Quasinormal modes of anti-de Sitter black holes

by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Maciej Zworski, Chair
Professor John Lott
Professor William Miller

Spring 2016

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Abstract

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Quasinormal frequencies are the damped modes of oscillation associated with linearized perturbations of a black hole. In this thesis we study the distribution of these frequencies in the complex plane for a class of solutions to the Einstein equations with negative cosmological constant; these are the so-called anti-de Sitter spacetimes. Particular emphasis is placed on the family of rotating Kerr–AdS black holes.

We begin by identifying quasinormal frequencies for massive scalar perturbations as poles of a certain meromorphic family of operators — this generalized resolvent is the inverse of $P(\lambda)$, which is obtained from the Klein–Gordon operator $\Box_g + m^2$ by replacing the stationary Killing field $T = \partial_t$ with $-i\lambda \in \mathbb{C}$. In trying to prove meromorphy of $P(\lambda)$ by analytic Fredholm theory, two complications arise.

1. The ellipticity of $P(\lambda)$ degenerates at the event horizon. In fact, for rotating Kerr–AdS spacetimes, the characteristic variety of $P(\lambda)$ enters the black hole exterior. This makes it difficult to prove the Fredholm property for $P(\lambda)$ without first understanding propagation of singularities phenomena.

2. Anti-de Sitter spacetimes are characterized by the existence of a conformally timelike boundary $\mathcal{I}$ at infinity. Although $P(\lambda)$ is elliptic near $\mathcal{I}$, it does not determine an elliptic boundary value problem in the usual sense. This is because solutions to $P(\lambda)u = 0$ near the boundary are not smooth, but rather have conormal singularities. From another perspective, after freezing coefficients at a boundary point, the resulting model operator on $\mathbb{R}_+$ is a type of singular Bessel operator.

The first item is addressed by exploiting special dynamical properties of $P(\lambda)$ near the event horizon, following a general microlocal framework exposed by Vasy [95]. This makes it possible to prove coercive estimates for $P(\lambda)$ (and its adjoint) microlocally near the characteristic variety.

To prove the Fredholm property, it remains to handle a neighborhood of the conformal boundary $\mathcal{I}$. There, the resolution is formulate a version of the Lopatinskiĭ condition for

$P(\lambda)$, where the one-dimensional model equation takes into account the singular behavior near $\mathcal{I}$. Once an elliptic boundary value problem (in the sense of Bessel operators) is set up, we deduce elliptic estimates which can be glued to the estimates near the characteristic variety.

The aforementioned estimates suffice to prove that $P(\lambda)$ is Fredholm on appropriate function spaces. Using energy estimates and stationarity of the Kerr–AdS spacetime, it is possible to prove that $P(\lambda)$ is invertible for $\operatorname{Im}\lambda > 0$ sufficiently large. This provides an effective characterization of quasinormal frequencies as the discrete, finite rank poles of $P(\lambda)^{-1}$, which can then be analyzed using tools from microlocal and functional analysis.

For Kerr–AdS spacetimes, the negative cosmological constant has the effect of confining certain null geodesics to elliptic orbits near infinity. The second part of the thesis focuses on how this stable trapping affects the distribution of quasinormal frequencies in the complex plane. Our main result is the existence of sequences of quasinormal frequencies converging exponentially to the real axis. These sequences can be viewed as an obstruction to uniform local energy decay.

The first step in showing existence of quasinormal frequencies is to construct exponentially accurate quasimodes, namely real sequences $\lambda_\ell \to \infty$ as $\ell \to \infty$ and functions u_ℓ such that

$$\|P(\lambda_\ell)u_\ell\| = \mathcal{O}(e^{-\ell/C}).$$

The quasimodes we construct (for the Schwarzschild–AdS spacetime) should be thought of as quantum realizations of the the stable trapping near infinity. We then adapt results in Euclidean scattering about the existence of scattering poles generated by quasimodes. This is based on resolvent estimates for $P(\lambda)^{-1}$.

To my family.

Contents

Acknowledgments

First and foremost I would like to acknowledge my advisor Maciej Zworski for suggesting this thesis problem and his untiring mentorship throughout graduate school. I am also grateful to Semyon Dyatlov, Jeffrey Galkowski, and Peter Hintz for their willingness to discuss my work and numerous mathematical insights.

Finally, I would like to thank my family for their constant support throughout many years of schooling.

Chapter 1

Introduction

1.1 Introduction

The study of quasinormal modes (QNMs) has proven useful in understanding long-time behavior of linearized perturbations throughout general relativity. These modes are solutions of the linear wave equation with harmonic time-dependence, subject to outgoing boundary conditions at event horizons. Associated to each QNM is a complex quasinormal frequency (QNF) which determines the time evolution of a QNM: the real part describes the mode of oscillation, while the imaginary part corresponds to exponential decay or growth in time.

The QNF spectrum depends on black hole parameters (such as cosmological constant, rotation speed, and mass), but not the precise nature of the perturbation. The distribution of QNFs in the complex plane is expected to dictate the return to equilibrium for linearized perturbations. This follows established tradition in scattering theory, where QNFs typically go by the name of scattering poles or resonances.

There has been a great deal of interest in the QNMs of asymptotically anti-de Sitter black holes, motivated both by developments in the AdS/CFT program and by closely related questions in classical gravitation [50, 70, 104]. Understanding perturbations of such black holes is a common thread in both the physics and mathematics literature.

According to the proposed holographic correspondence, a black hole in an AdS background is dual to a thermal state on the conformal boundary. Behavior of perturbations in the bulk therefore yields predictions on thermalization timescales for the dual gauge theory which are difficult to calculate within the strongly coupled field theory. It is also important to note that QNMs have a distinguished interpretation in the AdS/CFT correspondence [17, 61].

Additionally, a major unsolved problem in mathematical general relativity is the nonlinear instability of global anti-de Sitter space, in the sense that a generic perturbation of such a metric will grow and form a black hole [7, 9, 10, 11, 16, 24, 26]. If AdS is indeed unstable, a natural question is whether the endpoint of instability is a Kerr–AdS black hole. Both of these subjects have motivated substantial interest in the nonlinear instability (or stability) of Kerr–AdS [17, 24, 26, 55, 57, 56, 58, 59].

This thesis studies the particular case of scalar perturbations of Kerr–AdS black holes. The relevant linear equation to be solved is the Klein–Gordon equation

$$(\Box_g + \nu^2 - 9/4)\phi = 0 \text{ on } \mathcal{M}_0, \tag{1.1}$$

where $(\mathcal{M}_0, g)$ is the exterior of a Kerr–AdS black hole, and ν is an effective mass parameter which in appropriate units satisfies the Breitenlohner–Freedman unitarity bound $\nu > 0$. Our first goal is to provide a robust definition of QNFs for Kerr–AdS metrics which does not depend on any extra symmetries (separation of variables), and then show that the QNF spectrum forms a discrete subset of the complex plane. This means studying solutions to (1.1) of the form $\phi = e^{-i\lambda t^\star}u$, where $\lambda \in \mathbb{C}$ and u is a function on the time slice $\{t^\star = 0\}$ (here $t^\star$ is a time coordinate which is regular across the event horizon). The critical observation is that the outgoing condition is equivalent to a certain smoothness requirement for u at the event horizon.

We use recent advances in the microlocal study of wave equations on black hole backgrounds due to Vasy [95] to study global Fredholm properties of the time-independent operator

$$P(\lambda) = e^{i\lambda t^\star}(\Box_g + \nu^2 - 9/4)e^{-i\lambda t^\star}.$$

Upon verifying some dynamical assumptions on the null-geodesic flow of Kerr–AdS metrics, the approach of [95] provides certain estimates for $P(\lambda)$, at least away from the conformal boundary; there is no restriction on the rotation speed of the black hole.

Since the conformal boundary $\mathcal{I}$ on an asymptotically AdS spacetime is timelike, there is no reason for the set of QNFs to be discrete unless (1.1) is augmented by boundary conditions at $\mathcal{I}$. Choosing appropriate boundary conditions is a subtle point, depending on the effective mass ν. When $\nu \geq 1$, it suffices to rule out solutions which grow too rapidly near the conformal boundary. On the other hand, when $0 < \nu < 1$ the problem is underdetermined and boundary conditions must be imposed.

Furthermore, $P(\lambda)$ does not give rise to an elliptic boundary value problem in the usual sense, even if $0 < \nu < 1$. This is because $P(\lambda)$ has a type of Bessel structure, and solutions to the equation $P(\lambda)u = 0$ have only conormal regularity near $\mathcal{I}$. Another aim of this work is to develop a theory of boundary value problems for some Bessel-type operators. This theory applies to a large class of asymptotically anti-de Sitter spacetimes. Provided the boundary conditions satisfy a type of Lopatinskiĭ condition when $0 < \nu < 1$, we obtain elliptic estimates near the boundary. These boundary conditions account for the majority of those considered in the physics literature [3, 8, 13, 14, 25, 65, 105]. In particular, certain time-periodic boundary conditions are admissible.

Combining estimates near the boundary with those in the interior suffices to prove the Fredholm property for the stationary operator. The inverse of this operator forms a meromorphic family, and QNFs are then defined as poles of that family, with (finite) multiplicities given by the ranks of residues.

Next, we study the distribution of QNFs in the complex plane. Here, our thinking is guided by the existence of stably trapped geodesics near $\mathcal{I}$. In scattering problems, it is expected that trapping of classical trajectories should produce QNFs (or scattering poles or resonances) converging to the real axis; this is often referred to as the Lax-Phillips conjecture, see [72, Section V.3]. When trapping is weak, for instance in the sense of hyperbolicity, the general conjecture is not true as shown by Ikawa [63].

When the trapping is sufficiently strong that a construction of real quasimodes is possible, the works of Stefanov–Vodev [93], Tang–Zworski [94], and Stefanov [92] do show that there exist resonances close to the quasimodes. In our setting, (exponentially accurate) quasimodes refer to functions u_ℓ such that

$$\|P(\lambda_\ell)u_\ell\| = \mathcal{O}(e^{-\ell/C})$$

for a sequence of real frequencies $\lambda_\ell \to \infty$. These results were established in Euclidean scattering for compactly supported perturbations of the Laplacian, or more generally for perturbations which are dilation analytic near infinity [89, 90]. Another goal of this thesis is to prove an analogue of [94] for Kerr–AdS spacetimes: given any reasonable sequence of quasimodes, we would like to conclude the existence of QNFs nearby. This is achieved in Chapter 6.

To prove existence of QNFs converging exponentially to the real axis for rotating Kerr–AdS, we apply our result to the quasimodes found by Holzegel–Smulevici [56]. Their original motivation was not directly motivated by QNMs; instead, they constructed quasimodes to study lower bounds on uniform energy decay.

One downside is that these quasimodes lack a good description of their frequencies. In the non-rotating Schwarzschild–AdS case one can use an independent construction of quasimodes due to the author, where the spherical symmetry allows for precise spectral asymptotics. This refined construction is carried out in Chapter 5.

As a historical note, existence of QNFs converging to the real axis was first observed for the Schwarzschild–AdS solution through numerical and formal WKB analysis [26, 37]. In addition, the asymptotic relation between these QNFs and the spectrum of global AdS provides a link between the conjectured nonlinear instability of global AdS and that of Kerr–AdS [7, 9, 11, 10, 16, 21, 22, 26, 24, 55].

For the Kerr–AdS solution, it is more difficult to demonstrate the existence of long-lived QNMs. Due to the more complicated structure of the separated equations, a WKB analysis is harder to perform. Futhermore, the author is not aware of any numerical studies of QNFs for Kerr–AdS in the high frequency limit. Nevertheless, we are able to exhibit such QNFs for Kerr–AdS spacetimes without restrictions on the rotation speed.

Quasimodes, constructed either for Kerr–AdS or Schwarzschild–AdS, are quantum manifestations of elliptic trapping near infinity. A classical result of Ralston [83] (in obstacle scattering) shows that localized quasimodes of this type are an obstruction to uniform local energy decay. In fact, with a loss of finitely many derivatives the best decay rate one can obtain is logarithmic. This was proved for Kerr–AdS spacetimes in [56]. Since logarithmic upper bounds were previously established for Kerr–AdS [55], such a decay rate is more or less optimal; this slow decay lead to the original conjecture in [55] that Kerr–AdS is nonlinearly unstable.

Outline of the thesis

Here we briefly describe the contents of this thesis.

Chapter 2: In this chapter we define stationary, asymptotically anti-de Sitter (aAdS) spacetimes. We discuss global AdS space, as well as the Schwarzschild–AdS and Kerr–AdS families of black holes. In terms of a canonical product decomposition near the conformal boundary, we write down an expression for the stationary Klein–Gordon operator on an aAdS spacetime. This motivates the class of singular boundary value problems studied in Chapter 3.

Chapter 3: This chapter considers boundary value problems for a class of singular elliptic operators which appear naturally in the study of aAdS spacetimes. After formulating a Lopatinskiĭ–type condition, elliptic estimates are established near the boundary. The Fredholm property follows from additional hypotheses in the interior. This provides a rigorous framework for the mode analysis of aAdS spacetimes for the full range of boundary conditions considered in the physics literature. Completeness of eigenfunctions for some Bessel operator pencils with a spectral parameter in the boundary condition is shown, which has applications to the linear stability of certain aAdS spacetimes.

Chapter 4: The quasinormal frequencies of massive scalar fields on Kerr-AdS black holes are identified with poles of a certain meromorphic family of operators, once boundary conditions are specified at the conformal boundary. Consequently, the quasinormal frequencies form a discrete subset of the complex plane and the corresponding poles are of finite rank. This result holds for a broad class of elliptic boundary conditions, with no restrictions on the rotation speed of the black hole.

Chapter 5: In this chapter we produce quasimodes for the Klein–Gordon operator on Schwarzschild–AdS backgrounds subject to Dirichlet boundary conditions. This is done by considering an auxiliary eigenvalue problem whose eigenfunctions are concentrated near the conformal boundary. Because of the spherical symmetry, we are able to give a precise description of the frequencies at which these quasimodes exist.

Chapter 6: We construct an approximate inverse for $P(\lambda)$ modulo errors of Schatten-class. This allows us to estimate $P(\lambda)^{-1}$ away from QNFs. We also prove “self-adjoint” estimates in the upper half-plane using energy estimates, even though $P(\lambda)$ is far from self-adjoint. We then deduce the existence of QNFs converging exponentially to the real axis for Kerr–AdS spacetimes using the quasimode construction of [56]. Much more precise spectral asymptotics are available from our results in Chapter 5, provided one works with non-rotating black holes.

Previous work

Much of this thesis is contained in the author's previous papers and preprints. Chapters 3, 4, 5, and 6 are based on [40], [39], [42], and [41], respectively.

Chapter 2

Asymptotically anti-de Sitter spacetimes

This chapter introduces the notion of a stationary, asymptotically anti-de Sitter (aAdS) spacetime. The first example is global anti-de Sitter spacetime itself, whose metric behavior near infinity is the model for aAdS spacetimes in general.

To study the general case we review the ADM decomposition of a stationary Lorentzian metric and its d'Alembertian with respect to a foliation of the spacetime by spacelike hypersurfaces. The stationary d'Alembertian (depending on a complex spectral parameter) is defined by a Fourier transform, and formally reduces the wave equation to a spectral problem on a fixed time slice.

Next, we define stationary aAdS metrics and find a canonical product decomposition near the conformal boundary. In general the stationary Klein–Gordon equation is not an elliptic boundary value problem of the usual kind (apart from certain values of the Klein–Gordon mass). Instead, the operator has a Bessel structure in the conormal direction. This motivates an extension to Bessel operators of the elliptic boundary value problem apparatus studied in Chapter 3.

Finally, we define the Schwarzschild–AdS and Kerr–AdS families of spacetimes. These are solutions to the Einstein equations with negative cosmological constant which contain both an event horizon and an aAdS end. Spectral properties of the stationary Klein–Gordon operator on these spacetimes are studied in detail in Chapters 4, 5, 6.

2.1 Anti-de Sitter space

Hyperboloid model

Anti-de Sitter (AdS) spacetime is the unique maximally symmetric solution of the vacuum Einstein equation

$$\mathrm{Ric}_g + \Lambda g = 0$$

with negative cosmological constant $\Lambda < 0$. Equip $\mathbb{R}^{2,d} = \mathbb{R}^{d+2}$ with the metric

$$g_{\mathbb{R}^{2,d}} = -dz_1^2 - \cdots - dz_d^2 + dz_{d+1}^2 + dz_{d+2}^2$$

of signature $(2, d)$, where $(z_1, \ldots, z_{d+2})$ are Euclidean coordinates on $\mathbb{R}^{d+2}$. AdS_{d+1} is the hyperboloid

$$z_1^2 + \ldots + z_d^2 - z_{d+1}^2 - z_{d+2}^2 = -l^2$$

of radius $l > 0$ embedded in $\mathbb{R}^{2,d}$ with the metric induced by $g_{\mathbb{R}^{2,d}}$. The AdS radius l is related to the negative cosmological constant by the formula

$$\Lambda = -\frac{d(d-1)}{2l^2}. \tag{2.1}$$

Since $z_{d+1}^2 + z_{d+2}^2 \geq l^2$ on the hyperboloid, these last two variables can be written globally in polar coordinates

$$(z_{d+1}, z_{d+2}) = l \cdot Rt,$$

where $R \geq 1$ and $t \in \mathbb{S}^1$. This gives an embedding of AdS_{d+1} into $\mathbb{R}^n \times (0, \infty)_R \times \mathbb{S}^1_t$ as the hyperboloid

$$z_1^2 + \cdots + z_d^2 - l^2 R^2 = -l^2.$$

Moreover, the map $\mathbb{R}^d \times \mathbb{S}^1_t \to \mathbb{R}^d \times (0, \infty)_R \times \mathbb{S}^1_t$ given by

$$(z_1, \ldots, z_d, t) \mapsto (z_1, \ldots, z_d, R, t), \quad R^2 = \frac{z_1^2 + \ldots + z_d^2}{l^2} + 1 \tag{2.2}$$

is a diffeomorphism from $\mathbb{R}^d \times \mathbb{S}^1_t$ onto AdS_{d+1}.

Universal cover

In view of its topology, there exist closed timelike curves on AdS_{d+1}. For instance, parameterizing $t \in \mathbb{S}^1$ by $t(\alpha) = (\cos\alpha, \sin\alpha)$ yields the 2π-periodic timelike curve

$$\alpha \mapsto (0, \ldots, 0, t(\alpha)) \in \mathbb{R}^d \times \mathbb{S}^1_t.$$

From the perspective of evolution problems such as the wave equation, the existence of such curves is problematic. This can be avoided by working with the universal cover CAdS_{d+1}, obtained from AdS_{d+1} by replacing $t \in \mathbb{S}^1$ with $t \in \mathbb{R}$. Since we will only consider CAdS_{d+1}, henceforth AdS_{d+1} will be used to refer to this covering space, which has the topology $\mathbb{R}^d \times \mathbb{R}_t$.

Global coordinates

Away from the origin, $\mathbb{R}^d \setminus 0$ can be described by an additional set of polar coordinates

$$(z_1, \ldots, z_d) = r\omega,$$

where $r \in (0, \infty)$ and $\omega \in \mathbb{S}^{d-1}$. If R is given by (2.2), then $R^2 = (r/l)^2 + 1$. The AdS_{d+1} metric is given in (r, ω, t) coordinates by

$$g_{\mathrm{AdS}} = -\left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + \left(1 + \frac{r^2}{l^2}\right) dt^2 - r^2 d\omega^2.$$

These coordinates are typically referred to as *global coordinates*, although they have the typical degeneration at $r = 0$. The Lorentzian nature of g_{AdS} is evident in this coordinate system.

Bordification

AdS_{d+1} can be bordified by first identifying it with $\mathbb{R}^d \times \mathbb{R}_t$ via (2.2), and then radially compactifying $\mathbb{R}^d$ to a closed ball $\overline{\mathbb{R}^d}$ by gluing a sphere $\mathbb{S}^{d-1}$ at radial infinity. The function

$$s = \left(z_1^2 + \ldots + z_d^2\right)^{1/2}$$

is a boundary defining function for $\partial\overline{\mathbb{R}^d}$, and inverse polar coordinates $[0,1)_s \times \mathbb{S}^{d-1}_\omega \to \overline{\mathbb{R}^d}$ given by

$$(s, \omega) \mapsto s^{-1}\omega$$

gives a collar neighborhood of $\partial\overline{\mathbb{R}^d}$. The resulting bordification $\overline{\mathrm{AdS}}_{n+1}$ is diffeomorphic to $\overline{\mathbb{R}^d} \times \mathbb{R}_t$, and its boundary

$$\mathcal{I} = \partial\, \mathrm{AdS}_{d+1}$$

is diffeomorphic to $\mathbb{S}^{d-1} \times \mathbb{R}_t$. In (s, ω, t) coordinates, the metric can be rewritten in the form

$$g_{\mathrm{AdS}} = \frac{1}{s^2}\left(-(s^2 + l^{-2})^{-1}\, ds^2 + (s^2 + l^{-2})\, dt^2 - d\omega^2\right).$$

Observe that the conformal multiple $\bar{g}_{\mathrm{AdS}} = s^2 g_{\mathrm{AdS}}$ extends smoothly from AdS_{d+1} to $\overline{\mathrm{AdS}}_{n+1}$. Furthermore, the restriction of $\bar{g}_{\mathrm{AdS}}$ to the boundary is

$$l^{-2} dt^2 - d\omega^2,$$

which is again of Lorentzian signature; consequently the boundary is said to be conformally timelike.

Geodesic motion

AdS_{d+1} has the property that certain null-geodesics reach $\mathcal{I}$ in finite coordinate time as measured by t. Using spherical coordinates $(\theta_1, \ldots, \theta_{d-2}, \phi)$ on $\mathbb{S}^{d-1}$, we can restrict our attention to the equatorial set $\theta_1 = \ldots = \theta_{d-2} = \pi/2$. Corresponding to the Killing fields $T = \partial_t$ and $\Phi = \partial_\phi$ are the usual constants of motion along a null-geodesic γ:

$$E = g(T, \dot\gamma) = \left(1 + \frac{r^2}{l^2}\right) \dot t, \quad L = g(\Phi, \dot\gamma) = r^2 \dot\phi.$$

Combined with $g(\dot\gamma,\dot\gamma)=0$, this yields the equation

$$(\dot r)^2 = E^2 - \left(1+\frac{r^2}{l^2}\right)L^2$$

Since T is timelike and $\dot\gamma$ null, we have $E \geq 0$. If $L=0$ and $E>0$, there are radial null-geodesics with $r(\lambda)=E\lambda$ along an affine parameter λ. Plugging this into the equation for $\dot t$ gives

$$t = l\arctan(E\lambda).$$

In particular t approaches the finite value $\pi/2$ as $r\to\infty$. This indicates that boundary conditions must be specified along $\mathcal{I}$ in order to specify the evolution of, say, a wave propagating in AdS_{d+1}.

