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## ABSTRACT

This study investigates the quantum features of quantum entanglement (QE) in N-level atomic systems interacting with the thermal state subjected to varying Stark shift (SS) parameters. The N-level atomic system does not move in the presence of the SS interacting with the electromagnetic field. It is observed that a larger value of the SS parameter influences the quantum Fisher information (QFI) of 5-level atomic systems and increases the value of QFI of the 5-level atoms compared to 3- and 4-level atomic systems. The von Neumann entropy (VNE) also increases for the 5-level atoms at different values of the SS parameter. Hence, it is concluded that the SS enhances the VNE of a higher dimensional atomic system such as the 5-level atom. The QE of a higher dimensional atomic system such as the 5-level atom increases with the evolution of time at larger values of the SS parameter, and it also sustains as time progresses. The effect of the SS is more prominent on the 5-level atomic system. When an atom is in motion, in the presence of the SS interacting with the thermal state, the sudden death and birth of QE are observed. The SS environment is favorable for maintaining sudden death and birth of QE, so we can say in the case of a moving N-level atom interacting with the electromagnetic field, the SS supports the QE to maintain and sustain. Hence, non-moving and moving-N-level atomic systems interacting with the electromagnetic field in the presence of the SS are supportive of maintaining and sustaining the QE.

## I. INTRODUCTION

In the framework of quantum physics, notably in the study of quantum information theory and quantum thermodynamics, the relationship between electromagnetic field and thermal field is evident. In classical physics, thermal states refer to the distribution of thermal energy or heat within a system. In quantum physics, thermal states are described using concepts from statistical mechanics, where the behavior of large ensembles of particles is analyzed statistically. When quantum systems are in a thermal state, their quantum states can exhibit quantum entanglement (QE). The QE present in these thermal states can play a role in phenomena such as quantum phase transitions or in the behavior of quantum systems at finite temperatures. Thermal states describe the distribution of heat or energy in classical and quantum systems, and QE introduces a new layer of complexity by showing how quantum states can be interconnected in ways that classical physics cannot account for. The interplay between thermal states and QE is an active area of research in quantum information theory, quantum thermodynamics, and related fields. At temperature T, a source in thermal equilibrium emits a thermal state. Our understanding of the thermal state is therefore somewhat restricted because we only know the energy's mean value.[1,2] QE can always happen when a single qubit in a pure state interacts with an arbitrarily large system in any mixed state, as recently shown by Bose *et al.*[3] To do this, the authors modeled the interaction between a thermal state and a two-level atom. Using this paradigm, they studied the entangling of a qubit with a large system specified in an infinite-dimensional Hilbert space. When the system and thermal state get entangled, the system is reduced to a mixed state. In this work, we look into the following query: "Is it possible for qubits to become entangled in a thermal state, which is an extremely chaotic field?" The issues involved in entangling two atoms or qubits through their mutual interaction with a chaotic field may be entirely different from those involved in entangling an atom and a field. It is possible for both atoms to become entangled in the cavity field mode even if they are originally prepared in their exited states. The two models are not identical, though. In Ref. 4, cavity leakage and atomic spontaneous emission are not taken into account; in contrast, cavity leakage is the main factor in Ref. 5. In this situation, the interaction between the field and the atoms is limited to the strong-coupling regime. However, as Kim *et al.*[4] recently demonstrated, QE between two atoms can be partially induced by such an uncorrelated field. By monitoring the evolution of two identical two-level atoms interacting resonantly with a single-mode thermal state, they were able to calculate the QE between the atoms as a function of time *t*. It is demonstrated that the two atoms' initial states affect their level of QE. If one atom begins in the ground state and the other in the excited state, the thermal state can induce low-degree two-atom QE. Conversely, if both atoms are originally excited, resonant contact with a thermal state cannot produce any QE. Paternostro *et al.*[5] presented a strategy for the development and protection of two-qubit QE more recently. This method took into account the interaction between a leaky cavity in a single-mode thermal state and two-level atoms. Their work has led to the possibility of correctly modulating the atomic transition frequency.

The splitting and shifting of atoms' and molecules' spectral lines as a result of an external electric field is referred to as the Stark shift (SS). This phenomenon was first observed by Johannes Stark in 1913. When an atom or molecule is subjected to an electric field, the energy levels of its electrons are altered, causing changes in the wavelengths of light emitted or absorbed by the atom or molecule. In theoretical studies, researchers may consider systems of entangled atoms or molecules subjected to external electric fields. The presence of QE can affect how the SS manifests in such systems, potentially leading to unique spectroscopic signatures or interactions that differ from those of non-entangled systems. In these applications, researchers may explore how external fields, including electric fields that could induce SS, interact with entangled qubits (quantum bits) or quantum systems designed for information processing purposes. The SS can also be used as a tool to study entangled states or to manipulate quantum systems. For instance, researchers may investigate how the SS of entangled energy levels depends on the degree of QE or how it can be controlled to engineer specific quantum states. Quantum QE and the Stark effect in hydrogen-like atoms were studied by Safronova *et al.*, who also explored the influence of quantum QE on the SS in hydrogen-like atoms, providing theoretical insights into the interplay between QE and external electric fields.[6] Quantum QE and the Stark effect were studied by Vourdas, who also discussed the connection between quantum QE and the SS, presenting mathematical formalism and theoretical considerations related to entangled states in the context of quantum mechanics.[7] QE-assisted Stark effect was studied by S. Ghose, and B. C. Sanders investigated how QE between two qubits can enhance or modify the SS.[8]

When atoms or molecules are subjected to an external electric field (causing the SS), they are typically also influenced by their thermal environment. The thermal motion of particles in the environment can lead to fluctuations in the electric field experienced by the atoms or molecules, affecting the SS. This interplay between the external field and the thermal environment can be studied theoretically and experimentally to understand how thermal fluctuations impact spectroscopic measurements. It discusses how the thermal environment influences the SS and provides theoretical calculations for various scenarios.[9] The SS in a thermal radiation field was studied by A. Mohan, who explored the SS in the presence of a thermal radiation field, considering both theoretical models and experimental implications for spectroscopic measurements.[10]

