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# Curved WDVV equation and supersymmetric mechanics

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**Abstract.** We extend the relation between the Witten-Dijkgraaf-Verlinde-Verlinde equation and $N = 4$ supersymmetric mechanics to arbitrary curved spaces. The resulting curved WDVV equation is written in terms of the third rank Codazzi tensor. We provide the solutions of the curved WDVV equation for the $so(n)$ symmetric conformally flat metrics. We also explicitly demonstrate how each solution of the flat WDVV equation can be lifted up to the curved WDVV solution on the conformally flat spaces.

## 1. Introduction

One of the areas where the famous Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation [1] and [2] plays a significant role is the $n$-particle $N = 4$ supersymmetric mechanics. Indeed, it was firstly demonstrated in [3] that if we insist that the $N = 4$ supercharges $Q^a, \overline{Q}_b,\ (a, b = 1, 2)$

$$Q^a = p_i\,\psi^{ai} + \mathrm{i}\widetilde{F}_{ijk}\,\psi^{bi}\,\psi_b^j\,\bar{\psi}^{ak} \quad \overline{Q}_a = p_i\,\bar{\psi}_a^i + \mathrm{i}\widetilde{F}_{ijk}\,\bar{\psi}_b^i\,\bar{\psi}^{bj}\,\psi_a^k \tag{1.1}$$

form the $N = 4$ super Poincaré algebra

$$\left\{Q^a, Q^b\right\} = 0,\ \left\{\overline{Q}_a, \overline{Q}_b\right\} = 0, \qquad \left\{Q^a, \overline{Q}_b\right\} = \frac{\mathrm{i}}{2}\,\delta_b^a\,H, \tag{1.2}$$

then the totally symmetric over the indices structure functions $\widetilde{F}_{ijk}(x)$ entering into supercharges (1.1) have to obey the equations

$$\partial_i\widetilde{F}_{jkl} - \partial_j\widetilde{F}_{ikl} = 0, \quad \widetilde{F}_{ikp}\delta^{pq}\widetilde{F}_{jmq} - \widetilde{F}_{jkp}\delta^{pq}\widetilde{F}_{imq} = 0. \tag{1.3}$$

The second equation in (1.3) in which the evident solution of the first equation

$$\widetilde{F}_{ijk} = \frac{\partial^3}{\partial x^i\partial x^j\partial x^k}\widetilde{F} \tag{1.4}$$

is substituted, is just the WDVV equation.

While evaluating the brackets between the supercharges in (1.2), it was supposed that the basic variables obey the standard Dirac brackets

$$\left\{x^i, p_j\right\} = \delta_j^i,\ \ \left\{\psi^{ai}, \bar{\psi}_b^j\right\} = \frac{\mathrm{i}}{2}\delta_b^a\delta^{ij}. \tag{1.5}$$

Therefore, the bosonic part of the Hamiltonian $H$ in (1.2)

$$H = \delta^{ij} p_i p_j \tag{1.6}$$

describes the motion over a flat space. Clearly, this is not the most general case and the interesting question is what will be the analog of the WDVV equations (1.3) in the case of $N = 4$ supersymmetric mechanics defined on an arbitrary space with the metric $g_{ij}$? The main goal of this Letter is to answer this question by providing the corresponding curved WDVV equation and discussing its solutions.

## 2. Supercharges and generalized WDVV equation

In the case of the spaces with the metric $g_{ij}$ the Dirac brackets between the basic variables have to be written in a covariant way as

$$\begin{aligned}
&\{x^i, p_j\} = \delta^i_j, \;\; \{\psi^{ai}, \bar\psi^j_b\} = \frac{\mathrm{i}}{2}\delta^a_b\, g^{ij}, \;\; \{p_i, \psi^{aj}\} = \Gamma^j_{ik}\psi^{ak}, \;\; \{p_i, \bar\psi^j_a\} = \Gamma^j_{ik}\bar\psi^k_a, \\
&\{p_i, p_j\} = -2\mathrm{i} R_{ijkm}\psi^{ak}\bar\psi^m_a .
\end{aligned} \tag{2.1}$$

Here, $g_{ij}$ is a metric on the bosonic space, while the Levi-Civita connection $\Gamma^k_{ij}$ and the curvature tensor $R^i{}_{jkl}$ are defined as

$$\begin{aligned}
&\Gamma^k_{ij} = \frac{1}{2}\, g^{km}\left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right), \\
&R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^m_{jl}\,\Gamma^i_{mk} - \Gamma^m_{jk}\,\Gamma^i_{ml}.
\end{aligned} \tag{2.2}$$

It should be noted that the structure of the brackets (2.1) is completely fixed by the Jacobi identities as soon as we defined the first two brackets in (2.1).

