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Holographic complexity and two identities of action growth

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ABSTRACT

The recently proposed complexity-action conjecture allows one to calculate how fast one can produce a quantum state from a reference state in terms of the on-shell action of the dual AdS black hole at the Wheeler–DeWitt patch. We show that the action growth rate is given by the difference of the generalized enthalpy between the two corresponding horizons. The proof relies on the second identity that the surface-term contribution on a horizon is given by the product of the associated temperature and entropy.

1. Introduction

Holographic principle [1,2], the AdS/CFT correspondence [3] in particular, provides a powerful tool to study a strongly-coupled quantum theory at the boundary using a highly classical theory in the bulk. One area of research with widespread interest is relating [4–7] the quantum computational complexity [8], the minimum number of elementary operations needed to produce a state of interest from a reference state, to black hole physics. The most recent proposal is the complexity-action (CA) conjecture that the quantum complexity $\mathcal{C}$ of a boundary state is related to the corresponding bulk action $\mathcal{A}$ in the region called the Wheeler–DeWitt patch [9,10], namely

$$\mathcal{C} = \frac{\mathcal{A}}{\pi\hbar}\,. \tag{1}$$

This implies that how fast information can be stored may be computed by the growth rate of the on-shell action of the corresponding black hole.

The action of the Wheeler–DeWitt patch for anti-de Sitter (AdS) black holes is essentially evaluated over the spacetime volume between the outer and inner horizons [10]. (See [11,12] for further discussion on the global structure of the Wheeler–DeWitt patch.) The action growth for stationary AdS black holes with various charge or rotation parameters was computed [10,13]. For a variety of single-charged and/or single-rotation black holes, the answer takes the form [13]

$$\frac{d\mathcal{A}}{dt} = (M - \Omega J - \mu Q)_+ - (M - \Omega J - \mu\Omega)_-\,. \tag{2}$$

We have checked a great many further examples of AdS black holes in literature, including the static and rotating black holes in gauged STU models, and Kerr–AdS black holes with multiple rotations in general dimensions [14–22]. The general formula takes the form

$$\frac{d\mathcal{A}}{dt} = (M - \Omega^i J^i - \mu^\alpha Q^\alpha)_+ - (M - \Omega^i J^i - \mu^\alpha \Omega^\alpha)_-\,, \tag{3}$$

where the repeated indices imply summation. (In the appendix, we present two explicit examples.) The large number of examples we have checked indicate the formula is robust. The motivation of this paper is to give a formal proof. To do so, we find that the cumbersome formula (3) can be further abstracted to be

$$\frac{d\mathcal{A}}{dt} = (F + TS)_+ - (F + TS)_- = \mathcal{H}_+ - \mathcal{H}_-\,, \tag{4}$$

where F is the free energy obtained from the Euclidean action via the quantum statistic relation (QSR) [23], and $\mathcal{H} \equiv F + TS$ is the generalized enthalpy, whose terminology will be justified later.

Assuming that the QSR holds, the key to prove (4) is then the identity that the surface contribution to the action growth at each horizon is precisely the product of the associated Hawking temperature and entropy, namely

$$\frac{d\mathcal{A}^{\rm surf}}{dt}\Big|_\pm = T_\pm S_\pm\,. \tag{5}$$

The paper is organized as follows. In section 2, we establish the identities in two-derivative Einstein gravities. In section 3, we establish them in general higher-derivative gravities. We conclude the paper and give further discussions in section 4.

