# LIE ALGEBRAS AND BRAIDED GEOMETRY

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**Abstract.** We show that every Lie algebra or superLie algebra has a canonical braiding on it, and that in terms of this its enveloping algebra appears as a flat space with braided-commuting coordinate functions. This also gives a new point of view about $q$-Minkowski space which arises in a similar way as the enveloping algebra of the braided Lie algebra $gl_{2,q}$. Our point of view fixes the signature of the metric on $q$-Minkowski space and hence also of ordinary Minkowski space at $q = 1$. We also describe an abstract construction for left-invariant integration on any braided group.

**Key words:** Lie algebra – braided group – quantum group – $q$-Minkowski space – braided integration

## 1. Introduction

Braided geometry is a generalisation of ordinary geometry based on the idea of *braid statistics* between independent systems [1][2][3][4][5][6]. This includes as a special case the ideas of supergeometry but with the super-transposition $\Psi = \pm 1$ there replaced by a more general braiding where $\Psi^2 \neq \mathrm{id}$. Braided differentiation and integration on braided vector spaces, braided groups and braided Lie algebras are all known. Braided manifolds and braided Yang-Mills theory are in the pipeline. The main conclusion is that many constructions familiar in usual or supergeometry can be generalised to the braided case. Moreover, many constructions which are more commonly associated with quantum groups and the theory of $q$-deformations are more properly understood in these terms. There is a review article for physicists[7] as well as an introductory conference proceedings[8].

Here we would like to use some of this braided geometry to explore a basic conceptual problem that arises in quantum physics. The problem is that we think of a quantum algebra of observables on the one hand as a noncommutative version of the algebra of functions on phase space, or on the other hand as generated by the algebra of functions on configuration space and by the enveloping algebra $U(g)$ for $g$ a generalised momentum symmetry. These points of view are contradictory unless it happens that we

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can view $U(g)$ as like the algebra of functions on some space, the momentum part of phase space.

We will see in Section 2 that for any Lie algebra $g$, one can indeed view $U(g)$ as the algebra of functions on a braided version of $\mathbb{R}^n$. So the non-commutativity of this algebra, which we normally associate with differential operators and quantisation, can be thought of equally well as statistical non-commutativity like that of Grassmann variables, albeit with a braiding $\Psi$ rather than $\pm 1$. We call this phenomenon in which a Lie algebra or enveloping algebra of operators is thought of instead as the coordinate functions of some space, a *quantum-geometry transformation.* The very simplest example is $U(\mathbb{R}^n) = \mathbb{C}[x_1, x_2, \cdots, x_n]$ where the enveloping algebra of an Abelian Lie algebra is thought of instead as polynomials in some bosonic position coordinates $x_i$. This is the idea behind Fourier transforms and our quantum-geometry transformation is a generalisation of this.

In fact, we have already explored this idea in the context of quantum groups in [9][10], where it is related to Hopf algebra duality. We proposed the ability to make this transformation, which reverses the role of quantum and gravitational physics, as a guiding principle for physics at the Planck scale. Now we want to touch upon these same ideas in the context of Lie algebras and their generalisations. In fact, the above remarks apply just as well to superLie algebras and the braided Lie algebras introduced in [11]. In each case the enveloping algebra can be viewed instead as a braided version of flat space. We develop this in Section 3. It provides a new way to think about the definition of Lie algebras and braided Lie algebras.

In Section 4 we focus on the example of the braided Lie algebra $gl_{2,q}$. Its enveloping algebra recovers a natural definition of $q$-Minkowski space. The quantum-geometry transformation takes the subalgebra $U_q(su_2)$ to the mass-shell in $q$-Minkowski space. The signature of the metric is also fixed as a deformation of the Lorentzian one in this approach. As far as I know, the Euclidean metric on $\mathbb{R}^4$ cannot be deformed in the same way. Thus the ability to $q$-deform spacetime provides in this way a kind of regularity principle that physics should not be too much an artifact of setting $q = 1$. This is in addition to the more usual motivation for $q$-deformation in terms of regularising infinities in physics[12] and quantum corrections to geometry.

It is hoped that this note will serve as an introduction for physicists to braided geometry and to some of its motivation. The Appendix demonstrates some of the mathematical techniques behind braided groups and braided geometry. We give a self-contained account of braided integration. This provides in principle the integration on many $q$-deformed spaces.

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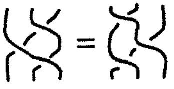
## 2. Canonical Braiding on any Lie Algebra

A braiding on a vector space $V$ is, by definition, a map $\Psi : V\otimes V \to V\otimes V$ such that

$$\Psi_{23}\circ\Psi_{12}\circ\Psi_{23} = \Psi_{12}\circ\Psi_{23}\circ\Psi_{12}, \qquad \text{i.e.} \tag{1}$$



where the suffices refer to the copy of $V$ in $V\otimes V\otimes V$. If one writes $\Psi = \times$ then this equation expresses that the two sides are topologically the same braid as shown.

The simplest example is when $V$ is $\mathbb{Z}_2$-graded and $\Psi(v\otimes w) = (-1)^{|v||w|} w\otimes v$ as in supersymmetry. Of course, in this example the exchange law is not truly braided since $\Psi^2 = \mathrm{id}$.

