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The charged black-hole bomb: A lower bound on the charge-to-mass ratio of the explosive scalar field

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ABSTRACT

The well-known superradiant amplification mechanism allows a charged scalar field of proper mass μ and electric charge q to extract the Coulomb energy of a charged Reissner–Nordström black hole. The rate of energy extraction can grow exponentially in time if the system is placed inside a reflecting cavity which prevents the charged scalar field from escaping to infinity. This composed black-hole-charged-scalar-field-mirror system is known as the *charged black-hole bomb*. Previous numerical studies of this composed physical system have shown that, in the linearized regime, the inequality $q/\mu > 1$ provides a necessary condition for the development of the superradiant instability. In the present paper we use analytical techniques to study the instability properties of the charged black-hole bomb in the regime of linearized scalar fields. In particular, we prove that the lower bound $\frac{q}{\mu} > \sqrt{\frac{r_m/r_- -1}{r_m/r_+ -1}}$ provides a necessary condition for the development of the superradiant instability in this composed physical system (here $r_\pm$ are the horizon radii of the charged Reissner–Nordström black hole and r_m is the radius of the confining mirror). This *analytically* derived lower bound on the superradiant instability regime of the composed black-hole-charged-scalar-field-mirror system is shown to agree with direct *numerical* computations of the instability spectrum.

1. Introduction

Kerr black holes may contain large amounts of rotational energies which can be released by bosonic fields that scatter off these spinning black holes. In this physical process, which is known as superradiant scattering [1–3], an incident bosonic field whose proper frequency lies in the superradiant regime [1–4]

$$0 < \omega < m\Omega_H \tag{1}$$

can extract the rotational energy and angular momentum of the spinning Kerr black hole (here m is the azimuthal harmonic index of the incident bosonic field and Ω_H is the angular velocity of the black-hole horizon).

The rate of energy extraction from the black hole can grow exponentially in time if the bosonic field is prevented from escaping to infinity. The required confinement mechanism can be provided either by a reflecting mirror which surrounds the black hole [2,5] or, for a massive bosonic field, by the mutual gravitational attraction between the central black hole and the extracting field [6,7].

It should be emphasized that not all bosonic modes trigger the black-hole superradiant instability. In particular, it was proved in [8] that the inequality

$$\mu < \sqrt{2}\cdot m\Omega_H \tag{2}$$

provides a necessary condition for the development of the superradiant instability in the composed Kerr-black-hole-massive-scalar-field system, where μ is the proper mass of the exploding scalar field.

As pointed out by Bekenstein [9], an analogous superradiant amplification of bosonic fields may occur when a *charged* field scatters off a *charged* black hole. In particular, a charged scalar field whose proper frequency lies in the superradiant regime [9]

$$0 < \omega < q\Phi_H \tag{3}$$

can extract the Coulomb energy and electric charge of a charged Reissner–Nordström (RN) black hole (here q is the charge coupling constant of the incident scalar field and Φ_H is the electric potential of the charged black hole).

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Interestingly, it was proved in [10] that, contrary to the spinning (Kerr) case, in the charged (RN) case the gravitational attraction between the black hole and the massive charged scalar field *cannot* provide the confinement mechanism which is required in order to trigger the black-hole superradiant instability. The charged black-hole bomb must therefore include a reflecting mirror which surrounds the black hole and prevents the amplified charged bosonic field from escaping to infinity [11–13].

In a very interesting work, Degollado et al. [11] have used numerical techniques to study the instability properties of the composed RN-black-hole-charged-scalar-field-mirror system. In particular, it was found in [11] that, in the linearized regime [14], the inequality

$$\frac{q}{\mu} > 1 \tag{4}$$

provides a necessary condition for the development of the superradiant instabilities in this charged system.

The main goal of the present paper is to explore the superradiant instability regime of the composed RN-linearized-charged-scalar-field-mirror system (the charged black-hole bomb) using analytical techniques. In particular, below we shall provide an *analytical* explanation for the characteristic inequality (4) observed *numerically* in the interesting study of Degollado et al. [11]. Moreover, in this paper we shall derive a stronger lower bound [see Eq. (45) below] on the dimensionless charge-to-mass ratio which characterizes the explosive charged massive scalar fields.

