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BPS states in $\mathcal{N} = 2$ supersymmetric G_2 and F_4 models

R. Ahl Laamara [a,b,c], O. Mellal [a,c], E.H. Saidi [a,c,*]

[a] *LPHE-MS, Science Faculty, Mohammed V University, Rabat, Morocco*
[b] *Centre Régional des Metiers de L'Education et de La Formation, Fès-Meknès, Morocco*
[c] *Centre of Physics and Mathematics, CPM, Morocco*

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Abstract

In BPS quiver theory of $\mathcal{N} = 2$ supersymmetric pure gauge models with gauge invariance $\mathcal{G}$, primitive BPS quivers $Q_0^{\mathcal{G}}$ are of two types: Q_0^{ADE} and Q_0^{BCFG}. In this study, we first show that Q_0^{ADE} have outer-automorphism symmetries inherited from the outer-automorphisms of the Dynkin diagrams of ADE Lie algebras. Then, we extend the usual folding operation of Dynkin diagrams $ADE \rightarrow BCFG$ to obtain the two following things: (i) relate Q_0^{BCFG} quivers and their mutations to the Q_0^{ADE} ones and their mutations; and (ii) link the BPS chambers of the $\mathcal{N} = 2$ ADE theories with the corresponding $BCFG$ ones. As an illustration of this construction, we derive the BPS and anti-BPS states of the strong chambers $\mathfrak{Q}_{stg}^{G_2}$ and $\mathfrak{Q}_{stg}^{F_4}$ of the 4d $\mathcal{N} = 2$ pure G_2 and F_4 gauge models.

1. Introduction

Few years ago, a BPS quiver theory has been proposed in [1,2] in order to build the complete set of BPS spectra of 4d $\mathcal{N} = 2$ supersymmetric quantum field theories (QFT_4) with gauge symmetry $\mathcal{G}$. This theory has been smoothly applied to supersymmetric ADE type gauge models, with and without hypermatter [3–9]; and to Gaiotto type theories describing the low energy limit of M5-branes wrapped on a punctured Riemann surface [10–13]; see also refs. [27–37] for

* Corresponding author at: LPHE-MS, Science Faculty, Mohammed V University, Rabat, Morocco.
E-mail address: h-saidi@fsr.ac.ma (E.H. Saidi).

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previous works and refs. [38–46] for other approaches to the $\mathcal{N}=2$ BPS spectra. An interesting generalisation of this BPS quiver construction beyond ADE type groups has been first proposed in [14] where particular BPS configurations have been described. In this study, we want to contribute to this BPS theory by developing a new manner to extend the smooth ADE-construction to the subclass of $\mathcal{N}=2$ supersymmetric pure gauge theories based on a non-simply laced type gauge invariance. Our way of doing relies on extending the usual folding method of Lie algebras, mapping ADE Dynkin diagrams into BCFG ones, to the BPS chambers $\mathfrak{Q}_{bps}^{ADE}$ of ADE-type theory in order to generate the $\mathfrak{Q}_{bps}^{BCFG}$ chambers for the case of non-simply laced type gauge symmetries.

To fix the ideas, recall that in the standard ADE type formulation of the quiver theory, the BPS/anti-BPS states of the 4d $\mathcal{N}=2$ QFT$_4$'s are composites of elementary BPS particles having electric–magnetic (EM) charges $\boldsymbol{\gamma}_i$. These charges are given by symplectic vectors $\boldsymbol{\gamma}_i=\left(\gamma_i^{\Lambda}\right)$ made of two component blocks: electric q_i^I components and magnetic p_{Ii} ones as follows

$$\gamma_i^{\Lambda}=\begin{pmatrix} q_i^I \\ p_{Ii} \end{pmatrix} \qquad , \qquad \gamma_{i\Lambda}=\left(p_{Ii},-q_i^I\right)$$

In the particular class of supersymmetric pure gauge theories with gauge symmetry $\mathcal{G}$ we will be interested in here, that is in absence of couplings to hypermatter, the basic particles are given by r elementary monopoles $\mathfrak{M}_i$ with EM charges $\boldsymbol{\gamma}_i^{\text{monopoles}}$, denoted below as $\mathbf{b}_i$, and r elementary dyons $\mathfrak{D}_i$ with EM charges $\boldsymbol{\gamma}_i^{\text{dyons}}$ denoted like $\mathbf{c}_i$. They are placed at nodes N_i of a primitive quiver $Q_0^{\mathcal{G}}$ with oriented links N_i-N_j given by the integral Dirac pairing

$$\boldsymbol{\gamma}_i\circ\boldsymbol{\gamma}_j=\mathcal{C}_{\Lambda\Upsilon}\gamma_i^{\Lambda}\gamma_j^{\Upsilon} \tag{1.1}$$

where $\mathcal{C}_{\Lambda\Upsilon}=-\mathcal{C}_{\Upsilon\Lambda}$ is the usual metric of real symplectic groups. The EM charge vectors $\boldsymbol{\gamma}_i$ and the intersection matrix $\left(\mathcal{A}_0^{\mathcal{G}}\right)_{ij}=\boldsymbol{\gamma}_i\circ\boldsymbol{\gamma}_j$ define the primitive quiver $Q_0^{\mathcal{G}}$ which, in some sense, resembles formally to the usual Dynkin diagram of Lie algebra of the gauge symmetry $\mathcal{G}$. The $\boldsymbol{\gamma}_i$'s and the $\mathcal{A}_0^{\mathcal{G}}$ play also an important role in the building of the BPS spectra of the $\mathcal{N}=2$ QFT$_4$. The BPS states $\mathcal{S}(\boldsymbol{\gamma})$ of the 4d $\mathcal{N}=2$ gauge theory and the corresponding anti-BPS ones $\mathcal{S}(-\boldsymbol{\gamma})$ are bound states made of the $\mathfrak{M}_i=\mathfrak{M}(\mathbf{b}_i)$'s and the $\mathfrak{D}_i=\mathfrak{D}(\mathbf{c}_i)$'s; the EM charges $\boldsymbol{\gamma}$ of BPS states $\mathcal{S}(\boldsymbol{\gamma})$ are given by positive integral linear combinations of the elementary $\boldsymbol{\gamma}_i$'s, that is EM charge vectors of the form $\boldsymbol{\gamma}=\sum_i N_i^+\boldsymbol{\gamma}_i$ with N_i^+ some positive integers. The symplectic charge vectors $\pm\boldsymbol{\gamma}$ are obtained *in practice* by the quiver mutation method whose algorithm may roughly be phrased into two main steps as follows: (i) First, start from the primitive $Q_0^{\mathcal{G}}$ and a given configuration of ordering of the arguments $\arg Z_i$ of the central charges $Z_i=Z\left(\boldsymbol{\gamma}_i\right)$ of the elementary particles $\boldsymbol{\gamma}_i$; an ordering of the phases of the complex central charges defines a BPS chamber $\mathfrak{Q}_{bps}^{\mathcal{G}}$ in the quiver theory, and has a nice representation in terms of rays in the Z-complex plane. (ii) Second, performs successive mutations $\boldsymbol{M}_n$ on the primitive $Q_0^{\mathcal{G}}$ generating descendant quivers $Q_n^{\mathcal{G}}=\boldsymbol{M}_n\left(Q_0^{\mathcal{G}}\right)$ with nodes occupied by new BPS states with EM charge vectors $\boldsymbol{\gamma}_i^{(n)}=(\boldsymbol{M}_n)_i^j\,\boldsymbol{\gamma}_j$; these $\boldsymbol{\gamma}_i^{(n)}$'s are precisely the linear positive (negative) integral $N_i^{\pm}\boldsymbol{\gamma}_i$ combinations of the elementary charges $\boldsymbol{\gamma}_i$ mentioned above. The building of BPS spectra using quiver mutation approach has been applied with success to $\mathcal{N}=2$ QFT$_4$ with ADE invariance; but, due to exotic properties such as diagonal links in Q_0^{BCFG} and 3-cycles as shown on Fig. 2, still needs more exploration for the class of non-simply laced type gauge symmetries.

In this paper, we consider the 4d $\mathcal{N}=2$ supersymmetric pure gauge models with finite dimensional non-simply laced BCFG type gauge symmetries, and develop further the method introduced in [15] for the building of BPS/anti-BPS states of this special subclass of $\mathcal{N}=2$ gauge theories. Here, we focus on the explicit construction of the CPT invariant BPS spectrum of the strong chambers $\mathfrak{Q}_{stg}^{G_2}$ and $\mathfrak{Q}_{stg}^{F_4}$ of the 4d $\mathcal{N}=2$ supersymmetric quantum field G_2 and F_4 gauge models; but, though it will not be explicitly detailed in present analysis, our method may be also applied to other $\mathcal{N}=2$ QFT_4's including those involving BPS quivers based on generalised Dynkin diagrams like the ones of affine Kac–Moody algebras encountered in $\mathcal{N}=2$ CFT_4 first considered in [16]; see also [17–19] for indefinite hyperbolic extensions. Our approach consists on starting from BPS quivers Q_0^{ADE} of type ADE and use *outer-automorphisms* to fold them into BPS quivers of type BCFG in quite similar manner as in the folding of ADE Dynkin diagrams to recover BCFG Dynkin ones [20–22]. In this way, the set of BPS states of 4d $\mathcal{N}=2$ pure gauge models with G_2 and F_4 gauge invariance gets related to the set of BPS states with D_4 and E_6 type gauge symmetries.

To achieve this goal, we proceed as follows: We first recall basic aspects of the primitive BPS quivers $Q_0^{\mathcal{G}}$ of $\mathcal{G}=ADE$ type; and show that they have outer-automorphism symmetries inherited from the outer-automorphisms of the Dynkin diagrams of Lie algebras. BPS graphs representing the $Q_0^{\mathcal{G}}$'s, which are given by the lists of Figs. 1 and 2 of section 2, are roughly speaking, a kind of a duplication of Dynkin diagrams of finite dimensional ADE Lie algebras; but with some specific properties to be exhibited at proper places. Like for the Dynkin diagram representing the Cartan matrix $\left(K^G\right)_{ij}=\boldsymbol{\alpha}_i^{\nu}.\boldsymbol{\alpha}_j$ of the Lie algebra of the gauge symmetry, the primitive BPS quiver $Q_0^{\mathcal{G}}$ is characterised by a "Cartan like" intersection matrix

$$\left(\mathcal{A}_0^{\mathcal{G}}\right)_{ij}=\boldsymbol{\gamma}_i\circ\boldsymbol{\gamma}_j$$

but based on Dirac pairing of the electric–magnetic (EM) charges of the BPS states [27–32].

Then, we build the pair of folding operators $\mathbf{f}_a^j$ and $\tilde{\mathbf{f}}_j^a$ (rectangular matrices) mapping ADE Dynkin diagrams to BCFG ones with the property $\mathbf{f}.\tilde{\mathbf{f}}=I_{id}$. After that we extend this folding method of Dynkin graphs to BPS quiver theory by constructing the generalised pair of folding operators $\mathbf{F}_A^J$ and $\tilde{\mathbf{F}}_J^A$ satisfying $\mathbf{F}.\tilde{\mathbf{F}}=I_{id}$ and allowing to generate BPS chambers $\mathfrak{Q}^{BCFG}$ out of the $\mathfrak{Q}^{ADE}$ ones. This generalised folding method constitutes a key ingredient in our way of doing. As an application, we derive the BPS states of the *strong* chambers of the 4d $\mathcal{N}=2$ supersymmetric pure G_2 and F_4 models as well as the group structures of the mutation sets $\left\{\boldsymbol{M}_n^{G_2}\right\}$ and $\left\{\boldsymbol{M}_n^{F_4}\right\}$. Recall that the Dynkin diagram of finite G_2 can be obtained by folding the three external nodes of the diagram of SO_8 reducing the rank of the Lie algebra from 4 to 2. Similarly, the Dynkin diagram of F_4 is obtained by folding nodes in the diagram of exceptional E_6.

The organisation of this paper is as follows: In section 2, we build the list of primitive BPS quivers $Q_0^{\mathcal{G}}$ of 4d $\mathcal{N}=2$ supersymmetric pure gauge theories and give their outer-automorphisms. In section 3, we study BPS states in $\mathcal{N}=2$ pure G_2 model; and build explicitly the set of BPS/anti-BPS states of the strong chamber $\mathfrak{Q}_{stg}^{G_2}$. In section 4, we do the same thing as in section 3; but for the exceptional F_4 gauge invariance. Section 5 is devoted to conclusion and comments. To complete this study, we give two Appendices A and B: the first appendix deals with the structure of the superpotentials associated with primitive quivers $Q_0^{SO_8}$, $Q_0^{G_2}$, $Q_0^{E_6}$, $Q_0^{F_4}$;

they have been added in order to make contact with general results in BPS quiver theory literature. The second appendix concerns the matrix realisation of the fundamental reflections r_i of the Coxeter groups generating the quiver mutations in the BPS strong chambers.

2. BPS quivers of $\mathcal{N} = 2$ QFT$_4$

In this section, we build the primitive BPS quivers $Q_0^{\mathcal{G}}$ of $\mathcal{N} = 2$ supersymmetric pure gauge theories with generic gauge symmetry $\mathcal{G}$. Since these $Q_0^{\mathcal{G}}$'s are intimately related with Dynkin diagrams of the Lie algebra of the gauge symmetry; we therefore split the list of BPS graphs into two subsets: (a) primitive quivers Q_0^{ADE} of type ADE; and (b) primitive quivers Q_0^{BCFG} of type BCFG. These primitive quivers together with an ordering of arg Z_i and mutations allow to build BPS states of supersymmetric pure gauge theories; the examples of G_2 and F_4 models will be explicitly studied in sections 3 and 4.

Before going into details, recall that BPS quivers in $\mathcal{N} = 2$ supersymmetric QFT$_4$ with gauge invariance $\mathcal{G}$ encode data on BPS states of the gauge theory. Depending on gauge coupling regime, we distinguish two particular chambers: the strong chamber $\mathfrak{Q}_{stg}^{\mathcal{G}}$ and the weak chamber $\mathfrak{Q}_{weak}^{\mathcal{G}}$. The content of these chambers can be generated by: (i) starting from $Q_0^{\mathcal{G}}$, with some ordering of the arguments arg Z_i of the central charges $Z\left(\boldsymbol{\gamma}_i\right)$ of the elementary BPS particles,

$$\arg Z_{i_1} > \arg Z_{i_2} > \ldots > \arg Z_{i_{2r}} \tag{2.1}$$

and (ii) performing *appropriate* and *successive* quiver mutations on primitive quiver. The length l of the successive mutations $\boldsymbol{M}_n$ may be closed, forming a finite cycle, or open and then infinite. It happens that the length l of the largest mutation is finite for the strong BPS chambers $\mathfrak{Q}_{stg}^{\mathcal{G}}$ of pure gauge theories; and it is infinite for weak chambers $\mathfrak{Q}_{weak}^{\mathcal{G}}$ which include the gauge particles as particular limits. Notice that a generic quiver mutation $\boldsymbol{M}_n$ has the structure

$$\boldsymbol{M}_n = r_{i_n} r_{i_{n-1}} \ldots r_{i_2} r_{i_1} \tag{2.2}$$

where the r_k's are non-commuting reflections ($r_k^2 = I_{id}$) generating a Coxeter group [23]. For infinite chambers $\mathfrak{Q}_{\inf}^{\mathcal{G}}$, there are infinitely many mutations; that is n a positive integer taking all possible values. But for finite BPS chambers $\mathfrak{Q}_{finite}^{\mathcal{G}}$, it happens that the successive reflections form a cycle with some length n_0; the largest mutation $\boldsymbol{M}_{n_0} = r_{i_{n_0}} r_{i_{n_0-1}} \ldots r_{i_2} r_{i_1}$ closes to the identity operator $\boldsymbol{M}_0 = I_{id}$; in other words

$$\boldsymbol{M}_{n_0} = I_{id} \qquad , \qquad \boldsymbol{M}_{n_0} = \boldsymbol{M}_{kn_0} = \boldsymbol{M}_0 \tag{2.3}$$

This cyclic property of mutations allows to determine exactly the BPS spectrum of the strong chambers; for explicit examples and calculations, see the analysis given in section 3 for the supersymmetric pure gauge models SO(8) and G_2; and in section 4 for the E_6 and F_4 models. For the general algorithm as well as illustrating examples including the method using quiver representations and induced superpotentials; see refs. [1,2] and appendix of [15]; see also refs. [5, 6] for explicit details using intersection matrix $\mathcal{A}_0^{\mathcal{G}}$.