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2. Action growth in Einstein gravity

We begin with Einstein gravity with minimally-coupled matter in general $D = n + 1$ dimensions. The action can be expressed as $\int dt\, L$, where the Lagrangian L consists of the bulk and boundary terms. We are interested in stationary black holes for which the on-shell Lagrangian L is time-independent. In other words, we have $\frac{d\mathcal{A}}{dt} = L$, with

$$L^{\rm bulk} = \frac{1}{16\pi}\int_{\mathcal{M}} d^n x\,\mathcal{L} = \frac{1}{16\pi}\int_{\mathcal{M}} d^n x\Big(\sqrt{-g}R - \mathcal{L}^{\rm mat}\Big),$$
$$L^{\rm surf} = L^{\rm GH} + L^{\rm ct}, \qquad L^{\rm GH} = \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^{n-1}x\sqrt{-h}K. \tag{6}$$

Here $K = h^{\mu\nu}K_{\mu\nu}$ is the trace of the second fundamental form $K_{\mu\nu} = h_\mu{}^\rho\nabla_\rho n_\nu$ and $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, with n^μ being the unit vector normal to the surface [23]. (Note that the cosmological constant Λ belongs to $\mathcal{L}^{\rm mat}$ in this paper.) For asymptotically AdS backgrounds, it is also necessary to introduce the counter terms [24]

$$L^{\rm ct} = \frac{1}{16\pi}\int_{\partial\mathcal{M}} d^{n-1}\sqrt{-h}\Big[-\frac{2(n-3)}{\ell} + \frac{\ell}{(n-4)}\mathcal{R}$$
$$+ \frac{\ell^3}{(n-6)(n-4)^2}(\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} - \frac{n-2}{4(n-3)}\mathcal{R}^2) + \cdots\Big], \tag{7}$$

where $\mathcal{R}^{\mu\nu\rho\sigma}$ and its contraction denote curvatures in the boundary metric $h_{\mu\nu}$, and ℓ is the AdS radius.

The QSR states that for black holes, the on-shell Euclidean action is $I_{\rm E} = F/T$, where the temperature T is the inverse of the period of the Euclidean time, and F is the thermodynamical free energy of the black holes [23]. To be specific, the QSR implies

$$-F = \frac{1}{16\pi}\int_+^\infty d^n x\mathcal{L} + L^{\rm GH}_\infty + L^{\rm ct}_\infty. \tag{8}$$

For Euclideanized black holes, there is only one boundary, located at the asymptotic infinity. The Euclidean Killing horizon is not a boundary but the middle of the bulk.

In order to compute the action growth of a black hole, we need to evaluate it in the original Minkowski signature. Since the event horizon is not geodesically complete, we need to count also the boundary contribution on the horizon. It is clear that all the polynomial invariants of $\mathcal{R}^{\mu\nu\rho\sigma}$ in (7) are finite and hence $L^{\rm ct}$ vanishes since $\sqrt{-h}$ vanishes on the horizon. Thus the on-shell action on and out of the horizon is

$$L_+ = \frac{1}{16\pi}\int_+^\infty d^n x\,\mathcal{L} + L^{\rm surf}_\infty - L^{\rm surf}_+ = -F - L^{\rm GH}_+. \tag{9}$$

The most general near-horizon geometry up to the relevant order takes the form

$$ds^2 = V\Big(\frac{dr^2}{4\pi T(r-r_0)} - 4\pi T(r-r_0)dt^2\Big)$$
$$+ g_{ij}(dy^i - \omega^i dt)(dy^j - \omega^j dt),$$
$$V = V(y) + \mathcal{O}(r-r_0), \quad g_{ij} = g^0_{ij}(y) + \mathcal{O}(r-r_0),$$
$$\omega^i = (\omega^0)^i + \mathcal{O}(r-r_0). \tag{10}$$

It is then straightforward to evaluate that

$$L^{\rm GH}_+ = TS, \qquad S = \tfrac{1}{4}\int d^{n-1}y\sqrt{\det(g^0_{ij})}. \tag{11}$$

(The detail demonstration will be given in section 3 for the more general higher-order gravity theories.) Here S, one-quarter of the horizon area, is precisely the Bekenstein–Hawking entropy. It is rather subtle to evaluate the boundary terms on the null surfaces like horizons approaching from the inside, and new contributions on the null surfaces were introduced in [11,12]. Analogous results of (11) involving the new contributions were also obtained in [25,26]. We shall comment on our approach presently.