2. Description of the system

We shall study the dynamics of a charged massive scalar field Ψ linearly coupled to a non-extremal charged RN black hole of mass M and electric charge Q. The charged RN black-hole spacetime is described by the line element [15,16]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) , \tag{5}$$

where the metric function $f(r)$ is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} . \tag{6}$$

The zeros of $f(r)$,

$$r_{\pm} = M \pm (M^2 - Q^2)^{1/2} , \tag{7}$$

determine the horizon radii of the charged RN black hole [15].

The dynamics of a test scalar field Ψ of proper mass μ and charge coupling constant q [17,18] in the background of the RN black-hole spacetime is governed by the Klein–Gordon wave equation [19–21]

$$[(\nabla^{\nu} - iqA^{\nu})(\nabla_{\nu} - iqA_{\nu}) - \mu^2]\Psi = 0 , \tag{8}$$

where $A_{\nu} = -\delta^0_{\nu} Q/r$ is the electromagnetic potential of the charged black hole. One can decouple the radial and angular parts of the scalar field Ψ and express it in the form

$$\Psi_{lm}(t, r, \theta, \phi) = e^{im\phi} S_{lm}(\theta) R_{lm}(r) e^{-i\omega t} , \tag{9}$$

where ω, l, and m are respectively the conserved frequency of the field mode and its angular harmonic indices [22,23].

It is worth noting that the sign of $\Im\omega$ in (9) determines the (in)stability properties of the scalar field mode: stable modes (that is, modes decaying in time) are characterized by $\Im\omega < 0$, whereas unstable modes (that is, modes growing exponentially in time) are characterized by $\Im\omega > 0$. Stationary modes with $\Im\omega = 0$ mark the boundary between stable and unstable solutions of the Klein–Gordon wave equation (8). These marginally stable field modes are characterized by the critical (marginal) frequency [see Eq. (3)]

$$\omega_{\text{c}} = \frac{qQ}{r_+} \tag{10}$$

for the superradiant scattering phenomenon [9].

Substituting the scalar field decomposition (9) into the Klein–Gordon wave equation (8) and using the line element (5) of the RN black-hole spacetime, one finds that the radial function $R(r)$ is determined by the characteristic radial equation [19–21]

$$\Delta\frac{d}{dr}\Big(\Delta\frac{dR}{dr}\Big) + UR = 0 , \tag{11}$$

where

$$\Delta \equiv r^2 f(r) \tag{12}$$

and

$$U \equiv (\omega r^2 - qQr)^2 - \Delta(\mu^2 r^2 + K_l) . \tag{13}$$

Here $K_l = l(l+1)$ (where m and $l \geq |m|$ are integers) are the characteristic eigenvalues of the angular function $S(\theta)$ [19–21].

The characteristic equation (11) for the radial eigenfunction $R(r)$ should be supplemented by the physical boundary condition of purely ingoing waves at the black-hole horizon [5–7,11]:

$$R \sim e^{-i(\omega - qQ/r_+)y} \quad \text{as } r \to r_+ \quad (y \to -\infty) , \tag{14}$$

where the radial coordinate y is determined by the relation $dy = dr/f(r)$ [see Eq. (18) below]. For field modes in the superradiant regime (3), the near-horizon boundary condition (14) corresponds to an outgoing flux of Coulomb energy and electric charge from the charged RN black hole [9–13]. In addition, the reflecting mirror which surrounds the composed black-hole-field system dictates the boundary condition [5,11–13]

$$R(r = r_{\text{m}}) = 0 \tag{15}$$

for the confined scalar field, where r_{m} is the radial location of the mirror.

3. The effective radial potential of the composed RN-black-hole-charged-massive-scalar-field system

The radial equation (11), together with the boundary conditions (14) and (15), determine a discrete family of complex field resonances $\{\omega_n(r_{\text{m}})\}$ [5,11,12,24]. As mentioned above, Degollado et al. [11] have performed a very interesting numerical study of these characteristic resonances of the composed RN-black-hole-charged-scalar-field-mirror system. In particular, Degollado et al. [11] have found numerically that unstable (exploding) charged field modes are characterized by the property $q > \mu$ [see Eq. (4)]. The main goal of the present paper is to provide an *analytical* explanation for this (*numerically* observed) characteristic inequality. Moreover, below we shall derive a stronger lower bound on the dimensionless charge-to-mass ratio of these explosive (unstable) charged massive scalar fields.