PROPOSITION 2.1. *Let $V = \mathbb{C}\oplus g$ and define the linear map*

$$\Psi(1\otimes 1) = 1\otimes 1, \quad \Psi(1\otimes\xi) = \xi\otimes 1, \quad \Psi(\xi\otimes 1) = 1\otimes\xi$$

$$\Psi(\xi\otimes\eta) = \eta\otimes\xi + [\xi,\eta]\otimes 1, \qquad \forall\xi,\eta\in g.$$

*Then $\Psi$ is a braiding* iff $[\ ,\ ] : g\otimes g\to g$ *obeys the Jacobi identity. It has minimal polynomial*

$$(\Psi^2 - \mathrm{id})(\Psi + \mathrm{id}) = 0 \tag{2}$$

iff $[\ ,\ ]$ *is non-zero and antisymmetric.*

This is an elementary computation. It says that the definition of a Lie algebra is mathematically completely equivalent to looking for a braiding of a certain form. We will use this principle to give a new point of view on the definition of a braided Lie algebra in the next section.

Now in the theory of supergeometry, the simplest examples of superspaces are supercommutative superalgebras. Thus for $\mathbb{R}^{n|m}$ some of the variables (the bosonic ones) commute and some (the Grassmann ones) anticommute etc. So the algebra is not commutative in the ordinary sense, but it is commutative in the super sense

$$\cdot\circ\Psi = \cdot \tag{3}$$

where $\Psi$ is included. Likewise, the universal enveloping algebra $U(g)$ for non-trivial Lie algebra $g$ is of course not commutative.

PROPOSITION 2.2. *The braiding in Proposition 2.1 extends to a braiding $\Psi : U(g)\otimes U(g) \to U(g)\otimes U(g)$ and $U(g)$ is indeed braided commutative in the sense of (3).*

The proof of this is easy enough at degree 2 for there it says that $\cdot \circ \Psi(\xi\otimes\eta) = \eta\xi + [\xi,\eta]$ is to equal $\xi\eta$, which is the defining relation of the universal enveloping algebra. So imposing the relations of braided-commutativity at order two and for the above braiding is mathematically equivalent to the usual definition of the enveloping algebra. The easiest way to prove the result to all orders is to prove it in complete generality for any Hopf algebra, of which $U(g)$ is an example with coproduct $\Delta\xi = \xi\otimes 1 + 1\otimes\xi$. If $H$ is a Hopf algebra then

$$\Psi(h\otimes g) = \sum \mathrm{Ad}_{h_{(1)}}(g)\otimes h_{(2)}, \qquad \mathrm{Ad}_h(g) = \sum h_{(1)} g S h_{(2)} \tag{4}$$

for all $h, g \in H$ is a braiding, and $H$ is braided commutative with respect to it in the sense of (3). Here $\Delta h = \sum h_{(1)}\otimes h_{(2)}$ is the coproduct of the Hopf algebra and $S$ is its antipode or 'inverse' operation.

We see that every enveloping algebra can be regarded as the algebra of functions on some braided space, and every quantum group too, with a suitable choice of braiding. This change in point of view in which an enveloping algebra gets regarded as a function algebra of some type is what we have called a quantum-geometry transformation in the introduction. Viewing a Lie algebra enveloping algebra in this way is significant for it means that the whole machinery of braided spaces and braided geometry[7], such as braided differential operators, etc can be applied. We will compute how one or two of these constructions look for our enveloping algebra.

In particular, given a braided algebra $B$ one has the braided tensor product $B\underline{\otimes}B$ between two copies[2]. This is an algebra in which the two copies do not commute but rather enjoy braid statistics. The product rule is

$$(a\otimes b)(c\otimes d) = a\Psi(b\otimes c)d \tag{5}$$

where we braid $b$ past $c$ and then multiply up. This is like the supertensor product of superalgebras. Here is an example of what this is good for:

PROPOSITION 2.3. *Let $B \doteq U(g)$ be regarded as a braided space as above. There is an algebra homomorphism $\underline{\Delta} : B \to B\underline{\otimes}B$ given by $\underline{\Delta}\xi = 1\otimes\xi - \xi\otimes 1$.*

Just as the usual coproduct corresponds to addition (e.g. of angular momentum), so this map corresponds to subtraction. In a dynamical context the usual addition provides a realisation of the centre of mass system in the tensor product of two systems, whereas the above map is more like the realisation of the reduced mass system in the (braided) tensor product. It has properties that one would expect for subtraction in relation to the addition. It also generalises to any quantum group with $\Delta(h) = Sh_{(1)} \otimes h_{(2)}$.

Now we come to a matrix version of the above results, in which we shall do a few concrete calculations. If we choose a basis $V = \{x_\mu\}$ and write

$$\Psi(x_\mu \otimes x_\nu) = x_\beta \otimes x_\alpha \mathbf{R}^\alpha{}_\mu{}^\beta{}_\nu$$

then the requirement for $\Psi$ to be a braiding is the celebrated Quantum Yang-Baxter Equation (QYBE) for $\mathbf{R}$.

