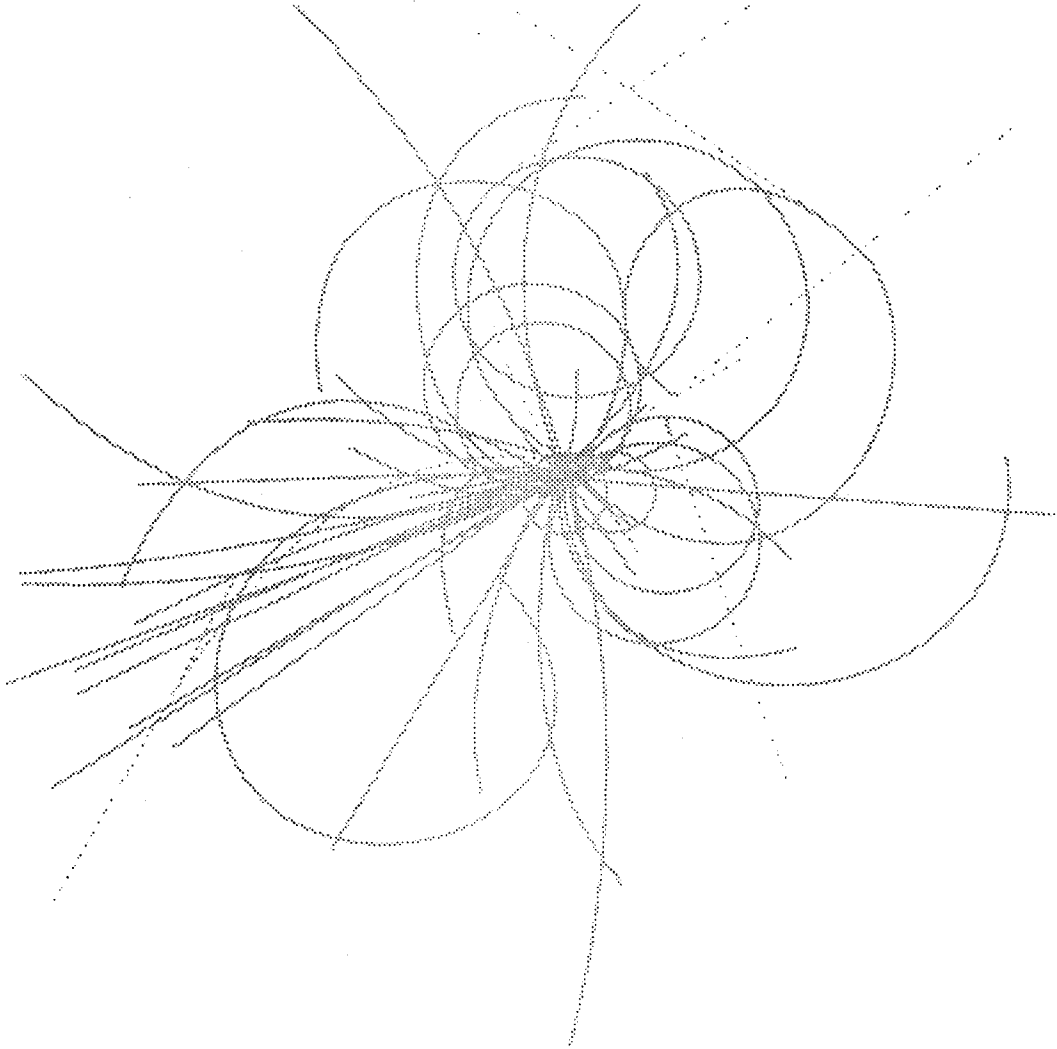


Superconducting Super Collider Laboratory



A Model for Local Current Decay in a Superconducting LR Circuit Chain

T. Garavaglia and S. Kauffmann

May 1991

**A Model For Local Current Decay
In A Superconducting LR Circuit Chain**

T. Garavaglia and S. K. Kauffmann

Superconducting Super Collider Laboratory, Dallas, Texas 75237*

May 1991

Solutions are given for the decay of the local current in both a finite and an infinite superconducting circuit consisting of a linear array of loops, each containing in series an inductance L and in parallel a resistance R . Numerical results obtained from these solutions are given, and the results for local current decay for both cases are compared. The properties of polynomials associated with the finite chain solution and two generating functions, along with the method of solution of the diffusion-type equation associated with the infinite chain model, are discussed in the appendices. These solutions are used to obtain insight into the nature of the time decay of the currents in the loops resulting from crossings of the strands forming the superconducting cable of the Superconduction Super Collider (SSC) dipole magnets.

I. INTRODUCTION

Within the SSC superconducting dipole magnet an unwanted persistent current survives after current powering the magnetic field has been reduced to injection values. This results in a residual magnetic field which contributes to multipole errors in the field of the dipole magnet. These errors, along with the usual multipole errors, can severely affect the dynamic aperture for the proton beam during the injection phase into the SSC. The effects of magnetic field errors on the dynamic aperture during the injection phase of the SSC have been investigated in a number of recent works [1] where it was seen for betatron oscillations in the non-uniform field regions that chaotic dynamical behavior as well as particle loss can result. It is well known that persistent currents within superconducting magnets lead to undesirable magnetic field distortion effects [2,3]. Recently, an additional magnetic field distortion effect, related to superconducting currents, has been observed [4], where a periodic pattern occurs along the magnet axis in the measurement of the sextupole and dipole components of the HERA dipole magnet. This periodic pattern is seen to be compatible with the transposition pitch of the Rutherford-type cable used to wind the dipole coils. It is observed that the amplitude of this periodic component is sensitive to the number of powering cycles through which the magnet has been taken and to the time duration for which the highest field is maintained. Even with zero current, this periodic pattern can persist for as long as 15 hours. Furthermore, slight shifts have been observed in the phase of oscillation of these periodic magnetic fields. It would be desirable to remove these currents as quickly as possible before an injection cycle. One method which may be useful for producing decay of the currents responsible for the periodic pattern would be to expose a portion of the superconducting cable to high temperature, resulting in inducing a resistance R_T into this part of the circuit.

In this paper a mathematical model is described and solved which gives some insight into this behavior. In this model, one end of a superconducting LR chain circuit carrying an initial current I_0 is opened, and expressions for the local current decay along the chain are obtained. Opening the circuit corresponds to placing a resistance $R_T \rightarrow \infty$ across one end of the circuit. Solutions showing the local current decay along the chain are found for both a finite chain and an infinite chain.

The remainder of this paper contains the detailed development of the two forms of this model. In Sec. II, the model for the finite chain is described, and the resulting solution for this case is given in Sec. III. In Sec. IV, the system of equations relevant to the finite chain is modified so as to produce a diffusion-type partial differential equation to represent the current behavior for an infinite chain, and the resulting solution for this equation is presented. Numerical results and a comparison of the results for both the finite chain and the infinite chain are given in Sec. V. The final section contains concluding remarks and a result obtained from the solution for the infinite chain which can be used to estimate the half-life decay time for the local currents along the chain. A mathematical discussion regarding the polynomials which occur in the solution for the finite chain and related generating functions is found in Appendix A. An explicit description of the method and of the solution for a finite chain which contains three circuit loops is given in Appendix B. This is done to illustrate the general method used to find the solution for a finite chain. In Appendix C, a method using a Green's function to obtain the solution for the infinite chain is given. This is an alternative to the method of solution described in Sec. IV.

II. MATHEMATICAL MODEL

The model consists of a pair of crossing superconducting strands to represent the

linear lattice produced by crossings of the strands in the strand array of a superconducting cable [5]. In the manufacture of the cable, initially an array of parallel strands is arranged to have the appearance of a flat strip. This strip is then folded, producing a number of crossings of the parallel strands. The geometry of these crossings is such that an array of loops is formed. The array for two parallel crossing strands can be represented as shown in FIG. 1. In the model used here, the two strands are shorted at both ends to form initially a closed circuit with a superconducting current I_o . If at time $t = 0_+$ a resistance R_T is placed across one end, currents will begin to flow within the loops formed by the crossing of the strands. It is assumed that a resistance appears across the copper-copper contact produced by each crossing of the strands. The strands cross in a regular periodic manner forming an array of loops. Each loop is characterized by an inductance L and, at each end, a resistance R resulting from the copper-copper contact. The circuit array for this geometry is shown in Fig. 2. This circuit has at the left end a time-dependent resistance of the form $R_T\theta(t)$ which is zero for $t < 0$ and equal to constant R_T for $t > 0$. The function $\theta(t)$ is the usual Heaviside step function. For $t < 0$, a superconducting current I_o flows through the outer loop, and it does not flow through the resistances R resulting from the high pressure contact of the strands.