In order to analyze the physical properties of the composed RN-black-hole-charged-scalar-field-mirror system, we shall first express the radial equation (11) for the charged massive scalar fields in the form of a Schrödinger-like wave equation. To this end, it proves useful to define the new radial function

$$\psi = rR , \tag{16}$$

in terms of which the radial equation (11) can be expressed in the form

$$\frac{d^2\psi}{dy^2} - V\psi = 0\,, \tag{17}$$

where the radial coordinate y is defined by the relation

$$dy = \frac{dr}{f(r)}\,. \tag{18}$$

The effective radial potential in (17) is given by

$$V = V(r; M, Q, \omega, q, \mu, l) = -\Big(\omega - \frac{qQ}{r}\Big)^2 + \frac{f(r)H(r)}{r^2}\,, \tag{19}$$

where

$$H(r; M, Q, \mu, l) = \mu^2 r^2 + l(l+1) + \frac{2M}{r} - \frac{2Q^2}{r^2}\,. \tag{20}$$

In the next section we shall analyze the near-horizon properties of the effective radial potential $V(r)$ that appears in the Schrödinger-like wave equation (17) for the charged massive scalar fields in the charged RN black-hole spacetime. We shall then use these properties in order to study the near-horizon spatial behavior of the radial eigenfunction ψ which characterizes the charged massive scalar fields.

4. The near-horizon behavior of the charged scalar eigenfunctions

Our main goal is to explore the onset of superradiant instabilities in the composed RN-black-hole-charged-scalar-field-mirror system. Thus, we shall henceforth analyze the behavior of the marginally stable (stationary) charged field modes (10) which mark the boundary of the superradiant instability regime [25]. In particular, in this section we shall study the near-horizon spatial behavior of the radial eigenfunction ψ which characterizes the stationary (marginally stable) resonances of the charged scalar fields in the charged RN black-hole spacetime. Specifically, we shall prove below that this characteristic function is a positive [26], increasing, and convex function in the near-horizon $\frac{r-r_+}{r_+-r_-} \ll 1$ region of the RN black-hole spacetime.

To that end, we shall first define the dimensionless variables

$$x \equiv \frac{r-r_+}{r_+} \quad ; \quad \tau \equiv \frac{r_+-r_-}{r_+}\,, \tag{21}$$

and study the near-horizon $x \ll \tau$ [27] behavior of the effective radial potential (19). Substituting the characteristic resonant frequency (10) of the marginally stable charged scalar fields into the expression (19) of the effective radial potential, one finds

$$r_+^2 V(x \to 0) = H(r_+)\tau \cdot x + O[(qQ)^2 x^2] \tag{22}$$

in the near-horizon region

$$x \ll \tau \times \frac{H(r_+)}{(qQ)^2}\,, \tag{23}$$

where [see Eq. (20)]

$$H(r = r_+) = \mu^2 r_+^2 + l(l+1) + 1 - \frac{Q^2}{r_+^2}\,. \tag{24}$$

Remembering that $1 - Q^2/r_+^2 > 0$, one finds the characteristic inequality

$$H(r = r_+) > 0 \tag{25}$$

for the massive charged scalar fields. Equations (22) and (25) imply that

$$V \geq 0 \tag{26}$$

in the near-horizon region (23).

Integrating the relation (18) in the near-horizon region,

$$x \ll \tau\,, \tag{27}$$

one finds

$$y = \frac{r_+}{\tau}\ln(x) + O(x)\,, \tag{28}$$

which implies [28]

$$x = e^{\tau y/r_+}[1 + O(e^{\tau y/r_+})]. \tag{29}$$

Taking cognizance of Eqs. (17), (22), and (29), one finds the near-horizon $x \ll \tau$ behavior

$$\frac{d^2\psi}{d\tilde{y}^2} - \frac{4H(r_+)}{\tau}e^{2\tilde{y}}\psi = 0 \tag{30}$$

of the Schrödinger-like wave equation (17), where

$$\tilde{y} \equiv \frac{\tau}{2r_+}y\,. \tag{31}$$

The physical solution [29] of the near-horizon Schrödinger-like wave equation (30) is given by the modified Bessel function of the first kind [30,31]:

$$\psi(y) = I_0\Big(2\sqrt{\frac{H(r_+)}{\tau}}e^{\tau y/2r_+}\Big)\,. \tag{32}$$