After this brief introduction of primitive quivers, mutations and BPS chambers, we turn now to give some details on the structure of the primitive quivers $Q_0^{\mathcal{G}}$ and useful aspects of their properties.

2.1. ADE type primitive quivers

Given a $\mathcal{N} = 2$ supersymmetric *pure* gauge theory with a *rank r* gauge symmetry $\mathcal{G}$ of ADE type, the corresponding primitive quiver $Q_0^{\mathcal{G}}$ is represented by one of the graphs shown on the Fig. 1. A generic graph consists of $2r$ nodes and $3r - 2$ links as briefly described below:

i) nodes N_i and links in Q_0^{ADE}

The $2r$ nodes of the primitive quiver refer to the elementary BPS states represented by $2r$ charge vectors $\boldsymbol{\gamma}_1, ..., \boldsymbol{\gamma}_{2r}$; half of these electric–magnetic (EM) charges, say $\{\boldsymbol{\gamma}_i\}_{1\leq i\leq r}$, are given by $\boldsymbol{b}_1, ..., \boldsymbol{b}_r$; and the other remaining half charge vectors $\{\boldsymbol{\gamma}_{i+r}\}_{1\leq i\leq r}$ are given by $\boldsymbol{c}_1, ..., \boldsymbol{c}_r$ as in Fig. 1. For the primitive quiver Q_0^{ADE}, the $\boldsymbol{b}_i$'s stand for the EM charge of the r elementary monopoles $\{\mathfrak{M}_1, ..., \mathfrak{M}_r\}$ and the $\boldsymbol{c}_i$'s for the EM of the elementary dyons $\{\mathfrak{D}_1, ..., \mathfrak{D}_r\}$ of the underlying supersymmetric gauge theory.

$$b_i^\Lambda = \begin{pmatrix} q_i^I \\ p_{Ii} \end{pmatrix} \quad , \quad c_i^\Lambda = \begin{pmatrix} q_{i+r}^I \\ p_{Ii+r} \end{pmatrix} \tag{2.4}$$

with $I = 1, ..., r$ and $\Lambda = 1, ..., 2r$; each one of above EM charges $\boldsymbol{\gamma}_i$ is then an $SP(2r, \mathbb{R})$ vector with components as (γ_i^Λ). In the case of $\mathcal{N} = 2$ supersymmetric *pure* gauge theory, these EM charge vectors read in terms of the simple roots $\vec{a}_1, ..., \vec{a}_r$ of the Lie algebra of the gauge symmetry $\mathcal{G}$ as follows

$$\boldsymbol{b}_i = \begin{pmatrix} \vec{0} \\ \vec{a}_i \end{pmatrix} \quad , \quad \boldsymbol{c}_i = \begin{pmatrix} \vec{a}_i \\ -\vec{a}_i \end{pmatrix} \tag{2.5}$$

they are $2r$-dimensional vectors with $2r \times 2r$ intersection matrix $\left(\mathcal{A}_0^{\mathcal{G}}\right)_{ij} = \boldsymbol{\gamma}_i \circ \boldsymbol{\gamma}_j$ given by (1.1) and reading explicitly by using the electric q_i^I and the magnetic p_{Ij} charges like $q_i^I p_{Ij} - q_j^I p_{Ii}$. In terms of the EM charges of the elementary monopoles $\boldsymbol{b}_i$ and dyons $\boldsymbol{c}_i$, this matrix can be also presented in four $r \times r$ blocks as follows

$$\mathcal{A}_0^{\mathcal{G}} = \begin{pmatrix} \boldsymbol{b}_i \circ \boldsymbol{b}_j & \boldsymbol{b}_i \circ \boldsymbol{c}_j \\ \boldsymbol{c}_i \circ \boldsymbol{b}_j & \boldsymbol{c}_i \circ \boldsymbol{c}_j \end{pmatrix} \tag{2.6}$$

For later use, notice the three following features useful in performing explicit calculations.

First, for ADE Lie algebras, the intersection matrix of the simple roots $\vec{a}_i$ is a symmetric matrix given by $\vec{a}_i.\vec{a}_j = K_{ij}$, it is the Cartan matrix of the Lie algebra underlying the gauge symmetry and is graphically represented by a Dynkin diagram (for short DD^{ADE}).

Second the Dirac pairings of $\boldsymbol{b}_i \circ \boldsymbol{b}_j$ vanishes identically and $\boldsymbol{b}_i \circ \boldsymbol{c}_i = -\boldsymbol{c}_i \circ \boldsymbol{b}_i$ reducing the content of the matrix (2.6). Third the simplest $\mathcal{A}_0^{\mathcal{G}}$ matrix corresponding to rank $r = 1$ is just the $\mathcal{A}_0^{SU_2}$ given by

$$\mathcal{A}_0^{SU_2} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \tag{2.7}$$

This is the elementary matrix in BPS quiver theory; it teaches us that the $(\boldsymbol{b}, \boldsymbol{c})$ pair is the building block in dealing with BPS states. The corresponding elementary quiver $Q_0^{SU_2}$ has two nodes that might be imagined as following from a "kind" of antisymmetric replication of the usual node of the Dynkin diagram of the SU(2) Lie algebra.

$$\mathcal{A}_0^{SU_2} = K^{SU_2}\varepsilon = 2\varepsilon \tag{2.8}$$

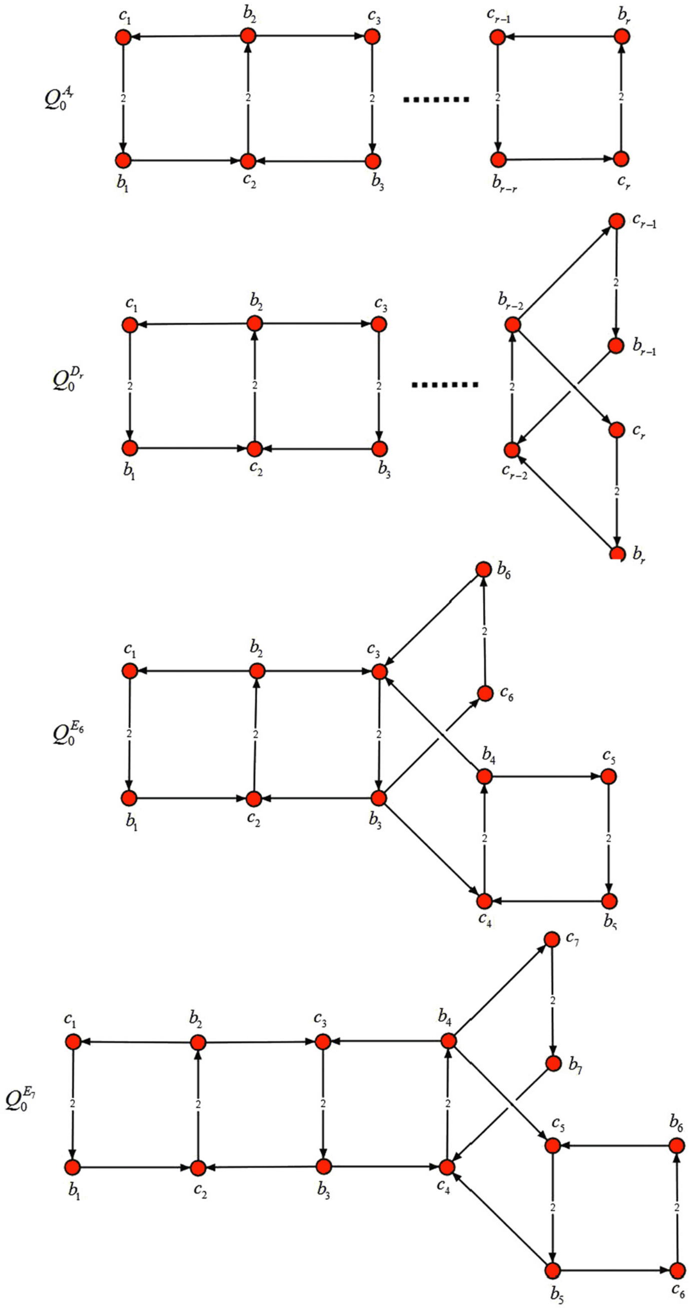

Fig. 1. BPS primitive quivers of type ADE in $\mathcal{N} = 2$ pure gauge theories.

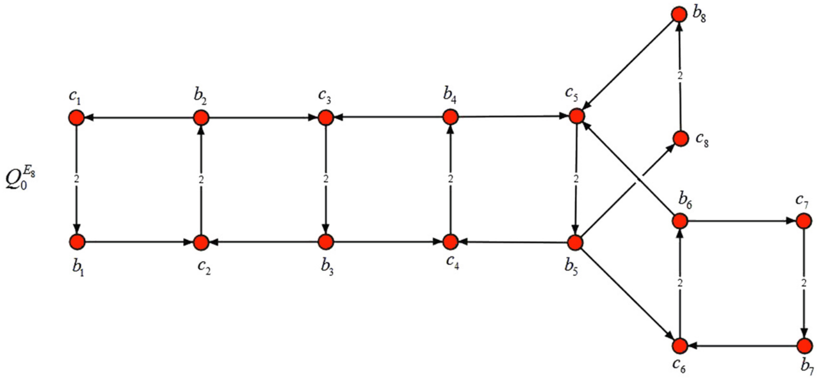

Fig. 1. (*continued*)

where

$$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.9}$$

Observe that the BPS quiver $Q_0^{SU_2}$ has two nodes and formally resembles to DD^{C_2}, the usual Dynkin diagram of the C_2 Lie algebra; the two objects are of course different; they are constructed by using two different pairing laws.

ii) More on links between $N_i - N_j$ nodes in Q_0^{ADE}

The $3r - 2$ links joining the nodes N_i of Q_0^{ADE} given by Fig. 1 are of two types: r oriented vertical links and $2(r-1)$ oriented horizontal ones. The r vertical links $l_1, ..., l_r$ join the two nodes of each of the r pairs $(\boldsymbol{b}_1, \boldsymbol{c}_1), ..., (\boldsymbol{b}_r, \boldsymbol{c}_r)$; they are oriented from the node $\boldsymbol{c}_i$ to the node $\boldsymbol{b}_i$ and they carry a charge given by the absolute value of the Dirac pairing $\boldsymbol{b}_i \circ \boldsymbol{c}_i$ which is equal to 2 as in (2.7). These links define r elementary sub-quivers as follows

$$\left(Q_0^{SU_2}\right)_1, \left(Q_0^{SU_2}\right)_2, ..., \left(Q_0^{SU_2}\right)_r \tag{2.10}$$

The $2(r-1)$ horizontal links l_{ij} join two nodes of different pairs $(\boldsymbol{b}_i, \boldsymbol{c}_i)$ and $\left(\boldsymbol{b}_j, \boldsymbol{c}_j\right)$; since for ADE Lie algebras the pairings $\boldsymbol{b}_i \circ \boldsymbol{b}_j = \boldsymbol{c}_i \circ \boldsymbol{c}_j = 0$, it follows that the intersection matrix $\mathcal{A}_0^G$ describing the primitive quiver Q_0^{ADE} which is given by (2.6) reduces to the $\boldsymbol{b}_i \circ \boldsymbol{c}_j$ off diagonal blocks. By using (2.5), we have $\boldsymbol{c}_i \circ \boldsymbol{b}_j \sim \vec{a}_i.\vec{a}_j$; and then the above intersection matrix becomes

$$\mathcal{A}_0^{\mathcal{G}} = \begin{pmatrix} 0 & -K^{\mathcal{G}} \\ K^{\mathcal{G}} & 0 \end{pmatrix} \tag{2.11}$$

where $K^{\mathcal{G}}$ is the $r \times r$ Cartan matrix of the ADE type gauge invariance $\mathcal{G}$ of the supersymmetric pure gauge theory. Eq. (2.11) captures the property behind the appearance of Q_0^{ADE} as a duplication of ADE Dynkin diagram encoding $K^{\mathcal{G}}$,

$$\mathcal{A}_0^{\mathcal{G}} = K^{\mathcal{G}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.12}$$

The ADE type graphs Q_0^{ADE} of the primitive quivers are as listed in Fig. 1. From the graphic representation of Q_0^{ADE}, one learns that some BPS quivers of ADE type have outer automorphisms leaving invariant their topology. When they exist, these outer automorphisms are given by discrete symmetries as follows

primitive quiver	gauge symmetry	outer-automorphism
$Q_0^{A_{2r}}$	SU_{2r+1}	–
$Q_0^{A_{2r-1}}$	SU_{2r}	$\mathbb{Z}_2$
$Q_0^{D_r}$	SO_{2r}	$\mathbb{Z}_2$
$Q_0^{D_4}$	SO_8	$\mathbb{Z}_2, \mathbb{Z}_3$
$Q_0^{E_6}$	E_6	$\mathbb{Z}_2$
$Q_0^{E_7}$	E_7	–
$Q_0^{E_8}$	E_8	–

(2.13)

they are similar to the ones we have in Dynkin diagrams of ADE Lie algebras.