III. SOLUTION FOR THE FINITE CHAIN

The voltage drop equations for this system of loops shown in Fig. 2 are

$$\begin{aligned}
 L\dot{I}_1(t) + R_T\theta(t)I_1(t) + R(I_1(t) - I_2(t)) &= 0 \\
 L\dot{I}_k(t) + R(I_k(t) - I_{k-1}(t)) + R(I_k(t) - I_{k+1}(t)) &= 0 \\
 (k = 2, 3, \dots, n-1) \\
 L\dot{I}_n(t) + R(I_n(t) - I_{n-1}(t)) &= 0.
 \end{aligned} \tag{3.1}$$

For $t < 0$, the solution to the above system of equations is

$$I_1(t) = I_2(t) = \dots = I_n(t) = I_o, \tag{3.2}$$

where I_o is the initial constant superconducting current in the closed outer loop. For $t \geq 0$, the Laplace transformed system of equations with the initial conditions (3.2) becomes

$$\begin{aligned}
 LsI_1(s) + R_T I_1(s) + R(I_1(s) - I_2(s)) &= LI_o \\
 LsI_k(s) + R(2I_k(s) - I_{k-1}(s) - I_{k+1}(s)) &= LI_o \\
 (k = 2, 3, \dots, n-1) \\
 LsI_n(s) + R(I_n(s) - I_{n-1}(s)) &= LI_o,
 \end{aligned} \tag{3.3}$$

where the Laplace transform of the current $I(t)_k$ is defined by

$$I(s)_k = \int_0^\infty I(t)_k e^{-st} dt. \tag{3.4}$$

It follows from the initial conditions (3.2) and the system of equations (3.3) that

$$I_k(s) = I_o/s, k = 1, 2, \dots, n. \tag{3.5}$$

The system of equations (3.3), along with the initial conditions (3.5), is equivalent to the system obtained from the first equation of (3.3) and from the relation

$$I_{n-k}(s) = P_k(x)I_n(s) - \left(\frac{P_k(x) - 1}{x}\right) \left(\frac{L}{R}\right) I_o, \tag{3.6}$$

where $k = 0, 1, 2, \dots, n - 1$ and where $x = (Ls/R)$. The polynomials $P_k(x)$ are defined by the relations

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x + 1 \\ P_{k+1}(x) &= (x + 2)P_k(x) - P_{k-1}(x) \\ k &= 1, 2, 3, \dots, n. \end{aligned} \tag{3.7}$$

The properties of these polynomials, along with two different generating functions, are described in Appendix A.

Although the system of equations (3.3) can be solved for finite R_T , the solution of practical interest in this paper is for the case when the superconducting circuit is opened at one end corresponding to the limit $R_T \rightarrow \infty$. For finite R_T the system of equations (3.3) can be solved with the aid of (3.6) which can be used to express $I_1(s), I_2(s), \dots, I_{n-1}(s)$ in terms of $I_n(s)$. The first of equations (3.3) is then used to find an expression for $I_n(s)$ which can be substituted again into (3.6). The case described in detail here is solved in the limit $R_T \rightarrow \infty$ which corresponds to an open circuit and which requires from the first of equations (3.3) that $I_1(s) = 0$. For this case the recursion relation (3.6) can be solved to give the Laplace transform of the current in the m^{th} loop of a finite chain of n loops as

$$I^n(s)_m = [1 - P_{n-m}(x)/P_{n-1}(x)]I_o/s. \tag{3.8}$$

This expression is found when $k = n - 1$ is substituted into (3.6) so that an expression for $I_n(s)$ is found when $I_1(s) = 0$. This result is substituted into (3.6) to find an expression for $I_{n-k}(s)$, and, with the replacement $k = n - m$, one finds (3.8). The time-dependent current in the m^{th} loop is found from the inverse Laplace transform,

$$I^n(t)_m = (1/2\pi i) \int_{\sigma-i\infty}^{\sigma+i\infty} I^n(s)_m e^{st} ds. \tag{3.9}$$

The poles of the integrand are the zeros x_j of the polynomial $P_{n-1}(x)$, and these roots are all real and negative. If an integration contour is chosen to enclose these poles, then the time-dependent current in the m^{th} loop is found to be

$$I^n(t)_m = \sum_{j=1}^{n-1} Res(x_j) e^{x_j t}. \quad (3.10)$$

The residue values are found from

$$Res(x_j) = \lim_{x \rightarrow x_j} (x - x_j) [1 - P_{n-m}(x)/P_{n-1}(x)] \frac{I_o L}{R x}. \quad (3.11)$$

To illustrate the solution method given in this section, an example is completely solved for the case with $n = 3$ (see Appendix B).

IV. THE INFINITE CHAIN

The results for an infinite chain can be found from the second of equations (3.1) when it is written in the form

$$\dot{I}_k(t) + (R/L)(\Delta x)^2 \left(\frac{2I_k(t) - I_{k-1}(t) - I_{k+1}(t)}{(\Delta x)^2} \right) = 0, \quad (4.1)$$

where Δx is the length of a loop. In the limit of small Δx , (4.1) becomes a diffusion equation

$$\frac{\partial^2 I(x, t)}{\partial x^2} - (L/R(\Delta x)^2) \frac{\partial I(x, t)}{\partial t} = 0, \quad (4.2)$$

where x is the linear position along the chain. The value of x at the beginning of the m^{th} loop is

$$x = (m - 1)\Delta x. \quad (4.3)$$

The boundary conditions for the partial differential equation (4.2) are

$$\begin{aligned} I(x, 0) &= I_o \\ I(0, t) &= 0, \quad t > 0. \end{aligned} \quad (4.4)$$

The general solution of (4.2) with these boundary conditions is of the form

$$I(x, t) = \int_{-\infty}^{\infty} A(k) \sin(kx) e^{-at} dk, \quad (4.5)$$

where

$$\begin{aligned} A(k) &= I_o / \pi k \\ a &= k^2 (R/L) (\Delta x)^2. \end{aligned} \quad (4.6)$$

The integral (4.5) can be evaluated if one first evaluates the partial derivative of $I(x, t)$ with respect to x and then evaluates the resulting integral by integrating with respect to k . The final result is obtained upon integration with respect to x . The partial derivative produces the integral

$$\frac{\partial I(x, t)}{\partial x} = (I_o / \pi) \int_{-\infty}^{\infty} \cos(kx) e^{-k^2 / 2 \sigma^2} dk, \quad (4.7)$$

where

$$\begin{aligned} \sigma &= 1 / \sqrt{2\tau} (\Delta x) \\ \tau &= (R/L)t. \end{aligned} \quad (4.8)$$

This integral can be evaluated with the aid of the identity

$$\cos(kx) = (e^{ikx} + e^{-ikx}) / 2 \quad (4.9)$$

and the Fourier transform

$$\int_{-\infty}^{\infty} e^{\pm ikx - k^2 / 2 \sigma^2} dk = \sqrt{2\pi} \sigma e^{-\sigma^2 x^2 / 2}. \quad (4.10)$$

As a result, one finds for the normalized current along the chain the expression

$$I(x, t) / I_o = (2 / \sqrt{\pi}) \int_0^{\sigma x / \sqrt{2}} e^{-y^2} dy = \operatorname{erf}(x / 2 \Delta x \sqrt{\tau}). \quad (4.11)$$

This result can be compared with (3.10) for the finite chain when (4.3) is used to replace x .