Let $g = \{x_i\}$ for $i = 1, 2, \cdots, n-1$ and let $x_0 = 1$ so that $V = \mathbb{C} \oplus g$. We use greek indices when the whole range $0, \cdots, n-1$ is intended. Then the content of Proposition 2.1 is that

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & c \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \tag{6}$$

where $I$ are identity matrices and $c^i{}_{jk}$ are the structure constants of $g$. The basis for $V \otimes V$ used here is $\{x_0 \otimes x_0, x_0 \otimes x_j, x_i \otimes x_0, x_i \otimes x_j\}$. Explicitly,

$$\mathbf{R}^0{}_i{}^k{}_j = c^k{}_{ij},\ \mathbf{R}^i{}_j{}^k{}_l = \delta^i{}_j \delta^k{}_l,\ \mathbf{R}^0{}_0{}^i{}_j = \delta^i{}_j = \mathbf{R}^i{}_j{}^0{}_0,\ \mathbf{R}^0{}_0{}^0{}_0 = 1$$

and zero for the rest. This obeys the QYBE *iff* $c$ obeys the Jacobi identity.

Next, given any R-matrix, the corresponding braided space $V\check{}(R)$ is the algebra with $x_i$ and 1 as generators and relations

$$x_\mu x_\nu = x_\beta x_\alpha \mathbf{R}^\alpha{}_\mu{}^\beta{}_\nu.$$

This defines a braided version of $\mathbb{R}^n$. Such a structure arises in many areas in physics and is often called the Zamolodchikov or exchange algebra. Putting in the form of our R-matrix (6) we recover the commutation relations

$$[\lambda, x_i] = 0,\ [x_i, x_j] = \lambda x_k c^k{}_{ij}$$

so that the associated braided space is our enveloping algebra $U(g)$ in a homogenised form where we add the central element $\lambda = x_0$ on the right hand side. This is a concrete version of Proposition 2.2.

In the point of view of quantum or braided linear algebra[4], this is just one of many other constructions. If the $\{x_\mu\}$ are like a row vector, then another algebra $V(R)$ defined by generators 1 and $\{p^\mu\}$ and relations

$$\mathbf{R}^\mu{}_\alpha{}^\nu{}_\beta p^\beta p^\alpha = p^\mu p^\nu$$

is more like a column vector. For our R-matrix above, this comes out as

$$[p^\mu, p^\nu] = 0.$$

There is also a notion of braided-quantum mechanics generalising the one-dimensional case $px - qxp = \hbar$ to any R-matrix. It is generated by vector and covector algebras and cross relations

$$p^\mu x_\nu - x_\alpha \mathbf{R}^\alpha{}_\nu{}^\mu{}_\beta p^\beta = \hbar \delta^\mu{}_\nu$$

as studied by several authors[13][5]. See also the contribution of A. Kempf at this conference. For our R-matrix (6), this comes out as

$$[p^i, x_j] = \lambda c^i{}_{jk} p^k + \hbar \delta^i{}_j, \quad [p^i, \lambda] = 0, \quad [\pi, x_i] = 0, \ [\pi, \lambda] = \hbar$$

where $\pi = p^0$. Some natural $xx$ and $pp$ relations in this context are with a certain matrix $R'$ rather than $R$, for in this case (or in the free case with no $xx$ or $pp$ relations) the general machinery in [5] says that one can represent $p^\mu$ by braided differentials $\frac{\partial}{\partial x_\mu}$ in analogy with usual quantum mechanics. One can likewise compute for our R-matrix (6) all the other R-matrix constructions for quantum groups and braided groups. On the quantum group side one has for example the usual quantum matrices $A(\mathbf{R})$. This comes out essentially as a matrix of $n$ copies of the homogenised Lie algebra, one for each row, and with each copy transforming as an adjoint tensor operator with respect to the others.

Finally, we note that all the constructions above work equally well if we begin with a superLie algebra. Now the canonical braiding is

$$\Psi(\xi \otimes \eta) = (-1)^{|\xi||\eta|} \eta \otimes \xi + [\xi, \eta] \otimes 1$$

and obeys (1) *iff* [ , ] now obeys the superJacobi identity. It obeys (2) *iff* [ , ] is graded-antisymmetric. The superenveloping algebra is once again characterised by (3). More generally, if $\Psi_0$ is any other symmetric braiding in the sense that $\Psi_0^2 = \mathrm{id}$ then for

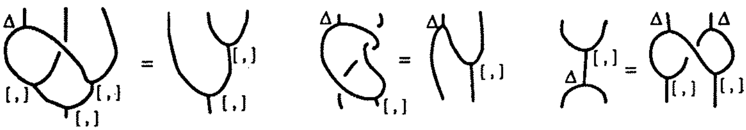
$$\Psi(\xi \otimes \eta) = \Psi_0(\xi \otimes \eta) + [\xi, \eta] \otimes 1$$

to obey (1) and (2) recovers the obvious axioms of a general $\Psi_0$-Lie algebra as in [14]. The corresponding matrix picture is

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & c \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & R_0 \end{pmatrix}$$

## 3. Braided-Lie Algebras

In this section we go beyond the super case and its obvious generalisations, to the case when our Lie algebra is of a type where the background $\Psi_0$ is itself truly braided. The axioms for such a braided Lie algebra have been introduced by the author in [11] and consist of a coalgebra $\mathcal{L}, \Delta, \epsilon$, a braiding $\Psi_0 = \times : \mathcal{L}\otimes\mathcal{L} \to \mathcal{L}\otimes\mathcal{L}$ and a map $[\ ,\ ] : \mathcal{L}\otimes\mathcal{L} \to \mathcal{L}$ such that



