look we also see that value of it at $t = 0$ equals to $-9/4 = -2.25$ and that UV limit is -6 in inverse square units of radius of AdS r_c. Plot of the radial dependence of the scalar potential (3.41) we present on the left panel of Fig. 3.2 using blue curve.

Fortunately enough for this simple form of warp factor (3.39) we can integrate as in equation (3.20) and the resulting scalar profile of a solution has the following radial

dependence

$$\Phi_{\text{sol}}(t) = \sqrt{6} \arctan e^t. \tag{3.42}$$

Inverting this relation for finding t as a function of Φ is an easy task equivalent to solving this simple transcendental equation for t. We may do this analytically or in the last part of the analysis we may resort to numerical results. Here we only want to add, that monotonically increasing function (3.42) takes values between 0 (which we have chosen as value of Φ_{IR}) and $\Phi_{\text{max}} = \frac{\sqrt{6}}{2}\pi \approx 3.85$ (which must be the value of the scalar field reached at UV FP) in a symmetric way around the origin. The red curve on the left panel of Fig. 3.2 shows, how this scalar profile changes, when we move in radial direction. From last relation (3.42) we get that

$$e^t = \tan\left(\frac{\sqrt{6}\,\Phi_{\text{sol}}}{6}\right). \tag{3.43}$$

Plugging this to the formula (3.41), we obtain an analytical shape of the scalar potential as the function of Φ given by the following combination of trigonometric functions:

$$V(\Phi) = -\frac{3}{2r_c^2}\sin^2\left(\frac{\sqrt{6}\,\Phi}{6}\right)\left(3 - \cos\left(\frac{\sqrt{6}\,\Phi}{3}\right)\right). \tag{3.44}$$

Using some trigonometric identities we rewrite this to the following simple form:

$$V(\Phi) = \frac{3}{8r_c^2}\left[-7 + 8\cos\left(\frac{\sqrt{6}\,\Phi}{3}\right) - \cos\left(\frac{2\sqrt{6}\,\Phi}{3}\right)\right]. \tag{3.45}$$

This function is shown on the plot placed on the right panel of Fig. 3.2 at least over the range of the scalar field values covered in the holographic flow, i.e. from $\Phi_{\text{IR}} = 0$ to $\Phi_{\text{UV}} = \Phi_{\text{max}}$. Therefore we have produced an analytical result for the scalar potential necessary to produce bulk geometry with conformal factor $A(r)$ given by (3.39) as a solution of the system consisted of Einstein gravity and bulk scalar fields.

We can describe quantitatively few features of this potential. Firstly despite that it is defined only for region of Φ between 0 and Φ_{max} we can make it periodic. The proper
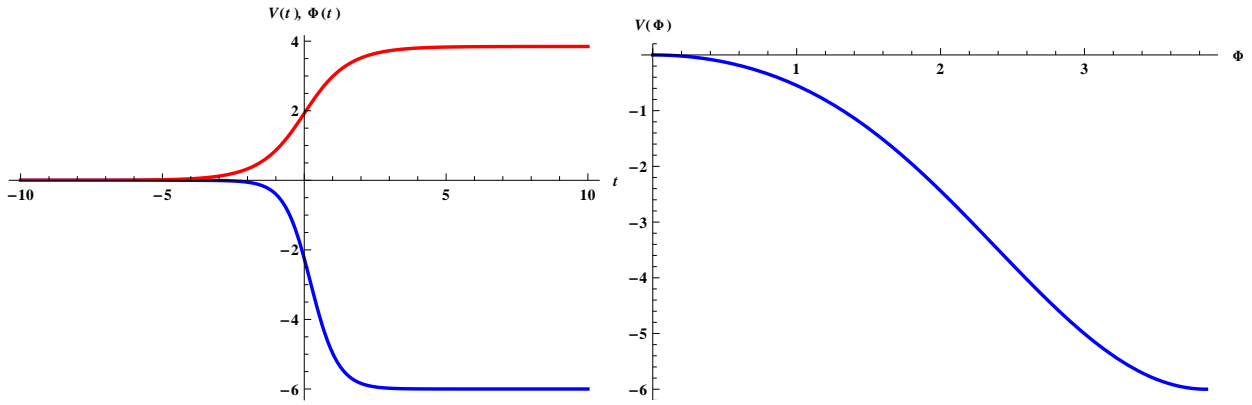
Figure 3.2: On the left: Radial dependence of the scalar potential $V(t)$ in blue and of the scalar profile Φ_{sol}, which is a solution of EOM, in red. On the right: Scalar potential V as a function of Φ. Scalar potentials are shown in units of r_c^{-2}.

period in this case is exactly equal to $2\Phi_{\text{max}}$ as we can read from explicit analytic formula (3.45). We chose the biggest proper period out of periods for trigonometric functions, which appeared there. As a result this was the period of the second term in the square bracket in this formula. Moreover we can extend potential's domain in such a way that the full potential would be symmetric around points with values of ordinates Φ_{UV} and Φ_{IR}. To do this we must reflect the plot valid for the holographic RG flow with respect to vertical axes at its critical points or simply extend naturally the domain of the trigonometric functions. We easily observe that the period of two trigonometric factors in (3.44) is identical and this is the result for the period of the whole potential, which is equal to $\sqrt{6}\pi \approx 7.70$. Obviously from the equation (3.20) we have the shift symmetry $\Phi \to \Phi + \Phi_0$ enjoyed by the scalar EOM related to the free choice of initial value of the field for IR region. For holographic purposes we chose $\Phi_{\text{IR}} = 0$, nevertheless we could equally well shift it by integer multiple of the period. Secondly we don't have the possibility of adjusting the constant in the potential, this is precisely determined by formula (3.18). This additive constant in the potential is related to the already determined by the properties of FP in UV, value of the cosmological constant in the bulk region V_{crit}. Therefore we have monotonically decreasing potential from value zero at $\Phi_{\text{IR}} = 0$ to the value $V_{\text{UV}} = -6r_c^{-2}$ at the edge of RG flow in

UV for $\Phi = \Phi_{\mathrm{max}}$. Despite the fact that we were able to produce an analytic results for the full form of the potential, it is still interesting and possible to consider limiting behaviours around the IR and UV critical points. We will obtain these by a perturbative expansion around these critical points. We define deviations of the scalar field value from its critical ones by $\delta\Phi_{\mathrm{UV,IR}} = \Phi - \Phi_{\mathrm{UV,IR}}$ respectively for IR and UV FP's. We easily verify that in expansions around the corresponding points odd powers of these deviations do not appear. We can cast potential in the vicinity thereof into the following form given by a series

$$V(\delta\Phi) = V_{\mathrm{crit}} + \frac{1}{2}m^2(\delta\Phi)^2 + \frac{1}{24}\lambda(\delta\Phi)^4 + \dots , \tag{3.46}$$

where formally infinite set of parameters V_{crit}, m^2, λ, ... parametrises dynamics of the scalar profile near the corresponding critical points of CFT's. We have obviously that V_{crit} is the value of the cosmological constant in the corresponding to FP of RG AdS vacuum. The mass parameter can be calculated from

$$m^2 = \left.\frac{\delta^2 V}{\delta\Phi^2}\right|_{\Phi=\Phi_{\mathrm{UV,IR}}} = \left.\frac{1}{\Phi'}\frac{d}{dr}\left(\frac{1}{\Phi'}\frac{dV}{dr}\right)\right|_{r\to\pm\infty} \tag{3.47}$$

and similarly with the fourth power of the operator $\frac{1}{\Phi'}\frac{d}{dr}$ for the quartic coupling λ. Here Φ' denotes the radial derivative of the scalar, which from (3.8) equals to $\sqrt{(d-1)A''}$, where all functions are understood as functions of the independent variable r. As the results of calculations we obtained for the masses $m_{\mathrm{IR}}^2 = -r_c^{-2}$, $m_{\mathrm{UV}}^2 = 3r_c^{-2}$ and for the quartic couplings $\lambda_{\mathrm{IR}} = -\frac{4}{3}$, $\lambda_{\mathrm{UV}} = -4$. We are not afraid of negative values of the quartic couplings here, because the standard problem of unboundedness of the potential doesn't show up here. Scalar potential, which we found, is defined only on finite interval of Φ, where it is bounded function and never reaches large negative values. This remains obviously true, if it is extended to be periodic. These negative values of quartic couplings are valid only in the neighbourhoods of critical points of the potential and must be understood as the first terms in series for the exact potential. We obviously see from the plot (and this also confirmed by the calculation of mass parameters), that the critical point in IR is unstable, whereas the other in UV exhibits stability.

3.4 Interpretation and discussion

Holographic interpretation of the findings from the last section is as follows. Standard reason for start of a nontrivial RG flow is that the boundary CFT is perturbed by some relevant operator with respect to UV critical point. It has a conformal dimension $\Delta < d$ in order to be a relevant deformation. In holographically dual gravity description this operator is dual to some bulk field. In the case of scalar deformation in CFT, this is precisely Φ, and the conformal dimension of the operator is related to the mass of the bulk scalar by the famous relation $\Delta(\Delta - d) = m^2$ (in units of r_c^{-2}). For asymptotic behaviour near UV we have two solutions for conformal dimension of deformation $\delta\Phi$, namely $\Delta_\pm = 2 \pm \sqrt{7}$. The scalar field near boundary will have two independent solutions

$$\delta\Phi = e^{t\Delta_-}\,\Phi_-(x) + e^{t\Delta_+}\,\Phi_+(x). \tag{3.48}$$

Now because Δ_{UV} are of opposite signs (as always for positive mass square parameter), we have one normalizable and one nonnormalizable mode of the scalar near UV boundary. Moreover we have that the standard choice $\Delta = \Delta_+$ of nonnormalizable mode corresponds to an irrelevant operator, which is implied by the fact that $\Delta_+ \approx 4.65 > d = 4$. This means that the vev for the dual operator in the boundary theory is represented by not vanishing coefficient $\Phi_+ \sim \langle\Phi\rangle_{\text{bdy}}$. However for normalizable mode $e^{t\Delta_-}$ we have decaying solution with the coefficient Φ_- proportional to the vanishing coupling for the dual operator in the boundary theory. For this mode we easily see that the conformal dimension $-\Delta_- \approx 0.65$ is precisely the exponent controlling asymptotic decay of this mode of $\delta\Phi$. In this spirit the operator, which triggers nontrivial RG flow from UV FP to a critical point in the IR is irrelevant from the viewpoint of UV FP and only its nonzero expectation value causes the flow.

Note that in the infrared limit we have $m_{\text{IR}}^2 < 0$ and hence the dual operator is always irrelevant there ($0 < \Delta < d$) from the perspective of IR FP. We have explicitly that $\Delta_\pm = 2 \pm \sqrt{3}$ and both modes in IR regime are normalizable and correspond to irrelevant operators. The standard choice for a surviving mode is $\Delta = \Delta_-$, which signifies, that the

dual operator is relevant. The leading contribution in the asymptotic decay of deviations $\delta\Phi$ means that the corresponding coupling is nonvanishing in the boundary theory. This is a standard holographic interpretation of the flow. From the boundary viewpoint we agree, that the deformation in IR is relevant, because the inclusion of threshold phenomena modifies the flow significantly by softening it. In UV we see, as noticed previously, that with running (3.38) we are already at FP and the deformation turning the flow into the direction of new IR FP is only irrelevant.

We can now come back shortly to the issue of a running of mass parameters in the boundary theory, which are not of the gravitational type. Using equations (3.23) and (3.39) we derive that all energy dependence is given by the root factor $\sqrt{1 + e^{2t}} = \sqrt{1 + \frac{k^2}{v_0^2}}$. Running of the Higgs vev is particularly simple here (because $k_0 = v_0$) and is expressed by formula $v(k) = \sqrt{v_0^2 + k^2}$. It is important to find UV limit of this formula. Here is the result $v(k) = k\left(1 + \frac{v_0^2}{2k^2}\right)$. We see explicitly that it scales asymptotically in UV like in a nontrivial FP regime. For comparison we can mention here the running of other mass parameter in a theory. It is given by $M = M_0\, e^{A(r)}$, so in terms of energy scale k it is expressed by the equation $M(k) = \frac{M_0}{v_0}\sqrt{1 + \frac{k^2}{v_0^2}}$. In the UV regime this simplifies to $k\frac{M_0}{v_0}\left(1 + \frac{v_0^2}{2k^2}\right)$. We derive the conclusion, that the running of all dimensionful matter couplings, no matter what is their initial value, enters UV FP regime around the same energy scale. This scale is given by the IR value of the Higgs vev v_0. These observations may harmonise with recent findings in [83]. There authors pointed out, that the only scale at which new physics beyond SM can reach nontrivial UV FP regime is the electroweak scale.

Here we discuss some issues related to the proposed modified RG flow of gravitational coupling in 4-dimensional theory and its holographic interpretation. Firstly in the holographic model with bulk phantomic scalar we were able to describe the dual RG flow geometry. We achieved softening of the flow in IR, by including relevant deformations caused by threshold phenomena. They hide some strongly interacting physics, because this IR part of the spacetime is dual to a flat holographic spacetime. To do this softening

of the flow in IR, we had to use warp factor, which was convex as a function of radial variable. Next the convexity of warp factor forced us to use the scalar field with wrong sign of the kinetic term. However the formal calculation in the holographic framework can still be carried on, even in this case. In the standard approach concavity of the warp factor is closely related to the famous c-theorem. This originates from the duality between local radius of curvature and the central charge of CFT. In holographic description of matter couplings we have always that $A'' < 0$, hence $\Delta A' < 0$ [47] and this perfectly agrees with the local and global version respectively, of c-theorem $\Delta c < 0$ between UV and IR, for references look at [59, 60]. For those holographic gravitational flows, which we considered in previous sections, we found disagreement with the standard c-theorem. Maybe the explanation for this is that gravitational interactions must be properly included and must modify somehow standard CFT from flat spacetime.

This holographic construction is only one, indeed very interesting and enlightening, way of describing effectively RG flows in real 4-dimensional spacetime. We do not have to attach physical reality to such holograms - they are good descriptions extending our insights for the physics of the boundary theory. We do not claim, that 5-dimensional bulk spacetime with phantomic scalar field is a real physical object, amenable to observations. This construction should be understood merely as a geometrization of the RG flow from 4-dimensional boundary theory. It happens that we achieved this mathematical construction by adding only one additional holographic dimension and this resembles very much ideas, which are present in AdS/CFT conjecture. Our phantom field has a nontrivial potential with two critical points (in the holographic domain) and the nontrivial RG flow corresponds to interpolating BPS domain wall solution for this potential. In our setup, where all functions depend on only one variable - radial coordinate, our model of phantom dynamics in such potential admits a nice classical mechanics analogy. A similar analogy occurs in the cosmological inflation, when the similar motion of a scalar field (inflaton) in the potential resembles much rolling down the potential by a material point with friction given by the Hubble parameter. In our case, from equation (3.3) in the phantom case, we see that when we interpret negative to radial coordinate as the time parameter and value of the scalar

field as the position coordinate, we obtain motion in the potential for a material point with a negative inertial mass. This means that the acceleration is in opposite direction to the applied force and the material point rolls up from $\Phi = \Phi_{\max}$ to $\Phi = 0$ during the time evolution. We also have a velocity-dependent friction term caused by the curvature of bulk spacetime. In this situation we can reverse direction of time and end up with perfectly reasonable dynamics of normal material point starting its evolution in the critical point in IR, rolling down and ending in UV with $\Phi = \Phi_{\max}$. This is one interpretation of our RG flow and its holographically dual description in terms of gravitation and phantomic scalar in 5-dimensional bulk. We saw in the previous section, that thanks to the holographic interpretation, we could find some interesting features of the gravitational RG flows. The examples here are the dimensionalities and characters of the operators, which caused the non-trivial RG flows from UV to IR and which deformed IR gaussian FP.

At the end we talk about some comments regarding asymptotic safety and its holographic interpretation in the light of just presented construction. A gaussian FP of RG flow of gravitational coupling in IR is to be understood as noninteracting free CFT on flat spacetime, so it satisfies the description given above. At this FP Newton's constant G rescaled by the square of the typical momenta tends to zero, gravity is decoupled, however in the holo-dual AdS-spacelike curvature is infinite (there 5-dimensional Planck mass M doesn't run). This CFT in IR FP is deformed by adding irrelevant operator (from the IR FP perspective) $m_P^2 R$ to the free action and therefore the gravitational couplings are turned on. The nontrivial RG running for them starts. The holo-dual is no longer AdS spacetime, but more complicated holographic RG geometry. Going towards UV we integrate in interacting degrees of freedom - higher energetic gravitational modes, so the corresponding c-function (if possible to construct) should grow monotonically. However as we saw from holographic approach, this doesn't happen in the gravitational case, but presumably usual arguments don't apply here. In UV limit we enter another FP region of RG flow, different from one in (3.25). However there is an important difference, because 4-dimensional Newton's constant in units of momenta tends to a constant and Planck constant grows without a bound. The UV theory is scale invariant; if at FP, it may be

conformally invariant, but surely it is not a standard CFT on flat spacetime.

Part II

Classicalization and Quantum Effective Action

Chapter 4

Classicalization in nonlinear sigma model

4.1 Introduction

The nonlinear sigma model and Einstein's theory of gravity have many similar features. At the kinematical level, both theories have nonlinear configuration spaces, which make their dynamics necessarily nonlinear too. There is no "zero field" limit and the quantization procedure can be based on the use of the background field method. In both cases the degrees of freedom can be viewed as Goldstone bosons [1] and their interactions involve derivatives. Due to the nonpolynomial nature of the action, it is natural to think of the fundamental fields as being dimensionless. Aside from a vacuum term, the Lagrangian can be expanded as

$$S = \sum_k \sum_n \bar{g}_{k,n} \mathcal{O}_{k,n} \ , \tag{4.1}$$

where $\mathcal{O}_{k,n}$ is an operator containing k derivatives and n powers of the fields. [2] This

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[1] In the case of gravity this is explained for example in [64].

[2] Usually k must be even.

operator can be naturally a sum of finitely many monomial terms in fundamental fields of the theory. In four spacetime dimensions and in natural units the coefficients $\bar{g}_{2,n}$ have dimension of mass squared and $\bar{g}_{4,n}$ are dimensionless. In order to define a perturbative expansion with a canonically normalized kinetic term, one usually redefines the fluctuation field by a factor $\sqrt{g_{2,2}} = m$. Then one finds that the role of the perturbative coupling is played by $1/m$. It has dimension of length, so these theories are power counting nonrenormalizable. Perhaps more urgently, perturbative scattering amplitudes grow like powers of momentum and exceed the unitarity bound for momenta comparable to m. In fact, it is more correct to say that the perturbative expansion parameter is the dimensionless ratio p/m, where p is a typical momentum of the process under study, so that the perturbative treatment is useful up to momenta of order m. The standard view is then to regard these theories as effective field theories, valid at energy and momentum scales below m.