In many areas, including estimate theory, Fisher information is an essential parameter.[11–13] The quantity of knowledge a quantum can absorb regarding an unknown parameter in a quantum state is quantified by quantum Fisher information, or QFI. Due to the wide range of scientific areas that have gained interest from precise estimation of the values of parameters characterizing an underlying physical state, parameter estimation theory has grown swiftly.[11,14,15] The quantum Fisher information (QFI), which is the inverse of the classical Fisher information (CFI)[16] and whose supremum is used to quantify the lower bound of the mean-square error of the unbiased estimator about the unknown parameter,[17–19] is the cornerstone of quantum parameter estimation theory. Interestingly, QFI is relevant not only to describe estimation precision but also for quantifying how well nearby quantum states can be distinguished statistically,[20,21] the study of quantum correlations,[22–24] the effects that are non-Markovian in nature,[25–28] to analyze the quantum phase transition,[23–32] and the quantum speed limit.[33–36] Quantum Fisher information (QFI) is a captivating concept in quantum physics. It is like the secret sauce that flavors our quantum measurements, especially when we are estimating parameters or phases. The QFI quantifies how precisely we can estimate a parameter (such as a phase) using a given quantum state. Quantum Fisher Information (QFI) plays a crucial role in various aspects of quantum physics and information theory. QFI is a fundamental tool in quantum parameter estimation, where it helps determine the precision with which one can estimate a parameter of a quantum state.[20,37] It provides a bound on the minimum error achievable in estimating this parameter, known as the Cramér–Rao bound. This is crucial in experiments where high-precision measurements are needed. In quantum metrology,[38,39] QFI quantifies the sensitivity of a quantum system to changes in a parameter, which is essential for developing techniques that surpass classical measurement limits. For instance, it can help design quantum sensors that are more accurate than their classical counterparts. QFI can be used to assess how distinguishable two quantum states are, which has implications for quantum information processing tasks such as quantum computing and communication.[40] In quantum thermodynamics, QFI helps in understanding the thermodynamic properties of quantum systems. It can provide bounds on the precision of temperature measurements and energy estimations in quantum systems.[30] QFI can be used to study quantum phase transitions by analyzing how quantum correlations change as parameters are varied. This helps in understanding the nature of phase transitions in quantum systems.[41] In essence, QFI is a versatile and powerful tool that provides deep insights into the limits of measurement precision, the nature of quantum states, and the performance of quantum technologies. It bridges theoretical and practical aspects of quantum physics, making it indispensable in both fundamental research and applied quantum technology.

Entangled states often exhibit higher QFI in quantum systems than non-entangled states. This higher QFI indicates that entangled states are more sensitive to parameter changes, making them valuable for precise measurements. Entangled states can achieve measurement precisions that surpass the standard quantum limit, which is achievable with non-entangled states. This is a direct consequence of the higher QFI associated with entangled states. QFI can also be used to detect QE. For instance, if the QFI for a certain parameter exceeds a certain threshold, it indicates the presence of QE in the quantum system. The QFI of an entangled system often scales with the number of entangled particles. For example, in an N-particle entangled state, the QFI can scale as N2N^2, indicating a quadratic improvement in sensitivity compared to N for non-entangled states. While QFI is not a direct measure of QE, it is strongly influenced by the presence of QE. Higher QFI values generally indicate greater QE and greater sensitivity to parameter changes, making entangled states highly valuable for quantum metrology and precision measurements. Calculating the Quantum Fisher Information (QFI) for N-level atoms under the influence of the thermal state and the Stark effect can provide several important insights. QFI measures the sensitivity of the quantum state to changes in external parameters. By evaluating QFI, we can understand how the N-level atoms respond to variations in the thermal state and the Stark effect, revealing how these external fields influence the system's dynamics. Higher values of QFI often indicate greater QE. By analyzing the QFI, we can infer the degree of QE induced by the thermal state and Stark effect in the N-level atomic system. This helps one understand the QE properties and the potential for quantum information processing. The thermal state introduces decoherence, which can degrade QE and coherence in the system. By studying the QFI, we can quantify the impact of decoherence on the N-level atoms, providing insights into how thermal fluctuations affect the system's quantum properties. QFI is crucial for quantum metrology as it determines the precision of parameter estimation. In the context of N-level atoms, QFI can help optimize measurement strategies and improve the accuracy of estimating parameters such as field or interaction strengths in the Stark effect. Comparing QFI under different conditions (e.g., with and without the Stark effect) allows us to analyze how the system's behavior changes. This comparison helps us understand the Stark effect's role in modifying the system's energy levels and overall quantum state.

Weijun Wu and co-workers studied QFI as a multipartite QE witness.[42] It is investigated that the QFI, a witness for genuinely multipartite QE, becomes measurable for thermal ensembles via the dynamic susceptibility, i.e., with resources readily available in present cold atomic gas and condensed-matter experiments.[36] This moreover establishes a connection between multipartite QE and many-body correlations contained in response functions, with immediate implications close to quantum phase transitions. There, the quantum Fisher information becomes universal, allowing us to identify strongly entangled phase transitions with a divergent multipartite QE. The relation between QFI and multiparticle QE was studied by Philipp Hyllus and co-workers.[21]

The von Neumann entropy (VNE) is a fundamental concept in quantum mechanics and quantum information theory. It is a measure of the uncertainty or disorder associated with a quantum state. VNE quantifies the amount of information or uncertainty in a quantum state. A quantum state is described by the density matrix $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$, where Tr denotes the trace operation and log is the matrix logarithm.[43] In thermodynamics, VNE is analogous to the classical concept of entropy. It provides a measure of the disorder or the number of microstates corresponding to a macrostate of the system. In particular, it plays a role in understanding the thermodynamic properties of quantum systems.[44,45] VNE measures the amount of information contained in a quantum state and is crucial in the study of quantum information theory.[45] It helps in quantifying the information gain or loss when a quantum state is subjected to various operations.[46] In quantum communication, von Neumann entropy is used to understand the capacity of quantum channels and the efficiency of quantum coding schemes.[47] The von Neumann entropy is used to quantify QE in quantum systems. For a bipartite quantum system with density matrix $\rho^{AB}$, the entropy of the reduced density matrix $S(\rho_A)$ or $S(\rho_B)$ quantifies the QE between subsystems $A$ and $B$.[47] For pure states of a bipartite system, the VNE of the reduced density matrix of one subsystem is a measure of the QE of the state.[48] This is particularly useful for defining and understanding various QE measures. In summary, VNE is fundamental in understanding quantum states, quantum communication, and QE. It provides a bridge between classical thermodynamics and quantum mechanics, helping to quantify information, entropy, and the relationships between quantum subsystems.

Determining the QE dynamics of the N-level atomic system under the influence of the SS, both in the presence and absence of a moving atom, is the aim of the current investigation. We investigate how the SS affects the dynamics of the atomic systems in QFI and VNE for 3-, 4-, and 5-level atoms that are in motion and those that are not. It is evident the SS is predominant during the quantum system's temporal growth. The SS significantly affects QFI dynamics. Furthermore, the SS affects the VNE more strongly when there is no motion of the atom. The von Neumann entropy is a crucial concept in quantum statistical mechanics, where it helps describe the thermodynamic properties of quantum systems in equilibrium. It relates to concepts such as temperature and free energy in the quantum context.

The format of the document is as follows: In Sec. II, the model of the system under the effect of the thermal state of moving 3-, 4-, and 5-level atomic systems impacted by the SS is shown along with the model Hamiltonian and interaction dynamics. The discussion and numerical results are provided in Sec. III. Section IV gives a brief conclusion.

## II. HAMILTONIAN MODEL

We study the system of moving 3-, 4-, and 5-level atoms present in the thermal state with the SS, and we study the cascade configuration of the system.

