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SLATER'S THEOREM FOR MAGNETIC MIRROR BOUNDARY PERTURBATIONS

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A general form of Slater's formula, taking into account perturbations of Electric as well as Magnetic Mirror boundaries of a microwave cavity, is presented. The frequency shift induced by the Magnetic Mirror perturbation has the opposite sign to that of the Electric one.

KEY WORDS: Microwave cavity, boundary perturbation, magnetic mirror, frequency shift, cavity tuning

1 INTRODUCTION

The procedure shown in Figure 1 is often used to compute frequency shifts induced by "small" deformations of a cavity under the effects of various forces: helium bath pressure, thermal stresses, tuning or Lorentz forces.[1,2]

Such a scheme is very useful to evaluate the effect of "small" perturbations of the cavity shape because it provides good and fast estimates of the frequency shift without need for time consuming field codes that may, moreover, introduce errors of the same order of the frequency shift itself because of their finite discretization of boundary walls. The scheme is however not as simple as it appears at a first glance. For example a not completely rigorous application of Slater's theorem may produce an overestimate of the frequency shift, in particular when a cavity oscillating in a π – mode is shortened under the effects of the perturbation. One must remember in particular that calculating the frequency of an accelerating π – mode with a field code requires imposing Magnetic Mirror boundary conditions on the iris planes.

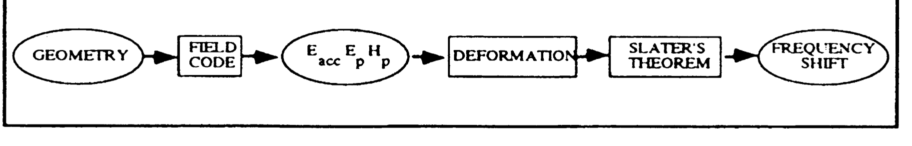

FIGURE 1: Procedure to compute "small" frequency shifts.

In the next section a general derivation of Slater's theorem, taking into account Electric (EM) and Magnetic Mirror (MM) boundary conditions, is discussed.

2 DISCUSSION OF SLATER'S THEOREM

The frequency shift of a resonant mode under the effect of a cavity wall perturbation is calculated from the well-known Slater's formula:[3]

$$\frac{\delta\omega_\nu}{\omega_\nu} = \frac{1}{4\bar{U}} \int\limits_{\delta v^*} \left(\varepsilon_0 E^2 - \mu_0 H^2\right) dv \tag{1}$$

where:

$$\bar{U} = \frac{1}{4} \int\limits_{V} \left(\varepsilon_0 E^2 + \mu_0 H^2\right) dv \tag{2}$$

is the average energy stored in the cavity volume V and δv^* is the volume variation caused by the deformation at the neighborhood surface S representing the cavity wall. It is important to stress that S is a perfectly conducting wall on which EM boundary condition: $\mathbf{n} \times \mathbf{E} = 0$ applies.

The formula (1) is incorrect for all perturbations of a MM boundary S' where $\mathbf{n} \times \mathbf{H} = 0$ as for example the case of iris plane displacement considering the π_TM$_{010}$ accelerating mode. This situation can occur for perturbations leading to a shortening of a cavity, see Figure 2.

A more general Slater's formula taking into account the perturbation of EM as well as MM boundary is the following:

$$\frac{\delta\omega_\nu}{\omega_\nu} = \frac{1}{4\bar{U}} \int\limits_{\delta v^*} \left(\varepsilon_0 E^2 - \mu_0 H^2\right) dv + \frac{1}{4\bar{U}} \int\limits_{\delta v^{**}} \left(\mu_0 H^2 - \varepsilon_0 E^2\right) dv \tag{3}$$

The first term coincides with Equation (1) the second is valid on the iris plane (MM) and has the opposite sign to the first one.

Slater's theorem has been dealt with and generalized by many authors.[4,5,6,7,8,9,10,11] For example a useful formula for determining complex electric and magnetic susceptibilities by measuring the frequency shift produced by introducing a sample into a resonant cavity has been discussed by Hutcheon *et al.*.[7] They show that the Slater's perturbation formula for a perfectly conducting sample can be easily derived from their formula as a limiting case of a sample with $\mu_r \to 0$ (and $\varepsilon_r \to \infty$, see also References 9 and 11). This case includes the one of a local perturbation of the cavity metallic wall, that corresponds to a perturbation of EM boundary condition. The correspondence with the case of MM boundary perturbation is not evident, although it could be obtained as a limiting case of a sample with $\mu_r \to \infty$ and $\varepsilon_r \to 0$, (see the Appendix for further details).