In principle, however, it is possible that some of these theories may heal themselves of their perturbative problems. By including true quantum dynamical effects these theories may somehow overcome problems of violation of unitarity and nonrenormalizability [82]. One possibility is that the growth of the effective couplings such as p/m terminates in the ultraviolet limit. In field theory a growth of a relevant coupling without an upper bound doesn't make much sense. This is not the case, if the theory approaches a fixed point in the UV [65]. In particle physics and gravity this behaviour is also called "asymptotic safety" [14]. There is by now significant evidence for the existence of asymptotically safe RG trajectories in gravity, see for example [66]; some work has also been done for the nonlinear sigma models [67, 68] and in particular for the electroweak chiral model [69]. One expects that in such asymptotically safe theories the scattering amplitudes also stop growing and respect the unitarity bounds, although no complete calculation of this type has been performed so far.

More recently, a different idea has been proposed, namely that the growth of the scattering amplitudes is controlled by the formation of classical intermediate states. In this picture, which has been called "classicalization", a high energy quantum state with low occupation number would evolve into a classical state (called a "classicalon") with large

occupation number. This is conjectured to happen, when the radius of the configuration, during process like a collapse, becomes comparable to a characteristic radius r_* called "classicalization radius". The classicalization radius is a new length scale in the theory emerging from nonlinear dynamics. The important point is that r_* does not decrease with energy as one might naively think, but rather grows with it or at least tends to a constant. We will call these cases strong and weak classicalization, respectively. Therefore the standard paradigm of high energy physics, that with the increase of energy of colliding wavepackets we decrease the probing scale given by the corresponding Compton wavelength, breaks down here. As a result, when the energy of the incoming states becomes greater than the characteristic scale m, scattering is dominated by the formation of classicalons and the cross section tends to the classical geometrical value r_*^2. In such conditions quantum Compton wavelength ceases to be a resolving scale, instead its character is taken by the classicalization radius. One of the necessary prerequisite for classicalization in field theory is the high level of nonlinearity and corresponding self-sourcing. The idea of classicalization emerged first in the case of gravity, where the classicalons would correspond to black holes [70], but subsequently it has been recognized as a possible behaviour also in Goldstone bosons models [71, 72, 73, 74]. Other aspects of classicalization proposal have been considered in [75, 76, 77, 78].

In spite of the evident differences between asymptotic safety and classicalization, one wonders whether they might not be two ways of looking at the same phenomenon. If, for example, the amplitude for Goldstone boson scattering unitarizes at high energy without having to introduce new weakly coupled degrees of freedom, it would be surprising, if there existed two independent mechanisms by which Nature could achieve this. If two explanations are available, they might just be different descriptions of the same phenomenon. This is only a hypothesis, but still we believe, that in a fully consistent theory, Nature has chosen definitively one unique mechanism, at whose two faces maybe we are looking now.

Motivated in part by this question, in this chapter we discuss aspects of classicalization in the nonlinear sigma models. We extend previous analyses, done for scalar fields, by considering in some more detail the effect of the curvature of the target space. Much of the

work, that had been done previously, had concentrated on a simple model of a single scalar, and since a one dimensional space is flat, nontrivial interactions necessarily involve terms with more than two derivatives. When the target space is curved, there are infinitely many interaction terms already at the two derivative level. We analyse the effect of these terms first by themselves, and then in the presence of higher derivative interactions. In order to be able to discriminate the effect of positive and negative curvature we shall consider both spherical and hyperbolic target spaces. In a different vein for nonlinear sigma model with two derivatives and with positively curved target space, evidences in favour of nontrivial FP of RG have been found recently [19]. This is why, exactly this model is under our investigation in this chapter.

If one wants to compare classicalization to asymptotic safety, the first obvious difference is the fact that asymptotic safety is based on renormalization group running, which is a truly quantum effect. However classicalization, as the name may suggest, is related to the formation of classical states and according to the classical dynamical evolution equations. In order to disentangle classical from quantum effects we will work throughout in units where $\hbar$ is not set equal to one.

Various signatures of the classicalization were outlined in the literature [73, 74, 77]. The phenomenon of classicalization is of highly nonperturbative nature and hence various checks are useful to decide a priori about occurence or not of it. One of such check relies on a change of the characteristic of the nonlinear PDE. If the classical equations of motion are put in the quasi-linear form and the characteristic of them changes sign at some location, then this location is expected to be an onset of the classicalization. This way of UV completion is strongly based on the classical states. Hence another requirement is the existence and single-valuedness of the solutions in classical field theory, which must be defined on a whole spacetime. In this respect such classicalons are analogs to solitons, known in nonlinear physics. In this chapter we will use another method for deciding about the occurence of classicalization in our models. This method checks, whether the asymptotic behaviour of the solutions is changed and whether there is a significant deformations of the wave profile of the incoming packet in the scattering process. This approach was pioneered

in [72] and in the next sections we will follow closely these derivations.

In this chapter we consider the phenomenon of classicalization in nonlinear sigma models with both positive and negative target space curvature and with any number of derivatives. We will introduce and describe a weak form of classicalization, putting special attention to the dependence on the sign of the curvature. Nonlinear sigma models with higher derivatives actions are also analyzed, where a strong form of this phenomenon occurs and which is moreover independent of the sign of curvature. Finally we will argue that weak classicalization may actually be equivalent to asymptotic safety, whereas strong classicalization seems to be a genuinely different phenomenon. We also discuss possible ambiguities in the definition of the classical limit, which is in the very core of understanding physical mechanisms lying behind classicalization.

We conclude this introduction by outlining the content of the following sections. In section 2 we review the notion of classicalization in the case of a simple theory of a single Goldstone boson with arbitrary derivative interactions. In section 3 we discuss nonlinear sigma models with values in maximally symmetric spaces with both positive and negative curvature, and with two derivatives only. We find that a weak form of "classicalization" happens. In section 4 we extend the analysis to include higher derivative terms. There we find, that the classicalization radius grows with energy, regardless of the sign of the internal space's curvature. In section 5 we return to the comparison between classicalization and asymptotic safety and we draw our conclusions.

4.2   A single self-interacting Goldstone boson

In this section we begin by considering a model of a single Goldstone boson with higher derivative interaction lagrangian of the form:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{\alpha^{4(m-1)}}{2m}\left((\partial\phi)^2\right)^m \ . \tag{4.2}$$

Here m is an index counting the derivatives. The field has the canonical dimension

$M^{1/2} L^{-1/2}$ and the coupling α (which was called L_* in [71, 72, 73, 74]) has dimension $L^{3/4} M^{-1/4}$. Later we will comment on the effect on classicalization of the presence of terms with lower or higher number of derivatives, but for the moment we assume that (4.2), with a fixed m, is the only interaction. Despite that we are mostly interested in classicalization, which may occur in quantum dynamics of fields, here we will analyze it using methods and equations of classical field theory. The action given by (4.2) we view as the bare action of our model subject to quantization procedure. Performing the full non-perturbative calculation in quantum field theory is a very formidable task. Here we will only incorporate quantum ideas about Compton wavelength and the quantum resolution scale.

The equation of motion coming from the lagrangian (4.2) is

$$\Box\phi + \alpha^{4(m-1)} \partial^\mu \left[\partial_\mu\phi \left((\partial\phi)^2 \right)^{m-1} \right] = 0 \ . \tag{4.3}$$

In our setup we have wave incoming from infinity and approaching the centre of our coordinate frame. Assuming that free asymptotic states solving the equation $\Box\phi_0 = 0$ exist, the solution of the nonlinear equation (4.3) can be constructed perturbatively. We consider solutions with spherical symmetry. Then the divergence of a one-form v_μ is $\partial_0 v_0 - \frac{1}{r^2} \partial_r(r^2 v_r)$ and the d'Alembertian is $\Box = \partial_t^2 - \frac{1}{r^2} \partial_r(r^2 \partial_r)$.

The initial ingoing unperturbed free wave has the form $\phi_0(t,r) = \sqrt{\hbar}\,\psi(\omega(t+r))/r$, where $\psi(z) = A\sin(z) + B\cos(z)$ is a dimensionless harmonic function in one dimension. The general solution for the free massless wave equations of motion (without interactions) we obtain by superposing waves with different real coefficients A and B as well as with different frequencies ω. (We threw away other solutions of free equations due to boundary conditions at spatial infinity). However in further analysis, for simplicity, we will stick to a monochromatic wave. We will assume that the wavelength ω^{-1} is small compared to the radius r, so that we can think of this solution as a harmonic function with a slowly-varying r-dependent amplitude. At large distances the effect of the interaction is negligible (because of higher $1/r$ dependence of the interaction terms). This is why there the free wave solution

well approximates a solution of full theory and constitutes well-defined asymptotical state of the theory. When considering solutions with spherical symmetry approaching central region from radial infinity, it is natural to assume that the characteristic classical length of the configuration is given by radius r.

Our equations of motion are nonlinear, this means that even free initial wave, when approaching the centre develops a scattered component. We treat this scattering process perturbatively. The equation for the first order perturbation ϕ_1 is

$$\left(1 + \alpha^{4(m-1)}((\partial\phi_0)^2)^{m-1}\right) \Box\phi_1$$

$$+2(m-1)\alpha^{4(m-1)}((\partial\phi_0)^2)^{m-2}\left(\partial^\mu\phi_0\,\partial^\nu\phi_0\,\partial_\mu\partial_\nu\phi_1 + 2\partial_\nu\phi_0\,\partial^\nu\partial^\mu\phi_0\,\partial_\mu\phi_1\right)$$

$$= -2(m-1)\alpha^{4(m-1)}((\partial\phi_0)^2)^{m-2}\,\partial^\mu\phi_0\,\partial^\nu\phi_0\,\partial_\mu\partial_\nu\phi_0 \ . \tag{4.4}$$

We have written on the left hand side of the equation all the terms that contain derivatives of ϕ_1 and on the right a source term containing only ϕ_0. This equation is still quite complicated. However, we will see *a posteriori* that for the values of r, that we are interested in ($r \gg \omega^{-1}$ and $r \to \infty$), the terms on the l.h.s. that come from the interaction are small relative to $\Box\phi_1$. For our purposes it will therefore be sufficient to retain in the l.h.s. only the term $\Box\phi_1$. [3]

We make an ansatz for the form of the first perturbation:

$$\phi_1(t,r) = \sqrt{\hbar}\,f(r)\,\eta(\omega(t+r)) \ . \tag{4.5}$$

This ansatz preserves spherical symmetry of the configuration, and similarly to ϕ_0 we chose it in a separated form: oscillating function $\eta(z)$ and radially dependent amplitude $f(r)$. In the approximation $\omega r \gg 1$ we have that

$$\Box\phi_1 \simeq -\frac{2\omega\sqrt{\hbar}}{r}\,\eta'\,(fr)' \ , \tag{4.6}$$

_____

[3]Alternatively one could observe that as long as ϕ_1 is a small perturbation relative to ϕ_0, the terms on the l.h.s. coming from the interactions must be small relative to the source term on the r.h.s. .

where a prime denotes derivative of a function with respect to its argument. Then the equation for ϕ_1 in the leading order of our approximation is

$$-\frac{2\omega\sqrt{\hbar}}{r}\,\eta'\,(fr)' = -\frac{2^{m-1}(m-1)\alpha^{4(m-1)}\,\omega^m(\sqrt{\hbar})^{2m-1}}{r^{3m-1}}\,\psi^{m-1}\,\psi'^{m-2}\left[\psi\psi'' + 4\psi'^2\right]\ .\ (4.7)$$

We easily see, that two functions f and η of independent variables r and z in above equation separate thmeselves into two independent ordinary differential equations. The full solution of this equation can be expressed as

$$\phi_1 = -\frac{2^{m-1}\,\alpha^{4(m-1)}\,E^{m-1}\sqrt{\hbar}}{6r^{3m-2}}\,\eta(\omega(t+r)),\qquad(4.8)$$

where $E = \hbar\omega$ and $\eta(z) = \int^z \psi^{m-1}\,\psi'^{m-2}\left[\psi\psi'' + 4\psi'^2\right]dz'$. Note that for any m the integrand is an odd and periodic function with period 2π and such that the integral over one period is zero. Therefore the function η is again dimensionless and periodic with period 2π, which means that the scattered wave ϕ_1 has the same frequency as the incoming one. In the solution for $\eta(z)$ function we neglect the constant of integration. In the solution of radial equation for the function $f(r)$ we do the same, because such constant only renormalizes the amplitude of the initial wave. In solving (4.7) we have restricted ourselves to our ansatz (4.5) and *a posteriori* we confirm its validity. Invoking uniqueness theorems, well motivated by physical situation we are in, the form of the scattered component is to leading order of our approximations given by (4.8). This is the deformation of the incoming wave profile, that we were looking for. Now we will analyze it further.

Since $\eta \sim \psi \sim 1$, the ratio of the amplitudes of the first perturbation to the initial wave can be expressed as

$$|f(r)r| \simeq \frac{\alpha^{4(m-1)}\,2^{m-1}\,E^{m-1}}{r^{3(m-1)}} = \left(\frac{r_*}{r}\right)^{3(m-1)}\ ,\qquad(4.9)$$

where in the last step we defined the "classicalization radius" $r_* = \sqrt[3]{2\alpha^4 E}$. Notice that it does not depend on m. To obtain the ratio of amplitudes in our case, it is enough to consider only radial dependence of initial and perturbed wave. Another useful quantity

could be the ratio of energies stored in respective waves averaged in time to leading order in our approximations. It happens, that the squares of the oscillating parts averaged over one period are of the same order. This implies, that the ratio of energies is basically the square of the ratio (4.9). We can now see, that the interaction terms on the l.h.s. of (4.4) are indeed negligible. For example the second term in the first bracket is of order $(E \, \alpha^4/r^3)^{m-1}$, and when $r \gg r_*$ we have $(E \, \alpha^4/r^3) \ll 1$. Similar considerations apply to the other terms.

We thus find that the scattering process becomes important at distances of order r_*, where the ratio (4.9) is of order one. This behaviour must be put in contrast with the one in $\lambda\phi^4$ theory, where the characteristic radius is given roughly by λ/ω [72]. Normally one would expect, with quantum intuition, that a scattering process involving particles with energy E probes distances of order $\omega^{-1} = \hbar/E$. When collapsing wave approaches radial coordinate $r = r_*$, then the nonlinear modifications of the wave profile are so strong, that the information about structures resolved by Compton wavelength is completely unavailable. We can only read out the structures at the characteristic scale r_*. The radius r_* plays the role of the resolution scale and determines the characteristic momentum of the process as well as its cross section. This behaviour has been called "classicalization" in [71, 72, 73].

The meaning of the classicalization radius can be understood also as follows. First let us define a characteristic energy scale $E_* = \sqrt[4]{2} \, \hbar^{3/4} \, \alpha^{-1}$, for which classicalization radius equals to the Compton wavelenght $\hbar/E$. At low energy (i.e. $E \ll E_*$) the theory can be treated as an effective field theory. Due to the uncertainty relations, an incoming wave with energy E can only probe distances of order $\hbar/E$. When one gets close to the characteristic energy scale E_* one would normally expect the effective field theory to break down. What one sees here is that the scattered wave becomes significant at radius of order r_*, and therefore cannot resolve smaller distances. Since r_* grows with energy, there is a turnover energy where this bound becomes stronger than the one set by the uncertainty principle. At $E > E_*$ the resolving power decreases with energy. In this regime the scattering is dominated by the production of classical states with high occupation number, which will typically decay into many low energy particles [78]. The hard scattering of few particles into

few particles will be exponentially suppressed and unitarity will be restored [71, 72, 73, 74]. In this way classicalization may provide a form of UV completion of an effective field theory, that does not necessitate the introduction of new weakly coupled degrees of freedom.

The non-spherically symmetric case has been discussed in [76]. For mild deformations, it was found that the classicalization radius becomes smaller (larger) in regions where the curvature of the incoming wave is smaller (larger). Since the preceding arguments were order-of-magnitude estimates anyway, this does not change the conclusions. In the limiting case, when the incoming wavefronts are flat, the classicalization radius goes to zero and hence no classicalization occurs.

Let us now allow for the simultaneous presence of the interaction terms with different values of m. Motivated by effective field theory, we assume that all interactions are of the form

$$\mathcal{L}_{\text{int}} = \sum_m c_m \, \alpha^{4(m-1)} \left((\partial \phi)^2 \right)^m \tag{4.10}$$

To each interaction there corresponds a classicalization radius given by $r_*^3 = 2E \, \alpha^4 \, {}^{m-1}\!\!\sqrt{2m \, c_m}$. Which one of these scales plays the dominant role depends on the dimensionless coefficients c_m. If $c_m \sim 1/m$, as we assumed earlier, they are all of the same magnitude and therefore in principle all terms in the Lagrangian are equally important. On the other hand if $4c_2 > \sqrt{6c_3} > \sqrt[3]{8c_4} > \ldots$, then the corresponding r_* decreases with m, and the four-derivative term is the most important one. For large m one could assume that the coefficients c_m do not grow faster than exponentials of m ($c_m < a^m/2m$ for some $a > 1$). (This condition is quite reasonable for effective field theories.) Under these conditions the system will classicalize, when its size reaches the largest of all these possible classicalization radii and the higher derivative interactions will not play any significant role.

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4.3   Nonlinear sigma model with 2 derivatives

Now we start the analysis of nonlinear sigma models, which is the main task of this chapter. When there is more than one Goldstone boson, the internal space of them can be curved and moreover the theory admits interaction terms with just two derivatives. A standard way of describing the dynamics is to package the kinetic and the two-derivative interaction terms in the geometrical form

$$\mathcal{L} = \frac{1}{2} h_{ab}\, \partial_\mu \phi^a \partial^\mu \phi^b, \tag{4.11}$$

where h_{ab} is a metric in the target space. In full generality this metric is a function of coordinates on the internal space, here this role is played by field components ϕ^a. The coefficients of the Taylor expansion of the metric around a constant ϕ can be viewed as an infinite set of coupling constants. From this expansion we recover 2-derivative nonlinear interaction terms. We will consider real, maximally symmetric target spaces, for which all couplings are related and only the overall scale of the metric remains as a free parameter of the theory. In such case there exist coordinates such that

$$h_{ab} = \delta_{ab} \pm \frac{\phi_a \phi_b}{f_\phi^2 \mp \vec{\phi}^2} \ , \tag{4.12}$$

where the $+$ and $-$ signs correspond to positive and negative curvature of the target space (sphere and hyperboloid) respectively. In the above formula f_ϕ, which has the same dimensions as the field, has the meaning of radius of the sphere or hyperboloid in field space and $\vec{\phi}^2 = \vec{\phi} \cdot \vec{\phi} = \delta_{ab} \phi^a \phi^b$ is the usual flat Euclidean product. Moreover we used fields ϕ_a with covariant position of indices obtained by lowering them using the Kronecker delta symbol. Later we will work only with this definition and we will never use the true metric in the target space h_{ab} to lower indices on fields. In following derivation Lorentz indices will be suppressed, when this doesn't lead to confusion. We will use vector notation for denoting the components in the field space and the centerdot for a scalar product in this space. Exploiting the explicit form of the metric, the lagrangian (4.11) can be put in

the form

$$\mathcal{L} = \frac{1}{2}\left[(\partial\overrightarrow{\phi})^2 \pm \frac{(\overrightarrow{\phi}\cdot\partial\overrightarrow{\phi})^2}{f_\phi^2 \mp \overrightarrow{\phi}^2}\right] \tag{4.13}$$

The corresponding equations of motion are

$$\Box\phi^a \pm \frac{\phi^a\,\partial\left(\overrightarrow{\phi}\cdot\partial\overrightarrow{\phi}\right)}{f_\phi^2 \mp \overrightarrow{\phi}^2} \pm \frac{\phi^a\left(\overrightarrow{\phi}\cdot\partial\overrightarrow{\phi}\right)^2}{\left(f_\phi^2 \mp \overrightarrow{\phi}^2\right)^2} = 0 \tag{4.14}$$

We obtained them in a contravariant form as viewed from the flat internal space perspective. Therefore we treat the nonlinear structure in the kinetic term as the interaction, not as a geometry in the target space. Due to this paradigm our equations of motion in (4.14) are in a non-covariant form in a curved target space.