Assuming the rotating wave approximation, the system $\hat{H}_T$ is given by[49]

$$\hat{H}_T = \hat{H}_{Atom-Field} + \hat{H}_I, \tag{1}$$

where $\hat{H}_{Atom-Field}$ shows the atom that does not interact with the field Hamiltonian and $\hat{H}_I$ represents the coupling portion. Our writing for $\hat{H}_{Atom-Field}$ will be

$$\hat{H}_{Atom-Field} = \sum_j \omega_j \hat{\sigma}_{j,j} + \nu \hat{a}^\dagger \hat{a}\,, \tag{2}$$

where the $j$th level's $\hat{\sigma}_{j,j} = |j\rangle\langle j|$ represents the atomic population operator. Atomic population operators essentially refer to the operators that describe transitions between different energy levels of the atom. These transitions play a crucial role in QE dynamics. Here, $\hat{a}$ and $\hat{a}^\dagger$ are used for the non-resonant scenario, and the $\hat{H}_I$ is as follows:[49]

$$\hat{H}_I = \sum_{s=1}^{\mathrm{N}} \Omega(t)\Big[\hat{a}e^{-i\Delta_s t}\hat{\sigma}_{s,s+1} + (\hat{a}\,e^{-i\Delta_s t}\hat{\sigma}_{s,s+1})^\dagger\Big]. \tag{3}$$

In quantum systems, interactions play a crucial role. These interactions determine how particles influence each other. However, not all interactions are created equally. Some are central to the QE waltz, while others sit on the sidelines. We often study simple models in quantum mechanics and neglect some interaction terms. We might focus on the lead couple rather than the entire ensemble. Some interactions are negligible in specific contexts. By excluding less relevant terms, researchers make the math more manageable.

N = 1 depicts a two-level atom, and N = 3, 4, and 5 depict 3-, 4-, and 5-level atoms.

When it comes to the SS, $\hat{H}_I$ is provided by

$$\hat{H}_I = \sum_{s=1}^{\mathrm{N}} \Omega(t)\Big[\hat{a}e^{-i\Delta_s t}\hat{\sigma}_{s,s+1} + (\hat{a}\,e^{-i\Delta_s t}\hat{\sigma}_{s,s+1})^\dagger\Big] + \beta\hat{a}^\dagger\hat{a}. \tag{4}$$

The SS parameter is denoted by $\beta$. The final density matrix (DM), which is the DM generated by applying Eq. (4), includes the term atomic motion. Atomic mobility is modeled by the fundamental system Hamiltonian via the parameter $\Omega(t)$. However, for the purpose of studying information quantifiers such as QFI, the time evolution unitary operator controls the dynamics of the complete atom field interacting system to ascertain the time-dependent density matrix of the system. When dealing with time-dependent quantum systems, we often encounter the Heisenberg picture. In the Heisenberg picture, the operators (such as position, momentum, or field operators) evolve with time, while the states remain fixed. The detuning parameter is described as

$$\Delta_s = \nu - (\omega_s - \omega_{s+1}). \tag{5}$$

$\Omega(t)$ describes the moving atom in Eq. (4), where atom and field are coupled through constant $g$[50] and the motion of the atomic system is along the $z$-axis,

$$\begin{aligned} \Omega(t) &= g\ \sin(p\pi vt/L)\ , p \neq 0, \\ \Omega(t) &= g\,, p = 0, \end{aligned} \tag{6}$$

where $v$ is the atomic motion velocity, $L$ is the cavity length along the $z$-direction, and half of the mode's wavelengths in the cavity are denoted by the symbol $p$. Given the velocity of an atom, $v = gL/\pi$, we write

$$\Omega_1(t) = \int_0^t \Omega(\tau)d\tau = \frac{1}{p}(1 - \cos(p\pi vt/L) \quad \text{for} \quad p \neq 0, \tag{7}$$

$$= gt \quad \text{for} \quad p = 0. \tag{8}$$

In the presence of the thermal state, we write the initial state of the system of N-level system as

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + \exp(i\varphi)|0\rangle) \otimes \rho_f(0), \tag{9}$$

where $\rho_f(0)$ denotes the thermal state density matrix, which is given as

$$\rho_f(0) = \sum_{n=0}^{\infty} P(n)|n\rangle\langle n|, \tag{10}$$

with

$$P(n) = \frac{\bar{n}^n}{(\bar{n}+1)^{(n+1)}}, \tag{11}$$

where $|n\rangle$ indicates the number state and $\varphi$ represents the phase. With $k_B$ representing the Boltzmann constant, $\omega_f$ representing the cavity mode frequency, $T$ representing the temperature, and $\bar{n} = (e^{\hbar\omega_f}/k_B T - 1)^{-1}$ representing the average photon number, the DM expression is shown below:

$$\hat{\rho}(t) = \sum_{m,n}^{\mathrm{N}} |\psi_n(t)\rangle\langle\psi_n(t)|\hat{\rho}(t)|\psi_m(t)\rangle\langle\psi_m(t)|. \tag{12}$$

Here, we are considering an N- level atomic system with the density operator $\rho(\varphi)$. The spectral decomposition of the DM is described as

$$\rho_\varphi = \sum_K \lambda_K|k\rangle\langle k|. \tag{13}$$

The QFI, which is related to $\varphi$ for this DM, is represented by[51]

$$F_\varphi = \sum_k \frac{(\partial_\varphi\lambda_k)^2}{\lambda_k} + 2\sum_{k,k'} \frac{(\lambda_k - \lambda_{k'})^2}{(\lambda_k + \lambda_{k'})}|\langle k|\partial_\varphi k'\rangle|^2, \tag{14}$$

where $|k\rangle\lambda_k > 0$ and $\lambda_k + \lambda_{k'} > 0$. The first term in the above-mentioned equation represents the CFI, and the second term describes the QFI. The QFI is calculated by taking the trace over the field. Therefore, we will be able to represent the QFI of a bipartite density operator as $\rho_{AB.}$, which is related to $\varphi$ as shown in Ref. 27.

