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A general solution for rotating lattices of identical point vortices

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The rotating equilibrium solutions of N identical point vortices are the stationary energy states of a higher-dimensional object projected to the $x-y$ plane, in this case, an $(N-1)$-dimensional regular simplex. Parameterizing the point vortex Hamiltonian with this fundamental geometrical object leads to a simple bivector condition, which contains the equilibrium states and resolves the origin of the asymmetric solutions. This novel approach uses the geometric product and the multivector derivative, ∂_ψ, of the Hamiltonian in the geometric algebra, $Cl(N-1,0)$ to find a geometric condition that determines the equilibrium states. The resulting bivector equation is then used as the input to an optimizer, which rotates a simplex until the equilibrium condition is met, leading to a wealth of new solutions. If the vertices of the oriented simplex are projected to the x-axis, the values form the roots of the Hermite polynomials, $H_N(x)$, and obey the Stieltjes relations, capturing the collinear solutions. The vortex simplex exhibits a striking geometrical connection with the amplituhedron of quantum field theory and gives deep insight into the quantization of a classical system.

1. Introduction

Across the distance scale, rotating systems of vortices are one of nature's most profound structures. Vortex lattices are thought to exist in the superfluid cores of neutron stars [1] and have appeared in Bose–Einstein condensates [2] and the polar atmosphere of Jupiter.

Over the past century, a number of researchers have made great progress in analytical methods for vortex dynamics, and many of these modern approaches are described in the comprehensive book on the N-vortex problem by P. Newton [3]. In this article, we are concerned with a specific form of this N-body problem, rotating arrangements of identical point vortices in the unbounded plane, with fixed inter-particle distances.

These vortex crystal states have been studied extensively by researchers who have shown how the relative equilibrium problem touches on many branches of mathematics. There have been deep and lasting contributions to the vortex crystal problem, and many closed-form, exact solutions exist for configurations that possess some degree of symmetry. These gains were significant and required sophisticated analytic approaches, successfully developed by Aref *et al*. [4]. Despite these advances, basic questions remain unanswered concerning these structures. For example, is there a single closed-form or exact solution that captures all of the states, stable and otherwise? Aref [5] found through an elegant application of the fundamental theorem of algebra, a pair of differential equations for a polynomial whose roots correspond to the vortex positions. As he pointed out, they suffered from the fact that one coefficient was one of the roots of the system and was difficult to know unless the system had already been solved.

In a surprising development in 1998, using the numerical method of 'ghost vortices', Aref & Vainchtein [6] discovered asymmetric equilibria solutions for identical vortices. This seemed to contradict an intuitive sense that systems of point particles possessing only 'spin' should express at least some rotational symmetry. Until this interesting result, it was widely believed that all relative equilibrium solutions were characterized by at least one axis of symmetry. The motivation for this article is the search for the origin of these asymmetric equilibrium states.

Using a geometrical approach based on Clifford algebra (geometric algebra), a general solution to the N-body problem of identical strength vortices, which encompasses all of the solutions is demonstrated, thus resolving the origin of the asymmetrical states. It can also be shown that this novel approach leads to a straightforward algorithm that uses the rotations of a regular simplex in $\mathbb{R}^{N-1}$ to converge to a solution. This unifying geometric method complements previous analytical work, and we hope that new exact solutions emerge from a combination of both approaches.

In order to arrive at a deeper understanding of these systems, we turn our attention to Euclidean geometry in $\mathbb{R}^n$ and make use of the powerful coordinate-free methods in geometric algebra. The following results rely on the geometric product, where the two operators ($\cdot$, $\wedge$) are the inner and outer products, respectively, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are elements of the algebra. For the moment, it is best to imagine that $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in $\mathbb{R}^n$. We shall indicate vector objects in boldface type.

$$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b} \tag{1.1}$$

For the interested reader, excellent introductions to geometric algebra and its calculus can be found in the texts by Hestenes & Sobczyk [7], Doran & Lasenby [8] and Dorst *et al*. [9]. Using the ideas of Grassman and Clifford, we can gain deep insight into the nature of the quantization of a classical system.

2. Point vortex dynamics

Helmholtz [10] made a far-reaching approximation that reduced the problem of vortex motion to two dimensions, essentially confining all sources of vorticity to infinitely small points that interact. These new mathematical objects, which reside in two-dimensional manifolds, are point particles that induce the motion of another vortex particle through advection. Setting aside the derivation of the point vortex solution for the Euler equations, we will begin with the equations of point vortex motion in the $x-y$ plane, hereafter referred to as the $\boldsymbol{e}_{12}$ plane, where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the customary orthogonal unit vectors. In addition, let the squared distance between the particles be written as l^2_{ab}, where the index b is forbidden from taking on the value of a in all summands. Following Aref [11], we can write the equations of motion in the form, where Γ_i is a real number that represents the vorticity of a particle.

$$\frac{dx_a}{dt} = -\frac{1}{2\pi}\sum_{b=1}^{N}\frac{\Gamma_b(y_a - y_b)}{l_{ab}^2}$$
$$\frac{dy_a}{dt} = -\frac{1}{2\pi}\sum_{b=1}^{N}\frac{\Gamma_b(x_b - x_a)}{l_{ab}^2}$$

The 2N real equations given above can be combined into N complex differential equations

$$\overline{\frac{dz_a}{dt}} = \frac{1}{2\pi i}\sum_{b=1}^{N}\frac{\Gamma_b}{z_a - z_b}. \tag{2.1}$$

If the conjugate variables are defined to be $p_a = \Gamma_a y_a$ and $q_a = x_a$, the equations of motion are given by the following Hamiltonian:

$$H = -\frac{1}{4\pi}\sum_{a,b=1}^{N}\Gamma_a\Gamma_b \log|z_a - z_b|. \tag{2.2}$$

If we require the vortex particles to co-rotate in the complex plane and impose identical vorticities, substituting $z_a(t) = \lambda_a e^{2\pi i t}$ into equation (2.1) yields

$$\overline{z_a} = \sum_{b=1}^{N}\frac{\Gamma}{z_a - z_b}. \tag{2.3}$$

This is the relative equilibrium condition for identical point vortices in the complex plane, up to a rescaling. Although the restriction to identical vorticities might seem severe, there are many physical examples of vortex equilibria with this attribute. Bose–Einstein condensates with a source of angular momentum and vortices in liquid helium are two such well-known phenomena. They indicate that the relative equilibrium problem is one that is deeply related to quantum systems.

It is clear from equation (2.3) that every complex position, z_a, depends upon every other position, z_b, in the set, making a general solution difficult. For this system, the deep structure of these solutions has been elusive, but there is great insight to be gained by transforming this equation with the geometric product given by equation (1.1).

3. A brief introduction to geometric algebra

Geometric algebra emerges from the definition of the geometric product, which is the sum of a symmetric and an antisymmetric operator, where all the elements of the algebra form a much larger vector space. In a very real sense, geometric algebra, with its own idioms and concepts, completely subsumes tensor algebra, which also extends linear methods to higher-dimensional spaces. Geometric algebra gains its power by reinforcing geometric intuition for transformations on these smaller component subspaces, unlike tensor algebra, where the manipulations can feel far more abstract. Using the simple algebraic rules to interact with those subspaces renders many difficult calculations compact and can reveal previously unknown symmetries.

To understand the structure of the geometric product, we first examine the antisymmetric outer product $\boldsymbol{a} \wedge \boldsymbol{b}$ and observe that it introduces the notion of an oriented subspace, extending the concept beyond the familiar vectors.

We can first explore the algebra of $Cl(2,0)$ to get a sense of what the outer product might mean, along with the notion of oriented subspace, and then generalize to higher-dimensional realms to approach the point vortex problem. To begin our exploration, let a vector $\boldsymbol{v}$ lie in $\mathbb{R}^2$, and denote the basis vector for the x-axis as, $\boldsymbol{e}_1$, and similarly, the basis vector for the y-axis as $\boldsymbol{e}_2$. In geometric algebra, these basis vectors can be multiplied together in the following way using the geometric product:

$$\begin{aligned}\boldsymbol{e}_1\boldsymbol{e}_2 &= \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 + \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \\ &= \boldsymbol{e}_{12} \\ &= -\boldsymbol{e}_{21}\,. \end{aligned} \tag{3.1}$$

Here the (·) represents the familiar dot product and (∧) is the antisymmetric or outer product. From the definition it is clear that for the two orthogonal basis vectors, $\boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2$ since their dot product is zero. The product of these two objects is itself a new object, which is non-zero and has the property that if the factors are reversed, the sign changes. Formed from a pair of one-dimensional basis vectors, $\boldsymbol{e}_{12}$ is two-dimensional, just as the vector space $\mathbb{R}^2$ happens to be. In geometric algebra, this object is called a *bivector*, and since this particular *bivector* is formed from the outer product of two basis vectors, it is also called a *blade*. The simple diagram in figure 1 illustrates why it is described this way.

If the basis vectors are swapped in the product, the sense of the orientation changes with the sign. We also have $\boldsymbol{e}_{12}{}^2 = -1$, and we note that this has precisely the same algebraic property as i, the unit imaginary, although $\boldsymbol{e}_{12}$ has a geometric interpretation as an oriented subspace. As it happens, such two-dimensional blades are essential ingredients in the formulation of rotations in higher-dimensional space, in objects called *rotors*, yielding spinors and many other useful structures.

This is a good place to pause and take a look at the sort of objects we have grown on the familiar substrate of $\mathbb{R}^2$. Of course, there are scalars, α, objects of dimension zero. There are two basis vectors, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, each of dimension one. Finally, we have a bivector $\boldsymbol{e}_{12}$ of dimension two. So in the geometric algebra of $Cl(2,0)$, there are four basis elements, and a multivector, $\boldsymbol{w}$, in this algebra can be written in terms of scalar coefficients, α_i.

$$\boldsymbol{w} = \alpha_0 + \alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2 + \alpha_3\boldsymbol{e}_{12} \tag{3.2}$$

The geometric algebra of $Cl(2,0)$ contains all of the subspaces of $\mathbb{R}^2$, and we can easily see from the definition of the geometric product in equation (1.1) that the inverse of a vector $\boldsymbol{v}$ is given by $\boldsymbol{v}^{-1} = \boldsymbol{v}/\|\boldsymbol{v}\|^2$, since the outer product of a vector with itself is zero.