Before proceeding, notice that, because of the link between the BPS primitive $Q_0^{\mathcal{G}}$'s and Dynkin diagrams $DD^{\mathcal{G}}$ of finite Lie algebras of the gauge symmetry $\mathcal{G}$, the correspondence

$$Q_0^{ADE} \qquad \longleftrightarrow \qquad DD^{ADE} \tag{2.14}$$

given by eq. (2.13) may be naturally extended to the case affine Kac–Moody type Dynkin diagrams $DD^{\tilde{\mathcal{G}}}$ [24–26] like

$$Q_0^{\tilde{\mathcal{G}}} \qquad \longleftrightarrow \qquad DD^{\tilde{\mathcal{G}}} \tag{2.15}$$

For the case of simply laced affine Lie algebras $\tilde{\mathcal{G}} = \tilde{A}\tilde{D}\tilde{E}$; the corresponding affine type BPS quivers $Q_0^{\tilde{\mathcal{G}}}$ can be built in a similar manner as the ordinary Q_0^{ADE} BPS ones. In addition to the r pairs $(\boldsymbol{b}_i, \boldsymbol{c}_i)$, the affine quiver $Q_0^{\tilde{\mathcal{G}}}$ has an addition pair of elementary BPS states with EM given by $(\boldsymbol{b}_0, \boldsymbol{c}_0)$. As an illustrating example, let us describe briefly the *twisted* $\mathrm{SU}(2)_k$ Kac–Moody algebra with Kac–Moody level k. This is an infinite dimensional Lie algebra; its root system $\tilde{\Phi}$ is generated by two simple roots $\vec{a}_0$ and $\vec{a}_1$ with intersection matrix $\tilde{K}_{\mu\nu}^{SU_2} = \vec{a}_\mu.\vec{a}_\nu$ given by

$$\tilde{K}^{SU_2} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad , \qquad \det \tilde{K}^{SU_2} = 0 \tag{2.16}$$

By using the correspondence (2.15), we end with a primitive quiver $\tilde{Q}_0^{SU_2}$ with four nodes describing four elementary BPS particles with electric–magnetic charges $\boldsymbol{b}_0, \boldsymbol{b}_1, \boldsymbol{c}_0, \boldsymbol{c}_1$ and Dirac pairings given by the following generalised intersection matrix

$$\tilde{\boldsymbol{A}}_0^{SU_2} = \tilde{K}^{SU_2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.17}$$

As far as affine extension is concerned, recall that $\mathcal{N} = 2$ quiver gauge theories using generalised Cartan matrices $\tilde{K}$ and affine Kac–Moody diagrams $DD^{\tilde{\mathcal{G}}}$ are also present in the engineering of $\mathcal{N} = 2$ supersymmetric gauge theories; the $\tilde{K}$'s play an important role in: (i) the study 4D $\mathcal{N} = 2$ superconformal field theories in presence of bi-fundamental hypermatter; and (ii) the classification of these scale invariant theories [16,18].

With in mind the above general picture on simply laced ADE type $\mathcal{N} = 2$ models, we are now in position to address the extension to the case of non-simply laced type prototypes that we are interested in this study. In this generalisation, we will focus on the ordinary BCFG quivers; the description concerning generalised Dynkin diagrams of affine Kac–Moody type is straightforward; it is omitted.

2.2. BCFG type primitive quivers

In the $\mathcal{N}=2$ supersymmetric pure gauge theory with rank r gauge symmetry $\mathcal{G}$ of finite dimensional BCFG Lie algebra type, the primitive quivers $Q_0^{\mathcal{G}}$ are as depicted by the Fig. 2. These graphs consist of $2r$ nodes; but a different number of links compared with Q_0^{ADE}. In these quivers, one has, in addition to vertical and horizontal links, a diagonal link because for gauge symmetries with non-simply laced Lie algebras, the Dirac pairings $\boldsymbol{c}_i \circ \boldsymbol{c}_j$ are no longer zero. In this case, the intersection matrix $\mathcal{A}_0^{\mathcal{G}}$ of the BPS primitive quiver Q_0^{BCFG} has the form

$$\mathcal{A}_0^{\mathcal{G}} = \begin{pmatrix} 0_{r\times r} & -K^T \\ K & K^T - K \end{pmatrix} \tag{2.18}$$

where K is the Cartan matrix of the underlying gauge invariance. The matrix K

$$K_{ij}^{\mathcal{G}} = \frac{2\vec{\alpha}_i.\vec{\alpha}_j}{\vec{\alpha}_i.\vec{\alpha}_i} \tag{2.19}$$

is non-symmetric because simple roots for non-simply laced Lie algebras have two different lengths. By substituting $K_{ij}^{\mathcal{G}}$ in the $\mathcal{A}_0^{\mathcal{G}}$, we have for the example of the $\mathcal{G}=G_2$ gauge symmetry the following intersection matrix

$$\mathcal{A}_0^{G_2} = \begin{pmatrix} 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 2 & -1 & 0 & -2 \\ -3 & 2 & 2 & 0 \end{pmatrix} \tag{2.20}$$

Notice that for $\mathcal{N}=2$ supersymmetric pure gauge theory with BCFG gauge invariance, the EM charge vectors $\boldsymbol{\beta}_i$ and $\boldsymbol{\delta}_i$ of the r elementary monopoles $\{\mathfrak{M}_1, ..., \mathfrak{M}_r\}$ and the elementary dyons $\{\mathfrak{D}_1, ..., \mathfrak{D}_r\}$ read in terms of the simple roots $\vec{\alpha}_1, ..., \vec{\alpha}_r$ of the underlying supersymmetric gauge theory as follows

$$\boldsymbol{\beta}_i = \begin{pmatrix} \vec{0} \\ \frac{2}{\vec{\alpha}_i.\vec{\alpha}_i}\vec{\alpha}_i \end{pmatrix} \quad , \quad \boldsymbol{\delta}_i = \begin{pmatrix} \vec{\alpha}_i \\ -\frac{2}{\vec{\alpha}_i.\vec{\alpha}_i}\vec{\alpha}_i \end{pmatrix} \tag{2.21}$$

Notice also that in supersymmetric BCFG gauge models the ratio $2/\vec{\alpha}_i.\vec{\alpha}_i$ is not usually equal to one as in case of ADE. The list of the primitive quivers is given by the graphs of Fig. 2; the relationships between Q_0^{ADE} and Q_0^{BCFG} are as follows

primitive Q_0^{ADE}	folding	primitive Q_0^{BCFG}
$Q_0^{SU_{2r}}$	$\mathbb{Z}_2$	$Q_0^{SP_r}$
$Q_0^{SO_{2r}}$	$\mathbb{Z}_2$	$Q_0^{SO_{2r-1}}$
$Q_0^{SO_8}$	$\mathbb{Z}_3$	$Q_0^{G_2}$
$Q_0^{E_6}$	$\mathbb{Z}_2$	$Q_0^{F_4}$

(2.22)

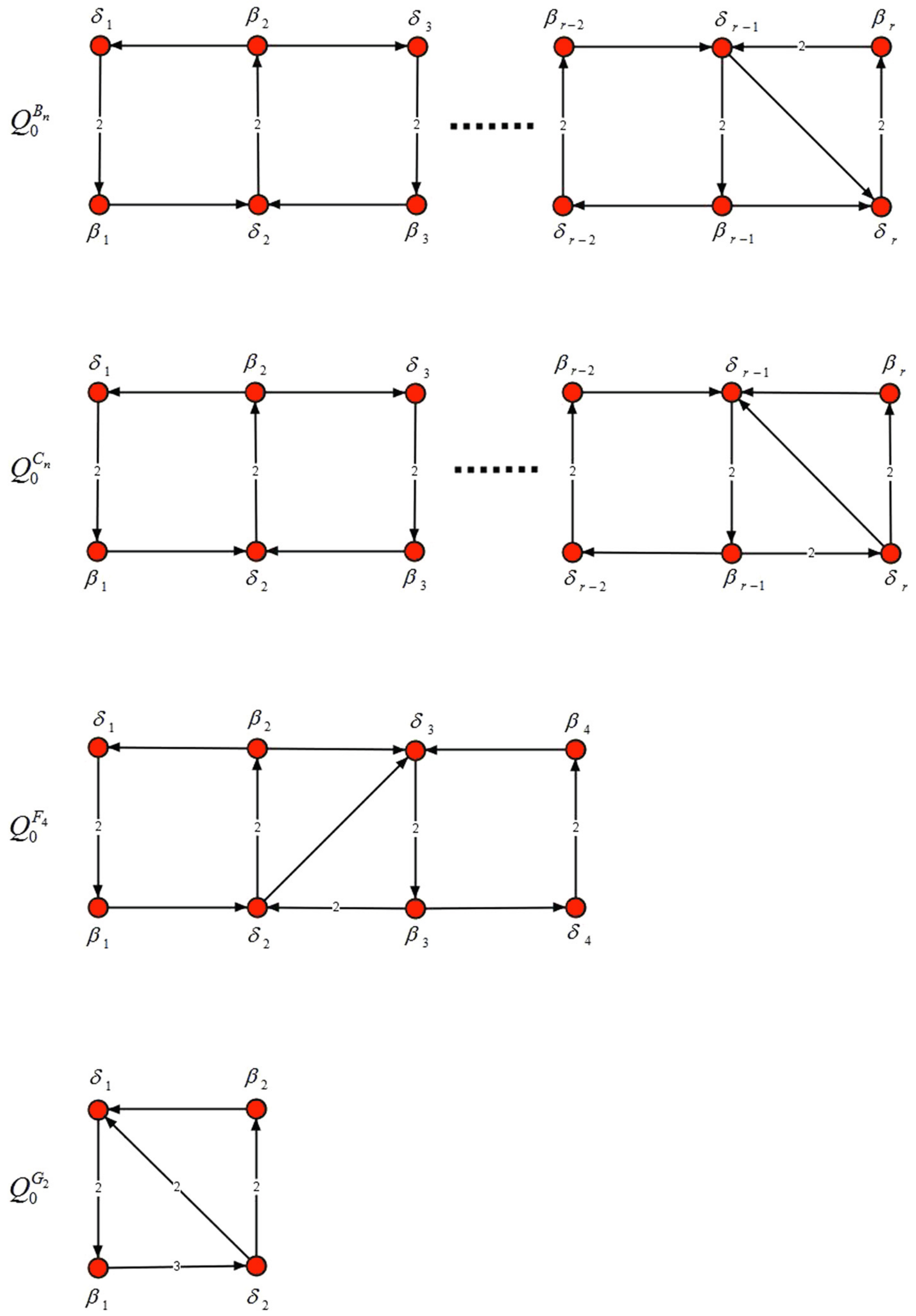

Fig. 2. BPS quivers of BCFG type; these graphs contains an extra diagonal link in addition to the horizontal and vertical links of quivers with ADE type.

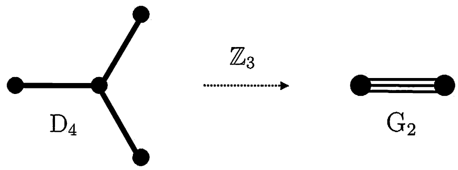

Fig. 3. Folding three nodes in SO(8) Dynkin diagram lead to the G_2 diagram.

They have been obtained by using the correspondence (2.14) as well as the following diagram showing that it possible to link Q_0^{BCFG} and Q_0^{ADE} by appropriate folding operations

$$\begin{array}{ccc} DD^{ADE} & \longleftrightarrow & Q_0^{ADE} \\ \downarrow^{folding} & & \downarrow^{folding} \\ DD^{BCFG} & \longleftrightarrow & Q_0^{BCFG} \end{array} \tag{2.23}$$

3. $\mathcal{N}=2$ pure G_2 theory

In this section, we construct the BPS states of the strong chamber of $\mathcal{N}=2$ supersymmetric pure G_2 theory. First we build the folding operators $\mathbf{f}$ and $\tilde{\mathbf{f}}$ mapping the Dynkin diagram of SO_8 down to the Dynkin diagram of G_2; see Fig. 3. Then, we extend this folding based construction to linking the primitive quivers $Q_0^{SO_8}$ and $Q_0^{G_2}$; this link is obtained by working out the generalisation of $\mathbf{f}$ and $\tilde{\mathbf{f}}$ denoted below like $\boldsymbol{F}$ and $\tilde{\boldsymbol{F}}$. Next, we derive the BPS spectrum of the strong chamber of $\mathfrak{Q}_{stg}^{G_2}$ of the supersymmetric theory with G_2 invariance.

3.1. *Dynkin diagram of G_2 as folded* DD_{SO_8}

To begin recall that the Lie algebra of the 14 dimensional G_2 gauge symmetry has 12 roots; the six positive roots are generated by the two simple $\vec{\alpha}_1$ and $\vec{\alpha}_2$ having different lengths taken here as $\vec{\alpha}_1.\vec{\alpha}_1=\frac{2}{3}$, $\vec{\alpha}_2.\vec{\alpha}_2=2$ and intersection like $\vec{\alpha}_1.\vec{\alpha}_2=-1$. The root system of G_2 is as follows

$$\begin{array}{lll} \pm\vec{\alpha}_1\ , & \pm(\vec{\alpha}_1+\vec{\alpha}_2)\ , & \pm(3\vec{\alpha}_1+\vec{\alpha}_2) \\ \pm\vec{\alpha}_2\ , & \pm(2\vec{\alpha}_1+\vec{\alpha}_2)\ , & \pm(3\vec{\alpha}_1+2\vec{\alpha}_2) \end{array} \tag{3.1}$$

For later use, we revisit some useful features concerning this system; in particular the issue regarding their link with the roots of SO(8) Lie algebra. We start by the Cartan matrix of G_2 given by $K_{ij}^{G_2}=\vec{\alpha}_i^{\nu}.\vec{\alpha}_j$; the two $\vec{\alpha}_i^{\nu}=\frac{2}{\vec{\alpha}_i.\vec{\alpha}_i}\vec{\alpha}_i$ are the coroots associated with the two $\vec{\alpha}_i$'s; the matrix $K_{ij}^{G_2}$ reads in terms of the usual Euclidean scalar product $\vec{\alpha}_i.\vec{\alpha}_j$ as follows

$$K_{ij}^{G_2}=\frac{2\vec{\alpha}_i.\vec{\alpha}_j}{\vec{\alpha}_i.\vec{\alpha}_i} \tag{3.2}$$

this is a non-symmetric integral 2×2 matrix which reads explicitly like

$$K_{G_2}=\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \tag{3.3}$$

This matrix is represented by the two nodes Dynkin diagram DD^{G_2} of Fig. 3, which in turns may be viewed as given by the folding of three nodes of the Dynkin diagram DD^{SO_8} of the SO(8) Lie algebra

$$DD^{G_2} = \frac{DD^{SO_8}}{\mathbb{Z}_3} \tag{3.4}$$

where the discrete $\mathbb{Z}_3$ is the outer-automorphism rotating the three external nodes of the DD^{SO_8} diagram; here $\mathbb{Z}_3$ is an abelian subgroup of the permutation group $\mathbb{S}_3$. Eq. (3.4) implies that the Cartan matrix $K_{JL}^{SO_8} = \vec{a}_J.\vec{a}_L$ of the of SO(8) Lie algebra namely

$$K_{SO_8} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \tag{3.5}$$

and the K_{G_2} one given by eq. (3.3) are related to each other by a pair of folding operators $\mathbf{f}$ and $\tilde{\mathbf{f}}$ with respective entries as $\left(f_i^J\right)$ and $(\tilde{f}_J^i)$. This means that the two simple roots of G_2 may be related in same way to the four real vectors simple roots $\vec{a}_J$ of the Lie algebra of SO(8). However, due to the difference between the dimensions of K_{SO_8} and K_{G_2}, the bridge between the K_{SO_8} and K_{G_2} is not unique; descending from K_{SO_8} down to K_{G_2} involves projections showing that, generally speaking, there are infinitely many ways to go from K_{SO_8} down to K_{G_2}. Despite this arbitrariness, one may nevertheless find a way to link the two matrices by imposing extra conditions to fix this arbitrariness. A manner to go from K_{SO_8} to K_{G_2} is by using the above mentioned two folding operators (rectangular matrices) $\mathbf{f} = \left(f_i^J\right)$ and a companion $\tilde{\mathbf{f}} = (\tilde{f}_J^i)$ defined as

$$\tilde{\mathbf{f}} = \mathbf{f}^T \left(\mathbf{f}\mathbf{f}^T\right)^{-1} \qquad , \qquad \mathbf{f}\tilde{\mathbf{f}} = I_{2\times 2} \tag{3.6}$$