For black holes that have an "inner" as well as an "outer" horizon, the first law of black hole "thermodynamics" is formally valid for both horizons. To be specific, we assume that these black hole solutions have the blackening factors of the form $f(r) = (r-r_+)(r-r_-)\tilde{f}(r)$, where $r_+ > r_-$ are the two real roots and $\tilde{f}(r)$ has no further real roots in $r \in (r_-, \infty)$. The horizon condition $f(r_+) = 0$ implies an algebraic relation between r_+ and the conserved quantities including mass, angular momenta and charges. The first law can then be established provided that this algebraic relation is valid. The first law as a mathematical expression, a priori, does not "know" that $r = r_+$ is the larger root for $f(r)$ and we can thus extend the first law formula to the inner horizon as well, provided that the thermodynamical quantities are now evaluated at $r = r_-$. The same can be said also for the QSR. The physical meaning of the "thermodynamics" in the inner horizon is not clear, but this is not important for the purpose of the paper. Introducing inner horizon "thermodynamics" only serves us as an intermediate step to calculate the action growth rate, whose final answer does not rely on the concept of the inner horizon thermodynamics.

For a concrete example, the Kerr–Newman–(AdS) black hole has two horizons, and we may label the quantities associated outer and inner horizons with "+" and "−" subscripts. The first law at each horizon takes the same form

$$dM = T_\pm dS_\pm + \mu_\pm dQ + \Omega_\pm dJ + V_\pm dP. \tag{12}$$

Here the pressure $P = -(D-2)\Lambda/(16\pi)$. For this reason, M is more appropriately called the enthalpy instead of the energy of AdS black holes [27,28]. The free energy associated with each horizon for the Kerr–Newman–(AdS) black hole can be formally computed using the QSR, giving rise to the Lagrangian in Minkowski signature as

$$L_\pm = \frac{1}{16\pi}\int_\pm^\infty d^n x\,\mathcal{L} + L^{\rm surf}_\infty - L^{\rm surf}_\pm$$
$$= -F_\pm - L^{\rm GH}_\pm = -(M - \mu_\pm Q - \Omega_\pm J). \tag{13}$$

In general the formulae (9) and (11) associated with the outer horizon can be generalized to be valid for both horizons, yielding

$$L_\pm = -F_\pm - T_\pm S_\pm = -\mathcal{H}_\pm. \tag{14}$$

We refer to $\mathcal{H}$ as generalized enthalpy, since it is related to the enthalpy M by some Legendre transformation that does not involve either (T, S) or (P, V). It follows that the Lagrangian of the Wheeler–DeWitt patch is given by

$$L^{\rm WD} = L_- - L_+ = \frac{1}{16\pi}\int_-^+ d^n x\,\mathcal{L} + L^{\rm GH}_+ - L^{\rm GH}_- = \mathcal{H}_+ - \mathcal{H}_-. \tag{15}$$

We thus prove the identity (4). We now comment on our approach of evaluating the Gibbons–Hawking surface term. It follows from (10) that on the inner and outer horizons, we have $T_- < 0$ and $T_+ > 0$ respectively,[1] we choose to approach the horizon surfaces

[1] One may also adopt the convention that T is always chosen to be positive by modifying the first law, namely $dM = \pm T_\pm dS + \cdots$.

by taking the limit $r - r_\pm \to \pm 0$ respectively, in which cases the Gibbons–Hawking term is always evaluated on the time-like surfaces. It is clear that this is a smooth limit for both the bulk action and the Gibbons–Hawking term.