Wave equation

The d'Alembert operator in global coordinates is given by

$$\Box_{g_{\mathrm{AdS}}} = r^{1-d}D_r(r^{d-1}f(r)D_r) - r^{-2}\Delta_\omega - f(r)^{-1}D_t^2,$$

where $f(r)=1+(r/l)^2$ and Δ_ω is the nonpositive Laplacian on $\mathbb{S}^{d-1}$. The Klein–Gordon equation

$$\left(\Box_g + m^2\right)\phi = 0$$

can formally be analyzed by separation of variables. Let

$$\phi(t,r,\omega) = e^{-i\lambda t}\cdot r^{(d-1)/2}u(r)\cdot S_{m\ell}(\omega),$$

where $S_{m\ell}$ is a spherical harmonic on $\mathbb{S}^{d-1}$ with eigenvalue $\ell(\ell+d-2)$. Defining ν,σ by

$$\nu^2 = m^2 + d^2/4, \quad \sigma^2 = (2\ell+d-2)^2/4,$$

the equation $(\Box_g+m^2)\,\phi=0$ is equivalent to $(P_\ell-\lambda^2)u=0$, where

$$P_\ell = (fD_r)^2 + (\sigma^2-1/4)r^{-2}f + (\nu^2-1/4)f.$$

Now make the change of variables $r = l\cot z$, where $z\in(0,\pi/2)$. Then

$$P_\ell = D_z^2 + \frac{\sigma^2-1/4}{\cos^2 z} + \frac{\nu^2-1/4}{\sin^2 z},$$

which is a Schrödinger operator with potential on the interval $(0,\pi/2)$. Observe the symmetric form of P_ℓ at the endpoints $z=0,\pi/2$. Each of these endpoints is a regular singular point for P_ℓ, and the associated indicial roots are $1/2\pm\nu$ at $z=0$, and $1/2\pm\sigma$ at $z=\pi/2$. Elliptic theory for differential operators with this type of inverse-square singularity is studied in Chapter 3.

It should noted that the singularities at $z = 0, \pi/2$ have different interpretations. At $z = \pi/2$ (which corresponds to $r = 0$) the original metric is smooth. Since T is timelike, this means that the conjugated operator

$$e^{i\lambda t}\left(\Box_g + m^2\right)e^{-i\lambda t},$$

viewed as a differential operator on $\{t = 0\}$, is elliptic. By elliptic regularity, the solution

$$v(r,\omega) = r^{(d-1)/2}u(r)\cdot S_{m\ell}(\omega)$$

to the equation $\left(\Box_g + m^2\right)e^{i\lambda t}v = 0$ must be smooth up to $r = 0$. This means that if

$$u(r) \sim c_-\,\mathrm{arccot}(r/l)^{1/2-\sigma} + c_+\,\mathrm{arccot}(r/l)^{1/2+\sigma},$$

then $c_- = 0$. Indeed the function $r^{(d-1)/2}\,\mathrm{arccot}(r/l)^{1/2-\sigma}$ on $\mathbb{R}^d\setminus 0$ does not extend smoothly to $\mathbb{R}^d$ for any value of σ as above (with $\ell \geq 0$). This can be seen by repeatedly applying the Laplacian in polar coordinates.

On the other hand, there is no a priori reason to exclude the branch $z^{1/2-\nu}$ near $\mathcal{I}$ (corresponding to $z = 0$). It turns out that for $\nu \geq 1$, requiring solutions to have finite energy with respect to the geometric stress-energy tensor does indeed rule out this branch. This is not the case if $\nu \in (0,1)$, where it is possible to make sense of the energy for solutions behaving as $z^{1/2-\nu}$ — see Section 6.3. The dichotomy between $\nu \geq 1$ and $\nu \in (0,1)$ will be present throughout this thesis.

2.2 Stationary spacetimes and ADM formalism

Before defining asymptotically AdS spaces, we review some facts about stationary spacetimes and their ADM decompositions. This will allows us to write down the spectral family for the Klein–Gordon equation on an arbitrary stationary spacetime in a convenient form.

Foliations

Let (M,g) be a $d+1$ dimensional Lorentzian manifold with signature $(+,-,\ldots,-)$. Later we will consider manifolds with boundary, but for now it is safe to assume that $\partial M = \emptyset$. Suppose that there exists complete Killing vector field T along with a hypersurface $X \subseteq M$ such that

1. each integral curve of T intersects X exactly once,
2. X is spacelike with respect to g.

For any X with these properties, M is smoothly foliated the spacelike hypersurfaces

$$X_t = \exp(tT)(X), \quad \tau \in \mathbb{R},$$

where $\exp(tT) : M \to M$ is the flow of T. In particular, M is diffeomorphic to $\mathbb{R}_t \times X$ via the map

$$(t,p) \mapsto \exp(tT)p, \quad (t,p) \in \mathbb{R} \times X.$$

In this way the parameter $t \in \mathbb{R}$ along the flow is identified with a temporal function $\mathfrak{t} : M \to \mathbb{R}$ defined by

$$\mathfrak{t}(t,p) = t, \quad (t,p) \in \mathbb{R} \times X.$$

Here, temporal means that $d\mathfrak{t}$ is everywhere timelike, which is clear since $X_t = \{\mathfrak{t} = t\}$ and X_t is. Since T is Killing, M is said to be a stationary spacetime. For the remainder of this section we simply write $t = \mathfrak{t}$.

ADM decomposition

Let N_t denote the unit normal to X. Since dt is normal to X,

$$N_t = A\, dt^\sharp, \quad A = g^{-1}(dt, dt)^{-1/2},$$

where $A > 0$ is called the lapse function. Although T is not necessarily normal to X, it is transversal by definition — therefore there exists a vector field W tangent to X, called the shift vector, such that

$$T = AN_t + W. \tag{2.3}$$

This also shows that $A = g(T, N_t)$. Let h denote the (positive definite) metric induced on X. Under our identification

$$TM = T\mathbb{R}_t \oplus TX,$$

where $T\mathbb{R}_t$ is spanned by T. Then the metric g is determined by

$$\begin{cases} g(T,T) = -h(W,W) + A^2, \\ g(T,V) = -h(W,V), \\ g(V,V) = -h(V,V) \end{cases} \tag{2.4}$$

for any $V \in TX$. Conversely, suppose we are given M diffeomorphic to $\mathbb{R}_t \times X$ for some manifold X. Fix a function $A > 0$, along with a vector field W and Riemannian metric h on X. Then the data (A, h, W), lifted to M, uniquely determine a stationary metric g on M via the formulas (6.3).

If (x^i) are local coordinates on X, then (t, x^i) provide local coordinates on M, and $T = \partial_t$. In these coordinates,

$$g = A^2\, dt^2 - h_{ij}(dx^i + W^i\, dt)(dx^j + W^j\, dt), \tag{2.5}$$

where lowercase Latin letters denote spatial indices on X. The dual metric is neatly expressed in a coordinate free manner by

$$g^{-1} = A^{-2}\,(T - W) \otimes (T - W) - h^{-1}.$$

Furthermore, the determinant of g is related to that of h by the formula $|\det g| = A^2 \cdot \det h$.

d'Alembert operator

The d'Alembert operator can be written in the form

$$\Box_g = P_0 + P_1 \cdot D_t + P_2 \cdot D_t^2$$

where P_k is a differential operator of order $2-k$ on X. In local (t, x^i) coordinates:

$$\begin{aligned}\Box_g = &- A^{-2} D_t^2 + A^{-2}\, W^i D_i D_t + A^{-1}(\det h)^{-1/2}\, D_i \left(A^{-1}(\det h)^{1/2}\, W^i\, D_t\right)\\ &+ A^{-1}(\det h)^{-1/2}(D_i(A^{-1}(\det h)^{1/2}(h^{ij} - A^{-2} W^i W^j))D_j).\end{aligned}$$

Therefore

$$\begin{cases} P_1 = -iA^{-2}(W + W^*) - iA^{-1}W(A^{-1}), \\ P_2 = -A^{-2}, \end{cases} \tag{2.6}$$

where W^* is the formal adjoint of W with respect to the volume form induced by h on X. Writing down a coordinate-free expression for P_0 is not as nice, but its coordinate expression is

$$P_0 = A^{-1}(\det h)^{-1/2}(D_i(A^{-1}(\det h)^{1/2}(h^{ij} - A^{-2} W^i W^j))D_j).$$

We are also interested in the Klein–Gordon operator $\Box_g + m^2$.

Hyperbolicity

The spacelike nature of X implies strict hyperbolicity of $\Box_g$ with respect to the hypersurfaces X_t [60, Chapter 24]. Begin by letting

$$G(p, \zeta) = g^{-1}(\zeta, \zeta), \quad \zeta \in T_p^* M,$$

which is the principal symbol of $-\Box_g$ as a second order operator on M. If $\zeta \in T^*M \setminus 0$ is not collinear to dt, then the quadratic polynomial

$$\tau \mapsto G(p, \zeta + \tau\, dt)$$

has two real distinct roots as can be seen by writing ζ as the sum of a vector in $\mathbb{R} \cdot dt$ and its g-orthocomplement; this is the definition of strict hyperbolicity. In particular, $dG \neq 0$ on Σ, where

$$\Sigma = \{G = 0\} \setminus 0$$

is the lightcone or characteristic set. Thus Σ is a smooth, codimension one, conic submanifold of $T^*M \setminus 0$. Furthermore,

$$\Sigma \cap \{g^{-1}(\zeta, dt) = 0\} = \emptyset$$

since dt is timelike and hence G is nonpositive on the latter hyperplane. This implies that Σ is a disjoint union $\Sigma = \Sigma_+ \cup \Sigma_-$ of the future and past directed lightcones

$$\Sigma_\pm = \Sigma \cap \{\mp g^{-1}(\zeta, dt) > 0\}.$$

The apparently counterintuitive choice of sign is made for consistency purposes later on. With the identification $T^*M = T^*\mathbb{R}_t \oplus T^*X$, where $T^*\mathbb{R}_t$ is spanned by dt, any covector $\xi \in T^*M$ can be written uniquely as

$$\zeta = \xi + \tau\, dt \tag{2.7}$$

with $\xi \in T^*X$ and $\tau \in \mathbb{R}$. In coordinates,

$$G(p, \xi + \tau\, dt) = A^{-2}\tau^2 - 2A^{-2}W^i\xi_i\tau + \left(A^{-2}W^iW^j - h^{ij}\right)\xi_i\xi_j,$$

where we write $\xi = \xi_i\, dx^i$.

Lemma 2.2.1. *If T is timelike at $p \in M$ and $\langle T, \zeta\rangle = 0$, then $G(p,\zeta) < 0$ for each $\zeta \in T_p^*M \setminus 0$.*

Proof. The condition $\langle T, \zeta\rangle = 0$ implies that $\zeta \in T_p^*M \setminus 0$ is orthogonal to a timelike vector, hence G is negative definite on $\mathbb{R}\cdot\zeta$. □

From a different perspective, the condition $\langle T, \zeta\rangle$ implies $\tau = 0$ in (2.7), hence $G(p,\zeta) < 0$ is equivalent to $A^{-2}W \otimes W - h^{-1}$ being negative definite when T is timelike. Next, we let τ assume complex values.

Lemma 2.2.2. *If $\operatorname{Im}\tau \neq 0$, then $G(p, \zeta + \tau\, dt) \neq 0$ for each $\zeta \in T_p^*M$.*

Proof. By homogeneity it suffices to assume $\tau = i$, in which case

$$G(p, \zeta + i\, dt) = G(p,\zeta) - G(p, dt) + 2ig^{-1}(\zeta, dt)$$

If $\operatorname{Im} G(p, \zeta + i\, dt) = 0$, then $\operatorname{Re} G(p, \zeta + i\, dt) < 0$ since dt is timelike. □

The stationary operator

Since $\Box_g$ commutes with T, we may define the stationary d'Alembertian $\widehat{\Box}_g(\lambda)$ as an operator on Σ by the formula

$$\widehat{\Box}_g(\lambda)u = e^{i\lambda t}\,\Box_g\left(e^{-i\lambda t}u\right), \quad u \in \mathcal{D}'(\Sigma)$$

depending on the parameter $\lambda \in \mathbb{C}$. In terms of the representation (2.6), this amounts to replacing D_t with $-\lambda \in \mathbb{C}$,

$$\widehat{\Box}_g(\lambda) = P_0 - \lambda P_1 + \lambda^2 P_2.$$

Properties of $\widehat{\Box}_g(\lambda)$, or more generally the stationary Klein–Gordon operator $\widehat{\Box}_g(\lambda) + m^2$, occupy the central part of this thesis. In coordinates, the principal symbol of $\widehat{\Box}_g(\lambda)$ in the sense of standard microlocal analysis is

$$p_0(x,\xi) = \left(h^{ij} - A^{-2}W^iW^j\right)\xi_i\xi_j,$$

where $\xi = \xi_i \, dx^i \in T^*X$. This corresponds to $-G(x, \xi + 0 \cdot dt)$. In view of the Lemma 2.2.1,

$$p_0(x, \xi) > 0$$

for $\xi \in T^*_x X \setminus 0$ whenever T is timelike at x. In other words, p_0 is elliptic in the region where T is timelike.

There is also the parameter-dependent principal symbol of $\widehat{\Box}_g(\lambda)$ which takes into account the behavior as $|\lambda| \to \infty$ (possibly in the complex plane). This is just the function

$$p(x, \xi; \lambda) = -G(x, \xi - \lambda \, dt)$$

depending on λ. According to Lemma 2.2.2, if $\operatorname{Im} \lambda \neq 0$, then $G(x, \xi - \lambda \, dt) \neq 0$.

2.3 Asymptotically anti-de Sitter spacetimes

In this section we define anti-de Sitter (aAdS) metrics. Then, a convenient expression for the Klein–Gordon equation is given with respect to a certain product decomposition near the conformal boundary. By means of a Fourier transform, the initial boundary value problem for the Klein–Gordon equation is reduced to the study of the boundary value problem for a stationary partial differential equation depending polynomially on the spectral parameter. The corresponding operator is a Bessel operator whose order ν depends on the Klein–Gordon parameter; the condition $\nu > 0$ translates into the well known Breitenlohner–Freedman bound.

Asymptotically simple ends

If (M, g) is a smooth Lorentzian manifold, then a connected open subset $U \subseteq M$ is said to be an **asymptotically simple end** if there exists

1. a Lorentzian manifold $(\overline{M}, \overline{g})$ with boundary ∂M and an embedding $M \to \overline{M}$ identifying M with an open subset of $\overline{M}$,
2. a connected boundary component $\mathcal{I} \subseteq \partial M$ consisting of limit points of U, such that $\mathcal{I}$ is timelike for $\overline{g}$,
3. a boundary defining function $\rho \in C^\infty(\overline{M})$ for $\mathcal{I}$ such that $\overline{g} = \rho^2 g$ in U.

Observe that the notion of an asymptotically simple end is independent of the choice of boundary defining function. Indeed, if ρ is replaced by $e^w \rho$ for some $w \in C^\infty(\overline{M})$, it suffices to replace $(\overline{M}, \overline{g})$ with $(\overline{M}, e^{2w} \, \overline{g})$.

Lemma 2.3.1 ([38, Section 3.1])**.** *Suppose that (M, g) solves the vacuum Einstein equation*

$$\operatorname{Ric}_g + \Lambda g = 0.$$

If M has an asymptotically simple end with conformal boundary $\mathcal{I}$, then $\mathcal{I}$ is automatically timelike.

When M is not Einstein, the last condition 3 must be imposed. The restriction of $\overline{g}$ to $T\mathcal{I}$ is a Lorentzian metric on $\mathcal{I}$, and its conformal class is denoted by $[\overline{g}|_{T\mathcal{I}}]$. This conformal class is independent of the choice of boundary defining function, since replacing ρ by $e^w\rho$ modifies the boundary metric only by a conformal factor e^{2w}.

Asymptotically anti-de Sitter ends

Let (M, g) be a smooth Lorentzian manifold with an asymptotically simple end $U \subseteq M$. If

$$\bar{g}^{-1}(d\rho, d\rho) = -1$$

on $\mathcal{I}$, then U is said to be an **asymptotically anti-de Sitter (aAdS) end**. The aAdS property is independent of the choice of boundary defining function used to define $\bar{g} = \rho^2 g$.

Stationary aAdS spacetimes

Next, we consider asymptotically simple spacetimes with a preferred foliation by spacelike surfaces. Begin by assuming there exists a complete vector field T on $\overline{M}$ which is tangent to ∂M and Killing on M with respect to g. Furthermore, suppose there exists a hypersurface $\overline{X} \subseteq \overline{M}$ such that

1. each integral curve of T intersects $\overline{X}$ exactly once,
2. $\overline{X}$ is spacelike with respect to $\overline{g}$.

The spacelike nature of $\overline{\Sigma}$ does not depend on the choice of boundary defining function, and moreover

$$X := \overline{X} \cap M$$

is spacelike with respect to g. Since $\mathcal{I}$ is timelike, $\overline{X}$ necessarily intersects $\mathcal{I}$ transversally; therefore $\overline{X}$ is a manifold with boundary component $\partial X = \overline{X} \cap \mathcal{I}$; this is a slight abuse of notation since $\overline{X}$ may have additional boundary components which do not intersect $\mathcal{I}$.

As in Section 2.2, we let $\overline{X}_t = \exp(tT)\overline{X}$, where

$$\exp(tT) : \overline{M} \to \overline{M}$$

is the flow of T. Then $\overline{M}$ is smoothly foliated by the spacelike surfaces $\overline{\Sigma}_t$, and t defines a temporal function such that $T = \partial_t$.

It is always possible to choose a boundary defining function ρ such that $T\rho = 0$. We say that ρ is stationary. This can be done by choosing an arbitrary boundary defining function $\rho \in C^\infty(M)$, restricting ρ to $\overline{\Sigma}$, and then extending the resulting function to $\overline{M}$ as a constant along the integral curves of T. Observe that if ρ is stationary, then $\overline{g} = \rho^2 g$ is a stationary metric in the sense that T is Killing for $\overline{g}$. We also say that a tensor field on $\mathcal{I}$ is stationary if its Lie derivative with respect to T vanishes; this is a well defined notion since T is tangent to $\mathcal{I}$.

Lemma 2.3.2. *Let ρ be a boundary defining function for $\mathcal{I}$, and set $\overline{g} = \rho^2 g$. If $\gamma_0 \in [\overline{g}|_{T\mathcal{I}}]$ is stationary, then there exists a unique stationary boundary defining function x such that $x^2 g|_{T\mathcal{I}} = \gamma_0$.*

Proof. By definition, there exists $w \in C^\infty(\overline{M})$ such that $\gamma_0 = e^{2w}\, \overline{g}|_{T\mathcal{I}}$. Define x on $\overline{M}$ by

$$x(\exp(tT)p) = (e^w \rho)(p), \quad p \in \overline{X}.$$

Then $x^2 g|_{T_{\partial X}\mathcal{I}} = \gamma_0|_{\partial X}$, and by stationarity this also holds on $\mathcal{I}$. □

Given a stationary representative $\gamma_0 \in [\overline{g}|_{T\mathcal{I}}]$, let x be given by Lemma 2.3.2. For notational convenience, let

$$\phi(s, \cdot) = \exp(sV)(\cdot)$$

denote the flow of the gradient vector field

$$V = \frac{1}{(x^2 g)^{-1}(dx, dx)} \cdot \operatorname{grad}_{x^2 g} x$$

This vector field commutes with T since both x and g are stationary. In particular, $\phi(s, \cdot)$ commutes with $\exp(sT)(\cdot)$.

Lemma 2.3.3. *If ∂X is compact, then there exists $\varepsilon > 0$ and an open neighborhood $\overline{\mathcal{C}}$ of $\mathcal{I}$ in $\overline{M}$ such that*

$$\phi : [0, \varepsilon)_s \times \mathcal{I} \to \overline{\mathcal{C}}$$

is a diffeomorphism.

Proof. Since ∂X is compact, there exists $\varepsilon > 0$ and an open neighborhood

$$U \simeq (-\delta, \delta)_t \times \partial X$$

of ∂X in $\mathcal{I}$ such that ϕ is a diffeomorphism from $[0, \varepsilon)_s \times U$ onto its range. This can be extended to a diffeomorphism from $[0, \varepsilon)_s \times \mathcal{I}$ onto its range by commuting ϕ with the t-translations $p \mapsto \exp(tT)p$. □

In particular, ϕ is a collar diffeomorphism compatible with x in the sense that x pulls back to the coordinate function $s \in [0, \varepsilon)$. The pullback of g to $[0, \varepsilon)_s \times \mathcal{I}$ by ϕ takes the form

$$\phi^* g = \frac{-ds^2 + \gamma}{s^2},$$

where γ is a smooth symmetric $(0, 2)$–tensor on $[0, \varepsilon)_s \times \mathcal{I}$ such that $\gamma|_{T\mathcal{I}} = \gamma_0$. By a slight abuse of notation, we identify $x = s$ and simply write

$$g = (-dx^2 + \gamma)/x^2.$$

Unless $\left(\operatorname{grad}_{x^2 g} x\right) t = 0$, it is not true that $\phi(\partial X)$ is contained in $\overline{X}$. To avoid confusion, we will use our original notation $\mathfrak{t}$ for the function on M such that

$$\mathfrak{t}(\overline{X}_t) = t$$

First we identify $\mathcal{I}$ with $\partial X \times \mathbb{R}_\tau$, corresponding to the foliation of $\mathcal{I}$ by $\overline{X}_\tau \cap \mathcal{I}$ for $\tau \in \mathbb{R}$. Points of $\mathcal{I}$ are pairs $(y, \tau) \in \partial X \times \mathbb{R}$. Observe that the parameter along the T flow restricted to $\mathcal{I}$ is now denoted by τ rather than t. By stationarity,

$$\mathfrak{t}(\phi(x, y, \tau)) = \mathfrak{t}(\phi(x, y, 0)) + \tau,$$

and of course also $\mathfrak{t}(\phi(0, y, \tau)) = \tau$. Therefore ϕ^{-1} maps each slice $\overline{\mathcal{C}} \cap \overline{X}_t$ onto the graph

$$\{(x, y, t - \mathfrak{t}(\phi(x, y, 0))) : (x, y) \in [0, \varepsilon) \times \partial X\}$$

Therefore the map

$$(x, y) \mapsto \phi(x, y, t - \mathfrak{t}(\phi(x, y, 0))$$

is a collar diffeomorphism $[0, \varepsilon)_x \times \partial X \to \overline{\mathcal{C}} \cap \overline{X}_t$ compatible with x.