As in the preceding section, we are going to look for perturbative solution in the form $\overrightarrow{\phi} = \overrightarrow{\phi}_0 + \overrightarrow{\phi}_1 + \ldots$, where $\overrightarrow{\phi}_0$ is a solution of the free wave equation: $\Box\overrightarrow{\phi}_0 = 0$. We will study to which extent in spacetime evolution we can treat $\overrightarrow{\phi}_1$ as a small perturbation solving approximately the nonlinear field equations with interactions. We will follow closely the analysis of the preceding section in a very much the same set-up with spherical incoming and scattered waves. In order to this, it is tempting to try and reduce the problem to a single-field problem by assuming that only one component of the field is nonzero. The equations of motion seem to retain much of their nonlinearity even in this case. This, however, is an illusion that can be easily undone by a field redefinition. For example, with a single-field ansatz ($\phi^1 = \phi$ and $\phi^{2,3,\cdots} = 0$) the Lagrangian (4.13) becomes

$$\frac{1}{2}\left(\partial\phi\right)^2 \frac{f_\phi^2}{f_\phi^2 \mp \phi^2} \tag{4.15}$$

and this can be recast as a free field Lagrangian for φ by the redefinition $\phi = f_\phi\sin\varphi$ (for the upper sign) or $\phi = f_\phi\sinh\varphi$ (for the lower sign). This means that, if we make a single-field ansatz we will not be able to detect effects due to curvature, which is one of our purposes. One-dimensional field space is diffeomorphic to a straight line and as such is not characterized by any curvature. We must consider multi-field ansatz, possibly with

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isotropy in a target space. This is an additional difficulty, we must overcome, when working with fields taking their values in the nontrivial internal space (nonlinear sigma model).

Without much loss of generality we will work with a general spherically symmetric unperturbed incoming wave $\phi_0^a(r,t) = \sqrt{\hbar}\,\psi^a(\omega(t+r))/r$, where we assume, that all components have the same frequency ω (monochromatic waves), and we assume $\omega r \gg 1$, as before. The first order perturbation will be written using the following form of the ansatz: $\phi_1^a(r,t) = \sqrt{\hbar}\,\eta^a(\omega(t+r))\,f(r)$. Later we will see, that it is consistent to assume that all components of ϕ_1^a have the same radial dependence. However we allow for different oscillating functions $\eta^a(z)$ for different components in field space.

Linearizing the field equation around $\overrightarrow{\phi}_0$ we find

$$
\delta^{ab}\,h_{bc}\,\Box\phi_1^c \pm \frac{2\phi_0^a}{f_\phi^2 - \overrightarrow{\phi}_0^2}\,h_{bc}\,\partial\phi_0^b\,\partial\phi_1^c \pm \frac{\phi_1^a}{f_\phi^2 - \overrightarrow{\phi}_0^2}\,h_{bc}\,\partial\phi_0^b\,\partial\phi_0^c
$$

$$
+\frac{2\phi_0^a}{\left(f_\phi^2 - \overrightarrow{\phi}_0^2\right)^2}\left[\left(\partial\overrightarrow{\phi}_0\right)^2\phi_0^b + \left(\overrightarrow{\phi}_0\cdot\partial^\mu\overrightarrow{\phi}_0\right)\partial_\mu\phi_0^b \pm 2\frac{\left(\overrightarrow{\phi}_0\cdot\partial\overrightarrow{\phi}_0\right)^2}{f_\phi^2 \mp \overrightarrow{\phi}_0^2}\,\phi_0^b\right]\phi_{1\,b}
$$

$$
= \mp\frac{\phi_0^a}{f_\phi^2 - \overrightarrow{\phi}_0^2}\,h_{bc}\,\partial\phi_0^b\,\partial\phi_0^c\ . \tag{4.16}
$$

Here the metric h_{ab} has to be regarded as a function of $\overrightarrow{\phi}_0$. We presented last equation in a mixed form, where we used covariant metric in the target space as well as noncovariant Euclidean products of fields. The reason for this is that such form of the linearized equations of motion emerges from fully covariant formalism in target space, when only the derivatives of the target metric are expressed in terms of fields. As we will see later it is useful to keep the metric field unexpanded. As in the preceding section, higher interaction terms on the l.h.s. can be neglected. We are left with the following form of the simplified equation for the first perturbation:

$$
\delta^{ab}\,h_{bc}\,\Box\phi_1^c = \mp\frac{\phi_0^a}{f_\phi^2 - \overrightarrow{\phi}_0^2}\,h_{bc}\,\partial\phi_0^b\,\partial\phi_0^c\ . \tag{4.17}
$$

To leading order in $1/r\omega$ we find equation in the target space covariant form:

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$$ -\frac{2\omega\sqrt{\hbar}}{r}(fr)'\eta^{a\,\prime} = \mp\frac{2\omega\hbar^{3/2}}{f_\phi^2\,r^4}\frac{\psi^a\left(\overrightarrow{\psi}\cdot\overrightarrow{\psi}'\right)\left(h_{bc}\,\psi^b\psi^c\right)^2}{\left(\overrightarrow{\psi}^2\right)^2}\;, \tag{4.18} $$

which is equivalent to the following equation, when we explicitly expand the target space metric

$$ -\frac{2\omega\sqrt{\hbar}}{r}(fr)'\eta^{a\,\prime} = \mp\frac{2\omega\hbar^{3/2}}{f_\phi^2\,r^4}\frac{\psi^a\left(\overrightarrow{\psi}\cdot\overrightarrow{\psi}'\right)}{\left(1\mp\frac{\hbar\overrightarrow{\psi}^2}{f_\phi^2\,r^2}\right)^2}\;. \tag{4.19} $$

We note right away that in contrast to equation (4.7) the ω-dependence will cancel out. Instead, the behaviour of the solution is governed by the new dimensionless parameter $f_\phi r/\sqrt{\hbar}$. As long as $f_\phi r/\sqrt{\hbar} \gg 1$, the denominator in the r.h.s. can be approximated by one and the equation can be solved by separation of variables. Now we can notice that in this case, after separation the radial equation for f is the same for all components of ϕ_1^a, therefore the choice $f^a(r) = f(r)$ is justified. The solution can be written in the form

$$ \phi_1^a = \mp\sqrt{\hbar}\frac{\hbar}{2f_\phi^2\,r^3}\eta^a(\omega(t+r))\;, \tag{4.20} $$

where $\eta^a(z) = \int^z \psi^a\overrightarrow{\psi}\cdot\overrightarrow{\psi}'dz'$. This first perturbation is again an oscillating function with r-dependent amplitude, but in contrast to the case of the preceding section (4.8), the amplitude of the oscillations of the scattered wave is independent of ω. The ratio between the amplitude of the first perturbation and the incoming wave is

$$ |f(r)r| = \frac{\hbar}{2f_\phi^2\,r^2} = \left(\frac{r_*}{r}\right)^2\;. \tag{4.21} $$

From the above expression we see that we can define a "classicalization radius" by

$$ r_* = \frac{\sqrt{\hbar}}{\sqrt{2}f_\phi} \tag{4.22} $$

independent of the frequency or energy of the incoming wave packet. Again, incoming waves with arbitrarily high frequency are unable to probe distances shorter than r_*, but in

contrast to the preceding case r_* does not increase with frequency. We thus have a weaker form of classicalization (compare [74]).

Let us now consider the effect of curvature, which (aside from the immaterial overall sign) is contained in the denominator of the r.h.s. of (4.19). We observe that since $0 \le \overrightarrow{\psi}^2 \le C$, for some constant C of order one, the effect of the denominator is to enhance the amplitude of the scattered wave for positive curvature (upper sign) and to decrease it for negative curvature (lower sign). In fact, with the positive curvature the amplitude reaches a pole for some $r \approx \sqrt{\hbar}/f_\phi$, strengthening the case for classicalization of the preceding analysis. In the case of negative curvature, rhe r.h.s. of (4.19) increases for decreasing radius, but tends to a constant for $r \to 0$. The argument for classicalization is considerably weaker in this case.

This can also be seen in another way. The approximation leading to solution in a form (4.20) corresponds to considering the theory with standard kinetic term and with interaction Lagrangian

$$\mathcal{L}_{\text{int}} = \pm \frac{(\overrightarrow{\phi} \cdot \partial \overrightarrow{\phi})^2}{2 f_\phi^2} . \tag{4.23}$$

Let us consider, what happens if we take as an interaction the next term in the expansion of the denominator of Lagrangian in equation (4.13)

$$\mathcal{L}_{\text{int}} = - \frac{\overrightarrow{\phi}^2 (\overrightarrow{\phi} \cdot \partial \overrightarrow{\phi})^2}{2 f_\phi^4} . \tag{4.24}$$

From here one finds instead of (4.19) the following approximate form of the equation of motion for the first perturbation:

$$-\frac{2\omega\sqrt{\hbar}}{r}(fr)'\eta^{a\,\prime} = -\frac{2\omega\,\hbar^{5/2}}{f_\phi^4\,r^6}\psi^a\,\overrightarrow{\psi}^2\,\overrightarrow{\psi}\cdot\overrightarrow{\psi}' , \tag{4.25}$$

whose solution has a radial dependence such that

$$|f(r)r| = \frac{\hbar^2}{2 f_\phi^4 r^4} . \tag{4.26}$$

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This corresponds again to a classicalization radius of order $\sqrt{\hbar}/f_\phi$. It is easy to see that this is true for all the terms in the expansion, but when one takes them all into account simultaneously, they appear all with negative sign, when the curvature is positive, but with alternating signs, when the curvature is negative. Therefore in the case of positive curvature of the target space, we have enhanced behaviour of the scattered wave amplitude signalizing the occurence of the classicalization. For sigma models with negatively curved internal space, these higher interaction terms are of the same order, but with alternating signs. In the effect there are no evidences for strong deformation of initial wave profile and classicalization does not occur. This dependence on the overall sign of interaction term in (4.23) is in agreement with general conclusions derived in [72, 73]. Nonlinear sigma models with these two different signs are very different also on the level of classical field theory solutions.

For nonlinear sigma model with two derivatives we showed, that only in a case of positive internal space curvature, classicalization happens. We must emphasize however that it happens not in a strong form (where classicalization radius depends and grows with the energy of the packet). In the case of two derivatives action, energy dependence is removed and classicalization radius is a fixed length scale. In this aspect situation can be similar to a linear sigma model with standard kinetic term written for a dimensionless fields. Then constant $\sqrt{\hbar}\, f_\phi^{-1}$ plays very similar role like r_* and is a fixed length scale. But we know that in this flat case theory is free, without interactions, and that there is no scattering. When target space is with positive curvature, then this length scale sets also the characteristic length of the classicalization phenomenon.

In the case of an incoming plane wave, the ratio of the first perturbation to the initial amplitude is independent both of ω and r. This gives no clue about classicalization. These considerations in the planar wave case confirms previous statement, that if classicalization holds for NSM with 2 derivatives, it is in the weak form.

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4.4 Nonlinear sigma model with 2 and 4 derivatives

Now we want to add higher derivative terms to action of nonlinear sigma model and check their impact on the analysis of scattering processes. In a maximally symmetric nonlinear sigma model with a two-derivative Lagrangian (4.13), a general four derivative interaction has the form

$$\mathcal{L}_{\text{int}}^{(4)} = g_4(\ell_1\, h_{ab}h_{cd} + \ell_2\, h_{ac}h_{bd})\partial_\mu\phi^a\,\partial^\mu\phi^b\,\partial_\nu\phi^c\,\partial^\nu\phi^d \ , \tag{4.27}$$

where ℓ_1 and ℓ_2 are dimensionless constants. In effective field theory framework we expect them to be of order one. Expanding the metrics h_{ab} in Taylor series would yield infinitely many monomial operators with coefficients $g_{4,n}$. For the sake of comparison to section §4.2 we could write $g_4 = \alpha^4$. In effective field theory one expects the coefficients of operators with different number of derivatives to be all proportional to powers of the same mass scale f_ϕ in natural units. Then we would write alternatively $g_4 = \hbar/f_\phi^4$. We will follow this notation here, but one can revert to α at any moment.

When this interaction is added to the two-derivative Lagrangian (4.13), applying the same ansatz for the fields as in the preceding section, neglecting $\overrightarrow{\phi}_1$ on the l.h.s. and expanding in inverse powers of ωr we get to the leading order the following linear equation for the first perturbation:

$$\Box\phi_1^a = \mp\frac{2\omega\hbar^{3/2}}{f_\phi^2\, r^4}\frac{\psi^a\, \overrightarrow{\psi}\cdot\overrightarrow{\psi}'}{\left(1\mp\frac{\hbar\overrightarrow{\psi}^2}{f_\phi^2 r^2}\right)^2} \tag{4.28}$$

$$-\frac{2\omega^2\hbar^{5/2}}{f_\phi^4\, r^5}\left[(\ell_1 + 3\ell_2)\psi^a\,\overrightarrow{\psi}'^2 + (3\ell_1 + 5\ell_2)\psi^{a\,\prime}\overrightarrow{\psi}'\cdot\overrightarrow{\psi} + (\ell_1 + \ell_2)\psi^a\overrightarrow{\psi}''\cdot\overrightarrow{\psi} + \ell_2\psi^{a\,\prime\prime}\overrightarrow{\psi}^2\right]$$

Note that in the four-derivative terms the ϕ-dependent part of the metric gives subleading contributions, so h_{ab} was already replaced by δ_{ab} in (4.28).

This equation can only be solved by separation of variables, if one of the two terms on the r.h.s of (4.28) can be neglected. However, we can get a reasonably good estimate of

the terms involved by simply setting equal to one all the fastly oscillating factors η in the l.h.s. and the terms involving ψ on the r.h.s.. The resulting equation for $f(r)r$ can then be easily integrated to yield

$$|f(r)r| = \mp \frac{\hbar}{2f_\phi^2 r^2} - \frac{E\hbar}{3f_\phi^4 r^3} - \frac{\hbar^2}{4f_\phi^4 r^4} + \ldots = \mp \left(\frac{r_{2*}}{r}\right)^2 - \left(\frac{r_{4*}}{r}\right)^3 - \left(\frac{r_{2*}}{r}\right)^4 + \ldots \quad (4.29)$$

where the first and third term come from the expansion of the two-derivative term and the second comes from the four-derivative term. Dots at the end of the formula correspond to higher powers of dimensionless ratios, which give subleading contributions. We have defined two classicalization radii by

$$r_{2*} = \sqrt{\frac{\hbar}{2f_\phi^2}} \qquad \text{and} \qquad r_{4*} = \sqrt[3]{\frac{E\hbar}{3f_\phi^4}} \; . \qquad (4.30)$$

All the terms in the expansion of the two-derivative term correspond to the same classicalization radius r_{2*}. These terms are dominant for $E < \sqrt{\hbar} f_\phi$. For higher energy the four-derivative terms dominate and the system behaves like some number of copies of the single Goldstone boson model of section §4.2, in the special case with $2m = 4$ derivatives. Note that if we use the notation $\alpha^4 = \hbar/f_\phi^4$, we find that $r_{4*} = \sqrt[3]{E\alpha^4/3}$, which is the same formula that we found in section §4.2. Strong classicalization occurs for $\omega > r_{4*}^{-1}$ regardless of the sign of the curvature. This means, that adding four-derivative interaction terms to the action of nonlinear sigma model makes it resembling very much at high energy the ordinary sigma model with four-derivative interactions. The feature of nonlinearity of kinetic term is not important in the context of high energy, when the highest order derivative interaction terms dominate. Therefore our system exhibits exactly the same properties as the system of single Goldston boson with higher derivatives studied in §4.2.

In the case of a plane incoming wave we also have to distinguish two regimes. When the two-derivative terms in (4.28) dominate, no clues of classicalization can be found, as in section §4.3. When the four-derivative term dominates classicalization does not occur, in agreement with the discussion in section §4.2 and with [76].

4.5 Classicalization vs. asymptotic safety

In the preceding sections we have analyzed a hypothetical scattering process in nonlinear sigma models with any number of derivatives and with positive, negative or zero target space curvature. We have found that quite generally, an incoming spherical wave satisfying the free wave equation will generate a strong scattered wave, when it reaches a size r_*, that depends in general on the couplings of the theory and on the initial energy. Contrary to naive expectation, this radius r_* either increases with energy or is independent of it. As discussed in [72], this is in sharp contrast to other field theories, such as a scalar with a potential interaction, where the scattered wave only becomes important at a radius of order $\hbar/E$. Following [71, 72, 73, 74], we call this phenomenon "classicalization", and for our purposes we distinguish a "weak classicalization", when r_* is independent of E, from "strong classicalization" when r_* grows with E. In both cases scattering processes cannot actually probe distances shorter than r_*. The scattering process is softened and there is a chance that, though perturbatively nonrenormalizable, the theory may actually be well behaved at high energy.

As already mentioned in the introduction, this sounds sufficiently similar to the program of asymptotic safety, that one may legitimately ask whether there is a relation between the two phenomena. To further motivate this expectation, let us recall that in order to avoid the complications due to redundant (or "inessential") couplings, in the discussion of asymptotic safety, it would be desirable to define the couplings directly in terms of physical observables [14]. Due to the difficulty of nonperturbatively computing observables in these theories, so far efforts have concentrated on the running of couplings defined as coefficients of operators in an effective Lagrangian. However, if there was a way of showing, for example, that certain amplitudes have the right behaviour as functions of energy, then one could show, that the couplings defined in terms of the corresponding exclusive cross sections would reach a fixed point. This would give truly operational definition of couplings, measured from experiment, not derived from some theoretical considerations. In this way classicalization could turn out to be a valuable alternative tool for studying some issues

about asymptotic safety.