The conclusion holds also for theories with non-minimally coupled matter. As a concrete example, we consider the Brans–Dick theory:

$$L^{\rm bulk} = \frac{1}{16\pi}\int d^n x\sqrt{-g}\,\phi R + \cdots\,,$$
$$L^{\rm surf} = \frac{1}{8\pi}\int d^{n-1}x\sqrt{-h}\,\phi K\,. \tag{16}$$

It is then straightforward to see that on the horizon with the near-horizon geometry (10) we have

$$L^{\rm surf}_{r=r_0} = T\times\frac{\phi(r_0)}{4}\int d^{n-1}y\sqrt{\det(g^0_{ij})}\,. \tag{17}$$

This is precisely the product of the temperature and entropy, which then leads directly to statement (4).

3. Higher derivative gravities

We now consider general classes of covariant gravities that are constructed from polynomial invariants of Riemann and matter tensors. Assuming that the QSR holds for black holes in these theories, it follows from the previous discussion that the key to establish (4) is the identity (5). The proof of (5) may appear to be difficult since the entropy in the general theory is expected to be given by the Wald entropy formula [29]

$$S = -\frac{1}{8}\int\limits_{\rm horizon} d^{n-1}x\sqrt{\hat h}\,\epsilon^{ab}\epsilon^{cd}\,T_{abcd}\,,\qquad T_{abcd}\equiv\frac{\partial\hat L}{\partial R^{abcd}}\,, \tag{18}$$

where $\hat L$ is defined by the bulk Lagrangian as $L^{\rm bulk} = \int d^n x\sqrt{-g}\,\hat L$, and ϵ^{ab} is the binormal to the bifurcation surface. For spherically-symmetric black holes in Einstein–Gauss–Bonnet theory, we find that the identity (5) can be shown using the results presented in [13]. For general theories with minimally-coupled matter, the surface term was obtained [30], given by

$$L^{\rm surf} = \frac{1}{8\pi}\int\limits_{\partial\mathcal{M}} d^{n-1}x\sqrt{-h}\,\frac{\partial\hat L}{\partial R^{abcd}}\,K^{ac}n^b n^d\,. \tag{19}$$

We expect this formula may also hold for theories with non-derivative matter couplings to curvatures, since then the matter fields can be treated as constants in the relevant terms. We now substitute the near-horizon geometry (10) into the above. For simplicity, we choose a coordinate gauge where $(\omega^0)^i = 0$, i.e. non-rotating on the horizon. Approaching from the outside of the horizon, the unit vector normal to the surface is

$$n = \frac{1}{\sqrt{g_{rr}}}\frac{\partial}{\partial r}\,. \tag{20}$$

It follows that

$$K_{tt} = \tfrac12 n^r\partial_r g_{tt} = -2\pi T V n^r\,. \tag{21}$$

Thus the surface term evaluating on the horizon gives

$$\begin{aligned}L^{\rm surf} &= \frac{1}{8\pi}\int\limits_{\mathcal{M}} d^{n-1}x\sqrt{-h}K_{\mu\nu}n_\rho n_\sigma T^{\mu\rho\nu\sigma}\\ &= \frac{1}{8\pi}\int\limits_{r=r_0} d^{n-2}x\sqrt{-g_{tt}}\sqrt{\hat h}K_{tt}(n_r)^2 T^{trtr}\\ &= -\frac{1}{8}T\int\limits_{r=r_0} d^{n-2}x\sqrt{\hat h}\,2T^{\bar0\bar1\bar0\bar1} = TS\,.\end{aligned} \tag{22}$$

Here $(\bar0,\bar1)$ are the tangent indices associated with the (t,r) coordinates respectively. Note that the contributions from K_{0i} and K_{ij} terms are subleading to $\mathcal{O}(r-r_0)$ in the non-rotating frame of the horizon. The identity (4) then follows directly. (It is interesting to note that integrating over Euclidean time of $L^{\rm surf}_+$ gives precisely the entropy, providing a new method of computing the entropy.)