V. NUMERICAL RESULTS

Numerical results for both the finite and the infinite chain can be obtained from the Fortran code EIGDIF.FOR which requires values m , the specific loop considered, and n , the number of loops in a chain. Graphs as a function of the dimensionless quantity $\tau = (R/L)t$ for the normalized local currents $I^n(\tau)_m/I_o$ for the finite chain and $I(\tau)_m/I_o$ for the infinite chain are produced in the Top Drawer [6] files FINCHN.TOP and INFCHN.TOP respectively. The Fortran code EIGDIF.FOR makes use of the CERN Program Library [7] subroutines EISST1 and ERF. The roots of the polynomials are found by determining the eigenvalues of the real $(n-1) \times (n-1)$ symmetric tri-diagonal matrix

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & -1 & 1 \end{pmatrix} \quad (5.1)$$

resulting from the system of equations (3.1) which can be written as

$$\mathbf{M}\mathbf{I}(t) = -(L/R)d\mathbf{I}(t)/dt, \quad (5.2)$$

where $\mathbf{I}(t)$ is the $n-1$ dimensional vector

$$\mathbf{I}(t) = \begin{pmatrix} I_2(t) \\ I_3(t) \\ \vdots \\ I_n(t) \end{pmatrix}. \quad (5.3)$$

When the circuit is opened at the beginning of the first loop, one has $I(t)_1 = 0$ for

$t > 0$. At $t = 0$, $\mathbf{I}(t = 0)$ is the $n - 1$ dimensional vector

$$\mathbf{I}(t = 0) = I_o \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}. \quad (5.4)$$

Assuming for (5.2) a solution of the form

$$\mathbf{I}(t) = \mathbf{I}e^{st},$$

one finds the eigenvalues for this system are the roots of the polynomial $P_{n-1}(x)$ defined by (3.7).

Examples of the decay as a function of τ of the normalized local currents for both the finite chain and the infinite chain are given in Figs. 3 through 12, where excellent numerical agreement is seen for the decay of the normalized local current in the m^{th} loop for both types of chains when the condition $1 < m \ll n$ is satisfied. Fig. 3 shows the normalized local current decay as a function of τ in the second loop, $m = 2$, for the case of the finite chain with $n = 200$ loops. This is to be compared with the result shown in Fig. 4 for the decay of the normalized local current in the second, $m = 2$, loop of an infinite chain. One can see that there is good agreement between these results except for values of τ near zero. This is to be expected as a result of the two different representations of the second derivative of the local current used in the two different forms of the model, where the finite difference representation appropriate for the finite chain is replaced with a continuous representation of the second derivative in the infinite chain.

Fig. 5 for the finite chain and Fig. 6 for the infinite chain compare the results for the case of the decay of the normalized local current for the loop with $m = 25$

when the finite chain has $n = 200$. The excellent agreement seen between these two results is expected because the number of loops beyond 25 for the finite chain is well approximated by an infinite chain.

For the case with $m = 50$ and $n = 400$, one sees in Fig. 7 for the finite chain and in Fig. 8 for the infinite chain good agreement except for large values of τ where the normalized local current in the finite chain decays more rapidly than in the infinite chain; however, these differences start to appear only after 80% of the local current has decayed. Similar results are seen for the case $m = 100$ and $n = 800$ as shown in Fig. 9 for the finite chain and in Fig. 10 for the infinite chain. Here the agreement is excellent, and the results display a scaling behavior when compared with Fig. 3 through 8. Larger differences are expected to be seen when the results for the finite chain with $n = 800$ (Fig. 11) and the infinite chain (Fig. 12) are compared for the case where the local current decay in the $m = 700$ loop occurs in a region approaching the end of the finite chain. This is a case where the approximation of an infinite number of loops beyond the loop of observation is not valid. However, this case which shows large differences provides an opportunity to test experimentally the validity of the solutions associated with the finite chain and with the infinite chain.

VI. CONCLUSIONS

In conclusion, one can see that the approximations associated with an infinite chain give a good representation of the behavior of the decay of the local current if the chain is such that there are a large number of loops remaining in the finite chain beyond the local loop of consideration. The principal approximation made for the infinite chain is that the second finite difference equation appearing in the recursive loop equations (3.1) for the finite chain can be made to represent the second derivative

of the local current $I(x, t)$ in the infinite chain. Additionally, in the infinite chain model, one neglects the differences in the loop equations for the end loops. These end loops are important for determining the eigenvalues associated with the matrix (5.1) which give the roots appearing in the finite chain result (3.10).

An important result of the infinite chain approximation is the prediction of the half-life, $t_{1/2}$, for the decay of the local current in the m^{th} loop. This can be found from (4.11) when $I((m - 1)\Delta x, \tau_{1/2})/I_o = 1/2$. This leads to the formula for the decay half-life

$$t_{1/2} = (L/R)\tau_{1/2} = (L/R)((m - 1)/0.96)^2. \quad (6.1)$$

Estimates of the loop inductance L and of the loop resistance R have been made for the SSC superconducting strands, and these values [8] are $L = 4.5 \times 10^{-8} \text{ hy}$ and $R = 7 \times 10^{-6} \Omega$. This gives an estimate for the half-life for the local current in the m^{th} loop as

$$t_{1/2} = 0.007(m - 1)^2 \text{ s}. \quad (6.2)$$

As a final remark, it should be noted that the model of two crossing strands with resistances appearing at the copper-copper contacts as described in Sec. II is a great simplification of the complexity of the many strands and crossings that are actually in a superconducting cable; however, it appears reasonable to assume that the principal contributions to the magnetic field will result from the nearest neighbour current loops directed along the length of the superconducting cable and in the principal direction of current flow. These loops result from the crossing of the parallel strands which make up the cable. Conclusions regarding usefulness of this model must await experimental measurements of the magnetic field decay along the SSC superconducting cable. However, measurement of the decay times of the local

currents along a lattice of the type shown in Fig. 2 formed from two crossing superconducting strands would be of interest for a better understanding of the properties of superconducting circuits.

ACKNOWLEDGEMENTS

We are grateful to R. Stiening for suggesting the need for this study and to R. Meinke for comments related to the nature of the periodic pattern in the persistent current field of the HERA dipole magnets.

APPENDIX A: POLYNOMIAL PROPERTIES AND GENERATING FUNCTIONS

The polynomials defined by (3.7) can be found from the generating function

$$G(\alpha, x) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} P_n(x). \quad (\text{A.1})$$

The generating function satisfies the conditions

$$G(0, x) = 1, \quad (\text{A.2})$$

$$G(\alpha, 0) = e^\alpha, \quad (\text{A.3})$$

and

$$\left(\frac{\partial}{\partial \alpha} G(\alpha, x) \right)_{\alpha=0} = x + 1. \quad (\text{A.4})$$

Using the recursion relation(3.7) to form the sum

$$\sum_{k=1}^{\infty} P_{k+1}(x) \frac{\alpha^k}{k!} = (x+2) \sum_{k=1}^{\infty} P_k(x) \frac{\alpha^k}{k!} - \sum_{k=1}^{\infty} P_{k-1}(x) \frac{\alpha^k}{k!}, \quad (\text{A.5})$$

one observes that

$$\begin{aligned} & \frac{\partial}{\partial \alpha} (G(\alpha, x) - 1 - (x+1)\alpha) \\ &= (x+2)(G(\alpha, x) - 1) - \int_0^\infty d\alpha' G(\alpha', x). \end{aligned} \quad (\text{A.6})$$

After differentiation with respect to α , this becomes the second order differential equation

$$\frac{\partial^2}{\partial \alpha^2} G(\alpha, x) - (x+2) \frac{\partial}{\partial \alpha} G(\alpha, x) + G(\alpha, x) = 0. \quad (\text{A.7})$$

The solution to this equation satisfying the conditions (A.2) through (A.4) may be written in the form

$$G(\alpha, x) = A(x)e^{\alpha b+(x)} + (1 - A(x))e^{\alpha b-(x)}. \quad (\text{A.8})$$

Substituting this form of the solution into the differential equation (A.7), one finds that $b_{\pm}(x)$ is a solution to the quadratic equation

$$b_{\pm}^2(x) - (x + 2)b_{\pm}(x) + 1 = 0. \quad (A.9)$$

The condition (A.4) is used to find

$$A(x) = \frac{1}{2} + \frac{1}{2} \left(\frac{x}{x+4} \right)^{1/2}. \quad (A.10)$$