Since asymptotic safety, if realized in nature, is clearly a quantum phenomenon, the first priority is to understand, whether there is a way of viewing also classicalization as a quantum phenomenon, in spite of its name. We believe, that the distinction between classical and quantum phenomena is not as clear cut as it seems. The real world is quantum in nature and classical behaviour can only emerge in certain limits, but there are ambiguities in the way these limits are taken. We refer to [84] for a recent discussion of this issue in the context of QED. In order to introduce the issue in the context of the nonlinear sigma model, let us go back to the parametrization where the fields φ^a are dimensionless (which is natural in view of the fact that they appear as arguments in nonpolynomial metric functions h_{ab}, which has a geometrical meaning). The action can be expanded schematically as in (4.1), where $\mathcal{O}_{k,n} \sim \int \partial^k \varphi^n$ contains k derivatives and n powers of the field φ. The dimensions of the couplings $\bar{g}_{k,n}$ are $M\,L^{k-3}$, independent of n. For the sake of perturbation theory, one has to separate the kinetic term from the interactions. Defining a canonically normalized field $\phi^a = \varphi^a \sqrt{g_{2,2}}$, of dimension $\sqrt{M/L}$, the action becomes

$$S = \int \left[(\partial \phi)^2 + \sum_k \sum_{n>2} g_{k,n} \partial^k \phi^n \right] \tag{4.31}$$

where $g_{k,n} = \bar{g}_{k,n} (\sqrt{g_{2,2}})^n$ have dimension $M^{1-n/2}\,L^{k-3+n/2}$. There is a theorem to the effect that higher derivative corrections to the propagator can be eliminated by field redefinitions, order by order in perturbation theory [85], so we may assume, without loss of generality, that $g_{k,2} = 0$ for $k > 2$. Assuming that a Z_2 symmetry forbids the appearance of odd powers of the field, the lowest interaction would be of the form $g_{2,4}\phi^2(\partial\phi)^2$. Let us define $g_{2,4} = f_\phi^{-2}$, where f_ϕ has the same dimensions as the field (it can be viewed as a kind of VEV). Global symmetry then implies that $g_{2,n} \sim f_\phi^{2-n}$ (see for example (4.13)). In effective field theory it seems reasonable to assume that all dimensionful couplings are proportional to powers of f_ϕ. (This is particularly clear in natural units, where f_ϕ can be viewed as a natural mass scale, and all couplings are proportional to powers of this mass.) Then we would write $g_{k,n} = c_{k,n} f_\phi^{4-k-n} \hbar^{k/2-1}$, where $c_{k,n}$ are dimensionless.

One can define different notions of classical limit, depending on which couplings are being kept fixed. If one takes $\hbar \to 0$ keeping $g_{k,n}$ fixed, one obtains a classical field theory with all the higher derivative terms; if one takes $\hbar \to 0$ keeping f_ϕ and the $c_{k,n}$ fixed one gets a classical field theory with the two-derivative terms only. How one defines the classical limit obviously affects the interpretation of classicalization. In the former limit the classicalization radius, when $k > 2$, is $(g_{k,n})^{\frac{2}{n+2k-6}} E^{\frac{n-2}{n+2k-6}}$ independent of $\hbar$ and is therefore a truly classical notion [71]. In the latter limit, reexpressing $g_{k,n}$ in terms of f_ϕ and $\hbar$, the classicalization radius goes to zero and should therefore be regarded as a quantum effect. The classicalization radius found in section §4.3, for the case $k = 2$, is truly of quantum nature regardless which limit is taken. However as pointed out in [80, 81] the emergence of the classicalization radius has to be understood as the macroscopic effect of a quantum nonlinear dynamics of microscopic constituents of the system under question. Hence according to authors of [80, 81] it has quantum origin.

Another potential source of ambiguity in the definition of the classical limit is the question whether E or ω is to be held fixed [84]. In the latter case again the classicalization radius $\sqrt[3]{\alpha^4 \hbar \omega}$ vanishes in the classical limit. Since in this paper we are mainly interested in scattering experiments, where the momenta of the external particles are known and fixed, it seems more appropriate to stick to the case when E is kept fixed in the classical limit. Furthermore, writing the couplings in terms of powers of a single coupling f_ϕ is motivated by perturbative arguments. Since both asymptotic safety and classicalization are nonperturbative notions, it is perhaps more appropriate to stick to the generic parameterization (4.31) and to consider all couplings $g_{k,n}$ as truly independent. This is the notion of classical limit which is implicitly assumed in [71, 72, 73, 74].

We now restrict ourselves to this particular notion of classical limit, and we try to extract some conclusions from the results of the preceding sections. From the given expressions for r_* we see that the weak classicalization, that was found in the two-derivative models of section §4.3 is a quantum phenomenon, whereas the strong classicalization of the higher derivative models of sections §4.2 and §4.4 are genuinely classical effects. There is therefore a chance that weak classicalization has something to do with asymptotic safety,

whereas strong classicalization seems to be a genuinely different effect. There are then some other suggestive facts. It was found in [67] that in the two-derivative truncation of the nonlinear sigma model a non-trivial fixed point exists for positive curvature, but not for negative curvature. This seems to agree with the result in section §4.3, according to which the argument for (weak) classicalization is much more robust in the positive curvature case than in the negative curvature case. On the other hand, no non-trivial fixed point seems to exist in the S^1-valued nonlinear sigma model, which corresponds to the single Goldstone boson model of section §4.2 [68]. And furthermore, we have found in section §4.4 that strong classicalization is completely insensitive to the sign of the curvature. Finally, returning to natural units, the amplitude for scattering of two particles into two particles in the two-derivative model with positive curvature behaves like p^2/f_ϕ^2, where p is the momentum transfer. Since the latter is asymptotically of order $r_*^{-1} \sim f_\phi$, the amplitude tends to a constant, as one would expect in an asymptotically safe theory.

In the case of gravity, it has been argued that classicalization is intimately related to the notion of a minimal length [70]. This seems to be in contrast to the notion of a field theoretic UV completion, where one talks of "arbitrarily high energy scales". In fact it had already been noted that in a certain sense a notion of minimal length is present in an asymptotically safe theory of gravity [86]. We refer to [87] for further discussion of this point.

We may comment here on the importance of weak and strong version of classicalization. In [81] authors noticed, that the self-completion of a nonrenormalizable theory by classicalization manifests itself as the increase of number of degrees of freedom with energy in a classical configuration. It is this increase that replaces the notion of the usual Wilsonian renormalization standard viewed as integration in some new weakly interacting degrees of freedom. Moreover only in the case of strong classicalization $r_* = r_*(E)$, we get a growing with energy number of degrees of freedom present in the configuration and the portrait of the classicalon as a soliton can be correct. The conclusion is again, that weak and strong classicalization are very different and that way of UV completion by strong classicalization doesn't have features similar with those present in asymptotically safe scenario.

All these facts reinforce the hypothesis, that weak classicalization may be a direct manifestation of asymptotic safety in the scattering amplitudes whereas strong classicalization, if true, would be a different kind of effect. We also observe that, if we assume equivalence between weak classicalization and asymptotic safety, the absence of classicalization in the case of plane waves suggests that momentum transfer is more important than total energy in these matters. In order to substantiate the preceding conclusions one would need to directly calculate some amplitudes in an asymptotically safe theory.

Chapter 5

1-loop effective action in system of gravitating scalar

5.1 Truncation ansatz and 'inverse propagator'

In this chapter we will compute 1-loop effective action in a system, where we have standard Einstein-Hilbert gravitation and minimally coupled scalar field. Standard computation, known in the literature, are mainly based on perturbative quantization methods and they exploit Feynman diagrams techniques. Here we will follow a different route. Namely we will obtain 1-loop quantum effective action as the effect of integrating average effective action along the flow trajectory from UV down to IR limit. Moreover in the core of our calculation we will use non-local heat kernel techniques to evaluate some functional traces. We will pay special attention to the appearance of nonlocal terms in the quantum effective action.

Now we want to introduce the notion of the average effective action (EAA). The EAA is a scale-dependent generalisation of the standard effective action that interpolates smoothly between the bare action for $k \to \infty$ and the standard quantum effective action for $k \to 0$. In this way, we avoid the problems of performing the functional integral. Instead they are converted into the problem of integrating the exact flow of the EAA from the UV to

the IR. The EAA formalism deals naturally with several different aspects of quantum field theories. One aspect is related to the discovery of non-Gaussian fixed points of the RG flow. In particular, the EAA framework is a useful setting to search for Asymptotically Safe theories, i.e. theories valid up to arbitrarily high energy scales. A second aspect, in which the EAA reveals its big usefulness, is the domain of nonperturbative calculations. In fact, the exact flow that EAA satisfies is a valuable starting point for inventing new approximation schemes.

In EAA the crucial point is the separation between high and small energy modes of quantum fields. The elimination of higher energy modes is performed by separating the low energy modes, to be integrated out, from the high modes in a covariant way. To do this we introduce a cutoff action constructed using the covariant d'Alambertian, that respects the symmetries of the underlying theory. In full generality in order to construct EAA we add to the bare action S an infrared (IR) "cutoff" or regulator term ΔS_k of the form:

$$\Delta S_k = \frac{1}{2} \int d^d x \sqrt{g} \phi R_k(\Box) \phi \,. \tag{5.1}$$

In above formula the operator kernel R_k is chosen in such a way to suppress the field modes ϕ_n, eigenfunctions of the covariant second differential operator $\Box$, with eigenvalues smaller than the cutoff scale $\nu_n < k^2$. Generic fields of our quantum field theory are denoted here by ϕ. We will call ΔS_k the cutoff action. The functional form of the cutoff kernels $R_k(z)$ is arbitrary except for the requirements that they should be monotonically decreasing functions in both z and k arguments, i.e. rigorously that $R_k(z) \to 0$ for $z \gg k^2$ and that $R_k(z) \to k^2$ for $z \ll k^2$. It is important to consider two limits of EAA. First in the IR limit ($k = 0$) quantum effective action is obtained. On the other hand, when $k \to \infty$, then EAA equals to the bare action of considered quantum theory. In this way we obtain the scale dependent generalisation of the standard effective action, which interpolates between the two.

Quantum gravity gives unambiguous predictions at low energy in the framework of effective field theories. The low energetic action contains only the simplest Einstein-Hilbert

term (with a possibility of adding a cosmological constant, which we however neglect here). In this effective theory there exist observables, which do not depend on the particular way of UV completion. They are genuine predictions of quantum gravity. The quantum divergences, which must be absorbed in the renormalization procedure, are contained in local, but not universal terms in the quantum effective action. We are mainly interested in nonlocal term in quantum effective action. The reason for this is that they are universal terms in low-energetic effective field theory of quantum gravity [95, 91]. They do not depend on any specific way of UV completion of gravity. There are different ways, by which, one can obtain quantum effective action in infrared limit. However it is without any doubt that low- energetic predictions of quantum gravity are calculable and solid, regardless of any complicated dynamics, which saves the theory in UV. In our method for integration the RG flow we will use exact (also known as functional) Renormalization Group equations. In integration of RG flow of average effective action such nonlocal terms originate from the part of integration done for the lowest momentum scales.

At the end of this chapter we will try to draw a comparison with a similar computation done in the perturbative framework [96]. Our computation we will finally perform entirely in four spacetime dimension, however in the first sections we will be more general, working with spacetime of any dimensionality.

We will use the following ansatz for the form of the action of our system

$$
\begin{aligned}
S \;=\; & \int d^d x \sqrt{g}\left[\frac{1}{K^2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right] \\
& -\frac{1}{2K^2\alpha}\int d^d x \sqrt{g}\chi^2 \\
& +\int d^d x \sqrt{g}\,\bar{C}_\mu\left(-\Box\delta^\mu_\nu - R^\mu_\nu\right)C^\nu\,.
\end{aligned}
\tag{5.2}
$$

where d'Alambertian is given by $\Box = \nabla_\mu\nabla^\mu$. Due to the gauge diffeomorphism symmetry present in the system we are forced to introduce gauge fixing conditions necessary for perturbative quantization of the system:

$$\chi_\mu = \nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h \,. \tag{5.3}$$

Moreover another consequence of this gauge redundancy in the system is that for consistency, we also had to add vector ghosts denoted by C_μ in the third line of (5.2). In our computation we use the background field method and we take the metric perturbations in the form $h_{\mu\nu} = \delta g_{\mu\nu}$ and in contracted version $h = g^{\mu\nu} h_{\mu\nu}$. All covariant derivatives are with respect to the background metric. As we can see in the action (5.2) we included minimally coupled scalar field ϕ and we allow for the existence of potential $V(\phi)$ for it. Gravitational coupling appears there as K, which has the inverse energy dimension. In the gravitational part of the action R is a curvature invariant built out of the full metric $g_{\mu\nu}$ and R^μ_ν corresponding Ricci tensor. Additionally constant α is a gauge parameter in our gauge fixing condition.

Now we are going to compute the bilinear part in fluctuations of action S, because this is the main ingredient if we target on finding the explicit form of the covariant d'Alambertian operator. The bilinear part in the metric field of the gravitational part S_g of the effective action is given by:

$$
\begin{aligned}
\frac{1}{2}\delta^2 S_g \;=\; & \int d^4x \sqrt{g}\, \frac{1}{K^2}\left\{ \frac{1}{4}h^{\alpha\beta}\Box h_{\alpha\beta} + \left(-\frac{1}{4}+\frac{1}{8\alpha}\right) h\Box h + \right. \\
& + \frac{1}{2}\left(-1+\frac{1}{\alpha}\right)\left(h_{\mu\nu}\nabla^\mu\nabla^\alpha h^\nu_\alpha - h_{\alpha\beta}\nabla^\alpha\nabla^\beta h\right) + \left(\frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right) R \\
& \left. - \frac{1}{2}h_{\mu\nu}R^{\mu\nu}h + \frac{1}{2}h_{\mu\rho}R^{\mu\nu}h^\rho_\nu + \frac{1}{2}h_{\mu\nu}R^{\mu\alpha\nu\beta}h_{\alpha\beta} \right\}
\end{aligned}
\tag{5.4}
$$

We are not interested in the ghost part here and we will not report corresponding results for this part of the action. However we present the ghost functional derivative, which is equal to

$$\frac{\delta^2 S}{\delta C_\mu(x)\delta\bar{C}^\nu(x')} = -\Box\delta^\mu_\nu - R^\mu_\nu \,. \tag{5.5}$$

Fixing $\alpha = 1$ the second functional derivative of the gravitational part of the effective action takes the following minimal form (summarised in pairs of indices (μ,ν) and (α,β)):

$$\frac{\delta^2 S_g}{\delta h_{\mu\nu}(x)\delta h_{\alpha\beta}(x')} = \frac{1}{K^2}\left\{C^{\mu\nu,\alpha\beta}\Box - R^{\alpha(\mu\nu)\beta} - \frac{1}{2}g^{\mu\nu}R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R^{\mu\nu} + \right.$$

$$\left. + \frac{1}{4}g^{\alpha\beta}g^{\mu\nu}R + g^{(\mu(\alpha}R^{\beta)\nu)} - \frac{1}{2}g^{\mu(\alpha}g^{\beta)\nu}R\right\}, \tag{5.6}$$

where $C^{\mu\nu,\alpha\beta} = -\frac{1}{2}\left(\frac{1}{2}g^{\mu\nu}g^{\alpha\beta} - g^{\mu(\alpha}g^{\beta)\nu}\right)$ is the contravariant DeWitt metric tensor.

In what follows, we will derive the operator of second variation needed for computation of Schwinger-DeWitt technique, simultaneously correcting the misprints, which appeared in [98]. This computation we will keep in general dimensionality, only later we will restrict ourselves to $d = 4$. This is the novel feature of this work. The variation of the scalar field away from the background field ϕ we denote by f. We note here the second variation of the matter action $S_m = -\int d^d x\sqrt{g}\left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right]$ with respect to all fluctuating fields $\psi^A = (h_{\mu\nu}, f)$ given by

$$\frac{\delta^2 S_m}{\delta f^2} = \Box - V'', \tag{5.7}$$

$$\frac{\delta^2 S_m}{\delta h_{\mu\nu}\delta f} = -\frac{1}{4}g^{\mu\nu}\left(\nabla^\alpha\phi\right)\nabla_\alpha + \frac{1}{2}\left(\nabla^{(\mu}\phi\right)\nabla^{\nu)} - \frac{1}{2}g^{\mu\nu}V', \tag{5.8}$$

$$\frac{\delta^2 S_m}{\delta f\delta h_{\mu\nu}} = +\frac{1}{4}g^{\mu\nu}\left(\nabla^\alpha\phi\right)\nabla_\alpha - \frac{1}{2}\left(\nabla^{(\mu}\phi\right)\nabla^{\nu)} + \frac{1}{4}g^{\mu\nu}\Box\phi - \frac{1}{2}\nabla^\mu\nabla^\nu\phi - \frac{1}{2}g^{\mu\nu}V' \tag{5.9}$$

$$\frac{\delta^2 S_m}{\delta h_{\mu\nu}\delta h_{\rho\sigma}} = VC^{\mu\nu,\rho\sigma} + \frac{1}{2}\left[C^{\mu\nu,\rho\sigma}\left(\nabla\phi\right)^2 + \frac{1}{2}g^{\mu\nu}\left(\nabla^\rho\phi\right)\left(\nabla^\sigma\phi\right) + \right.$$

$$\left. + \frac{1}{2}g^{\rho\sigma}\left(\nabla^\mu\phi\right)\left(\nabla^\nu\phi\right) - 2g^{(\mu(\rho}\left(\nabla^{\sigma)}\phi\right)\left(\nabla^{\nu)}\phi\right)\right]. \tag{5.10}$$

The linear matrix-differential operator $F_{AB}(\nabla)$ defined by the relation $\delta^2 S = \frac{1}{2}\int d^4 x\sqrt{g}\psi^A F_{AB}\psi^B$ is given by

$$F_{AB}(\nabla) = C_{AB}\Box + 2\Gamma^\sigma{}_{AB}\nabla_\sigma + W_{AB}, \tag{5.11}$$

where we order terms by number of covariant derivatives. Operator F_{AB} is obviously equal to the second variation $\frac{\delta^2 S}{\delta\psi^A\delta\psi^B}$. The indices A, B take only two value 1 (for graviton) or 2 (for scalar) and all matrices with such indices have tensorial character with respect to

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diffeomorphism transformations. Due to the different dimensionality of considered fluctuations the energy dimensions of entries of matrix $F_{AB}(\nabla)$ are different. The action of the operator $F_{AB}(\nabla)$ on fluctuations is schematically depicted by $(h_{\mu\nu}, f)\, F \begin{pmatrix} h_{\alpha\beta} \\ f \end{pmatrix}$. By explicit calculation, we get expressions for all tensors appearing in (5.11):

$$C_{AB} = \begin{pmatrix} \frac{1}{K^2} C^{\mu\nu,\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix} , \Gamma^{\sigma}{}_{AB} = \begin{pmatrix} 0 & C^{\mu\nu,\sigma\rho}\nabla_{\rho}\phi \\ -C^{\alpha\beta,\sigma\rho}\nabla_{\rho}\phi & 0 \end{pmatrix} \text{ and} \tag{5.12}$$

$$W_{AB} = \begin{pmatrix} C^{\mu\nu,\rho\sigma} H_{\rho\sigma}{}^{\alpha\beta} & -\frac{1}{2}g^{\mu\nu}V' \\ -2C^{\alpha\beta,\rho\sigma}\nabla_{\rho}\nabla_{\sigma}\phi - \frac{1}{2}g^{\alpha\beta}V' & -V'' \end{pmatrix}. \tag{5.13}$$