4. Conclusions and discussions

In this paper, we showed for general covariant theories that the bulk Lagrangian for stationary black holes within the inner and outer horizons satisfied

$$(L^{\rm bulk})^+_- \equiv \int_-^+ d^n x\,\mathcal{L}^{\rm bulk} = F_+ - F_-\,. \tag{23}$$

Furthermore, we found that the surface contribution on each horizon took the form

$$L^{\rm surf}_\pm = T_\pm S_\pm\,. \tag{24}$$

(The identities are valid for both asymptotically AdS or flat black holes.) Together, they give rise to the growth rate of the action of Wheeler–DeWitt patch, given by (4). The validity of the identities relies on the two assumptions. The first is that the QSR is valid and the second is that the Wald entropy formula correctly computes the entropy of the black hole. Both assumptions could become problematic in higher-derivative theories with non-minimally coupled derivative matters, such as Horndeski gravity [31,32]. It is of great interest to investigate these theories in this context.

One test of the CA conjecture is to compare the bound for information storage in computational science, with that of black holes, since black holes are expected to be the fastest computers [9]. By studying the thermofield double states, the bound was proposed in [9]; it can be paraphrased for the general case as

$$\frac{d\mathcal{A}}{dt} = (F+TS)_+ - (F+TS)_- \le 2(F+TS)_+ - 2(F+TS)^{\rm gs}\,, \tag{25}$$

where the superscript "gs" denotes some appropriate ground state. In the limit of the neutral and static black holes with only single horizon, the ground state is the AdS vacuum. The bound is indeed saturated by the Schwarzschild–AdS black holes. For charged or rotating black holes with two horizons, the ground state is naturally the (zero-temperature) extremal black hole of the same charge and/or angular momenta. For example, the extremal Reissner–Nordstrøm–AdS black hole with horizon radius r_0 has $\mathcal{H}^{\rm ext} = (F+TS)^{\rm ext} = -r_0^3/\ell^2$. The bound (25) however can be violated for small black holes ($r_0 \ll \ell$). One explanation given in [9] is that stringy effects may not be ignored for small black holes. We find that in the asymptotically-flat limit ($\ell\to\infty$), $H^{\rm ext} = 0$ and the bound (25) is saturated precisely. In this limit, the black hole mass M can be correctly interpreted as thermal internal energy. For AdS black holes, on the other hand, M should be interpreted as enthalpy, since the cosmological constant acts as the pressure of the system. The relation between complexity and black hole volume was discussed in [12,25]. These leads to a tantalizing possibility that the volume and pressure of the AdS black holes may play a role in resolving puzzle of violation of the complexity bound.

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Appendix A. Two explicit examples

In the introduction, we mentioned that we have checked the formula (3) for a variety of AdS black holes. In this appendix, we present two explicit examples.

A.1. $D = 5$ Kerr–AdS black holes

The $D = 5$ Kerr–AdS black holes is an Einstein metric from the action

$$\mathcal{A} = \frac{1}{16\pi}\int \sqrt{-g}\big(R + 12g^2\big)d^5x + \frac{1}{8\pi}\int_{\partial\mathcal{M}} \sqrt{-h}Kd^4x. \tag{26}$$

The solution is given by [17]. The relevant thermodynamical quantities were obtained in [33], given by

$$\begin{aligned} M &= \frac{\pi m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b)}{4\Xi_a^2\Xi_b^2}, \quad J_a = \frac{\pi ma}{2\Xi_a^2\Xi_b}, \quad J_b = \frac{\pi mb}{2\Xi_b^2\Xi_a}, \\ \Omega_a &= \frac{a(1+g^2r_+^2)}{r_+^2+a^2}, \quad \Omega_b = \frac{b(1+g^2r_+^2)}{r_+^2+b^2}, \\ S &= \frac{\pi^2(r_+^2+a^2)(r_+^2+b^2)}{2r_+\Xi_a\Xi_b} \\ \kappa &= r_+(1+g^2r_+^2)\left(\frac{1}{r_+^2+a^2} + \frac{1}{r_+^2+b^2}\right) - \frac{1}{r_+}, \quad T = \frac{\kappa}{2\pi}. \end{aligned} \tag{27}$$