The final form of $G(\alpha, x)$ is

$$G(\alpha, x) = e^{\alpha[\frac{x+2}{2}]} [\cosh((\alpha/2)\sqrt{x(x+4)}) + \sqrt{\frac{x}{x+4}} \sinh((\alpha/2)\sqrt{x(x+4)})]. \quad (A.11)$$

The polynomials $P_k(x)$ are found from

$$P_k(x) = \left(\frac{1}{k!} \frac{\partial^k}{\partial \alpha^k} G(\alpha, x) \right)_{\alpha=0}. \quad (A.12)$$

The first seven polynomials are

$$\begin{aligned}
P_0(x) &= 1. \\
P_1(x) &= X + 1. \\
P_2(x) &= X^2 + 3.X + 1. \\
P_3(x) &= X^3 + 5.X^2 + 6.X + 1. \\
P_4(x) &= X^4 + 7.X^3 + 15.X^2 + 10.X + 1. \\
P_5(x) &= X^5 + 9.X^4 + 28.X^3 + 35.X^2 + 15.X + 1. \\
P_6(x) &= X^6 + 11.X^5 + 45.X^4 + 84.X^3 + 70.X^2 + 21.X + 1.
\end{aligned} \tag{A.13}$$

The roots of these polynomial are between 0 and -4 , and they tend to cluster near -4 . A graph of the polynomial $P_{40}(x)$ illustrating this property is shown in Fig. 13.

These polynomials are related to the Chebyshev polynomials of the second kind [9]. They satisfy the relations

$$\begin{aligned}
P_n(x) &= U_n(y) - U_{n-1}(y), \quad n \geq 1 \\
U_n(y) &= \sum_0^n P_n(x), \quad x = 2y - 2,
\end{aligned} \tag{A.14}$$

with

$$P_n(0) = 1, \quad U_n(1) = n + 1, \quad P_0(x) = U_0(x) = 1. \tag{A.15}$$

They can also be obtained from the generating function

$$G_2(\beta, x) = \frac{(1 - \beta)}{1 - (x + 2)\beta + \beta^2} = \sum_{n=0}^{\infty} P_n(x)\beta^n. \tag{A.16}$$

APPENDIX B: EXAMPLE FOR THE FINITE CHAIN

An example of the results for the finite chain can be used to illustrate the general method described in Sec. III. This is done by considering the case for $n = 3$. The equations for $t > 0$ from (3.1) are

$$\begin{aligned} I^3(t)_1 &= 0 \\ L\dot{I}^3(t)_2 + R(2I^3(t)_2 - I^3(t)_1 - I^3(t)_3) &= 0 \\ L\dot{I}^3(t)_3 + R(I^3(t)_3 - I^3(t)_2) &= 0. \end{aligned} \tag{B.1}$$

The Laplace transformed solutions for $m = 1, 2$, and 3 are

$$\begin{aligned} I^3(s)_1 &= 0 \\ I^3(s)_2 &= \left[\frac{(Ls/R) + 2}{(Ls/R)^2 + 3(Ls/R) + 1} \right] (LI_o/R) \\ I^3(s)_3 &= \left[\frac{(Ls/R) + 3}{(Ls/R)^2 + 3(Ls/R) + 1} \right] (LI_o/R). \end{aligned} \tag{B.2}$$

The roots of $P_2(x)$ are at

$$x_{\pm} = -\frac{3 \pm \sqrt{5}}{2}. \tag{B.3}$$

Using (3.10), the time-dependent solutions are found to be

$$\begin{aligned} I^3(t)_1 &= 0, \\ I^3(t)_2 &= \left[(\sqrt{5} - 1)e^{-(\frac{3+\sqrt{5}}{2})(Rt/L)} + (\sqrt{5} + 1)e^{-(\frac{3-\sqrt{5}}{2})(Rt/L)} \right] (I_o/2\sqrt{5}), \end{aligned}$$

and

$$I^3(t)_3 = \left[(\sqrt{5} - 3)e^{-(\frac{3+\sqrt{5}}{2})(Rt/L)} + (\sqrt{5} + 3)e^{-(\frac{3-\sqrt{5}}{2})(Rt/L)} \right] (I_o/2\sqrt{5}). \tag{B.4}$$

It is seen that these reproduce the correct initial conditions $I^3(0)_2 = I^3(0)_3 = I_o$ and that the loop equations (B.1) are satisfied.

APPENDIX C: GREEN'S FUNCTION FOR THE DIFFUSION EQUATION

The diffusion equation (4.2) has a solution that can be written in terms of a Green's function $G(x, x', t)$ such that

$$I(x, t) = \int_{-\infty}^{\infty} G(x, x', t) I(x', 0) dx'. \quad (C.1)$$

The Green's function is a solution of the differential equation

$$-(R(\Delta x)^2/L) \frac{\partial^2 G(x, x', t)}{\partial x^2} + \frac{\partial G(x, x', t)}{\partial t} = \delta(x - x') \delta(t). \quad (C.2)$$

It is easy to see with the replacements $t \rightarrow it$, and $1/2m = R(\Delta x)^2/L$ that (4.2) can be transformed into the free particle Schrödinger equation ($\hbar = 1$) and that (C.2) can be transformed into the corresponding Green's function equation. The Green's function for this case can be found from the identity

$$\int_{-\infty}^{\infty} (x|U(t)\hat{x}U(-t)|x'')(x''|U(t)|x') dx'' = x'(x|U(t)|x'). \quad (C.3)$$

The Green's function for this Schrödinger equation is

$$\mathcal{G}(x, x', t) = (x|U(t)|x')\theta(t) \quad (C.4)$$

with

$$U(t) = e^{-it\hat{p}^2/2m} \quad (C.5)$$

and with $\hat{p} = -i\partial/\partial x$. From the commutation relation

$$[\hat{x}, \hat{p}] = i, \quad (C.6)$$

one finds, using

$$\hat{x}(t) = U(t)\hat{x}(0)U(-t) = \hat{p}t/m + \hat{x}(0), \quad (C.7)$$

$$(x|\hat{x}(0)|x'') = \delta(x - x'')x,$$

and

$$(x|\hat{p}(0)|x'') = \delta(x - x'')(-i\partial/\partial x), \quad (C.8)$$

the differential equation

$$-i\frac{\partial}{\partial x}\mathcal{G}(x, x', t) = (m/t)(x - x')\mathcal{G}(x, x', t). \quad (C.9)$$

The solution of this differential equation can be written as

$$\mathcal{G}(x, x', t) = \mathcal{G}(0, x', t)e^{i(m/2t)(x^2 - 2xx')}. \quad (C.10)$$

The Green's function $\mathcal{G}(x, x', t)$ and its complex conjugate satisfy the property

$$\mathcal{G}(x', x, t) = \mathcal{G}^*(x, x', -t) \quad (C.11)$$

so that

$$\mathcal{G}(x, x', t) = \mathcal{G}(0, 0, t)e^{i(m/2t)(x-x')^2}. \quad (C.12)$$

The value of $\mathcal{G}(0, 0, t)$ is found upon using the momentum space representation to write

$$\mathcal{G}(0, 0, t) = \mathcal{G}(x, x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(p^2/2m)t} dp. \quad (C.13)$$

When integrated this gives the result for the free particle Green's function

$$\mathcal{G}(x, x', t) = \sqrt{\frac{m}{2\pi it}} e^{i(m/2t)(x-x')^2} \theta(t). \quad (C.14)$$

This result can now be used to obtain the Green's function for the diffusion equation (4.2) so that the solution satisfies the boundary conditions (4.4). Upon

making the replacement for m as before and $it \rightarrow t$, the Green's function appropriate for the boundary conditions is found to be

$$G(x, x', t) = \frac{1}{\sqrt{2\pi\bar{\sigma}}} \left(e^{-\frac{(x-x')^2}{2\bar{\sigma}^2}} - e^{-\frac{(x+x')^2}{2\bar{\sigma}^2}} \right) \theta(t) \quad (C.15)$$

with

$$\bar{\sigma} = \sqrt{2\tau}\Delta x.$$

When this is substituted into (C.1) with $I(x', 0) = I_o\theta(x')$, one finds, using the change of variables $y = x - x'$ and $z = x + x'$, that

$$\begin{aligned} I(x, t)/I_o &= -\frac{1}{\sqrt{2\pi\bar{\sigma}}} \left[\int_x^0 e^{-\frac{y^2}{2\bar{\sigma}^2}} dy + \int_x^0 e^{-\frac{z^2}{2\bar{\sigma}^2}} dz \right. \\ &\quad \left. + \int_0^{-\infty} e^{-\frac{y^2}{2\bar{\sigma}^2}} dy + \int_0^{\infty} e^{-\frac{z^2}{2\bar{\sigma}^2}} dz \right] \\ &= (2/\sqrt{\pi}) \int_0^x e^{-\frac{z^2}{2\bar{\sigma}^2}} d(z/\sqrt{2\bar{\sigma}}), \end{aligned} \quad (C.16)$$

which agrees with (4.11).