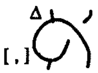

Here $\Delta : \mathcal{L} \to \mathcal{L}\otimes\mathcal{L}$ should be coassociative in an obvious sense and $\epsilon : \mathcal{L} \to \mathbb{C}$ should be a counit and obey $\epsilon \circ [\ ,\ ] = \epsilon\otimes\epsilon$. Note that an ordinary Lie algebra obeys these axioms if one puts $[1,\xi] = \xi$, $[\xi, 1] = 0$ and

$$\mathcal{L} = \mathbb{C}\oplus g, \quad \Delta 1 = 1\otimes 1,\ \epsilon 1 = 1, \quad \Delta\xi = \xi\otimes 1 + 1\otimes\xi,\ \epsilon\xi = 0.$$

So this structure $\Delta, \epsilon$ is implicit for an ordinary Lie algebra but we never think about it because it has this standard form. The same is true for superLie algebras, etc. But for examples of the truly braided type we need to take a more general form.

THEOREM 3.1. *Let $\mathcal{L}, \Delta, \epsilon$ be a coalgebra and $\Psi_0 = \times$ a compatible braiding. Then $[\ ,\ ]$ defines a braided Lie algebra implies that*

$\Psi$ =




*is a braiding. The braided enveloping algebra $U(\mathcal{L})$ is generated by* 1 *and* $\mathcal{L}$ *with the relations (3) of braided commutativity.*

The proof of this uses the same diagrammatic techniques as for braided groups[7]. We shall see some of these techniques in action in the Appendix. Here we content ourselves with the description of a general class of examples from [11]. They are of matrix type where

$$\mathcal{L} = \mathbb{C}^{n^2} = \{u^i{}_j\}, \quad \Delta u^i{}_j = u^i{}_k \otimes u^k{}_j, \quad \epsilon u^i{}_j = \delta^i{}_j.$$

The only data we need is a matrix solution $R \in M_n \otimes M_n$ of the QYBE which is bi-invertible. The 'second inverse' here is $\widetilde{R}$ and is characterised by

$$\widetilde{R}^i{}_a{}^b{}_l R^a{}_j{}^k{}_b = \delta^i{}_j \delta^k{}_l = R^i{}_a{}^b{}_l \widetilde{R}^a{}_j{}^k{}_b.$$

We write $I = (i_0, i_1)$ etc as multi-indices. Then[15][11]

$$\Psi_0(u_J \otimes u_L) = u_K \otimes u_I R_0{}^I{}_J{}^K{}_L, \quad [u_I, u_J] = u_K c^K{}_{IJ}$$

$$R_0{}^I{}_J{}^K{}_L = R^{j_0}{}_a{}^d{}_{k_0} R^{-1a}{}_{i_0}{}^{k_1}{}_b R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d$$

$$c^K{}_{IJ} = \widetilde{R}^a{}_{i_1}{}^{j_0}{}_b R^{-1b}{}_{k_0}{}^{i_0}{}_c R^{k_1}{}_e{}^c{}_d R^d{}_a{}^e{}_{j_1}$$

is a braided Lie algebra. We changed conventions here from [11] to lower indices for the $\{u_I\}$ in order to maintain compatibility with Section 2. The associated canonical braiding from Theorem 3.1 is

$$\Psi(u_J \otimes u_L) = u_K \otimes u_I \mathbf{R}^I{}_J{}^K{}_L$$

$$\mathbf{R}^I{}_J{}^K{}_L = R^{-1d}{}_{k_0}{}^{j_0}{}_a R^{k_1}{}_b{}^a{}_{i_0} R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d.$$

The braided enveloping algebra $U(\mathcal{L})$ is given by taking $\mathbf{u} = \{u^i{}_j\}$ as generators and imposing $\cdot \circ \Psi = \cdot$. So this is the algebra

$$u_J u_L = u_K u_I \mathbf{R}^I{}_J{}^K{}_L, \quad \text{i.e.} \quad R_{21}\mathbf{u}_1 R_{12}\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}_1 R_{12} \tag{7}$$

where the second puts two of the $R$'s to the left and uses a popular notation.

Our construction of braided Lie algebras works over the whole moduli space of bi-invertible solutions $R$. Inside this moduli space is a subvariety of so-called triangular solutions where $R_{21}R = 1$. On this subvariety one has $\Psi_0^2 = \mathrm{id}$ and our braided Lie algebras are not truly braided. They reduce in this case to the more obvious notion of $\Psi_0$-Lie algebras as at the end of the last section after one takes a suitable scaling limit. To see this, we parametrise $R$ in such a way that as a parameter $q \to 1$, we land on the triangular subvariety. We also change variables to $\chi_I = u_I - \delta_I$ where $\delta_I = \delta^{i_0}{}_{i_1}$. The braided enveloping algebra then looks like

$$\chi_J \chi_L - \chi_K \chi_I \mathbf{R}^I{}_J{}^K{}_L = \chi_K \left(\delta_I \mathbf{R}^I{}_J{}^K{}_L - \delta_J \delta^K{}_L\right) \tag{8}$$

and as $q \to 1$ the right hand side vanishes. But if we rescale $\chi$ to $\bar{\chi} = (q^2-1)^{-1}\chi$ say, then the effective structure constants for $\bar{\chi}$ can have a finite limit and indeed they become those of a usual, super, etc. Lie algebra depending on the point on the triangular subvariety that we are landing at. Meanwhile, the coproduct

$$\Delta\bar{\chi} = \bar{\chi} \otimes 1 + 1 \otimes \bar{\chi} + (q^2-1)\bar{\chi} \otimes \bar{\chi}, \quad \epsilon\bar{\chi} = 0$$

becomes our standard one. In this way, ordinary, super, etc. Lie algebras are the semiclassical limits of braided Lie algebras as we approach the triangular subvariety. They are therefore all unified and interpolated by our notion of braided Lie algebras. Incidentally, this shows why the classification of all solutions of the QYBE is such a hard problem: it includes the classification of all Lie algebras, superLie algebras and more generally, of braided-Lie algebras. Usual quantum enveloping algebras also fit into this picture[11].