Special boundary defining functions

In the next lemma, we find a canonical form for the metric near $\mathcal{I}$ in terms of a special boundary defining function.

Lemma 2.3.4. *If ∂X is compact and $\gamma_0 \in [\bar{g}|_{T\partial M}]$ is stationary, then there exists a unique stationary boundary defining function x such that*

1. $x^2 g|_{T\mathcal{I}} = \gamma_0$,
2. $(x^2 g)^{-1}(dx, dx) = -1$ *in a neighborhood of* $\mathcal{I}$.

Proof. The proof of [46, Lemma 5.2] goes through essentially unchanged. Let ρ be any stationary boundary defining function such that $\rho^2 g|_{T\mathcal{I}} = \gamma_0$. Write $\bar{g} = \rho^2 g$, and look for x in the form

$$x = e^w \rho.$$

Then the equation $|dx|^2_{x^2 g} = -1$ is equivalent to

$$-1 = |d(e^w \rho)|^2_{e^{2w}\bar{g}} = \bar{g}(d\rho, d\rho) + 2\rho\, \bar{g}^{-1}(dw, d\rho) + \rho^2 \bar{g}^{-1}(dw, dw). \tag{2.8}$$

Dividing this equation by ρ,

$$2(\operatorname{grad}_{\bar{g}} \rho) w + \rho \bar{g}^{-1}(dw, dw) = -\rho^{-1}(1 + \bar{g}^{-1}(d\rho, d\rho)), \tag{2.9}$$

which is a first order Hamilton–Jacobi equation. If y^A are local coordinates on ∂X, then $(q^i) = (\rho, y^A, \tau)$ are local coordinates on $\overline{\mathcal{C}}$ via the collar diffeomorphism induced by ρ. Then (2.9) is of the form

$$F(dw, \rho, y) = 2\bar{g}^{\,i\rho}\,\partial_i w + \rho\,\bar{g}^{\,ij}\,\partial_i w\,\partial_j w + f(\rho, y) = 0$$

for some function f. The noncharacteristic condition is $\partial_{\sigma_0} F(\sigma, s, y)|_{\mathcal{I}} \neq 0$, where (σ_i) are variables dual to (q^i) (so σ_0 is dual to ρ). This follows since $\nabla_{\bar{g}}\rho$ is transverse to $\mathcal{I}$:

$$\bar{g}^{\,\rho\rho} = \big(\mathrm{grad}_{\bar{g}}\,\rho\big)\,\rho \neq 0.$$

Therefore (2.9) can be solved uniquely with the initial condition $w|_{\mathcal{I}} = 0$ in a neighborhood of $\mathcal{I}$ diffeomorphic to $[0,\varepsilon)_x \times \mathcal{I}$: by compactness, this can be done in a neighborhood of ∂X, and since the solution does not depend on t is can be extended along the integral curves of T.

□

If x satisfies the condition described in Lemma 2.3.4, then x is said to be a special boundary defining function. The integral curves of $\mathrm{grad}_{x^2 g}\, x$ are geodesics near $\mathcal{I}$, so the Gauss lemma implies that they are orthogonal to the hypersurfaces $\{x = \text{constant}\}$. If ϕ is the induced collar diffeomorphism, then

$$\phi^*(g) = \frac{-dx^2 + \gamma(x)}{x^2} \tag{2.10}$$

on $[0,\varepsilon)_x \times \mathcal{I}$. Here $x \mapsto \gamma(x)$ is a smooth family of stationary Lorentzian metrics on ∂M such that $\gamma(0) = \gamma_0$.

Almost even metrics

The metric g as in Lemma 2.3.4 is said to even modulo $\mathcal{O}(x^3)$ (in the sense of Guillarmou [48]) if there exists a two-tensor γ_1 on ∂M such that

$$\gamma(x) = \gamma_0 + x^2\gamma_1 + \mathcal{O}(x^3).$$

As in [48, Proposition 2.1], this evenness property is instrinsic to the conformal class $[\bar{g}|_{T\mathcal{I}}]$ in the sense that it does not depend on the particular representative γ_0 such that $\gamma(0) = \gamma_0$.

The fundamental class of aAdS metrics which are even modulo $\mathcal{O}(x^3)$ are the Einstein aAdS metrics. This was studied in great detail by Fefferman–Graham [36]. We only mention the following fact:

Lemma 2.3.5. *Suppose that (M, g) solves the vacuum Einstein equation*

$$\mathrm{Ric}_g + \Lambda g = 0.$$

If M has an aAdS end with conformal boundary $\mathcal{I}$, then g is even modulo $\mathcal{O}(x^3)$ near $\mathcal{I}$.

See [2, Section 2] for more information about this. The Einstein condition also enforces additional conditions on the expansion of $\gamma(s)$ which are not exploited here — in the asymptotically hyperbolic setting, see Mazzeo–Pacard [78, Section 2].

Finding the special boundary defining function for an arbitrary aAdS spacetime (and therefore also verifying the evenness property) may take some work. The following criterion is frequently easier to verify for aAdS spacetimes.

Lemma 2.3.6. *Suppose that ρ is stationary, and let ϕ denote the diffeomorphism considered in Lemma 2.3.3 associated with ρ. If $\rho^2 g$ has an expansion*

$$\phi^*(\rho^2 g) = -ds^2 + \gamma_0 + \mathcal{O}(s^2),$$

where γ_0 is a stationary Lorentzian metric on ∂M, then g is even modulo $\mathcal{O}(x^3)$.

Proof. Let x denote the unique stationary boundary defining function associated to the representative $\rho^2 g$. Write $x = e^w \rho$ for some w. Then (2.8) implies that after dividing by ρ,

$$2\partial_\rho w = \mathcal{O}(\rho),$$

so $\partial_\rho w = 0$ at the boundary. The result can then be deduced from an explicit computation in local coordinates, see also Lemma 3.1.3. □

Klein–Gordon operator

Given a special boundary defining function, there are two natural sets of coordinates $\mathcal{I}$. The first arises from the map

$$(x, y, \tau) \mapsto \phi(x, y, \tau).$$

These coordinates do not respect the foliation by surfaces of constant t in the sense that $\phi(\overline{X}_\tau \cap \mathcal{I})$ is not contained in $\overline{X}_\tau$. However, these coordinates do have the property that ∂_y, ∂_τ are orthogonal to ∂_x. The second set of coordinates are (x, y, t) arising from

$$(x, y, t) \mapsto \phi(x, y, t - \mathfrak{t}(\phi(x, y, 0))),$$

where t is the ADM time coordinate. We may then view (x, y) as coordinates on $\overline{X}_t$ (near $\mathcal{I}$). It is easiest to write down the d'Alembertian in the first set of coordinates. We have

$$\Box_g = x^2 D_x^2 + i(d - 1 + e(x)) x D_x + x^2 \Box_{\gamma(x)},$$

where $x \mapsto e(x)$ is a smooth family of functions on $\mathcal{I}$ satisfying

1. $Te(x) = 0$,
2. $e(x) = x^2 e_0 + \mathcal{O}(x^3)$ for some $e_0 \in C^\infty(\mathcal{I})$.

Indeed, $e(x) = -(1/2)x\partial_x \log(\det \gamma(x))$, and $\det \gamma(x) = \det \gamma_0 + \mathcal{O}(x^2)$. By means of a direct calculation,

$$x^{-(d-1)/2}\left(\Box_g + \nu^2 - d^2/4\right)x^{(d-1)/2} = x^2\left(\Box_{x^2 g} + (\nu^2 - 1/4)x^{-2}\right),$$

where now

$$\Box_{x^2 g} = D_x^2 + ix^{-1}e(x)D_x + \Box_{\gamma(x)}.$$

To write down the corresponding ADM expression for $\Box_g$ near $\mathcal{I}$, we temporarily write

$$\tilde{x} = \tilde{x}(x,y,\tau) = x, \quad \tilde{y} = \tilde{y}(x,y,\tau) = y, \quad t = t(x,y,\tau) = \tau + \mathfrak{t}(\phi(x,y,0)).$$

In that case,

$$\partial_x = \partial_{\tilde{x}} + \frac{\partial t}{\partial x}\partial_t, \quad \partial_y = \partial_{\tilde{y}} + \frac{\partial t}{\partial y}\partial_t, \quad \partial_\tau = \partial_t.$$

We clearly have $\partial_{\tilde{y}} = \partial_y$ on $\mathcal{I}$. Furthermore, by definition,

$$\frac{\partial \mathfrak{t}}{\partial x} = \partial_x(\mathfrak{t}(\phi(x,y,0))) = (\operatorname{grad}_{x^2 g} x)\,\mathfrak{t}. \tag{2.11}$$

The quantity (2.11) vanishes precisely when $\overline{X}$ meets $\mathcal{I}$ orthogonally with respect to $x^2 g$ (which does not depend on the choice of boundary defining function). Dropping the tildes, we subsequently can write

$$\begin{aligned}\Box_{x^2 g} = {} & (D_x + (\operatorname{grad}_{x^2 g} x)\,\mathfrak{t}\cdot D_t)^2 + ix^{-1}e(x)(D_x + (\operatorname{grad}_{x^2 g} x)\,\mathfrak{t}\cdot D_t) \\ & + Q_0(x,y,D_y) + Q_1(x,y,D_y)D_t + Q_2(x,y)D_t^2\end{aligned} \tag{2.12}$$

in the coordinates (x,y,t). If $x = x_0$ is fixed, then Q_k are invariantly defined differential operators of order $2-k$ on $\overline{X}_t \cap \{x = x_0\}$; in fact,

$$Q_0(x_0,y,D_y) + Q_1(x_0,y,D_y)D_t + Q_2(x_0,y)D_t^2$$

is the ADM decomposition of $\Box_{\gamma(x_0)}$ with respect to the foliation of $\{x = x_0\}$ by the submanifolds $\overline{X}_t \cap \{x = x_0\}$.

Next, we define the operator

$$P_g = \Box_{x^2 g} + (\nu^2 - 1/4)x^{-2}, \tag{2.13}$$

which up to a conjugation is x^{-2} times the Klein–Gordon operator $\Box_g + \nu^2 - d^2/4$. The corresponding stationary operator is

$$P(\lambda) = e^{i\lambda t}P_g\, e^{-i\lambda t}, \tag{2.14}$$

acting on X. The next lemma gives the precise structure of $P(\lambda)$ as a Bessel operator, studied in Chapter 3.

Lemma 2.3.7. *Let* $\Phi : [0, \varepsilon) \times \partial X \to \overline{X}_t$ *be given by*

$$\Phi(x, y) = \phi(x, y, -\mathfrak{t}(x, y, 0)).$$

If $\overline{X}$ *meets* $\mathcal{I}$ *orthogonally with respect to* $x^2 g$*, then*

$$\Phi^* P(\lambda) = D_x^2 + (\nu^2 - 1/4)x^{-2} + B(x, y, D_y; \lambda) D_x + A(x, y, D_y; \lambda),$$

where $A \in \mathrm{Diff}^2(\partial X)$*,* $B \in \mathrm{Diff}^1(\partial X)$ *are parameter-dependent differential operators on* ∂X *depending smoothly on* $x \in [0, \varepsilon)$*, such that* $B(0, y, D_y; \lambda) = 0$*.*

Proof. This follows from (2.12), noting that both $x^{-1}e(x)$ and $(\mathrm{grad}_{x^2 g}\, x)\,\mathfrak{t}$ vanish at $\mathcal{I}$. □

Formally disregarding the singular but "lower order" x^{-2} term, the principal symbol of $\Box_{x^2 g}$ at $x = 0$ is

$$\xi^2 + Q_0(0, y, \eta), \tag{2.15}$$

whereas its parameter-dependent principal symbol is

$$\xi^2 + Q_0(0, y, \eta) - Q_1(x, y, \eta)\lambda + Q_2(0, y)\lambda^2. \tag{2.16}$$

Here we are letting ξ denote the variable dual to x, and η those dual to y. By Lemma 2.2.1, if T is timelike at boundary point $p \in \partial X$, then (2.15) is positive definite in (ξ, η). Similarly, since dt is timelike, by Lemma 2.2.2 we see that (2.16) does not vanish for $\mathrm{Im}\, \lambda \neq 0$.

2.4 Schwarzschild–AdS space

The simplest AdS black holes belong to the Schwarzschild–AdS family of solutions. These are the unique static, spherically symmetric solutions to the vacuum Einstein equations with negative cosmological constant and spherical horizon.

Remark 1. *Unlike their nonnegative cosmological counterparts, there exist AdS black holes with non-spherical horizon topologies, but these are not considered here.*

The $d + 1$ dimensional Schwarzschild–AdS spacetime with curvature radius l is

$$M = \mathbb{R}_t \times (r_+, \infty)_r \times \mathbb{S}^{d-1}_\omega$$

equipped with the metric

$$g = f(r)\, dt^2 - f(r)^{-1} dr^2 - r^2\, d\omega^2, \quad f(r) = r^2/l^2 + 1 - \mu r^{2-d},$$

where $d\omega^2$ is the round metric on $\mathbb{S}^{d-1}$. The parameter $\mu > 0$ appearing in the definition of $f(r)$ is related to the black hole mass by

$$\mathrm{mass} = \frac{(d-1)A_{d-1}}{16\pi}\mu,$$

where A_{d-1} is the volume of the unit $(d-1)$ sphere. As for AdS_{d+1}, the radius is related to the negative cosmological constant by (2.1). The radial coordinate r ranges over (r_+, ∞), where r_+ is the positive root of f. Observe that f is monotone increasing for $r > 0$, and $f \to \pm\infty$ as $r \to \infty$ and $r \to 0$ respectively; therefore r_+ certainly exists and is unique. This also means that the nondegeneracy assumption

$$f'(r_+) > 0 \tag{2.17}$$

is automatically satisfied. In more sophisticated language, this means that the surface gravity of the Killing horizon $\mathcal{H}^+ = \{r = r_+\}$ is positive [98, Section 2]

Up to a constant multiple of the metric, we may assume that $l = 1$. The scaling transformation

$$l \mapsto kl, \quad \mu \mapsto k\mu, \quad r \mapsto kr, \quad t \mapsto kt$$

induces the conformal change $g \mapsto k^2 g$. For the remainder of this section, we may therefore assume that $l = 1$ by setting $k = l^{-1}$.

Schwarzschild–AdS as an aAdS spacetime

Rewriting the metric in terms of $s = r^{-1}$ shows that $s^2 g$ extends smoothly from M to

$$\overline{M} = \mathbb{R}_t \times [0, r_+^{-1})_s \times \mathbb{S}^{d-1}_\omega,$$

where now $s \in C^\infty(\overline{M})$ is a boundary defining function for $\mathcal{I} = \{s = 0\}$. The boundary metric with respect to s is

$$s^2 g|_{\mathcal{I}} = l^{-2}\, dt^2 - d\omega^2,$$

which is the same as for AdS_{d+1}. It is also clear that $s^{-2} g^{-1}(ds, ds) \to -1$ as $s \to 0$, so $\mathcal{I}$ is an aAdS boundary. In fact, using Lemma 2.3.3 it is easy to see that g is even modulo $\mathcal{O}(x^3)$.

Extended spacetime

The metric coefficient of dr^2 blows up at $\mathcal{H}^+ = \{r = r_+\}$. This apparent singularity can be removed by an appropriate change of variables. Define a new time coordinate

$$t^\star = t + r^\star(r),$$

where $r^\star$ satisfies

$$\partial_r r^\star(r) = \frac{1}{f(r)} + \frac{1}{1+r^2}, \quad r^\star(\infty) = 0.$$

In the ingoing Eddington–Finkelstein coordinates $(t^\star, r, \omega)$,

$$g = f(r)\,(dt^\star)^2 - \frac{2\mu}{r^{d-2}(1+r^2)} dt^\star dr - \frac{r^2 + 1 + \mu r^{d-2}}{(1+r^2)^2} dr^2 - r^2 d\omega^2.$$

This expression is clearly smooth, and in fact extends as a Lorentzian metric up to $r = r_+ - \delta$ for $\delta > 0$ sufficiently small. We denote this extended spacetime by

$$M_\delta = \mathbb{R}_{t^\star} \times (r_+ - \delta) \times \mathbb{S}^{d-1}$$

Let $T = \partial_{t^\star}$ (which equals ∂_t on M), which is Killing for g. Clearly M_δ is foliated by the slices of constant $t^\star$, and each of these surfaces is spacelike. To see this, we calculate the inverse metric

$$g^{-1} = \frac{1 + r^2 + \mu r^{2-d}}{(1+r^2)^2}\, \partial_{t^\star}^2 - \frac{2\mu r^{2-d}}{1+r^2}\, \partial_{t^\star}\partial_r - f(r)\, \partial_r^2 - r^{-2}\, d\omega^{-2}.$$

In particular, we clearly have that

$$g^{-1}(dt^\star, dt^\star) = \frac{1 + r^2 + \mu r^{2-d}}{(1+r^2)^2} > 0.$$

In order to apply general results from Section 2.3, in particular Lemma 2.3.7, we should also check that $\{t^\star = 0\}$ intersects $\mathcal{I}$ orthogonally. With $s = r^{-1}$ a boundary defining function, this follows from

$$g^{-1}(dt^\star, ds) = -s^2 g^{-2}(dt^\star, dr) = \mu s^{d+2}/(1+s^2),$$

which vanishes at $s = 0$.

2.5 Kerr–AdS spacetime

The Kerr–AdS metric is determined by three parameters (Λ, M, a), where $\Lambda < 0$ is the negative cosmological constant, $M > 0$ is the black hole mass, and $a \in \mathbb{R}$ is the angular momentum per unit mass. Given parameters (Λ, M, a), let $l^2 = 3/|\Lambda|$ and introduce the quantities

$$\Delta_r = (r^2 + a^2)\left(1 + \frac{r^2}{l^2}\right) - 2Mr; \quad \Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta;$$

$$\varrho^2 = r^2 + a^2\cos^2\theta; \quad \alpha = \frac{a^2}{l^2}.$$

The following lemma concerns the location of roots of Δ_r.

Lemma 2.5.1. *Any real root of Δ_r must be nonnegative, and there at most two real roots. If $a = 0$, then Δ_r always has a unique positive root.*

Proof. When $a = 0$ it is clear that Δ_r has a unique positive root, and furthermore $\partial_r \Delta_r(r) > 0$ for $r > 0$.

On the other hand, if $a \neq 0$ then $\Delta_r(0) > 0$ and $\partial_r \Delta_r(0) < 0$. At the same time, $\Delta_r(r) \to \infty$. Since $\partial_r^2 \Delta_r > 0$, when $a \neq 0$ any real root of Δ_r must be positive, and there are at most two real roots. □

We assume that Δ_r has at least one real root, and we let r_+ denote the largest of these. We also assume the nondegeneracy condition

$$\Delta_r'(r_+) > 0. \tag{2.18}$$

The Kerr–AdS metric is given in Boyer–Lindquist coordinates by

$$\begin{aligned} g = & -\varrho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) - \frac{\Delta_\theta \sin^2\theta}{\varrho^2(1-\alpha)^2} \left(a\, dt - (r^2+a^2)\, d\phi \right)^2 \\ & + \frac{\Delta_r}{\varrho^2(1-\alpha)^2} \left(dt - a\sin^2\theta\, d\phi \right)^2. \end{aligned}$$

In this expression $(\theta, \phi) \in (0, \pi) \times (\mathbb{R}/2\pi\mathbb{Z})$ are spherical coordinates on $\mathbb{S}^2$, while $t \in \mathbb{R}$ and $r \in (r_+, \infty)$. The dual metric g^{-1} is given by

$$\begin{aligned} g^{-1} = & -\frac{\Delta_r}{\varrho^2}\partial_r^2 - \frac{\Delta_\theta}{\varrho^2}\partial_\theta^2 - \frac{(1-\alpha)^2}{\varrho^2 \Delta_\theta \sin^2\theta} \left(a\sin^2\theta \partial_t + \partial_\phi \right)^2 \\ & + \frac{(1-\alpha)^2}{\varrho^2 \Delta_r} \left((r^2+a^2)\partial_t + a\partial_\phi \right)^2. \end{aligned}$$

As in the Schwarzschild–AdS case, we will henceforth assume that $l = 1$ by rescaling (equivalently, $\Lambda = -3$); the only difference is that we must also scale $a \mapsto ka$.

There are two apparent singularities: at the poles $\theta \in \{0, \pi\}$ where $\sin^2\theta = 0$, and at $r = r_+$ where $1/\Delta_r$ blows up. The former is an artifact of spherical coordinates. To see that the metric is indeed smooth up to the poles, introduce Cartesian coordinates

$$x_1 = \sin\theta\cos\phi, \quad x_2 = \sin\theta\sin\phi.$$

Since $\sin^2\theta\, d\phi = x_1\, dx_2 - x_2\, dx_1$ and the latter is smooth up to $x_1 = x_2 = 0$, it suffices to consider only angular terms where $d\phi$ is not a multiple of $\sin^2\theta$. The only such term is

$$-\frac{\varrho^2}{\Delta_\theta}\, d\theta^2 - \frac{\Delta_\theta \sin^2\theta}{\varrho^2(1-\alpha)^2}(r^2+a^2)^2\, d\phi^2.$$

Since $r^2 + a^2 = \varrho^2 + a^2\sin^2\theta$ and $\Delta_\theta = (1-\alpha) + a^2\sin^2\theta$, this is further reduced to considering

$$-\frac{\varrho^2}{\Delta_\theta} \left(d\theta^2 + \sin^2\, d\phi^2 \right),$$

which is the metric on $\mathbb{S}^2$ times a quantity smooth up to the poles, hence is smooth itself.