Using the well-known properties of the modified Bessel function I_0 [30], one finds from (32) that the radial eigenfunction ψ, which characterizes the charged massive scalar fields in the charged RN black-hole spacetime, is a positive, increasing, and convex function in the near-horizon region [see Eqs. (23) and (27)]

$$x \ll \tau \times \min\{1, H(r_+)/(qQ)^2\}\,. \tag{33}$$

That is,

$$\{\psi > 0 \ \ \text{and} \ \ \frac{d\psi}{dy} > 0 \ \ \text{and} \ \ \frac{d^2\psi}{dy^2} > 0\} \quad \text{for} \quad 0 < x \ll \tau \times \min\{1, H(r_+)/(qQ)^2\}\,. \tag{34}$$

Taking cognizance of the characteristic near-horizon spatial behavior (34) of the radial eigenfunction ψ [32], together with the boundary condition (15) which is dictated by the presence of the reflecting mirror, one concludes that the radial eigenfunction ψ must have (at least) one maximum point, $x = x_{\text{max}}$, between the black-hole horizon [where ψ is a positive and increasing function, see (34)] and the reflecting mirror [where ψ vanishes, see (15)]. We note, in particular, that the radial eigenfunction ψ is characterized by the relations

$$\{\psi > 0 \ \ \text{and} \ \ \frac{d^2\psi}{dy^2} < 0\} \quad \text{for} \quad x = x_{\text{max}} \tag{35}$$

at the maximum point $x = x_{\text{max}}$.

5. The superradiant instability regime of the charged black-hole bomb

It the previous section we have proved that the radial eigenfunction ψ, which characterizes the confined charged scalar fields in the charged RN black-hole spacetime, must have (at least) one maximum point, $r = r_{\text{max}}$, between the black-hole horizon and the reflecting mirror. That is,

$$r_+ < r_{\text{max}} < r_{\text{m}} . \tag{36}$$

Taking cognizance of Eqs. (17) and (35), one finds that the effective radial potential is characterized by the relation

$$V(r = r_{\text{max}}) < 0 \tag{37}$$

at this maximum point. We shall now use this characteristic inequality in order to derive a generic bound on the superradiant instability regime of the charged black-hole bomb.

Substituting the characteristic resonant frequency (10) of the marginally stable charged scalar fields into the expression (19) of the effective radial potential, one finds the relation

$$V(r = r_{\text{max}}; \omega = \omega_{\text{c}}) = \frac{r_{\text{max}} - r_+}{r^2_{\text{max}}}\Big[\frac{r_{\text{max}} - r_-}{r^2_{\text{max}}}H(r_{\text{max}}) - (qQ)^2\frac{r_{\text{max}} - r_+}{r^2_+}\Big] , \tag{38}$$

which yields the inequality [see (37)]

$$\frac{qQ}{r_+} > \sqrt{\frac{r_{\text{max}} - r_-}{r_{\text{max}} - r_+}\cdot\frac{H(r_{\text{max}})}{r^2_{\text{max}}}} . \tag{39}$$

Using the inequality $r_{\text{max}} < r_{\text{m}}$ [see Eq. (36)], one finds

$$\frac{r_{\text{max}} - r_-}{r_{\text{max}} - r_+} > \frac{r_{\text{m}} - r_-}{r_{\text{m}} - r_+} \tag{40}$$

and

$$\frac{l(l+1)}{r^2_{\text{max}}} > \frac{l(l+1)}{r^2_{\text{m}}} . \tag{41}$$

In addition, for charged RN black holes the expression $2M/r^3 - 2Q^2/r^4$ is a concave function whose maximum is located at $r = 4Q^2/3M$. One can therefore write [33]

$$\frac{2M}{r^3_{\text{max}}} - \frac{2Q^2}{r^4_{\text{max}}} > \mathcal{F} \equiv \begin{cases} \frac{r_+ - r_-}{r^3_+} & \text{for} \quad r_{\text{m}} \leq 4Q^2/3M ; \\ \min\{\frac{r_+ - r_-}{r^3_+}, \frac{2M}{r^3_{\text{m}}} - \frac{2Q^2}{r^4_{\text{m}}}\} & \text{for} \quad r_{\text{m}} > 4Q^2/3M . \end{cases} \tag{42}$$