REFERENCES

- * Operated by the Universities Research Association, Inc., for the
U. S. Department of Energy under Contract No. DE-AC02- 89ER40486.
- [1] T. Garavaglia, S. K. Kauffmann, R. Stiening, and D. Ritson, in
Supercollider 2, edited by M. McAshan
(Plenum Press, New York, 1990), pp. 39-84.
- [2] E. W. Collings, K. R. Marken Jr., and M. D. Sumption, in
Supercollider 2, edited by M. McAshan
(Plenum Press, New York, 1990), pp. 581-592.
- [3] *Superconducting Super Collider Conceptual Design*,
SSC-SR-2020, edited by J. D. Jackson (SSC Central Design Group,
Berkeley, 1986), pp. 47-51 and pp. 277-282.
- [4] H. Brück et al., *Observation of a Periodic Pattern in
the Persistent-Current Fields of the Superconducting HERA Dipole
Magnets*, HERA preprint (1991).
- [5] See Ref. 2 and Ref. 3.
- [6] *Top Drawer*, (SLAC Computation Group, Stanford California, 1980).
- [7] *CERN Program Library*, (CERN Computer Center,
Geneva, Switzerland, 1989).
- [8] R. Stiening, private communication.
- [9] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series,
and Products*, 2nd ed. (Academic, New York, 1980), p. 1032.

FIGURES

FIG. 1. Two parallel strands in the SSC dipole superconducting cable.

FIG. 2. The equivalent electrical circuit for two crossing parallel strands in the SSC dipole superconducting cable.

FIG. 3. The local current decay in loop 2 for a finite chain of 200 loops.

FIG. 4. The local current decay in loop 2 for an infinite chain.

FIG. 5. The local current decay in loop 25 for a finite chain of 200 loops.

FIG. 6. The local current decay in loop 25 for an infinite chain.

FIG. 7. The local current decay in loop 50 for a finite chain of 400 loops.

FIG. 8. The local current decay in loop 50 for an infinite chain.

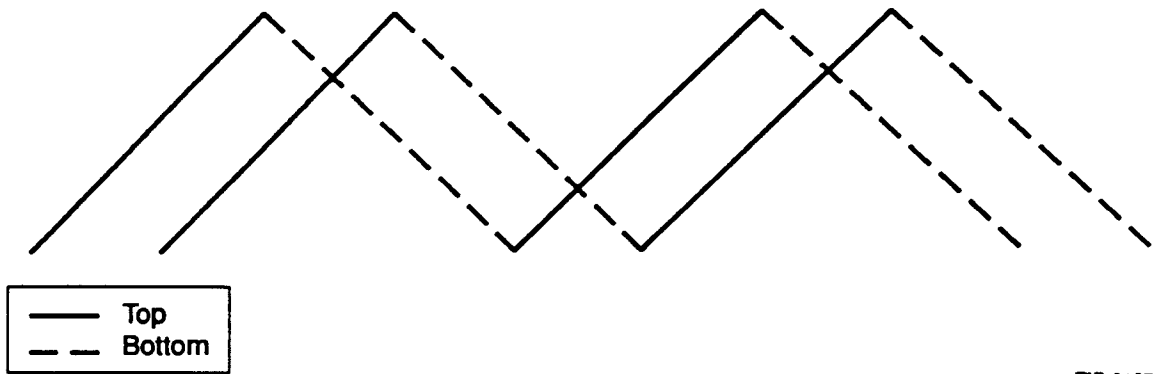
FIG. 9. The local current decay in loop 100 for a finite chain of 800 loops.

FIG. 10. The local current decay in loop 100 for an infinite chain.

FIG. 11. The local current decay in loop 700 for a finite chain of 800 loops.

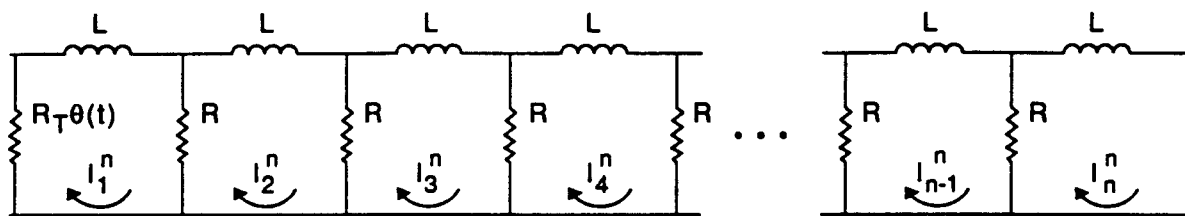
FIG. 12. The local current decay in loop 700 for an infinite chain.

FIG. 13. The zeros of the polynomial $P_{40}(x)$.