Extended spacetime

The singularity at the event horizon $\mathcal{H}^+ = \{r = r_+\}$ can be remedied by a change of variables similar to the Schwarzschild–AdS case. Set

$$t^\star = t + F_t(r); \quad \phi^\star = \phi + F_\phi(r), \tag{2.19}$$

where F_t, F_ϕ are smooth functions on (r_+, ∞) such that

$$\partial_r F_t(r) = \frac{1-\alpha}{\Delta_r}(r^2+a^2) + f_+(r), \quad \partial_r F_\phi(r) = a\frac{1-\alpha}{\Delta_r}. \tag{2.20}$$

Here f_+ is an arbitrary smooth function, and F_t, F_ϕ are chosen to vanish at infinity. The dual metric in $(t^\star, r, \theta, \phi^\star)$ coordinates reads

$$\begin{aligned} \varrho^2 g^{-1} = &-\Delta_r\left(\partial_r + f_+\partial_{t^\star}\right)^2 - \Delta_\theta \partial_\theta^2 - 2(1-\alpha)\left(\partial_r + f_+\partial_{t^\star}\right)\left((r^2+a^2)\partial_{t^\star} + a\partial_{\phi^\star}\right) \\ &- \frac{(1-\alpha)^2}{\Delta_\theta \sin^2\theta}\left(a\sin^2\theta\partial_{t^\star} + \partial_{\phi^\star}\right)^2, \end{aligned} \tag{2.21}$$

which is smooth up to $r = r_+$. Given $\delta \geq 0$ set

$$X_\delta = (r_+ - \delta, \infty) \times \mathbb{S}^2, \quad \mathcal{M}_\delta = X_\delta \times \mathbb{R}_{t^\star}.$$

If $\delta > 0$ is sufficiently small, then g admits an extension (as a metric) from $\mathcal{M}_0$ to $\mathcal{M}_\delta$.

There are two Killing fields on $\mathcal{M}_\delta$ corresponding to stationarity and axisymmetry of Kerr–AdS:

$$T = \partial_t = \partial_{t^\star}, \quad \Phi = \partial_\phi = \partial_{\phi^\star}.$$

The extended region $\mathcal{M}_\delta$ is smoothly foliated by the hypersurfaces $\{t^\star = \tau\}$ for $\tau \in \mathbb{R}$. We can also identify $\{t^\star = 0\}$ with X_δ, and then the foliation is obtained by translating X_δ along the integral curves of T. With this identification, we need X_δ to be spacelike; this condition depends on the function f_+ used to extend the metric. One explicit choice with this property is

$$f_+(r) = \frac{\alpha - 1}{r^2 + 1}. \tag{2.22}$$

The hypersurface $\mathcal{H}^+$ is null since dr is null there. Furthermore, $\mathcal{H}^+$ is a Killing horizon generated by the vector field

$$K = T + \frac{a}{r_+^2 + a^2}\Phi,$$

as it is easy to check that K is null on $\mathcal{H}^+$ as well. In particular there exists $\kappa \in \mathbb{R}$ such that

$$\operatorname{grad}_g g(K,K) = 2\kappa K$$

on $\mathcal{H}^+$. Examining the $\partial_{t^\star}$ component of the above equation on the horizon gives the value

$$\kappa = \frac{\partial_r \Delta_r(r_+)}{2(1-\alpha)(r_+^2 + a^2)}, \tag{2.23}$$

which is strictly positive under our assumption (2.18) on r_+.

Kerr–AdS as an asymptotically anti-de Sitter spacetime

To analyze the behavior of g for large r, introduce a new radial coordinate $s = r^{-1}$ for large values of r. In fact s is a well defined function on the entirety of $\mathcal{M}_\delta$ since $r_+ > 0$. Let

$$\mathcal{I} = \{s = 0\}$$

denote the conformal boundary. Noting that

$$\varrho^2 = s^{-2} + \mathcal{O}(1), \quad \Delta_r = s^{-4} + \mathcal{O}(s^{-2}),$$

it follows that $s^2 g$ has a smooth extension to $\overline{\mathcal{M}}_\delta = \mathcal{M}_\delta \cup \mathcal{I}$. Thus $s^2 g$ may be written as

$$s^2 g = -ds^2 + \gamma,$$

where γ is a $(0,2)$-tensor near the conformal boundary $\mathcal{I}$ such that $\gamma|_{\mathcal{I}}$ does not depend on ds. In terms of (t, s, θ, ϕ) and $s \geq 0$ small,

$$\gamma = \frac{d\theta^2}{\Delta_\theta} - \frac{\Delta_\theta \sin^2\theta}{(1-\alpha)^2} d\phi^2 + (1-a^2)^{-2}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \mathcal{O}(s^2),$$

where the $\mathcal{O}$-term denotes a $(0,2)$-tensor on $\mathcal{M}$ with $\mathcal{O}(s^2)$ coefficients. Note that ds is spacelike for $s^2 g|_{\mathcal{I}}$ and $\gamma|_{T\mathcal{I}}$ is a Lorentzian metric on $\mathcal{I}$. The leading term in the expansion of γ is in fact a Lorentzian metric γ_0 on $\mathcal{I}$; both T and dt are timelike for γ_0. According to Lemma 2.3.3, g is even modulo $\mathcal{O}(x^3)$. We can also use (2.21) and (2.22) to see that $\mathcal{I}$ meets the surfaces of constant $t^\star$ orthogonally, hence Lemma 2.3.7 is applicable.

Chapter 3

Elliptic boundary value problems for Bessel operators

The study of linear fields on asymptotically anti-de Sitter (aAdS) spaces has stimulated new interest in boundary value problems for a class of singular elliptic equations, wherein the operator $D_x^2 + (\nu^2 - 1/4)x^{-2}$ acts on one of the variables [33, 54, 59, 57, 96, 100, 99]. To formulate this class of operators more precisely, consider a product manifold $[0, \varepsilon) \times \partial X$, where ∂X is compact. The model for what we call a Bessel operator has the form

$$P(x, y, D_x, D_y) = D_x^2 + (\nu^2 - 1/4)x^{-2} + A(x, y, D_y),$$

where $(x, y) \in (0, \varepsilon) \times \partial X$ and A is a family of second order differential operators on ∂X depending smoothly on $x \in [0, \varepsilon)$. The parameter ν is required to be real and strictly positive. In the study of linear waves on aAdS spacetimes, ν is related to the mass of a scalar field — see Section 3.2 for more details. The condition $\nu > 0$ corresponds to the Breitenlohner–Freedman bound [13, 14].

Boundary data for this problem are formally defined by the following weighted restrictions:

$$\gamma_- u = x^{\nu - 1/2} u|_{\partial X}, \quad \gamma_+ u = x^{1-2\nu} \partial_x (x^{\nu - 1/2} u)|_{\partial X}.$$

Some care is needed to give precise meaning to these restrictions — see Section 3.3, along with an earlier discussion in Warnick [100]. The boundary operators in this paper are of the form $T = T^- \gamma_- + T^+ \gamma_+$, where T^-, T^+ are differential operators on ∂X of order at most one and zero respectively. This paper is concerned with solvability of the boundary value problem

$$\begin{cases} Pu = f \text{ on } X \\ Tu = g \text{ on } \partial X \end{cases} \tag{3.1}$$

when $0 < \nu < 1$, and the simpler equation

$$Pu = f \text{ on } X \tag{3.2}$$

when $\nu \geq 1$. No boundary conditions are imposed when $\nu \geq 1$, since in that case u must satisfy a Dirichlet boundary condition. The difference between the cases $0 < \nu < 1$ and $\nu \geq 1$ is explained in more detail in the introduction to Section 3.4.

Ellipticity of the Bessel operator P is defined in Section 3.1. As in the study of smooth boundary value problems, there is also a notion of ellipticity for (3.1), given by a natural Lopatinskiĭ condition on the pair (P, T). This condition is introduced in Section 3.4. Elliptic estimates are proved in Theorem 1 of Section 3.4. When the operators P, T depend polynomially on a spectral parameter λ, there is a notion of parameter-ellipticity for both P and the boundary value problem (3.1). Theorem 2 of Section 3.4 provides elliptic estimates in terms of parameter-dependent norms, which are uniform as $|\lambda| \to \infty$ in the cone of ellipticity.

For the global problem, consider a compact manifold $\overline{X}$ where $[0, \varepsilon) \times \partial X$ is identified with a collar neighborhood of ∂X. Suppose that the restriction of P to this collar is a Bessel operator — see Section 3.1 for details. As in the case of smooth boundary value problems, estimates for P near ∂X may often be combined with estimates in the interior X to establish the Fredholm property (including some cases where P fails to be everywhere elliptic on X). In Section 3.5, a sufficient condition of this type is discussed. Furthermore, in the presence of a spectral parameter λ, unique solvability is established for λ in the cone of ellipticity provided $|\lambda|$ is sufficiently large.

For stationary aAdS spacetimes with compact time slices and an everywhere timelike Killing field, Section 3.6 describes a class of boundary conditions which yield a complete set of normal modes associated to a discrete set of eigenvalues. Of particular interest are time-periodic boundary conditions which depend on ∂_t (hence depend on the spectral parameter after a Fourier transform). This is important for the study of modes with transparent or dissipative boundary conditions, along with superradiance phenomena [3, 59, 103].

In Chapter 4, discreteness of quasinormal frequencies is also established for massive fields on Kerr–AdS black holes with arbitrary rotation speed. These frequencies replace eigenvalues in scattering problems [17, 26, 39, 61, 70, 99]. When $0 < \nu < 1$, arbitrary boundary conditions satisfying the Lopatinskiĭ condition may be imposed on the field (although of course one does not have any completeness statement).

Our approach is inspired by the texts of Roitberg [84] and Kozlov–Mazya–Rossman [71]. This approach is particularly suited to the singular nature of Bessel operators, and allows for the study of boundary value problems in low regularity spaces as needed in applications to general relativity — see Section 3.5. All the methods are classical, using only homogeneity properties of differential operators. The key is exploiting the theory of “twisted” derivatives as first emphasized in [100]. This is based on the classical observation that the one-dimensional Bessel operator $D_x^2 + (\nu^2 - 1/4)x^{-2}$ admits a factorization as the product of a first order order operator and its adjoint; this first order operator is then treated as an elementary derivative.

Using a variational approach, Holzegel [54], Warnick [100, 99], and Holzegel–Warnick [58] derive some of the same (or similar) elliptic estimates. However, only the “classical” self-adjoint boundary conditions are handled when $0 < \nu < 1$; these are the Dirichlet ($T = \gamma_-$) and Robin boundary conditions ($T = \gamma_+ + \beta\gamma_-$ with real-valued β). The approach taken here accounts for a larger class of non-self-adjoint boundary conditions, which is optimal among differential boundary conditions.

The results of this paper should also be compared to earlier works of Vasy [96], Holzegel [54] on aAdS spaces, where a more restrictive measure is used to define the space of square integrable functions. In those works, the square integrability condition is equivalent to the generalized Dirichlet boundary condition. This limits the range of applications, since different boundary conditions are employed throughout the physics literature on aAdS spaces [3, 8, 13, 14, 25, 65, 105]

There is also a general microlocal approach to degenerate boundary value problems developed by Mazzeo–Melrose [80], Mazzeo [77], Mazzeo–Vertman [79], among many others. The work of Vasy [96] on aAdS spaces makes use of this technology. In particular, the elliptic theory in [79] could likely reproduce the results of this paper, and is also applicable to much more general classes of elliptic operators. On the other hand, the approach developed here is directly motivated by the physics literature. For instance, the Sobolev spaces used in this paper were originally defined in [100] to give precise meaning to the energy renormalization implicit in the work of Breitenlohner–Freedman [13]. There is also a simplicity advantage in using physical space methods, rather than a more sophisticated microlocal approach.

Bessel operators of the precise kind studied here arise in numerous contexts outside of general relativity with negative cosmological constant, both mathematical and physical. See for instance the monograph of Kipriyanov [69] and the substantial literature surrounding "generalized axially symmetric potentials" [34, 44, 62, 101, 106].

3.1 Preliminaries

Conventions for differential operators

If P is a smooth differential operator on a manifold Y, then in local coordinates,

$$P = \sum_{|\alpha| \leq m} a_\alpha(y) D_y^\alpha. \tag{3.3}$$

In that case the order of P is said to no greater than m. The smallest m for which there exists a nonzero coefficient a_α, $|\alpha| = m$ in some coordinate representation is the order of P, written $\mathrm{ord}(P)$. If the order of P is no greater than m, the symbol $\sigma_m(P)$ of P with respect to m is the polynomial function on T^*Y given in local coordinates by

$$\sigma_m(P)(y,\eta) = \sum_{|\alpha|=m} a_\alpha(y)\eta^\alpha, \quad (y,\eta) \in T^*Y.$$

Thus if m is strictly larger than the order of P, then $\sigma_m(P) = 0$. The space of differential operators of order no greater than m is denoted $\mathrm{Diff}^m(Y)$. If P has order no greater than m with $m < 0$, then $P = 0$; conversely if $P = 0$, then P can assigned any negative order. This convention will be useful throughout Section 3.4.

The class of parameter-dependent differential operators on a manifold Y is defined as follows: $P \in \mathrm{Diff}^m_{(\lambda)}(Y)$ if in local coordinates,

$$P(y, D_y; \lambda) = \sum_{j+|\alpha| \leq m} a_{\alpha,j}(x)\lambda^j D_y^\alpha.$$

The parameter-dependent order $\mathrm{ord}^{(\lambda)}(P)$ is defined by assigning to λ^j the same weight as a derivative of order j. Thus the parameter-dependent principal symbol of P is given by

$$\sigma_m^{(\lambda)}(A) = \sum_{j+|\alpha|=m} a_{\alpha,j}(t)\lambda^j \eta^\alpha, \quad (y, \eta, \lambda) \in T^*Y \times \mathbb{C}.$$

If $Y = \mathbb{T}^n$ and $s \in \mathbb{R}$, the parameter-dependent Sobolev norms on Y are defined by

$$|||u|||^2_{H^s(\mathbb{T}^n)} = \sum_{q \in \mathbb{Z}^n} (1 + |\lambda|^2 + |q|^2)^s \, |\hat{u}(q)|^2,$$

where $\hat{u}(q)$ are the Fourier coefficients of u. This definition can be extended to an arbitrary compact manifold Y so that any parameter-dependent operator $P \in \mathrm{Diff}^m_{(\lambda)}(Y)$ is bounded $P : H^s(Y) \to H^{s-m}(Y)$ uniformly with respect to $|\lambda|$ in these norms.

Manifolds with boundary

Let $\overline{X} = X \cup \partial X$ denote an n-dimensional manifold with compact boundary ∂X and interior X. A boundary defining function for ∂X is a function $x \in C^\infty(\overline{X})$ satisfying

$$x^{-1}(0) = \partial X, \quad x > 0 \text{ on } X, \quad dx|_{\partial X} \neq 0.$$

Given x, there exists an open subset $\overline{W} \supseteq \partial X$, a number $\varepsilon > 0$, and a diffeomorphism $\phi : [0, \varepsilon) \times \partial X \to \overline{W}$ such that $x \circ \phi$ agrees with the projection $[0, \varepsilon) \times \partial X$. A collar of this type is said to be compatible with x. A compatible collar can be constructed as follows: choose a Riemannian metric g on $\overline{X}$ and consider the unit normal vector field $N = \mathrm{grad}_g\, x / g^{-1}(dx, dx)$. This is well defined on $\{0 \leq x < \varepsilon\}$ provided $\varepsilon > 0$ is chosen sufficiently small to ensure $dx \neq 0$. Then let

$$\phi(s, y) = \exp(sN)(y)$$

where $\exp(sN)$ is the flow of N for $s \in [0, \varepsilon)$ and $y \in \partial X$. Unless otherwise specified, a manifold with boundary $\overline{X}$ will always refer to $\overline{X}$ equipped with a distinguished boundary defining function x and a choice of compatible collar diffeomorphism ϕ.

Bessel operators

Given $\nu \in \mathbb{R}$, formally define the differential operator ∂_ν by the formula

$$\partial_\nu = \partial_x + (\nu - 1/2)x^{-1} = x^{1/2-\nu}\partial_x x^{\nu-1/2}.$$

Furthermore, let $\partial_\nu^* = -x^{\nu-1/2}\partial_x x^{1/2-\nu}$, which is the formal adjoint of ∂_ν with respect to Lebesgue measure on $\mathbb{R}_+$. Similarly, let $D_\nu = -i\partial_\nu$ and $D_\nu^* = i\partial_\nu^*$. Note that

$$|D_\nu|^2 := D_\nu^* D_\nu = D_x^2 + (\nu^2 - 1/4)x^{-2}$$

is the one-dimensional Bessel operator in Schrödinger form.

Definition 3.1.1. *Let $\overline{X}$ denote a manifold with compact boundary. A differential operator $P \in \mathrm{Diff}^2(X)$ is called a Bessel operator of order $\nu > 0$ if there exist*

$$A = A(x, y, D_y) \in \mathrm{Diff}^2(\partial X), \quad B = B(x, y, D_y) \in \mathrm{Diff}^1(\partial X),$$

depending smoothly on $x \in [0, \varepsilon)$, such that $B(0, y, D_y) = 0$ and

$$\phi^* P = |D_\nu|^2 + B(x, y, D_y)D_\nu + A(x, y, D_y) \tag{3.4}$$

The set of such operators is denoted by $\mathrm{Bess}_\nu(X)$.

Remark 2. *The requirement that $|D_\nu|^2$ appears with unit coefficient is not at all essential. If the coefficient is a positive function smooth up to $x = 0$, then the quotient of P by this coefficient is a Bessel operator as above, and this normalization does not affect any of the arguments in Sections 3.4, 3.5.*

The class of Bessel operators depends strongly on the pair (x, ϕ), where x is a boundary defining function and ϕ is a collar diffeomorphism compatible with x in the sense of Section 3.1. To make this explicit, write $\mathrm{Bess}_\nu(X; x, \phi)$.

Lemma 3.1.2. *Let x, ρ be two boundary defining functions with associated collar diffeomorphisms ϕ_x, ϕ_ρ. Given $p \in \overline{W}_x$, write*

$$\phi_x^{-1}(p) = (x(p), y(p)) \in [0, \varepsilon) \times \partial X.$$

Let $\Phi = \phi_x^{-1} \circ \phi_\rho$, so that $\Phi(t, z) = ((x \circ \phi_\rho)(t, z), (y \circ \phi_\rho)(t, z))$ for each $(t, z) \in [0, \varepsilon) \times \partial X$. If

$$\partial_t^2 (x \circ \phi_\rho)(0, \cdot) = 0, \quad \partial_t (y \circ \phi_\rho)(0, \cdot) = 0 \tag{3.5}$$

and $P \in \mathrm{Bess}_\nu(X; \rho, \phi_\rho)$, then there exists a positive function $f \in C^\infty(\overline{W}_x \cap \overline{W}_\rho)$ such that $fP \in \mathrm{Bess}(X; x, \phi_x)$.

Proof. The condition (3.5) gives

$$(x \circ \phi_\rho)(t,z) = t\partial_t(x \circ \phi_\rho)(0,z) + \mathcal{O}(t^3), \quad (y \circ \phi_\rho)(t,z) = (y \circ \phi_\rho)(0,z) + \mathcal{O}(t^2).$$

The result can be established from the composition rule for derivatives in local coordinates. □

Lemma 3.1.3. *Suppose that x is a boundary defining function, and that ϕ_x is given by the flow of $\operatorname{grad}_g x/g^{-1}(dx,dx)$ with respect to a fixed metric g defined near ∂X. Let $\rho = e^w x$ and $h = e^{2w}g$, where w is smooth near ∂X and $\partial_x w(0) = 0$. If ϕ_ρ is given by the flow of $\operatorname{grad}_h \rho/h^{-1}(d\rho,d\rho)$, then $\Phi := \phi_x^{-1} \circ \phi_\rho$ satisfies (3.5).*

Proof. First write

$$x(\phi_\rho(t,z)) = e^{-w(\phi_\rho(t,z))}\rho(\phi_\rho(t,z)) = e^{-w(\phi_\rho(t,z))}t,$$

which gives

$$\partial_t^2(x \circ \phi_\rho)(t,z) = e^{-w(\phi_\rho(t,z))}\left(-2\partial_t(w \circ \phi_\rho) + t\partial_t(w \circ \phi_\rho)^2 + t\partial_t^2(w \circ \phi_\rho)\right)(t,z).$$

This quantity vanishes at $t = 0$ provided $\partial_t(w \circ \phi_\rho)(0,z) = 0$, as assumed in the lemma. Next, write

$$\operatorname{grad}_h \rho = xe^w \operatorname{grad}_h w + e^w \operatorname{grad}_h x = e^{-w}(x \operatorname{grad}_g w + \operatorname{grad}_g x).$$

When restricted to ∂X we therefore have

$$\operatorname{grad}_h \rho = e^{-w}\operatorname{grad}_g x, \quad \frac{\operatorname{grad}_h \rho}{h^{-1}(d\rho,d\rho)} = e^w \frac{\operatorname{grad}_g x}{g^{-1}(dx,dx)}.$$

Thus $\partial_t(y \circ \phi_\rho)(0,z)$ is proportional to $(\operatorname{grad}_h \rho)_z y$ for any $z \in \partial X$, since $\phi_\rho(0,z) = z \in \partial X$. Further, $(\operatorname{grad}_h \rho)_z y$ is proportional to $(\operatorname{grad}_g x)_z y$, which is proportional to $\partial_t(y \circ \phi_x)(0,z)$. But $(y \circ \phi_x)(t,z) = z$ for all t, so the derivative of this constant function vanishes, finishing the proof. □

In Lemma 3.1.3 the relation $h = e^{2w}g$ can be replaced by $h = e^v g$ for any smooth v, but in practice h will be related to g as stated.

A (smooth, positive) density μ on $\overline{X}$ is said to be of product type near ∂X if

$$\phi^*\mu = |dx| \otimes \mu_{\partial X}$$

for a fixed density $\mu_{\partial X}$ on ∂X. It is of course always possible to choose a density of product type near ∂X. This is useful in light of the next lemma. If $\overline{X}$ is compact, then $L^2(X)$ may be defined as the space of square integrable functions with respect to any smooth density μ on $\overline{X}$, in particular one of product type near ∂X.

Lemma 3.1.4. *Suppose that μ is of product type near ∂X. If $P \in \mathrm{Bess}_\nu(X)$, then $P^* \in \mathrm{Bess}_\nu(X)$, where P^* is the formal adjoint of P with respect to μ.*

Proof. The pullback of P^* to $(0,\varepsilon) \times \partial X$ is given by

$$|D_\nu|^2 + D_\nu^* B^* + A^*,$$

where A^*, B^* are the formal adjoints of A, B with respect to $\mu_{\partial X}$.