We want to emphasise that the matrix W is not symmetric in indices (A, B). The tensorial expression $H_{\rho\sigma}{}^{\alpha\beta}$ (which in [98] was called $P_{\rho\sigma}{}^{\alpha\beta}$) equals to

$$\frac{1}{K^2}\left[-2R_{(\rho}{}^{\alpha\beta}{}_{\sigma)} + 2\delta^{(\alpha}_{(\rho}R^{\beta)}_{\sigma)} - \delta^{\alpha\beta}_{\rho\sigma}R - \frac{2}{d-2}g_{\rho\sigma}R^{\alpha\beta} - g^{\alpha\beta}R_{\rho\sigma} + \frac{1}{d-2}g_{\rho\sigma}g^{\alpha\beta}R\right] +$$
$$+\frac{1}{2}\delta^{\alpha\beta}_{\rho\sigma}\left(\nabla\phi\right)^2 - 2\delta^{(\alpha}_{(\rho}\left(\nabla_{\sigma)}\phi\right)\left(\nabla^{\beta)}\phi\right) + V\delta^{\alpha\beta}_{\rho\sigma} + \frac{1}{d-2}g_{\rho\sigma}\left(\nabla^{\alpha}\phi\right)\left(\nabla^{\beta}\phi\right) + \tag{5.14}$$
$$+\frac{1}{2}g^{\alpha\beta}\left(\nabla_{\rho}\phi\right)\left(\nabla_{\sigma}\phi\right) - \frac{1}{2(d-2)}g^{\alpha\beta}g_{\rho\sigma}\left(\nabla\phi\right)^2 ,$$

and therefore in the result of contraction we have that

$$C^{\mu\nu,\rho\sigma}H_{\rho\sigma}{}^{\alpha\beta} = \frac{1}{K^2}\left[-R^{\alpha(\mu\nu)\beta} - \frac{1}{2}g^{\mu\nu}R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R^{\mu\nu} + \frac{1}{4}g^{\alpha\beta}g^{\mu\nu}R + g^{(\mu(\alpha}R^{\beta)\nu)} - \frac{1}{2}g^{\mu(\alpha}g^{\beta)\nu}R\right] +$$
$$+\frac{1}{2}\left[C^{\mu\nu,\alpha\beta}\left(\nabla\phi\right)^2 + \frac{1}{2}g^{\mu\nu}\left(\nabla^{\alpha}\phi\right)\left(\nabla^{\beta}\phi\right) + \frac{1}{2}g^{\alpha\beta}\left(\nabla^{\mu}\phi\right)\left(\nabla^{\nu}\phi\right) - \tag{5.15}$$
$$-2g^{(\mu(\alpha}\left(\nabla^{\beta)}\phi\right)\left(\nabla^{\nu)}\phi\right)\right] + VC^{\mu\nu,\alpha\beta} .$$

Later we will need functional determinant of the operator F and such quantity is well-defined (independent of chosen vector basis), if it has mixed position of indices. In order to achieve this we multiply $F_{AB}(\nabla)$ by the inverse matrix C^{DA} ($C^{DA}C_{AB} = \delta^D_B$), which is equal to

$$C^{DA} = \begin{pmatrix} K^2 C_{\kappa\lambda,\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} \tag{5.16}$$

with $C_{\kappa\lambda,\mu\nu} = g_{\kappa\mu}g_{\lambda\nu} + g_{\kappa\nu}g_{\lambda\mu} - \frac{2}{d-2}g_{\kappa\lambda}g_{\mu\nu}$ (this is not the version of $C^{\kappa\lambda,\mu\nu}$ with covariant indices lowered by covariant metric tensor g, even for $d = 4$). The matrix C^{DA} plays the role of the contravariant metric in the vector space of fluctuations. Additionally we define a set of hatted quantities: $\hat{I} = \delta_B^A$, $\hat{\Gamma}^\sigma = \Gamma^{\sigma D}{}_B = C^{DA}\Gamma^\sigma{}_{AB}$, $\hat{F}(\nabla) = F(\nabla)^D{}_B = C^{DA}F(\nabla)_{AB}$ and $\hat{W} = W^D{}_B = C^{DA}W_{AB}$. With these definitions we have that

$$\hat{F}(\nabla) = \hat{I}\Box + 2\hat{\Gamma}^\sigma\nabla_\sigma + \hat{W}. \tag{5.17}$$

It is much easier to compute determinants of the differential operators, which are in the minimal form (no piece with one covariant derivative). We can use a new covariant derivative $\mathcal{D}_\mu = \nabla_\mu + \hat{\Gamma}_\mu$, which is the old one ∇_μ shifted by the covariant vector $\hat{\Gamma}_\mu = g_{\mu\nu}\hat{\Gamma}^\nu$. With this trick we absorb the part linear in derivative operators in $\hat{F}(\nabla)$. Then our operator takes the following minimal form

$$\hat{F}(\mathcal{D}) = \hat{I}g^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu + \hat{P} - \frac{1}{6}\hat{I}R, \tag{5.18}$$

where the scalar curvature R of the metric g was extracted for reasons of convenience. Now newly defined operator $\hat{P}$ is expressed by the relation $\hat{P} = \hat{W} - \left(\nabla_\sigma\hat{\Gamma}^\sigma\right) - \hat{\Gamma}_\sigma\hat{\Gamma}^\sigma + \frac{1}{6}\hat{I}R$. The energy dimensions of diagonal elements of $\hat{P}$ are equal to E^2, while for P_{12} it is E and for P_{21} is E^3. The matrices of $\hat{\Gamma}^\sigma$ and $\hat{W}$ act on fluctuations as given schematically by $\left(\tilde{h}^{\kappa\lambda}, f\right)\hat{F}\begin{pmatrix} h_{\alpha\beta} \\ f \end{pmatrix}$ with $\tilde{h}^{\kappa\lambda} = K^{-2}h_{\mu\nu}C^{\mu\nu,\kappa\lambda}$. And they look as follows

$$\hat{\Gamma}^\sigma = \begin{pmatrix} 0 & K^2\delta_{\kappa\lambda}^{\sigma\rho}\nabla_\rho\phi \\ -C^{\alpha\beta,\sigma\rho}\nabla_\rho\phi & 0 \end{pmatrix} \quad \text{and} \tag{5.19}$$

$$\hat{W} = \begin{pmatrix} K^2 H_{\kappa\lambda}{}^{\alpha\beta} & \frac{2}{d-2}K^2 g_{\kappa\lambda}V' \\ -2C^{\alpha\beta,\rho\sigma}\nabla_\rho\nabla_\sigma\phi - \frac{1}{2}g^{\alpha\beta}V' & -V'' \end{pmatrix}. \tag{5.20}$$

We have interesting expressions for $\nabla_\sigma\hat{\Gamma}^\sigma$ and $\hat{\Gamma}_\sigma\hat{\Gamma}^\sigma$ explicitly equal to

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$$\nabla_\sigma \hat{\Gamma}^\sigma = \begin{pmatrix} 0 & K^2 \nabla_\kappa \nabla_\lambda \phi \\ -C^{\alpha\beta,\sigma\rho} \nabla_\sigma \nabla_\rho \phi & 0 \end{pmatrix} \quad \text{and} \tag{5.21}$$

$$\hat{\Gamma}_\sigma \hat{\Gamma}^\sigma = \begin{pmatrix} -\frac{1}{4} K^2 \left(2\delta_{(\kappa}^{(\alpha} \left(\nabla^{\beta)} \phi \right) \left(\nabla_{\lambda)} \phi \right) - g^{\alpha\beta} \left(\nabla_\kappa \phi \right) \left(\nabla_\lambda \phi \right) \right) & 0 \\ 0 & -K^2 \left(\nabla \phi \right)^2 \end{pmatrix}. \tag{5.22}$$

With this in mind we obtain the following matrix form of the $\hat{P}$ operator (we change the indices pair (κ, λ) to (μ, ν)):

$$\hat{P} = \begin{pmatrix} A_{\mu\nu}{}^{\alpha\beta} + \frac{1}{6} R \delta_{\mu\nu}^{\alpha\beta} & B_{\mu\nu} \\ E^{\alpha\beta} & D + \frac{1}{6} R \end{pmatrix}. \tag{5.23}$$

The coefficient functions are given below

$$A_{\mu\nu}{}^{\alpha\beta} = K^2 H_{\mu\nu}{}^{\alpha\beta} + \frac{1}{2} K^2 \delta_{(\mu}^{(\alpha} \left(\nabla^{\beta)} \phi \right) \left(\nabla_{\nu)} \phi \right) - \frac{1}{4} K^2 g^{\alpha\beta} \left(\nabla_\mu \phi \right) \left(\nabla_\nu \phi \right), \tag{5.24}$$

$$B_{\mu\nu} = \frac{2}{d-2} K^2 g_{\mu\nu} V' - K^2 \nabla_\mu \nabla_\nu \phi, \tag{5.25}$$

$$E^{\alpha\beta} = -\frac{1}{2} \nabla^\alpha \nabla^\beta \phi + \frac{1}{4} g^{\alpha\beta} \Box \phi - \frac{1}{2} g^{\alpha\beta} V' \quad \text{and} \tag{5.26}$$

$$D = -V'' + K^2 \left(\nabla \phi \right)^2. \tag{5.27}$$

Note that the coefficient $\frac{1}{2}$ in front of the second derivative of the scalar potential in coefficient D was incorrect in [98].

Now we can compute the generalised curvature defined as the commutator of shifted covariant derivatives $[\mathcal{D}_\alpha, \mathcal{D}_\beta] \psi = \hat{\mathcal{R}}_{\alpha\beta} \psi$, where $\hat{\mathcal{R}}_{\alpha\beta} = \mathcal{R}_{\alpha\beta}{}^A{}_B$ is understood as a 2x2 matrix. Using the definitions of $\mathcal{D}_\alpha$, we get the relation $\hat{\mathcal{R}}_{\alpha\beta} = \hat{\mathcal{R}}_{\alpha\beta}^0 + 2\nabla_{[\alpha} \hat{\Gamma}_{\beta]} + 2\hat{\Gamma}_{[\alpha} \hat{\Gamma}_{\beta]}$, where $\hat{\mathcal{R}}_{\alpha\beta}^0$ is the curvature for the ordinary spacetime covariant derivatives ∇_α in the matrix form. Only the $(1,1)$ element of the latter matrix is nonvanishing (when acting on

a tensor of metric fluctuations $h_{\rho\tau}$) and equals to $[\nabla_\alpha, \nabla_\beta] h_{\rho\tau} = \mathcal{O}_{\rho\tau}{}^{\mu\nu}{}_{\alpha\beta} h_{\mu\nu}$. An operator $\mathcal{O}$ is expressed by the Riemann tensor according to the formula $\mathcal{O}_{\rho\tau}{}^{\mu\nu}{}_{\alpha\beta} = R_{\alpha\beta\rho}{}^{(\mu}\delta_\tau^{\nu)} + R_{\alpha\beta\tau}{}^{(\mu}\delta_\rho^{\nu)}$. Covariant derivative commute when acting on a scalar, so all other components of $\hat{\mathcal{R}}^0_{\alpha\beta}$ are zero. The generalised curvature acts in the following way on the fluctuations $\left(\tilde{h}^{\rho\tau}, f\right) \hat{\mathcal{R}}_{\alpha\beta} \begin{pmatrix} h_{\mu\nu} \\ f \end{pmatrix}$. Now we write explicitly expressions appearing in the expansion of the generalised curvature. We have that

$$\nabla_{[\alpha}\hat{\Gamma}_{\beta]} = \begin{pmatrix} 0 & K^2\delta^{\sigma\lambda}_{\rho\tau} g_{\sigma[\beta}\nabla_{\alpha]}\nabla_\lambda\phi \\ -C^{\mu\nu,\sigma\lambda} g_{\sigma[\beta}\nabla_{\alpha]}\nabla_\lambda\phi & 0 \end{pmatrix} \text{ and} \qquad (5.28)$$

$$\hat{\Gamma}_{[\alpha}\hat{\Gamma}_{\beta]} = \Gamma_{[\alpha}{}^A{}_B\Gamma_{\beta]}{}^B{}_C = \begin{pmatrix} -K^2\delta^{\sigma\lambda}_{\rho\tau}C^{\mu\nu,\varepsilon\kappa} g_{\sigma[\alpha}g_{\beta]\kappa}\left(\nabla_\lambda\phi\right)\left(\nabla_\varepsilon\phi\right) & 0 \\ 0 & -K^2C^{\varepsilon\kappa,\sigma\lambda} g_{\sigma[\alpha}g_{\beta]\kappa}\left(\nabla_\lambda\phi\right)\left(\nabla_\varepsilon\phi\right) \end{pmatrix}$$
$$(5.29)$$

The last low entry in $\hat{\Gamma}_{[\alpha}\hat{\Gamma}_{\beta]}$ is equal to zero, because after doing the change of names of dummy indices $(\varepsilon, \kappa) \leftrightarrow (\sigma, \lambda)$ we get this term equal to $-K^2C^{\varepsilon\kappa,\sigma\lambda} g_{\sigma[\beta}g_{\alpha]\kappa}\left(\nabla_\lambda\phi\right)\left(\nabla_\varepsilon\phi\right)$. And this means that this expression is symmetric in α, β indices. Adopting the following convention for writing the matrix operator $\hat{\mathcal{R}}_{\alpha\beta}$:

$$\hat{\mathcal{R}}_{\alpha\beta} = \begin{pmatrix} X_{\rho\tau}{}^{\mu\nu}{}_{\alpha\beta} & Y_{\rho\tau,\alpha\beta} \\ Z^{\mu\nu}{}_{\alpha\beta} & 0 \end{pmatrix} \qquad (5.30)$$

we can read out the expression for X, Y and Z functions. Namely we have

$$X_{\rho\tau}{}^{\mu\nu}{}_{\alpha\beta} = -2\delta^{(\mu}_{(\rho}R^{\nu)}{}_{\tau)\alpha\beta} + 2k^2\delta^{\varepsilon\kappa}_{\rho\tau}C^{\mu\nu,\sigma\lambda} g_{\sigma[\alpha}g_{\beta]\kappa}\left(\nabla_\lambda\phi\right)\left(\nabla_\varepsilon\phi\right), \qquad (5.31)$$

$$Y_{\rho\tau,\alpha\beta} = -2k^2\delta^{\sigma\lambda}_{\rho\tau} g_{\sigma[\alpha}\nabla_{\beta]}\nabla_\lambda\phi \quad \text{and} \qquad (5.32)$$

$$Z^{\mu\nu}{}_{\alpha\beta} = -2C^{\mu\nu,\sigma\lambda} g_{\sigma[\beta}\nabla_{\alpha]}\nabla_\lambda\phi. \qquad (5.33)$$

(In formulas (2.39-2.41) of [98] the overall sign was incorrect!). The energy dimensions of these entries are respectively: E^2, E and E^3. Up to this moment every computation was done under the assumption of the general dimensionality d of spacetime.

5.2 Local terms of one-loop effective action

First we will look for local terms in 1-loop effective action for our system. They are related to UV divergences of the theory. In general these divergences give rise to the renormalization of couplings in front of local terms. They are not universal and depend on the precise way of UV completion. However we assume, that the bare action is given by (5.2). At one loop order the quantum effective action is given by the integral

$$\Gamma[\phi, g] = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr}\, e^{-s\hat{S}^{(2)}} \,, \tag{5.34}$$

where $\text{Tr}\, e^{-s\hat{S}^{(2)}}$ is the functional trace of some differential operator, which we are going to compute with the heat kernel techniques. For our applications in the exponent of heat kernel we use inverse propagator, found in the previous section, denoted here by $\hat{S}^{(2)}$ (second variational derivative of the action S with respect to all fluctuating fields). This operator, as other quantities with a hat over, is a matrix in field space of gravitons and scalar field perturbations. In order to find logarithmically divergent part of one-loop effective action to second order in curvature we can use the Schwinger-DeWitt method for quadratic operators:

$$\text{Tr}\, e^{-s\hat{S}^{(2)}} = \frac{1}{(4\pi s)^{d/2}} \int d^d x \sqrt{g}\, \text{tr} \left\{ \hat{1} + s\hat{P} + s^2 \left[\frac{1}{2}\hat{P}^2 + \frac{1}{12}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu} + \right. \right.$$
$$\left. \left. + \frac{1}{180}\text{Riem}^2\hat{1} - \frac{1}{180}R_{\mu\nu}R^{\mu\nu}\hat{1} \right] \right\} \,. \tag{5.35}$$

We will restrict ourselves to second order contribution in operators $\hat{P}$, $\hat{\mathcal{R}}_{\mu\nu}$ and gravitational curvatures. (We don't consider here application of this method to the ghost part of the action, because we are mainly interested in nonminimally coupled matter terms.)

Using Schwinger-DeWitt technique we reduced the functional trace to matrix traces. In what follows, small traces denote the traces done in field space (of 2x2 matrices). One finds particular traces in forms given below. For a trace of quadratic scalar operator $\hat{P}^2$ we find:

$$
\begin{aligned}
\frac{1}{2}\mathrm{tr}\hat{P}^2 =\ & \frac{3}{2}\mathrm{Riem}^2 - 3\mathrm{Ric}^2 + \frac{119}{72}R^2 \\
& + \frac{11}{8}K^4\nabla_\alpha\phi\nabla^\alpha\phi\nabla_\beta\phi\nabla^\beta\phi - \frac{1}{4}K^2\nabla^2\phi\nabla^2\phi + \frac{1}{2}K^2\nabla_\beta\nabla_\alpha\phi\nabla^\beta\nabla^\alpha\phi \\
& + K^2\left(K^2 V(\phi) - \frac{5}{12}R - V''(\phi)\right)\nabla_\alpha\phi\nabla^\alpha\phi \\
& + K^2 V'(\phi)\nabla^2\phi + 5k^4 V^2(\phi) - \frac{13}{3}K^2 R V(\phi) \\
& - 2k^2 V'(\phi)^2 - \frac{R}{6}V''(\phi) + \frac{(V''(\phi))^2}{2}\ .
\end{aligned}
\tag{5.36}
$$

The trace of the contracted square of the generalised curvature $\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu}$ amounts to:

$$
\begin{aligned}
\frac{1}{12}\mathrm{tr}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu} =\ & -\frac{1}{2}\mathrm{Riem}^2 + \frac{1}{6}K^2 R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi \\
& - \frac{1}{8}K^4\nabla_\alpha\phi\nabla^\alpha\phi\nabla_\beta\phi\nabla^\beta\phi + \frac{1}{12}K^2\nabla^2\phi\nabla^2\phi \\
& - \frac{1}{3}K^2\nabla_\beta\nabla_\alpha\phi\nabla^\beta\nabla^\alpha\phi + \frac{1}{12}K^2 R\,\nabla_\alpha\phi\nabla^\alpha\phi
\end{aligned}
\tag{5.37}
$$

Finally we report here for completeness the trace of the unity matrix $\hat{1}$ in field space:

$$
\mathrm{tr}\hat{1} = \frac{d(d+1)}{2} + 1 = 11 \quad \text{for } d = 4\ .
\tag{5.38}
$$

The last result is equal to the sum of the dimensionality of space of symmetric tensors, i.e. metric perturbations and one-dimensional scalar perturbation.