TIP-01878

Figure 1



TIP-01879

Figure 2

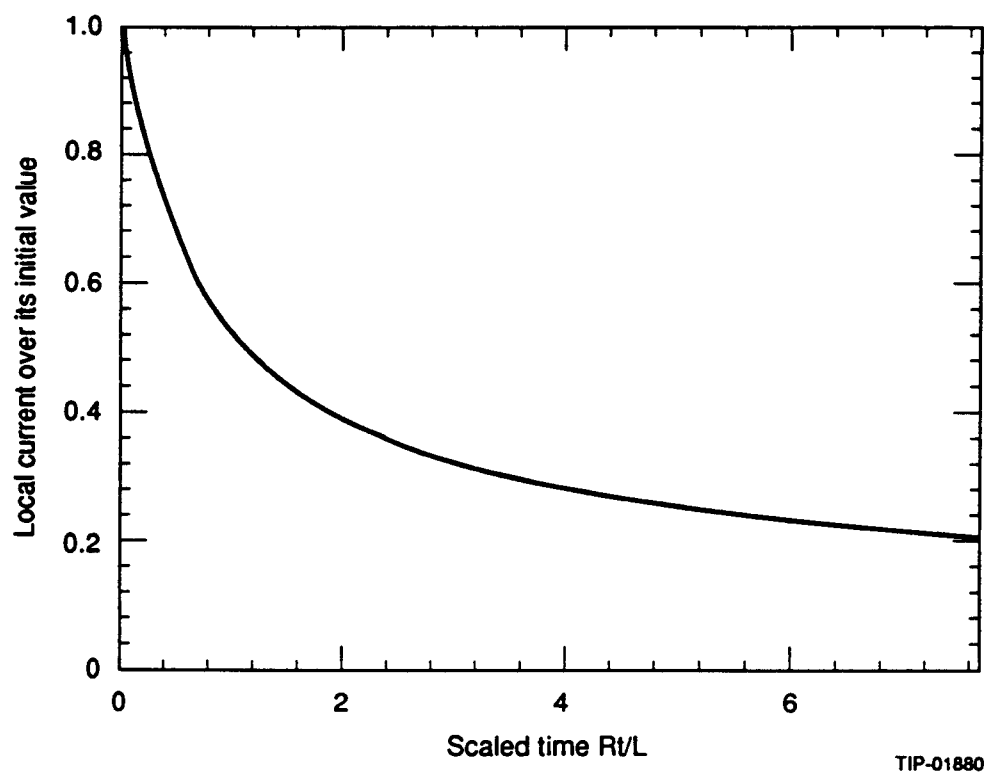


Figure 3

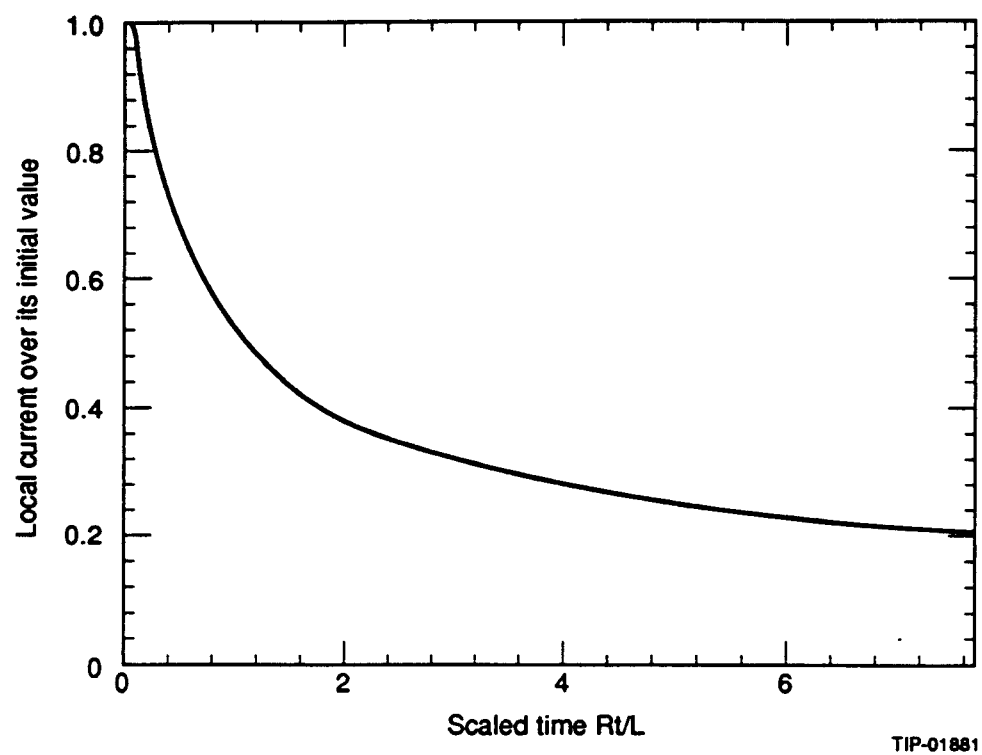


Figure 4

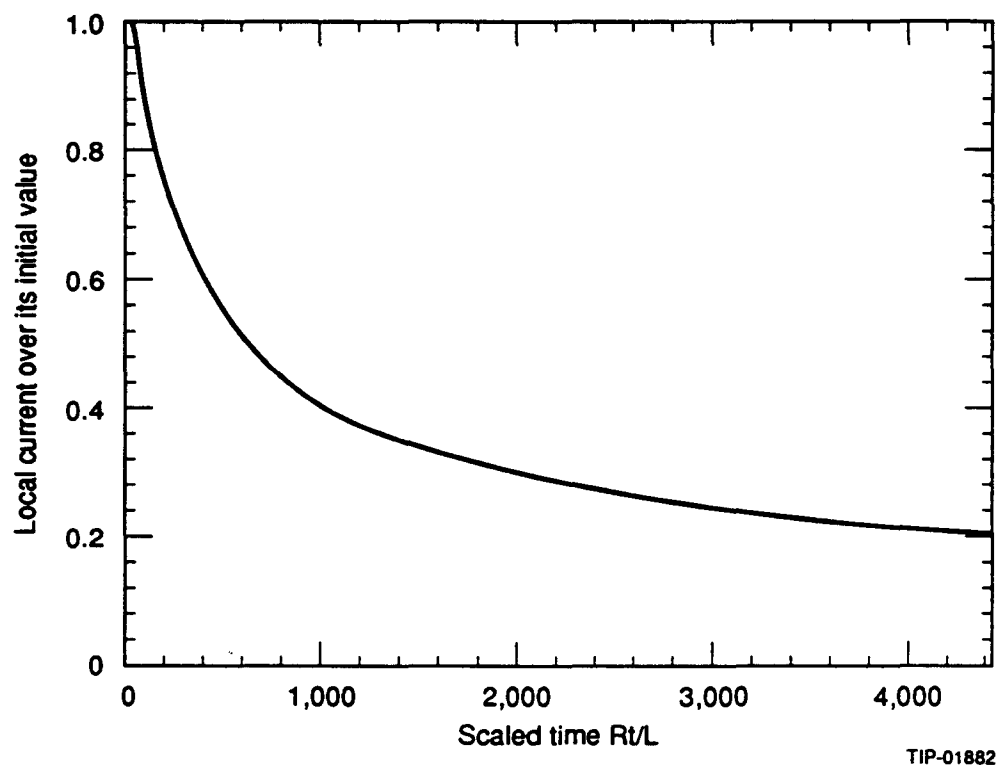


Figure 5

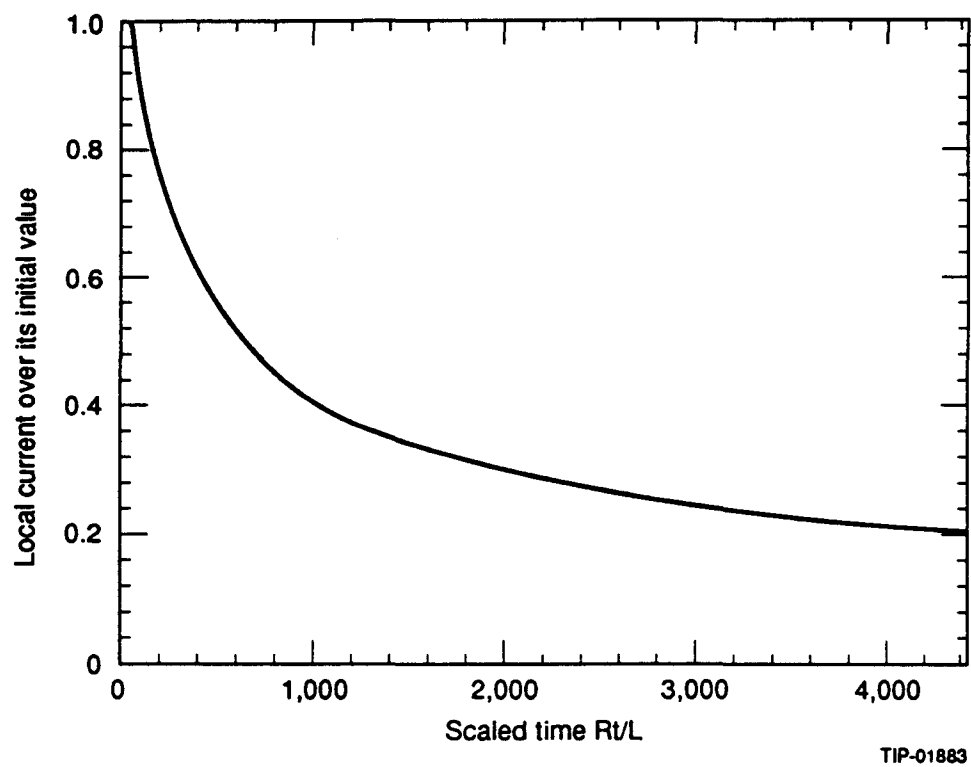
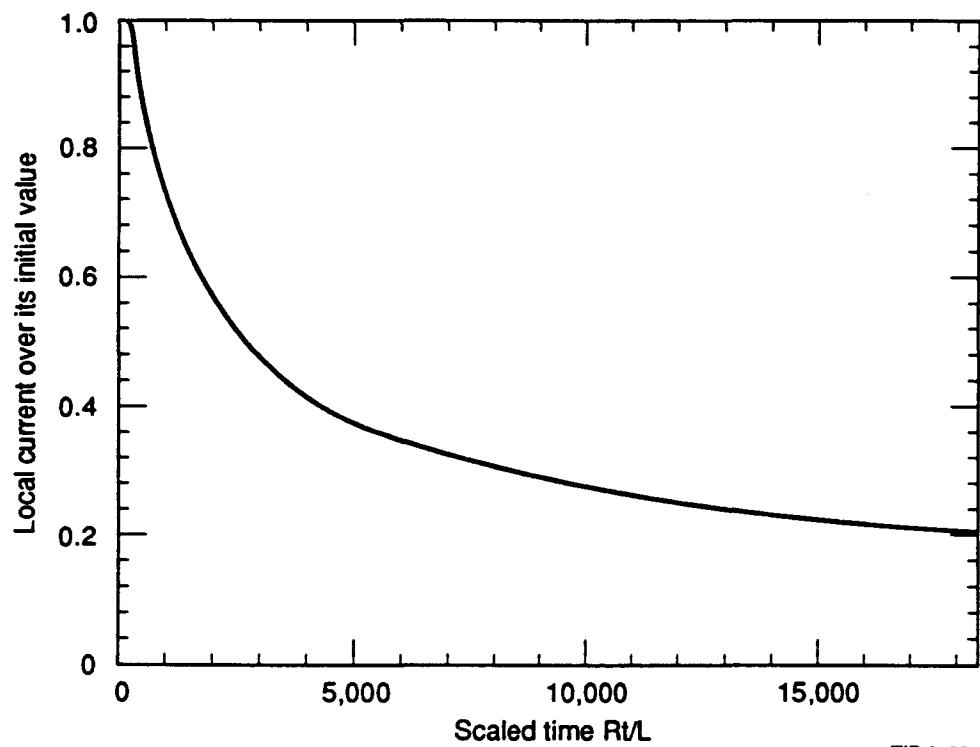