On the other hand, $D_\nu^* B^* = B^* D_\nu^* + [D_x, B^*]$. Furthermore, since $B = xB_1$ for a first order operator B_1 on ∂X depending smoothly on $x \in [0,\varepsilon) \times \partial X$, it follows that

$$B^* D_\nu^* = xB_1^* D_\nu^* = B_1^*(xD_x - i(1/2 - \nu)) = B^* D_\nu - i(1/2-\nu)B_1^*,$$

which completes the proof since the multiple of B_1^* as well as $[D_x, B^*]$ may be absorbed into A^*. □

For the local theory, it is convenient to work on $\mathbb{T}^n_+ = \mathbb{R}_+ \times \mathbb{T}^{n-1}$, where $\mathbb{T}^{n-1} = (\mathbb{R}/2\pi\mathbb{Z})^{n-1}$. The set of Bessel operators on $\mathbb{T}^n_+$ is defined with respect to the canonical product decomposition on $\mathbb{T}^n_+$. Thus $P \in \mathrm{Bess}_\nu(\mathbb{T}^n_+)$ if

$$P(x,y,D_\nu,D_y) = |D_\nu|^2 + \sum_{|\beta|\le 1} b_\beta(x,y) D_y^\beta D_\nu + \sum_{|\alpha|\le 2} a_\alpha(x,y) D_y^\alpha$$

for $b_\beta \in xC^\infty(\overline{\mathbb{T}^n_+})$ and $a_\alpha \in C^\infty(\overline{\mathbb{T}^n_+})$. When working on $\mathbb{T}^n_+$, the functions b_β, a_α are referred to as the *coefficients* of P.

Fix a coordinate chart $Y \subseteq \partial X$ and a diffeomorphism $\theta : Y \to V$, where V is an open subset of $\mathbb{T}^{n-1}$. Setting

$$U = \phi([0,\varepsilon) \times Y),$$

the map $\psi : U \to [0,\varepsilon) \times V$ given by $\psi = (1 \times \theta) \circ \phi^{-1}$ defines a boundary coordinate chart on $\overline{X}$. Given $P \in \mathrm{Bess}_\nu(X)$, there clearly exists $P_U \in \mathrm{Bess}_\nu(\mathbb{T}^n_+)$ such that

$$Pu = P_U(u \circ \psi)$$

for each $u \in C_c^\infty(U^\circ)$. Furthermore, it is always possible to arrange it so that the coefficients of P_U (in the sense of the previous paragraph) are constant outside a compact subset of $\overline{\mathbb{T}^n_+}$.

Ellipticity and the boundary symbol

Given $P \in \mathrm{Bess}_\nu(X)$ which near ∂X has the form

$$P = |D_\nu|^2 + BD_\nu + A,$$

let $A_0(y, D_y) = A(0, y, D_y)$. Ellipticity of P at a point $p \in \partial X$ is defined via the function

$$\xi^2 + \sigma_2(A_0)(p,\eta), \tag{3.6}$$

which is a homogeneous polynomial of degree two in $(\xi,\eta) \in \mathbb{R} \times T_p^* \partial X$.

Definition 3.1.5. *The Bessel operator $P \in \mathrm{Bess}_\nu(X)$ is said to be (properly) elliptic at $p \in \partial X$ if for each $\eta \in T_p^*\partial X \setminus 0$ the polynomial*

$$\xi \mapsto \xi^2 + \sigma_2(A_0)(p,\eta) \tag{3.7}$$

has no real roots.

Ellipticity at $p \in \partial X$ is equivalent to the statement that the homogeneous polynomial $\xi^2 + \sigma_2(A_0)(p,\eta)$ is elliptic in (ξ,η). Thus, ellipticity implies the existence of nonreal roots $\pm\xi(p,\eta)$, where $\operatorname{Im}\xi(p,\eta) < 0$ by convention.

For each $(y,\eta) \in T^*\partial X\setminus 0$, the symbol $\sigma_2(A_0)(y,\eta)$ determines a family of one dimensional Bessel operators given by

$$\widehat{P}_{(y,\eta)} = |D_\nu|^2 + \sigma_2(A_0)(y,\eta). \tag{3.8}$$

The operator $\widehat{P}_{(y,\eta)}$ is called the boundary symbol operator of P. Let $\mathcal{M}_+(y,\eta)$ denote the space of solutions to the equation

$$\widehat{P}_{(y,\eta)} u = 0$$

which are bounded as $x \to \infty$. Ellipticity at $p \in \partial X$ implies that $\dim \mathcal{M}_+(p,\eta) = 1$ for each $\eta \in T_p^*\partial X \setminus 0$. Indeed, the space of solutions to $\widehat{P}_{(p,\eta)} u = 0$ is spanned by the modified Bessel functions

$$\left\{ x^{1/2} K_\nu(i\xi(p,\eta)x),\, x^{1/2} I_\nu(i\xi(p,\eta)x) \right\}.$$

Since $\operatorname{Re} i\xi(p,\eta) > 0$, it follows that

$$x^{1/2} K_\nu(i\xi(p,\eta)x) = \mathcal{O}\left(e^{-x/C}\right), \quad x \to \infty,$$

while the second solution grows exponentially [82, Chapter 7.8]. Thus only the first solution can possible lie in $\mathcal{M}_+(p,\eta)$.

Parameter-dependent Bessel operators

Definition 3.1.6. *Let $\overline{X}$ denote a compact manifold with boundary as in Section 3.1. A differential operator $P(\lambda) \in \mathrm{Diff}^2_{(\lambda)}(X)$ is called a parameter-dependent Bessel operator of order $\nu > 0$ if there exist*

$$A(\lambda) = A(x,y,D_y;\lambda) \in \mathrm{Diff}^2_{(\lambda)}(\partial X), \quad B(\lambda) = B(x,y,D_y;\lambda) \in \mathrm{Diff}^1_{(\lambda)}(\partial X)$$

depending smoothly on $x \in [0,\varepsilon_x)$, such that $B(0,y,D_y;\lambda) = 0$ and

$$\phi_x^* P(\lambda) = |D_\nu|^2 + B(x,y,D_y;\lambda)D_\nu + A(x,y,D_y;\lambda) \tag{3.9}$$

The set of such operators is denoted by $\mathrm{Bess}^{(\lambda)}_\nu(X)$.

Ellipticity with parameter is defined by replacing the standard principal symbol of A with its parameter-dependent version. Begin by fixing an angular sector $\Lambda \subseteq \mathbb{C}$.

Definition 3.1.7. *A parameter-dependent Bessel operator $P(\lambda)$ is said to be (properly) parameter-elliptic with respect to Λ at $p \in \partial X$ if for each $(\eta, \lambda) \in T_p^*\partial X \times \Lambda \setminus 0$, the polynomial*

$$\xi \mapsto \xi^2 + \sigma_2^{(\lambda)}(A_0)(p, \eta; \lambda) \tag{3.10}$$

has no real roots.

Similarly, for $(y, \eta, \lambda) \in T^*\partial X \times \Lambda \setminus 0$, define

$$\widehat{P}_{(y,\eta;\lambda)} = |D_\nu|^2 + \sigma_2^{(\lambda)}(A_0)(y, \eta; \lambda),$$

and then let $\mathcal{M}_+(y, \eta; \lambda)$ denote the space of solutions to $\widehat{P}_{(y,\eta;\lambda)}u = 0$ which are bounded as $x \to \infty$. As before, this space is one-dimensional.

3.2 Motivation: asymptotically anti-de Sitter spacetimes

The main motivation for the introduction of Bessel operators is to study the stationary Klein–Gordon equation on a stationary aAdS spacetime as in Section 2.3.

Lemma 3.2.1. *Let $(\overline{M}, g)$ denote a stationary aAdS spacetime with conformal boundary $\mathcal{I}$. Suppose that x is a special boundary defining function for $\mathcal{I}$, and that g is even modulo $\mathcal{O}(x^3)$. Define the rescaled and conjugated stationary Klein–Gordon operator $P(\lambda)$ on X as in (2.14). If $\overline{X}$ intersects $\mathcal{I}$ orthogonally, Then $P(\lambda)$ is a parameter-dependent Bessel operator on X of order ν with respect to x.*

Proof. This follows from Lemma 2.3.7: since $x^{-1}e(x)D_x$ and $x^{-1}e(x)D_\nu$ differ by a smooth multiplication operator, the difference can be absorbed into the terms not involving differentiation in x. □

If T is timelike at $\mathcal{I}$, then according to the discussion following Lemma 2.3.7, the Bessel operator $P(\lambda)$ is elliptic at ∂X. Furthermore, if dt is timelike, then $P(\lambda)$ is parameter-elliptic with respect to any angular sector $\Lambda \subseteq \mathbb{C}$ disjoint from $\mathbb{R} \setminus 0$.

3.3 Function spaces and mapping properties

The purpose of this section is to define Sobolev-type spaces $\mathcal{H}^s$ based on the elementary derivatives ∂_ν and $|D_\nu|^2$, both on $\mathbb{T}^n_+$ and on a manifold with boundary. Finally, it is shown that Bessel operators act continuously between these spaces.

The exposition is closest to that of [100], where these "twisted" Sobolev spaces were first introduced in the context of aAdS geometry. The relationship between $\mathcal{H}^1$ and certain weighted Sobolev spaces was exploited both in [100] and also in the closely related study of asymptotically hyperbolic spaces in [23]. Similar spaces are also defined in [66], and in [67, 108] as related to the Hankel transform.

Throughout this section the spaces $L^2(\mathbb{T}^n_+)$ and $L^2(\mathbb{T}^{n-1})$ are defined with respect to ordinary Lebesgue measure, and $H^m(\mathbb{T}^{n-1})$ denotes the standard Sobolev space of order m on $\mathbb{T}^{n-1}$. The notation $\mathcal{H}^0(\mathbb{T}^n_+) := L^2(\mathbb{T}^n_+)$ is also frequently used.

The weighted space $H^1_\mu(\mathbb{T}^n_+)$

Given $\mu \in \mathbb{R}$, let

$$H^1_\mu(\mathbb{T}^n_+) = \{u \in \mathscr{D}'(\mathbb{T}^n_+) : x^{\frac{\mu}{2}}\partial^\alpha u \in L^2(\mathbb{T}^n_+) \text{ for } |\alpha| \leq 1\},$$

which is a Hilbert space under the norm

$$\|u\|^2_{H^1_\mu(\mathbb{T}^n_+)} = \sum_{|\alpha|\leq 1} \|x^{\frac{\mu}{2}}\partial^\alpha u\|^2_{L^2(\mathbb{T}^n_+)}.$$

Furthermore, let $\mathring{H}^1_\mu(\mathbb{T}^n_+)$ denote the closure of $C^\infty_c(\mathbb{T}^n_+)$ in H^1_μ. These spaces are well studied, see Lions [74], Grisvard [47] for example.

Lemma 3.3.1. *The following hold for $\mu \in \mathbb{R}$.*

1. *If $|\mu| < 1$, then $C^\infty_c(\overline{\mathbb{T}^n_+})$ is dense in $H^1_\mu(\mathbb{T}^n_+)$.*
2. *If $|\mu| \geq 1$, then $H^1_\mu(\mathbb{T}^n_+) = \mathring{H}^1_\mu(\mathbb{T}^n_+)$.*

Proof. Proofs of these facts may be found in [47, 74]. □

Given a Hilbert space E, let $H^1_\mu(\mathbb{R}_+; E)$ denote the Hilbert space of E-valued distributions $u \in \mathscr{D}'(\mathbb{R}_+; E)$ such that

$$x^{\frac{\mu}{2}} u \in L^2(\mathbb{R}_+; E), \quad x^{\frac{\mu}{2}} u' \in L^2(\mathbb{R}_+; E),$$

equipped with obvious norm. The Sobolev embedding theorem in this setting, [47, Proposition 1.1'], says that $H^1_\mu(\mathbb{R}_+; E) \hookrightarrow C^0(\overline{\mathbb{R}_+}; E)$ for $\mu < 1$, thus the map $u \mapsto u(0)$ is continuous $H^1_\mu(\mathbb{R}_+; E) \to E$. Since $H^1_\mu(\mathbb{T}^n_+) \subseteq H^1_\mu(\mathbb{R}_+; L^2(\mathbb{T}^{n-1}))$, it follows that any $u \in H^1_\mu(\mathbb{T}^n_+)$ admits a trace

$$u \mapsto u|_{\mathbb{T}^{n-1}} \in L^2(\mathbb{T}^{n-1}). \tag{3.11}$$

Furthermore, the kernel of $u \mapsto u|_{\mathbb{T}^{n-1}}$ is $\mathring{H}^1_\mu(\mathbb{T}^n_+)$ — see [47, Proposition 1.2]. The next lemma improves upon the regularity of this restriction.

Lemma 3.3.2. *Suppose that $\mu < 1$. Then the restriction*

$$u \mapsto u|_{\mathbb{T}^{n-1}}, \quad u \in C^\infty_c(\overline{\mathbb{T}^n_+})$$

extends uniquely to continuous map $\gamma : H^1_\mu(\mathbb{T}^n_+) \to H^{(1-\mu)/2}(\mathbb{T}^{n-1})$, and furthermore γ admits a continuous right inverse.

Sketch of proof. By the Sobolev embedding, any $\varphi \in C_c^\infty(\overline{\mathbb{R}_+}) \subseteq H^1_\mu(\mathbb{R}_+)$ admits an estimate of the form

$$|\varphi(0)|^2 \leq C \int_{\mathbb{R}_+} x^\mu \left(|\varphi|^2 + |\varphi'|^2\right) dx.$$

Apply this inequality to the function $\varphi(sx)$, and then choose s (depending on φ) satisfying

$$\int_{\mathbb{R}} x^\mu |\varphi|^2 \, dx = s^2 \int_{\mathbb{R}} x^\mu |\varphi'|^2 \, dx.$$

This yields the estimate

$$|\varphi(0)|^2 \leq 2C \left(\int_{\mathbb{R}_+} x^\mu |\varphi|^2 dx \right)^{(1-\mu)/2} \left(\int_{\mathbb{R}_+} x^\mu |\varphi'|^2 dx \right)^{(1+\mu)/2}. \tag{3.12}$$

Now consider $u \in H^1_\mu(\mathbb{T}^n_+)$ and let $\hat{u}(q)$ denote its Fourier coefficients, where $q \in \mathbb{Z}^{n-1}$. It suffices to apply the inequality (3.12) to $\hat{u}(q)$, which lies in $H^1_\mu(\mathbb{R}_+)$ for each $q \in \mathbb{Z}^{n-1}$. Multiplying (3.12) by $\langle q \rangle^{1-\mu}$ and summing over all q, it follows that

$$\|\gamma u\|_{H^{(1-\mu)/2}} \leq C \|u\|_{H^1_\mu(\mathbb{R}_+)}.$$

The unique continuation of γ follows from the density of $C_c^\infty(\overline{\mathbb{T}^n_+})$ in $H^1_\mu(\mathbb{T}^n_+)$. That γ admits a right inverse is also straightforward, see Lemma 3.7.3 for a closely related result. □

The trace $u \mapsto \gamma u$ defined in Lemma 3.3.2 agrees with the restriction given by (3.11) since they both agree on the dense set $C_c^\infty(\overline{\mathbb{T}^n_+})$.

The space $\mathcal{H}^1(\mathbb{T}^n_+)$

Given $\nu \in \mathbb{R}$, define

$$\mathcal{H}^1(\mathbb{T}^n_+) = \{u \in \mathscr{D}'(\mathbb{T}^n_+) : \partial_\nu^j \partial_y^\alpha u \in L^2(\mathbb{T}^n_+) \text{ for } j + |\alpha| \leq 1\},$$

where $\partial_\nu^j \partial_y^\alpha u$ is taken in the sense of distributions on $\mathbb{T}^n_+$. Then $\mathcal{H}^1(\mathbb{T}^n_+)$ is a Hilbert space when equipped with the norm

$$\|u\|^2_{\mathcal{H}^1(\mathbb{T}^n_+)} = \sum_{j+|\alpha|\leq 1} \|\partial_\nu^j \partial_y^\alpha u\|^2_{L^2(\mathbb{T}^n_+)},$$

The space $\mathcal{H}^1_*(\mathbb{T}^n_+)$ is defined analogously by replacing ∂_ν with its formal adjoint ∂_ν^*. Let $\mathring{\mathcal{H}}^1(\mathbb{T}^n_+)$ denote the closure of $C_c^\infty(\mathbb{T}^n_+)$ in $\mathcal{H}^1(\mathbb{T}^n_+)$, and similarly for $\mathring{\mathcal{H}}^1_*(\mathbb{T}^n_+)$.

Lemma 3.3.3. *If $\nu \neq 0$, then $\mathring{\mathcal{H}}^1(\mathbb{T}^n_+) = \mathring{H}^1(\mathbb{T}^n_+)$ with an equivalence of norms.*

Proof. Let $u \in C_c^\infty(\mathbb{T}^n_+)$. Note that

$$0 \leq \langle \partial_\nu u, \partial_\nu u\rangle_{\mathbb{T}^n_+} = \|\partial_x u\|^2_{L^2(\mathbb{T}^n_+)} + (\nu^2 - 1/4)\|x^{-1}u\|^2_{L^2(\mathbb{T}^n_+)},$$

so Hardy's inequality in one dimension follows by plugging in $\nu = 0$. From this and for any $\nu \in \mathbb{R}$,

$$\|\partial_\nu u\|_{L^2(\mathbb{T}^n_+)} \leq \|\partial_x u\|_{L^2(\mathbb{T}^n_+)} + |\nu - 1/2|\|x^{-1}u\|_{L^2(\mathbb{T}^n_+)} \leq C_\nu \|\partial_x u\|_{L^2(\mathbb{T}^n_+)}.$$

Conversely, If $|\nu| \geq 1/2$, then $\nu^2 - 1/4 \geq 0$, so

$$\|\partial_x u\|_{L^2(\mathbb{T}^n_\sharp)} \leq \|\partial_\nu u\|_{L^2(\mathbb{T}^n_\sharp)}.$$

If $0 < |\nu| < 1/2$, then $\nu^2 - 1/4 < 0$, so

$$(4\nu^2 - 1)\|\partial_x u\|^2_{L^2(\mathbb{T}^n_+)} \leq (\nu^2 - 1/4)\|x^{-1}u\|_{L^2(\mathbb{T}^n_+)}.$$

This also gives $4\nu^2\|\partial_x u\|^2_{L^2(\mathbb{T}^n_+)} \leq \|\partial_\nu u\|^2_{L^2(\mathbb{T}^n_+)}$. In either case, adding the $L^2(\mathbb{T}^n_+)$ norms of $\partial_y^\alpha u$ for $|\alpha| \leq 1$ shows that

$$C_\nu^{-1}\|u\|_{H^1(\mathbb{T}^n_+)} \leq \|u\|_{\mathcal{H}^1(\mathbb{T}^n_+)} \leq C_\nu\|u\|_{H^1(\mathbb{T}^n_+)}$$

for a constant $C_\nu > 0$ depending on $\nu \neq 0$. The result follows from the density of $C_c^\infty(\mathbb{T}^n_+)$ in both spaces. □

The basic observation concerning $\mathcal{H}^1(\mathbb{T}^n_+)$ is that the map $\mathscr{D}'(\mathbb{T}^n_+) \to \mathscr{D}'(\mathbb{T}^n_+)$ given by $u \mapsto x^{\nu-1/2}u$ restricts to an isometric isomorphism

$$\mathcal{H}^1(\mathbb{T}^n_+) \to H^1_{1-2\nu}(\mathbb{T}^n_+).$$

It follows from Lemma 3.3.1 that $x^{1/2-\nu}C^\infty(\overline{\mathbb{T}^n_+})$ is dense in $\mathcal{H}^1(\mathbb{T}^n_+)$ if $0 < \nu < 1$, and $\mathcal{H}^1(\mathbb{T}^n_+) = \mathring{\mathcal{H}}^1(\mathbb{T}^n_+)$ if $\nu \geq 1$. Using Lemma 3.3.2, it is also possible to define weighted traces of $\mathcal{H}^1(\mathbb{T}^n_+)$ functions, as will be explained in Section 3.3.

The space $\mathcal{H}^2(\mathbb{T}^n_+)$

Given $\nu > 0$, define

$$\mathcal{H}^2(\mathbb{T}^n_+) = \{u \in \mathcal{H}^1(\mathbb{T}^n_+) : \partial_\nu u \in \mathcal{H}^1_*(\mathbb{T}^n_+), \text{ and } \partial_y^\alpha u \in \mathcal{H}^1(\mathbb{T}^n_+) \text{ for } |\alpha| \leq 1\}.$$

Then $\mathcal{H}^2(\mathbb{T}^n_+)$ becomes a Hilbert space when equipped with the norm

$$\|u\|^2_{\mathcal{H}^2(\mathbb{T}^n_+)} = \|\partial_\nu^*\partial_\nu u\|^2_{L^2(\mathbb{T}^n_+)} + \sum_{|\alpha|\leq 1}\|\partial_y^\alpha u\|^2_{\mathcal{H}^1(\mathbb{T}^n_+)}. \tag{3.13}$$

Although $x^{1/2-\nu}C_c^\infty(\mathbb{T}^n_+)$ is dense in $\mathcal{H}^1(\mathbb{T}^n_+)$ when $0 < \nu < 1$, this is not the case for $\mathcal{H}^2(\mathbb{T}^n_+)$. In fact, $x^{1/2-\nu}C_c^\infty(\overline{\mathbb{T}^n_+})$ is not contained in $\mathcal{H}^2(\mathbb{T}^n_+)$ unless $\nu = 1/2$. An appropriate dense space of smooth functions is defined in Section 3.3.

Weighted traces

It follows from Lemma 3.3.2 that the weighted restriction

$$u \mapsto x^{\nu-1/2} u|_{\mathbb{T}^{n-1}}, \quad u \in x^{1/2-\nu} C_c^\infty(\overline{\mathbb{T}^n_+})$$

extends uniquely to a continuous map $\gamma_- : \mathcal{H}^1(\mathbb{T}^n_+) \to H^\nu(\mathbb{T}^{n-1})$, and furthermore γ_- admits a continuous right inverse. This is true for all $\nu > 0$. Similarly, there exists a weighted restricted

$$\gamma_-^* : \mathcal{H}^1_*(\mathbb{T}^n_+) \to H^{1-\nu}(\mathbb{T}^{n-1}),$$

initially defined for $u \in x^{\nu-1/2} C_c^\infty(\overline{\mathbb{T}^n_+})$ by $\gamma_-^* u = x^{1/2-\nu} u|_{\mathbb{T}^{n-1}}$. However, note that γ_-^* is now defined for $\nu < 1$ — indeed, $\mathcal{H}^1_*(\mathbb{T}^n_+)$ is isomorphic to $H^1_{2\nu-1}(\mathbb{T}^n_+)$, and the trace on $H^1_{2\nu-1}(\mathbb{T}^n_+)$ is only defined for $2\nu - 1 < 1$. Since $u \in \mathcal{H}^2(\mathbb{T}^n_+)$ implies $\partial_\nu u \in \mathcal{H}^1_*(\mathbb{T}^n_+)$, there exists a second trace

$$\gamma_+ : \mathcal{H}^2(\mathbb{T}^n_+) \to H^{1-\nu}(\mathbb{T}^{n-1})$$

given by the composition $\gamma_+ = \gamma_-^* \circ \partial_\nu$. The trace γ_+ therefore only exists for $0 < \nu < 1$, while γ_- is well defined for $\nu > 0$.