The on-shell Einstein–Hilbert bulk action is

$$\mathcal{A}_{\rm EH} = \frac{1}{16\pi}\int d^5x\sqrt{-g}(-8g^2). \tag{28}$$

Thus we have

$$\frac{d\mathcal{A}_{\rm EH}}{dt} = -\left.\frac{\pi g^2r^2(a^2+b^2+r^2)}{4\Xi_a\Xi_b}\right|_{r_-}^{r_+}. \tag{29}$$

From the definition of the extrinsic curvature, its trace K can be written as

$$K = \nabla_\mu n^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}n^\mu), \tag{30}$$

where the unit normal vector is given by (20). We can obtain the growth rate of GH boundary term

$$\frac{d\mathcal{A}_{\rm GH}}{dt} = \left.\frac{\pi(2g^2r^6 + (1+g^2a^2+g^2b^2)r^4 - a^2b^2)}{4\Xi_a\Xi_br^2}\right|_{r_-}^{r_+}. \tag{31}$$

The growth rate of the total action is then

$$\frac{d\mathcal{A}}{dt} = \left.\frac{\pi(r^4(1+g^2r^2) - a^2b^2)}{4\Xi_a\Xi_br^2}\right|_{r_-}^{r_+}. \tag{32}$$

After some algebra, we have

$$\frac{d\mathcal{A}}{dt} = (\Omega_{a-} - \Omega_{a+})J_a + (\Omega_{b-} - \Omega_{b+})J_b. \tag{33}$$

A.2. R-charged AdS black hole in $D = 4$

The bulk Lagrangian of the $D = 4$ gauged STU model is given by

$$e^{-1}\mathcal{L} = \frac{1}{16\pi}\left(R - \frac{1}{4}\sum_{I=1}^{4} e^{\vec{a}_I\cdot\vec{\phi}}F_I^2 - \frac{1}{2}\sum_{i=1}^{3}(\partial\phi_i)^2 - V(\phi_i)\right), \tag{34}$$

where

$$\begin{aligned} a_1 &= (1,1,1), \quad a_2 = (1,-1,-1), \\ a_3 &= (-1,1,-1), \quad a_4 = (-1,-1,1), \end{aligned} \tag{35}$$

and

$$V(\phi_i) = -2g^2\sum_i \cosh\phi_i. \tag{36}$$

The charged AdS black hole solution was constructed in [15,16]. The relevant thermodynamic quantities are given by

$$\begin{aligned} M &= m + \frac{1}{4}\sum_{I=1}^{4} q_I, \quad Q_I = \frac{1}{4}\sqrt{q_I(q_I+2m)}, \quad \Phi_I = \frac{\sqrt{q_I(q_I+2m)}}{r_+ + q_I} \\ T &= \frac{f'(r_+)}{4\pi}\prod_{I=1}^{4} H_I^{-1/2}(r_+), \quad S = \pi\prod_{I=1}^{4}\sqrt{r_+ + q_I} \end{aligned} \tag{37}$$

The action growth rate for bulk and surface terms is each very complicated and not worth presenting here. The total action growth however is relatively simple

$$\frac{d\mathcal{A}}{dt} = (r_+ - r_-)\left(1 + \frac{g^2}{4}(2\alpha + 3\beta + 4\gamma)\right), \tag{38}$$

where

$$\begin{aligned} \alpha &= q_1q_2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3q_4 \\ \beta &= (q_1+q_2+q_3+q_4)(r_+ + r_-) \\ \gamma &= r_+^2 + r_+r_- + r_-^2 \end{aligned} \tag{39}$$

It is then simple algebra to show that the action growth can be expressed in terms of thermodynamics quantities as follows

$$\frac{dA}{dt} = \sum_{I=1}^{4}(\Phi_{I-} - \Phi_{I+})Q_I. \tag{40}$$

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