The QFI expressed in terms of $\varphi$ for a bipartite $\rho_{AB.}$ is[27]

$$I_{QF}(t) = I(\varphi, t) = \mathrm{Tr}\left[\rho_{AB}(\varphi, t)\{L^2(\varphi, t)\}\right], \tag{15}$$

where the quantum score[52] SLD is represented by $L(\varphi, t)$, given by

$$\frac{\partial\rho(\varphi, t)}{\partial\varphi} = \frac{1}{2}[L(\varphi, t)\rho_{AB}(\varphi, t) + \rho_{AB}(\varphi, t)L(\varphi, t)]. \tag{16}$$

VNE is a fundamental concept in quantum information theory that measures the quantum uncertainty or mixedness of a quantum state. In the context of QE, it is used to quantify how much QE exists between subsystems of a composite quantum system. In similar vein, the VNE is described as

$$S_A = -\mathrm{Tr}(\rho_A \ln\rho_A) = -\sum_i r_i \ln r_i, \tag{17}$$

where the atomic DM $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ eigenvalues are denoted by $r_i$. $\rho$ is the density matrix of the quantum system. Tr denotes the trace, summing over the diagonal elements of the matrix. $\ln \rho$ is the matrix logarithm of $\rho$. The von Neumann entropy generalizes the classical Shannon entropy to the quantum domain and is used to measure the mixedness of a state, where $S(\rho) = 0$ for a pure state and $S(\rho) = 0$ for a mixed state. The von Neumann entropy (VNE) is primarily a good measure of QE for pure bipartite states. For pure states, it effectively quantifies the amount of QE between the two subsystems. However, for mixed states, the VNE does not solely measure QE. Instead, it captures the total correlations present in the system, which include both quantum QE and classical correlations. QE detection was carried out by Gühne and Tóth in 2009.[53] The foundations of quantum information for physical chemistry were investigated in Ref. 54. The review of basic aspects of QE, including its characterization, detection, distillation, and quantification, and various manifestations of QE via Bell inequalities, entropic inequalities, QE witnesses, and quantum cryptography are discussed in Ref. 55. A concise mathematical formulation of the multipartite QE phenomenon and its major features were investigated by Horodecki and co-workers.[56] Disorder enhanced exciton transport and quantum information spreading with the assistance of cavity QED were studied in Ref. 57.

The ID parameter $\gamma$ has time-inverse dimensions, while $p$ has length-dimensions.

We have performed numerical calculations as the system is bigger to solve analytically. We used computational languages and software for the numerical calculations. We have clearly described the Hamiltonian of the system and the interaction Hamiltonian. We also mention the state and the formulas for QE quantifiers such as QFI and VNE. To quantify QE, we construct the density matrix, and this density matrix is solved numerically. With the help of this density matrix, eigenvectors and eigenvalues of the density matrix are calculated. These eigenvectors and eigenvalues are used to calculate QFI and VNE, but this whole calculation is numerical.

## III. DISCUSSIONS AND NUMERICAL OUTCOMES

We consider both the static and moving cases of the system, and the presence of the thermal state. The system's level is SS with strength $\beta$. We solve the system dynamics numerically and have chosen a 0.1 time step size.

### A. VNE and QFI of N-level stationary Stark shifted atomic systems

This section presents the results for the quantum Fisher information (QFI) and von Neumann entropy (VNE) dynamical evolution for moving N-level atomic systems (3-, 4-, and 5-level atoms) interacting with the thermal state in the presence of the SS. We have taken the number of photons as $\alpha = 6$. For $\beta = 0.3$, the quantifiers' dynamic behavior is depicted in Fig. 1. A decline is seen in QFI as time increases, but this decrease has fluctuations in amplitude, which reduces as time progresses for $\varphi = 0$. At $\varphi = \pi/4$, the QFI has almost the same decreasing pattern with the evolution of time. The 4- and 5-level atoms have the same decrease in amplitude, but the 3-level atom has a slightly different pattern of decrease in amplitude as it has a small increase in amplitude compared to the 4-level atom. As time goes on, the N-level atomic system's VNE gets bigger. At $\varphi = 0$, the 5-level atom has a greater VNE than the 3- and 4-level atoms. It demonstrates that, in comparison with the 3- and 4-level atoms, the VNE of 5-level atoms is more influenced by the SS. As a result, we may state that a 5-level atom has a higher QE than the

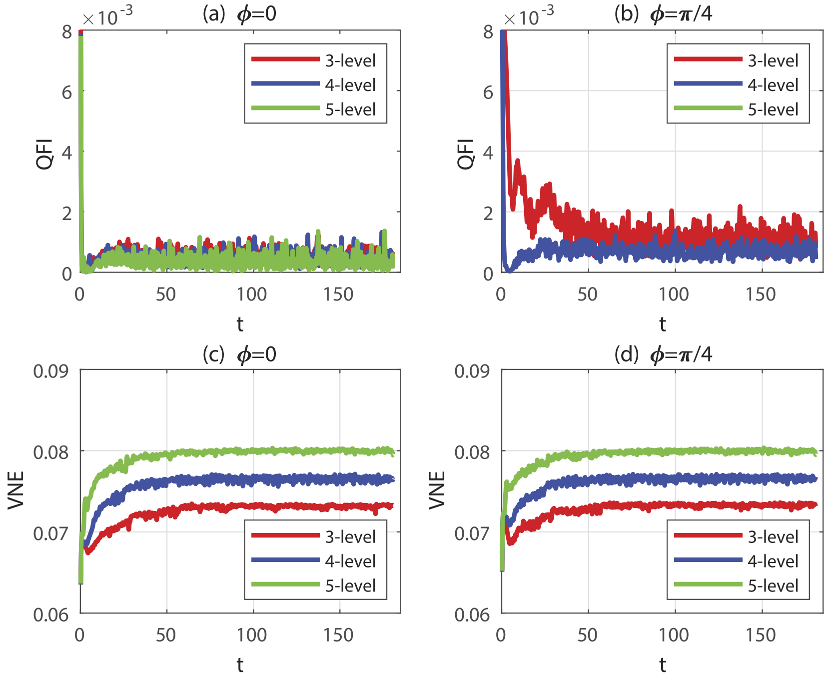

**FIG. 1.** The QFI (upper panel) and VNE (lower panel) as a function of time. $\alpha = 6$, $\beta = 0.3$, and $\phi = 0$ (left panel) and $\pi/4$ (right panel). $p = 0$. The VNE is along the $y$-axis, and $t$ is time along the $x$-axis.

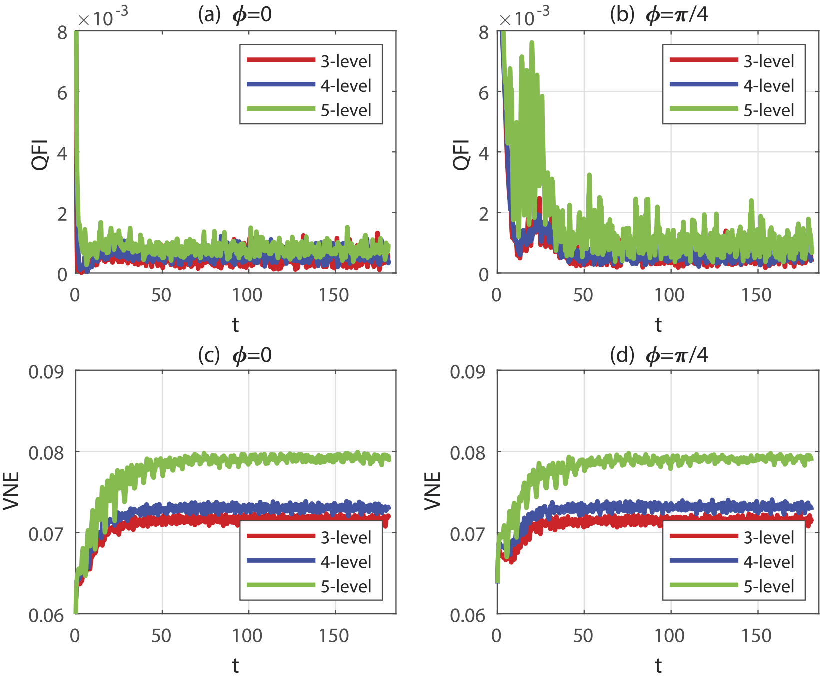

**FIG. 2.** The QFI (upper panel) and VNE (lower panel) as a function of time. $\alpha = 6$, $\beta = 1$, and $\phi = 0$ (left panel) and $\pi/4$ (right panel). $p = 0$. The VNE is along the $y$-axis, and $t$ is time along the $x$-axis.