To construct the supercharges $Q^a, \overline{Q}_b$, we will use the same Ansatz (1.1) :

$$Q^a = p_i\, \psi^{ai} + \mathrm{i} F_{ijk}\, \psi^{bi}\, \psi^j_b\, \bar\psi^{ak}, \quad \overline{Q}_a = p_i\, \bar\psi^i_a + \mathrm{i} F_{ijk}\, \bar\psi^i_b\, \bar\psi^{bj}\, \psi^k_a\,, \tag{2.3}$$

where $F_{ijk}$ is an arbitrary, for the time being, function depending on the $n$-coordinates $x^i$ and is, by construction, symmetric over the first two indices. Now it is a matter of straightforward calculations to check that the conditions that these supercharges span the $N = 4$ super Poincaré algebra (1.2), with the new Dirac brackets (2.1) taken into account, result in the following equations on the function $F_{ijk}$:

$$F_{ijk} - F_{ikj} = 0, \;\Rightarrow\;\; F_{ijk} \text{ is totally symmetric over the indices} \tag{2.4}$$

$$\nabla_i F_{jkm} - \nabla_j F_{ikm} = 0, \tag{2.5}$$

$$F_{ikp} g^{pq} F_{jmq} - F_{jkp} g^{pq} F_{imq} + R_{ijkm} = 0, \tag{2.6}$$

where, as usual,

$$\nabla_i F_{jkl} = \partial_i F_{jkl} - \Gamma^m_{ij} F_{klm} - \Gamma^m_{ik} F_{jlm} - \Gamma^m_{il} F_{jkm}. \tag{2.7}$$

Once the equations (2.4)-(2.6) are satisfied, the Hamiltonian $H$ acquires the form

$$H = g^{ij} p_i p_j - 2\left[\nabla_i F_{kmj} + \nabla_m F_{ijk} + 2R_{imjk}\right]\psi^{ci}\bar\psi_c{}^m\,\psi^{dj}\bar\psi_d{}^k. \tag{2.8}$$

Equations (2.4)-(2.6) are just the curved WDVV equations we were looking for.

## 3. Particular solutions of the curved WDVV equation

The serious problem with construction of the solutions of the curved WDVV equations (2.4)-(2.6) is that even for the first equation (2.5), which defines the so called third rank Codazzi tensor, the general solution, to the best of our knowledge, is known only for the constant curvature spaces [4]

$$\begin{aligned}
F_{ijk} \;=\; & \frac{1}{3}\big(\nabla_i\nabla_j\nabla_k F + \nabla_j\nabla_i\nabla_k F + \nabla_k\nabla_i\nabla_j F\big) + \frac{4R}{3n(n-1)}\big(g_{ij}\nabla_k F + g_{ik}\nabla_j F + g_{jk}\nabla_i F\big) = \\
& \nabla_i\nabla_j\nabla_k F + \frac{R}{n(n-1)}\big(2g_{jk}\partial_i F + g_{ij}\partial_k F + g_{ik}\partial_j F\big), \quad R = g^{jk}R^i_{jik} = const.
\end{aligned} \tag{3.1}$$

The pre-potential $F$ has to be determined by the second equation (2.6). Clearly, this is a very complicated task. In what follows, we will present particular solutions of the curved WDVV equations for some special metrics $g_{ij}$.

### *3.1. Real Kähler metrics*

In a series of papers [5, 6] the many particle $N = 4$ supersymmetric mechanics was considered within the superfield formalism. One of the main results obtained in these papers is the conclusion that the resulting bosonic metrics is the real Kähler one, i.e.

$$g_{ij} = \frac{\partial^2}{\partial x^i \partial x^j} \mathcal{A}. \tag{3.2}$$

Let us analyze how this case fits in our scheme.

With the metrics (3.2), the Levi-Civita connection with all lower indices is fully symmetric, and the Riemann tensor does not contain the fourth and higher derivatives of $\mathcal{A}(x)$:

$$\Gamma_{ijk} = \frac{1}{2} \frac{\partial^3 A(x)}{\partial x^i \partial x^j \partial x^k}, \quad R_{ijkm} = g^{pq}\big(\Gamma_{imp}\Gamma_{jkq} - \Gamma_{ikp}\Gamma_{jmq}\big). \tag{3.3}$$

With such structure of the Levi-Civita connection and Riemann tensor, it is rather easy to check that the choice

$$F_{ijk} = \pm\Gamma_{ijk} \tag{3.4}$$

solves both curved WDVV equations (2.5), (2.6).