The role of each one of the $\mathbf{f}$ and $\tilde{\mathbf{f}}$ folding operators is as described in what follows:

a) The first folding operator $\mathbf{f}$ is a rectangular 2×4 matrix used to convert 4×4 matrix K_{SO_8} into the rectangular 2×4 matrix $\mathbf{f}.K_{SO_8}$. It converts as well the G_2 Cartan matrix like $K_{G_2}\mathbf{f}$. Explicitly, the 2×4 matrix $\mathbf{f}$ is needed to relate the two Cartan matrices and simples roots as follows

$$K_{G_2}.\mathbf{f} = \mathbf{f}.K_{SO_8} \qquad , \qquad \vec{\alpha}_i = \mathbf{f}_i^J \vec{a}_J \tag{3.7}$$

By thinking of the entries of the folding operator $\mathbf{f}$ as

$$\mathbf{f} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \tag{3.8}$$

with x_J and y_J numbers, eq. (3.7) leads to the following constraint relations

$$\begin{aligned} 2x_1 - y_1 &= 2x_1 - x_2 \\ 2x_2 - y_2 &= 2x_2 - x_1 - x_3 - x_4 \\ 2x_3 - y_3 &= 2x_3 - x_2 \\ 2x_4 - y_4 &= 2x_4 - x_2 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} 2y_1 - 3x_1 &= 2y_1 - y_2 \\ 2y_2 - 3x_2 &= 2y_2 - y_1 - y_3 - y_4 \\ 2y_3 - 3x_3 &= 2y_3 - y_2 \\ 2y_4 - 3x_4 &= 2y_4 - y_2 \end{aligned} \tag{3.10}$$

These constraint equations are invariant under the $\mathbb{S}_3$ discrete permutation group fixing the number x_2 and permuting (x_1, x_3, x_4) amongst themselves. The same symmetry transformations is valid for y_2, which is fixed, and (y_1, y_3, y_4) permuted among themselves. Using this symmetry, we can set

$$\begin{aligned} a = x_1 = x_3 = x_4 \quad &, \quad \xi = x_2 \\ b = y_1 = y_3 = y_4 \quad &, \quad \xi' = y_2 \end{aligned} \tag{3.11}$$

which, up on substituting back into (3.9)–(3.10), lead to $b = \xi$ and $\xi' = 3a$; and then a folding operator $\mathbf{f}$ as follows

$$\mathbf{f} = \begin{pmatrix} a & b & a & a \\ b & 3a & b & b \end{pmatrix} \tag{3.12}$$

It depends on two free parameters a and b that remain to be determined; one of them is fixed by the $\vec{\alpha}_i = \mathbf{f}_i^J \vec{a}_J$ and the normalisation of the lengths of two simple roots of G_2.

b) The second folding operator $\tilde{\mathbf{f}}$ is also a rectangular matrix, but of type 4×2; it behaves like $\mathbf{f}^T$ namely

$$\mathbf{f}^T = \begin{pmatrix} a & b \\ b & 3a \\ a & b \\ a & b \end{pmatrix} \tag{3.13}$$

it is needed to extract the K_{G_2} Cartan matrix from eq. (3.7). By multiplying, from the right, both sides of (3.7) by $\tilde{\mathbf{f}}$, we end with the following 2×2 matrix equation

$$K_{G_2}.\mathbf{f}\tilde{\mathbf{f}} = \mathbf{f}.K_{SO_8}.\tilde{\mathbf{f}} \tag{3.14}$$

To get K_{G_2} from above constraint relation with K_{SO_8}, we demand that the condition $\mathbf{f}\tilde{\mathbf{f}} = I_{2\times 2}$ leading to the folding relation

$$K_{G_2} = \mathbf{f}.K_{SO_8}.\tilde{\mathbf{f}} \quad , \quad \det \mathbf{f}\tilde{\mathbf{f}} = 1 \tag{3.15}$$

Notice that the condition $\mathbf{f}\tilde{\mathbf{f}} = I_{2\times 2}$ has infinitely many solutions; a particular solution is given by

$$\tilde{\mathbf{f}} = \mathbf{f}^T \left(\mathbf{f}\mathbf{f}^T\right)^{-1} \quad , \quad \det\left(\mathbf{f}\mathbf{f}^T\right) \neq 0 \tag{3.16}$$

Explicitly, we have

$$\tilde{\mathbf{f}} = \frac{1}{3a^2 - b^2} \begin{pmatrix} a & -\frac{b}{3} \\ -b & a \\ a & -\frac{b}{3} \\ a & -\frac{b}{3} \end{pmatrix} \tag{3.17}$$

By substituting in $\vec{\alpha}_i = \mathbf{f}_i^J \vec{a}_J$ and using the properties of simple roots both for G_2 and SO_8, in particular the ratio

$$\frac{\vec{\alpha}_1.\vec{\alpha}_1}{\vec{\alpha}_2.\vec{\alpha}_2} = \frac{1}{3} \tag{3.18}$$

we end with the condition $ab = 0$. Solving this constraint by taking $a = 0$, we then have

$$\mathbf{f} = \begin{pmatrix} 0 & b & 0 & 0 \\ b & 0 & b & b \end{pmatrix} \quad , \quad \tilde{\mathbf{f}} = \begin{pmatrix} 0 & \frac{1}{3b} \\ \frac{1}{b} & 0 \\ 0 & \frac{1}{3b} \\ 0 & \frac{1}{3b} \end{pmatrix} \tag{3.19}$$

where b appears as a scaling parameter like $\mathbf{f} = b\mathbf{f}_0$ and $\tilde{\mathbf{f}} = \frac{1}{b}\tilde{\mathbf{f}}_0$ with

$$\mathbf{f_0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad , \quad \tilde{\mathbf{f}}_\mathbf{0} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 0 \\ 0 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \tag{3.20}$$

Below, we use the folding matrix operators $\mathbf{f_0}$ and $\tilde{\mathbf{f}}_\mathbf{0}$; and for convenience, we will drop out the extra index; i.e.: $\mathbf{f_0} \equiv \mathbf{f}$ and $\tilde{\mathbf{f}}_\mathbf{0} \equiv \tilde{\mathbf{f}}$.

3.2. BPS states of $\mathfrak{Q}_{stg}^{G_2}$

To obtain the BPS states of the strong chamber $\mathfrak{Q}_{stg}^{G_2}$ of the $\mathcal{N} = 2$ supersymmetric pure G_2 theory, we use the two following things:
(i) the extension of the idea of folding operators $\mathbf{f}$ and $\tilde{\mathbf{f}}$, relating the Dynkin diagrams DD_{G_2} and DD_{SO_8}, to the two primitive quivers $Q_0^{G_2}$ and $Q_0^{SO_8}$ of the supersymmetric pure gauge models with G_2 and $SO(8)$ gauge symmetries. Recall that $Q_0^{G_2}$ and $Q_0^{SO_8}$ are roughly speaking duplications of DD_{G_2} and DD_{SO_8}.
(ii) the knowledge of the BPS states of the strong chamber $\mathfrak{Q}_{stg}^{SO_8}$ of the supersymmetric pure SO_8 theory. There, the BPS states are obtained by mutating the primitive quiver $Q_0^{SO_8}$; that is by performing transformations like

$$\boldsymbol{M}_n : Q_0^{SO_8} \to Q_n^{SO_8} \tag{3.21}$$

where the $\boldsymbol{M}_n$ mutation operators are as in (2.2).

In this subsection, we first describe briefly how the machinery works for the derivation of the strong chamber $\mathfrak{Q}_{stg}^{SO_8}$; and turn after to build the $\mathfrak{Q}_{stg}^{G_2}$ by using the extended folding method.

3.2.1. Strong chamber $\mathfrak{Q}_{stg}^{SO_8}$

To begin, recall that the set of mutations $\{\boldsymbol{M}_n\} \equiv \boldsymbol{G}_{stg}^{SO_8}$ of the strong chamber $\mathfrak{Q}_{stg}^{SO_8}$ is given by the Coxeter group $\boldsymbol{G}_{stg}^{SO_8}$ generated by 8 fundamental reflections $r_1, ..., r_8$. These reflections obey the property

$$\left(r_i r_j\right)^{m_{ij}} = I_{8\times 8} \tag{3.22}$$

where the positive m_{ij} integers are given by the Coxeter matrix [6,23]. Recall also that in order to get the BPS/anti-BPS states of $\mathfrak{Q}_{stg}^{SO_8}$, it is enough to use a subgroup $H_{stg}^{SO_8}$ of the Coxeter $\boldsymbol{G}_{stg}^{SO_8}$; this subgroup is generated by two particular non-commuting operators L_1 and L_2 given by the composition of four r_i reflections ($r_i^2 = I_{id}$) like

$$L_1 = r_4 r_3 r_2 r_1 \quad , \quad L_2 = r_8 r_7 r_6 r_5 \tag{3.23}$$

These composed reflections are also reflections ($L_1^2 = L_2^2 = I_{id}$); they correspond, on the BPS states building side, to taking the arguments $\arg Z(\boldsymbol{\gamma}_i)$ of the central charges $Z(\boldsymbol{\gamma}_i)$ of the elementary monopoles $\mathfrak{M}_i$ and dyons $\mathfrak{D}_i$ as follows

$$\begin{array}{llll} \arg Z(\mathbf{b}_1) & = \arg Z(\mathbf{b}_2) & = \arg Z(\mathbf{b}_3) & = \arg Z(\mathbf{b}_3) \\ \arg Z(\mathbf{c}_1) & = \arg Z(\mathbf{c}_2) & = \arg Z(\mathbf{c}_3) & = \arg Z(\mathbf{c}_3) \end{array} \tag{3.24}$$

together with the ordering

$$\arg Z(\mathbf{c}_i) > \arg Z(\mathbf{b}_i) \tag{3.25}$$

The matrix realisations of the generators L_1 and L_2 on the space of EM charges ($\mathbf{b}_1, .., \mathbf{b}_4$, $\mathbf{c}_1, .., \mathbf{c}_4$) have a remarkable form as shown below

$$L_1 = \begin{pmatrix} I_{4\times4} & R \\ 0_{4\times4} & -I_{4\times4} \end{pmatrix} \quad , \quad L_2 = \begin{pmatrix} -I_{4\times4} & 0_{4\times4} \\ R & I_{4\times4} \end{pmatrix} \tag{3.26}$$

where the 4×4 matrix R is related to the Cartan matrix like $R = 2I_{4\times4} - K_{SO_8}$; explicitly the R-matrix is as follows

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{3.27}$$

The 8×8 matrix generators (3.26) satisfy the properties

$$L_1^2 = L_2^2 = I_{8\times8} \quad , \quad (L_2L_1)^6 = I_{8\times8} \tag{3.28}$$

teaching us a set of interesting information on the structure of the subgroup $H_{stg}^{SO_8}$; in particular the three following things:

- First, the subgroup $H_{stg}^{SO_8}$ is a finite discrete subgroup with matrix elements M_n given by particular monomials of the generators like

$$M_{2k} = (L_2L_1)^k \quad , \quad M_{2k+1} = L_1 M_{2k} \tag{3.29}$$

  with integer $k \geq 0$ and $M_0 = I_{8\times8}$.
- Second, because of the property $M_{12} = I_{8\times8}$ and the remarkable relation $M_6 = -I_{8\times8}$, the cardinality of the subgroup $H_{stg}^{SO_8}$ is equal to *12*; and is given by

$$H_{stg}^{SO_8} = \{\pm I_{id}, \pm M_1, \pm M_2, \pm M_3, \pm M_4, \pm M_5\} \tag{3.30}$$

  $H_{stg}^{SO_8}$ is isomorphic to the dihedral group Dih_{12} [6,15]. Knowing the explicit expressions of L_1 and L_2 which are as in (3.26), we can write down the explicit expressions of all elements in (3.30).
- Third, the BPS/anti-BPS states of the strong chamber $\mathfrak{Q}_{stg}^{SO_8}$ can be read from the rows of the M_n matrices of $H_{stg}^{SO_8}$. The identity $M_0 = I_{8\times8}$ gives precisely the elementary monopoles and elementary dyons.
  By performing the first mutation $M_1 = L_1$: $Q_0^{SO_8} \to Q_1^{SO_8}$, the resulting quiver $Q_1^{SO_8}$ has eight new BPS states $\boldsymbol{\gamma}_i^{(1)}$ with EM charges directly read from the 8 rows of the matrix representation of L_1 namely

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.31}$$

The first row of this matrix namely $(1, 0, 0, 0, 0, 1, 0, 0)$ means that a BPS state with EM charge $\boldsymbol{\gamma}_1^{(1)} = \boldsymbol{b}_1 + \boldsymbol{c}_2$ has been generated by the L_1 mutation of $Q_0^{SO_8}$; the second row $(0, 1, 0, 0, 1, 0, 1, 1)$ gives another new BPS state with EM charge $\boldsymbol{\gamma}_2^{(1)} = \boldsymbol{b}_2 + \boldsymbol{c}_1 + \boldsymbol{c}_3 + \boldsymbol{c}_4$; and so on. The BPS/anti-BPS states generated by the first mutation are as follows

$$\begin{array}{ll} \boldsymbol{b}_1 + \boldsymbol{c}_2 & -\boldsymbol{c}_1 \\ \boldsymbol{b}_2 + \boldsymbol{c}_1 + \boldsymbol{c}_3 + \boldsymbol{c}_4 & -\boldsymbol{c}_2 \\ \boldsymbol{b}_3 + \boldsymbol{c}_2 & -\boldsymbol{c}_3 \\ \boldsymbol{b}_4 + \boldsymbol{c}_2 & -\boldsymbol{c}_4 \end{array} \tag{3.32}$$

By performing the eleven M_n mutations in (3.30) which corresponds just to building the elements of the subgroup $H_{stg}^{SO_8}$, we obtain the list of the *48* BPS states of $\mathfrak{Q}_{stg}^{SO_8}$; it reads as follows

$$\begin{array}{lll} \pm b_1 & \pm(b_1 + c_2) & \pm(b_1 + b_4 + c_2) \\ \pm b_2 & \pm(b_3 + c_2) & \pm(b_1 + b_3 + c_2) \\ \pm b_3 & \pm(b_4 + c_2) & \pm(b_2 + c_3 + c_4) \\ \pm b_4 & \pm(b_2 + c_1) & \pm(b_3 + b_4 + c_2) \\ \pm c_1 & \pm(b_2 + c_3) & \pm(b_1 + b_3 + b_4 + c_2) \\ \pm c_2 & \pm(b_2 + c_4) & \pm(b_1 + b_3 + b_4 + 2c_2) \\ \pm c_3 & \pm(b_2 + c_1 + c_4) & \pm(2b_2 + c_1 + c_3 + c_4) \\ \pm c_4 & \pm(b_2 + c_1 + c_3) & \pm(b_2 + c_1 + c_3 + c_4) \end{array} \tag{3.33}$$

With this construction of the BPS/anti-BPS states of $\mathfrak{Q}_{stg}^{SO_8}$ in mind, we turn now to build the BPS spectrum of the strong chamber $\mathfrak{Q}_{stg}^{G_2}$ of the supersymmetric gauge theory with gauge symmetry G_2.