Now we are going to compute traces of the various matrix-valued operators appearing in the second variation of the action Γ_k. We have after summation that

$$
\begin{aligned}
\mathrm{tr}b_4 =\ & \frac{11}{180}\left(\mathrm{Riem}^2 - \mathrm{Ric}^2\right) + \frac{1}{2}\mathrm{tr}P^2 + \frac{1}{12}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu} \\
=\ & \frac{191}{180}\mathrm{Riem}^2 - \frac{551}{180}\mathrm{Ric}^2 + \frac{119}{72}R^2
\end{aligned}
$$

$$+5K^4V^2(\phi) - 2K^2V'(\phi)^2 - \frac{13}{3}K^2RV(\phi) - \frac{R}{6}V''(\phi) + \frac{(V''(\phi))^2}{2}$$

$$+\frac{5}{4}K^4\nabla_\alpha\phi\nabla^\alpha\phi\nabla_\beta\phi\nabla^\beta\phi \qquad (5.39)$$

$$-\frac{1}{6}K^2\nabla^2\phi\nabla^2\phi + \frac{1}{6}K^2\nabla_\beta\nabla_\alpha\phi\nabla^\beta\nabla^\alpha\phi + \frac{1}{6}K^2R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi$$

$$+K^2\left(K^2V(\phi) - \frac{1}{3}R - V''(\phi)\right)\nabla_\alpha\phi\nabla^\alpha\phi$$

$$+K^2V'(\phi)\nabla^2\phi\,,$$

where by b_4 we denoted traditionally the expansion coefficients of matrix trace in front of the second power of s parameter. For any scalar field we have the following identity originating from the commutation of the second covariant derivatives acting upon it:

$$\frac{1}{6}R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi - \frac{1}{6}\nabla^2\phi\nabla^2\phi + \frac{1}{6}\nabla_\beta\nabla_\alpha\phi\nabla^\beta\nabla^\alpha\phi = 0\,, \qquad (5.40)$$

Next we use the scalar equation of motion derived from the standard action. Here it is given by

$$\nabla^2\phi - V'(\phi) = 0\,. \qquad (5.41)$$

Then one is able to derive the following identity valid onshell (For the moment we put scalar perturbations onshell, while keeping graviton perturbations completely general).

$$K^2V'(\phi)\nabla^2\phi - 2k^2V'(\phi)^2 = -K^2V'(\phi)^2\,. \qquad (5.42)$$

Exploiting this in our formula for $\mathrm{tr}\,b_4$ leads to the simplification:

$$
\begin{aligned}
\mathrm{tr}b_4 \;=\; & \frac{191}{180}\mathrm{Riem}^2 - \frac{551}{180}\mathrm{Ric}^2 + \frac{119}{72}R^2 \\
& +\frac{5}{4}K^4\nabla_\alpha\phi\nabla^\alpha\phi\nabla_\beta\phi\nabla^\beta\phi \\
& +K^2\left(-\frac{1}{3}R + K^2V(\phi) - 2V''(\phi)\right)\nabla_\alpha\phi\nabla^\alpha\phi \qquad (5.43) \\
& -\frac{13}{3}K^2RV(\phi) - \frac{R}{6}V''(\phi) + 5k^4V^2(\phi) - 2k^2V'(\phi)^2 + \frac{(V''(\phi))^2}{2}\,.
\end{aligned}
$$

Now we decide about the form of the scalar potential $V(\phi)$. If we set it to contain only the mass term $V(\phi) = \frac{m^2}{2}\phi^2$, then we have:

$$
\begin{aligned}
\mathrm{tr}b_4 \;=\; & \frac{191}{180}\mathrm{Riem}^2 - \frac{551}{180}\mathrm{Ric}^2 + \frac{119}{72}R^2 \\
& + \frac{5}{4}K^4\nabla_\alpha\phi\nabla^\alpha\phi\nabla_\beta\phi\nabla^\beta\phi \\
& + K^2\left(-\frac{1}{3}R + \frac{1}{2}K^2m^2\phi^2 - 2m^2\right)\nabla_\alpha\phi\nabla^\alpha\phi \\
& - \frac{13}{6}K^2 R\, m^2\phi^2 - \frac{1}{6}m^2 R + \frac{5}{4}K^4 m^4\phi^4 - 2k^2m^4\phi^2 + \frac{1}{2}m^4 \,.
\end{aligned}
\tag{5.44}
$$

In the last step we can put gravitational excitations on shell. For our result this means, that we can use Euler identity relating squares of Riemann, Ricci and scalar curvatures according to the formula $\mathrm{Riem}^2 = 4\mathrm{Ric}^2 - R^2 + E$. We neglect the difference term E, because it is a total derivative. After doing this, we arrive at a final result for the trace of b_4 coefficient:

$$
\begin{aligned}
\mathrm{tr}b_4 \;=\; & \frac{213}{180}\mathrm{Ric}^2 + \frac{213}{360}R^2 \\
& + \frac{5}{4}K^4\nabla_\alpha\phi\nabla^\alpha\phi\nabla_\beta\phi\nabla^\beta\phi \\
& + K^2\left(-\frac{1}{3}R + \frac{1}{2}K^2m^2\phi^2 - 2m^2\right)\nabla_\alpha\phi\nabla^\alpha\phi \\
& - \frac{13}{6}K^2 R\, m^2\phi^2 - \frac{1}{6}m^2 R + \frac{5}{4}K^4 m^4\phi^4 - 2k^2m^4\phi^2 + \frac{1}{2}m^4 \,.
\end{aligned}
\tag{5.45}
$$

This is the form of one-loop local terms for scalar field with mass (non self-interacting) minimally coupled to Einstein-Hilbert gravitation in four spacetime dimensions. We obtained an agreement with previous results found by others for local part of divergent part of one-loop effective action [98, 99]. The contribution from the ghost part of the action has the impact only on the first two coefficients (in front of the quadratic curvature invariants). This is so, because of the initial gauge choice we adopted, where the scalars do not appear. However in this and in the later derivations we focus on the monomials from the matter part, where the scalar field ϕ is present.

5.3 Nonlocal terms and exact RG flow equations

In order to go beyond Schwinger-DeWitt technique and find form of nonlocal part of one-loop action we insert nonlocal structure functions. They are functions of s parameter and box operator $\Box = \nabla^\mu \nabla_\mu$ (acting under the integral) appearing in the combination $-s\Box$. We insert these structure functions between two matrix operators present at the second order as in the detailed formula below

$$\frac{1}{(4\pi s)^{d/2}} \int d^dx \sqrt{g} s^2 \mathrm{tr} \left\{ \left[\hat{P} f_P \left(-s\Box \right) \hat{P} + \hat{\mathcal{R}}_{\mu\nu} f_{\mathcal{R}} \left(-s\Box \right) \hat{\mathcal{R}}^{\mu\nu} + \right.\right.$$
$$\left.\left. + \hat{P} f_{PR} \left(-s\Box \right) R + R f_R \left(-s\Box \right) R \hat{1} + R_{\mu\nu} f_{Ric} \left(-s\Box \right) R^{\mu\nu} \hat{1} \right] + ... \right\} . \qquad (5.46)$$

It must be emphasised, that the leading order in s contribution is equal to constants as written in the formulas (5.35) in section above (for $\hat{P}R$ operator this constant vanishes). Moreover we have used the Euler identity relating contribution of the square of Riemann tensor to quadratic expression in Ricci tensor and scalar according to the formula $E = \mathrm{Riem}^2 - 4\mathrm{Ric}^2 + R^2$. By E we denote Euler characteristics of the spacetime manifold - this is a topological quantity and doesn't influence local dynamics in the bulk of spacetime. The traces of matrix terms of order curvature square are modified with respect to expressions given in previous section by the appearance of structure functions f_P, $f_{\mathcal{R}}$, f_{PR}, f_R and f_{Ric}. We have the results for the trace of the quadratic scalar operator $\hat{P}^2$:

$$\begin{aligned}
\mathrm{tr} \hat{P} f_P \hat{P} = \; & 3R_{\mu\nu} f_P R^{\mu\nu} + \frac{11}{72} R f_P R \\
& + \frac{11}{8} K^4 (\nabla_\alpha \phi \nabla^\alpha \phi) f_P (\nabla_\beta \phi \nabla^\beta \phi) - \frac{1}{4} K^2 (\nabla^2 \phi) f_P (\nabla^2 \phi) + \frac{1}{2} K^2 (\nabla_\beta \nabla_\alpha \phi) f_P (\nabla^\beta \nabla^\alpha \phi) \\
& + K^4 V(\phi) f_P (\nabla_\alpha \phi \nabla^\alpha \phi) - \frac{5}{12} K^2 R f_P (\nabla_\alpha \phi \nabla^\alpha \phi) - K^2 V''(\phi) f_P (\nabla_\alpha \phi \nabla^\alpha \phi) \qquad (5.47) \\
& + K^2 V'(\phi) f_P (\nabla^2 \phi) + 5k^4 V(\phi) f_P V(\phi) - \frac{13}{3} K^2 R f_P V(\phi) \\
& - 2k^2 V'(\phi) f_P V'(\phi) - \frac{R}{6} f_P V''(\phi) + \frac{1}{2} V''(\phi) f_P V''(\phi) .
\end{aligned}$$

The trace of the contracted square of the generalised curvature $\hat{\mathcal{R}}_{\mu\nu} \hat{\mathcal{R}}^{\mu\nu}$ amounts to:

$$\mathrm{tr}\hat{\mathcal{R}}_{\mu\nu}f_{\mathcal{R}}\hat{\mathcal{R}}^{\mu\nu} = -2R_{\mu\nu}f_{\mathcal{R}}R^{\mu\nu} + \frac{1}{2}Rf_{\mathcal{R}}R + \frac{1}{6}K^2R_{\alpha\beta}f_{\mathcal{R}}(\nabla^\alpha\phi\nabla^\beta\phi)$$
$$-\frac{1}{8}K^4(\nabla_\alpha\phi\nabla^\alpha\phi)f_{\mathcal{R}}(\nabla_\beta\phi\nabla^\beta\phi) + \frac{1}{12}K^2(\nabla^2\phi)f_{\mathcal{R}}(\nabla^2\phi) \quad (5.48)$$
$$-\frac{1}{3}K^2(\nabla_\beta\nabla_\alpha\phi)f_{\mathcal{R}}(\nabla^\beta\nabla^\alpha\phi) + \frac{1}{12}K^2R\,f_{\mathcal{R}}(\nabla_\alpha\phi\nabla^\alpha\phi)\ .$$

At the end we report here for completeness the trace of the new operator $\hat{P}R$ equal to:

$$\mathrm{tr}\hat{P}f_{PR}R = 2k^2R\,f_{PR}(\nabla_\alpha\phi\nabla^\alpha\phi) + 10k^2R\,f_{PR}V(\phi)$$
$$-Rf_{PR}V''(\phi) - \frac{25}{6}Rf_{PR}R\ . \quad (5.49)$$

We proceed in a very similar way like in the last section. This time the only difference is that we have to take care of nonlocal structure functions. Our previous results are still valid, when we restrict ourselves to first terms in the expansion of these formfactors. After summing all the terms we find the nonlocal equivalent of $\mathrm{tr}\,b_4$:

$$\frac{11}{180}\left(3R_{\mu\nu}f_{Ric}R^{\mu\nu} - Rf_RR\right) + \mathrm{tr}\hat{P}f_P\hat{P} + \mathrm{tr}\hat{P}f_{PR}R + \mathrm{tr}\hat{\mathcal{R}}_{\mu\nu}f_{\mathcal{R}}\hat{\mathcal{R}}^{\mu\nu} =$$
$$= R_{\mu\nu}\left[\frac{33}{180}f_{Ric} + 6f_P - 24f_{\mathcal{R}}\right]R^{\mu\nu}$$
$$+R\left[-\frac{11}{180}f_R + \frac{11}{36}f_P + 6f_{\mathcal{R}} - \frac{25}{6}f_{PR}\right]R$$
$$+K^4(\nabla_\alpha\phi\nabla^\alpha\phi)\left[\frac{11}{4}f_P - \frac{3}{2}f_{\mathcal{R}}\right](\nabla_\beta\phi\nabla^\beta\phi)$$
$$+K^2(\nabla^2\phi)\left[-\frac{1}{2}f_P + f_{\mathcal{R}}\right](\nabla^2\phi)$$
$$+K^2(\nabla_\beta\nabla_\alpha\phi)\left[f_P - 4f_{\mathcal{R}}\right](\nabla^\beta\nabla^\alpha\phi)$$
$$+K^2R\left[-\frac{5}{6}f_P + f_{\mathcal{R}} + 2f_{PR}\right](\nabla_\alpha\phi\nabla^\alpha\phi) \quad (5.50)$$
$$+K^2R_{\alpha\beta}\left[2f_{\mathcal{R}}\right](\nabla^\alpha\phi\nabla^\beta\phi)$$
$$+K^4V(\phi)\left[2f_P\right](\nabla_\alpha\phi\nabla^\alpha\phi) - K^2V''(\phi)\left[2f_P\right](\nabla_\alpha\phi\nabla^\alpha\phi)$$
$$+K^2V'(\phi)\left[2f_P\right](\nabla^2\phi) + K^4V(\phi)\left[10f_P\right]V(\phi)$$

$$K^2 R \left[-\frac{26}{3} f_P + 10 f_{PR} \right] V(\phi) - K^2 V'(\phi) \left[4 f_P \right] V'(\phi)$$

$$R \left[-\frac{1}{3} f_P - f_{PR} \right] V''(\phi) + V''(\phi) \left[f_P \right] V''(\phi) \, .$$

If we set the scalar potential to $V(\phi) = \frac{m^2}{2} \phi^2$ and use scalar equations of motion in this case, then we have some cancellations. We have even more cancellations and simplifications, because obviously we have, that $\Box m^2 = 0$ and this means, that the structure functions with nonlocal pieces cannot be inserted between mass powers in mass terms. When we exploit this fact, we arrive at the following final expression:

$$\frac{11}{180} \left(3 R_{\mu\nu} f_{Ric} R^{\mu\nu} - R f_R R \right) + \mathrm{tr} \hat{P} f_P \hat{P} + \mathrm{tr} \hat{P} f_{PR} R + \mathrm{tr} \hat{\mathcal{R}}_{\mu\nu} f_{\mathcal{R}} \hat{\mathcal{R}}^{\mu\nu} =$$

$$= R_{\mu\nu} \left[\frac{33}{180} f_{Ric} + 6 f_P - 24 f_{\mathcal{R}} \right] R^{\mu\nu} + R \left[-\frac{11}{180} f_R + \frac{11}{36} f_P + 6 f_{\mathcal{R}} - \frac{25}{6} f_{PR} \right] R$$

$$+ K^4 (\nabla_\alpha \phi \nabla^\alpha \phi) \left[\frac{11}{4} f_P - \frac{3}{2} f_{\mathcal{R}} \right] (\nabla_\beta \phi \nabla^\beta \phi) + K^4 m^2 \phi^2 \left[f_P \right] (\nabla_\alpha \phi \nabla^\alpha \phi)$$

$$+ K^4 m^4 \phi^2 \left[\frac{5}{2} f_P \right] \phi^2 + K^2 m^4 \phi \left[-\frac{5}{2} f_P + f_{\mathcal{R}} \right] \phi \qquad (5.51)$$

$$+ K^2 (\nabla_\beta \nabla_\alpha \phi) \left[f_P - 4 f_{\mathcal{R}} \right] (\nabla^\beta \nabla^\alpha \phi) + K^2 R \left[-\frac{5}{6} f_P + f_{\mathcal{R}} + 2 f_{PR} \right] (\nabla_\alpha \phi \nabla^\alpha \phi)$$

$$+ K^2 R_{\alpha\beta} \left[2 f_{\mathcal{R}} \right] (\nabla^\alpha \phi \nabla^\beta \phi) + K^2 m^2 R \left[-\frac{13}{3} f_P + 5 f_{PR} \right] \phi^2 \, .$$

Now we want to consider the exact RG flow of EAA, which will be denoted here by $\bar{\Gamma}_k$. As the ansatz for it we choose the expression above, understood that all the couplings and structure functions now acquire dependence on the momentum scale k. The exact RG flow equation for the background effective average action (bEAA) is the following

$$\partial_t \bar{\Gamma}_k [\phi, g] = \frac{1}{2} \mathrm{Tr} \frac{\partial_t R_k (-\mathcal{D}^2) - \eta_k R(-\mathcal{D}^2)}{-\mathcal{D}^2 + R_k (-\mathcal{D}^2)} - \mathrm{Tr} \frac{\partial_t R_k (\Delta_{gh})}{\Delta_{gh} + R_k (\Delta_{gh})} \, . \qquad (5.52)$$