(a) Vectors, reflections and rotations

Using only vectors and simple algebra, we can first construct a reflection, then reflect once again to create an arbitrary rotation in a hyperplane. These operations, which are highly compact, can be built up from familiar vectors and bound together by the geometric product. Following §2.6 in Doran & Lasenby [8], we first examine a simple reflection. Let $\boldsymbol{n}$ be a unit vector and $\boldsymbol{a}$ be an arbitrary vector. We want to decompose $\boldsymbol{a}$ into two orthogonal pieces, $\hat{\boldsymbol{a}}$ for the perpendicular component and $\check{\boldsymbol{a}}$ for the component parallel to $\boldsymbol{n}$.

$$\begin{aligned}\boldsymbol{a} &= \boldsymbol{n}^2\boldsymbol{a} \\ &= \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{a} + \boldsymbol{n}\wedge\boldsymbol{a}) \\ &= \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{a}) + \boldsymbol{n}(\boldsymbol{n}\wedge\boldsymbol{a}) \\ &= \check{\boldsymbol{a}} + \hat{\boldsymbol{a}}\end{aligned}$$

How can we construct a reflection of the vector, $\boldsymbol{a}$, through the orthogonal plane to the unit vector, $\boldsymbol{n}$? Let us denote the reflected vector by $\boldsymbol{a}'$as depicted in figure 2. The geometric algebraic approach gives a very elegant representation of the reflection of $\boldsymbol{a}$ through the plane normal to $\boldsymbol{n}$. We have from the definition of the geometric product

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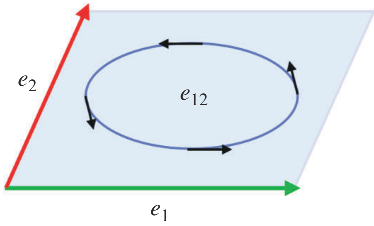


Figure 1. The bivector $\boldsymbol{e_{12}}$ formed from two orthogonal vectors, $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$.

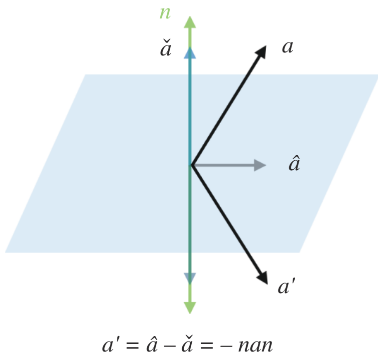


Figure 2. The reflection of a vector $\boldsymbol{a}$ is given by the sandwich product of a unit vector $\boldsymbol{n}$.

$$\begin{aligned}\boldsymbol{a}' &= \hat{\boldsymbol{a}} - \check{\boldsymbol{a}}\\ &= -\boldsymbol{n}\boldsymbol{n}\wedge\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{n}\boldsymbol{n}\\ &= -(\boldsymbol{n}\cdot\boldsymbol{a})\boldsymbol{n} - (\boldsymbol{n}\wedge\boldsymbol{a})\boldsymbol{n}\\ &= -\boldsymbol{n}\boldsymbol{a}\boldsymbol{n}\end{aligned}. \tag{3.3}$$

Using the full geometric algebra, a reflection through the plane orthogonal to $\boldsymbol{n}$ is a conjugation of the vector $\boldsymbol{a}$ by a unit vector, $\boldsymbol{n}$, a transformation that does not change the length of $\boldsymbol{a}$. What happens when we choose another unit vector $\boldsymbol{m}$ and reflect $\boldsymbol{a}'$ through an orthogonal plane to $\boldsymbol{m}$, using our conjugation in equation (3.3)? Our new vector $\boldsymbol{a}''$ has the form

$$\boldsymbol{a}'' = (\boldsymbol{m}\boldsymbol{n})\,\boldsymbol{a}\,(\boldsymbol{n}\boldsymbol{m})\,. \tag{3.4}$$

This new algebraic object $\psi = \boldsymbol{m}\boldsymbol{n}$ causes a rotation of the vector $\boldsymbol{a}$, is composed of a pair of reflections and is called a *rotor*. The vector $\boldsymbol{a}''$ can be written as the result of a sandwich product where the $(\tilde{\cdot})$ represents the *reversion* operator. The action of the reversion operator simply reverses the orders of the factors in a product. If we let $\psi = \boldsymbol{m}\boldsymbol{n}$, the equation can be written

$$\boldsymbol{a}'' = \psi\boldsymbol{a}\tilde{\psi}\,. \tag{3.5}$$

It is important to note that ψ is the product of invertible vectors and thus has an inverse. To find this inverse, we calculate the product of the rotor ψ with its reversion

$$\psi\tilde{\psi} = (\boldsymbol{m}\boldsymbol{n})(\boldsymbol{n}\boldsymbol{m}) = 1\,.$$

Hence, the reversion of this rotor is its inverse. In general, if the vectors that make up the rotor are not unit vectors, this will not be true. Such an object is called a *versor* more generally, although in this article we only consider versors composed of even numbers of vectors, which correspond to rotations. If ψ is a versor, it will still rotate $\boldsymbol{a}$, but there will be a scaling factor introduced, analogous to multiplying by a complex number that does not have a unit norm. In full algebra, multivector inverses can be somewhat involved, but versors, composed of invertible vectors, have simple inverses. It would be very useful indeed to be able to take the derivative of a scalar function with respect to a rotor. As it turns out, we can do precisely that with the multivector derivative.

(b) The scalar product of a pair of k-blades

In our usual vector algebra, there is the familiar inner product that takes two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, and returns a real number, $\alpha = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos(\theta)$. In geometric algebra, it is possible to define a scalar product ($*$), which has the same properties for k-blades. Such a product should depend on all the dot products of the factors in the outer products, $\boldsymbol{A} = \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge .. \wedge \boldsymbol{a}_k$ and $\boldsymbol{B} = \boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge .. \wedge \boldsymbol{b}_k$. The definition of the scalar product should reduce to the familiar dot product between vectors, and indeed the determinant meets that requirement for $\boldsymbol{A} * \boldsymbol{B}$.

$$(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \ldots \wedge \boldsymbol{a}_k) * (\boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge \ldots \wedge \boldsymbol{b}_k) = \det \begin{bmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{b}_k & \cdots & \boldsymbol{a}_1 \cdot \boldsymbol{b}_1 \\ \vdots & \ddots & \vdots \\ \boldsymbol{a}_k \cdot \boldsymbol{b}_k & \cdots & \boldsymbol{a}_k \cdot \boldsymbol{b}_1 \end{bmatrix}.$$

The squared norm of a k-blade is given by $\|\boldsymbol{A}\|^2 = \boldsymbol{A} * \tilde{\boldsymbol{A}}$, and we note that the scalar product of two blades with differing numbers of factors is zero.

(c) Geometric calculus and the multivector derivative

In order to follow the calculations using the multivector derivative, ∂_X, we present a few basic definitions first. The algebra, $Cl(n, 0)$, is built up from subspaces of increasing dimension, called grades, using the outer product, $\boldsymbol{a} \wedge \boldsymbol{b}$. In general, $Cl(n, 0)$ has 2^n components. A grade projection operator, $\langle \boldsymbol{X} \rangle_k$, returns the components of $\boldsymbol{X}$ of dimension or grade k. For example, if $\boldsymbol{X} = \alpha + \beta \boldsymbol{e}_1 + \delta \boldsymbol{e}_2 + \gamma \boldsymbol{e}_{12}$, the grade projection will be $\langle \boldsymbol{X} \rangle_1 = \beta \boldsymbol{e}_1 + \delta \boldsymbol{e}_2$. Projection to scalar quantities, an operation that is frequently used to derive the results which follow, is denoted by angle brackets without an integer subscript.

To quickly arrive at a definition of the multivector derivative, we first define the directional derivative of a multivector-valued function, $\boldsymbol{F}$, which is analogous to $a \cdot \nabla$, where ∇ is the customary gradient operator and $\boldsymbol{a}$ is a vector.

$$\boldsymbol{A} * \partial_X \boldsymbol{F}(\boldsymbol{X}) \equiv \lim_{\epsilon \to 0} \frac{\boldsymbol{F}(\boldsymbol{X} + \epsilon \boldsymbol{A}) - \boldsymbol{F}(\boldsymbol{X})}{\epsilon}. \tag{3.6}$$

Next, we write the multivector derivative in terms of its action on the basis elements of the algebra and the reciprocal frames. A surprising number of identities flow out of this powerful linear operator, of which we will only use a small number.

$$\partial_X = \sum_J \boldsymbol{e}^J (\boldsymbol{e}_J * \partial_X). \tag{3.7}$$

Three identities that are helpful in computing the multivector derivatives of scalar projections are

$$\partial_X \langle \boldsymbol{X}\boldsymbol{M} \rangle = P_X[\boldsymbol{M}] \tag{3.8}$$

$$\partial_X \langle \tilde{\boldsymbol{X}}\boldsymbol{M} \rangle = P_X(\tilde{\boldsymbol{M}}) \tag{3.9}$$

$$\partial_X \langle \boldsymbol{M}\boldsymbol{X}^{-1} \rangle = -\boldsymbol{X}^{-1} P_X(\boldsymbol{M}) \boldsymbol{X}^{-1}, \tag{3.10}$$

where P_X is the projection operator to the manifold where $\boldsymbol{X}$ takes its values. For the purposes of the point vortex problem, the projection operator is the identity. One convenient symmetry of multivector differentiation of a scalar projection is the ability to permute factors within the brackets. In the final identity below, we can make use of this property and the identities (3.8) and (3.10) to find an expression central to taking the derivative of a rotation. Lasenby *et al.* [12] demonstrated how we can also greatly simplify the process for free differentiation by writing a rotation ψ as an even product of (not necessarily unit length) invertible vectors, a *versor*, then performing the differentiation with respect to ψ. Let $\boldsymbol{X} = \psi$, a versor, and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors. From the identities given above, we find

$$\partial_\psi \langle \psi \boldsymbol{u} \psi^{-1} \boldsymbol{v} \rangle = 2\psi^{-1}(\psi \boldsymbol{u} \psi^{-1} \wedge \boldsymbol{v}). \tag{3.11}$$

4. Relative equilibrium and the geometric product

To gain insight into the relative equilibrium problem, we begin by rewriting the set of N complex, first-order differential equations given in equation (2.3), with vorticities in equation (2.3) set to unity

$$\overline{z_a} = \sum_{b=1}^{N} \frac{1}{z_a - z_b}. \tag{4.1}$$

Using the geometric product, it is straightforward to transform complex numbers in the system above to vectors by identifying the unit imaginary i with $\boldsymbol{e_{12}}$. Writing $z_a = x_a + y_a \boldsymbol{e_{12}}$, while noting that $\boldsymbol{e_{12}}$ has identical algebraic properties to i, $\boldsymbol{e_{12}}^2 = (\boldsymbol{e_1} \wedge \boldsymbol{e_2})^2 = -1$, and substituting into equation (4.1) yields

$$x_a - y_a \boldsymbol{e_{12}} = \sum_{b=1}^{N} \frac{1}{(x_a - x_b) + (y_a - y_b)\boldsymbol{e_{12}}}. \tag{4.2}$$

Taking the reversion, which is equivalent to complex conjugation, of both sides yields

$$\begin{aligned} x_a + y_a \boldsymbol{e_{12}} &= \sum_{b=1}^{N} \frac{1}{(x_a - x_b) - (y_a - y_b)\boldsymbol{e_{12}}} \\ &= \sum_{b=1}^{N} \frac{(x_a - x_b) + (y_a - y_b)\boldsymbol{e_{12}}}{(x_a - x_b)^2 + (y_a - y_b)^2}. \end{aligned}$$

Then, multiplying on the left by the basis vector $\boldsymbol{e_1}$ and letting $v_a = x_a \boldsymbol{e_1} + y_a \boldsymbol{e_2}$

$$\begin{aligned} x_a \boldsymbol{e_1} + y_a \boldsymbol{e_2} &= \sum_{b=1}^{N} \frac{(x_a - x_b)\boldsymbol{e_1} + (y_a - y_b)\boldsymbol{e_2}}{(x_a - x_b)^2 + (y_a - y_b)^2} \\ \boldsymbol{v_a} &= \sum_{b=1}^{N} \frac{1}{\boldsymbol{v_a} - \boldsymbol{v_b}} \end{aligned} \tag{4.3}$$

After performing this algebraic transformation, what appeared at first to be a system of N equations in the complex plane in equation (2.3) is nothing more than a system of N *vector* equations in $\mathbb{R}^2$. The first geometric insight we obtain is that each equilibrium solution has the property that every point vortex position vector, $\boldsymbol{v_a}$, is the sum of the inverted edges it makes with all the other vectors in the set, $\{\boldsymbol{v_b}\}_{b=1}^{N-1}$. If all $\{\boldsymbol{v_a}\}_{a=1}^{N}$ lie in $\mathbb{R}^1$, these quantities are well known, having been first explored by Stieltjes [13] in his study of Coulomb charges placed on a line and correspond to the roots of the Hermite polynomials, $H_N(x)$. It will emerge that these position vectors are the projections of a higher-dimensional object, a regular simplex, Δ^{N-1}.