3- or 4-level atom, but for all kinds of atomic systems, it saturates. The 3-, 4-, and 5-level atoms all have higher QE because of the SS; however, the system with the 5-level atom is most affected. In the presence of the SS, QE rises with time but becomes saturated as time goes on. A higher dimensional atomic system, such as the 5-level atom, exhibits a more pronounced increase in QE than do 3- and 4-level atoms. As time progresses, the rise in VNE for the N-level system reaches saturation. The system's QFI shown in Fig. 2 at $\beta = 1$ is about the same as that at $\beta = 0.3$. The 3-, 4-, and 5-level atoms at $\varphi = 0$ and $\varphi = \pi/4$ exhibit considerable changing behavior in the VNE. The main difference is that the QFI behavior at $\varphi = \pi/4$ is the same as that at $\beta = 0.3$, where there is a minor increase in the amplitude of fluctuation. The 5-level atoms have larger VNE than the 3- and 4-level atoms, whereas the 4-level atoms have somewhat higher VNE than the 3-level atoms. It means the SS increases the QE of the 5-level atoms but the QE of the 3- and 4-level atoms is almost the same at $\beta = 1$ as compared to $\beta = 0.3$. The system's QFI shown in Fig. 3 at $\beta = 3$ diminishes with increasing time at $\varphi = 0$ and $\pi/4$, whereas the 5-level atom's amplitude slightly increases at $\varphi = \pi/4$. In comparison to 3- and 4-level atoms, the VNE of the 5-level atoms increases at $\varphi = 0$ and $\pi/4$, but at $\varphi = \pi/4$, the VNE of the 3- and 4-level atoms is nearly equal. It is observed in Figs. 1–3 that for a static system in the presence of SS, its QFI decreases as time progresses. The amplitude of fluctuations in the QFI at $\beta = 1$ and 3 is greater than that at $\beta = 0.3$. In comparison to the 3- and 4-level atomic systems, the QFI of the higher-dimensional atomic system is affected by a bigger SS parameter, which also raises the value of the QFI of the 5-level atom. The VNE also increases for the 5-level atoms at different values of the SS parameter. Therefore, it may be said that the 5-level atom has a greater VNE in the presence of the SS. For the 5-level atom, for instance, the QE is maintained over time and rises with the evolution of time for larger SS parameter values. The N-level atomic system's QE is reinforced and preserved. The SS effect is stronger in larger-N systems. When the atom is not moving in the presence of the thermal state, enhancing the QE is possible in the SS environment.

## B. VNE and QFI of N-level moving Stark shifted atomic systems

The dynamics of the N-level atomic system moving in the presence of the SS is examined in this section. We examine the scenarios in which the SS parameter is increased and examine the system's QFI and VNE dynamics. The dynamics of the moving N-level system in the presence of the SS for $\beta = 0.3, 1$, and 3 are shown in Figs. 4–6, respectively. For all values of $\beta$, we observe the periodic behavior of the QFI and VNE for both phase factors of the state. The 3-, 4-, and 5-level atoms in the QFI scenario at $\beta = 0.3$ at $\varphi = 0$ exhibit periodic behavior with the same amplitude. Still, at $\varphi = \pi/4$, the amplitude of oscillation of the 4- and 5-level atom is much smaller than that of the 3-level atom. The amplitude of oscillation of a 4-level atom is a little greater than that of the 5-level atom. Hence, it means that the SS influences the 4- and 5-level atoms at a large scale and it reduces the amplitude of N = 4 and 5 as compared to N = 3. For the case of VNE at $\beta = 0.3$ and $\varphi = 0$, the amplitude of periodic oscillation of the 5-level atom is greater than 3- and 4-level atoms, and the same pattern is almost observed at $\varphi = \pi/4$. here the only difference is that the periodic oscillation of the 4-level atom is less than that of both 3- and 5-level atoms. For the case of QFI at $\beta = 1$ and $\varphi = 0$, the 3-, 4-, and 5-level atoms show the same periodic behavior of oscillations. Still, at $\varphi = \pi/4$, the amplitude of oscillation of the 4- and 5-level atoms is

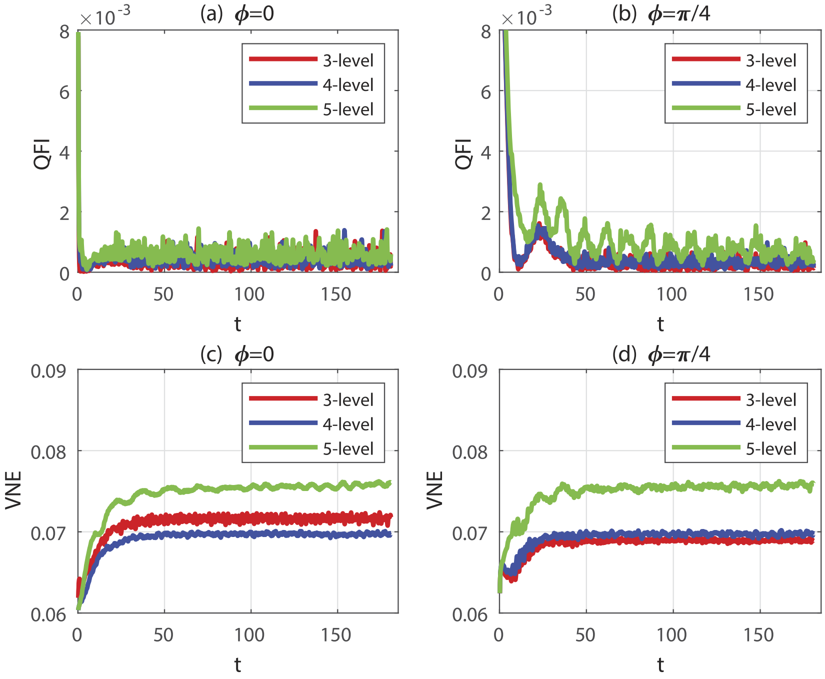

**FIG. 3.** The QFI (upper panel) and VNE (lower panel) as a function of time. $\alpha = 6$, $\beta = 3$, and $\phi = 0$ (left panel) and $\pi/4$ (right panel). $p = 0$. The VNE is along the $y$-axis, and $t$ is time along the $x$-axis.