So the braided enveloping algebra in the form (8) looks like an enveloping algebra but in the form (7) it looks like the coordinate functions on a braided commutative space. This is our quantum-geometry transformation again, in a braided form.

In fact, these quadratic algebras (7) and the matrices $R_0, \mathbf{R}$ were introduced by the author in [2] exactly as a braided analogue $B(R)$ of the algebra of functions on $M_n$. They are the *braided matrices* associated to $R$. We recall that the more well-known quantum matrices $A(R)$ have a matrix of non-commuting coordinate functions forming a bialgebra or quantum group[16]. Likewise, $B(R)$ is a braided-bialgebra or braided group. The difference is that the matrix coproduct above extends to an algebra homomorphism

$$\Delta : B(R) \to B(R)\underline{\otimes}B(R) \tag{9}$$

provided we take for $\underline{\otimes}$ the braided tensor product algebra (5). This is like the definition of a supermatrix, but with general braid statistics.

## 4. $q$-Minkowski Space

There are many approaches to what $q$-Minkowski space should be. Here we describe our own approach coming out of braided geometry[17]. Generally speaking, our approach to $q$-deforming physics is to introduce $q$ as a parameter controlling braid statistics but with the geometry otherwise remaining commutative. Since usual Minkowski space can be thought of as $2 \times 2$ hermitian matrices, we naturally propose that $q$-Minkowski space should be the algebra of $2 \times 2$ braided hermitian matrices. This is broadly compatible with the pioneering approach of [18][19], who were motivated by the possibility of spinors when defining their $q$-Lorentz group. On the other hand, we understand directly the full structure of $q$-Minkowski space first and come to the $q$-Lorentz group etc. only later as a quantum group that acts covariantly on it.

We take the well-known R-matrix associated to the Jones knot polynomial and the quantum plane,

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{10}$$

and in this case we have the braided matrix algebra $BM_q(2)$ with generators and relations computed in [2] as $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$qd + q^{-1}a \text{ central}, \ ba = q^2ab, \ ac = q^2ca, \ bc = cb + (1-q^{-2})a(d-a).$$

The braid statistics from $\Psi_0$ has $qd + q^{-1}a$ bosonic but the others mixing among themselves. The content of the braided matrix property (9) is that we can multiply two copies $\mathbf{u}, \mathbf{u}'$ as

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

provided we remember the corresponding braid statistics. We also showed in [2] that our algebra has a multiplicative braided determinant $\mathrm{BDET}(\mathbf{u}) = ad - q^2cb$. It is bosonic and central.

Next, we studied $*$-structures on braided matrices in [17]. For real $q$, we have

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

so that these matrices are naturally hermitian. One has also

$$\tau \circ (* \otimes *) \circ \Delta = \Delta \circ *$$

where $\tau$ denotes ordinary transposition. This is what one would expect since the coproduct corresponds to matrix multiplication and $(A \cdot B)^\dagger = B \cdot A$ for ordinary hermitian matrices $A, B$. We denote the braided matrix bialgebra $BM_q(2)$ with this $*$-structure by $BH_q(2)$, the algebra of *braided hermitian matrices.* Note that the situation here is in sharp contrast to the usual axioms of $*$-quantum groups, where hermitian quantum matrices cannot be formulated. BDET is self-adjoint.

All of this makes this particular algebra ideally suited to serve as $q$-Minkowski space. So we define $q$-Minkowski space as $BH_q(2)$. The generators

$$x_0 = qd + q^{-1}a, \quad x_1 = \frac{b+c}{2}, \quad x_2 = \frac{b-c}{2i}, \quad x_3 = d - a$$

are some natural self-adjoint spacetime coordinates while BDET becomes

$$\frac{q^2}{(q^2+1)^2}x_0^2 - q^2x_1^2 - q^2x_2^2 - \frac{(q^4+1)q^2}{2(q^2+1)^2}x_3^2 + \left(\frac{q^2-1}{q^2+1}\right)^2 \frac{q}{2}x_0x_3$$

and provides a real $q$-deformed Lorentz metric.