From Eqs. (20), (41), and (42), one finds the lower bound

$$\frac{H(r_{\text{max}})}{r^2_{\text{max}}} > \mu^2 + \frac{l(l+1)}{r^2_{\text{m}}} + \mathcal{F} . \tag{43}$$

Substituting the inequalities (40) and (43) into (39), one can write the lower bound on the dimensionless quantity qQ in terms of the physical parameters $\{r_\pm, r_{\text{m}}\}$ of the black hole and its confining mirror:

$$qQ > \sqrt{\frac{r_{\text{m}} - r_-}{r_{\text{m}} - r_+}\cdot\Big[\mu^2 + \frac{l(l+1)}{r^2_{\text{m}}} + \mathcal{F}\Big]r^2_+} . \tag{44}$$

It is worth emphasizing that the analytically derived lower bound (44) provides a necessary condition for the development of the superradiant instabilities in the composed RN-black-hole-charged-scalar-field-mirror system [34,35].

Table 1
The superradiant instability regime of the composed RN-black-hole-charged-scalar-field-mirror system (the charged black-hole bomb). We display the dimensionless ratio $(qQ)^{\text{stat}}/(qQ)^{\text{bound}}$, where $(qQ)^{\text{stat}}$ is the *numerically* computed [11] value of the quantity qQ which corresponds to the stationary (marginally stable) charged scalar configurations [36], and $(qQ)^{\text{bound}}$ is the *analytically* derived lower bound on the superradiant instability regime given by Eq. (44). The data presented is for the case $M\mu = 0.3, Mq = 0.36$, and $l = 1$. One finds that the superradiant instability regime of the charged black-hole bomb is characterized by the relation $(qQ)^{\text{stat}}/(qQ)^{\text{bound}} > 1$, in agreement with the analytically derived lower bound (44).

Q/M	0.990	0.997	0.999
$(qQ)^{\text{stat}}/(qQ)^{\text{bound}}$	1.03	1.09	1.12

6. Numerical confirmation

We shall now verify the validity of the analytically derived lower bound (44) on the superradiant instability regime of the charged black-hole bomb. The instability spectrum of this composed RN-black-hole-charged-massive-scalar-field-mirror system was investigated numerically in [11]. In Table 1 we display the dimensionless ratio $(qQ)^{\text{stat}}/(qQ)^{\text{bound}}$, where $(qQ)^{\text{stat}}$ is the *numerically* computed [11] value of the quantity qQ which corresponds to the stationary (marginally stable) charged scalar configurations [36], and $(qQ)^{\text{bound}}$ is the *analytically* derived lower bound on the superradiant instability regime given by Eq. (44). One finds from Table 1 that the charged black-hole bomb is characterized by the relation $(qQ)^{\text{stat}}/(qQ)^{\text{bound}} > 1$, in agreement with the analytically derived lower bound (44).

7. Summary and discussion

We have studied analytically the superradiant instability regime of the charged black-hole bomb. This physical system is composed of a charged massive scalar field which, on the one hand, extracts the Coulomb energy of a charged Reissner–Nordström black hole and, on the other hand, is prevented from escaping to infinity by a reflecting mirror which surrounds the black hole. We have proved that in order for the superradiant instability to develop in this composed charged black-hole bomb, the dimensionless quantity qQ of the black-hole-field system must be bounded from below as in (44).

In a very interesting study, Degollado et al. [11] have used numerical techniques to study the instability spectrum of the charged black-hole bomb. In particular, it was found in [11] that the inequality $q/\mu > 1$ [see Eq. (4)] provides a necessary condition for the development of the superradiant instability in this composed system. We can now provide an *analytical* explanation for this *numerically* observed [11] necessary condition: Using the relation $Q^2 = r_+r_-$, one finds from (44) the compact lower bound [37]

$$\frac{q}{\mu} > \sqrt{\frac{r_{\text{m}}/r_- - 1}{r_{\text{m}}/r_+ - 1}} > 1 \tag{45}$$

on the dimensionless charge-to-mass ratio of the scalar fields in the explosive (unstable) regime of the charged black-hole bomb [38,39]. It is worth emphasizing that this lower bound provides a necessary condition for the development of the superradiant instabilities in the composed RN-black-hole-charged-scalar-field-mirror system [40].