5. Projections of simplex vertices and relative equilibrium

We first begin with the observation that a single point is a trivial example of relative equilibrium, and the same is true for a pair of points. It is easy to see that both solutions correspond to the simplices, Δ^0 and Δ^1.

The last of our trivial examples is an equilateral triangle, Δ^2, oriented such that one edge is parallel to $\boldsymbol{e_1}$ or the x-axis, of vertex length $\sqrt{2}$. This triangle in figure 3 has the vertex projections, $\sqrt{2}\cos(\pi/6) = \pm\sqrt{3/2}$ and 0, onto $\mathbb{R}^1$, which are nothing more than the roots of $H_3(x)$. We can ascend the dimensional ladder several more rungs to see that the pattern does continue to hold. In figure 4*a*, a tetrahedron is oriented precisely so that its vertices project to the roots of $H_4(x)$ on $\mathbb{R}^1$. In figure 4*b*,*c*, Δ^3 generates the pair of two-dimensional solutions to the equilibrium equation (4.3), a square and a centred triangle in $\mathbb{R}^2$. In the final diagram, figure 4*d*, the vertices of the pentachoron, Δ^4, are projected to the roots of $H_5(x)$.

6. A remarkable simplex invariant

In order to show that a set of identical point vortices in relative equilibrium can be lifted to a simplex, Δ^{N-1}, we first describe a geometric invariant for simplices, one that can be matched with an invariant for the point vortices. It is this pair of coinciding invariants that permits the solution of this point vortex problem for every N.

We can illustrate this simplex invariant using an equilateral triangle, Δ^2, and project its vertices to the $\boldsymbol{e_1}$ axis (x-axis). Let us first form the squares of the projections of the vertex vectors to the x-axis, allowing θ to be a rotation angle. Furthermore, let us set the vertex length as follows: $|\boldsymbol{r_a}| = (N-1)/2 = 1$. Let the $\{\check{\boldsymbol{r}}_{\boldsymbol{a}}\}_{a=1}^{N}$ represent the projected vertex vectors. For the squares of the projections we can write

$$\begin{aligned} \check{\boldsymbol{r}}_{\boldsymbol{1}}^2 &= \cos^2(\theta) \\ \check{\boldsymbol{r}}_{\boldsymbol{2}}^2 &= \cos^2(\theta + 2\pi/3) \, . \\ \check{\boldsymbol{r}}_{\boldsymbol{3}}^2 &= \cos^2(\theta + 4\pi/3) \end{aligned}$$

From standard trigonometric identities, we have

$$\begin{aligned} \sum_{a=1}^{3} \check{\boldsymbol{r}}_{\boldsymbol{a}}^2 &= \cos^2(\theta) + \sin^2(\pi/6 - \theta) + \sin^2(\pi/6 + \theta) \\ &= N(N-1)/4 = 3/2 \end{aligned} \, .$$

We see that the sum of the squared magnitudes of the vertex projections $\{\check{\boldsymbol{r}}_{\boldsymbol{a}}\}_{a=1}^{N}$ to the $\boldsymbol{e_1}$ axis is a constant and does not depend on the orientation of the triangle, as depicted in figure 5. In fact, the sum of the squared projections of the simplex vertices is equal with respect to all coordinate axes, and this holds true for regular simplices as we ascend the dimensional ladder.

As it happens, this geometric invariant has a long and rich mathematical history. Schläfi [15] named the set of vectors that possess this property *eutactic* and Coxeter [16] further demonstrated that such vectors inherit this property directly from the symmetry group of the underlying object. Eastwood & Penrose [17] used the eutactic property and Hadwiger's principal theorem to obtain an elegant equation for the projection of a simplex to the complex plane. To demonstrate that regular simplices possess this property, we will adopt a constructive approach with an expanded proof of Schläfi [15], who proved that the vertices of all regular polytopes are eutactic.

In the following section, we will construct a sequence of regular simplices, increasing in dimension, whose squared vertex projections to each coordinate axis, when summed, are

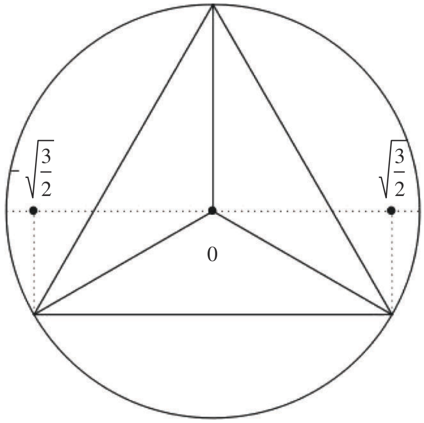


Figure 3. An equilateral triangle with vertex projections that coincide with the roots of $H_3(x)$. This is a relative equilibrium solution for identical point vortices.

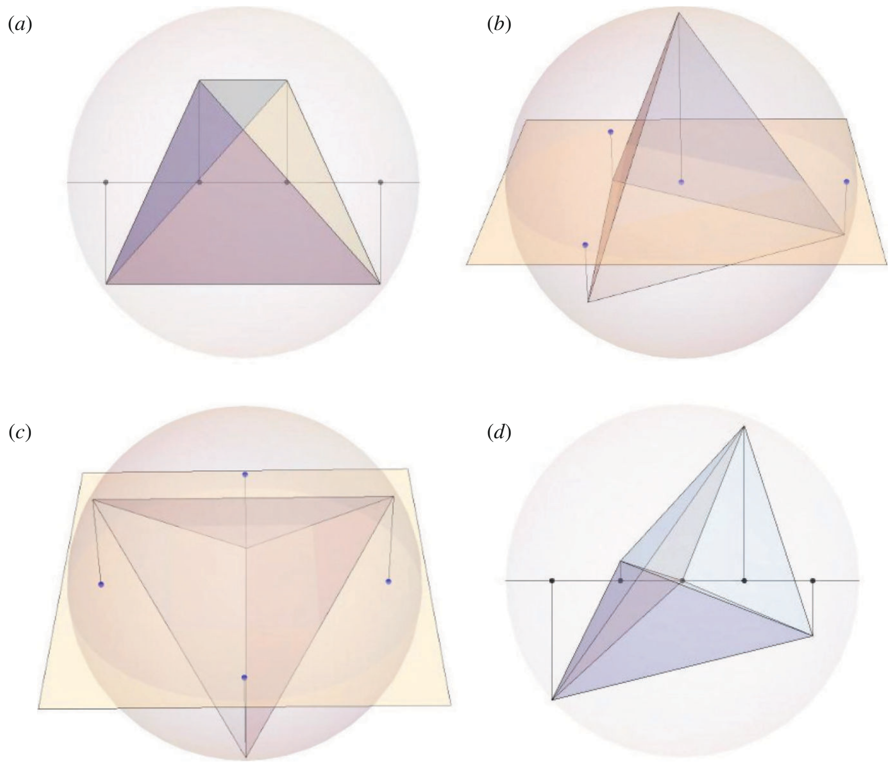


Figure 4. Regular simplices are oriented so that vertices project to equilibrium solutions [14]. These solutions are exact. (*a*) $N = 4, r = 3/\sqrt{2}$; (*b*) $N = 4, r = 3/2$; (*c*) $N = 4, r = 3/2$; (*d*) $N = 5, r = \sqrt{8}$.

equal to $N(N-1)/4$, where N represents the number of vertices. We then use a simple matrix calculation and demonstrate that rotations do not alter the quantity. To show that this projection property is invariant, we form the coordinate matrix S_{N-1}, where the columns are the N vertex vectors, $\{\boldsymbol{r_a}\}_{a=1}^{N}$. This matrix has dimensions $(N-1) \times N$. For invariance, we observe that it is enough to show that if $W_{N-1} = S_{N-1}S_{N-1}^T$

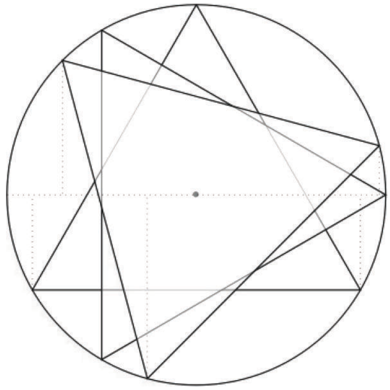


Figure 5. The sum of the squared vertex projections of an equilateral triangle does not depend on the rotation angle.

$$W_{N-1} = \frac{N(N-1)}{4} I_{N-1}, \tag{6.1}$$

where I_{N-1} represents the $(N-1) \times (N-1)$ identity matrix. It is instructive to form the matrix S_2 for our example, Δ^2, allowing the columns to be the vertex vectors, and then easily compute W_2 to show that it has the form in equation (6.1).

$$S_2 = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \tag{6.2}$$

$$W_2 = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} = S_2 S_2^T. \tag{6.3}$$

To prove that a regular simplex of arbitrary dimension is eutactic, we create a sequence of regular simplices increasing in dimension according to a simple recursion process. We also introduce the following notation for the projections of a simplex vertex to the coordinate axis, $\boldsymbol{e_j}$

$$\check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{j}} = P_{\boldsymbol{e}_j}[\boldsymbol{r_a}], \tag{6.4}$$

where the matrix elements of W_{N-1} for a regular simplex are given by

$$w_{j,l} = \sum_{a=1}^{N} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{j}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{l}}. \tag{6.5}$$

It is also useful to note that W_{N-1} is a symmetric matrix and that our quantities of interest are given by

$$w_{j,\,j} = \sum_{a=1}^{N} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{j}}^{\;2} = \sum_{a=1}^{N} (P_{\boldsymbol{e}_j}[\boldsymbol{r_a}])^2. \tag{6.6}$$