Definition 3.3.4. *Given $\nu > 0$, let $\mathcal{F}_\nu$ denote the following spaces of functions.*

1. *If $0 < \nu < 1$, then $\mathcal{F}_\nu$ consists of $u \in C^\infty(\mathbb{T}^n_+)$ of the form*

$$u(x, y) = x^{1/2-\nu} u_-(x^2, y) + x^{1/2+\nu} u_+(x^2, y) \tag{3.14}$$

   for some $u_\pm \in C_c^\infty(\overline{\mathbb{T}^n_+})$.

2. *If $\nu \geq 1$, then $\mathcal{F}_\nu = C_c^\infty(\mathbb{T}^n_+)$.*

Note that $\mathcal{F}_\nu$ is contained in $\mathcal{H}^s(\mathbb{T}^n_+)$ for each $s = 0, 1, 2$. If $0 < \nu < 1$, then $\mathcal{F}_\nu$ is not typically contained in $x^{1/2-\nu} C_c^\infty(\overline{\mathbb{T}^n_+})$ (unless $\nu = 1/2$); on the other hand, traces of $u \in \mathcal{F}_\nu$ are still easily computed from the definitions.

Lemma 3.3.5. *Suppose that $0 < \nu < 1$. If $u \in \mathcal{F}_\nu$ satisfies (3.14), then*

$$\gamma_- u = u_-(0, \cdot), \quad \gamma_+ u = 2\nu u_+(0, \cdot) \tag{3.15}$$

Proof. If $u \in \mathcal{F}_\nu$ satisfies (3.14), then $x^{\nu-1/2} u(x, y) = u_-(x^2, y) + x^{2\nu} u_+(x^2, y)$. Since this function is continuous on $\mathbb{R}_+$ with values in $C^\infty(\mathbb{T}^{n-1})$, it follows that $\gamma_- u = u_-(0, \cdot)$, see the remark after Lemma 3.3.2. A similar argument shows that $\gamma_+ u = 2\nu u_+(0, \cdot)$. □

Lemma 3.3.6. *Suppose that $\nu > 0$ and $s = 0, 1, 2$. Then $\mathcal{F}_\nu$ is dense in $\mathcal{H}^s(\mathbb{T}^n_+)$.*

Proof. A proof is provided in Appendix 3.7. □

Proposition 3.3.7. *Suppose that* $0 < \nu < 1$*. Then there exist unique continuous maps*

$$\gamma_{\mp} : \mathcal{H}^s(\mathbb{T}^n_+) \to H^{s-1\pm\nu}(\mathbb{T}^{n-1})$$

such that if $u \in \mathcal{F}_\nu$ *satisfies* (3.14)*, then* $\gamma_- u = u_-(0,\cdot)$ *and* $\gamma_+ u = 2\nu u_+(0,\cdot)$*. Here* γ_- *is defined for* $s = 1, 2$*, while* γ_+ *is only defined for* $s = 2$*.*

Proof. Combining Lemma 3.3.6 with Lemma 3.3.5 shows that the map

$$u \mapsto u_-(0,\cdot), \quad u \in \mathcal{F}_\nu$$

admits a unique extension $\mathcal{H}^s(\mathbb{T}^n_+) \to H^\nu(\mathbb{T}^{n-1})$ for $s = 1, 2$. The additional regularity $\gamma_- u \in H^{1+\nu}(\mathbb{T}^{n-1})$ for $u \in \mathcal{H}^2(\mathbb{T}^n_+)$ follows from the equality $\gamma_\pm \partial_y^\alpha u = \partial_y^\alpha \gamma_\pm u$ for $u \in \mathcal{F}_\nu$ and each multiindex α. Similarly, the map

$$u \mapsto 2\nu u_+(0,\cdot), \quad u \in \mathcal{F}_\nu$$

admits a unique extension $\mathcal{H}^2(\mathbb{T}^n_+) \to H^{1-\nu}(\mathbb{T}^{n-1})$.

□

Remark 3. *As noted above,* γ_- *can be defined for all* $\nu > 0$ *(rather than just* $0 < \nu < 1$*) but this fact is only ever used in Lemma 3.7.5 of the Appendix.*

Dual spaces

Throughout, $\mathcal{H}^0(\mathbb{T}^n_+) = L^2(\mathbb{T}^n_+)$ is identified with its own antidual $\mathcal{H}^0(\mathbb{T}^n_+)'$ via the Riesz representation. Given $s = 1, 2$, let

$$\mathcal{H}^{-s}(\mathbb{T}^n_+) = \mathcal{H}^s(\mathbb{T}^n_+)'$$

denote the corresponding antiduals. Since the inclusion $\iota : \mathcal{H}^s(\mathbb{T}^n_+) \hookrightarrow \mathcal{H}^0(\mathbb{T}^n_+)$ is dense, $\mathcal{H}^0(\mathbb{T}^n_+)$ is identified with a dense subspace of $\mathcal{H}^{-s}(\mathbb{T}^n_+)$ via the map $\iota^* : \mathcal{H}^0(\mathbb{T}^n_+) \hookrightarrow \mathcal{H}^{-s}(\mathbb{T}^n_+)$. Thus if $s \geq 0$ and $u, v \in \mathcal{H}^s(\mathbb{T}^n_+)$, then the image $\iota^* u$ in $\mathcal{H}^{-s}(\mathbb{T}^n_+)$ acts on v via the $\mathcal{H}^0(\mathbb{T}^n_+)$ pairing

$$\iota^* u(v) = \langle u, v \rangle_{\mathbb{T}^n_+}.$$

Because $\mathcal{H}^s(\mathbb{T}^n_+)$ is dense in $\mathcal{H}^{-s}(\mathbb{T}^n_+)$, there is no ambiguity in using the notation

$$\langle f, v \rangle_{\mathbb{T}^n_+} := f(v), \quad f \in \mathcal{H}^{-s}(\mathbb{T}^n_+),\ v \in \mathcal{H}^s(\mathbb{T}^n_+)$$

in general.

A Fourier characterization

Given $s = 0, 1, 2$, any $u \in \mathcal{H}^s(\mathbb{T}^n_+)$ has well defined Fourier coefficients

$$\hat{u}(q) = (2\pi)^{-(n-1)/2} \int_{[-\pi,\pi]^{n-1}} e^{-i\langle q,y\rangle} u(\cdot, y)\, dy, \quad q \in \mathbb{Z}^{n-1}.$$

It is easily seen $\hat{u}(q) \in \mathcal{H}^s(\mathbb{R}_+)$ for each fixed $q \in \mathbb{Z}^{n-1}$.

This may be extended uniquely by duality: given $f \in \mathcal{H}^{-s}(\mathbb{T}^n_+)$, let $\hat{f}(q) \in \mathcal{H}^{-s}(\mathbb{R}_+)$ denote the functional

$$\langle \hat{f}(q), v\rangle_{\mathbb{T}^n_+} = (2\pi)^{-(n-1)/2} \langle f, e^{i\langle q,y\rangle} v\rangle_{\mathbb{T}^n_+}, \tag{3.16}$$

where $v \in \mathcal{H}^s(\mathbb{R}_+)$. Given $\tau > 0$ and $u \in \mathcal{F}_\nu$, let

$$(S_\tau u)(x, y) = u(\tau x, y) \tag{3.17}$$

denote the action of dilation in the normal variable. This clearly extends to a bounded map $S_\tau : \mathcal{H}^s(\mathbb{T}^n_+) \to \mathcal{H}^s(\mathbb{T}^n_+)$ for $s = 0, 1, 2$. Furthermore, S_τ may be extended uniquely to $\mathcal{H}^{-s}(\mathbb{T}^n_+)$ by duality: given $f \in \mathcal{H}^{-s}(\mathbb{T}^n_+)$, define

$$\langle S_\tau f, v\rangle_{\mathbb{T}^n_+} = \tau^{-1} \langle f, S_{\tau^{-1}} v\rangle_{\mathbb{T}^n_+}$$

for $v \in \mathcal{H}^s(\mathbb{T}^n_+)$.

Lemma 3.3.8. *Given* $s = 0, \pm 1, \pm 2$,

$$\|u\|^2_{\mathcal{H}^s(\mathbb{T}^n_+)} = \sum_{q \in \mathbb{Z}^{n-1}} \langle q\rangle^{2s-1} \|S_{\langle q\rangle^{-1}} \hat{u}(q)\|^2_{\mathcal{H}^s(\mathbb{R}_+)}.$$

for each $u \in \mathcal{F}_\nu$.

Proof. When $s \geq 0$ this follows immediately from Parseval and Fubini's theorems. When $s < 0$, the proof is exactly the same as in [71, Lemma 2.3.1]. □

The space $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$

If $\underline{t} = (t_1, \ldots, t_k)$, define

$$H^{\underline{t}}(\mathbb{T}^{n-1}) := \prod_{j=1}^{k} H^{t_k}(\mathbb{T}^{n-1}).$$

Keeping this notation in mind, let $\underline{\nu} = (1 - \nu, 1 + \nu)$ and then set

$$\underline{\gamma} = \begin{pmatrix} \gamma_- \\ \gamma_+ \end{pmatrix}. \tag{3.18}$$

Following [71, 84], define the following spaces for $0 < \nu < 1$. Given $s = 0, \pm 1, \pm 2$, let $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$ denote the set of all

$$(u, \phi_-, \phi_+) \in \mathcal{H}^s(\mathbb{T}^n_+) \times H^{s-\underline{\nu}}(\mathbb{T}^{n-1})$$

such that

1. $\phi_- = \gamma_- u$ and $\phi_+ = \gamma_+ u$ if $s = 2$,
2. $\phi_- = \gamma_- u$ and ϕ_+ is arbitrary if $s = 1$,
3. $\phi_\pm$ are arbitrary if $s \leq 0$.

A typical element of $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$ will be denoted $(u, \underline{\phi})$, where $\underline{\phi} = (\phi_-, \phi_+)$. The norm of $(u, \underline{\phi})$ is given by

$$\|(u, \underline{\phi})\|^2_{\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)} = \|u\|^2_{\mathcal{H}^s(\mathbb{T}^n_+)} + \|\underline{\phi}\|^2_{H^{s-\underline{\nu}}(\mathbb{T}^{n-1})}.$$

If $s = 2$, then $u \mapsto (u, \underline{\gamma})$ provides an isomorphism

$$\mathcal{H}^2(\mathbb{T}^n_+) \to \widetilde{\mathcal{H}}^2(\mathbb{T}^n_+).$$

On the other hand, if $s \leq 1$, then the two spaces $\mathcal{H}^s(\mathbb{T}^n_+)$, $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$ are fundamentally different.

Lemma 3.3.9. *Suppose that $0 < \nu < 1$. Then for each $s = 0, \pm 1, \pm 2$, the set*

$$\{(u, \underline{\gamma} u) : u \in \mathcal{F}_\nu\}$$

is dense in $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$.

Proof. It suffices to prove this for $s \geq 0$, since $\widetilde{\mathcal{H}}^0(\mathbb{T}^n_+)$ is dense in $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$ if $s < 0$. Assuming that $s \geq 0$ and $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$, choose $u_n \in \mathcal{F}_\nu$ such that $u_n \to u$ in $\mathcal{H}^s(\mathbb{T}^n_+)$. If $\chi \in C_c^\infty(\overline{\mathbb{R}_+})$ satisfies $\chi = 1$ near $x = 0$, define

$$u_{n,\varepsilon} = u_n - \left(x^{1/2-\nu}(\gamma_- u_n - \phi_-) + (2\nu)^{-1} x^{1/2+\nu}(\gamma_+ u_n - \phi_+)\right) \chi(\varepsilon^{-1} x).$$

Clearly $u_{n,\varepsilon} \in \mathcal{F}_\nu$ and $\gamma_\pm u_{n,\varepsilon} = \phi_\pm$. Furthermore, since $s \geq 0$, it is particularly easy to check that $u_{n,\varepsilon} \to u_n$ in $\mathcal{H}^s(\mathbb{T}^n_+)$ for n fixed and $\varepsilon \to 0$. Thus it is possible find a sequence $\varepsilon_n \to 0$ such that $u_{n,\varepsilon_n} \to u$ in $\mathcal{H}^s(\mathbb{T}^n_+)$ as $n \to \infty$, and hence

$$(u_{n,\varepsilon_n}, \underline{\gamma} u_{n,\varepsilon_n}) \to (u, \underline{\phi})$$

in $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$. □

Recall from Section 3.3 the dilation S_τ, defined by (3.17). Note that

$$(\gamma_- \circ S_\tau)u = \tau^{1/2-\nu}\gamma_- u, \quad (\gamma_+ \circ S_\tau)u = \tau^{1/2+\nu}\gamma_+ u$$

for each $\tau > 0$ and $u \in \mathcal{F}_\nu$. It follows that S_τ may be extended uniquely to $\widetilde{\mathcal{H}}^{s,k}(\mathbb{T}^n_+)$ by defining

$$S_\tau(u, \underline{\phi}) = (S_\tau u, \tau^{1/2-\nu}\phi_-, \tau^{1/2+\nu}\phi_+).$$

It follows from Lemma 3.3.8 and the usual Fourier characterization of $H^m(\mathbb{T}^{n-1})$ that

$$\|(u, \underline{\phi})\|^2_{\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)} = \sum_{q \in \mathbb{Z}^{n-1}} \langle q \rangle^{2s-1} \|S_{\langle q \rangle^{-1}}(\hat{u}(q), \hat{\underline{\phi}}(q))\|^2_{\mathcal{H}^s(\mathbb{R}_+)}$$

for each $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$.

Parameter-dependent norms

When considering the action of parameter-dependent Bessel operators, one must consider modified norms on the spaces defined so far. Given $s = 0, 1, 2$ and $u \in \mathcal{H}^s(\mathbb{T}^n_+)$, let

$$|||u|||^2_{\mathcal{H}^s(\mathbb{T}^n_+)} = \sum_{j=0}^{s} |\lambda|^{2(s-j)} \|u\|^2_{\mathcal{H}^s(\mathbb{T}^n_+)}.$$

Furthermore, if $f \in \mathcal{H}^{-s}(\mathbb{T}^n_+)$, let

$$|||f|||_{\mathcal{H}^{-s}(\mathbb{T}^n_+)} = \sup\{|\langle f, u\rangle_{\mathbb{T}^n_+}| : |||v|||_{\mathcal{H}^s(\mathbb{T}^n_+)} = 1\}.$$

Recall the standard parameter-dependent norms $|||v|||_{\mathcal{H}^m(\mathbb{T}^{n-1})}$ on $H^m(\mathbb{T}^{n-1})$ as in Section 3.1. Given $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$, set

$$|||(u, \underline{\phi})|||^2_{\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)} = |||u|||^2_{\mathcal{H}^s(\mathbb{T}^n_+)} + |||\underline{\phi}|||^2_{H^{s-\underline{\nu}}(\mathbb{T}^{n-1})}.$$

These parameter-dependent norms have the property that there exists $C > 0$ independent of λ such that

$$|||u|||_{\mathcal{H}^{s-1}(\mathbb{T}^n_+)} \leq C|\lambda|^{-1}|||u|||_{\mathcal{H}^s(\mathbb{T}^n_+)}, \quad |||(u, \underline{\phi})|||_{\widetilde{\mathcal{H}}^{s-1}(\mathbb{T}^n_+)} \leq C|\lambda|^{-1}|||(u, \underline{\phi})|||_{\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)}$$

for $u \in \mathcal{H}^s(\mathbb{T}^n_+)$ and $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$, respectively.

Mapping properties

In this section, mapping properties of Bessel operators on $\mathbb{T}^n_+$ are examined. The analogue of Green's formulas is established, which allows the extension of P to spaces with low regularity. Recall from Section 3.1 that $P \in \mathrm{Bess}_\nu(\mathbb{T}^n_+)$ means that

$$P = |D_\nu|^2 + B(x, y, D_y)D_\nu + A(x, y, D_y), \tag{3.19}$$

where $B \in \mathrm{Diff}^1(\mathbb{T}^{n-1})$, $A \in \mathrm{Diff}^2(\mathbb{T}^{n-1})$ depend smoothly on $x \in \overline{\mathbb{R}_+}$ and $B(0, y, D_y) = 0$. Throughout this section, assume that the coefficients of A, B are constant outside a compact subset of $\overline{\mathbb{T}^n_+}$. The boundedness of each term in P will be exmined individually.

Before proceeding, it is necessary to consider certain multipliers of $\mathcal{H}^s(\mathbb{T}^n_+)$ when $s \geq 0$. The commutation relations

$$[\partial_\nu, \varphi] = \partial_x\varphi = [\partial_\nu^*, \varphi], \quad [|D_\nu|^2, \varphi] = -\partial_x^2\varphi - 2(\partial_x\varphi)\partial_x. \tag{3.20}$$

will be used throughout the following lemma.

Lemma 3.3.10. *Suppose that $\varphi \in C^\infty(\overline{\mathbb{T}^n_+})$ is bounded along with all of its derivatives, and consider multiplication by φ as a continuous map $\mathscr{D}'(\mathbb{T}^n_+) \to \mathscr{D}'(\mathbb{T}^n_+)$.*

1. *For $s = 0, 1$, multiplication by φ restricts to a continuous map*

$$\mathcal{H}^s(\mathbb{T}^n_+) \to \mathcal{H}^s(\mathbb{T}^n_+).$$

2. *If $\partial_x\varphi|_{\mathbb{T}^{n-1}} = 0$, then multiplication by φ restricts to a continuous map*

$$\mathcal{H}^2(\mathbb{T}^n_+) \to \mathcal{H}^2(\mathbb{T}^n_+).$$

In either of these two cases,

$$\|\varphi u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} \leq \|\varphi\|_{C^0(\overline{\mathbb{T}^n_+})}\|u\|_{\mathcal{H}^s} + C_s\|u\|_{\mathcal{H}^{s-1}(\mathbb{T}^n_+)}, \tag{3.21}$$

where $C_s \geq 0$ depends on the first s derivatives of φ, and $C_0 = 0$.

Proof. The continuity statement is obvious for $s = 0$. For $s = 1$ it follows from the first commutator formula (3.20). When $s = 2$, the additional condition $\partial_x\phi|_{\mathbb{T}^{n-1}} = 0$ is needed to ensure that

$$u \mapsto (\partial_x\varphi)\partial_x u$$

is bounded $\mathcal{H}^1(\mathbb{T}^n_+) \to \mathcal{H}^0(\mathbb{T}^n_+)$: the vanishing of $\partial_x\phi$ implies $(\partial_x\varphi)\partial_x = (\partial_x\varphi)\partial_\nu$ modulo multiplication by a smooth function, which acts continuously by the first part. The estimate (3.21) follows as well from (3.20). $\square$

Remark 4. *Lemma 3.3.10 result may also be extended to $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$ by defining*

$$\varphi(u, \underline{\phi}) := (\varphi u, \varphi|_{\mathbb{T}^{n-1}}\underline{\phi}),$$

and using that standard Sobolev spaces on $\mathbb{T}^{n-1}$ are closed under multiplication by smooth functions.

Remark 5. *The hypotheses of Lemma 3.3.10 can not be improved when $s = 2$. In other words $\mathcal{H}^2(\mathbb{T}^n_+)$ is not closed under multiplication by arbitrary $C^\infty_c(\overline{\mathbb{T}^n_+})$ functions. On the other hand, if $\varphi \in C^\infty_c(\overline{\mathbb{T}^n_+})$ is constant in a neighborhood of $\mathbb{T}^{n-1}$, then $\mathcal{H}^s(\mathbb{T}^n_+)$ is closed under multiplication by φ for each $s = 0, 1, 2$.*

Now consider the term $|D_\nu|^2$, which is clearly bounded

$$|D_\nu|^2 : \mathcal{H}^2(\mathbb{T}^n_+) \to \mathcal{H}^0(\mathbb{T}^n_+).$$

The distinction between $0 < \nu < 1$ and $\nu \geq 1$ plays an important role when extending this action. Suppose that $0 < \nu < 1$, and let J denote the usual symplectic matrix,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the following formulae are valid for each $u, v \in \mathcal{F}_\nu$:

$$\left\langle |D_\nu|^2 u, v \right\rangle_{\mathbb{T}^n_+} = \left\langle u, |D_\nu|^2 v \right\rangle_{\mathbb{T}^n_+} + \left\langle \underline{\gamma} u, J \underline{\gamma} v \right\rangle_{\mathbb{T}^{n-1}}, \tag{3.22}$$

$$\left\langle |D_\nu|^2 u, v \right\rangle_{\mathbb{T}^n_+} = \langle D_\nu u, D_\nu v \rangle_{\mathbb{T}^n_+} + \langle \gamma_+ u, \gamma_- v \rangle_{\mathbb{T}^{n-1}} \tag{3.23}$$

Since $\mathcal{F}_\nu$ is dense, (3.22) is valid for $v \in \mathcal{H}^2(\mathbb{T}^n_+)$, and (3.23) is valid for $v \in \mathcal{H}^1(\mathbb{T}^n_+)$.

Lemma 3.3.11. *Let $0 < \nu < 1$ and $s = 0, 1, 2$. Then there exists $C > 0$ such that*

$$\| |D_\nu|^2 u \|_{\mathcal{H}^{s-2}(\mathbb{T}^n_+)} \leq C \| (u, \underline{\gamma} u) \|_{\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)}.$$

for each $u \in \mathcal{F}_\nu$.

Proof. For $s = 2$ this follows since the norms $\|u\|_{\mathcal{H}^2(\mathbb{T}^n_+)}$ and $\|(u, \underline{\gamma} u)\|_{\widetilde{\mathcal{H}}^2(\mathbb{T}^n_+)}$ are equivalent for each $u \in \mathcal{F}_\nu$. The case $s = 1$ follows from (3.23), and the case $s = 0$ follows from (3.22). □

As a consequence of Lemma 3.3.11, the map $(u, \underline{\gamma} u) \mapsto |D_\nu|^2 u$, $u \in \mathcal{F}_\nu$ admits a unique extension as a bounded operator

$$|D_\nu|^2 : \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+) \to \mathcal{H}^{s-2}(\mathbb{T}^n_+)$$

for $s = 0, 1, 2$ and $0 < \nu < 1$. The situation is simpler when $\nu \geq 1$, since in that case $\mathcal{F}_\nu = C^\infty_c(\mathbb{T}^n_+)$ is dense in $\mathcal{H}^s(\mathbb{T}^n_+)$. The analogues of (3.22), (3.23) are given by

$$\left\langle |D_\nu|^2 u, v \right\rangle_{\mathbb{T}^n_+} = \left\langle u, |D_\nu|^2 v \right\rangle_{\mathbb{T}^n_+}, \tag{3.24}$$

$$\left\langle |D_\nu|^2 u, v \right\rangle_{\mathbb{T}^n_+} = \langle D_\nu u, D_\nu v \rangle_{\mathbb{T}^n_+}, \tag{3.25}$$

valid for each $u, v \in \mathcal{F}_\nu$. The analogue of Lemma 3.3.11 is the following.