3.2.2. *Strong chamber* $\mathfrak{Q}_{stg}^{G_2}$

By using the method of quiver folding induced from folding of the DD_{SO_8} down to DD_{G_2} as in eq. (2.23), the primitive quiver $Q_0^{SO_8}$ can be folded into the primitive quiver $Q_0^{G_2}$. The same feature holds for the mutation subgroup group $H_{stg}^{SO_8}$, used above for constructing $\mathfrak{Q}_{stg}^{SO_8}$, which gets then mapped to a group $H_{stg}^{G_2}$. This set $H_{stg}^{G_2}$ should be also thought of as a subgroup of the Coxeter group $\boldsymbol{G}_{stg}^{G_2}$ in the same manner as $H_{stg}^{SO_8}$ is a subgroup of $\boldsymbol{G}_{stg}^{SO_8}$; that is:

$$\begin{array}{ccc} H_{stg}^{SO_8} & \hookrightarrow & \boldsymbol{G}_{stg}^{SO_8} \\ \downarrow & & \downarrow \\ H_{stg}^{G_2} & \hookrightarrow & \boldsymbol{G}_{stg}^{G_2} \end{array} \tag{3.34}$$

Having $Q_0^{G_2}$ and $H_{stg}^{G_2}$, we can therefore build the strong chamber $\mathfrak{Q}_{stg}^{G_2}$ just by repeating the same steps done for constructing the BPS strong chamber $\mathfrak{Q}_{stg}^{SO_8}$. To get the structure of the set $H_{stg}^{G_2}$, notice that like the subgroup $H_{stg}^{SO_8}$, it is as well generated by two operators $\mathcal{L}_1$ and $\mathcal{L}_2$ related to above L_1 and L_2 by two folding matrix operators $\boldsymbol{F}$ and $\tilde{\boldsymbol{F}}$ as follows

$$\mathcal{L}_i = \boldsymbol{F}.L_i.\tilde{\boldsymbol{F}} \tag{3.35}$$

It happens that the $\boldsymbol{F}$ and $\tilde{\boldsymbol{F}}$ have much to do with the $\mathbf{f}$ and $\tilde{\mathbf{f}}$ used in relating the Dynkin diagrams of G_2 and SO(8); the main difference is that these $\boldsymbol{F}$ and $\tilde{\boldsymbol{F}}$ have the double dimensions compared to $\mathbf{f}$ and $\tilde{\mathbf{f}}$ in agreement with eqs. (2.11)–(2.18); the folding operator $\boldsymbol{F}$ is given by a 4×8 matrix and its companion $\tilde{\boldsymbol{F}}$ is a 8×4 matrix; in same way as $\boldsymbol{F}^T$. Recall that $\mathbf{f}$ is a 2×4 matrix operator acting on the simple roots of SO(8); and $\tilde{\mathbf{f}}$ is a 4×2 matrix. The explicit relationship between the pair $\left(\boldsymbol{F}, \tilde{\boldsymbol{F}}\right)$ and the pair $\left(\mathbf{f}, \tilde{\mathbf{f}}\right)$ reads like

$$\boldsymbol{F} = \begin{pmatrix} \mathbf{f} & 0 \\ 0 & \mathbf{f} \end{pmatrix} \quad , \quad \tilde{\boldsymbol{F}} = \begin{pmatrix} \tilde{\mathbf{f}} & 0 \\ 0 & \tilde{\mathbf{f}} \end{pmatrix} \tag{3.36}$$

These folding operators obey the property $\boldsymbol{F}\tilde{\boldsymbol{F}} = I_{4\times4}$ which is induced from $\mathbf{f}\tilde{\mathbf{f}} = I_{2\times2}$ of eq. (3.6). Explicitly, the two generators $\mathcal{L}_1$ and $\mathcal{L}_2$ of the group $H_{stg}^{G_2}$ are given by

$$\mathcal{L}_1 = \begin{pmatrix} I_{2\times2} & \mathcal{R} \\ 0_{2\times2} & -I_{2\times2} \end{pmatrix} \quad , \quad \mathcal{L}_2 = \begin{pmatrix} -I_{2\times2} & 0_{2\times2} \\ \mathcal{R} & I_{2\times2} \end{pmatrix} \tag{3.37}$$

with $\mathcal{R}$ a 2×2 matrix induced by the folding mapping. This matrix $\mathcal{R}$ is related to the previous 4×4 matrix R, of the SO(8) gauge theory as in eq. (3.27), by the following transformation as

$$\mathcal{R} = \mathbf{f}.R.\tilde{\mathbf{f}} \tag{3.38}$$

The explicit expression of the $\mathcal{R}$ matrix is given by

$$\mathcal{R} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} \quad , \quad \mathcal{R} = 2I_{2\times2} - K_{G_2} \tag{3.39}$$

The relationship between $\mathcal{R}$ and R (3.38) results from $K_{G_2} = \mathbf{f}.K_{SO_8}.\tilde{\mathbf{f}}$ by proceeding as follows: First splitting the Cartan matrices K_{SO_8} and K_{G_2} by exhibiting the identity matrices $I_{2\times2}$ and $I_{4\times4}$ like

$$K_{G_2} = 2I_{2\times2} - \mathcal{R} \quad , \quad K_{SO_8} = 2I_{4\times4} - R \tag{3.40}$$

Then calculating the folding of the SO(8) Cartan matrix $\mathbf{f}.K_{SO_8}.\tilde{\mathbf{f}}$ by substituting K_{SO_8} in terms of the R-matrix; this gives

$$K_{G_2} = 2\mathbf{f}\,\tilde{\mathbf{f}} - \mathbf{f}.R.\tilde{\mathbf{f}} \tag{3.41}$$

Moreover, by using the property $\mathbf{f}\tilde{\mathbf{f}} = I_{2\times2}$, the above relation reduces to $K_{G_2} = 2I_{2\times2} - \mathbf{f}.R.\tilde{\mathbf{f}}$. By equating with $K_{G_2} = 2I_{2\times2} - \mathcal{R}$, we obtain $\mathbf{f}.R.\tilde{\mathbf{f}} = \mathcal{R}$.

Having the explicit expressions of the $\mathcal{L}_1$ and $\mathcal{L}_2$ generators of $H_{stg}^{G_2}$, we can now build the mutation elements $\boldsymbol{N}_m$ of this set by proceeding in similar manner as for $H_{stg}^{SO_8}$. We find the following properties:

• *composite $\mathcal{L}_1$ and $\mathcal{L}_2$ are reflections*
The non-commuting generators $\mathcal{L}_1$ and $\mathcal{L}_2$ are also reflections; they satisfy

$$\mathcal{L}_1^2 = I_{4\times 4} \qquad , \qquad \mathcal{L}_2^2 = I_{4\times 4} \tag{3.42}$$

showing in turns that the $\boldsymbol{N}_m$ elements $H_{stg}^{G_2}$ have the form

$$\boldsymbol{N}_{2k} = (\mathcal{L}_2\mathcal{L}_1)^k \qquad , \qquad \boldsymbol{N}_{2k+1} = \mathcal{L}_1 \boldsymbol{N}_{2k} \tag{3.43}$$

• *the subgroup $H_{stg}^{G_2}$*
The generators $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy as well the properties

$$\begin{aligned} \boldsymbol{N}_{12} &= (\mathcal{L}_2\mathcal{L}_1)^6 = +I_{4\times 4} \\ \boldsymbol{N}_6 &= (\mathcal{L}_2\mathcal{L}_1)^3 = -I_{4\times 4} \end{aligned} \tag{3.44}$$

indicating that $H_{stg}^{G_2}$ is a finite discrete group with cardinality *12* as follows

$$H_{stg}^{G_2} = \{\pm I_{id}, \pm N_1, \pm N_2, \pm N_3, \pm N_4, \pm N_5\} \tag{3.45}$$

it is isomorphic to a 4×4 matrix representation of Dih_{12}.
• *the strong chamber $\mathfrak{Q}_{stg}^{G_2}$*
The BPS/anti-BPS states of the strong chamber $\mathfrak{Q}_{stg}^{G_2}$ can be read from the rows of the $\boldsymbol{N}_n$ matrices of $H_{stg}^{G_2}$. The identity $\boldsymbol{N}_0 = I_{8\times 8}$ gives precisely the EM charges $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ of two elementary monopoles and the EM charges $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$ of the two elementary dyons.

By performing the first mutation $\boldsymbol{N}_1 = \mathcal{L}_1$ on the primitive quiver $Q_0^{G_2}$ of the G2 theory, that is $\boldsymbol{N}_1$: $Q_0^{G_2} \to Q_1^{G_2}$, the resulting quiver $Q_1^{G_2}$ has four new BPS states with EM charges $\boldsymbol{\gamma}_i^{(1)}$ directly read from the four rows of the matrix representation of $\boldsymbol{N}_1$ namely

$$\boldsymbol{N}_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.46}$$

The first row of the above mutation matrix namely $(1, 0, 0, 1)$ means that a BPS state with EM charge $\boldsymbol{\gamma}_1^{(1)} = \boldsymbol{\beta}_1 + \boldsymbol{\delta}_2$ has been generated by the first N_1 mutation of $Q_0^{G_2}$; the second row $(0, 1, 3, 0)$ gives another new BPS state with EM charge $\boldsymbol{\gamma}_2^{(1)} = \boldsymbol{\beta}_2 + 3\boldsymbol{\delta}_1$; the third and fourth give the EM charges of the anti-dyons. The BPS/anti-BPS states generated by the first mutation on $Q_0^{G_2}$ are as follows

$$\begin{array}{ll} \boldsymbol{\beta}_1 + \boldsymbol{\delta}_2 & -\boldsymbol{\delta}_1 \\ \boldsymbol{\beta}_2 + 3\boldsymbol{\delta}_1 & -\boldsymbol{\delta}_2 \end{array} \tag{3.47}$$

By performing the eleven $\boldsymbol{N}_n$ mutations of $H_{stg}^{G_2}$ of (3.45), we obtain the list of the 24 BPS/anti-BPS states of $\mathfrak{Q}_{stg}^{G_2}$; it reads as follows

$$\begin{array}{lll} \pm\beta_1 & \pm(\beta_1 + \delta_2) & \pm(3\beta_1 + 2\delta_2) \\ \pm\beta_2 & \pm(\beta_2 + \delta_1) & \pm(2\beta_2 + 3\delta_1) \\ \pm\delta_1 & \pm(2\beta_1 + \delta_2) & \pm(3\beta_1 + \delta_2) \\ \pm\delta_2 & \pm(\beta_2 + 2\delta_1) & \pm(\beta_2 + 3\delta_1) \end{array} \tag{3.48}$$

4. $\mathcal{N} = 2$ pure F_4 theory

In this section, we construct the BPS states of the strong chamber $\mathfrak{Q}_{stg}^{F_4}$ of the $\mathcal{N} = 2$ pure F_4 theory by proceeding in same manner as for the G_2 theory of previous section. First, we build the folding $(\mathbf{f}, \tilde{\mathbf{f}})$ operators mapping DD_{E_6} down to the Dynkin diagram DD_{F_4}. Then, we extend this folding approach to linking the two primitive quivers $Q_0^{E_6}$ and $Q_0^{F_4}$ by working out the explicit expression of the extended $(\mathcal{F}, \tilde{\mathcal{F}})$ folding operators. After that we build the BPS states of the strong chamber of $\mathfrak{Q}_{stg}^{F_4}$ of the supersymmetric pure F_4 gauge model.

4.1. DD_{F_4} *as folded* DD_{E_6}

We begin by recalling that the *52* dimensional Lie algebra of the F_4 gauge symmetry has *48* roots; the *24* positive roots are generated by four simple $\vec{\alpha}_1$, $\vec{\alpha}_2$, $\vec{\alpha}_3$, $\vec{\alpha}_4$ with two possible lengths generally taken like $\|\vec{\alpha}_1\|^2 = \|\vec{\alpha}_2\|^2 = 2$ and $\|\vec{\alpha}_3\|^2 = \|\vec{\alpha}_4\|^2 = 1$ as well as $\vec{\alpha}_i.\vec{\alpha}_{i+1} = -1$. A weaker normalisation of these simple roots corresponds to thinking of their lengths in terms of ratios like

$$\frac{\|\vec{\alpha}_1\|^2}{\|\vec{\alpha}_2\|^2} = 1 \quad , \quad \frac{\|\vec{\alpha}_3\|^2}{\|\vec{\alpha}_4\|^2} = 1 \quad , \quad \frac{\|\vec{\alpha}_1\|^2}{\|\vec{\alpha}_3\|^2} = 2 \tag{4.1}$$

These four simple roots may be expressed in terms of the six $\vec{a}_1, ..., \vec{a}_6$ simple roots of the E_6 Lie algebra with one length $\|\vec{a}_K\|^2 = 2$ as follows

$$\vec{\alpha}_i = f_i^K \, \vec{a}_K \tag{4.2}$$

where f_i^K is a folding 4×6 matrix operator. By solving the normalisation constraints of the four simple $\vec{\alpha}_i$'s like

$$\begin{array}{llll} \vec{\alpha}_1 &= \frac{1}{q}\,(\vec{a}_1 + \vec{a}_5) \quad , & \vec{\alpha}_3 &= \frac{1}{q}\vec{a}_3 \\ \vec{\alpha}_2 &= \frac{1}{q}\,(\vec{a}_2 + \vec{a}_4) \quad , & \vec{\alpha}_4 &= \frac{1}{q}\vec{a}_6 \end{array} \tag{4.3}$$

where q is a non-zero real number, it results $\|\vec{\alpha}_1\|^2 = \|\vec{\alpha}_2\|^2 = \frac{4}{q^2}$ and $\|\vec{\alpha}_3\|^2 = \|\vec{\alpha}_4\|^2 = \frac{2}{q^2}$; if choosing $q = \sqrt{2}$ we rediscover the normalisation $\|\vec{\alpha}_1\|^2 = \|\vec{\alpha}_2\|^2 = 2$ and $\|\vec{\alpha}_3\|^2 = \|\vec{\alpha}_4\|^2 = 1$. Therefore, the folding matrix operator $\mathbf{f}$ takes the generic form

$$\mathbf{f} = \frac{1}{q}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.4}$$

depending on a scaling parameter q. From this matrix operator, we can determine the explicit expression of its companion $\tilde{\mathbf{f}}$ given by $\mathbf{f}^T \left(\mathbf{f}\mathbf{f}^T\right)^{-1}$ and reading as follows

$$\tilde{\mathbf{f}} = \frac{q}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{4.5}$$

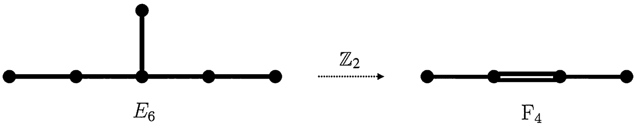

Fig. 4. Dynkin diagram of F_4 by folding the $\mathbb{Z}_2$ symmetric nodes in E_6 diagram.