7. Iterative construction of regular simplices and the simplex invariant

To create a sequence of regular simplices, $\{\boldsymbol{\sigma_k}\}_{k=1}^{n}$, centred at the origin, we begin with a seed, $\boldsymbol{\sigma_1}$, corresponding to Δ^1 aligned with the $\boldsymbol{e_1}$ axis. At each step of the process, a new regular simplex is generated with dimension one greater than the previous simplex. Of course, each simplex of a given dimension contains regular simplices of all lower dimensions. Accordingly, we begin the sequence with the trivial simplex

$$\boldsymbol{\sigma_1} = \{-\frac{\boldsymbol{e}_1}{2}, \frac{\boldsymbol{e}_1}{2}\} = \{\boldsymbol{r}_1, \boldsymbol{r}_2\}\,. \tag{7.1}$$

Certainly, the trivial simplex $\boldsymbol{\sigma_1}$ has the property that $\sum_{a=1}^{2} \check{\boldsymbol{r}}_{\boldsymbol{a}}^{\,2} = N(N-1)/4 = 1/2$. To construct $\boldsymbol{\sigma_2}$, we add another orthogonal coordinate, $\boldsymbol{e}_2$, then scale $\boldsymbol{\sigma_1}$ by $\sqrt{1+2}$. The next step is to shift $\boldsymbol{\sigma_1}$ 'downward' by 1/2 along the $\boldsymbol{e}_2$ axis. The final step is simply to add the last vertex perpendicular to $\boldsymbol{\sigma_1}$ scaled by $\frac{2}{2}$, yielding

$$\boldsymbol{\sigma_2} = \{-\frac{\sqrt{3}}{2}\boldsymbol{e}_1 - \frac{1}{2}\boldsymbol{e}_2, \frac{\sqrt{3}}{2}\boldsymbol{e}_1 - \frac{1}{2}\boldsymbol{e}_2, \boldsymbol{e}_2\} \tag{7.2}$$

$$= \{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3\}\,. \tag{7.3}$$

It is easy to see that this process can be continued indefinitely, expanding space by an orthogonal coordinate at each step, with the first three $\boldsymbol{\sigma_k}$ depicted in figure 6. This scaling, shifting and vertex-adding recurrence relation can be written compactly if we allow $T_{-\frac{1}{2}}$ to be the translation operator, which shifts $\boldsymbol{\sigma_k}$ down along the $\boldsymbol{e_{k+1}}$ axis such that, $T_{-\frac{1}{2}}[\boldsymbol{r_a}] = \boldsymbol{r_a} - \frac{1}{2}\boldsymbol{e_{k+1}}$ where $\boldsymbol{r_a}$ is a vertex belonging to $\boldsymbol{\sigma_k}$

$$\boldsymbol{\sigma_{k+1}} = \{T_{-\frac{1}{2}}[\sqrt{\frac{k+2}{k}}\boldsymbol{\sigma_k}], (\frac{k+1}{2})\boldsymbol{e_{k+1}}\}\,. \tag{7.4}$$

Each $\boldsymbol{\sigma_k}$ can be considered as an ordered list of vertex vectors, with scaling and translation in the recurrence relation applied to each vertex in the list. Using equation (7.4) to build up regular simplices, we can show the invariance of the sum of the squares of the projected vertices to each coordinate axis. In the following proof, we have set the vertex length of the simplices to be $|\boldsymbol{r_a}| = (N-1)/2$, a choice that does not affect invariance.

Theorem 7.1 (Schläfi). *The sum of the squares of the projected vertices of a regular simplex, centred at the origin, to a coordinate axis,* $\boldsymbol{e_j}$, *are equal and invariant under rotations of the simplex. In other words, the vertices of a regular simplex are eutactic.*

Proof. We can begin a straightforward inductive proof at $k = 2$. Recalling that the sum of the squares of the projections to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ of $\boldsymbol{\sigma_2}$ are separately equal to

$$\sum_{a=1}^{3} \check{\boldsymbol{r}}_{\boldsymbol{a},j}^{2} = \frac{(k+1)k}{4} = \frac{N(N-1)}{4} = \frac{3}{2}\,. \tag{7.5}$$

We assume this to be true for the projections of the vertices to each coordinate axis for the simplex $\boldsymbol{\sigma_k}$. We must show that the projections of the vertices of the next simplex in the sequence, $\boldsymbol{\sigma_{k+1}}$, have this eutactic property.

We can begin by calculating the diagonal entries of W_{N-1}, the $w_{j,j}$. The two cases we need to consider are when the vertices are projected to a coordinate axis from the previous set in the recurrence relation, $\{\boldsymbol{e_j}\}_{j=1}^{k}$, and when they are projected to the new axis, $\boldsymbol{e_{k+1}}$. We will need to show that

$$\sum_{a=1}^{k+2} \check{\boldsymbol{r}}_{\boldsymbol{a}}^{2} = \frac{(k+2)(k+1)}{4} = \frac{N(N-1)}{4}\,. \tag{7.6}$$

From the inductive assumption for $\boldsymbol{\sigma_k}$, we have

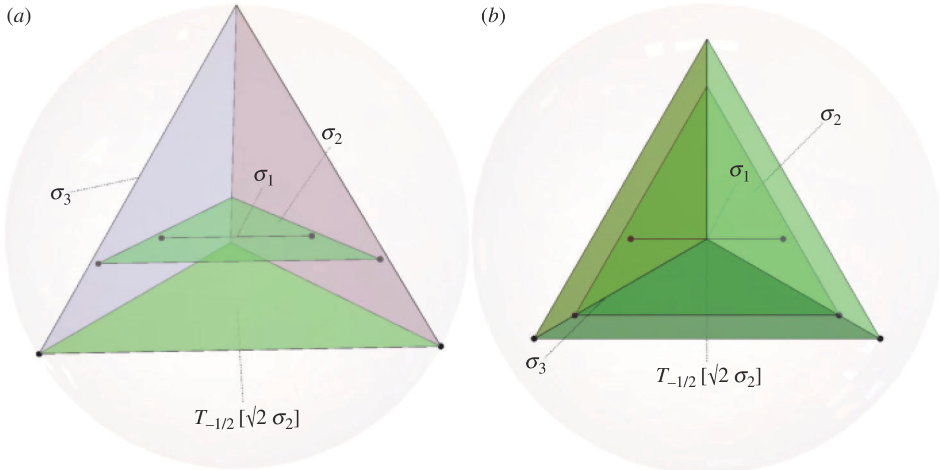


Figure 6. Two views showing the generation of the first three simplices, ($\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$, $\boldsymbol{\sigma_3}$) described in the recurrence relation (equation (7.4)). (*a*) A view of the objects with the $\boldsymbol{e_3}$ axis aligned to the top of the page. (*b*) A view of the simplices looking down along the $\boldsymbol{e_3}$ axis that is perpendicular to the page. The black dots represent the behaviour of a pair of vertices under the transformation in equation (7.4).

$$\sum_{a=1}^{k+1} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{j}}^{\boldsymbol{2}} = \frac{(k+1)k}{4}. \tag{7.7}$$

For the first case, recall that $\boldsymbol{r}_{\boldsymbol{k}+2}$, our most recently added vertex, is orthogonal to the $\{\boldsymbol{e_j}\}_{j=1}^{k}$ and does not contribute to the sum. The other $(k+1)$ vertices of $\boldsymbol{\sigma}_{\boldsymbol{k}+1}$, $\{\boldsymbol{r_a}\}_{a=1}^{k+1}$ are just the scaled and shifted vertices of $\boldsymbol{\sigma_k}$. After scaling the $\boldsymbol{\sigma_k}$ vertices from equation (7.4), the sum of the squared projections yields

$$\sum_{a=1}^{k+2} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{j}}^{\boldsymbol{2}} = \frac{(k+2)}{k}\frac{(k+1)k}{4} = \frac{(k+2)(k+1)}{4} = \frac{N(N-1)}{4}. \tag{7.8}$$

For the second case, we calculate the sum of the squared projections of the $\{\boldsymbol{r_a}\}_{a=1}^{k+2}$ to the remaining $\boldsymbol{e}_{\boldsymbol{k}+1}$ axis. It is a simple matter to calculate from the recurrence relation as $(k+1)$ of these have $-(1/2)\boldsymbol{e}_{\boldsymbol{k}+1}$ projections with $\boldsymbol{e}_{\boldsymbol{k}+1}$. The remaining vertex and the sum are given by

$$\boldsymbol{r}_{\boldsymbol{k}+2} = \frac{(k+1)}{2}\boldsymbol{e}_{\boldsymbol{k}+1} \tag{7.9}$$

$$\sum_{a=1}^{k+2} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{k}+1}^{\boldsymbol{2}} = \frac{k+1}{4} + \left(\frac{k+1}{2}\right)^2 = \frac{(k+2)(k+1)}{4} = \frac{N(N-1)}{4}. \tag{7.10}$$

We now have the diagonal entries of W_{N-1} described in equation (6.1), where $w_{jj} = N(N-1)/4$.

For the off-diagonal entries, we know from equation (6.3) that the assertion is true for $k=2$. We must show that if $j \neq l$ and assuming the relation holds for $\boldsymbol{\sigma_k}$

$$w_{j,l} = \sum_{a=1}^{k+2} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{j}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{l}} = 0. \tag{7.11}$$

Again, there are only two cases we need to consider when computing the sum given above. The first case is where the set of $\{\boldsymbol{r_a}\}_{a=1}^{k+2}$ vertex vectors is projected to an axis, $\boldsymbol{e_j}$ or $\boldsymbol{e_l}$ from the previous collection of axes as before, the set $\{\boldsymbol{e_m}\}_{m=1}^{k}$

$$w_{j,l} = \sum_{a=1}^{k+2} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{j}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{l}} = \sum_{a=1}^{k+1} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{j}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{l}} + \check{\boldsymbol{r}}_{\boldsymbol{k+2},\,\boldsymbol{j}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{k+2},\,\boldsymbol{l}}. \tag{7.12}$$

The first term on the right-hand side is zero due to the inductive assumption. The second term contains the new vertex, $\boldsymbol{r}_{\boldsymbol{k+2}}$ projected to a member of the previous set of coordinate axes, but this must be zero since $\boldsymbol{r}_{\boldsymbol{k+2}}$ is orthogonal to any $\boldsymbol{e}_{\boldsymbol{j}}$ or $\boldsymbol{e}_{\boldsymbol{l}}$ in the set $\{\boldsymbol{e}_{\boldsymbol{m}}\}_{m=1}^{k}$ by construction.