In the above formula $\mathcal{D}$ is a general operator of the covariant derivative and R_k are cutoff kernels (suitably chosen functions of momenta to suppress the contributions from high energy modes in the path integral). We explicitly split the graviton and scalar part from the ghost part in this equation. Our exact RG flow equation describes the change of the

bEAA under the infinitesimal change of the RG logarithmic scale $t = \log k$. The r.h.s. of this equation expresses itself by functional traces of some differential operators and the RG time derivatives of cutoff kernels. We note that in the denominator we have differential part $\mathcal{D}^2$ of our inverse propagator operator (second variation) (5.18), . The r.h.s. of the flow equation is then (neglecting the ghost contribution) and writing all terms

$$\partial_t \bar{\Gamma}_k[\phi, g] = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g} \left\{ R_{\mu\nu} \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_a(s\Box) \right] R^{\mu\nu} + \right.$$

$$+ R \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_b(s\Box) \right] R + K^4 (\nabla_\alpha \phi \nabla^\alpha \phi) \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_c(s\Box) \right] (\nabla_\beta \phi \nabla^\beta \phi) +$$

$$+ K^4 m^2 \phi^2 \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_d(s\Box) \right] (\nabla_\alpha \phi \nabla^\alpha \phi) + K^4 m^4 \phi^2 \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_e(s\Box) \right] \phi^2 +$$

$$+ K^2 m^4 \phi \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_f(s\Box) \right] \phi + K^2 (\nabla_\beta \nabla_\alpha \phi) \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_g(s\Box) \right] (\nabla^\beta \nabla^\alpha \phi) (5.53)$$

$$+ K^2 R \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_h(s\Box) \right] (\nabla_\alpha \phi \nabla^\alpha \phi) + K^2 R_{\alpha\beta} \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_j(s\Box) \right] (\nabla^\alpha \phi \nabla^\beta \phi) +$$

$$\left. + K^2 m^2 R \left[\int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, \tilde{f}_l(s\Box) \right] \phi^2 \right\} .$$

where the functions $\tilde{f}_a(x), \tilde{f}_b(x), ..., \tilde{f}_l(x)$ were derived combining non-local heat kernel structure functions. In the above equation we enlisted all monomial terms, which appeared in (5.51). In the $\{ R_{\mu\nu} R^{\mu\nu}, R^2, K^4 (\nabla_\alpha \phi \nabla^\alpha \phi)^2, K^4 m^2 \phi^2 (\nabla_\alpha \phi \nabla^\alpha \phi), K^4 m^4 \phi^4, K^2 m^4 \phi^2,$ $K^2 (\nabla_\beta \nabla_\alpha \phi)^2, K^2 R (\nabla_\alpha \phi \nabla^\alpha \phi), K^2 R_{\alpha\beta} (\nabla^\alpha \phi \nabla^\beta \phi), K^2 m^2 R \phi^2 \}$ basis as in (5.51), the corresponding functions for each monomial read explicitly:

$$\tilde{f}_a = \frac{33}{180} f_{Ric} + 6 f_P - 24 f_{\mathcal{R}} \tag{5.54}$$

$$\tilde{f}_b = -\frac{11}{180} f_R + \frac{11}{36} f_P + 6 f_{\mathcal{R}} - \frac{25}{6} f_{PR} \tag{5.55}$$

$$\tilde{f}_c = \frac{11}{4} f_P - \frac{3}{2} f_{\mathcal{R}} \tag{5.56}$$

$$\tilde{f}_d = f_P \tag{5.57}$$

$$\tilde{f}_e = \frac{5}{2} f_P \tag{5.58}$$

$$\tilde{f}_f = -\frac{5}{2} f_P + f_{\mathcal{R}} \tag{5.59}$$

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$$\tilde{f}_g = f_P - 4f_{\mathcal{R}} \tag{5.60}$$

$$\tilde{f}_h = -\frac{5}{6}f_P + f_{\mathcal{R}} + 2f_{PR} \tag{5.61}$$

$$\tilde{f}_j = 2f_{\mathcal{R}} \tag{5.62}$$

$$\tilde{f}_l = -\frac{13}{3}f_P + 5f_{PR} \tag{5.63}$$

In [90] another basis for structure functions was used. The transformation between them are linear and are given below:

$$f_P = \phi_4 \tag{5.64}$$

$$f_{\mathcal{R}} = \phi_5 \tag{5.65}$$

$$f_{PR} = \phi_3 \tag{5.66}$$

$$f_R = -180\phi_2 \tag{5.67}$$

$$f_{Ric} = 60\phi_1 \tag{5.68}$$

$$\tag{5.69}$$

Above structure functions for Lagrangian monomials can be rewritten using $\phi_1, \ldots, \phi_5$ structural functions coefficients (being linear combination of f_P, $f_{\mathcal{R}}$, f_{PR}, f_R and f_{Ric}). Moreover we apply identity (5.40) to reduce one term. Then the form of the quadratic part of the effective action is given by

$$\frac{11}{180} \left(3R_{\mu\nu} f_{Ric} R^{\mu\nu} - R f_R R \right) + \operatorname{tr}\hat{P} f_P \hat{P} + \operatorname{tr}\hat{P} f_{PR} R + \operatorname{tr}\hat{\mathcal{R}}_{\mu\nu} f_{\mathcal{R}} \hat{\mathcal{R}}^{\mu\nu} =$$
$$= \left(11\phi_1 + 6\phi_4 - 24\phi_5 \right) \operatorname{Ric}^2 + \left(11\phi_2 - \frac{25}{6}\phi_3 + \frac{11}{36}\phi_4 + 6\phi_5 \right) R^2 +$$
$$+ \frac{5}{2}\phi_4 K^4 m^4 \phi^4 - \left(\frac{5}{2}\phi_4 - 3\phi_5 \right) K^2 m^4 \phi^2 + \left(5\phi_3 - \frac{13}{3}\phi_4 \right) K^2 m^2 R\phi^2 - \tag{5.70}$$
$$- \left(\phi_3 + \frac{1}{3}\phi_4 \right) m^2 R + \phi_4 m^4 + \left(\frac{11}{4}\phi_4 - \frac{3}{2}\phi_5 \right) K^4 \left((\nabla\phi)^2 \right)^2 +$$
$$+ \left(\phi_4 - 6\phi_5 \right) K^2 \left(\nabla_\alpha \nabla_\beta \phi \right)^2 + K^2 \left(\phi_4 K^2 m^2 \phi^2 + \left(2\phi_3 - \frac{5}{6}\phi_4 + \phi_5 \right) R - 2m^2 \right) (\nabla\phi)^2.$$

(If we use modified version of the operator $\hat{P}$, where the mass for the scalar field is treated exactly, not perturbatively, then instead of the last numerical coefficient 2 in the last line we have coefficient equal to twice the fourth structure functions $2\phi_4$.)

Since now we are already in $d = 4$. From formula (5.70) we read the coefficients of Lagrangian monomials in the different (extended) basis for formfactors. We have them explicitly:

$$f_a = 11\phi_1 + 6\phi_4 - 24\phi_5 \tag{5.71}$$

$$f_b = 11\phi_2 - \frac{25}{6}\phi_3 + \frac{11}{36}\phi_4 + 6\phi_5 \tag{5.72}$$

$$f_c = \frac{5}{2}\phi_4 \tag{5.73}$$

$$f_d = -\left(\frac{5}{2}\phi_4 - 3\phi_5\right) \tag{5.74}$$

$$f_e = 5\phi_3 - \frac{13}{3}\phi_4 \tag{5.75}$$

$$[f_f = -\left(\phi_3 + \frac{1}{3}\phi_4\right)] \tag{5.76}$$

$$[f_g = \phi_4] \tag{5.77}$$

$$f_h = \frac{11}{4}\phi_4 - \frac{3}{2}\phi_5 \tag{5.78}$$

$$f_j = \phi_4 - 6\phi_5 \tag{5.79}$$

$$f_l = \phi_4 \tag{5.80}$$

$$f_m = 2\phi_3 - \frac{5}{6}\phi_4 + \phi_5 \tag{5.81}$$

$$[f_n = 2(-2\phi_4)] \tag{5.82}$$

Inside square brackets were written formfactors, for which only the constant term matters. This is, because, when d'Alambertian operator acts on such expression, where the structure functions are inserted in, it gives zero (presence of mass terms). Structure functions for such monomials do not contain any non-local part. Now we are working in the following basis of 12 Lagrangian monomials $\{R_{\mu\nu}R^{\mu\nu}, R^2, K^4m^4\phi^4, K^2m^4\phi^2, K^2m^2R\phi^2, m^2R, m^4, K^4((\nabla\phi)^2)^2, K^2(\nabla_\alpha\nabla_\beta\phi)^2, K^4m^2\phi^2(\nabla\phi)^2, K^2R(\nabla\phi)^2, K^2m^2(\nabla\phi)^2\}$. Out of these monomials only 9 allow for the nonlocal form-factors functions depending on the operator $\square$.

With the definition of the basic heat kernel non-local form factor $f(x)$

$$f(x) = \int_0^1 d\xi e^{-\xi(1-\xi)x} \tag{5.83}$$

and recalling the relations between structure functions and $f(x)$

$$\phi_1(x) = \frac{f(x) - 1 + \frac{1}{6}x}{x^2} \tag{5.84}$$

$$\phi_2(x) = \frac{1}{8}\left[\frac{1}{36}f(x) + \frac{f(x) - 1}{3x} - \frac{f(x) - 1 + \frac{1}{6}x}{x^2}\right] \tag{5.85}$$

$$\phi_3(x) = \frac{1}{12}f(x) + \frac{1}{2}\frac{f(x) - 1}{x} \tag{5.86}$$

$$\phi_4(x) = \frac{1}{2}f(x) \tag{5.87}$$

$$\phi_5(x) = -\frac{1}{2}\frac{f(x) - 1}{x} \tag{5.88}$$

we can extract the running of the structure functions (k is the momentum scale here!) given by the following equation

$$\partial_t f_{I,k}(-\Box) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty ds\, \tilde{h}(s)\, s^{2-\frac{d}{2}}\, f_I(sx)\Big|_{x=-\Box}, \tag{5.89}$$

for $I = a, b, ..., l(n)$. Index I counts the number of possible monomials present in (5.51). We reserve letter k for momentum scale here and therefore it is excluded from the possible values of the index I. The letter i is also excluded, in order not to confuse with other notation.

Firstly our 9 non-local structure functions written in terms of $f(x)$ function are given below:

$$f_a = 3f(x) + \frac{-61 + 72f(x)}{6x} + \frac{11(f(x) - 1)}{x^2} \tag{5.90}$$

$$f_b = -\frac{5}{32}f(x) + \frac{211 - 222f(x)}{48x} - \frac{11(f(x) - 1)}{8x^2} \tag{5.91}$$

$$f_c = \frac{5}{4}f(x) \tag{5.92}$$

$$f_d = -\frac{5}{4}f(x) - \frac{3(f(x)-1)}{2x} \tag{5.93}$$

$$f_e = -\frac{7}{4}f(x) + \frac{5(f(x)-1)}{2x} \tag{5.94}$$

$$f_h = \frac{11}{8}f(x) + \frac{3(f(x)-1)}{4x} \tag{5.95}$$

$$f_i = \frac{1}{2}f(x) + \frac{3(f(x)-1)}{x} \tag{5.96}$$

$$f_l = \frac{1}{2}f(x) \tag{5.97}$$

$$f_m = -\frac{1}{4}f(x) + \frac{f(x)-1}{2x}. \tag{5.98}$$

For each value of index I, above corresponding expression (5.89) can be rewritten in terms of a combination of Q-functionals inside parametric integrals. We have for example

$$
\begin{aligned}
(4\pi)^2\,\partial_t F_{b,\,k}(x) &= -\frac{5}{32}\int_0^1 d\xi\,Q_0\left[h_k\left(z+x\xi(1-\xi)\right)\right] - \frac{37}{8x}\int_0^1 d\xi\,Q_1\left[h_k\left(z+x\xi(1-\xi)\right)\right] + \\
&+ \frac{211}{48x}Q_1\left[h_k\left(z\right)\right] - \frac{11}{8x^2}\left\{\int_0^1 d\xi\,Q_2\left[h_k\left(z+x\xi(1-\xi)\right)\right] - Q_2\left[h_k\left(z\right)\right]\right\}
\end{aligned}
\tag{5.99}
$$

The arguments of Q-functionals are given by the expression $h_k(z) = \frac{\partial_t R_k(z)}{z + R_k(z)}$. Now it is the moment, we have to specify the cutoff function $R_k(z)$. We use the optimised cutoff shape function $R_k(z) = (k^2 - z)\theta(k^2 - z)$, as proposed in [35]. Next we use explicitly the values of the Q-functionals as computed below

$$Q_0\left[h_k(z)\right] = 2 \tag{5.100}$$

$$Q_1\left[h_k(z)\right] = 2k^2 \tag{5.101}$$

$$Q_2\left[h_k(z)\right] = k^4 \tag{5.102}$$

and their integrals over ξ variable inside the $h_k(z)$ functions:

$$\int_0^1 d\xi\,Q_0\left[h_k\left(z+x\xi(1-\xi)\right)\right] = 2\left[1 - \sqrt{1 - \frac{4}{u}}\theta(u-4)\right] \tag{5.103}$$

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$$\int_0^1 d\xi \, Q_1 \left[h_k \left(z + x\xi(1-\xi) \right) \right] = 2k^2 \left[1 - \frac{u}{6} + \frac{u}{6} \sqrt{1 - \frac{4}{u}}^3 \theta(u-4) \right] \qquad (5.104)$$

$$\int_0^1 d\xi \, Q_2 \left[h_k \left(z + x\xi(1-\xi) \right) \right] = 2k^4 \left[\frac{1}{2} - \frac{u}{6} + \frac{u^2}{60} - \frac{u^2}{60} \sqrt{1 - \frac{4}{u}}^5 \theta(u-4) \right], \qquad (5.105)$$

where $u = \frac{x}{k^2}$. We have now, after the integration, a set of equations, which can be generally put in the following general form

$$\partial_t f_{I,k}(x) = \frac{1}{(4\pi)^2} g_I \left(\frac{x}{k^2} \right), \qquad (5.106)$$

where the functions $g_I(u)$ are given for each monomial term respectively by

$$g_a(u) = \frac{71}{30} + \left(-\frac{71}{30} - \frac{196}{15u} - \frac{88}{15u^2} \right) \sqrt{1 - \frac{4}{u}} \theta(u-4) \qquad (5.107)$$

$$g_b(u) = \frac{71}{60} + \left(-\frac{71}{60} + \frac{29}{5u} + \frac{11}{15u^2} \right) \sqrt{1 - \frac{4}{u}} \theta(u-4) \qquad (5.108)$$

$$g_c(u) = \frac{5}{2} - \frac{5}{2} \sqrt{1 - \frac{4}{u}} \theta(u-4) \qquad (5.109)$$

$$g_d(u) = -2 + \left(2 + \frac{2}{u} \right) \sqrt{1 - \frac{4}{u}} \theta(u-4) \qquad (5.110)$$

$$g_e(u) = -\frac{13}{3} + \left(\frac{13}{3} - \frac{10}{3u} \right) \sqrt{1 - \frac{4}{u}} \theta(u-4) \qquad (5.111)$$

$$g_h(u) = \frac{5}{2} + \left(-\frac{5}{2} - \frac{1}{u} \right) \sqrt{1 - \frac{4}{u}} \theta(u-4) \qquad (5.112)$$

$$g_i(u) = -\frac{4}{u} \sqrt{1 - \frac{4}{u}} \theta(u-4) \qquad (5.113)$$

$$g_l(u) = 1 - \sqrt{1 - \frac{4}{u}} \theta(u-4) \qquad (5.114)$$

$$g_m(u) = -\frac{2}{3} + \left(\frac{2}{3} - \frac{2}{3u}\right)\sqrt{1 - \frac{4}{u}}\theta(u - 4).$$ (5.115)

We now integrate the flow equations from a UV scale Λ down to a generic IR scale k. We have schematically that

$$f_{I,\Lambda}(x) - f_{I,k}(x) = \frac{1}{(4\pi)^2} \int_k^\Lambda \frac{dk'}{k'} g_I\left(\frac{x}{k'^2}\right)$$ (5.116)

and after going to u variable we get

$$f_{I,\Lambda}(x) - f_{I,k}(x) = \frac{1}{(4\pi)^2} \int_{x/\Lambda^2}^{x/k^2} \frac{du}{2u} g_I(u).$$ (5.117)

In functions $g_I(u)$ we can isolate constant part $g_{I,0}$ in perturbative expansion in u around $u = 0$, by the relation $g_I(u) = g_{I,0} + \tilde{g}_I(u)$. The constants $g_{I,0}$ are equal to $g_{a,0} = \frac{71}{30}$, $g_{b,0} = \frac{71}{60}$, $g_{c,0} = \frac{5}{2}$, $g_{d,0} = -2$, $g_{e,0} = -\frac{13}{3}$, $g_{h,0} = \frac{5}{2}$, $g_{l,0} = 1$, $g_{m,0} = -\frac{2}{3}$. (Only those nonvanishing were listed here). We isolate the logarithmic divergences in the following schematic way

$$f_{I,\Lambda}(x) - f_{I,k}(x) = \frac{1}{(4\pi)^2} g_{I,0}\left(\log\frac{\Lambda}{k_0} + \log\frac{k_0}{k}\right) + \frac{1}{(4\pi)^2} \int_{x/\Lambda^2}^{x/k^2} \frac{du}{2u} \tilde{g}_I(u).$$ (5.118)

We can renormalize the theory, imposing the following UV boundary conditions for the flow of formfactors:

$$f_{I,\Lambda}(x) = \frac{1}{(4\pi)^2} g_{I,0} \log\frac{\Lambda}{k_0} + c_I,$$ (5.119)

where c_I's are possible finite renormalizations. The general form of the $\tilde{g}_I(u)$ function is as follows:

$$\tilde{g}_I(u) = \left(A_I + \frac{B_I}{u} + \frac{C_I}{u^2}\right)\sqrt{1 - \frac{4}{u}}\theta(u - 4).$$ (5.120)

Therefore the integral $\int_{x/\Lambda^2}^{x/k^2} \frac{du}{u} \tilde{g}_I(u)$ amounts to $\int_4^{x/k^2} \left(\frac{A_I}{u} + \frac{B_I}{u^2} + \frac{C_I}{u^3}\right)\sqrt{1 - \frac{4}{u}}du$. This integral solved equals to

$$2A_I \log \left(\frac{1 + \sqrt{1 - \frac{4}{u}}}{2} \right) + \left(-2A_I + \frac{B_I}{6} + \frac{C_I}{60} \right) \sqrt{1 - \frac{4}{u}} +$$

$$+ \left(-\frac{2B_I}{3} + \frac{C_I}{30} \right) \frac{\sqrt{1 - \frac{4}{u}}}{u} - \frac{2C_I}{5u^2} \sqrt{1 - \frac{4}{u}} + A_I \log u \Bigg|_{u = x/k^2} . \tag{5.121}$$

It always happens, that the coefficient A_I is the negative of $g_{I,0}$. With this simplification in mind we have the following answer for the structure functions $f_{I,k}(x)$ at momentum scale k:

$$f_{I,k}(x) = \frac{1}{32\pi^2} \left\{ \left[-2A_I \log \left(\frac{1 + \sqrt{1 - \frac{4k^2}{x}}}{2} \right) + \left(2A_I - \frac{B_I}{6} - \frac{C_I}{60} \right) \sqrt{1 - \frac{4k^2}{x}} \right.\right.$$

$$+ \left(\frac{2B_I}{3} - \frac{C_I}{30} \right) \frac{k^2 \sqrt{1 - \frac{4k^2}{x}}}{x} + \frac{2C_I k^4}{5x^2} \sqrt{1 - \frac{4k^2}{x}} \tag{5.122}$$

$$\left.\left. - A_I \log \left(\frac{x}{k_0^2} \right) \right] \theta(x - 4k^2) - A_I \log \left(\frac{k^2}{k_0^2} \right) \theta(4k^2 - x) \right\} + c_I.$$

The finite renormalization constants c_I can be chosen to be equal precisely to $-\frac{1}{32\pi^2} \left(2A_I - \frac{B_I}{6} - \frac{C_I}{60} \right)$, hence we don't get any cosmological constant. We skip here the explicit form of the structure functions for each nine cases. They can be easily recovered from the general expression above, after plugging corresponding values of coefficients A_I, B_I and C_I for each value of the index I.

We are interested in the effective action $\Gamma = \Gamma|_{k=0}$. In general form the limits of form-factors in one-loop quantum effective action ($k \to 0$) are equal to

$$f_{I,0}(x) = -\frac{A_i}{32\pi^2} \log \left(\frac{x}{k_0^2} \right) = \frac{g_{I,0}}{32\pi^2} \log \left(\frac{x}{k_0^2} \right), \tag{5.123}$$

therefore the explicit form of this action is

$$\bar{\Gamma}_0 \big|_{\mathcal{R}^2} = \frac{1}{32\pi^2} \int d^4x \sqrt{g} \left\{ \frac{71}{30} R_{\mu\nu} \log \left(\frac{-\Box}{k_0^2} \right) R^{\mu\nu} + \frac{71}{60} R \log \left(\frac{-\Box}{k_0^2} \right) R + \right.$$

$$+\frac{5}{2}K^4 m^4 \phi^2 \log\left(\frac{-\Box}{k_0^2}\right)\phi^2 - 2K^2 m^4 \phi \log\left(\frac{-\Box}{k_0^2}\right)\phi \qquad (5.124)$$

$$-\frac{13}{3}K^2 m^2 R \log\left(\frac{-\Box}{k_0^2}\right)\phi^2 - \frac{1}{6}m^2 R + \frac{1}{2}m^4 + \frac{5}{2}K^4 (\nabla\phi)^2 \log\left(\frac{-\Box}{k_0^2}\right)(\nabla\phi)^2$$

$$+K^4 m^2 \phi^2 \log\left(\frac{-\Box}{k_0^2}\right)(\nabla\phi)^2 - \frac{2}{3}K^2 R \log\left(\frac{-\Box}{k_0^2}\right)(\nabla\phi)^2 - K^2 m^2 (\nabla\phi)^2\Big\},$$

where we have also added the nonlogarithmic contributions coming from constant terms proportional to mass.