We have therefore demonstrated the general formula (3) starting directly from the Slater treatment. We recall now for completeness the main steps by a simplified notation.[12]

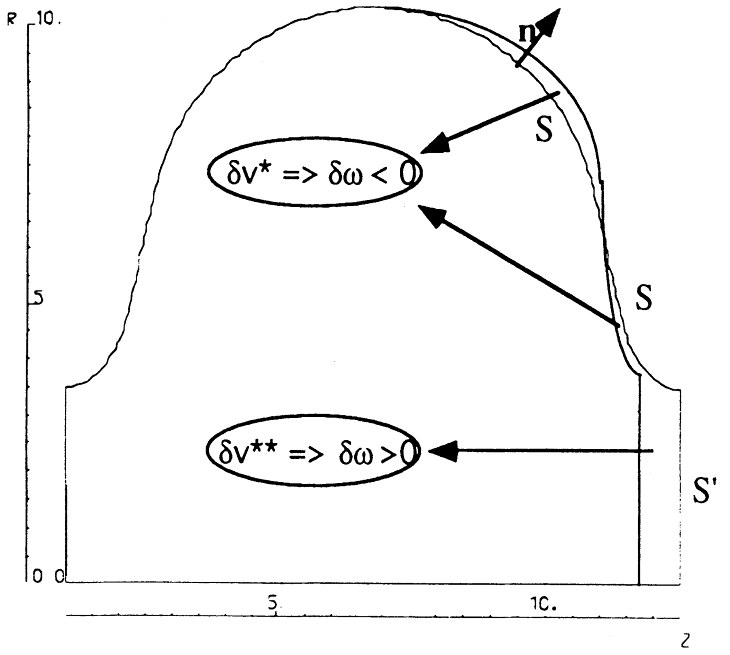

FIGURE 2: A schematic drawing of a cavity perturbation. Slater's theorem must be applied not only in regions δv^* where $\delta\omega<0$ but also in region δv^{**} where $\delta\omega>0$ in accelerating π-mode.

Let **E** and **H** be written in terms of the complex cavity field functions $\{\mathbf{e}_\nu, \mathbf{h}_\nu\}$ within a volume V as follows:

$$\mathbf{E}(x, y, z, t) = \Sigma_\nu a_\nu(t)\mathbf{e}_\nu(x, y, z)e^{j\omega_\nu t} \quad \mathbf{H}(x, y, z, t) = \Sigma_\nu b_\nu(t)\mathbf{h}_\nu(x, y, z)e^{j\omega_\nu t} \tag{4}$$

where $\{\mathbf{e}_\nu, \mathbf{h}_\nu\}$ must satisfy the following equations:

$$\left[\nabla^2 + k_\nu^2\right]\mathbf{e}_\nu = 0 \qquad \left[\nabla^2 + k_\nu^2\right]\mathbf{h}_\nu = 0 \tag{5}$$

within a volume V, where $k_\nu = \omega_\nu/c$. For solenoidal fields we have:

$$\nabla \bullet \mathbf{e}_\nu = 0 \qquad \nabla \bullet \mathbf{h}_\nu = 0 \tag{6}$$

and the boundary conditions are (EM over a surface S and MM over another surface S'):

$$\mathbf{n} \times \mathbf{e}_\nu = 0 \text{ over } S \quad \mathbf{n} \times \mathbf{h}_\nu = 0 \text{ over } S' \tag{7a}$$

$$\mathbf{n} \cdot \mathbf{h}_\nu = 0 \text{ over} S \quad \mathbf{n} \cdot \mathbf{e}_\nu = 0 \text{ over} S' \tag{7b}$$

where **n** is a unit vector pointing *outward* from the cavity surface. The normalization conditions are:

$$\int_V \mathbf{e}_\nu \cdot \mathbf{e}^*_\mu dv = \delta_{\nu\mu} \qquad \int_V \mathbf{h}_\nu \cdot \mathbf{h}^*_\mu dv = \delta_{\nu\mu} \tag{8}$$

and the coefficients $a_\nu(t)$, $b_\nu(t)$ in Equation (4) are given by:

$$a_\nu = \int_V \mathbf{E} \cdot \mathbf{e}^*_\nu dv \qquad b_\nu = \int_V \mathbf{H} \cdot \mathbf{h}^*_\nu dv \tag{9}$$