The second case for the off-diagonal entries is where one set of vertex vectors in the dot product is projected to the $\boldsymbol{e}_{\boldsymbol{k+1}}$ axis, $\{\check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{k+1}}\}_{a=1}^{k+2}$. We have

$$w_{k+1,l} = \sum_{a=1}^{k+2} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{k+1}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{l}} = \sum_{a=1}^{k+1} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{k+1}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{l}} + \check{\boldsymbol{r}}_{\boldsymbol{k+2},\,\boldsymbol{k+1}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{k+2},\,\boldsymbol{l}}. \tag{7.13}$$

The second term on the right-hand side is easily seen to be zero, as we noted that the $\boldsymbol{r}_{\boldsymbol{k+2}}$ vertex is perpendicular to the set of $\{\boldsymbol{e}_{\boldsymbol{m}}\}_{m=1}^{k}$, which means that $\check{\boldsymbol{r}}_{\boldsymbol{k+2},\,\boldsymbol{l}} = 0$. The first term is straightforward to calculate

$$\sum_{a=1}^{k+1} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{k+1}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{l}} = -\frac{1}{2}\boldsymbol{e}_{\boldsymbol{k+1}} \cdot \left(\sum_{a=1}^{k+1} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{l}}\right) = 0. \tag{7.14}$$

From the recurrence relation (7.4), we have $\{\check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{k+1}}\}_{a=1}^{k+1} = -(1/2)\boldsymbol{e}_{\boldsymbol{k+1}}$. In the summand on the right-hand side, these are the shifted and scaled vertices from the regular simplex $\boldsymbol{\sigma}_{\boldsymbol{k}}$, projected to an $\boldsymbol{e}_{\boldsymbol{l}}$ from the previous set of coordinate axes, $\{\boldsymbol{e}_{\boldsymbol{m}}\}_{m=1}^{k}$*. By regularity, this sum is zero. Hence, for $j \neq l$

$$w_{j,l} = \sum_{a=1}^{k+2} \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{j}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{a},\,\boldsymbol{l}} = 0$$

Since the diagonal and off-diagonal entries of W_{N-1} have all been computed, we have

$$w_{j,j} = \frac{N(N-1)}{4}$$

$$W_{N-1} = S_{N-1}S_{N-1}^{T} = \frac{N(N-1)}{4} I_{N-1}.$$

To simplify notation, we can now drop the subscript of $(N-1)$ on the matrices. If we denote the projection to the $\boldsymbol{e}_{12}$ plane with P represented as a vector with $(N-1)$ components and q as the sum of the squared projections of the simplex vertices of σ_{N-1} to $\boldsymbol{e}_{12}$, we can write

$$P = [1 \quad 1 \quad 0 \ldots 0] \tag{7.15}$$

$$q = PSS^{T}P^{T} = PWP^{T} \tag{7.16}$$

$$= PRSS^{T}R^{T}P^{T}. \tag{7.17}$$

In equation (7.17), a rotation R applied to S commutes with the matrix W, a scalar multiple of the identity, leaving the sum of the squares of the coordinate projections of the simplex vertices unchanged. Therefore, the sum of the squared projections of the $\{\boldsymbol{r}_{\boldsymbol{a}}\}_{a=1}^{N}$ to the plane $\boldsymbol{e}_{12}$ for the regular simplex $\boldsymbol{\sigma}_{\boldsymbol{N-1}}$ is invariant with respect to rotations and equal to

$$q = \frac{N(N-1)}{4} + \frac{N(N-1)}{4} = \frac{N(N-1)}{2}. \tag{7.18}$$

∎

We will see below that this number exactly matches a point vortex invariant for identical, unit-strength vortices in relative equilibrium, which is all the more remarkable because both numbers emerge in very different contexts. We should also observe for completeness, that for a vertex length given by λ, an identical argument to that given above for vertex projections to a single coordinate axis gives

$$\sum_{a=1}^{N} \check{\boldsymbol{r}}_{\boldsymbol{a}}^{2} = \lambda^2 \frac{N}{N-1}. \tag{7.19}$$

8. Vortex invariants

Point vortices that satisfy the relative equilibrium condition and have identical strengths have a hierarchy of interesting invariants or moments, described by Aref & van Buren [18], of which we will use the first two. Let the vortex vectors in relative equilibrium be given by $\{\boldsymbol{v}_{\boldsymbol{a}}\}_{a=1}^{N}$. The first vortex invariant is straightforward, and the second can be derived using straightforward algebraic means. Let the $\{\boldsymbol{v}_{\boldsymbol{a}}\}_{a=1}^{N}$ be vortex vectors in the plane and $\{\boldsymbol{r}_{\boldsymbol{a}}\}_{a=1}^{N}$ be vertex vectors of a regular simplex. Furthermore, continuing with our notational convention, we will use the check symbol ($\check{}$) to denote a vector's projection from $\mathbb{R}^{N-1}$ to the plane $\boldsymbol{e}_{12}$

$$N_1 = \sum_{a=1}^{N} \boldsymbol{v}_{\boldsymbol{a}} = \sum_{a,b=1}^{N} \frac{1}{\boldsymbol{v}_{\boldsymbol{a}} - \boldsymbol{v}_{\boldsymbol{b}}} = 0 \tag{8.1}$$

$$M_1 = \sum_{a=1}^{N} \boldsymbol{v}_{\boldsymbol{a}}^2 = \sum_{a,b=1}^{N} \frac{\boldsymbol{v}_{\boldsymbol{a}}}{\boldsymbol{v}_{\boldsymbol{a}} - \boldsymbol{v}_{\boldsymbol{b}}} = \frac{N(N-1)}{2}. \tag{8.2}$$

As for the simplex vertices, we also have the corresponding equalities with the appropriate choice of λ

$$N_1 = \sum_{a=1}^{N} \boldsymbol{r}_{\boldsymbol{a}} = \sum_{a=1}^{N} \check{\boldsymbol{r}}_{\boldsymbol{a}} = 0 \tag{8.3}$$

$$M_1 = \sum_{a=1}^{N} \check{\boldsymbol{r}}_{\boldsymbol{a}}^{2} = \frac{N(N-1)}{2}. \tag{8.4}$$

We have added another interesting invariant to the collection, N_1, which captures how both the projected vertices and the vortex vectors sum to zero. For vertices, this is a direct result of regularity, while for vortices it emerges from the equilibrium condition. We have already seen that for a value of $(N-1)/2$, λ in equation (7.19) coincides with the vortex invariant, M_1. The vertex length λ, now taken as a coupling constant, takes the values below depending on whether the simplex invariant corresponds to the M_1 invariant for vortices in $\boldsymbol{e}_1$ or the plane $\boldsymbol{e}_{12}$ respectively.

$$\lambda = \left\{\frac{(N-1)}{\sqrt{2}}, \frac{(N-1)}{2}\right\}. \tag{8.5}$$

With the invariance of the sum of the squares of the vertex projections in hand, we have enough information to write a lifting theorem. Our approach will use an argument based on linear maps between vector spaces and does not directly invoke geometric algebra. Using the invariance described above, we will show that a rotation of a simplex in $\mathbb{R}^{N-1}$ is a transformation that preserves the coinciding invariants of the two systems, the projected vertices of the simplex and the point vortex vectors.

9. The lifting theorem

Theorem 9.1 (Lifting Theorem). *For every system of N unit-strength point vortices in* $\mathbb{R}^2$*, which satisfy* $\boldsymbol{v_a} = \sum_{b=1}^{N} \frac{1}{\boldsymbol{v_a} - \boldsymbol{v_b}}$*, a regular simplex,* Δ^{N-1}*, with vertex length,* $|\boldsymbol{r_a}| = (N-1)/2$ *can be oriented such that the orthogonal projection of its N vertices will coincide with the vortex positions. This also holds if the vortex positions are confined to* $\mathbb{R}^1$*, with vertex length* $|\boldsymbol{r_a}| = (N-1)/\sqrt{2}$*.*

Proof. To show that the vertices of a correctly oriented simplex, Δ^{N-1}, project to a point vortex relative equilibrium for a given N, we begin by forming the following vectors, $\boldsymbol{x}^{\Delta}$ and $\boldsymbol{y}^{\Delta}$, from the projections of the vertex vectors $\{\boldsymbol{r_a}\}_{a=1}^{N}$ of the suitably scaled simplex to the $(\boldsymbol{e_1}, \boldsymbol{e_2})$ axes in $\mathbb{R}^2$, and a corresponding pair of vectors formed in the same manner for the relative equilibrium solution. Let $P_{\boldsymbol{e_1}}[*]$ and $P_{\boldsymbol{e_2}}[*]$ be the projection operators to $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$.

The projection operator on the simplex vertices produces a vector $\boldsymbol{x}^{\Delta}$, which is an ordered list of the $\boldsymbol{e_1}$-component of each vertex, $\{\boldsymbol{r_a}\}_{a=1}^{N}$ in Δ^{N-1}, with $\boldsymbol{y}^{\Delta}$ being formed in a similar way for the $\boldsymbol{e_2}$-components.

$$\boldsymbol{x}^{\Delta} = \{x_a\}_{a=1}^{N} = \{P_{\boldsymbol{e_1}}[\boldsymbol{r_a}]\}_{a=1}^{N} \tag{9.1}$$

$$\boldsymbol{y}^{\Delta} = \{y_a\}_{a=1}^{N} = \{P_{\boldsymbol{e_2}}[\boldsymbol{r_a}]\}_{a=1}^{N}. \tag{9.2}$$

We note that the squared magnitudes of these vectors composed of $\boldsymbol{e_1}$-components and $\boldsymbol{e_2}$-components are equal to the same constant from theorem 7.1 given above.

$$\|\boldsymbol{x}^{\Delta}\|^2 = \|\boldsymbol{y}^{\Delta}\|^2 = \frac{N(N-1)}{4} \tag{9.3}$$

$$\|\boldsymbol{x}^{\Delta}\|^2 + \|\boldsymbol{y}^{\Delta}\|^2 = \frac{N(N-1)}{2}. \tag{9.4}$$

Similarly, we can form another pair of vectors, $\boldsymbol{x}^{\dagger}$ and $\boldsymbol{y}^{\dagger}$ from the $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ components of the point vortex vectors that comprise the equilibrium solution

$$\boldsymbol{x}^{\dagger} = \{x_a^{\dagger}\}_{a=1}^{N} \tag{9.5}$$

$$\boldsymbol{y}^{\dagger} = \{y_a^{\dagger}\}_{a=1}^{N}. \tag{9.6}$$

These vectors obey the same invariants as the vertex vector projections

$$\|\boldsymbol{x}^{\dagger}\|^2 = \|\boldsymbol{y}^{\dagger}\|^2 = \frac{N(N-1)}{4} \tag{9.7}$$

$$\|\boldsymbol{x}^{\dagger}\|^2 + \|\boldsymbol{y}^{\dagger}\|^2 = \frac{N(N-1)}{2}. \tag{9.8}$$

First, recall that the following is also true for both the coefficients of the solution $\boldsymbol{e_1}$-component vector, $\boldsymbol{x}^{\dagger}$, and the vertex $\boldsymbol{e_1}$-component vector, $\boldsymbol{x}^{\Delta}$.