This $q$-Minkowski space has plenty of geometry associated to it, some of which we describe now. It is evident from the description of braided matrices (7) that they can be viewed if we want as a 4-dimensional row vector algebra of the same general type as the $\{x_\mu\}$ in Section 2. They therefore transform as usual under the action of the corresponding quantum matrices $A(\mathbf{R})$. Thus,

$$u_J \to u_I \Lambda^I{}_J \tag{11}$$

is an algebra homomorphism (we have a right comodule algebra) under the $4 \times 4$ matrix quantum group

$$\mathbf{R}^I{}_A{}^K{}_B \Lambda^A{}_J \Lambda^B{}_L = \Lambda^K{}_B \Lambda^I{}_A \mathbf{R}^A{}_J{}^B{}_L, \quad \Delta\Lambda^I{}_J = \Lambda^I{}_A \otimes \Lambda^A{}_J$$

This quantum group provides the basis for a $q$-Lorentz group in our picture. It has a $*$-algebra structure

$$\Lambda^I{}_J{}^* = \Lambda^{(i_1,i_0)}{}_{(j_1,j_0)}$$

and the coaction and coproduct are $*$-algebra homomorphisms. We have taken the quantum group line here because it is more familiar. There is an equally good braided Lorentz group based on $B(\mathbf{R})$ acting in the same way as a braided comodule algebra.

Moreover, the quantum Lorentz group here maps into the dual of the Drinfeld quantum double[20] with the result that our approach is indeed compatible with other proposals based on spinors[18][21]. Thus, our $A(\mathbf{R})$ can be realised in the quantum group $A(R)\bowtie A(R)$ introduced in [22] and generated by two copies of the $2\times 2$ quantum matrices. We take these in the form $\mathbf{t}\in A(R)$ and $\mathbf{t}^\dagger\in A(R_{21})$ say, with mutual relations and $*$-structure

$$t^i{}_a R^a{}_j{}^k{}_b t^{\dagger b}{}_l = t^{\dagger k}{}_b R^i{}_a{}^b{}_l t^a{}_j,\quad t^j{}_i{}^* = t^{\dagger i}{}_j,\qquad \text{i.e.,}\quad \mathbf{t}_1 R\mathbf{t}_2^\dagger = \mathbf{t}_2^\dagger R\mathbf{t}_1.$$

The abstract picture behind $A(R)\bowtie A(R)$ as a $*$-quantum group was found in [3] as well as its relation to the quantum double. One should use the inverse-transpose of the dual-quasitriangular structure found there in Proposition 12. The realisation and the resulting $2\times 2$ matrix form of the Lorentz transformation (11) is

$$\Lambda^I{}_J = t^{\dagger j_0}{}_{i_0} t^{i_1}{}_{j_1},\quad u^i{}_j\to u^a{}_b t^{\dagger i}{}_a t^b{}_j,\qquad \text{i.e.,}\quad \mathbf{u}\to\mathbf{t}^\dagger\mathbf{u}\mathbf{t}.$$

These constructions all work for any R-matrix of real type. For (10), one should think of our two copies of $2\times 2$ quantum matrices as the analogue of the complexification $SL(2,\mathbb{C})$ of $SU(2)$. Then the diagonal action $\mathbf{u}\to\mathbf{t}^{-1}\mathbf{u}\mathbf{t}$ when $\mathbf{t}$ is unitary defines an action of the quantum group $SU_q(2)$. This in turn is the double-cover of rotations, which appears here as $SO_q(3)\subset SU_q(2)$, the subHopf algebra generated by expressions quadratic in the $\mathbf{t}$.

All the usual geometrical ideas likewise go though without difficulty. For example, the mass-shell or Lorentzian sphere in $q$-Minkowski space is defined by adding the relation

$$\mathrm{BDET}(\mathbf{u}) = 1 \tag{12}$$

and is preserved under the $SO_q(3)$ action as one would expect. There are also vector fields on $q$-Minkowski space for translation[11], and for Lorentz transformation from (11). The action of the rotational vectors generates the quantum group $U_q(su_2)$ as

$$X_+\triangleright\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}-q^{\frac{3}{2}}c & -q^{\frac{1}{2}}(d-a)\\ 0 & q^{-\frac{1}{2}}c\end{pmatrix}\to[\begin{pmatrix}a & b\\ c & d\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}]$$

$$X_-\triangleright\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}q^{\frac{1}{2}}b & 0\\ q^{-\frac{1}{2}}(d-a) & -q^{-\frac{3}{2}}b\end{pmatrix}\to[\begin{pmatrix}a & b\\ c & d\end{pmatrix},\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}]$$

$$H\triangleright\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}0 & -2b\\ 2c & 0\end{pmatrix}\to[\begin{pmatrix}a & b\\ c & d\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}]$$

where the limits are as $q \to 1$ and are as one would expect.

Another interesting feature is that this mass-shell or Lorentzian sphere forms a braided group. This parallels the way that the Euclidean sphere in the $2 \times 2$ quantum matrices $M_q(2)$ is the quantum group $SU_q(2)$. The big difference is the $*$-structure or signature. In fact, this is part of a general phenomenon. Just as most familiar groups have supergoup analogues, there is a general procedure in [1] called *transmutation* which turns a quantum group into a braided group in a systematic way. The formulae at the lowest level are

$$u^i{}_j = t^i{}_j, \; u^i{}_j u^k{}_l = t^a{}_b t^d{}_l R^i{}_a{}^c{}_d \widetilde{R}^b{}_j{}^k{}_c, \quad \text{i.e.,} \quad \mathbf{u} = \mathbf{t}, \; \mathbf{u}_1 R \mathbf{u}_2 = R \mathbf{t}_1 \mathbf{t}_2$$

etc. and come out of category theory. We also gave a direct quantum groups point of view to them in [15]. Finally we found in [17] that this transmutation from quantum geometry to braided geometry also has the side-effect in general of taking us from the unitary picture (our sphere in Euclidean space) to the hermitian picture (our Lorentzian sphere). This is the abstract reason why only braided matrices and not quantum matrices can serve in the $q$-deformed picture if we want the Lorentzian signature. One does not see this constraint at $q = 1$.