### *3.2. so(n)-invariant metrics*

Careful analysis of equation (2.5) shows that it admits a series of solutions in case of the conformally flat metrics depending on $x^2$ only, i.e. in the case when

$$g_{ij} = \frac{1}{f^2[x^2]} \delta_{ij}. \tag{3.5}$$

Indeed, starting from the following Ansatz on the $F_{ijk}$:

$$(F_1)_{ijk} = A[x^2]\delta_{ii'}\delta_{jj'}\delta_{kk'} x^{i'} x^{j'} x^{k'} + B[x^2]\left(\delta_{ij}\delta_{kk'} x^{k'} + \delta_{jk}\delta_{ii'} x^{i'} + \delta_{ik}\delta_{jj'} x^{j'}\right), \tag{3.6}$$

one may check that equation (2.5) is satisfied if the functions $A$ and $B$ obey the equation

$$A\left(2f'x^2 - f\right) + 8Bf' + 2fB' = 0. \tag{3.7}$$

Here, prime means differentiation over $x^2$.

Besides the solution (3.7), there are also solutions which generalize any solution $\widetilde{F}_{ijk}$ of the flat WDVV equations (1.3) which additionally obey the condition (see, e.g., (2.15) from [13])

$$x^i \widetilde{F}_{ijk} = \delta_{jk}. \tag{3.8}$$

This new solution of equation (2.5) reads

$$(F_2)_{ijk} = \frac{1}{f^2} \widetilde{F}_{ijk}. \tag{3.9}$$

Combining now these two solutions, one may check that the second curved WDVV equation (2.6) is satisfied by the choice

$$F_{ijk} = A\delta_{ii'}\delta_{jj'}\delta_{kk'} x^{i'} x^{j'} x^{k'} + B\left(\delta_{ij}\delta_{kk'} x^{k'} + \delta_{jk}\delta_{ii'} x^{i'} + \delta_{ik}\delta_{jj'} x^{j'}\right) + \frac{1}{f^2}\widetilde{F}_{ijk} \tag{3.10}$$

if the functions $A$ and $B$ are further constrained to obey the equations

$$A + f^2 B\left(x^2 A + B\right) = -4\frac{f''}{f^3}, \quad B\left(2 + x^2 f^2 B\right) = \frac{4f'}{f^4}\left(x^2 f' - f\right). \tag{3.11}$$

One may check that equation (3.7) is satisfied automatically if the functions $A$ and $B$ obey the equations (3.11).

Equations (3.11) may be easily solved as

$$A = -\frac{f^5 B^2 + 4f''}{f^3\left(1 + x^2 f^2 B\right)}, \quad B = -\frac{f \pm \left(f - 2x^2 f'\right)}{x^2 f^3}. \tag{3.12}$$

Thus, for the $so(n)$-invariant conformally flat metrics (3.5) the curved WDVV equations (2.4), (2.5), (2.6) admit a quite general solution (3.10).

## 4. Conclusion

In the present Letter we have used the relation between the WDVV equation and $n$-particle $N = 4$ supersymmetric mechanics to find the curved WDVV equations (2.4)-(2.6) defined on arbitrary spaces with the metric $g_{ij}$. In these generalized equations, the third derivative of the pre-potential $\widetilde{F}_{ijk} = \partial^3_{ijk}\widetilde{F}$ is replaced by the third rank Codazzi tensor $F_{ijk}$, while the WDVV equation itself acquires the non-trivial r.h.s. equal to the curvature tensor. Due to existing problems with the explicit structure of the third rank Codazzi tensors on arbitrary spaces, we have found the solutions of the curved WDVV equations on the $so(n)$-invariant conformally flat spaces. A nice feature of these solutions is that they include an arbitrary solution of the flat WDVV equation. Thus, any solution found for the flat WDVV equations in [7]-[11] (with the constraint (3.8) being satisfied) can be lifted up to the solution of the curved WDVV equations.

As concerns the applications of the obtained results to the $N = 4$ supersymmetric mechanics, the urgent task is to extend the supercharges by the potential terms (as it was done in [3], [12], [13] for the flat space case) and to find admissible potentials. This task will be considered elsewhere.

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