From Maxwell's equations we have:

$$\nabla \times \mathbf{e}_\nu = k_\nu \mathbf{h}_\nu \qquad \nabla \times \mathbf{h}_\nu = k_\nu \mathbf{e}_\nu \tag{10}$$

and the average stored energy is:

$$\bar{U} = \frac{1}{2}\varepsilon_0 \Sigma_\nu |a_\nu|^2 = \frac{1}{2}\mu_0 \Sigma_\nu |b_\nu|^2 \tag{11}$$

Slater obtained for each component of the field amplitude the following wave equation:[3]

$$\frac{1}{c^2}\frac{d^2 a_\nu}{dt^2} + k_\nu^2 a_\nu = \mu_0 \frac{d}{dt}\int_{\delta s^*} (\mathbf{n} \times \mathbf{H}) \cdot \mathbf{e}^*_\nu ds - k_\nu \int_{\delta s^{**}} (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{h}^*_\nu ds \tag{12}$$

where δs^* and δs^{**} are closed surfaces obtained joining the perturbed and the unperturbed surfaces of the EM and MM type respectively. In the last equation are taken into account the contributions of the surface current $\mathbf{J}_e = \mathbf{n} \times \mathbf{H}$ (first integral) and of the fictitious magnetic surface current $\mathbf{J}_m = \mathbf{n} \times \mathbf{E}$ (second integral), which appear on the perturbed surfaces at the discontinuity of the tangential component of **H** and **E** fields respectively.

Setting now

$$\mathbf{E} = a_\nu(t)\mathbf{e}_\nu \qquad \mathbf{H} = b_\nu(t)\mathbf{h}_\nu \tag{13}$$

the surface integrals in (12) become:

$$\begin{aligned}
\int_{\delta s^{**}} (\mathbf{n} \times a_\nu \mathbf{e}_\nu) \cdot \mathbf{h}_\nu^* ds &= a_\nu \int_{\delta s^{**}} \mathbf{n} \cdot \left(\mathbf{e}_\nu \times \mathbf{h}_\nu^*\right) ds \\
&= a_\nu \int_{\delta v^{**}} \nabla \cdot \left(\mathbf{e}_\nu \times \mathbf{h}_\nu^*\right) dv \\
&= a_\nu \int_{\delta v^{**}} \left(\mathbf{h}_\nu^* \cdot \nabla \times \mathbf{e}_\nu - \mathbf{e}_\nu \cdot \nabla \times \mathbf{h}_\nu^*\right) dv \\
&= a_\nu \int_{\delta v^{**}} \left(\mathbf{h}_\nu^* \cdot k_\nu \mathbf{h}_\nu - \mathbf{e}_\nu \cdot k_\nu \mathbf{e}_\nu^*\right) dv \\
&= k_\nu a_\nu \int_{\delta v_{**}} \left(h_\nu^2 - e_\nu^2\right) dv
\end{aligned} \tag{14}$$

where one applied the divergence theorem to the volume δv^{**} enclosed by the surface δs^{**} and:

$$\begin{aligned}
\int_{\delta s^*} (\mathbf{n} \times b_\nu \mathbf{h}_\nu) \cdot \mathbf{e}_\nu^* ds &= k_\nu b_\nu \int_{\delta v^*} \left(e_\nu^2 - h_\nu^2\right) dv \\
&= j\omega_\nu \varepsilon_0 a_\nu \int_{\delta v^*} \left(e_\nu^2 - h_\nu^2\right) dv
\end{aligned} \tag{15}$$

where δv^* is the volume enclosed by the surface δs^* and where one used the equality $k_\nu b_\nu = j\omega_\nu \varepsilon_0 a_\nu$ as derived from Maxwell's equations for a source free cavity.