We note that the coefficients in equation (5.70) are related to those in (5.124) in an algebraic way. Finally we give the shortcuts assignments, which could give us the form of the quantum effective action just from the form of $\mathrm{tr}b_4$ with non-local heat kernel for this particular choice of the cutoff. Therefore they are not universal. We saw that only $g_{I,0}$ terms contribute to quantum effective action. If knowing all of them, we don't have to do any integral over momentum. On the other hand the contributions to $g_{I,0}$ come entirely from ξ-integrals according to the following assignments:

$$\int_0^1 d\xi\, Q_0\left[h_k\left(z + x\xi(1-\xi)\right)\right] \;\to\; 2 \qquad (5.125)$$

$$\int_0^1 d\xi\, Q_1\left[h_k\left(z + x\xi(1-\xi)\right)\right] \;\to\; -\frac{1}{3} \qquad (5.126)$$

$$\int_0^1 d\xi\, Q_2\left[h_k\left(z + x\xi(1-\xi)\right)\right] \;\to\; \frac{1}{30}. \qquad (5.127)$$

On the level of expressions with functions $f(x)$ the nonvanishing contributions come only from:

$$f(x) \;\to\; 2 \qquad (5.128)$$

$$\frac{f(x)}{x} \;\to\; -\frac{1}{3} \qquad (5.129)$$

$$\frac{f(x)}{x^2} \;\to\; \frac{1}{30} \qquad (5.130)$$

and this originates from the following assignments in terms of $\phi_{1,\dots,5}$ functions:

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$$\phi_1 \rightarrow \frac{1}{30} \tag{5.131}$$

$$\phi_2 \rightarrow -\frac{1}{90} \tag{5.132}$$

$$\phi_3 \rightarrow 0 \tag{5.133}$$

$$\phi_4 \rightarrow 1 \tag{5.134}$$

$$\phi_5 \rightarrow \frac{1}{6}. \tag{5.135}$$

If we have the expression for $\operatorname{tr} b_4$ with non-local structure functions ϕ_i in monomials, the shortest way to get quantum effective action is to use the above shortcut assignments.

5.4   Flat space limit and formfactors

The goal of this section is to compute one-loop corrections to three-point vertex from quantum effective action. In the last section we computed it to the second order in operators of heat kernel and we arrived at a nonanalytic expression with low-energetic logarithms. We want to consider the simplest vertex of interaction within our theory. This is a vertex with one gravitons and two scalar field. That's why we shall compute the third variational derivative with respect to mentioned fluctuations. At the end we specify flat gravitational background and vanishing background scalar field. Such third variational derivative equals to double derivative of the matter energy-momentum tensor over scalar. We also prefer to write the expression for the vertex in the momentum space.

The tree-level action on the general spacetime is:

$$S[\phi, g] = \int d^d x \sqrt{g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right] \tag{5.136}$$

We calculate the energy momentum tensor by varying the action with respect to the metric and we find

$$\delta S[\phi, g] = \int d^d x \sqrt{g} \left[\frac{1}{4} g^{\alpha\beta} (\partial \phi)^2 - \frac{1}{2} \partial^\alpha \phi \partial^\beta \phi + \frac{1}{2} V(\phi) g^{\alpha\beta} \right] h_{\alpha\beta}. \tag{5.137}$$

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Hence the expression for the energy-momentum tensor is

$$T^{\mu\nu} = g^{\mu\nu} \left[\frac{1}{2} (\partial\phi)^2 + V(\phi) \right] - \partial^\mu\phi\partial^\nu\phi . \tag{5.138}$$

On flat Euclidean space ($\phi = 0$, $g_{\mu\nu} = \delta_{\mu\nu}$) and in momentum representation we have:

$$\left. \frac{\delta^2 T_x^{\mu\nu}}{\delta\phi_{x_1}\delta\phi_{x_2}} \right|_{\phi=0,\, g_{\mu\nu}=\delta_{\mu\nu}} \rightarrow -p_1^\mu p_2^\nu - p_1^\nu p_2^\mu - \delta^{\mu\nu} \left[-p_1 \cdot p_2 - V''(\phi) \right] . \tag{5.139}$$

In the above formula we used the following substitutions for the derivatives of delta functions in momentum space. We assume that particles 1 and 3 are ingoing, while 2 is the only one outgoing out of the considered vertex. For ingoing particles' momentum we take $\partial_{x,\alpha}\delta_{x,x_1} \rightarrow ip_{1,\alpha}$ and $\partial_{x,\alpha}\delta_{x,x_3} \rightarrow ip_{3,\alpha}$ and for outgoing $\partial_{x,\alpha}\delta_{x,x_2} \rightarrow -ip_{2,\alpha}$.

Besides this in the local part of the effective action, we have the following three types of operators

$$\mathcal{O}_0 = \int d^d x \sqrt{g} \, \phi^2 \tag{5.140}$$

$$\mathcal{O}_2 = \int d^d x \sqrt{g} \, R \, \phi^2 \tag{5.141}$$

$$\mathcal{O}_3 = \int d^d x \sqrt{g} \, R \, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{5.142}$$

and corresponding vertices in momentum space amount to

$$\frac{\delta^3 \mathcal{O}_{0,x}}{\delta\phi_{x_1}\delta\phi_{x_2}\delta g_{\mu\nu,x_3}} = \delta^{\mu\nu} , \tag{5.143}$$

$$\frac{\delta^3 \mathcal{O}_{2,x}}{\delta\phi_{x_1}\delta\phi_{x_2}\delta g_{\mu\nu,x_3}} \rightarrow 2 \left[p_3^2\delta^{\mu\nu} - p_3^\mu p_3^\nu \right] \text{ and} \tag{5.144}$$

$$\frac{\delta^3 \mathcal{O}_{3,x}}{\delta\phi_{x_1}\delta\phi_{x_2}\delta g_{\mu\nu,x_3}} \rightarrow 2 \left[p_3^2\delta^{\mu\nu} - p_3^\mu p_3^\nu \right] (p_1 \cdot p_2) . \tag{5.145}$$

We used various kinematical relations between momenta (two ingoing ones and one for outgoing graviton) to put the formulas in the above final forms.

In our quantum effective action, calculated using non-local heat kernel technique, we are interested in operators, which give nonvanishing contribution to the vertex of our interest. Such monomials must contain precisely two powers of scalar fields (may be under covariant derivatives) and not more than two gravitational curvatures. We easily see, that from (5.124), three operators satisfy this criterion. They are listed below.

$$\tilde{\mathcal{O}}_1 = -2 \int d^4x \sqrt{g} K^2 m^4 \phi \log\left(\frac{-\Box}{k_0^2}\right) \phi \tag{5.146}$$

$$\tilde{\mathcal{O}}_2 = -\frac{13}{3} \int d^4x \sqrt{g} K^2 m^2 R \log\left(\frac{-\Box}{k_0^2}\right) \phi^2 \tag{5.147}$$

$$\tilde{\mathcal{O}}_3 = -\frac{2}{3} \int d^4x \sqrt{g} K^2 R \log\left(\frac{-\Box}{k_0^2}\right) (\nabla\phi)^2 \tag{5.148}$$

The computation of 3-rd variational derivative for $\tilde{\mathcal{O}}_1$ gives as follows:

$$\frac{\delta^3 \tilde{\mathcal{O}}_{1,x}}{\delta\phi_{x_1}\delta\phi_{x_2}\delta g_{\mu\nu,x_3}} \rightarrow (-2)K^2 m^4 \left[\frac{\delta^{\mu\nu}}{2}\log\left(\frac{p_1^2 p_2^2}{k_0^4}\right) - \frac{p_2^\mu p_2^\nu}{p_1^2} - \frac{p_1^\mu p_1^\nu}{p_2^2} + \frac{p_3^\mu p_2^\nu}{2p_1^2} - \frac{p_3^\mu p_1^\nu}{2p_2^2} + \right.$$
$$\left. + \frac{p_2^\mu p_3^\nu}{2p_1^2} - \frac{p_1^\mu p_3^\nu}{2p_2^2} - \frac{\delta^{\mu\nu}(p_3 \cdot p_2)}{2p_1^2} + \frac{\delta^{\mu\nu}(p_3 \cdot p_1)}{2p_2^2}\right]. \tag{5.149}$$

We used the fact that

$$\delta \log\left(\frac{-\Box}{k_0^2}\right) = \frac{\delta\Box}{\Box} = \frac{1}{\Box}\left(-h_{\mu\nu}\nabla^\mu\nabla^\nu - (\nabla^\mu h_{\mu\nu})\nabla^\nu + \frac{1}{2}(\nabla_\alpha h)\nabla^\alpha\right), \tag{5.150}$$

where the last two terms come from the variation of the second covariant derivative acting on the scalar. To other two vertices (coming from operators $\tilde{\mathcal{O}}_2$ and $\tilde{\mathcal{O}}_3$) part with the variation of the logarithm doesn't contribute, because it is multiplied by scalar curvature R and so vanishes in flat spacetime limit. We have respectively on flat spacetime that

$$\frac{\delta^3 \tilde{\mathcal{O}}_{2,x}}{\delta\phi_{x_1}\delta\phi_{x_2}\delta g_{\mu\nu,x_3}} \rightarrow -\frac{26}{3}K^2 m^2 \delta_{x,x_1}\delta_{x,x_2}\log\left(\frac{-\Box}{k_0^2}\right)\left(-\delta^{\mu\nu}\partial^2 \delta_{x,x_3} + \partial^\mu\partial^\nu\delta_{x,x_3}\right) \quad \text{and}$$
$$\tag{5.151}$$

$$\frac{\delta^3 \tilde{\mathcal{O}}_{3,x}}{\delta\phi_{x_1}\delta\phi_{x_2}\delta g_{\mu\nu,x_3}} \rightarrow -\frac{4}{3}K^2 \partial_\alpha \delta_{x,x_1}\partial^\alpha \delta_{x,x_2}\log\left(\frac{-\Box}{k_0^2}\right)\left(-\delta^{\mu\nu}\partial^2\delta_{x,x_3} + \partial^\mu\partial^\nu\delta_{x,x_3}\right) \quad (5.152)$$

We integrate by parts in the above two expressions to flip the logarithm of box operator to act only on δ_{x,x_3} (third particle). This is justified by the conservation of momentum for the vertex and we perturb around flat spacetime, where ordinary momentum is conserved. Corresponding vertices exhibit similar structure to (5.144) and (5.145) multiplied by two characteristic logarithms. Namely we find that

$$\tilde{\mathcal{O}}_2 \rightarrow -\frac{26}{3}K^2 m^2 \left[p_3^2 \delta^{\mu\nu} - p_3^\mu p_3^\nu\right]\log\left(\frac{p_3^2}{k_0^2}\right) \quad (5.153)$$

$$\tilde{\mathcal{O}}_3 \rightarrow -\frac{4}{3}K^2 \left[p_3^2 \delta^{\mu\nu} - p_3^\mu p_3^\nu\right](p_1 \cdot p_2)\log\left(\frac{p_3^2}{k_0^2}\right) . \quad (5.154)$$

There is also a local nonlogarithmic term in the quantum effective action $-\int d^4x\sqrt{g}K^2 m^2 (\nabla\phi)^2$, which is of our interest. Corresponding to it vertex has the following structure:

$$-K^2 m^2 \left[\delta^{\mu\nu}(p_1 \cdot p_2) - 2p_1^{(\mu}p_2^{\nu)}\right] = -K^2 m^2 \left[-2P^\mu P^\nu + m^2\delta^{\mu\nu} - \frac{1}{2}\left(q^2\delta^{\mu\nu} - q^\mu q^\nu\right)\right] \quad (5.155)$$

Summing all these contributions we can write the form of three-point vertex coming from our form of quantum nonlocal effective action to one loop.

$$\begin{aligned}
\Gamma^{(2,1)}_{p_1,p_2,p_3}[0,\delta]^{\mu\nu} &= -\frac{K^2}{32\pi^2}\left\{\delta^{\mu\nu}m^4\left[1 + 4\log\left(\frac{m}{k_0}\right)\right] - 6m^2 P^\mu P^\nu \right.\\
&\quad + \left. \left[q^2\delta^{\mu\nu} - q^\mu q^\nu\right]\left(-\frac{3}{2}m^2 + \left(10m^2 - \frac{2}{3}q^2\right)\log\left(\frac{q^2}{k_0^2}\right)\right)\right\}, \quad (5.156)
\end{aligned}$$

where we defined a momentum transfer fourvector $q = p_3 = p_2 - p_1$ and a characteristic momentum of the process $P = \frac{1}{2}(p_1 + p_2)$. Moreover we used on-shell conditions for scalar lines.

The most general form of the three-point vertex with two scalars and one graviton is on the flat spacetime tightly constrained by Poincarè symmetry. Additional requirement

is put by the transversality of the vertex function, when one contracts with one index on the graviton field. This leads to the expression

$$\Gamma^{(2,1)}_{p_1,p_2,p_3}[0,\delta]^{\mu\nu} = -\frac{K^2}{32\pi^2}\left\{2P^\mu P^\nu F_1(q^2) + \left[q^2\delta^{\mu\nu} - q^\mu q^\nu\right]F_2(q^2)\right\},\tag{5.157}$$

where the formfactors $F_1(q^2)$ and $F_2(q^2)$ appeared as functions of only invariant quantity q^2. Now comparing above formula with (5.156) we get the explicit form of the gravitational form-factors $F_1(q^2)$ and $F_2(q^2)$:

$$F_1(q^2) = -3m^2\tag{5.158}$$
$$F_2(q^2) = -\frac{3}{2}m^2 + \left(10m^2 - \frac{2}{3}q^2\right)\log\left(\frac{q^2}{m^2}\right)\tag{5.159}$$

We also set the reference scale equal to the mass of the scalar $k_0 = m$ and neglect the constant term proportional to $\delta^{\mu\nu}$.

It is necessary to continue the same computation, but for the third order in generalised heat kernel curvatures. The reason for this is that the simplest vertex in interacting theory must contain three lines and hence it corresponds to the third variational derivative. When we set the background fields to vanishing values, the contribution from the third order doesn't vanish. Only for the order of derivatives higher than three, we have basically no contribution to three-point vertex. Additionally the mass parameter of the scalar particle must be treated exactly to all orders. However it seems, that the second order computation is not enough to capture the full result and this is only a part of the final result. Here for completeness we show the results for form-factors computed by other methods in perturbative effective field theory of gravity.

$$F_1(q^2) = 1 + \frac{K^2}{32\pi^2}q^2\left(-\frac{3}{4}\log(-q^2) + \frac{1}{16}\frac{\pi^2 m}{\sqrt{-q^2}}\right)\tag{5.160}$$
$$F_2(q^2) = \frac{K^2}{32\pi^2}m^2\left(-\frac{4}{3}\log(-q^2) + \frac{7}{8}\frac{\pi^2 m}{\sqrt{-q^2}}\right)\tag{5.161}$$

This was the result of one-loop computation carried out using Feynman diagram technique and first reported in [95].

Part III

Chapter 6

Conclusions and summary

In this thesis we touched on many issues. These topics may seem to be unrelated, however the common point is their relation to Quantum Gravity and Renormalization Group methods. Despite the fact, that Quantum Gravity is very vast field of research, we tried to concentrate on this approach to it, which uses RG methods. We also attempted to show this particular approach as seen from different perspectives. This is the reason, why we studied holography, classicalization and effective field theory of gravitational interactions. This opens up the possibility that these powerful machineries could be brought to bear on the issue of asymptotic safety. Here we want to summarise, what was obtained in this research program and described in this thesis.

In the first part we concentrated on relations between holographic and exact functional RG flows. By considering simple Randall-Sundrum setup, with AdS_5 spacetime in the bulk, we were able to find agreement between two flows. The common similarities of the flows were noticed for matter as well as for gravitational couplings. We found, that the best agreement was at high energy in 4 dimensions, where holographic AdS spacetime corresponded to our theory under RG flow in the vicinity of the nontrivial fixed point. However to account for threshold phenomena in the infrared limit, we had to modify holographic flow by introducing some sources. Then we went on constructing a 5-dimensional holographic model, which must be understood as a geometrization of the 4-dimensional RG

flow in the theory living on a brane. We achieved this by adding minimally coupled scalar field to Einstein-Hilbert gravitation in the holographic bulk and solving resulting system of classical coupled equation of motion. We found implicitly scalar profile for every RG flow of gravitational coupling and for particular interpolating RG flow explicitly. Due to the nature of running of 4-dimensional Planck mass, we had to choose a scalar field with negative kinetic term in 5d action. In this way we discover a fully-fledged 5-dimensional description of the physics described in different language by 4d theory, where also gravitation was present and dynamical. This was the novelty of this work. In a sense we used holography in a very similar way like it is done for matter couplings in the framework of AdS/CFT correspondence. Next using ideas from this conjecture we were able to derive some interesting facts about gravitational RG flows and asymptotic safety in the ultraviolet limit.

We devoted the fourth chapter for studying the phenomenon of classicalization. Our target model was a nonlinear sigma model, which shares a lot of common features with 4-dimensional gravity, but at the same time is much simpler. We studied maximally symmetric target spaces with positive and negative curvatures. The results for classicalization depended strongly on the sign of the curvature and also on the number of derivatives present in the action. For 2-derivatives (only nonlinear kinetic term) and model on a sphere, we presented evidences in favour of weak classicalization. Our analysis of model on a hyperboloid was inconclusive. In the case of four derivatives, we noticed the occurrence of strong (standard) classicalization in similarity with model of single Goldstone bosons. At the end we motivated the conjecture that weak classicalization is related to asymptotic safety, because both have quantum origin.

In the last main chapter the issue of low-energetic quantum gravitational theory was discussed. We recalled the significance and origin of the local and non-local terms present in the quantum effective action. We concentrated on the latter, because they are universal genuine prediction at low energy. We obtained the first few terms in an expansion in powers of curvature of the quantum effective action for the system of minimally gravitationally coupled scalar field in four spacetime dimension using a novel method. Namely

we integrated the flow of the effective average action over RG trajectory from UV down to IR. Hence we derived the equations for non-local formfactors in quantum effective action. The last step consisted of taking the flat spacetime limit in obtained covariant quantum action and deriving the form of the simplest vertex with the inclusion of one-loop quantum corrections. In this way we got flat spacetime formfactors of the gravitational interactions with scalars and were able to compare them with perturbative computation, which used Feynman diagrams techniques.

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