Lemma 3.3.12. *Let $\nu \geq 1$ and $s = 0, 1, 2$. Then there exists $C > 0$ such that*

$$\||D_\nu|^2 u\|_{\mathcal{H}^{s-2}(\mathbb{T}^n_+)} \leq C\|u\|_{\mathcal{H}^s(\mathbb{T}^n_+)}$$

for each $u \in \mathcal{F}_\nu$.

From Lemma 3.3.12, it follows that the map $u \mapsto |D_\nu|^2 u, u \in \mathcal{F}_\nu$ admits a unique continuous extension as a bounded operator $|D_\nu|^2 : \mathcal{H}^s(\mathbb{T}^n_+) \to \mathcal{H}^{s-2}(\mathbb{T}^n_+)$ for $s = 0, 1, 2$ and $\nu \geq 1$.

Next consider a typical term in BD_ν. Such a term may be written as $b(x, y)D_y^\beta D_\nu$, where $b \in xC^\infty(\overline{\mathbb{T}^n_+})$ is constant for x large and $|\beta| \leq 1$. The following result holds for all $\nu > 0$, since there are no boundary terms when integrating by parts.

Lemma 3.3.13. *Suppose that $b \in xC^\infty(\overline{\mathbb{T}^n_+})$ is constant for x large and $|\beta| \leq 1$. Then*

$$bD_y^\beta D_\nu : \mathcal{H}^s(\mathbb{T}^n_+) \to \mathcal{H}^{s-|\beta|-1}(\mathbb{T}^n_+)$$

is bounded for each $s = 0, 1, 2$. Furthermore, there exists $c > 0$ depending on s, β and $C \geq 0$ depending on b, s, β, r such that

$$\|bD_y^\beta D_\nu u\|_{\mathcal{H}^{s-|\beta|-1}(\mathbb{T}^n_+)} \leq cr\|b\|_{C^1(\overline{\mathbb{T}^n_+})}\|u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} + C\|u\|_{\mathcal{H}^{s-1}(\mathbb{T}^n_+)}. \tag{3.26}$$

for each $u \in \mathcal{H}^s(\mathbb{T}^n_+)$ such that $\operatorname{supp} u \subseteq \{0 \leq x \leq r\}$.

Proof. The boundedness result is clear for $s = 2$. For $s = 0, 1$, it follows from the same considerations as in Lemma 3.1.4: define $B = bD_y^\beta$, and note that $B = B_1 x = xB_1$ where B_1 is smooth up to $x = 0$. Thus

$$BD_\nu = D_\nu^* B + i(1 - 2\nu)B_1 + [B, D_x]$$

So for each $u, v \in \mathcal{F}_\nu$,

$$\langle BD_\nu u, v\rangle_{\mathbb{T}^{n-1}} = \langle D_\nu u, B^* v\rangle = \langle u, B^* D_\nu v - i(1 - 2\nu)B_1^* v + [B, D_x]^* v\rangle_{\mathbb{T}^n_+}.$$

The first equality implies boundedness for $s = 1$, while the second implies boundedness for $s = 2$.

Similarly, (3.26) clearly holds for $s = 2$. To prove the other cases, begin by writing $b = xb_1$, where b_1 is smooth up to $x = 0$. Also define $q = [D_y^\beta, b]$, and $q = xq_1$, so the functions q, q_1 are smooth up to $x = 0$ (and vanish if $|\beta| = 0$).

(1) If $s = 1$, then for $u, v \in \mathcal{F}_\nu$,

$$\left\langle bD_y^\beta D_\nu u, v\right\rangle_{\mathbb{T}^n_+} = \left\langle bD_\nu u, D_y^\beta v\right\rangle_{\mathbb{T}^n_+} - \langle u, D_\nu \overline{q}\, v - i(1 - 2\nu)\overline{q_1} v\rangle_{\mathbb{T}^n_+}.$$

Thus

$$\|bD_y^\beta u\|_{\mathcal{H}^{s-|\beta|-1}(\mathbb{T}^n_+)} \leq \|bD_\nu u\|_{\mathcal{H}^0(\mathbb{T}^n_+)} + C\|u\|_{\mathcal{H}^0(\mathbb{T}^n_+)},$$

whence the result follows.

(2) Similarly for $s = 0$, if $u, v \in \mathcal{F}_\nu$, then

$$\left\langle bD_y^\beta D_\nu u, v\right\rangle_{\mathbb{T}^n_+} = \left\langle bu, D_\nu^* D_y^\beta v\right\rangle_{\mathbb{T}^n_+} - \left\langle u, D_\nu^* \,\overline{q}\, v + [D_x, b]^* D_y^\beta v\right\rangle_{\mathbb{T}^n_+}.$$

To handle the first term write

$$\left\langle bu, D_\nu^* D_y^\beta v\right\rangle = \left\langle b_1 u, x D_\nu^* D_y^\beta v\right\rangle = \left\langle bu, (D_\nu - i(1-2\nu)) D_y^\beta v\right\rangle,$$

which gives exactly

$$|\left\langle bu, D_\nu^* D_y^\beta v\right\rangle_{\mathbb{T}^n_+}| \leq \|bu\|_{\mathcal{H}^0(\mathbb{T}^n_+)} \|v\|_{\mathcal{H}^{1+|\beta|}(\mathbb{T}^n_+)},$$

as desired. Now consider the second term, which in absolute value is bounded by

$$|\left\langle u, D_\nu^* \,\overline{q}\, v + \overline{q}\, D_y^\beta v\right\rangle_{\mathbb{T}^n_+}| \leq \|u\|_{\mathcal{H}^{-1}(\mathbb{T}^n_+)} \left(\|D_\nu^* \,\overline{q}\, v\|_{\mathcal{H}^1(\mathbb{T}^n_+)} + \|\overline{q}\, D_y^\beta v\|_{\mathcal{H}^1(\mathbb{T}^n_+)} \right).$$

The second term in parentheses on the right hand side is bounded by a constant times $\|v\|_{\mathcal{H}^{1+|\beta|}(\mathbb{T}^n_+)}$ according to Lemma 3.3.10. For the first term on the right hand side, the only part of the $\mathcal{H}^1(\mathbb{T}^n_+)$ norm which can't be handled as above is the summand $\|D_\nu D_\nu^* \,\overline{q}\, v\|_{L^2(\mathbb{T}^n_+)}$. For this,

$$D_\nu D_\nu^* \, x\overline{q_1} = D_\nu \left(x D_\nu - i(2-2\nu) \right) \overline{q_1} = x|D_\nu|^2 \overline{q_1} - i(3-2\nu) D_\nu \overline{q_1}.$$

Using (3.20),

$$x|D_\nu|^2 \overline{q_1} = x\overline{q_1} |D_\nu|^2 - 2x(\partial_x \overline{q_1})\partial_\nu - x(\partial_x^2 \overline{q_1}) - (1-2\nu)\partial_x(\overline{q_1}),$$

which is bounded $\mathcal{H}^2(\mathbb{T}^n_+) \to \mathcal{H}^0(\mathbb{T}^n_+)$. This shows that

$$\|D_\nu^* \,\overline{q}\, v\|_{\mathcal{H}^1(\mathbb{T}^n_+)} \leq C \|v\|_{\mathcal{H}^{1+|\beta|}(\mathbb{T}^n_+)},$$

which completes the proof.

□

Remark 6. *Lemma 3.3.13 implies that $bD_y^\beta D_\nu$ is also bounded $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+) \to \mathcal{H}^{s-|\beta|-1}(\mathbb{T}^n_+)$ since the projection $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+) \to \mathcal{H}^s(\mathbb{T}^n_+)$ onto the first factor is continuous.*

Finally, a typical term in the operator A can be written as $a(x,y)D_y^\alpha$, where $|\alpha| \leq 2$ and $a \in C^\infty(\overline{\mathbb{T}^n_+})$ is constant outside a compact subset of $\overline{\mathbb{T}^n_+}$.

Lemma 3.3.14. *Suppose that $a \in C^\infty(\overline{\mathbb{T}^n_+})$ is constant for x large.*

1. *If $s = 0, 1$ and $|\alpha| \leq 2$, then $aD_y^\alpha : \mathcal{H}^s(\mathbb{T}^n_+) \to \mathcal{H}^{s-|\alpha|}(\mathbb{T}^n_+)$ is bounded.*

2. *If $s = 0, 1, 2$ and $0 < |\alpha| \leq 2$, then $aD_y^\alpha : \mathcal{H}^s(\mathbb{T}^n_+) \to \mathcal{H}^{s-|\alpha|}(\mathbb{T}^n_+)$ is bounded*

Furthermore, suppose that $a(0,p) = 0$ *for* $p \in \mathbb{T}^{n-1}$. *Then there exists* $c > 0$ *depending on* s, α *and* $C \geq 0$ *depending on* a, s, α, r *such that in each of the above cases,*

$$\|aD_y^\alpha u\|_{\mathcal{H}^{s-|\alpha|}(\mathbb{T}^n_+)} \leq cr\|a\|_{C^1(\overline{\mathbb{T}^n_+})}\|u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} + C\|u\|_{\mathcal{H}^{s-1}(\mathbb{T}^n_+)}. \tag{3.27}$$

for each $u \in \mathcal{H}^s(\mathbb{T}^n_+)$ *such that* $\operatorname{supp} u \subseteq \{(x,y) \in \overline{\mathbb{T}^n_+} : |x| + |y-p| < r\}$.

Proof. (1) First suppose that $s = 0, 1$. The boundedness result is clear if $s = 1$ and $|\alpha| \leq 1$ or $s = 0$ and $|\alpha| = 0$. Otherwise, suppose that $s = 1$ and $|\alpha| = 2$. Write $aD_y^\alpha = \sum_{|\gamma|=1} D_y^\gamma A_\gamma$ for smooth tangential operators $A_\gamma(y, D_y)$ of order at most one. Then for each $u, v \in \mathcal{F}_\nu$ and $|\gamma| = 1$,

$$|\left\langle D_y^\gamma A_\gamma u, v\right\rangle_{\mathbb{T}^n_+}| = |\left\langle A_\gamma u, D_y^\gamma v\right\rangle_{\mathbb{T}^n_+}| \leq C\|u\|_{\mathcal{H}^1(\mathbb{T}^n_+)}\|v\|_{\mathcal{H}^1(\mathbb{T}^n_+)}.$$

On the other hand, suppose that $s = 0$. Then

$$|\left\langle aD_y^\alpha u, v\right\rangle_{\mathbb{T}^n_+}| = |\left\langle u, D_y^\alpha av\right\rangle_{\mathbb{T}^n_+}| \leq C\|u\|_{\mathcal{H}^0(\mathbb{T}^n_+)}\|v\|_{\mathcal{H}^{|\alpha|}(\mathbb{T}^n_+)}$$

for $1 \leq |\alpha| \leq 2$.

(2) The only case not handled above is $s = 2$, in which case it follows from Lemma 3.3.10 that aD_y^α is bounded $\mathcal{H}^2(\mathbb{T}^n_+) \to \mathcal{H}^{s-|\alpha|}(\mathbb{T}^n_+)$ provided $|\alpha| \neq 0$.

(3) This follows from the same arguments as in (1) and (2).

□

To summarize the above discussion, write $A = \sum_{|\alpha|\leq 2} a_\alpha D_y^\alpha$ (non uniquely) in the form

$$A = \sum_{|\alpha|\leq 1} D_y^\alpha A_\alpha$$

for some $A_\alpha \in \operatorname{Diff}^1(\mathbb{T}^{n-1})$ which depends smoothly on $x \in \mathbb{R}_+$. Recall that P^* is also a Bessel operator, according Lemma 3.1.4. If $0 < \nu < 1$, then there are the two Green's formulas

$$\langle Pu, v\rangle_{\mathbb{T}^n_+} = \langle u, P^*v\rangle_{\mathbb{T}^n_+} + \langle \underline{\gamma} u, J\underline{\gamma} v\rangle_{\mathbb{T}^{n-1}}, \tag{3.28}$$

$$\langle Pu, v\rangle_{\mathbb{T}^n_+} = \langle D_\nu u, D_\nu v\rangle_{\mathbb{T}^n_+} + \langle D_\nu u, B^* v\rangle_{\mathbb{T}^n_+} + \sum_{|\alpha|\leq 1}\left\langle A_\alpha u, D_y^\alpha v\right\rangle_{\mathbb{T}^n_+} + \langle \gamma_+ u, \gamma_- v\rangle_{\mathbb{T}^{n-1}}, \tag{3.29}$$

valid for each $u, v \in \mathcal{F}_\nu$.

Lemma 3.3.15. *Let* $0 < \nu < 1$ *and* $s = 0, 1, 2$. *Then there exists* $C > 0$ *depending on* s *such that*

$$\|Pu\|_{\mathcal{H}^{s-2}(\mathbb{T}^n_+)} \leq C\|(u, \underline{\gamma} u)\|_{\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)}$$

for each $u \in \mathcal{F}_\nu$. *Thus the map* $(u, \underline{\gamma} u) \mapsto Pu$, $u \in \mathcal{F}_\nu$ *admits a unique extension as a bounded operator*

$$P : \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+) \to \mathcal{H}^{s-2}(\mathbb{T}^n_+)$$

for $s = 0, 1, 2$ *and* $0 < \nu < 1$. *When* $s = 0, 1$, *this extension is determined by* (3.28), (3.29).

Proof. This is a direct consequence of Lemmas 3.3.11, 3.3.13, 3.3.14. □

The situation is simpler when $\nu \geq 1$: the analogues of (3.28), (3.29) are given by

$$\langle Pu, v \rangle_{\mathbb{T}^n_+} = \langle u, P^* v \rangle_{\mathbb{T}^n_+} \tag{3.30}$$

$$\langle Pu, v \rangle_{\mathbb{T}^n_+} = \langle D_\nu u, D_\nu v \rangle_{\mathbb{T}^n_+} + \langle D_\nu, B^* v \rangle_{\mathbb{T}^n_+} + \sum_{|\alpha| \leq 1} \left\langle A_\alpha u, D^\alpha_y v \right\rangle_{\mathbb{T}^n_+}, \tag{3.31}$$

valid for each $u, v \in \mathcal{F}_\nu$. As before, (3.30) is in fact valid for $v \in \mathcal{H}^2(\mathbb{T}^n_+)$, while (3.31) is valid for $v \in \mathcal{H}^1(\mathbb{T}^n_+)$.

Lemma 3.3.16. *Let $\nu \geq 1$ and $s = 0, 1, 2$. Then there exists $C > 0$ such that*

$$\|Pu\|_{\mathcal{H}^{s-2}(\mathbb{T}^n_+)} \leq C \|u\|_{\mathcal{H}^s(\mathbb{T}^n_+)}.$$

for each $u \in \mathcal{F}_\nu$. the map $u \mapsto Pu$, $u \in \mathcal{F}_\nu$ admits a unique extension as a bounded operator

$$P : \mathcal{H}^s(\mathbb{T}^n_+) \to \mathcal{H}^{s-2}(\mathbb{T}^n_+)$$

for $s = 0, 1, 2$ and $\nu \geq 1$. When $s = 0, 1$ this extension is determined by (3.30), (3.31). The action of P on $\mathcal{H}^s(\mathbb{T}^n_+)$ is simply the restriction of $P : \mathscr{D}'(\mathbb{T}^n_+) \to \mathscr{D}'(\mathbb{T}^n_+)$ to $\mathcal{H}^s(\mathbb{T}^n_+)$.

Proof. This is a direct consequence of Lemmas 3.3.12, 3.3.14, 3.3.13. □

Suppose that $0 < \nu < 1$. If $s = 0, 1$, then an element $f \in \mathcal{H}^{s-2}(\mathbb{T}^n_+)$ is not uniquely determined by a distribution in $\mathscr{D}'(\mathbb{T}^n_+)$. On the other hand, f may certainly be restricted to a functional on $\mathring{\mathcal{H}}^s(\mathbb{T}^n_+)$, which is determined uniquely by a distribution since $C^\infty_c(\mathbb{T}^n_+)$ is dense in this space by definition. Given $s = 0, 1, 2$ and $u \in \mathcal{H}^s(\mathbb{T}^n_+)$, $f \in \mathcal{H}^{s-2}(\mathbb{T}^n_+)$, the equation $Pu = f$ can be interpreted in this weak sense, namely

$$\langle u, P^* v \rangle_X = \langle f, v \rangle_X$$

for all $v \in C^\infty_c(\mathbb{T}^n_+) \subseteq \mathring{\mathcal{H}}^{2-s}(\mathbb{T}^n_+)$. For $s = 2$ this is just the statement that $Pu = f$ in distributions. Furthermore, if $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$ and $P(u, \underline{\phi}) = f$, then $Pu = f$ weakly.

Now suppose that $P \in \mathrm{Bess}^{(\lambda)}_\nu(\mathbb{T}^n_+)$ is a parameter-dependent Bessel operator. Recalling the definition of the parameter-dependent norms as in Section 3.3, it is straightforward to show that the following hold.

1. If $0 < \nu < 1$ and $s = 0, 1, 2$, then there exists $C > 0$ such that

$$|\!|\!| P(\lambda)(u, \underline{\phi}) |\!|\!|_{\mathcal{H}^{s-2}(\mathbb{T}^n_+)} \leq C |\!|\!| (u, \underline{\phi}) |\!|\!|_{\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)}$$

   for each $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$.

2. If $\nu \geq 1$ and $s = 0, 1, 2$, then there exists $C > 0$ such that

$$|\!|\!| P(\lambda) u |\!|\!|_{\mathcal{H}^{s-2}(\mathbb{T}^n_+)} \leq C |\!|\!| u |\!|\!|_{\mathcal{H}^s(\mathbb{T}^n_+)}$$

   for each $u \in \mathcal{H}^s(\mathbb{T}^n_+)$.

There are also straightforward extensions of Lemmas 3.3.11, 3.3.12, 3.3.13, 3.3.14 for parameter-dependent norms.

Function spaces on a manifold

Consider a compact manifold with boundary $\overline{X}$, equipped with a distinguished boundary defining function x and collar diffeomorphism ϕ as in Section 3.1.

Definition 3.3.17. *Given $\nu > 0$, let $\mathcal{F}_\nu(X)$ denote the following spaces of functions.*

1. *If $0 < \nu < 1$, then $\mathcal{F}_\nu(X)$ consists of $u \in C^\infty(X)$ such that*
$$(u \circ \phi)(x, y) = x^{1/2-\nu} u_-(x^2, y) + x^{1/2+\nu} u_+(x^2, y) \tag{3.32}$$
for some $u_\pm \in C^\infty([0, \sqrt{\varepsilon}) \times \partial X)$.
2. *If $\nu \geq 1$, then $\mathcal{F}_\nu = C_c^\infty(X)$.*

Fix a finite open cover $\overline{X} = \bigcup_i U_i$ by coordinate charts (U_i, ψ_i), such that either
$$U_i \cap \partial X = \emptyset, \quad \psi_i : U_i \to \psi_i(U_i) \subseteq \mathbb{T}^n,$$
or if $U_i \cap \partial X \neq \emptyset$ then
$$U_i = \phi([0, \varepsilon) \times Y_i), \quad \psi_i = (1 \times \theta_i) \circ \phi^{-1}$$
for a coordinate chart (Y_i, θ_i) on ∂X. This of course implies that $\partial X = \bigcup_i Y_i$, where the union is taken over all i such that $U_i \cap \partial X \neq \emptyset$. Now take a partition of unity of the form
$$\sum_i \chi_i^2 = 1, \quad \chi_i \in C_c^\infty(U_i),$$
with the additional property that if $U_i \cap \partial X \neq \emptyset$, then χ_i has the form
$$\chi_i = (\alpha \beta_i) \circ \phi^{-1},$$
for functions $\alpha \in C_c^\infty([0, \varepsilon))$, $\beta_i \in C_c(Y_i)$, where $\alpha = 1$ near $x = 0$. Note that if $u \in \mathcal{F}_\nu(X)$ then $\chi_i u$ may be identified with an element of $\mathcal{F}_\nu$ via the coordinate map ψ_i. Keeping this in mind, define
$$\|u\|_{i,\mathcal{H}^s(X)} := \|(\chi_i u) \circ \psi_i^{-1}\|_{\mathcal{H}^s(\mathbb{T}^n_+)},$$
for $s = 0, \pm 1, \pm 2$ and $u \in \mathcal{F}_\nu(X)$.