$$\sum_{a=1}^{N} x_a^{\Delta} = 0 \tag{9.9}$$

$$\sum_{a=1}^{N} x_a^{\dagger} = 0. \tag{9.10}$$

The first equation follows from the fact that the sum of the vertex vectors of a regular simplex is zero, so the sum of the orthogonal projections is also zero. The second equation follows from the relative equilibrium condition for point vortices. Of course, there is a corresponding pair of equations for the e_2-components. Note that the N-th coefficient of bold vectors defined above is dependent on the other coefficients

$$\boldsymbol{x}^{\boldsymbol{\Delta}} = \{x_1^{\Delta}, \ldots, x_{N-1}^{\Delta}, -\sum_{a=1}^{N-1} x_a^{\Delta}\} \tag{9.11}$$

$$\boldsymbol{x}^{\boldsymbol{\dagger}} = \{x_1^{\dagger}, \ldots, x_{N-1}^{\dagger}, -\sum_{a=1}^{N-1} x_a^{\dagger}\}. \tag{9.12}$$

Consider the pair of vectors $(\boldsymbol{x}^{\boldsymbol{\Delta}}, \boldsymbol{x}^{\dagger})$ in R^N and observe that there exists a linear function that carries $\boldsymbol{x}^{\boldsymbol{\Delta}}$ to $\boldsymbol{x}^{\dagger}$, defined entirely by its action on $(N-1)\boldsymbol{e_1}$ components of $\boldsymbol{x}^{\boldsymbol{\Delta}}$. We will denote this map as $\boldsymbol{\tau_x}$ and observe that it acts on $\mathbb{R}^{N-1}$. Here, $\boldsymbol{\tau_x}$ is taken to be in matrix form. For the $\boldsymbol{e}_2$-components, we can form another linear function, $\boldsymbol{\tau_y}$, which carries the vector $\boldsymbol{y}^{\boldsymbol{\Delta}}$ to $\boldsymbol{y}^{\dagger}$.

$$\boldsymbol{x}^{\boldsymbol{\dagger}} = \boldsymbol{\tau_x}\boldsymbol{x}^{\boldsymbol{\Delta}} \tag{9.13}$$

$$\boldsymbol{y}^{\boldsymbol{\dagger}} = \boldsymbol{\tau_y}\boldsymbol{y}^{\boldsymbol{\Delta}}. \tag{9.14}$$

Since $\boldsymbol{\tau_x}$ acts on the $\boldsymbol{e_1}$-components of the simplex, there must be an induced map, also linear, which acts in R^{N-1} on its vertices, $\{\boldsymbol{r_a}\}_{a=1}^{N}$. Let us denote this with the symbol $\boldsymbol{\rho}_x$. For the purposes of clarity in the diagram given below, Δ^{N-1} is simply denoted as $\boldsymbol{\Delta}$ and $\boldsymbol{A}$ is the image of Δ^{N-1} under $\boldsymbol{\psi} = (\boldsymbol{\rho_x}, \boldsymbol{\rho_y})$, where $\boldsymbol{\psi}$ represents the induced pair of maps that carry the vertex vectors to their image in $\boldsymbol{A}$. The vector spaces $(K^{\Delta}, K^{\dagger})$ contain $(\boldsymbol{x}^{\Delta}, \boldsymbol{y}^{\Delta})$ and $(\boldsymbol{x}^{\dagger}, \boldsymbol{y}^{\dagger})$, respectively.

$$\begin{array}{ccc} \Delta & \xrightarrow{\ \boldsymbol{\psi}\ } & A \\ {\scriptstyle P_{\boldsymbol{e1},\boldsymbol{e2}}}\downarrow & & \downarrow{\scriptstyle P_{\boldsymbol{e1},\boldsymbol{e2}}} \\ K^{\Delta} & \xrightarrow[\ \boldsymbol{\tau}\ ]{} & K^{\dagger} \end{array}$$

Essentially, the approach is to show that a rotation satisfies the constraints placed on the linear transformation $\boldsymbol{\psi}$ from $\boldsymbol{\Delta}$ to $\boldsymbol{A}$ by the invariants described above.

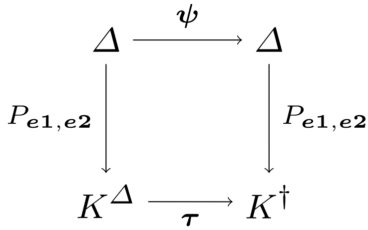
The most general linear transformation from $\mathbb{R}^{N-1}$ to $\mathbb{R}^{N-1}$ has the form

$$\boldsymbol{\psi} = \boldsymbol{M}[\boldsymbol{r_a}] + \boldsymbol{T}. \tag{9.15}$$

First, we examine the action of a translation, $\boldsymbol{T}$, on the vertices of Δ^{N-1}. If a translation affects the projected components of Δ^{N-1}, the projected vectors of $\boldsymbol{A}$ would contradict equations (9.10) and (9.7); hence, those components of the translation $\boldsymbol{T}$ must be zero. If the translation acts only on the orthogonal components of the vertices, it will not affect the projections; therefore, we can set $\boldsymbol{T} = \boldsymbol{0}$.

Moving onto the map $\boldsymbol{M}$, let us make the assumption for the moment that $\boldsymbol{M}$ is non-singular and allow it to be a scaling transformation of the $\{\boldsymbol{r_a}\}_{a=1}^{N}$. Note that if $\boldsymbol{M}$ is a scaling transformation that affects the projected components of Δ^{N-1}, this would contradict equation (9.7). If $\boldsymbol{M}$ scales only the orthogonal components, the solutions remain unchanged. We can therefore set the scaling constants to be 1. It follows that $\boldsymbol{M}$ is an isometry. As we have already ruled out a translation, the non-singular $\boldsymbol{M}$ must therefore be a rotation.

Our last task is to consider what the action of a singular $\boldsymbol{M}$ has on the vertices of Δ^{N-1}. Since a singular transformation, $\boldsymbol{\psi}$, most generally, is the composition of a non-singular transformation and a projection, it is sufficient to allow $\boldsymbol{\psi}$ to be a simple projection operator. If a projection affects the projected components of Δ^{N-1}, then it must change the length of $\boldsymbol{x}^\dagger$ or $\boldsymbol{y}^\dagger$ and contradict equation (9.7). Once again, any change to the orthogonal components of the vertices of Δ^{N-1} does not affect the solution; therefore, we can consider the map $\boldsymbol{M}$ to be non-singular. From the above observations for a non-singular map, $\boldsymbol{M}$, must be a rotation, which means that the induced map, $\boldsymbol{\psi}$, can be a rotation of the simplex. The commutative diagram becomes



where the induced map $\boldsymbol{\psi}$ is a rotation in $\mathbb{R}^{N-1}$. ∎

10. Bivector equation for relative equilibrium

Theorem 10.1 (Bivector equation for relative equilibrium). *If N unit-strength vortices are in relative equilibrium, the vertices of the corresponding simplex, Δ^{N-1}, will satisfy the geometric relationship given by $\sum_{a,b=1}^{N} \hat{\boldsymbol{r}}_{\boldsymbol{ab}}/\check{\boldsymbol{r}}_{\boldsymbol{ab}} = 0$, where the $\{\hat{\boldsymbol{r}}_{\boldsymbol{ab}}\}_{a,b=1}^{N}$ represents the rejection, the component of the simplex edge that is orthogonal to the subspace containing the vortices, either $\mathbb{R}^1$ or $\mathbb{R}^2$, and the $\{\check{\boldsymbol{r}}_{\boldsymbol{ab}}\}_{a,b=1}^{N}$ are the edge projections to the target subspace. The vertex projections, $\{\check{\boldsymbol{r}}_{\boldsymbol{a}}\}_{a=1}^{N}$, are the vertex positions in the rotating frame.*

Proof. The first task is to find the stationary points of the point vortex Hamiltonian with respect to a rotation ψ. To this end, we define the following projections: Let the vertices of the simplex be given by $\{\boldsymbol{r_a}\}_{a=1}^{N}$, and the orthogonal projection to the $\boldsymbol{e}_{12}$ plane by the operator P, such that

$$P_{\mathbf{e}_{12}}[\boldsymbol{r_a} - \boldsymbol{r_b})] = \check{\boldsymbol{r}}_{\boldsymbol{a}} - \check{\boldsymbol{r}}_{\boldsymbol{b}} = \check{\boldsymbol{r}}_{\boldsymbol{ab}}\,. \tag{10.1}$$

To introduce the action of a rotation on the vertices of the simplex, let

$$\boldsymbol{r}'_{\boldsymbol{a}} = \psi \boldsymbol{r_a} \psi^{-1} \tag{10.2}$$

$$\check{\boldsymbol{r}}'_{\boldsymbol{ab}} = P_{\mathbf{e}_{12}}[\psi \boldsymbol{r_a} \psi^{-1}]\,. \tag{10.3}$$

We can now rewrite the point vortex Hamiltonian (2.2) in terms of the projected simplex vertices with the $\Gamma_a = 1$.

$$H = -\frac{1}{4\pi} \sum_{a,b=1}^{N} \log |\check{\boldsymbol{r}}_{\boldsymbol{ab}}|\,. \tag{10.4}$$

What follows is just a formal application of the multivector derivative to a scalar function, using elementary algebraic operations and identities in the geometric calculus for $Cl(N-1,0)$. If we denote the scalar product with $(*)$, the stationary states of H with respect to a rotation ψ will satisfy

$$\partial_\psi H = \partial_\psi \left(-\frac{1}{4\pi} \sum_{a,b=1}^{N} \log |\check{\boldsymbol{r}}'_{\boldsymbol{ab}}| \right) = 0\,. \tag{10.5}$$

Applying the multivector derivative to the *log* function and using the chain rule gives the expression

$$\sum_{a,b=1}^{N} \partial_\psi \log |\check{\boldsymbol{r}}'_{\boldsymbol{ab}}| = \sum_{a,b=1}^{N} (\partial_\psi (P_{e_{12}}[\psi \boldsymbol{r}_{\boldsymbol{ab}} \psi^{-1})])) * (\check{\boldsymbol{r}}'_{\boldsymbol{ab}})^{-1}. \tag{10.6}$$