Figure 6



TIP-01884

Figure 7

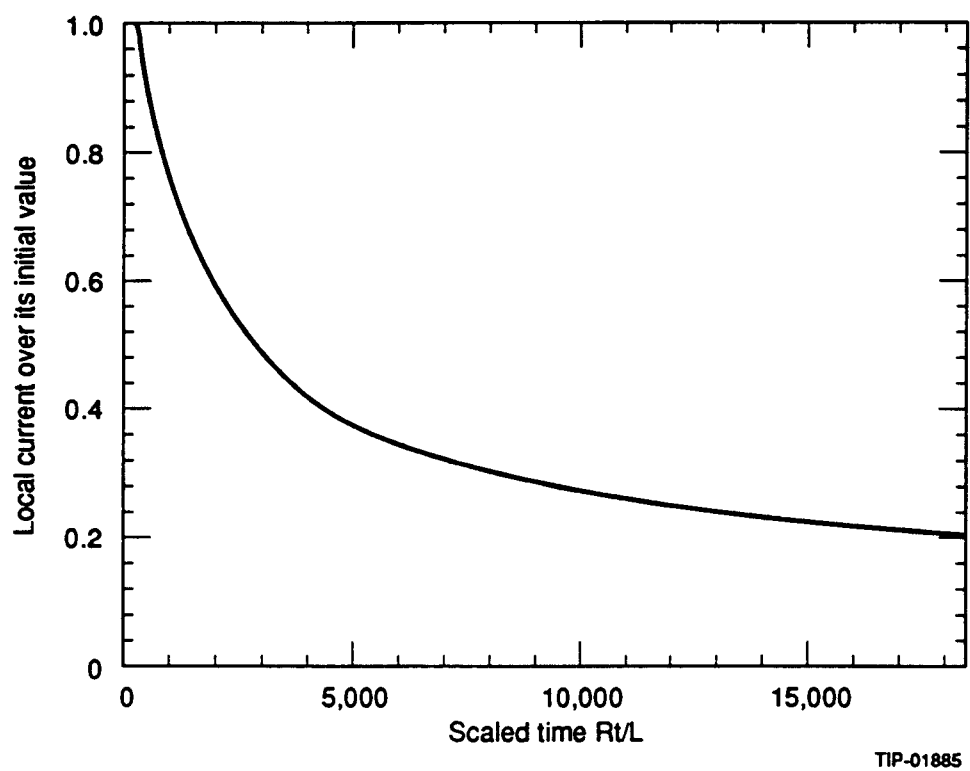


Figure 8

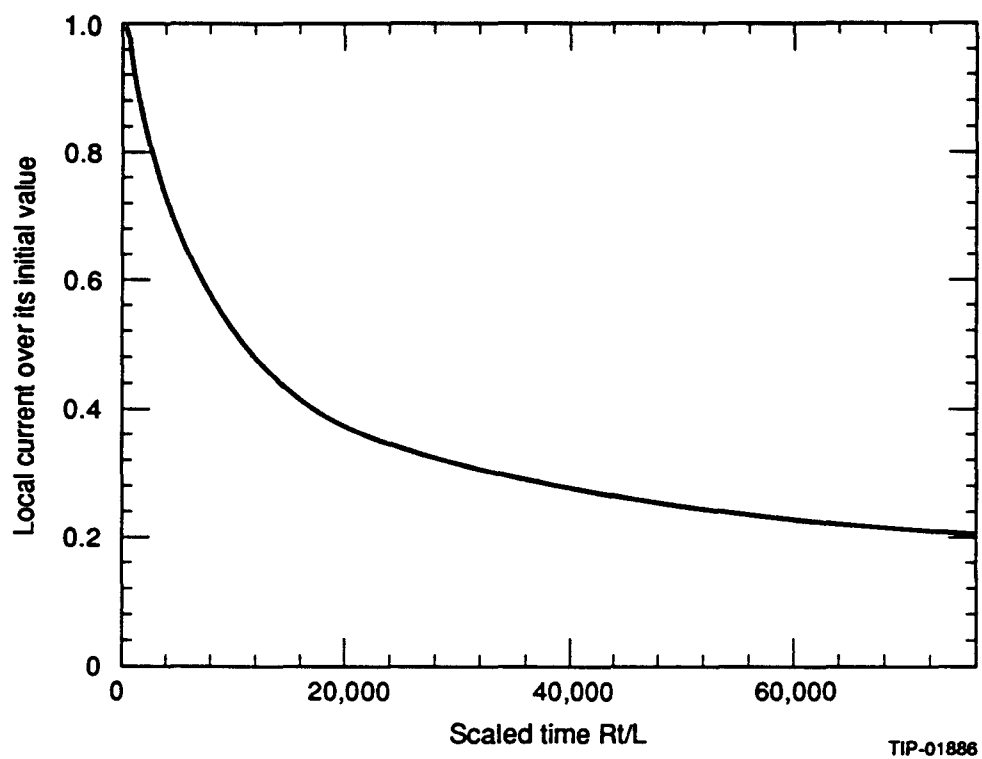


Figure 9

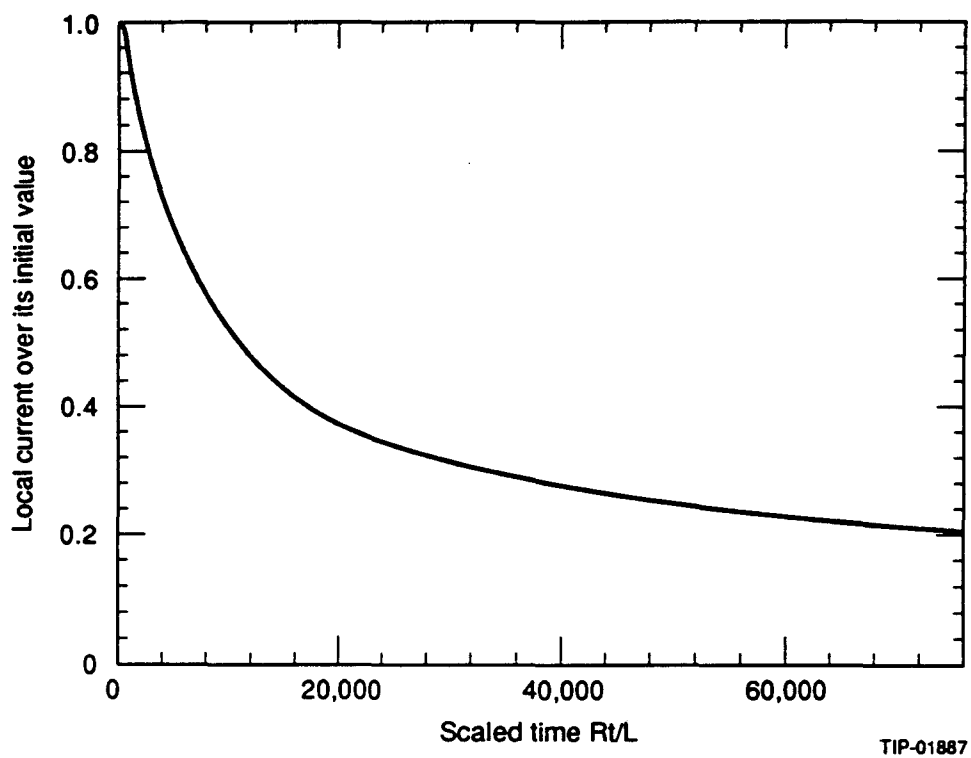