Thus far, we have treated the composed charged black-hole bomb at the *classical* level. It should be emphasized, however, that the well known Schwinger *quantum* pair-production mechanism [41–44] restricts the physical parameters of the composed RN-black-hole-charged-massive-scalar-field system. In particular, this vacuum polarization effect sets the upper bound [41–45] $E_+ \ll E_c \equiv \mu^2/q\hbar$ on the strength of the black-hole electric field (here $E_+ = Q/r_+^2$ is the electric field at the horizon of the charged RN black hole). The quantum production of charged particle/anti-particle pairs in the charged black-hole spacetime (the Schwinger discharge of the RN black hole) therefore sets the upper bound $qQ \ll \mu^2 r_+^2$ on the physical parameters of the composed black-hole-field system. Taking cognizance of Eq. (44) [46] one finds that, in the superradiant explosive regime, the dimensionless quantity qQ is restricted by the two inequalities

$$\mu r_+ < qQ \ll \mu^2 r_+^2 \ . \tag{46}$$

The two inequalities in (46) imply that, in physically acceptable situations [47], the explosive charged massive scalar fields must be characterized by the strong inequalities

$$1 \ll \mu r_+ < qQ \ . \tag{47}$$

It is worth noting that the physical restriction $\mu r_+ \gg 1$ [see (47)] imposed by the quantum Schwinger pair-production mechanism (the vacuum polarization effect) implies that, in physically acceptable situations [47], the lower bound (44) is well approximated by the lower bound (45) [48].

Finally, it is worth emphasizing again that in this study we have treated the charged massive scalar fields at the linear level. Our analytical results are therefore expected to be valid in the early stages of the development of the superradiant instability (that is, in the *ignition* stage of the black-hole bomb), when the external charged scalar fields are weak and can still be regarded as perturbation fields on the background of the charged RN black-hole spacetime. As we demonstrated explicitly in this paper, the main advantage of this perturbative (linearized) approach stems from the fact that the physical properties of the composed RN-black-hole-charged-massive-scalar-field-mirror system (the charged black-hole bomb) can be explored *analytically* in the linear regime. It should be emphasized, however, that the late-time (non-linear) development of the superradiant instability can only be tackled with *numerical* techniques, as recently done in the interesting numerical work of Sanchis-Gual et al. [49].

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[14] That is, in the regime of test (linearized) charged scalar fields propagating on the background of the charged RN black-hole spacetime.
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[16] Here r is the Schwarzschild areal coordinate.
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[18] We shall assume, without loss of generality, that $q > 0$ and $Q > 0$.
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[22] Here l and m are respectively the spherical harmonic index and the azimuthal harmonic index of the field mode.
[23] We shall henceforth omit the angular harmonic indices $\{l, m\}$ for brevity.
[24] It is worth noting that, contrary to the caged (bounded) field modes that we consider here, which are characterized by the boundary condition (15) at the finite radial location $r = r_m$ of the confining mirror, the more familiar quasinormal resonances of a charged scalar field in the background of an asymptotically flat RN black-hole spacetime are characterized by an outgoing boundary condition at spatial infinity $r \to \infty$;
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[25] Note that these stationary field modes, which are characterized by the critical (marginal) frequency (10) for the superradiant scattering phenomenon, mark the boundary between stable ($\Im\omega < 0$) and unstable ($\Im\omega > 0$) resonances of the confined scalar fields.
[26] One can assume $R(r = r_+) \geq 0$ without loss of generality.
[27] We shall henceforth consider non-extremal black holes which are characterized by $\tau > 0$.
[28] It is worth noting that the near-horizon $x \ll \tau$ region corresponds to $\tau y/r_+ \to -\infty$ [see Eq. (28)]. This implies $e^{\tau y/r_+} \to 0$ in the near-horizon region.
[29] That is, the radial solution which respects the physically motivated boundary condition (14).
[30] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1970.

[31] Here we have used equations (9.1.54) and (9.6.3) of [30].

[32] It is worth emphasizing again that the inequalities presented in (34) imply that the radial eigenfunction ψ is a positive and increasing function in the vicinity of the black-hole horizon [that is, in the near-horizon $x \ll \tau \times \min\{1, H(r_+)/(qQ)^2\}$ region].