As the summand on the right-hand side is a scalar quantity and the derivative operator only acts on the left-most factor before the scalar product (it has already emerged from the unwrapping of the *log* function as part of the chain rule), we can rewrite the expression using the scalar projection operator. Once inside the scalar projection brackets, the projection operator $P_{e_{12}}$ can be discarded, as the scalar product is with a vector, $(\check{\boldsymbol{r}}'_{\boldsymbol{ab}})^{-1}$, which lies in $\boldsymbol{e}_{12}$. Below, we differentiate with respect to ψ while holding $(\check{\boldsymbol{r}}'_{\boldsymbol{ab}})^{-1}$ constant, making use of the scalar symmetry of interchanging factors within scalar projection brackets. For convenience, we denote $(\check{\boldsymbol{r}}'_{\boldsymbol{ab}})^{-1} = \boldsymbol{g}_{\boldsymbol{ab}}$. Additionally, an overdot shall indicate which factor the multivector derivative is acting upon. For example, $\dot{\partial}_\psi \dot{\psi} X \psi$ indicates that the multivector derivative is acting on the left-most factor. The right-hand side of equation (10.6) becomes

$$\sum_{a,b=1}^{N} \partial_\psi \langle \psi \boldsymbol{r}_{\boldsymbol{ab}} \psi^{-1} \boldsymbol{g}_{\boldsymbol{ab}} \rangle = \sum_{a,b=1}^{N} \dot{\partial}_\psi \langle \dot{\psi}(\boldsymbol{r}_{\boldsymbol{ab}} \psi^{-1} \boldsymbol{g}_{\boldsymbol{ab}}) \rangle + \dot{\partial}_\psi \langle \dot{\psi}^{-1} (\boldsymbol{g}_{\boldsymbol{ab}} \psi \boldsymbol{r}_{\boldsymbol{ab}}) \rangle \tag{10.7}$$

$$= \sum_{a,b=1}^{N} (\boldsymbol{r}_{\boldsymbol{ab}} \psi^{-1} \boldsymbol{g}_{\boldsymbol{ab}} - \psi^{-1} (\boldsymbol{g}_{\boldsymbol{ab}} \psi \boldsymbol{r}_{\boldsymbol{ab}}) \psi^{-1}) \tag{10.8}$$

$$= 2\psi^{-1} \sum_{a,b=1}^{N} (\psi \boldsymbol{r}_{\boldsymbol{ab}} \psi^{-1}) \wedge \boldsymbol{g}_{\boldsymbol{ab}} \tag{10.9}$$

$$= 2\psi^{-1} \sum_{a,b=1}^{N} \boldsymbol{r}'_{\boldsymbol{ab}} \wedge (\check{\boldsymbol{r}}'_{\boldsymbol{ab}})^{-1}. \tag{10.10}$$

In equation (10.8), we used the identities for the multivector derivative of a scalar projection given by equations (3.8) and (3.10) earlier. In equation (10.9), we used equation (3.11) to obtain the derivative with respect to a rotation ψ. For one further simplification, recall that $\boldsymbol{r}_{\boldsymbol{ab}}' = \hat{\boldsymbol{r}}'_{\boldsymbol{ab}} + \check{\boldsymbol{r}}_{\boldsymbol{ab}}'$. Since the left-most factor in the summand is constant and only the orthogonal $\hat{\boldsymbol{r}}'_{\boldsymbol{ab}}$ components contribute to the sum, the expression can be simplified and the primes can be dropped

$$\sum_{a,b=1}^{N} \hat{\boldsymbol{r}}_{\boldsymbol{ab}} \wedge (\check{\boldsymbol{r}}_{\boldsymbol{ab}})^{-1} = 0. \tag{10.11}$$

Or more succinctly, since $\hat{\boldsymbol{r}}_{\boldsymbol{ab}} \cdot \check{\boldsymbol{r}}_{\boldsymbol{ab}} = 0$

$$\sum_{a,b=1}^{N} \frac{\hat{\boldsymbol{r}}_{\boldsymbol{ab}}}{\check{\boldsymbol{r}}_{\boldsymbol{ab}}} = 0. \tag{10.12}$$

Now that we have an equation for the stationary points of the simplex Hamiltonian with respect to a rotation ψ, we can employ the lifting theorem 9.1 to show that a system in relative equilibrium (4.3) will satisfy (10.12). To begin, we are going to make good use of the eutactic property of the simplex vertices. If we differentiate the following expression with $\check{\boldsymbol{r}}'_{\boldsymbol{a}}$, representing rotated simplex vertices, and setting the right-hand side to zero from the eutactic property of the simplex

$$\partial_\psi \left(\sum_{a=1}^{N} \check{\boldsymbol{r}}'^2_{\boldsymbol{a}} \right) = \sum_{a=1}^{N} \partial_\psi \langle \check{\boldsymbol{r}}'_{\boldsymbol{a}} \check{\boldsymbol{r}}'_{\boldsymbol{a}} \rangle = 0. \tag{10.13}$$

We obtain a compact equation using a process very similar to the derivation of the bivector equation (10.12)

$$\sum_{a=1}^{N} \hat{\boldsymbol{r}}_{\boldsymbol{a}} \wedge \check{\boldsymbol{r}}_{\boldsymbol{a}} = 0\,. \tag{10.14}$$

Let a system of point vortices in $\mathbb{R}^2$ satisfy the vector equilibrium condition (4.3).

$$\boldsymbol{v}_{\boldsymbol{a}} = \sum_{b=1}^{N} \frac{1}{\boldsymbol{v}_{\boldsymbol{a}} - \boldsymbol{v}_{\boldsymbol{b}}}\,. \tag{10.15}$$

Using the lifting theorem 9.1 we can exchange vortex positions for projected vertices and use the outer product to obtain

$$\check{\boldsymbol{r}}_{\boldsymbol{a}} = \sum_{b=1}^{N} \frac{1}{\check{\boldsymbol{r}}_{\boldsymbol{a}} - \check{\boldsymbol{r}}_{\boldsymbol{b}}} \tag{10.16}$$

$$\hat{\boldsymbol{r}}_{\boldsymbol{a}} \wedge \check{\boldsymbol{r}}_{\boldsymbol{a}} = \sum_{b=1}^{N} \frac{\hat{\boldsymbol{r}}_{\boldsymbol{a}}}{\check{\boldsymbol{r}}_{\boldsymbol{a}} - \check{\boldsymbol{r}}_{\boldsymbol{b}}} \tag{10.17}$$

$$\sum_{a=1}^{N} \hat{\boldsymbol{r}}_{\boldsymbol{a}} \wedge \check{\boldsymbol{r}}_{\boldsymbol{a}} = \sum_{a,b=1}^{N} \frac{\hat{\boldsymbol{r}}_{\boldsymbol{a}}}{\check{\boldsymbol{r}}_{\boldsymbol{a}} - \check{\boldsymbol{r}}_{\boldsymbol{b}}} = 0. \tag{10.18}$$

In equation (10.18), we used the compact identity for simplex vertices, (10.14). Note that swapping the indices on the right-hand side of equation (10.18) gives

$$\sum_{a,b=1}^{N} \frac{\hat{\boldsymbol{r}}_{\boldsymbol{a}}}{\check{\boldsymbol{r}}_{\boldsymbol{a}} - \check{\boldsymbol{r}}_{\boldsymbol{b}}} = \sum_{a,b=1}^{N} \frac{\hat{\boldsymbol{r}}_{\boldsymbol{b}}}{\check{\boldsymbol{r}}_{\boldsymbol{b}} - \check{\boldsymbol{r}}_{\boldsymbol{a}}}\,. \tag{10.19}$$

Since both these above sums vanish, we have

$$\sum_{a,b=1}^{N} \frac{\hat{\boldsymbol{r}}_{\boldsymbol{a}}}{\check{\boldsymbol{r}}_{\boldsymbol{a}} - \check{\boldsymbol{r}}_{\boldsymbol{b}}} - \sum_{a,b=1}^{N} \frac{\hat{\boldsymbol{r}}_{\boldsymbol{b}}}{\check{\boldsymbol{r}}_{\boldsymbol{a}} - \check{\boldsymbol{r}}_{\boldsymbol{b}}} = 0 \tag{10.20}$$

$$\sum_{a,b=1}^{N} \frac{\hat{\boldsymbol{r}}_{\boldsymbol{ab}}}{\check{\boldsymbol{r}}_{\boldsymbol{ab}}} = 0 \tag{10.21}$$

∎

An interesting interpretation of the bivector equation arises when we note that each term in the sum is simply the bivector tangent of θ_{ab}, the angle the simplex edge, $\boldsymbol{r}_{\boldsymbol{ab}}$, makes with the $\boldsymbol{e}_{12}$ plane, where

$$\left\| \frac{\hat{\boldsymbol{r}}_{\boldsymbol{ab}}}{\check{\boldsymbol{r}}_{\boldsymbol{ab}}} \right\| = |\tan(\theta_{ab})|\,. \tag{10.22}$$

One way of looking at the bivector tangent terms is to see their magnitudes as a measure of the *observability* of the interaction between a pair of vortices, where the component of the interaction that lives in the dual space, the orthogonal complement to the plane, $\boldsymbol{e}_{12}$, does not contribute to the energy of the system. If the system is in a relative equilibrium state, the bivectors will cancel. As there are no potentials present, the bivector tangent blade cancellation is a fundamental feature of the system's quantization, a beautiful internal symmetry with a well-defined physical meaning.

What is the precise relationship between this bivector equation and relative equilibrium? The bivector equation is the result of finding a condition for a stationary state of the Hamiltonian, parametrized by the rotations and projections of a simplex, which must hold true at the corresponding equilibrium orientation. We should also note that a discussion of whether the stationary states are relative minima will bring into focus the question of the stability of solutions, which is beyond the scope of this article. As it turns out, not all relative equilibria are stable, and, in fact, many of them are not.

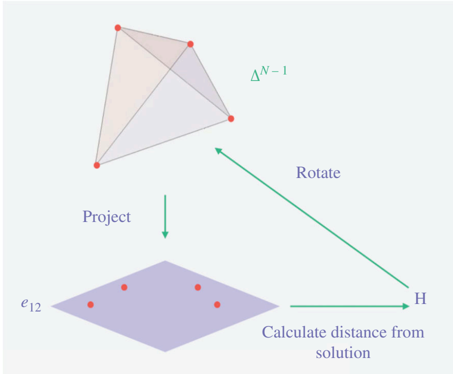


Figure 7. The steps the **solver** takes to converge to a solution.

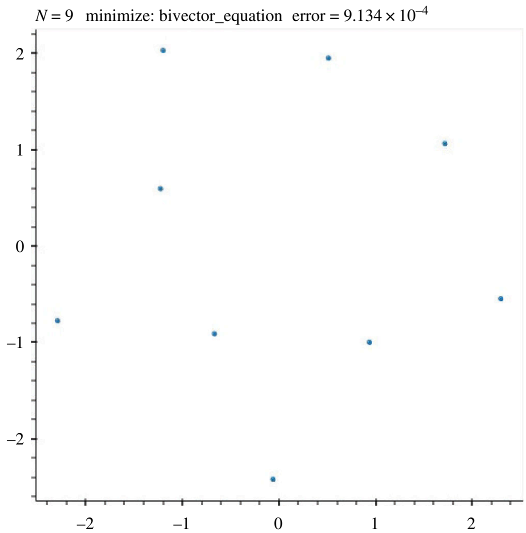


Figure 8. Output of **solver** [14], after rotating a regular simplex, Δ^8, and projecting its vertices, using the bivector equation (10.12) as an objective function.

11. Using the bivector equation to find relative equilibrium solutions

The lifting theorem 9.1 and the bivector equation (10.12) suggest that we can undertake a search for equilibrium solutions making use of only rotations and projections of a simplex to the $\boldsymbol{e}_{12}$ plane. Fortunately, there are very good Clifford algebra Python libraries, which serve this purpose. Using the sci-py optimizer library, the Python **clifford** [19] library, and a random set of initial rotation angles, the **solver** [14] program rotates the simplex, Δ^{N-1}, until it finds projected vertex positions that satisfy the bivector equation, as depicted in figure 7. This approach reveals previously unknown solutions and also captures examples of relative equilibria with no axis of symmetry as described by Aref & Vainchtein [6]. The equilibria in figures 8 and 9 were found using the **solver** program initialized with a set of random angles. **Solver** then used the bivector (10.12) as an objective function, optimizing with a sequential least squares algorithm, to approach a solution. The equilibrium state for $N = 6$ in figure 10 was found in the same manner. Here, a simplex, Δ^5, is shown to be orthogonally projected to $\boldsymbol{e}_{12}$ over contour lines generated in Mathematica.