Figure 10

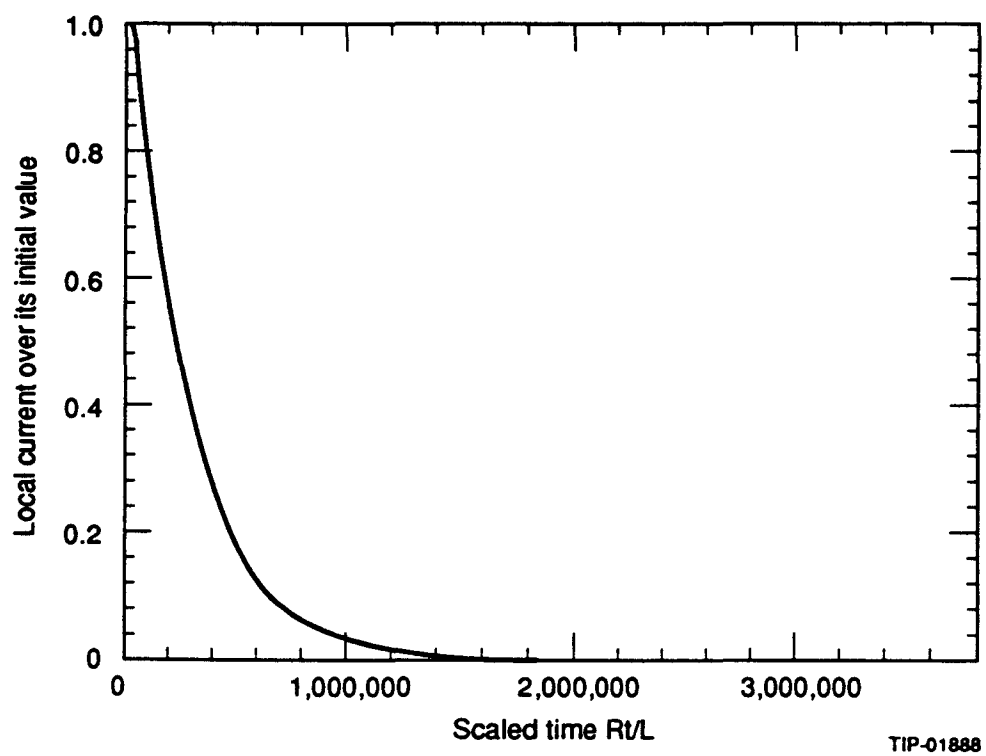


Figure 11

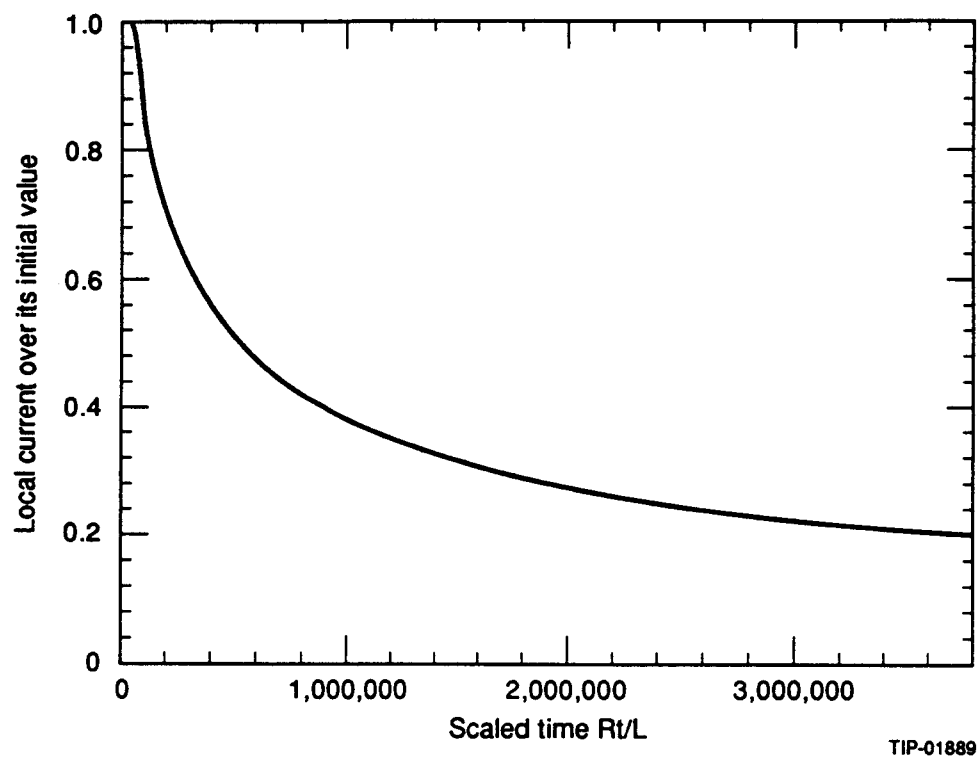


Figure 12

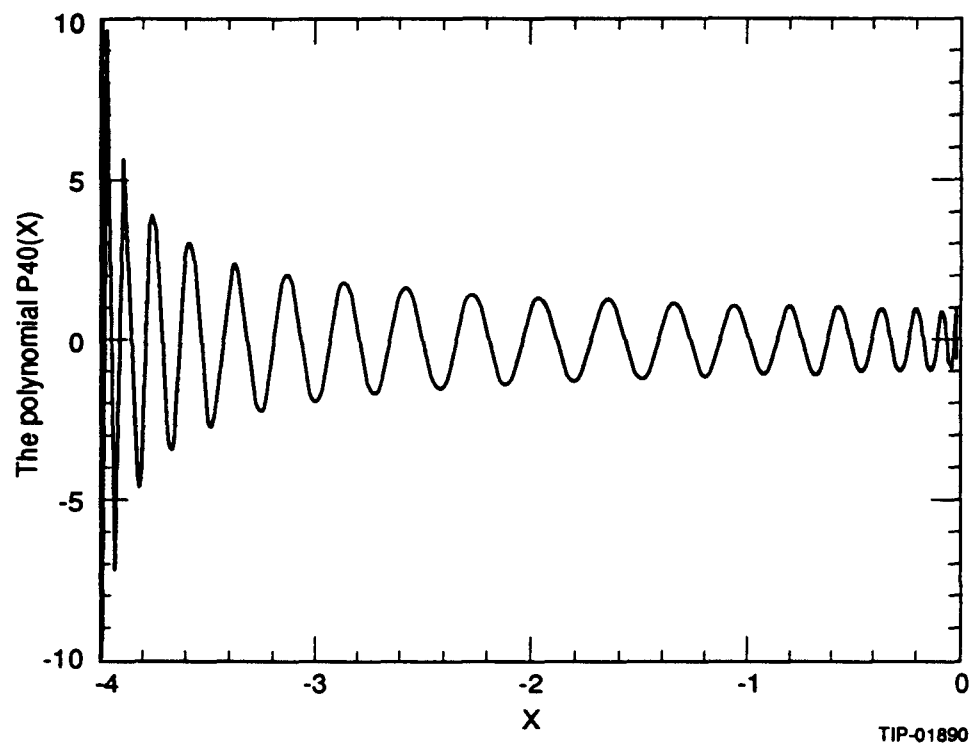


Figure 13