The $\mathbf{f}$ and $\tilde{\mathbf{f}}$ obey the property $\mathbf{f}\tilde{\mathbf{f}} = I_{4\times 4}$. Following the same approach used in section 3 by applying the folding operations to the Cartan matrix K_{E_6} of the exceptional E_6 Lie algebra, we obtain the Cartan matrix K_{F_4} of the exceptional K_{F_4} Lie algebra

$$K_{F_4} = \mathbf{f}.K_{E_6}.\tilde{\mathbf{f}} \tag{4.6}$$

This relation is somehow an illustration of the folding depicted in Fig. 4 and it can be explicitly checked by using (4.4)–(4.5) and

$$K_{E_6} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \tag{4.7}$$

and

$$K_{F_4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{4.8}$$

4.2. BPS states of $\mathfrak{Q}_{stg}^{F_4}$

To obtain the BPS states of the strong chamber $\mathfrak{Q}_{stg}^{F_4}$ of the $\mathcal{N} = 2$ supersymmetric pure F_4 theory, we proceed as in subsection § 3.2. We use the two following data:

- the folding operators $\mathbf{f}$ and $\tilde{\mathbf{f}}$, given by eqs. (4.4)–(4.5), relating the Dynkin diagrams of F_4 and E_6, to link the two primitive quivers $Q_0^{F_4}$ and $Q_0^{E_6}$ of the supersymmetric pure gauge models with F_4 and E_6 gauge symmetries; and
- the BPS states of the strong chamber $\mathfrak{Q}_{stg}^{E_6}$ of the supersymmetric pure E_6 theory. There, the BPS states

$$\boldsymbol{\gamma}_{m,n} = \sum m_i \boldsymbol{b}_i + n_i \boldsymbol{c}_i \tag{4.9}$$

  of $\mathfrak{Q}_{stg}^{E_6}$ are obtained by mutating the primitive quiver $Q_0^{E_6}$; that is by performing the transformations $\boldsymbol{M}_n : Q_0^{E_6} \to Q_n^{E_6}$.

4.2.1. Strong chamber $\mathfrak{Q}_{stg}^{E_6}$

The content of the chamber $\mathfrak{Q}_{stg}^{E_6}$ is obtained by mutating the primitive quiver $Q_0^{E_6}$. Generally speaking the set of the quiver mutations $\{\mathcal{M}_n\} \equiv \boldsymbol{G}_{stg}^{E_6}$ of the strong chamber $\mathfrak{Q}_{stg}^{E_6}$ is given by the Coxeter group of $\boldsymbol{G}_{stg}^{E_6}$ generated by 12 fundamental reflections $r_1, ..., r_{12}$ obeying

$$\left(r_i r_j\right)^{m_{ij}} = I_{12\times 12} \tag{4.10}$$

where the positive m_{ij} integers are given by the Coxeter matrix [6]. In practice, the content of $\mathfrak{Q}_{stg}^{E_6}$ can be derived by restricting to a subgroup $H_{stg}^{E_6} \equiv \{\boldsymbol{M}_n\}$ of the Coxeter $\boldsymbol{G}_{stg}^{E_6}$. This subgroup $H_{stg}^{E_6}$ is generated by two particular composite reflection operators L_1 and L_2 given by

$$L_1 = r_6 r_5 r_4 r_3 r_2 r_1 \qquad , \qquad L_2 = r_{12} r_{11} r_{10} r_9 r_8 r_7 \tag{4.11}$$

The matrix realisation of these two non-commuting generators is as follows

$$L_1 = \begin{pmatrix} I_{6\times 6} & R \\ 0_{6\times 6} & -I_{6\times 6} \end{pmatrix} \qquad , \qquad L_2 = \begin{pmatrix} -I_{6\times 6} & 0_{6\times 6} \\ R & I_{6\times 6} \end{pmatrix} \tag{4.12}$$

with 6×6 matrix R given in term of the Cartan matrix of the exceptional Lie algebra by $2I_{6\times 6} - K_{E_6}$ and reads explicitly like

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.13}$$

The composite reflections L_1 and L_2 of eq. (4.12) obey obviously the property $L_1^2 = L_2^2 = I_{6\times 6}$; they generate the subgroup $H_{stg}^{E_6}$ of the Coxeter group $\boldsymbol{G}_{stg}^{E_6}$. As for the group $H_{stg}^{SO_8}$ of the SO_8 theory, the elements of the set $H_{stg}^{E_6}$ have quite similar structure as (3.29); $H_{stg}^{E_6}$ has 24 mutations matrices M_n of the form

$$M_{2k} = (L_2 L_1)^k \qquad , \qquad M_{2k+1} = L_1 M_{2k} \tag{4.14}$$

The finite value of the cardinality of $H_{stg}^{E_6}$ follows from the property

$$M_{24} = (L_2 L_1)^{12} = I_{12\times 12} \tag{4.15}$$

leading to

$$H_{stg}^{E_6} = \{M_{2k}, M_{2k+1}\}_{0\leq k\leq 11} \tag{4.16}$$

By using (4.12), one can write down the explicit expressions of the M_{2k} and M_{2k+1} matrix mutations. These explicit expressions allow to write down the $2\,(78-6) = 144$ BPS/anti-BPS states of the strong chamber $\mathfrak{Q}_{stg}^{E_6}$ of the $\mathcal{N} = 2$ supersymmetric pure E_6 theory; the full list can be found in [6].

4.2.2. $\mathfrak{Q}_{stg}^{F_4}$ from the folding of $\mathfrak{Q}_{stg}^{E_6}$

By using our quiver folding method, the primitive $Q_0^{E_6}$ gets mapped to the primitive $Q_0^{F_4}$; and the mutation set $H_{stg}^{E_6}$ is mapped to $H_{stg}^{F_4}$. The last group is generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ related to above L_1 and L_2 as

$$\mathcal{L}_i = \mathcal{F}.L_i.\tilde{\mathcal{F}} \tag{4.17}$$

with

$$\mathcal{F} = \begin{pmatrix} \mathbf{f} & 0 \\ 0 & \mathbf{f} \end{pmatrix} \qquad , \qquad \tilde{\mathcal{F}} = \begin{pmatrix} \tilde{\mathbf{f}} & 0 \\ 0 & \tilde{\mathbf{f}} \end{pmatrix} \tag{4.18}$$

The non-commuting matrix representations $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by

$$\mathcal{L}_1 = \begin{pmatrix} I_{4\times4} & \mathcal{R} \\ 0_{6\times6} & -I_{4\times4} \end{pmatrix} \quad , \quad \mathcal{L}_2 = \begin{pmatrix} -I_{4\times4} & 0_{4\times4} \\ \mathcal{R} & I_{4\times4} \end{pmatrix} \tag{4.19}$$

with 4×4 matrix $\mathcal{R} = 2I_{4\times4} - K_{F_4}$. This matrix reads explicitly as follows

$$\mathcal{R} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.20}$$

Like for $H_{stg}^{E_6}$, the generators of $H_{stg}^{F_4}$ obey as well the properties

$$(\mathcal{L}_1)^2 = (\mathcal{L}_2)^2 = I_{4\times4} \quad , \quad (\mathcal{L}_2\mathcal{L}_1)^{12} = I_{4\times4} \tag{4.21}$$

teaching us that $H_{stg}^{F_4}$ is also a representation of Dih_{24}. The 24 mutation matrices $\boldsymbol{N}_n$ of this set are given by

$$\begin{aligned} \pm\boldsymbol{N}_{2k} &= \pm(\mathcal{L}_2\mathcal{L}_1)^k \\ \pm\boldsymbol{N}_{2k+1} &= \pm\mathcal{L}_1\boldsymbol{N}_{2k} \end{aligned} \tag{4.22}$$

Applying quiver mutations, we can work out explicitly the full list of BPS states of the $\mathfrak{Q}_{stg}^{F_4}$ strong chamber of the supersymmetric pure F_4 gauge theory. In addition to the four elementary monopoles $\boldsymbol{\beta}_i$ and the four elementary dyons $\boldsymbol{\delta}_i$ making the primitive quiver $Q_0^{F_4}$, the mutations

$$\boldsymbol{N}_n : Q_0^{F_4} \to Q_n^{F_4} \tag{4.23}$$

allow to generate the other BPS/anti-BPS states. For example, the first mutation $\boldsymbol{N}_1 : Q_0^{F_4} \to Q_1^{F_4}$ generated by the mutation matrix

$$\boldsymbol{N}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.24}$$

leads to the following BPS states

$$\begin{array}{ll} -\boldsymbol{\delta}_1 & \boldsymbol{\beta}_1 + \boldsymbol{\delta}_2 \\ -\boldsymbol{\delta}_2 & \boldsymbol{\beta}_2 + \boldsymbol{\delta}_1 + 2\boldsymbol{\delta}_3 \\ -\boldsymbol{\delta}_3 & \boldsymbol{\beta}_3 + \boldsymbol{\delta}_2 + \boldsymbol{\delta}_4 \\ -\boldsymbol{\delta}_4 & \boldsymbol{\beta}_4 + \boldsymbol{\delta}_3 \end{array} \tag{4.25}$$

The full list of BPS/anti-BPS states of the $\mathfrak{Q}_{stg}^{F_4}$ chamber is directly read form the matrix representation of the $\boldsymbol{N}_n$ elements (4.22) of $H_{stg}^{F_4}$; it contains $2\times24 = 48$ states. The BPS states with integral positive electric–magnetic charges are as listed here below; anti-BPS sates have opposite charges.

$$\begin{array}{lll} \beta_1 & \beta_1+\delta_2 & 2\beta_3+\delta_2+2\delta_4 \\ \beta_2 & \beta_2+\delta_1+2\delta_3 & 2\beta_2+2\beta_4+\delta_1+4\delta_3 \\ \beta_3 & \beta_3+\delta_2+\delta_4 & \beta_1+2\beta_3+2\delta_2+\delta_4 \\ \beta_4 & \beta_4+\delta_3 & \beta_2+\delta_1+\delta_3 \\ \delta_1 & \beta_2+2\delta_3 & \beta_2+2\beta_4+\delta_1+2\delta_3 \\ \delta_2 & \beta_1+2\beta_3+2\delta_2+2\delta_4 & \beta_1+4\beta_3+3\delta_2+2\delta_4 \\ \delta_3 & \beta_2+\delta_1+\beta_4+2\delta_3 & 2\beta_2+\beta_4+\delta_1+3\delta_3 \\ \delta_4 & \beta_3+\delta_2 & \beta_1+\beta_3+\delta_2+\delta_4 \end{array} \tag{4.26}$$

and

$$\begin{array}{lll} \beta_1+2\beta_3+2\delta_2 & \beta_1+2\beta_3+\delta_2+2\delta_4 & 2\beta_3+\delta_2 \\ 3\beta_2+2\beta_4+2\delta_1+4\delta_3 & 3\beta_2+2\beta_4+\delta_1+4\delta_3 & 2\beta_2+2\beta_4+\delta_1+2\delta_3 \\ \beta_1+3\beta_3+2\delta_2+2\delta_4 & \beta_1+3\beta_3+2\delta_2+\delta_4 & \beta_1+2\beta_3+\delta_2+\delta_4 \\ \beta_2+\beta_4+2\delta_3 & \beta_2+\beta_4+\delta_1+\delta_3 & \beta_2+\delta_3 \\ 2\beta_2+\delta_1+2\delta_3 & \beta_2+2\beta_4+2\delta_3 & \beta_2+\delta_1 \\ 2\beta_1+4\beta_3+3\delta_2+2\delta_4 & \beta_1+4\beta_3+2\delta_2+2\delta_4 & \beta_1+2\beta_3+\delta_2 \\ 2\beta_2+2\beta_4+\delta_1+3\delta_3 & 2\beta_2+\beta_4+\delta_1+2\delta_3 & \beta_2+\beta_4+\delta_3 \\ 2\beta_3+\delta_2+\delta_4 & \beta_1+\beta_3+\delta_2 & \beta_3+\delta_4 \end{array} \tag{4.27}$$

5. Conclusion and comments

In this paper, we have approached the construction of BPS states of 4d $\mathcal{N}=2$ supersymmetric pure gauge theories with gauge invariance $\mathcal{G}$ of non-simply laced BCFG type. To that purpose, we have proceeded in two main steps: First, we have remarked that BPS quivers $Q_0^{\mathcal{G}}$ of supersymmetric pure gauge theories are two types: (i) Q_0^{ADE} quivers of ADE-type; and (ii) Q_0^{BCFG} quivers of BCFG-type. This classification has been borrowed from the classification of the Dynkin diagrams of finite dimensional Lie algebras; this is because BPS quivers in 4d $\mathcal{N}=2$ supersymmetric pure gauge theories might be imagined as a duplication of Dynkin diagram of the Lie algebra of the underlying gauge symmetry. In the case of Dynkin diagrams, the basic node is given by K_{su_2}; and in the case of primitive quivers $Q_0^{\mathcal{G}}$ the basic object is $\mathcal{A}_0^{SU_2}$ as shown on following table; generic $Q_0^{\mathcal{G}}$'s correspond to intersecting of several $\mathcal{A}_0^{SU_2}$'s.

Gauge symmetry	Dynkin diagram	Matrix $\mathcal{A}_0^G$ of primitive Q_0^G
$SU(2)$	$K_{su_2}=2$	$\mathcal{A}_0^{SU_2}=\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}$
ADE	$K_{ADE}=K$	$\mathcal{A}_0^{ADE}=K\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$
$BCFG$	$K_{BCFG}=\tilde{K}$	$\mathcal{A}_0^{BCFG}=\begin{pmatrix}0 & -\tilde{K}^T\\ \tilde{K} & \tilde{K}^T-\tilde{K}\end{pmatrix}$

(5.1)

The structure of the various primitive quivers $Q_0^{\mathcal{G}}$ is explicitly exhibited on the lists given by Figs. 1 and 2; the quivers type Q_0^{BCFG} have a diagonal link in addition to the vertical and horizontal links appearing in the Q_0^{ADE} graphs; the diagonal link is therefore a special property of Q_0^{BCFG}.

In the second step, we have focused on $\mathcal{N}=2$ supersymmetric pure gauge theories with exceptional G_2 and F_4 gauge invariance. First, we have constructed the folding operators **f** and $\tilde{\mathbf{f}}$ mapping the simply laced Dynkin diagram of the Lie algebra of SO(8) down to the Dynkin diagram of the Lie algebra of the gauge symmetry G_2.

$$\mathbf{f}: DD_{SO_8} \to D_{G_2} \quad , \quad K_{G_2} = \mathbf{f} K_{SO_8} \tilde{\mathbf{f}} \tag{5.2}$$

with $\mathbf{f}\tilde{\mathbf{f}} = I_{id}$. Then, we have extended this construction to the BPS quivers; the desired quiver $Q_0^{G_2}$ is obtained by folding $Q_0^{SO_8}$ by help of the folding matrix operators **f** and $\tilde{\mathbf{f}}$. Knowing the BPS states of the strong chamber of $\mathfrak{Q}_{stg}^{SO_8}$, we have derived the BPS states of the $\mathfrak{Q}_{stg}^{G_2}$ chamber. We have also shown that these BPS states are completely controlled by a non-abelian group $H_{stg}^{G_2}$ isomorphic to 4×4 matrix representation of the dihedral group Dih_{12}.

After that, we have used the relationship between the Dynkin diagrams E_6 and F_4 to build the BPS states of the strong chambers of the supersymmetric pure F_4 gauge model. We have derived the explicit BPS/anti-BPS states content of the strong chamber of $\mathfrak{Q}_{stg}^{F_4}$. Here also this content is completely controlled by a non-abelian group $H_{stg}^{F_4}$ isomorphic to 8×8 matrix representation of the dihedral group Dih_{24}.

In the end of this study, we would like to notice that the lists of BPS quivers given by Figs. 1 and 2 is very remarkable; its similarity with Dynkin diagrams is very suggestive; it would be interesting to deepen this aspect by shedding more light on this correspondence and its generalisation to affine Kac–Moody type diagrams.

Appendix A. Quiver superpotentials

In this appendix, we give the chiral superfields and the superpotentials associated with the primitive BPS quivers Q_0^G of the $\mathcal{N}=2$ pure supersymmetric gauge models considered in this study; they concern those gauge group symmetries G given by the four following ones: SO(8), G_2, E_6 and F_4.