much smaller than that of the 3-level atom. The 4- and 5-level atoms have the same amplitude of oscillation, and there is no difference in the oscillation. The 3-level atomic system has more amplitude of periodic oscillation than the 4- and 5-level atomic systems. The VNE of 3-, 4-, and 5-level atoms shows periodic responses at $\varphi = 0$, but the amplitude of oscillation of 4-level atoms is greater than that of 5- and 3-level atoms. At $\varphi = \pi/4$, the amplitude of oscillation of 3-level atoms is greater than that at N = 4 and 5. Hence, periodic VNE behavior is observed for 3-, 4-, and 5-level atoms, but there is a slight difference in the amplitude of oscillation of 3-,

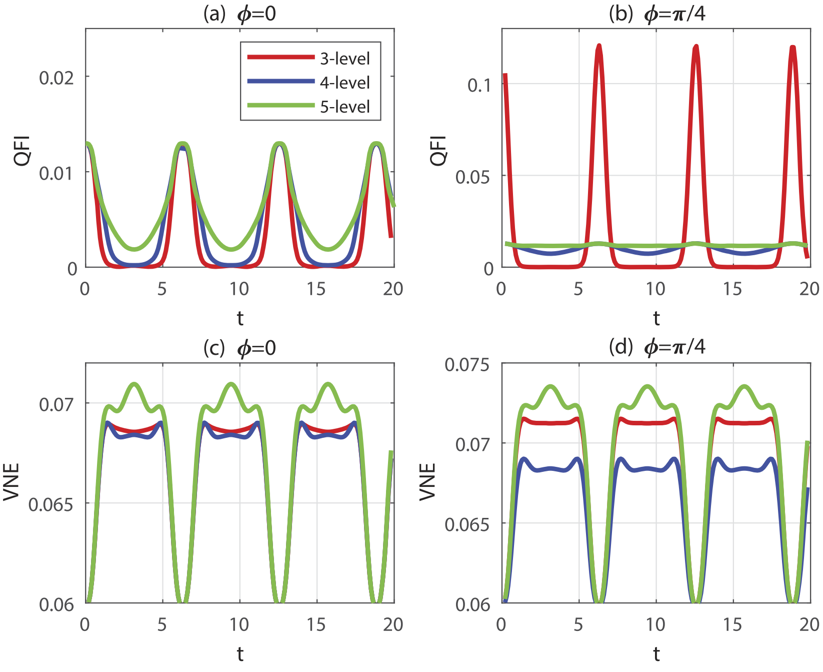

**FIG. 4.** The QFI (upper panel) and VNE (lower panel) as a function of time. $\alpha = 6$, $\beta = 0.3$, and $\phi = 0$ (left panel) and $\pi/4$ (right panel). $p = 1$. The VNE is along the $y$-axis, and $t$ is time along the $x$-axis.

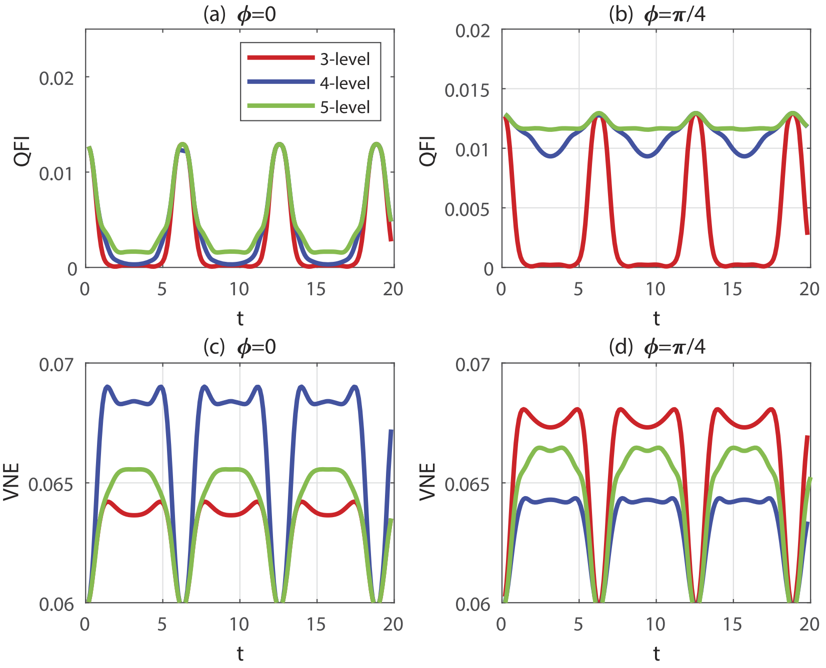

**FIG. 5.** The QFI (upper panel) and VNE (lower panel) as a function of time. $\alpha = 6$, $\beta = 1$, and $\phi = 0$ (left panel) and $\pi/4$ (right panel). $p = 1$. The VNE is along the $y$-axis, and $t$ is time along the $x$-axis.

4-, 5-level atoms. At $\beta = 0.3$, the QFI at $\varphi = 0$ is periodic for 3-, 4-, and 5-level atomic systems and their amplitude is the same, but at $\varphi = \pi/4$, the amplitude of oscillation of the 3-level is much greater than that of 4- and 5-level atoms. The amplitude of oscillation of 4- and 5-level atoms is very small compared to that of 3-level atoms. It is determined that, with only minor amplitude variations, the periodic behavior of oscillation for the 3-, 4-, and 5-level atoms nearly exhibits the same increasing value of the SS parameter. The abrupt death and birth of QE are seen when the system is moving, although QFI exhibits diminishing periodic behavior for the 4- and 5-level atoms. In the case of moving atoms, the SS supports the abrupt death and birth of QE; nonetheless, QFI of the 4- and 5-level