More recently, U. Meyer in [23] has found an addition law for $q$-Minkowski space by introducing a new braiding suitable for the coaddition $\Delta \mathbf{u} = \mathbf{u} \otimes 1 + 1 \otimes \mathbf{u}$. The R-matrix for this braiding is different from $\mathbf{R}$ above and provides for a better $q$-Lorentz group with the quantum double appearing as its double cover. The addition law also provides for braided differential calculus according to the framework of [5] and, in principle, a translation-invariant integration as we shall see in the Appendix below.

This completes our introduction to the braided geometry of $q$-Minkowski space. On the other hand, we have seen in the last section that these braided hermitian matrices are also the braided enveloping algebra of the braided Lie algebra associated to our R-matrix. In our case this is the 4-dimensional braided Lie algebra $gl_{2,q}$. It has basis $h, x_+, x_-, \gamma$ with braided-Lie bracket

$$\begin{aligned}
&[h, x_+] = (q^{-2} + 1) q^{-2} x_+ = -q^{-2}[x_+, h] \\
&[h, x_-] = -(q^{-2} + 1) x_- = -q^2 [x_-, h] \\
&[x_+, x_-] = q^{-2} h = -[x_-, x_+] \\
&[h, h] = (q^{-4} - 1) h, \quad [\gamma, \begin{cases} h \\ x_+ \\ x_- \end{cases}] = (1 - q^{-4}) \begin{cases} h \\ x_+ \\ x_- \end{cases}
\end{aligned}$$

with zero for the others. We see that as $q \to 1$ the $\gamma$ mode decouples and we have the Lie algebra $su_2 \oplus u(1)$, but for $q \neq 1$ these are unified. There is also a braided Killing form[11] which is non-degenerate as long as $q \neq 1$. So $gl_{2,q}$ is an interesting braided-Lie algebra with potential applications in physics, such as in the unification of electroweak interactions in $q$-deformed Yang-Mills theory[24] with this as the gauge symmetry. Its $su_2$ part can also serve as differential operators of orbital angular momentum etc., along usual lines.

The quantum-geometry transformation thus connects these two conceptually quite distinct structures. Explicitly, it is

$$\begin{pmatrix} h \\ x_+ \\ x_- \\ \gamma \end{pmatrix} = (q^2-1)^{-1} \begin{pmatrix} a-d \\ c \\ b \\ q^{-2}a+d-(q^{-2}+1) \end{pmatrix}$$

and gives an isomorphism $U(gl_{2,q}) \cong BH_q(2)$. So, provided $q \neq 1$ there is only one braided group in the picture. From one point of view it is the algebra of functions on $q$-Minkowski space. From another point of view it is the enveloping algebra of a braided Lie algebra. But what we see at $q=1$ is two structures, depending on how we take the limit. If we work with $a,b,c,d$ then in the limit the algebra is the commutative algebra of functions on usual Minkowski space. If we work with $h, x_+, x_-, \gamma$ then the limit is the highly non-commutative enveloping algebra $U(su_2 \oplus u(1))$.

The quantum-geometry transform here is valid for $q \neq 1$ and maps Lie algebras and their properties to geometry. For example, what from the geometrical point of view is the mass-shell constraint (12) in $q$-Minkowski space, comes out from the Lie algebra or differential operator point of view as the quantum enveloping algebra $U_q(su_2)$. Explicitly, the quantum-geometry transform at this level becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^H & q^{-\frac{1}{2}}(q-q^{-1})q^{\frac{H}{2}}X_- \\ q^{-\frac{1}{2}}(q-q^{-1})X_+q^{\frac{H}{2}} & q^{-H}+q^{-1}(q-q^{-1})^2X_+X_- \end{pmatrix}.$$

This follows from some known results in the theory of quantum groups [16][25] by putting $\mathbf{u} = l^+Sl^-$. This connection with quantum groups is explained in full detail in [15], to which we refer the reader.

Likewise, what from the geometrical point of view is the time direction $x_0$ appears from the Lie algebra point of view as giving the $u(1)$ mode $\gamma$ which could appear in a gauge theory or which, for example, acts via [ , ] on $q$-Minkowski space by scaling of the space coordinates $\{x_i\}$. On the mass-shell it appears as the quadratic Casimir. In summary, $U(gl_{2,q})$ is both a braided enveloping algebra, such as an internal symmetry or an algebra of differential operators acting on $q$-Minkowski space, and can be identified with $q$-Minkowski space itself. Only remnants of this unification are visible when $q=1$. We have seen also that the ability to develop the $q$-deformed picture forces us from Euclidean space to Minkowski space.

We have not had room here to describe many other features of quantum and braided geometry. Notably, in [24] we introduced the theory of quantum group principal bundles and connections (gauge fields), including the example of a Dirac monopole on a $q$-sphere. Some of this machinery can be applied to $q$-Minkowski space. In short, a systematic $q$-deformed picture of the main ingredients of physics is emerging, as well as some unusual phenomena that are not very evident at the special point $q = 1$.