Introducing the above integrals in Equation (12), calculating the temporal derivatives $da_\nu/dt = j\omega a_\nu$ (where ω is the frequency of the perturbed cavity), and multiplying both sides of Equation (13) by a_ν^*, $(a_\nu \cdot a_\nu^* = |a_\nu|^2)$, one has:

$$\left[\omega_\nu^2 - \omega^2\right] |a_\nu|^2 = -\omega_\nu^2 |a_\nu|^2 \int_{\delta v^*} \left(e_\nu^2 - h_\nu^2\right) dv - \omega_\nu^2 |a_\nu|^2 \int_{\delta v^{**}} \left(h_\nu^2 - e_\nu^2\right) dv \tag{16}$$

To first order in $\delta\omega_\nu$, setting $\omega = \omega_\nu + \delta\omega_\nu$ one finally obtains:

$$\frac{\delta\omega_\nu}{\omega_\nu} = \frac{1}{2} \int_{\delta v^*} \left(e_\nu^2 - h_\nu^2\right) dv + \frac{1}{2} \int_{\delta v^{**}} \left(h_\nu^2 - e_\nu^2\right) dv \tag{17}$$

which because of Equations (11) and (13) is identical with Equation (3).

3 EXAMPLE

We have computed frequency and fields of a TESLA cavity with an improved[10] version of SUPERFISH[13] that performs a discretization of the Helmoltz equation over an irregular triangular mesh of up to 32000 points (in this example we used 20000 mesh points corresponding to 0.055 cm mesh size, 76 mesh points on the the boundary and a 10^{-8} frequency accuracy.[11]) The surface electric field amplitude is calculated from the relation: $E = (1/kr)\partial(rH)/\partial\ell$, ℓ being the path coordinate along the cavity boundary. Because the quantity $rH(r, z)$ is discretized up to the second order on the mesh points, a spline interpolation of the electric field was used to calculate the field to the second order also, instead of the previous linear interpolation.

As an example of the importance of using the complete Slater formula (3), note that shortening the TESLA cavity by 10^{-3} mm, by cutting off a slice at the iris plane — where Magnetic Mirror condition has been imposed — while leaving the rest of the cavity unchanged, gives according to Equation (3) a frequency shift of +2.03 KHz. A test run of SUPERFISH on the same shortened cavity gives a frequency shift of +2.06 KHz which agrees with the result of Equation (3).

The frequency shifts calculated with Equation (3) for various "cuts" on EM and MM planes are shown in Figure 3.

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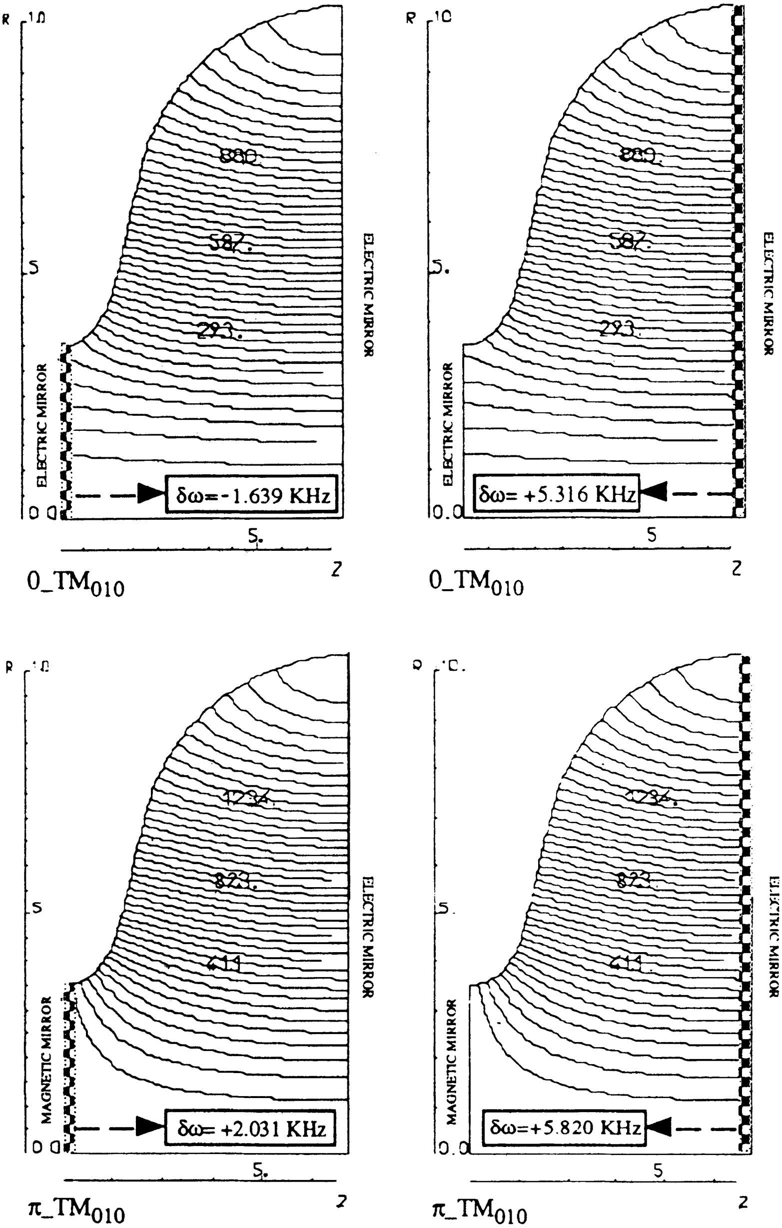