A.1. $\mathcal{N}=2$ supersymmetric $SO(8)$ and G_2 models

We first give the chiral superpotential $W_0^{SO_8}(\Phi)$ of the $\mathcal{N}=2$ supersymmetric pure SO(8) theory; then we turn to the derivation of the $W_0^{G_2}(\Phi)$ for the G_2 model obtained by folding method.

• *$SO(8)$ gauge model*

The primitive quiver $Q_0^{SO_8}$ of the pure SO(8) theory has six 4-cycles and fourteen superfields Φ as depicted on Fig. 5. By using the prescription of ref. [2] for building superpotentials, the $W_0^{SO_8}$ is a quartic chiral function given by

$$\begin{aligned} W_0^{SO_8} = & \left(A_1\phi_1^* A_2\phi_1 - B_1\phi_1^* B_2\phi_1\right) + \\ & \left(A_2\phi_2^* A_3\phi_2 - B_2\phi_2^* B_3\phi_2\right) + \\ & \left(A_2\phi_3^* A_4\phi_3 - B_2\phi_3^* B_4\phi_3\right) \end{aligned} \tag{A.1}$$

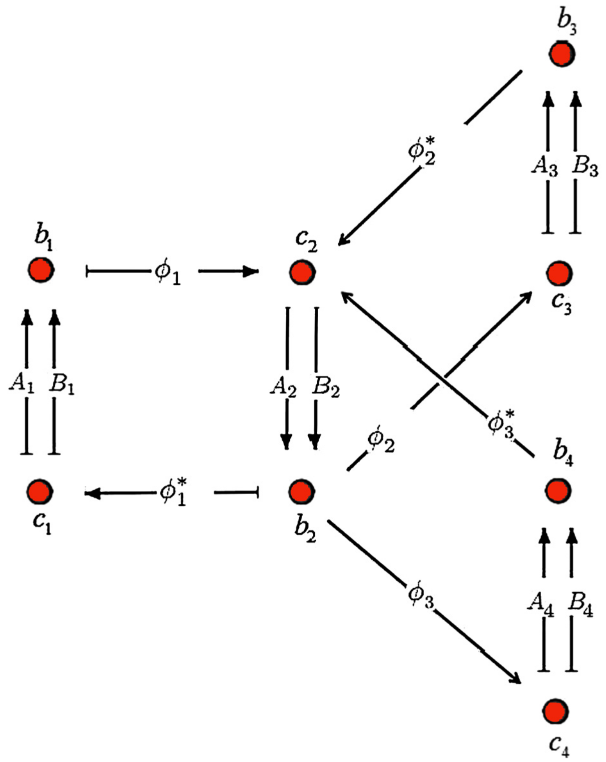

Fig. 5. Chiral superfields and cycles of primitive BPS quiver $Q_0^{SO_8}$.

The F-term equations, following from $W_0^{SO_8}$, are as follows

$$\begin{array}{llll} \phi_1^* A_2 \phi_1 = 0 & , & B_2 \phi_2^* \phi_2 & = 0 \\ A_2 \phi_2^* \phi_2 = 0 & , & B_2 \phi_3^* \phi_3 & = 0 \\ A_2 \phi_3^* \phi_3 = 0 & , & A_1 \phi_1^* \phi_1 + A_3 \phi_2^* \phi_2 + A_4 \phi_3^* \phi_3 & = 0 \\ \phi_1^* B_2 \phi_1 = 0 & , & B_1 \phi_1^* \phi_1 + B_3 \phi_2^* \phi_2 + B_4 \phi_3^* \phi_3 & = 0 \end{array} \tag{A.2}$$

and

$$\begin{array}{lll} (A_1 A_2 - B_1 B_2)\, \phi_1 = 0 & , & (A_1 A_2 - B_1 B_2)\, \phi_1^* = 0 \\ (A_2 A_3 - B_2 B_3)\, \phi_2 = 0 & , & (A_2 A_3 - B_2 B_3)\, \phi_2^* = 0 \\ (A_2 A_4 - B_2 B_4)\, \phi_3 = 0 & , & (A_2 A_4 - B_2 B_4)\, \phi_3^* = 0 \end{array} \tag{A.3}$$

The solutions of these relations define the moduli space $\mathcal{M}_\gamma^{SO_8}$ of the ground state of the supersymmetric quantum mechanics. Recall that $\mathcal{M}_\gamma$ is the space of solutions to the F-term equations subject to a stability condition modulo the action of the complexified gauge group $\prod_i Gl(n_i, \mathbb{C})$; for details see [1,2].

• G_2 *gauge model*

The primitive quiver $Q_0^{G_2}$ of the G_2 theory is given by Fig. 6; it involves 10 chiral superfields and has six cycles: two 4-cycles and four 3-cycles. By using the convention notation of [14], the superpotential of the G_2 theory reads as follows

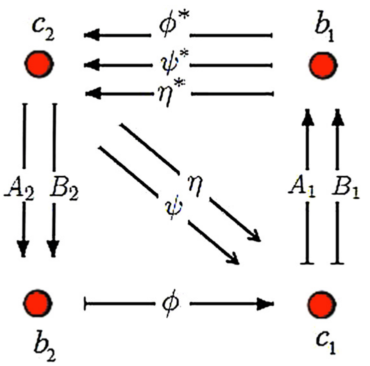

Fig. 6. Chiral superfields and cycles of primitive BPS quiver $Q_0^{G_2}$.

$$\begin{aligned} W_0^{G_2} =\ & A_2\phi^* B_1\phi - A_1\phi B_2\psi^* \\ & -A_1\psi\eta^* - A_1\eta\phi^* \\ & +B_1\psi\psi^* + B_1\eta\eta^* \end{aligned} \tag{A.4}$$

The F-term equations are given by

$$\begin{array}{llll} \phi^* B_1\phi = 0 & , & \phi B_2\psi^* + \psi\eta^* + \eta\phi^* = 0 \\ A_1\phi\psi^* = 0 & , & A_2\phi^*\phi + \psi\psi^* + \eta\eta^* = 0 \end{array} \tag{A.5}$$

and

$$\begin{array}{llll} B_1\eta - A_1\psi = 0 & , & A_2 B_1\phi - A_1\eta = 0 \\ B_1\psi - A_1\phi B_2 = 0 & , & A_2\phi^* B_1 - A_1 B_2\psi^* = 0 \end{array} \tag{A.6}$$

and the diagonal superfields lead to

$$B_1\eta^* - A_1\phi^* = 0 \quad , \quad B_1\psi^* - A_1\eta^* = 0 \tag{A.7}$$

Like for $\mathcal{M}_\gamma^{SO_8}$, these constraints define the moduli space $\mathcal{M}_\gamma^{G_2}$ of the ground state of the supersymmetric quantum mechanics.

A.2. $\mathcal{N} = 2$ supersymmetric E_6 and F_4 models

First, we consider superpotential $W_0^{E_6}$ of the pure E_6 gauge theory; and turn after to $W_0^{F_4}$ of the supersymmetric pure F_4 model obtained by the folding approach.

• E_6 *gauge model*

The primitive quiver $Q_0^{E_6}$ of this theory involves 22 chiral superfields and ten 4-cycles as shown on Fig. 7. The explicit expression of the superpotential $W_0^{E_6}$ reads as follows

$$W_0^{E_6} = \left(A_3\phi_5^* A_6\phi_5 - B_3\phi_5^* B_6\phi_5\right) + \sum_{i=1}^4 \left(A_i\phi_i^* A_{i+1}\phi_i - B_i\phi_i^* B_{i+1}\phi_i\right)$$

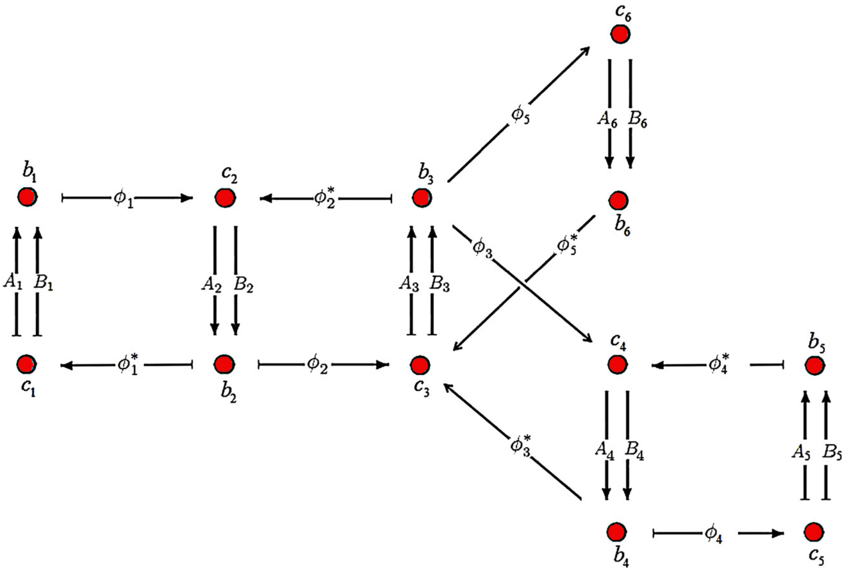

Fig. 7. Chiral superfields and cycles of primitive BPS quiver $Q_0^{E_6}$.

The corresponding F-term equations following from above $W_0^{E_6}$ are given by

$$\begin{array}{llll} \phi_1^* A_2 \phi_1 & = 0 \quad , & \phi_1^* B_2 \phi_1 & = 0 \\ A_4 \phi_4^* \phi_4 & = 0 \quad , & B_4 \phi_4^* \phi_4 & = 0 \\ A_3 \phi_5^* \phi_5 & = 0 \quad , & B_3 \phi_5^* \phi_5 & = 0 \\ A_1 \phi_1^* \phi_1 + \phi_2^* A_3 \phi_2 & = 0 \quad , & B_1 \phi_1^* \phi_1 + \phi_2^* B_3 \phi_2 & = 0 \\ A_2 \phi_2^* \phi_2 + \phi_3^* A_4 \phi_3 & = 0 \quad , & B_2 \phi_2^* \phi_2 + \phi_3^* B_4 \phi_3 & = 0 \\ A_3 \phi_3^* \phi_3 + \phi_4^* A_5 \phi_4 & = 0 \quad , & B_3 \phi_3^* \phi_3 + \phi_4^* B_5 \phi_4 & = 0 \end{array} \tag{A.8}$$

and

$$\begin{array}{llll} (A_1 A_2 - B_1 B_2)\,\phi_1 & = 0 \quad , & (A_1 A_2 - B_1 B_2)\,\phi_1^* & = 0 \\ (A_2 A_3 - B_2 B_3)\,\phi_2 & = 0 \quad , & (A_2 A_3 - B_2 B_3)\,\phi_2^* & = 0 \\ (A_3 A_4 - B_3 B_4)\,\phi_3 & = 0 \quad , & (A_3 A_4 - B_3 B_4)\,\phi_3^* & = 0 \\ (A_4 A_5 - B_4 B_5)\,\phi_4 & = 0 \quad , & (A_4 A_5 - B_4 B_5)\,\phi_4^* & = 0 \\ (A_3 A_6 - B_3 B_6)\,\phi_5 & = 0 \quad , & (A_3 A_6 - B_3 B_6)\,\phi_5^* & = 0 \end{array}$$

- *F_4 gauge model*

The primitive quiver $Q_0^{F_4}$ of this theory is given by Fig. 8; it has eight cycles containing six 4-cycles and two 3-cycles as shown in Fig. 8. By extending the construction of [2,14], the superpotential of the F_4 theory is given by

$$\begin{aligned} W_0^{F_4} = \; & \left(A_1 \phi_1^* B_2 \phi_1 - B_1 \phi_1^* A_2 \phi_1\right) + \\ & \left(A_2 \phi_2^* B_3 \phi_2 + A_3 \phi_2 B_2 \eta^* + A_3 \eta \phi_2^* + B_3 \eta \eta^*\right) \\ & \left(A_3 \phi_3^* B_4 \phi_3 - B_3 \phi_3^* A_4 \phi_3\right) \end{aligned} \tag{A.9}$$

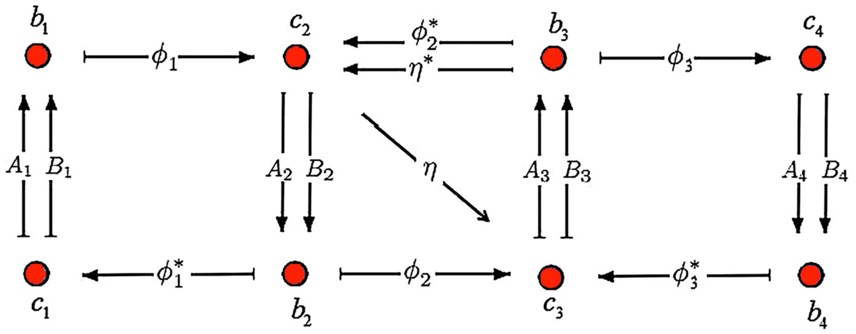

Fig. 8. Chiral superfields and cycles of primitive BPS quiver $Q_0^{F_4}$.

with F-term equations as

$$\begin{aligned}
&\phi_1^* B_2 \phi_1 = 0 \quad , \quad && \phi_1^* A_2 \phi_1 = 0 \\
&B_3 \phi_3^* \phi_3 = 0 \quad , \quad && A_3 \phi_3^* \phi_3 = 0 \\
&\phi_2^* B_3 \phi_2 - B_1 \phi_1^* \phi_1 = 0 \quad , \quad && A_1 \phi_1^* \phi_1 + A_3 \phi_2 \eta^* = 0 \\
&\phi_2 B_2 \eta^* + \eta \phi_2^* + \phi_3^* B_4 \phi_3 = 0 \quad , \quad && A_2 \phi_2^* \phi_2 + \eta \eta^* - \phi_3^* A_4 \phi_3 = 0
\end{aligned} \tag{A.10}$$

and

$$\begin{aligned}
&(A_1 B_2 - B_1 A_2)\, \phi_1^* = 0 \quad , \quad && (A_1 B_2 - B_1 A_2)\, \phi_1 = 0 \\
&A_2 \phi_2^* B_3 + A_3 B_2 \eta^* = 0 \quad , \quad && A_2 B_3 \phi_2 + A_3 \eta = 0 \\
&A_3 \phi_2^* + B_3 \eta^* = 0 \quad , \quad && A_3 \phi_2 B_2 + B_3 \eta = 0 \\
&(A_3 B_4 - B_3 A_4)\, \phi_3 = 0 \quad , \quad && (A_3 B_4 - B_3 A_4)\, \phi_3^* = 0
\end{aligned} \tag{A.11}$$

Appendix B. Mutations $H_{stg}^{\mathcal{G}}$ in chambers $\mathfrak{Q}_{stg}^{\mathcal{G}}$

In building the BPS states of the $\mathfrak{Q}_{stg}^{\mathcal{G}}$ strong chambers, we have used a subgroup $H_{stg}^{\mathcal{G}}$ of the Coxeter $\mathcal{G}_{stg}^{\mathcal{G}}$. The set $H_{stg}^{\mathcal{G}}$ is generated by two non-commuting reflections denoted in the core of paper by L_1 and L_2 and given by products type $\prod r_i$ with r_i standing for fundamental reflections r_i. In this appendix, we give the explicit expression of the fundamental mutations r_i generating $\mathcal{G}_{stg}^{\mathcal{G}}$.