## Appendix. Braided Integration

In this appendix we introduce the reader to some of the mathematical techniques of braided geometry by deriving here a formula for invariant integration. This is a problem that is of current interest and which was posed a couple of times at the conference. Since quantum planes, $q$-Minkowski space and many other $q$-deformed algebras are in fact braided groups, we can apply the general theory of braided groups. There are still some difficulties in interpreting and computing the formula for integration, which we offer as a challenge for the interested reader.

Our main goal is to demonstrate some diagrammatic techniques as used for the basic properties of braided groups in [7]. We refer there for full details of the methods and notation. As well as the result here, one can also prove Theorem 3.1 and the braided version of (4) using the same techniques.

Briefly, let us recall that a braided algebra $B$ is an algebra with a braiding $\Psi = \times$ mapping $B \otimes B \to B \otimes B$. There should also be a unit element, which we view as a map $\eta : \mathbb{C} \to B$. The algebra, and indeed all our maps, should be compatible with the braiding in an obvious way. We view it as like functions on a braided space. A braided group is such a braided algebra equipped also with a coproduct $\Delta : B \to B \underline{\otimes} B$ and counit $\epsilon : B \to \mathbb{C}$. This is like the definition of a quantum group with the key difference that $B \underline{\otimes} B$ is defined with braid statistics as in (5). We saw some concrete examples in the form of the braided matrices in Sections 3 and 4. Likewise, some quantum planes are also braided groups with coaddition[3]. We are using the term 'braided group' quite loosely here. In general, there should also be an antipode $S : B \to B$ obeying axioms like the usual ones. One can also ask for some braided-commutativity as in [2] but we do not need this here.

Crucial for us is the diagrammatic notation in which $\Delta = \curlywedge$ and $\cdot = \curlyvee$. We also suppose that our braided group has a dual $B^*$ and denote the evaluation map $\mathrm{ev} : B^* \otimes B \to \mathbb{C}$ and coevaluation map $\mathrm{coev} : \mathbb{C} \to B \otimes B^*$ by $\mathrm{ev} = \cup$ and $\mathrm{coev} = \cap$. In concrete terms, ev is usual evaluation and $\mathrm{coev}(\lambda) = \lambda \sum e_a \otimes f^a$ for a basis $\{e_a\}$ and dual basis $\{f^a\}$.

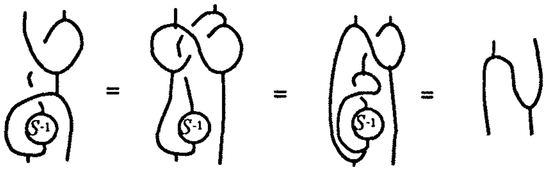
Our goal is to find a map $\int : B \to \mathbb{C}$ which assigns to a 'function' in $B$ a number, and which is translation invariant under the group law. Classically this means $\int b(h(\ )) = \int b$ for all $h$ in our group. We find correspondingly

$$\int = \underline{\mathrm{Tr}}\, L \circ S^2 =$$

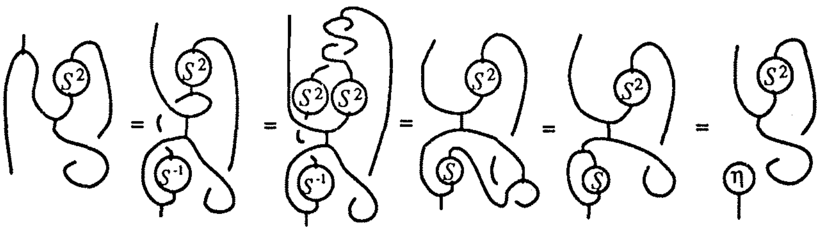

$$(\mathrm{id} \otimes \int)\Delta = 1 \int$$

where the first is our definition of $\int$ and the second is its translation-invariance property. Here $\underline{\mathrm{Tr}}$ is the braided trace as in [11] and $L$ is left multiplication, which gives the diagrammatic form shown.

A similar formula applies for ordinary quantum groups, and we will use a similar strategy of proof. We note that braided integrals have also been studied in [26] but our proof will be different. Our first step in the proof is a lemma. We assume that $S$ is invertible, then




where the first equality is the property that $\Delta$ is an algebra homomorphism to the braided tensor product algebra $B\underline{\otimes}B$. The second equality uses associativity and coassociativity of the product and coproduct. The last equality then cancels the inverse-antipode as explained in [7]. Then




where the first equality is our lemma and the second uses that $S$ is a braided antialgebra homomorphism. Now pick up the coproduct at the top of the third expression and push it down and to the left (not changing the topology), giving the fourth expression. Now we use coassociativity and cancel the antipode loop. We obtain the desired left-invariance of the integral.

Thus we have a nice formula for the invariant integral on a braided group. The braided trace plays the role of 'averaging'. The formula should, however, be viewed with care because it could easily happen that it gives identically zero or infinity and may well require a renormalisation to get a finite answer. To see the nature of this problem, let $G$ be an ordinary finite group and take a basis of delta-functions $\{\delta_g\}$. The dual basis is the the set of group elements themselves. Then the formula gives

$$\int b = \sum_g \langle g, b\delta_g \rangle = \sum_g b(g)\delta_g(g).$$

In the continuous case this gives $\delta(0)$ times the usual integral. One can evaluate the trace in any convenient basis. It would be interesting to find a suitable basis in the case of the quantum plane or $q$-Minkowski space and likewise evaluate this integral. This is a direction for further work.

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