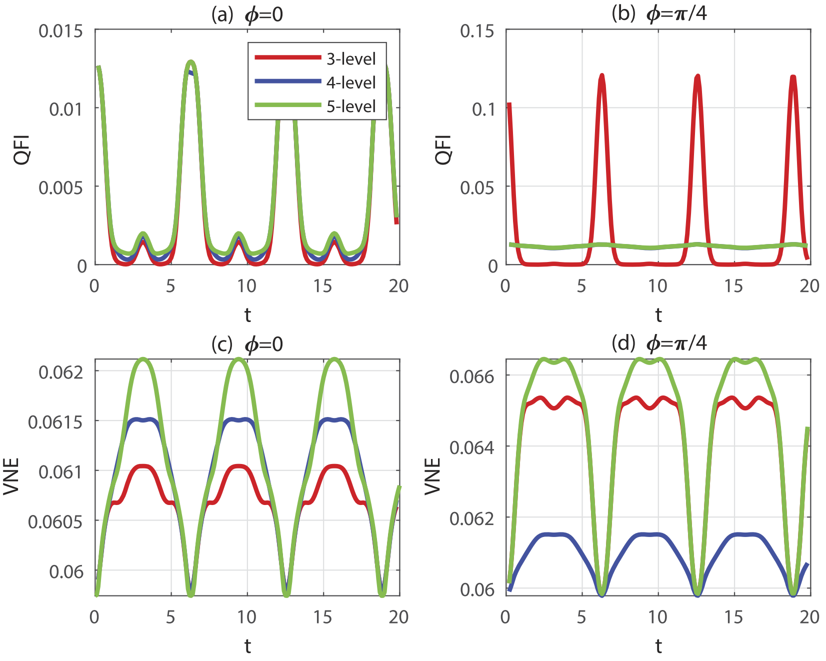

**FIG. 6.** The QFI (upper panel) and VNE (lower panel) as a function of time. $\alpha = 6$, $\beta = 3$, and $\phi = 0$ (left panel) and $\pi/4$ (right panel). $p = 1$. The VNE is along the $y$-axis, and $t$ is time along the $x$-axis.

atoms reduces in comparison to that of the 3-level atoms. Therefore, we can conclude that for a moving N-level atom, the SS supports the QE's sustainability and maintenance. The QFI of N = 4, 5 is less than that of N = 3, and the SS system is favorable for maintaining the sudden death and birth of QE. Consequently, the QE is sustained and maintained by the moving and non-moving N-level atomic systems under the SS.

The Stark effect influences quantum systems at a fundamental level. The Stark effect, which refers to the shifting and splitting of energy levels of atoms or molecules in an external electric field, can have a significant impact on the quantum states involved in QE. The Stark effect alters the energy levels of the quantum states involved in the entangled system, potentially changing the dynamics of the system. This shift may influence the interaction between the qubits or particles in the entangled state. The presence of an external electric field might induce changes in coherence, affecting how the entangled states evolve. Depending on the field strength and the nature of the system, it could either enhance or degrade the degree of QE. In strong electric fields, nonlinear Stark effects can lead to more complex changes in QE, possibly creating or destroying correlations in unexpected ways. Since QE is a critical resource for quantum information tasks, understanding the Stark effect's role could be essential for fine-tuning the system's behavior, especially in controlled experimental setups.

## IV. CONCLUSIONS

In conclusion, in the thermal state, we have found some interesting characteristics in the QE dynamics of the N-level system that experiences SS. The susceptibility of QE to outside perturbations is seen by the reduction in QE magnitude with the increasing SS parameter. The intricate structure of quantum systems is highlighted by the complex dynamics of the VNE and QFI, which fluctuate with respect to N-levels, phase factors, and the values of the SS parameter.

Larger values of the SS parameter have been found to affect the QFI of 5-level atomic systems in static systems, which also boosts the QFI of the 5-level system in comparison to the 3- and 4-level systems. The VNE also increases for the 5-level atom with different values of the SS parameter. As a result, the 5-level atoms may be considered to have a larger VNE with the SS. The 5-level atom experiences a continuous increase in QE with increasing values of the SS parameter throughout time. The N-level system's QE is strengthened and preserved by the SS, and the 5 atomic systems exhibit a greater effect of the SS. The SS environment helps to improve the QE when the atom is 5-level. It is established that the periodic behavior of oscillation for the 3-, 4-, and 5-level atoms almost always shows the same growing value of the SS parameter, with only small amplitude differences. When the system is moving in the presence of the SS, QE abruptly dies and is revived again, while QFI shows diminishing periodic behavior for the 4- and 5-level atoms. The SS supports the abrupt death and birth of QE for moving atoms; however, QFI decreases for 4- and 5-level atoms compared to 3-level atoms. As a result, the 4- and 5-level atoms have lower QFIs than the 3-level atom. Thus, we may infer that the SS facilitates the upkeep and sustenance of the QE in the case of a moving N-level atom interacting with the thermal state. The conditions in the SS are ideal for sustaining abrupt death and birth of QE. Thus, in the presence of the SS, both moving and stationary N-level atomic systems interacting with the thermal state maintain and prolong the QE.

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

### Author Contributions

**S. Jamal Anwar**: Conceptualization (equal); Writing – original draft (equal). **M. Ibrahim**: Software (equal); Writing – review & editing (equal). **M. Khalid Khan**: Supervision (equal). **Haifa S. Alqannas**: Validation (equal); Visualization (equal); Writing – review & editing (equal). **S. Abdel-Khalek**: Supervision (equal); Validation (equal).

## DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

## REFERENCES

[1] V. Buzek, G. Adam, and G. Drobny, "Quantum state reconstruction and detection of quantum coherences on different observation levels," Phys. Rev. A **54**, 804 (1996).

[2] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics* (Oxford University Press, Oxford, 1997).

[3] S. Bose, I. Fuentes-Guridi, P. L. Knight, and V. Vedral, "Subsystem purity as an enforcer of entanglement," Phys. Rev. Lett. **87**, 050401 (2001).

[4] M. S. Kim, J. Lee, D. Ahn, and P. L. Knight, "Entanglement induced by a single-mode heat environment," Phys. Rev. A **65**, 040101 (2002).

[5] M. Paternostro, M. S. Tame, G. M. Palma, and M. S. Kim, "Entanglement generation and protection by detuning modulation," Phys. Rev. A **74**, 052317 (2006).

[6] M. S. Safronova, C. J. Williams, and C. W. Clark, "Relativistic many-body calculations of electric-dipole matrix elements, lifetimes, and polarizabilities in rubidium," Phys. Rev. A **69**, 022509 (2004).

[7] A. Vourdas, "Propagation of a quantum state in a continuum of coupled oscillators and applications to entanglement," J. Phys. A: Math. Gen. **38**, 9859 (2005).

[8] F. Buscemi, P. Bordone, and A. Bertoni, "Entanglement dynamics of electron–electron scattering in low-dimensional semiconductor systems," Phys. Rev. A **73**, 052312 (2006).

[9] A. I. Milstein and O. P. Sushkov, "Parity nonconservation in heavy atoms: The radiative correction enhanced by the strong electric field of the nucleus," Phys. Rev. A **66**, 022108 (2002).

[10] H. R. Verma *et al.*, "Photon-induced L-shell X-ray intensity ratios for $_{74}$W and $_{80}$Hg in the energy range 17 ≤ E ≤ 47 keV," J. Phys. B: At. Mol. Phys. **18**, 1133 (1985).

[11] R. A. Fisher, "Theory of statistical estimation," Math. Proc. Cambridge Philos. Soc. **22**(5), 700–725 (1925).

[12] R. A. Fisher, *The General Theory of Natural Selection* (Clarendon Press, 1928).