B.1. Fundamental reflections of $\mathcal{G}_{stg}^{SO_8}$ and $\mathcal{G}_{stg}^{G_2}$

Here, we give explicit details regarding fundamental reflections of the Coxeter groups $\mathcal{G}_{stg}^{SO_8}$ and $\mathcal{G}_{stg}^{G_2}$ as well as on their subgroups $H_{stg}^{SO_8}$ and $H_{stg}^{G_2}$ used in our analysis of section 3.

B.1.1. Fundamental reflections of $\mathcal{G}_{stg}^{SO_8}$

The set of mutations of the strong chamber of 4d $\mathcal{N}=2$ supersymmetric pure SO(8) gauge model is a group generated by 8 fundamental non-commuting reflections r_i acting on the primitive quiver $Q_0^{SO_8}$. For convenience, we split these basic reflections into two subsets; one subset with 4 reflections associated with the four elementary dyons; they are denoted like $r_1=t_1$, $r_2=t_2$, $r_3=t_3$, $r_4=t_4$; and the remaining four others associated with the four elementary monopoles; they are denoted like $r_5=s_1$, $r_6=s_2$, $r_7=s_3$, $r_8=s_4$. The reflections t_i and s_i are realised by 8×8 matrices as follows

$$t_k^{so_8}=\begin{pmatrix} I_{4\times 4} & R_k \\ 0_{4\times 4} & \mathcal{E}_k \end{pmatrix} \quad , \quad s_k^{so_8}=\begin{pmatrix} \mathcal{E}_k & 0_{4\times 4} \\ R_k & I_{4\times 4} \end{pmatrix} \tag{B.1}$$

with $k=1,2,3,4$. The $\mathcal{E}_k$ is a 4×4 diagonal matrix with components $(\mathcal{E}_k)_{kk}=-1$, $(\mathcal{E}_k)_{ii}=1$ for $i\neq k$ and zero elsewhere; in a condensed manner it reads as

$$(\mathcal{E}_k)_{ij}=(-1)^{\delta_{ik}}\,\delta_{ij} \tag{B.2}$$

The R_k is a 4×4 matrix related to the matrix R of (3.27) like $\delta_{kj}R_{ij}$ and therefore to the $K_{ij}^{SO_8}$ Cartan matrix as follows

$$(R_k)_{ij}=\delta_{kj}\left(2\delta_{ij}-K_{ij}^{SO_8}\right) \tag{B.3}$$

The two first $t_1^{so_8}$ and $t_2^{so_8}$ matrices and their $s_1^{so_8}$ and $s_2^{so_8}$ homologue are as follows

$$t_1^{so_8}=\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad t_2^{so_8}=\begin{pmatrix} 1&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \tag{B.4}$$

and

$$s_1^{so_8}=\begin{pmatrix} -1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad s_2^{so_8}=\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&1&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&1 \end{pmatrix} \tag{B.5}$$

The set of quiver mutation $\mathcal{G}_{stg}^{SO_8}$ generated by the eight r_i's has a Coxeter group structure; the generators are non-commuting and satisfy

$$\left(r_i r_j\right)^{m_{ij}^{SO_8}} = I_{id}^{SO_8} \tag{B.6}$$

integers $m_{ij}^{SO_8}$ given by the Coxeter 8×8 matrix

$$M^{SO_8} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 3 & 2 & 2 \\ 2 & 1 & 2 & 2 & 3 & 2 & 3 & 3 \\ 2 & 2 & 1 & 2 & 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 & 3 & 2 & 2 \\ 2 & 3 & 2 & 2 & 1 & 2 & 2 & 2 \\ 3 & 2 & 3 & 3 & 2 & 1 & 2 & 2 \\ 2 & 3 & 2 & 2 & 2 & 2 & 1 & 2 \\ 2 & 3 & 2 & 2 & 2 & 2 & 2 & 1 \end{pmatrix} \tag{B.7}$$

By using the 4×4 matrix J,

$$J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{B.8}$$

we can express it like

$$M^{SO_8} = \begin{pmatrix} 2J - I & 2J + R \\ 2J + R & 2J - I \end{pmatrix} \tag{B.9}$$

To generate BPS states in the strong chamber $\mathfrak{Q}_{stg}^{SO_8}$ of the supersymmetric pure SO(8) gauge theory, we have used the two composite mutation operators $L_1^{SO_8} = r_4 r_3 r_2 r_1$ and $L_2^{SO_8} = r_8 r_7 r_6 r_5$; these are non-commuting reflections generating a subgroup $H_{stg}^{SO_8} \simeq Dih_{12}$ of the Coxeter $\boldsymbol{G}_{stg}^{SO_8}$.

B.1.2. Fundamental reflections of $\mathcal{G}_{stg}^{G_2}$

The set $\boldsymbol{G}_{stg}^{G_2}$ of mutations of the strong chamber of 4d $\mathcal{N} = 2$ supersymmetric pure G_2 gauge model is a group generated by 4 non-commuting reflections: $r_1 = t_1$, $r_2 = t_2$ generators for the two elementary dyons in the primitive quiver; and $r_3 = s_1$, $r_4 = s_2$ for the elementary monopoles. These reflections are realised by 4×4 matrices like

$$t_k^{G_2} = \begin{pmatrix} I_2 & \mathcal{R}_k \\ 0_{2\times 2} & \mathcal{E}_k \end{pmatrix} \quad , \quad s_k^{G_2} = \begin{pmatrix} \mathcal{E}_k & 0_{2\times 2} \\ \mathcal{R}_k & I_2 \end{pmatrix} \tag{B.10}$$

with $k = 1, 2$. The $\mathcal{E}_k$ is a 2×2 matrix with entries $(-1)^{\delta_{ik}}\, \delta_{ij}$ and the $(\mathcal{R}_k)_{ij}$ is related to the matrix $\mathcal{R}_{ij}$ of eq. (3.39) like $\delta_{kj} \mathcal{R}_{ij}$. Explicitly, we have

$$t_1^{G_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad t_2^{G_2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{B.11}$$

and

$$s_1^{G_2} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix} \quad , \quad s_2^{G_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.12}$$

The set of mutation $\boldsymbol{G}_{stg}^{G_2}$ has a Coxeter group structure with generators t_i satisfying $\left(r_i r_j\right)^{m_{ij}^{G_2}} = I_{id}^{G_2}$ where the $m_{ij}^{G_2}$ integers are the entries of the Coxeter matrix

$$M^{G_2} = \begin{pmatrix} 1 & 2 & 2 & 6 \\ 2 & 1 & 6 & 2 \\ 2 & 6 & 1 & 2 \\ 6 & 2 & 2 & 1 \end{pmatrix} \tag{B.13}$$

These reflections are related to the $\boldsymbol{G}_{stg}^{so_8}$ ones by using Folding matrices $\boldsymbol{F}$ and $\tilde{\boldsymbol{F}}$ (3.36) obeying $\boldsymbol{F}\tilde{\boldsymbol{F}} = I_{id}$; we have:

$$\begin{aligned} r_1^{G_2} &= \boldsymbol{F} r_2^{so_8} \tilde{\boldsymbol{F}} \\ r_2^{G_2} &= \boldsymbol{F}\left(r_4^{so_8} r_3^{so_8} r_1^{so_8}\right)\tilde{\boldsymbol{F}} \\ s_1^{G_2} &= \boldsymbol{F} s_2^{so_8} \tilde{\boldsymbol{F}} \\ s_2^{G_2} &= \boldsymbol{F}\left(s_4^{so_8} s_3^{so_8} s_1^{so_8}\right)\tilde{\boldsymbol{F}} \end{aligned} \tag{B.14}$$

To generate BPS states in the strong chamber $\mathfrak{Q}_{stg}^{G_2}$ of the 4d $\mathcal{N} = 2$ supersymmetric pure G_2 gauge theory, we have used the two composite mutation operators $\mathcal{L}_1 = r_2 r_1$ and $\mathcal{L}_2 = r_4 r_3$.

B.2. *Fundamental reflections of $\boldsymbol{G}_{stg}^{E_6}$ and $\boldsymbol{G}_{stg}^{F_4}$*

In this subsection, we give explicit details regarding the fundamental reflections of $\boldsymbol{G}_{stg}^{E_6}$, $\boldsymbol{G}_{stg}^{F_4}$ and their subgroups $H_{stg}^{E_6}$ and $H_{stg}^{F_4}$ used in section 4.

B.2.1. *Quiver mutation set $\boldsymbol{G}_{stg}^{E_6}$*

The set of mutations of the strong chamber $\mathfrak{Q}_{stg}^{E_6}$ of 4d $\mathcal{N} = 2$ supersymmetric pure E_6 gauge model is a group generated by *12* fundamental reflections; six of them $r_1 = t_1$, $r_2 = t_2$, $r_3 = t_3$, $r_4 = t_4$, $r_5 = t_5$, $r_6 = t_6$ associated with the elementary dyons; and the other six $r_7 = s_1$, $r_8 = s_2$, $r_9 = s_3$, $r_{10} = s_4$, $r_{11} = s_5$, $r_{12} = s_6$ with the elementary monopoles. As in the case of SO_8 gauge model, these basic reflections can be realised by 12×12 matrices as follows:

$$t_k^{E_6} = \begin{pmatrix} I_6 & R_k \\ 0_{6\times 6} & \mathcal{E}_k \end{pmatrix} \quad , \quad s_k^{E_6} = \begin{pmatrix} \mathcal{E}_k & 0_{6\times 6} \\ R_k & I_6 \end{pmatrix} \tag{B.15}$$

with $k = 1, 2, 3, 4, 5, 6$. The $\mathcal{E}_k$ and R_k are given by

$$\begin{aligned} (\mathcal{E}_k)_{ij} &= (-1)^{\delta_{ik}}\,\delta_{ij} \\ (R_k)_{ij} &= \delta_{kj}\left(2\delta_{ij} - K_{ij}^{E_6}\right) \end{aligned} \tag{B.16}$$

As an example, we give

$$t_1^{E_6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.17}$$

and

$$s_1^{E_6} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.18}$$

The set of quiver mutation $\mathcal{G}_{stg}^{E_6}$ has a Coxeter group structure, with generators satisfying the following features $\left(r_i r_j\right)^{m_{ij}^{E_6}} = I_{id}^{E_6}$ where $m_{ij}^{E_6}$ are the entries of the Coxeter 12×12 matrix

$$M^{E_6} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 2 & 2 & 3 & 2 & 3 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 2 & 3 & 2 & 3 & 2 & 3 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 3 & 2 & 3 & 2 \\ 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\ 3 & 2 & 3 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 3 & 2 & 3 & 2 & 3 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 2 & 3 & 2 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \end{pmatrix} \tag{B.19}$$

By using the 6×6 matrix J

$$J = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{B.20}$$

we can express M^{E_6} like

$$M^{E_6} = \begin{pmatrix} 2J - I & 2J + R \\ 2J + R & 2J - I \end{pmatrix} \tag{B.21}$$

To generate BPS states in thee strong chamber of the supersymmetric pure E_6 gauge theory, we have used the two composite mutation operators $L_1^{E_6} = r_6 r_5 r_4 r_3 r_2 r_1$, $L_2^{E_6} = r_{12} r_{11} r_{10} r_9 r_8 r_7$ generating a subgroup $H_{stg}^{E_6} \simeq Dih_{24}$ of the Coxeter $\mathcal{G}_{stg}^{E_6}$.

B.2.2. *Quiver mutation set* $\boldsymbol{G}_{stg}^{F_4}$

The set $\boldsymbol{G}_{stg}^{F_4}$ of mutations of the strong chamber of 4d $\mathcal{N} = 2$ supersymmetric pure F_4 gauge model is a group generated by 8 reflections: $r_1 = t_1$, $r_2 = t_2$, $r_3 = t_3$, $r_4 = t_4$ generators for the elementary dyons in the primitive quiver; and $r_5 = s_1$, $r_6 = s_2$, $r_7 = s_3$, $r_8 = s_4$ for corresponding monopoles. These reflections are realised by 4×4 matrices like

$$t_k^{G_2} = \begin{pmatrix} I_4 & \mathcal{R}_k \\ 0_{4\times4} & \mathcal{E}_k \end{pmatrix} \quad , \quad s_k^{G_2} = \begin{pmatrix} \mathcal{E}_k & 0_{4\times4} \\ \mathcal{R}_k & I_4 \end{pmatrix} \tag{B.22}$$

where $(\mathcal{E}_k)_{ij} = (-1)^{\delta_{ik}} \delta_{ij}$ and $(\mathcal{R}_k)_{ij} = \delta_{kj} \mathcal{R}_{ij}$. As examples, we have

$$t_1^{F_4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.23}$$

and

$$s_1^{F_4} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.24}$$

The set of mutation $\boldsymbol{G}_{stg}^{F_4}$ has a Coxeter group structure with generators r_i satisfying $(r_i r_j)^{m_{ij}^{F_4}} = I_{id}^{F_4}$ where the $m_{ij}^{F_4}$ integers are the entries of the Coxeter matrix

$$M^{F_4} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 3 & 2 & 2 \\ 2 & 1 & 2 & 2 & 3 & 2 & 4 & 2 \\ 2 & 2 & 1 & 2 & 2 & 4 & 2 & 3 \\ 2 & 2 & 2 & 1 & 2 & 2 & 3 & 2 \\ 2 & 3 & 2 & 2 & 1 & 2 & 2 & 2 \\ 3 & 2 & 4 & 2 & 2 & 1 & 2 & 2 \\ 2 & 4 & 2 & 3 & 2 & 2 & 1 & 2 \\ 2 & 2 & 3 & 2 & 2 & 2 & 2 & 1 \end{pmatrix} \tag{B.25}$$

These reflections are related to the $\mathcal{G}_{stg}^{E_6}$ ones by using Folding matrices $\mathcal{F}$ and $\tilde{\mathcal{F}}$ (4.18) obeying $\mathcal{F}\tilde{\mathcal{F}} = I$; these matrices have been explicitly constructed in section 4 of present study; we have:

$$\begin{aligned} r_1^{F_4} &= \mathcal{F}\left(r_5^{E_6} r_1^{E_6}\right)\tilde{\mathcal{F}} \quad &, \quad s_1^{F_4} &= \mathcal{F}\left(s_5^{E_6} s_1^{E_6}\right)\tilde{\mathcal{F}} \\ r_2^{F_4} &= \mathcal{F}\left(r_4^{E_6} r_2^{E_6}\right)\tilde{\mathcal{F}} \quad &, \quad s_2^{F_4} &= \mathcal{F}\left(s_4^{E_6} s_2^{E_6}\right)\tilde{\mathcal{F}} \\ r_3^{F_4} &= \mathcal{F} r_3^{E_6}\tilde{\mathcal{F}} \quad &, \quad s_3^{F_4} &= \mathcal{F} s_3^{E_6}\tilde{\mathcal{F}} \\ r_4^{F_4} &= \mathcal{F} r_6^{E_6}\tilde{\mathcal{F}} \quad &, \quad s_4^{F_4} &= \mathcal{F} s_6^{E_6}\tilde{\mathcal{F}} \end{aligned} \tag{B.26}$$

To generate BPS states in the strong chamber of 4d $\mathcal{N} = 2$ supersymmetric pure F_4 gauge theory, we have used the two composite mutation operators $\mathcal{L}_1 = r_4 r_3 r_2 r_1$ and $\mathcal{L}_2 = r_8 r_7 r_6 r_5$.

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