FIGURE 3: Generalized Slater's formula (3) results: frequency shifts induced by shortening the TESLA cavity by 10^{-3} mm, by cutting off a slice at either the iris or at the equator plane — where different boundary condition are imposed — leaving the rest of the cavity unchanged. (Unperturbed frequencies are: 0_TM_{010} = 1276.667 MHz and π_TM_{010} = 1301.016 MHz).

APPENDIX

The following formula has been discussed by Hutcheon:[7]

$$\frac{\delta\omega_\nu}{\omega_\nu} = \frac{1}{4\bar{U}} \int_{\delta v} \left(\chi_m \mu_0 \mathbf{H}_s \cdot \mathbf{H}_o^* + \chi_e \varepsilon_0 \mathbf{E}_s \cdot \mathbf{E}_o^*\right) dv \tag{18}$$

where:

$$\chi_e = \varepsilon_r - 1 \qquad \chi_m = \mu_r - 1 \tag{19}$$

are the electric and magnetic susceptibilities, $\mathbf{H}_o$ and $\mathbf{E}_o$ are the unperturbed fields, $\mathbf{H}_s = G_m \mathbf{H}_o$ and $\mathbf{E}_s = G_e \mathbf{E}_o$ are the fields inside the perturbing sample which are related to the unperturbed fields through the coefficients:

$$G_{e,m} = \frac{1}{1 + \chi_{e,m} F_{e,m}} \tag{20}$$

where $F_{e,m}$ are sample form factors ($F_{e,m} = 1/3$ for a sphere). The case of a wall displaced approximately parallel to itself[11] corresponds to $G_{e,m} = 1$.

Substituting $\mathbf{E}_s \cdot \mathbf{E}_o^* = G_e E_o^2$ and $\mathbf{H}_s \quad \mathbf{H}_o^* = G_m H_o^2$ in Equation (18) one has:

$$\frac{\delta\omega_\nu}{\omega_\nu} = \frac{1}{4\bar{U}} \int_{\delta v} \left(\chi_m \mu_0 G_m H_o^2 + \chi_e \varepsilon_0 G_e E_o^2\right) dv \tag{21}$$

Two limiting cases are included in the previous formula (21):

1) Metallic sample[7,9,11] with $\varepsilon_r \to \infty$ and $\mu_r \to 0$

$$\frac{\delta\omega_\nu}{\omega_\nu} = \frac{1}{4\bar{U}} \int_{\delta v} \left(\varepsilon_0 E_o^2 - \mu_0 G_m H_o^2\right) dv \tag{22}$$

setting $G_m = 1$ it corresponds to the original Slater's formula (1) for a EM boundary perturbation

2) Ideal sample with $\mu_r \to \infty$ and $\varepsilon_r \to 0$

$$\frac{\delta\omega_\nu}{\omega_\nu} = \frac{1}{4\bar{U}} \int_{\delta v} \left(\mu_0 H_o^2 - \varepsilon_0 G_e E_o^2\right) dv \tag{23}$$

setting $G_e = 1$ it corresponds to the case of a MM boundary perturbation formula (3)