[33] Here we have used the fact that $2M/r^3 - 2Q^2/r^4$ is a monotonic increasing function in the regime $r < 4Q^2/3M$. Thus, $2M/r^3_{\text{max}} - 2Q^2/r^4_{\text{max}} > 2M/r^3_+ - 2Q^2/r^4_+ = (r_+ - r_-)/r^3_+$ in the regime $r_+ < r_{\text{max}} < r_{\text{m}} \leq 4Q^2/3M$. On the other hand, the expression $2M/r^3 - 2Q^2/r^4$ is a monotonic decreasing function in the regime $r > 4Q^2/3M$. Thus, for $r_{\text{m}} > 4Q^2/3M$ one cannot conclude that r_{max} is located in the increasing region of $2M/r^3 - 2Q^2/r^4$, in which case one can only write $2M/r^3_{\text{max}} - 2Q^2/r^4_{\text{max}} > \min\{2M/r^3_+ - 2Q^2/r^4_+, 2M/r^3_{\text{m}} - 2Q^2/r^4_{\text{m}}\}$.

[34] It should be emphasized that in the present study we have used the (*real*-valued) critical (threshold) frequency for the superradiant scattering phenomenon in the RN black-hole spacetime [see Eq. (10)]. The marginally stable (stationary) scalar field modes, which mark the onset of the superradiant instabilities in the composed black-hole-charged-scalar-field-mirror system, are characterized by the real-valued ($\Im\omega = 0$) resonant frequency (10), in which case the effective radial potential (19) of the composed black-hole-field system is also a real-valued function. As we explicitly shown in this paper, one can study analytically the physical properties of the Schrödinger-like wave equation (17) with the *real*-valued radial potential (19) which characterizes the marginally stable field configurations. It should be emphasized that unstable field modes in the superradiant regime $\Re\omega < qQ/r_+$ are characterized by *complex* eigenfrequencies with $\Im\omega > 0$, in which case the effective radial potential (19) of the Schrödinger-like wave equation (17) is a *complex*-valued function. The study of second order differential equations with complex-valued radial potentials is usually a very complicated task which is beyond the scope of the present paper. The extrapolation of condition (44) which characterizes the marginally stable (stationary) field modes to the superradiant modes is relying on the comparison with the numerical data of [11].

[35] It is worth emphasizing that we have proved in this paper that the lower bound (44) on the dimensionless quantity qQ characterizes the marginally stable (stationary) field modes (10). In addition, we know that the RN-mirror system is *stable* to uncharged field modes with $qQ = 0 < (qQ)^{\text{bound}}$, see Eq. (3) [here $(qQ)^{\text{bound}}$ denotes the r.h.s of (44)]. It is therefore natural to expect (though it should be emphasized that we have not proved it here) that the *unstable* charged field modes are characterized by the opposite inequality $qQ > (qQ)^{\text{bound}}$.

[36] These marginally stable (stationary) field configurations [with $\omega = \omega_{\text{c}}$, see Eq. (10)] mark the onset of superradiant instabilities in the composed RN-black-hole-charged-massive-scalar-field-mirror system. In particular, the explosive (unstable) regime of the charged black-hole bomb is characterized by confined charged field configurations whose dimensionless quantity qQ lies in the regime $qQ > (qQ)^{\text{stat}}$ (see also [11]).

[37] Here we have used the inequalities $l(l+1) \geq 0$ and $\mathcal{F} \geq 0$ [see Eq. (42)] in Eq. (44).

[38] Note that the analytically derived lower bound (45) on the dimensionless charge-to-mass ratio of the explosive scalar fields can be approximated by the simple inequality $q/\mu > 1$ in the regime $r_{\text{m}}/r_+ \gg 1$ of large mirror radii.

[39] Note that the lower bound (45) on the dimensionless charge-to-mass ratio of the explosive scalar fields can also be expressed as a lower bound on the dimensionless radius of the confining mirror: $\frac{r_{\text{m}}}{r_+} > \frac{(q/\mu)^2-1}{(q/\mu)^2-r_+/r_-}$.

[40] It is worth emphasizing that the lower bound (44) is stronger than the lower bound (45).

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[46] Note, in particular, that the r.h.s of (44) is larger than μr_+, which implies $qQ > \mu r_+$ in the superradiant explosive regime of the charged black-hole bomb.

[47] That is, taking into account the well known quantum Schwinger pair-production mechanism in the charged black-hole spacetime [41–44].

[48] It is worth emphasizing again that the lower bound (44) is stronger than the lower bound (45).

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