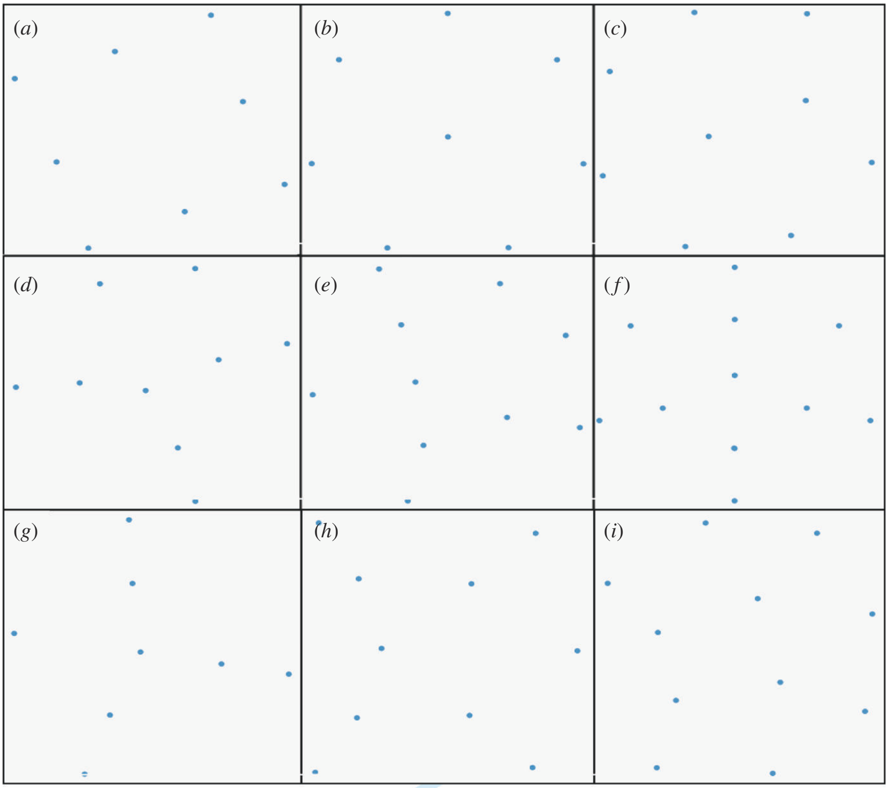


Figure 9. Vortex equilibria found by rotating a simplex, Δ^{N-1}, and projecting vertices, using **solver** [14]. (*a*) $N = 8$, (*b*) $N = 8$, (*c*) $N = 9$, (*d*) $N = 9$, (*e*) $N = 10$, (*f*) $N = 11$, (*g*) $N = 8$, (*h*) $N = 10$ and (*i*) $N = 11$.

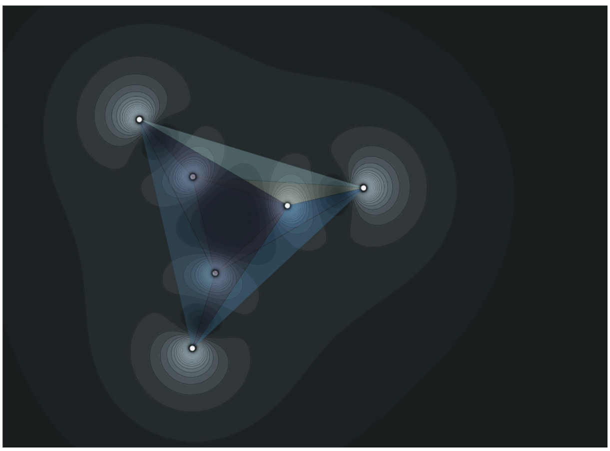


Figure 10. A relative equilibrium state for $N = 6$ with contour lines and superimposed orthogonal projection of Δ^5, found using **solver** [14].

Employing only rotations of a rigid higher-dimensional object, the **solver** successfully generates solutions with high degrees of symmetry, those with only a single axis of symmetry and those with none.

There is another surprising aspect of the lifting theorem 9.1. If the vertex length coupling constant, λ, is taken to be a free parameter, which determines the angular velocity of the system, and we set $z = e^{2\pi i\Omega t}$ and substitute into the equations of motion, we can recover a result surprisingly close to the Feynman relation for quantized vortices. We have

$$\Omega\overline{z_a} = \sum_{b=1}^{N} \frac{\Gamma}{z_a - z_b} \tag{11.1}$$

$$\Omega\boldsymbol{v_a} = \sum_{b=1}^{N} \frac{\Gamma}{\boldsymbol{v_a} - \boldsymbol{v_b}}. \tag{11.2}$$

If the $\{\check{\boldsymbol{r}}_{\boldsymbol{a}}\}_{a=1}^{N}$ are the projections of the simplex vertices and $\lambda = \sqrt{(N-1)/2}$, we obtain from equation (7.19)

$$\frac{\Gamma N(N-1)}{2\Omega} = \sum_{a=1}^{N} \check{\boldsymbol{r}}_a^2 = \lambda^2 \frac{2N}{N-1} \tag{11.3}$$

$$\Omega = \Gamma(N-1)/2. \tag{11.4}$$

This expression differs from the Feynman vorticity relation only by a fixed constant that occurs because a point vortex system consisting of a single point will not rotate. Point vortex systems spin under their own forces, while in the case of the system Feynman describes, the angular momentum is supplied from the experimental apparatus. Equation (11.4) suggests that a simplex, Δ^{N-1}, scaled in this way, could describe the geometry of quantized vortices well.

12. Conclusion

In the point vortex relative equilibrium problem, we have an N-body system that separates in a remarkable manner using the elementary transformations of geometric algebra. Furthermore, we have found a simple bivector equation for stationary states. Using only rotations and orthogonal projections of a higher-dimensional object to minimize energy, the space of solutions is revealed. In our classical problem, the observables are the fixed distances between the point vortices. Even so, it is useful to see each equilibrium solution as a distinct quantum state, with simplex rotations generating all the possible states for N identical particles.

A compelling similarity appears between this simplex approach and the amplituhedron of N. Arkani-Hamed & Trnka [20], in which the positive Grassmannian is used to calculate scattering amplitudes in 4*SYM* theory. Although scattering is manifestly not an equilibrium process, the point vortex simplex, Δ^{N-1}, can be placed in the same class of objects as the amplituhedron, a higher-dimensional object that yields observables under the action of various operators. The simplex approach could be a promising source of solutions for the Gross–Pitaevskii equation for quantized vortices in Bose–Einstein condensates. Lastly, a synthetic approach, which makes use of both the analytic advances and the geometric structure, could produce new classes of exact solutions, a direction for future research.

Data accessibility. The *vortex solve*r software and a set of generated solutions are available from the compressed *tar* file, `vortex_solver.tar.gz`. The directories, `vortex_solver/solver` and `vortex_solver/solutions`, contain the software and the data, respectively. The compressed *tar* archive resides at the following Zenodo link: [21]

Declaration of AI use. I have not used AI-assisted technologies in creating this article.

Authors' contributions. P.R.: conceptualization, formal analysis, funding acquisition, investigation, methodology, software, visualization, writing—original draft.

Conflict of interests. I declare I have no competing interests.

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References

1. Ruderman M. 1970 Long period oscillations in rotating neutron stars. *Nature* **225**, 619–620. (doi:10.1038/225619a0)
2. Abo-Shaeer JR, Raman C, Vogels JM, Ketterle W. 2001 Observation of vortex lattices in Bose-Einstein condensates. *Science* **292**, 476–479. (doi:10.1126/science.1060182)
3. Newton PK. 2001 *The N-vortex problem*. New York, NY: Springer.
4. Aref H, Newton PK, Stremler MA, Tokeida T, Vainchtain DL. 1986 Vortex crystals. *Adv. App. Mech.* **39**, 1–79.
5. Aref H. 2011 Relative equilibria of point vortices and the fundamental theorem of algebra. *Proc. R. Soc. A.* **467**, 2168–2184. (doi:10.1098/rspa.2010.0580)
6. Aref H, Vainchtein DL. 1998 Point vortices exhibit asymmetric equilibria. *Nature* **392**, 669–670. (doi:10.1038/33827)
7. Hestenes D, Sobczyk G. 1985 *Clifford algebra to geometric calculus. a unified language for mathematics and physics*. Dordrecht: Springer.
8. Doran C, Lasenby A. 2003 *Geometric algebra for physicists*. Cambridge, UK: Cambridge University Press.
9. Dorst L, Fontijn D, Mann S. 2007 *Geometric algebra for computer science*. Burlington, MA: Morgan-Kaufmann. (doi:10.1016/B978-012369465-2/50004-9)
10. von Helmholtz H. 1858 On integrals of the hydrodynamic equations that correspond to vortex motions. *J. Reine Angew. Math.* **55**, 22–55. (doi:10.1080/14786446708639824)
11. Aref H. 2007 Point vortex dynamics: a classical mathematics playground. *J. Math. Phys.* **48**, 065401. (doi:10.1063/1.2425103)
12. Lasenby J, Fitzgerald WJ, Lasenby AN, Doran CJL. 1998 New geometric methods for computer vision. *Int. J. Comput. Vis.* **26**, 191–213. (doi:10.1023/A:1007901028047)
13. Stieltjes TJ. 1885 Sur certains polynomes qui verifient une equation differentielle. *Acta. Math.* **6**, 321–326. (doi:10.1007/BF02400421)
14. Robertson P. 2022 Vortex solver: a program for finding relative equilibrium states of identical point vortices. Zenodo. See https://zenodo.org/records/12775490.
15. Schlafi L. 1901 *Theorie der vielfachen kontinuität*, pp. 134–138. Basel, Switzerland: Springer Basel AG.
16. Coxeter HSM. 1948 *Regular polytopes*, pp. 250–262. London, UK: Methuen.
17. Eastwood M, Penrose R. 2000 Drawing with complex numbers. *Math. Intell.* **22**, 8–13. (doi:10.1007/BF03026760)
18. Aref H, van Buren M. 2005 Vortex triple rings. *Phys. Fluids* **17**, 057104. (doi:10.1063/1.1898143)
19. Hadfield H *et al.* Clifford python module. Zenodo. See https://zenodo.org/records/12775490.
20. Arkani-Hamed N, Trnka J. 2014 The amplituhedron. *J. High Energ. Phys.* **30**, 1–32. (doi:10.1007/JHEP10(2014)030)
21. Robertson P. 2022 Figures.nb. Zenodo. See https://zenodo.org/records/12775490.