[13] R. A. Fisher, "Two new properties of mathematical likelihood," Proc. R. Soc. London, Ser. A **144**(852), 285–307 (1934).

[14] V. Giovannetti, S. Lloyd, and L. Maccone, "Advances in quantum metrology," Nat. Photonics **5**, 222 (2011).

[15] J. Liu, H. Yuan, X.-M. Lu, and X. Wang, "Quantum Fisher information matrix and multiparameter estimation," J. Phys. A: Math. Theor. **53**, 023001 (2020).

[16] C. W. Helstrom, "Quantum detection and estimation theory," J. Stat. Phys. **1**, 231 (1969).

[17] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Springer Science and Business Media, 2011), Vol. 1.

[18] M. G. Paris, "Quantum estimation for quantum technology," Int. J. Quantum Inf. **7**, 125 (2009).

[19] W. K. Wootters, "Statistical distance and Hilbert space," Phys. Rev. D **23**, 357 (1981).

[20] S. L. Braunstein and C. M. Caves, "Statistical distance and the geometry of quantum states," Phys. Rev. Lett. **72**, 3439 (1994).

[21] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, "Fisher information and multiparticle entanglement," Phys. Rev. A **85**, 022321 (2012).

[22] G. Tóth, "Multipartite entanglement and high-precision metrology," Phys. Rev. A **85**, 022322 (2012).

[23] Z. Hradil, J. Rehacek, L. Sánchez-Soto, and B.-G. Englert, "Quantum Fisher information with coherence," Optica **6**, 1437 (2019).

[24] S. Banerjee, J. Erdmenger, and D. Sarkar, "Connecting Fisher information to bulk QE in holography," J. High Energy Phys. **8**, 1 (2018).

[25] G. Tóth and I. Apellaniz, "Quantum metrology from a quantum information science perspective," J. Phys. A: Math. Theor. **47**, 424006 (2014).

[26] I. de Vega and D. Alonso, "Dynamics of non-Markovian open quantum systems," Rev. Mod. Phys. **89**, 015001 (2017).

[27] X.-M. Lu, X. Wang, and C. Sun, "Quantum Fisher information flow and non-Markovian processes of open systems," Phys. Rev. A **82**, 042103 (2010).

[28] H. Song, S. Luo, and Y. Hong, "Quantum non-Markovianity based on the Fisher-information matrix," Phys. Rev. A **91**, 042110 (2015).

[29] T.-L. Wang, L.-N. Wu, W. Yang, G.-R. Jin, N. Lambert, and F. Nori, "Quantum Fisher information as a signature of the superradiant quantum phase transition," New J. Phys. **16**, 063039 (2014).

[30] U. Marzolino and T. Prosen, "Fisher information approach to nonequilibrium phase transitions in a quantum XXZ spin chain with boundary noise," Phys. Rev. B **96**, 104402 (2017).

[31] J. Ma and X. Wang, "Fisher information and spin squeezing in the Lipkin–Meshkov–Glick model," Phys. Rev. A **80**, 012318 (2009).

[32] Z. Sun, J. Ma, X.-M. Lu, and X. Wang, "Fisher information in a quantum-critical environment," Phys. Rev. A **82**, 022306 (2010).

[33] S. Deffner and S. Campbell, "Quantum speed limits: From Heisenberg's uncertainty principle to optimal quantum control," J. Phys. A: Math. Theor. **50**, 453001 (2017).

[34] M. Gessner and A. Smerzi, "Statistical speed of quantum states: Generalized quantum Fisher information and Schatten speed," Phys. Rev. A **97**, 022109 (2018).

[35] A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, "Quantum speed limits in open system dynamics," Phys. Rev. Lett. **110**, 050403 (2013).

[36] M. Beau and A. del Campo, "Nonlinear quantum metrology of many-body open systems," Phys. Rev. Lett. **119**, 010403 (2017); P. Hauke, M. Heyl, L. Tagliacozzo, and P. Zoller, "Measuring multipartite QE through dynamic susceptibilities," Nat. Phys. **12**(1), 778–782 (2016).

[37] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, 1976).

[38] V. Giovannetti, S. Lloyd, and L. Maccone, "Quantum-enhanced measurements: Beating the standard quantum limit," Science **306**(5700), 1330–1336 (2004).

[39] C. L. Degen, F. Reinhard, and P. Cappellaro, "Quantum sensing," Rev. Mod. Phys. **89**(3), 035002 (2017).

[40] Y.-C. Liu, Y.-B. Cheng, X.-B. Pan, Z.-Z. Sun, D. Pan, and G.-L. Long, "Quantum integrated sensing and communication via entanglement," Phys. Rev. Appl. **22**, 034051 (2024).

[41] X. Liu, W. Cheng, and J. M. Liu, "Renormalization-group approach to quantum Fisher information in an XY model with staggered Dzyaloshinskii-Moriya interaction," Sci. Rep. **6**, 19359 (2016).

[42] W. Wu *et al.*, "Molecular QE witness by absorption spectroscopy in cavity QED," arXiv:2408.14782 (2024).

[43] J. Von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, 1955).

[44] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Butterworth-Heinemann, 1980).

[45] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2010).

[46] J. Eisert, M. Cramer, and M. B. Plenio, "Colloquium: Area laws for the entanglement entropy," Rev. Mod. Phys. **82**(1), 277–306 (2010).

[47] V. Vedral, "The role of relative entropy in quantum information theory," Rev. Mod. Phys. **74**(1), 197–234 (2002).

[48] W. H. Zurek, "Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse," Phys. Rev. D **24**(6), 1516–1525 (1981).

[49] S. Abdel-Khalek, "Quantum Fisher information for moving three-level atom," Quantum Inf. Process. **12**, 3761 (2013).

[50] N. A. Enaki and N. Ciobanu, "Quantum trapping conditions for three-level atom flying through bimodal cavity field," J. Mod. Opt. **55**, 1557–1569 (2008).

[51] W. K. Wootters, "Entanglement of formation and concurrence," Quantum Inf. Comput. **1**, 27 (2001).

[52] O. E. Barndorff-Nielsen, R. D. Gill, and P. E. Jupp, "On quantum statistical inference," J. R. Stat. Soc. Ser. B: Stat. Methodol. **65**, 775–804 (2003).

[53] O. Gühne and G. Tóth, "QE detection," Phys. Rep. **474**(1–6), 1–75 (2009).

[54] W. Wu and G. D. Scholes, "Foundations of quantum information for physical chemistry," J. Phys. Chem. Lett. **15**, 4056–4069 (2024).

[55] R. Horodecki *et al.*, "Quantum entanglement," Rev. Mod. Phys. **81**(2), 865–942 (2009).

[56] P. Horodecki, Ł. Rudnicki, and K. Yczkowski, "Multipartite QE," arXiv:2409.04566 (2024).

[57] W. Wu, A. N. Hejazi, and G. Scholes, "Disorder-enhanced exciton transport and quantum information QED," arXiv:2409.06900 (2024).