Definition 3.3.18. *Given $s = 0, \pm 1, \pm 2$, let*
$$\|u\|^2_{\mathcal{H}^s(X)} = \sum_i \|u\|^2_{i,\mathcal{H}^s(X)}.$$
Then define
$$\mathcal{H}^s(X) = \textit{closure of } \mathcal{F}_\nu(X) \textit{ in the } \mathcal{H}^s(X) \textit{ norm.}$$

To prove that $\mathcal{H}^s(X)$ is independent of the choice of covering U_i and partition of unity χ_i, the following two results are needed

Lemma 3.3.19. *Let Y, Y' be open subsets of $\mathbb{T}^{n-1}$, and suppose that $\Phi : Y \to Y'$ is a diffeomorphism between them. Suppose that $K \subseteq \overline{\mathbb{R}_+} \times Y$ is compact. Then for each $s = 0, \pm 1, \pm 2$ there exists $C > 0$ such that*

$$C^{-1}\|u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} \leq \|u \circ (1 \times \Phi)\|_{\mathcal{H}^s(\mathbb{T}^n_+)} \leq C\|u\|_{\mathcal{H}^s(\mathbb{T}^n_+)}$$

for each $u \in \mathcal{F}_\nu$ with $\operatorname{supp} u \subseteq K' := (1 \times \Phi)(K)$,

Proof. For $s = 0, 1, 2$ this follows immediately from the change of variables formula in the tangential direction. The cases $s = -1, -2$ follow by duality: choose $\alpha \in C_c^\infty(\overline{\mathbb{R}_+})$ and $\beta \in C_c^\infty(Y')$ such that $\chi := \alpha\beta = 1$ on K'. Then for each $v \in \mathcal{F}_\nu$,

$$\frac{|\langle u, v\rangle_{\mathbb{T}^n_+}|}{\|v\|_{\mathcal{H}^{-s}(\mathbb{T}^n_+)}} \leq C_1^{-1}\frac{|\langle u, \chi v\rangle_{\mathbb{T}^n_+}|}{\|\chi v\|_{\mathcal{H}^{-s}(\mathbb{T}^n_+)}},$$

where $\|\chi v\|_{\mathcal{H}^{-s}(\mathbb{T}^n_+)} \leq C_1\|v\|_{\mathcal{H}^{-s}(\mathbb{T}^n_+)}$. Thus

$$\|u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} \leq C_1^{-1} \sup_{\|w\|_{\mathcal{H}^{-s}(\mathbb{T}^n_+)}=1} |\langle u, w\rangle_{\mathbb{T}^n_+}|,$$

where the supremum is taken over all $w \in \mathcal{F}_\nu$ such that $\operatorname{supp} w \subseteq K'$. Then

$$\langle u, w\rangle_{\mathbb{T}^n_+} = \langle u \circ (1 \times \Phi), w_1\rangle_{\mathbb{T}^n_+},$$

where $w_1(x, y) = J(y)(w \circ (1 \times \Phi))(x, y)$ and J is the Jacobian determinant of $1 \times \Phi$. Since J depends only on y,

$$\|w_1\|_{\mathcal{H}^{-s}(\mathbb{T}^n_+)} \leq C_2\|w\|_{\mathcal{H}^{-s}(\mathbb{T}^n_+)},$$

which shows that $\|u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} \leq C\|u \circ (1 \times \Phi)\|_{\mathcal{H}^s(\mathbb{T}^n_+)}$ for some $C > 0$. The same argument is now applied with Φ replaced by Φ^{-1} to conclude the reverse inequality. □

Remark 7. *More generally, the space $\mathcal{H}^s(\mathbb{T}^n_+)$ are invariant under diffeomorphisms Φ satisfying the conditions in Lemma 3.1.2*

Before stating the next result, recall that for M a (non-compact) manifold without boundary, the standard Sobolev spaces $H^s_{\mathrm{loc}}(M)$, $H^s_{\mathrm{comp}}(M)$ are defined as follows: Fix a locally finite open cover $M = \bigcup_j X_j$ by coordinate charts (X_j, ψ_j) where $\psi_j : X_j \to \mathbb{R}^n$, and a subordinate partition of unity $\sum_j \chi_j^2 = 1$ where $\chi_j \in C_c^\infty(X_j)$. Then $H^s_{\mathrm{comp}}(M)$ is defined as all compactly supported distributions $u \in \mathcal{E}'(M)$ such that the norm

$$\|u\|^2_{H^s(M)} := \sum_j \|(\chi_j u) \circ \psi_j\|^2_{H^s(\mathbb{R}^{n-1})}. \tag{3.33}$$

is finite. The local spaces $H^2_{\mathrm{loc}}(M)$ then consist of distributions $u \in \mathcal{D}'(M)$ such that $\chi u \in H^s_{\mathrm{comp}}(M)$ for each $\chi \in C_c^\infty(M)$. Let $H^s_K(M)$ denote the space of all $u \in H^s_{\mathrm{comp}}(M)$ whose supports are contained in a fixed closed set $K \subset M$. If M is compact, then $H^s_K(M)$ is complete under the norm 3.33. Furthermore, these spaces do not depend on any of the choices used to define them.

Lemma 3.3.20. *Let K be a compact subset of $\mathbb{T}^n_+$. Then for each $s = 0, \pm 1, \pm 2$ there exists $C > 0$ such that*

$$C^{-1}\|u\|_{H^s(\mathbb{T}^n_+)} \leq \|u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} \leq C\|u\|_{H^s(\mathbb{T}^n_+)}$$

for each $u \in C_c^\infty(\mathbb{T}^n_+)$ such that $\operatorname{supp} u \subseteq K$.

Proof. This is again straightforward for $s = 0, 1, 2$, while the cases $s = -1, -2$ follow by duality. □

The combination of Lemmas 3.3.19, 3.3.20 show that the spaces $\mathcal{H}^s(X)$ do not depend on any of the choices used to define them.

Lemma 3.3.21. *Fix a density on $\overline{X}$ of product type near ∂X. Let $\langle \cdot, \cdot \rangle_X$ denotes the inner product on $L^2(X; \mu)$. For each $s = 0, \pm 1, \pm 2$*

$$|\langle u, v \rangle_X| \leq C\|u\|_{\mathcal{H}^s(X)}\|v\|_{\mathcal{H}^{-s}(X)},$$

where $u, v \in \mathcal{F}_\nu$. Furthermore, $\langle \cdot, \cdot \rangle_X$ extends to a nondegenerate pairing $\mathcal{H}^s(X) \times \mathcal{H}^{-s}(X) \to \mathbb{C}$.

Proof. This can be reduced to the case on $\mathbb{T}^n_+$ via the coordinate charts (U_i, ψ_i) and partition of unity χ_i used to define $\mathcal{H}^s(X)$ (it is here that choosing a quadratic partition of unity is particularly convenient) □

Thus $\mathcal{H}^{-s}(X)$ is naturally identified with the antidual of $\mathcal{H}^s(X)$ via the inner product induced by μ on $\mathcal{H}^0(X)$. When $0 < \nu < 1$, it is also possible to show that the maps

$$u \mapsto u_-(0, \cdot), \quad u \mapsto 2\nu u_+(0, \cdot)$$

for $u \in \mathcal{F}_\nu(X)$ satisfying (3.32) admit continuous extensions $\gamma_\mp$ such that

$$\gamma_\mp : \mathcal{H}^s(X) \to H^{s-1\pm\nu}(\partial X).$$

It is understood that γ_- exists for $s = 1, 2$, while γ_+ exists for $s = 2$. The spaces $\widetilde{\mathcal{H}}^s(X)$ are then defined exactly as in Section 3.3.

Lemma 3.3.22. *Let $P \in \mathrm{Bess}_\nu(X)$. Then the following hold.*

1. *If $0 < \nu < 1$ and $s = 0, 1, 2$, then there exists $C > 0$ such that*

$$\|Pu\|_{\mathcal{H}^{s-2}(X)} \leq C\|(u, \underline{\gamma} u)\|_{\widetilde{\mathcal{H}}^s(X)}.$$

   for each $u \in \mathcal{F}_\nu(X)$.

2. *If $\nu \geq 1$ and $s = 0, 1, 2$, then there exists $C > 0$ such that*

$$\|Pu\|_{\mathcal{H}^{s-2}(X)} \leq C\|u\|_{\mathcal{H}^s(X)}.$$

   for each $u \in \mathcal{F}_\nu(X)$.

As in Section 3.3, it follows that $(u, \underline{\gamma} u) \mapsto Pu$ admits a unique extension to $\widetilde{\mathcal{H}}^s(X)$ for $0 < \nu < 1$, and $u \mapsto Pu$ has a unique continuous extension to $\mathcal{H}^s(X)$ for $\nu \geq 1$.

The traces $\gamma_\pm$ can be formulated in terms of the boundary defining function x. This is clear for γ_-, which is just the restriction of $x^{\nu-1/2}u$ to the boundary. Treating ∂_x as a coordinate vector field near ∂X, then γ_+ is the restriction of $x^{1-2\nu}\partial_x(x^{\nu-1/2}u)$ to the boundary.

Lemma 3.3.23. *Suppose that x and ρ are boundary defining functions satisfying the conditions in Lemma 3.1.3. Then the traces $\gamma_\pm$ defined with respect to x agree with those defined with respect to ρ.*

Proof. This can be checked in local coordinates using Lemmas 3.1.2, 3.1.3. However, note that the spaces $\mathcal{F}_\nu(X)$ defined with respect to x and ρ do not agree. □

The parameter-dependent norms on $\mathcal{H}^s(X)$ are defined by replacing $\|\cdot\|_{\mathcal{H}^s(\mathbb{T}^n_+)}$ with $|\!|\!|\cdot|\!|\!|_{\mathcal{H}^s(\mathbb{T}^n_+)}$ in Definition 3.3.18, and similarly for $\widetilde{\mathcal{H}}^s(X)$. Then P is uniformly bounded in λ with respect to these norms.

The following compactness result is established in Section 3.8. A different proof can be found in [58, Section 6]. It is used in Section 3.5 to prove the Fredholm property for certain boundary value problems.

Lemma 3.3.24. *[58, Section 6] Let $\nu > 0$ and μ be a density of product type near ∂X.*

1. *The inclusion $\mathcal{H}^1(X) \hookrightarrow \mathcal{H}^0(X)$ is compact.*

2. *The injection $\mathcal{H}^0(X) \hookrightarrow \mathcal{H}^{-1}(X)$ induced by the $L^2(X;\mu)$ inner product is compact.*

3. *If $0 < \nu < 1$, then $\widetilde{\mathcal{H}}^1(X) \hookrightarrow \widetilde{\mathcal{H}}^0(\mathbb{T}^n_+)$ and the injection $\widetilde{\mathcal{H}}^0(X) \hookrightarrow \widetilde{\mathcal{H}}^1(X)'$ induced by the $L^2(X;\mu)$ and $L^2(\partial X;\mu_{\partial X})$ inner products are compact.*

Proof. (1) For a proof, see the Lemma 3.8.2. The other cases (2), (3) follow by duality. □

Finally, it is important to consider the action of standard pseudodifferential operators whose Schwartz kernels are compactly supported in $X \times X$. If $Q \in \Psi^m(X)$ is compactly supported, then there exists a compact subset $K \subseteq X$ such that $\operatorname{supp} Qu \subseteq K$ for each $u \in C^\infty(X)$.

Lemma 3.3.25. *Suppose that $Q \in \Psi^2(X)$ is compactly supported. Then there exists a compact subset $K \subseteq X$ such that for each $s = 0, \pm 1, \pm 2$, the map*

$$u \mapsto Qu, \quad u \in \mathcal{F}_\nu$$

extends uniquely to a bounded map

$$Q : \begin{cases} \widetilde{\mathcal{H}}^s(X) \to H^{s-2}_K(X) & \text{if } 0 < \nu < 1, \\ \mathcal{H}^s(X) \to H^{s-2}_K(X) & \text{if } \nu \geq 1. \end{cases}$$

Proof. It suffices to prove this in local coordinates, where the result follows from Lemma 3.3.20. □

Remark 8. *It is also necessary to consider the class of (compactly supported) parameter-dependent pseudodifferential operators, denoted here by $\Psi^{m,(\lambda)}(X)$ — see [87, Chapter II.9] for this class of operators, or [111, Chapter 4], [27, Chapter 7] for an equivalent semiclassical description. In that case, if $Q \in \Psi^{2,(\lambda)}(X)$, then the boundedness result of Lemma 3.3.25 holds uniformly for the appropriate parameter-dependent norms.*

Graph norms

Throughout this section, assume that $0 < \nu < 1$. Following [84, Chapter 6.1], an alternative characterization of the spaces $\widetilde{\mathcal{H}}^s(X)$ is given. Given $s = 0, 1, 2$ and a Bessel operator P, define the norm

$$\|u\|_{\mathcal{H}^s_P(X)} = \|u\|_{\mathcal{H}^s(X)} + \|Pu\|_{\mathcal{H}^{s-2}(X)}$$

for $u \in \mathcal{F}_\nu(X)$.

Lemma 3.3.26. *Give $s = 0, 1, 2$ there exists $C > 0$ such that*

$$C^{-1}\|u\|_{\mathcal{H}^s_P(X)} \leq \|(u, \underline{\gamma} u)\|_{\widetilde{\mathcal{H}}^s(X)} \leq C^{-1}\|u\|_{\mathcal{H}^s_P(X)}$$

for each $u \in \mathcal{F}_\nu(X)$.

Proof. The first inequality above holds according to Lemma 3.3.15. For the converse, first recall that the trace map $\underline{\gamma} : \widetilde{\mathcal{H}}^s(X) \to H^{s-\underline{\nu}}(\partial X)$ has a continuous right inverse $\mathcal{K}$, which furthermore maps $C^\infty(\partial X) \times C^\infty(\partial X) \to \mathcal{F}_\nu(X)$ — see Lemma 3.7.3. Fix $u \in \mathcal{F}_\nu(X)$ and define the linear form ℓ on $C^\infty(\partial X) \times C^\infty(\partial X)$ by

$$\ell(\underline{\psi}) = \langle Pu, v\rangle_X - \langle u, P^* v\rangle_X\,,$$

where $v \in \mathcal{F}_\nu(X)$ is any element satisfying $\underline{\gamma} v = \underline{\psi}$. The form ℓ is well defined, since by Green's formula it is independent of the choice of v; in particular, it is possible to take $v = \mathcal{K}(\underline{\psi})$. Here the duality is induced by a fixed density of product type near ∂X. Then

$$|\ell(\underline{\psi})| \leq C_1(\|Pu\|_{\mathcal{H}^{s-2}(X)} + \|u\|_{\mathcal{H}^s(X)})\|\mathcal{K}(\underline{\psi})\|_{\widetilde{\mathcal{H}}^{2-s}(X)} \leq C_2\|u\|_{\mathcal{H}^s_P(X)}\|\underline{\psi}\|_{H^{2-s-\underline{\nu}}(\partial X)}.$$

By Hahn–Banach and the Riesz theorem, there exists a unique $\underline{\phi} \in H^{s-2+\underline{\nu}}(\partial X)$ such that for each $u, v \in \mathcal{F}_\nu(X)$,

$$\langle Pu, v \rangle_X - \langle u, P^* v \rangle_X = \langle \underline{\phi}, \underline{\gamma} v \rangle_{\partial X}, \quad \|\underline{\phi}\|_{H^{s-2+\underline{\nu}}(\partial X)} \leq C_2 \|u\|_{\mathcal{H}^s_P(X)}.$$

The constant C_2 in the latter inequality is independent of $u \in \mathcal{F}_\nu(X)$. On the other hand, Green's formula implies that $\ell(\underline{\psi}) = -\langle J\underline{\gamma} u, \underline{\psi} \rangle_{\partial X}$ for all $\underline{\psi}$, so in particular $\underline{\phi} = -J\underline{\gamma} u$. Since

$$\|\underline{\gamma} u\|_{H^{s-\underline{\nu}}(\partial X)} = \|\underline{\phi}\|_{H^{s-2+\underline{\nu}}(\partial X)},$$

it follows that $\|(u, \underline{\gamma} u)\|_{\widetilde{\mathcal{H}}^s(X)} \leq C_2 \|u\|_{\mathcal{H}^s_P(X)}$. □

Let $\mathcal{H}^s_P(X)$ denote the closure of $\mathcal{F}_\nu(X)$ in the norm $\|\cdot\|_{\mathcal{H}^s_P(X)}$. Since $(u, \underline{\gamma} u)$, $u \in \mathcal{F}_\nu(X)$ is dense in $\widetilde{\mathcal{H}}^s(X)$, it follows from Lemma 3.3.26 that $\mathcal{H}^s_P(X)$ is naturally isomorphic to $\widetilde{\mathcal{H}}^s(X)$ via the closure of the map $u \mapsto (u, \underline{\gamma} u)$. Moreover, any element of $\mathcal{H}^s_P(X)$ can be identified with a unique pair (u, f), where $u \in \mathcal{H}^s(X)$, $f \in \mathcal{H}^{s-2}(X)$, and $Pu = f$ in the weak sense (described at the end of Section 3.3).

3.4 Elliptic boundary value problems

This section concerns boundary value problems for Bessel operators on a compact manifold with boundary $\overline{X}$ as in Section 3.1. When $0 < \nu < 1$, these are of them form

$$\begin{cases} Pu = f & \text{on } X \\ Tu = g & \text{on } \partial X. \end{cases}$$

Here $P \in \mathrm{Bess}_\nu(X)$ is Bessel operator which is elliptic in the sense of Section 3.1 on ∂X, and

$$T = T^+ \gamma_+ + T^- \gamma_-$$

for some differential operators $T^\pm$ on the boundary, to be specified in the next section. The boundary operator T is only relevant when $0 < \nu < 1$. When $\nu \geq 1$, one considers the simpler equation

$$Pu = f \text{ on } X.$$

To highlight the difference between the cases $0 < \nu < 1$ and $\nu \geq 1$, fix $p \in \partial X$ and consider the model equation on $\mathbb{R}_+$ determined by the boundary symbol operator,

$$\widehat{P}_{(p,\eta)} u = f. \tag{3.34}$$

referring to Section 3.1 for notation. Suppose that P is elliptic at $p \in \partial X$. Any two solutions to the equation (3.34) differ by an element of the kernel of $\widehat{P}_{(p,\eta)}$. If $u \in \ker \widehat{P}_{(p,\eta)}$ satisfies

$u \in L^2((1,\infty))$, then necessarily $u \in \mathcal{M}_+(p,\eta)$. On the other hand, if ν is not an integer, then

$$K_\nu(s) = \frac{\pi}{2}\frac{I_{-\nu}(s) - I_\nu(s)}{\sin(\nu\pi)}, \tag{3.35}$$

where I_ν is the modified Bessel function of the first kind [82, Chapter 7.8] (if ν is an integer, equality holds in the sense of limits). In particular, if $0 < \nu < 1$, then $I_{\pm\nu}(s) = \mathcal{O}(s^{\pm\nu})$. Consequently $\ker \widehat{P}_{(p,\eta)} \cap L^2(\mathbb{R}_+) = \mathcal{M}_+(p,\eta)$, and hence $\widehat{P}_{(p,\eta)}$ cannot be an isomorphism between any L^2 based spaces: in general, (3.34) must be augmented by boundary conditions so that the L^2 kernel is trivial. Of course, all of these observations are classical when $\nu = \frac{1}{2}$ (boundary value problems in the smooth setting).

This is in contrast to the situation when $\nu \geq 1$. In that case, $\sqrt{x}K_\nu(i\xi(p,\eta)x)$ is not square integrable near the origin, and so the L^2 kernel of $\widehat{P}_{(p,\eta)}$ is always trivial. Hence specifying f on the right hand side of (3.34) (in an appropriate function space) will uniquely determine a solution u. Thus in the case $\nu \geq 1$, it is not necessary to impose any boundary conditions apart from the square integrability requirement.

In the self-adjoint setting, the heuristic above is the limit point/limit circle criterion of Weyl on self-adjoint extensions of symmetric ordinary differential operators with regular singular points — see [109] for an exhaustive modern treatment, and [5, 65] for discussions in the context of AdS cosmology.

Boundary conditions

This section is only relevant in the case $0 < \nu < 1$. Choose differential operators

$$T^- \in \mathrm{Diff}^1(\partial X), \quad T^+ \in \mathrm{Diff}^0(\partial X),$$

noting that T_+ is just multiplication by a smooth function on ∂X. Then set

$$T = T^-\gamma_- + T^+\gamma_+.$$

A natural question is how to define the "leading order" term in T. Suppose that $\mu \in \{1-\nu, 2-\nu, 1+\nu\}$ and

$$\mathrm{ord}(T^-) - \nu \leq \mu - 1, \quad \mathrm{ord}(T^+) + \nu \leq \mu - 1. \tag{3.36}$$

Then T is said to have ν-order less than or equal to μ, written as $\mathrm{ord}_\nu(T) \leq \mu$. Note that if $\mathrm{ord}_\nu(T) \leq \mu$, then $B : \mathcal{H}^2(X) \to H^{2-\mu}(\partial X)$ is continuous.

Suppose that $\mu \in \{1-\nu, 2-\nu, 1+\nu\}$ and $\mathrm{ord}_\nu(T) \leq \mu$. Define the family of operators

$$\widehat{T}_{(y,\eta)} = \sigma_{\lceil \mu-1+\nu \rceil}(T^-)\gamma_- + \sigma_{\lceil \mu-1-\nu \rceil}(T^+)\gamma_+,$$

indexed by $(y,\eta) \in T^*\partial X$. Thus each $(y,\eta) \in T^*\partial X$ gives rise to a one-dimensional boundary operator $\widehat{T}_{(y,\eta)}$.

The boundary value problem

Although boundary value problems of the form (3.4) are ultimately of interest, for duality purposes it is convenient to consider a more general type of problem. Fix $J \in \mathbb{N}$, and choose

- $\mu_k \in \{1-\nu, 2-\nu, 1+\nu\}$ for $k \in \{1, \ldots, J+1\}$,
- numbers $\tau_j \in \mathbb{R}$ for $j \in \{1, \ldots J\}$, not necessarily integers.

Let $T = (T_1, \ldots, T_{J+1})^\top$ denote a $(J+1) \times 1$ matrix of boundary operators, such that $\mathrm{ord}_\nu(T_k) \leq \mu_k$. Furthermore, for each $k \in \{1, \ldots, J+1\}$ and $j \in \{1, \ldots, J\}$, suppose $C_{k,j} \in \mathrm{Diff}^*(\partial X)$ is a differential operator on ∂X such that

$$\mathrm{ord}(C_{k,j}) \leq \tau_j + \mu_k.$$

Let C denote the $(J+1) \times J$ matrix with entries $C_{k,j}$. Given these prerequisites, consider the modified boundary value problem

$$\begin{cases} Pu = f & \text{on } X \\ Tu + C\underline{u} = \underline{g} & \text{on } \partial X, \end{cases} \tag{3.37}$$

where $\underline{u} = (u_1, \ldots u_J)$, $\underline{g} = (g_1, \ldots, g_{J+1})$ are collections of functions on ∂X. In order to associate an operator to this problem, note that Tu may be written in the form

$$Tu = G\underline{\gamma}u,$$

where G is the $(J+1) \times 2$ matrix

$$G = \begin{pmatrix} T_1^- & T_1^+ \\ \vdots & \vdots \\ T_{J+1}^- & T_{J+1}^+ \end{pmatrix}.$$

Throughout, it is always understood that G is associated with T in this way. Finally, set $\underline{\mu} = (\mu_1, \ldots, \mu_{J+1})$ and $\underline{\tau} = (\tau_1, \ldots, \tau_J)$. Then let $\mathscr{P}$ denote the map

$$\mathscr{P}(u, \underline{\phi}, \underline{u}) = (P(u, \underline{\phi}), G\underline{\phi} + C\underline{u}).$$

This is also written as $\mathscr{P} = \{P, T, C\}$.

Lemma 3.4.1. *The map $\mathscr{P} = \{P, T, C\}$ is bounded*

$$\mathscr{P} : \widetilde{\mathcal{H}}^s(X) \times H^{s+\underline{\tau}}(\partial X) \to \mathcal{H}^{s-2}(X) \times H^{s-\underline{\mu}}(\partial X)$$

for each $s = 0, 1, 2$.

Proof. The mapping properties follows from the results of Section 3.3. □

The adjoint boundary value problem

Fix a density μ which is of product type near ∂X. Let P^* denote the formal $L^2(X;\mu)$ adjoint of P; then P^* is also a Bessel operator in light of Lemma 3.1.4. Let C^*, G^* denote the formal $L^2(\partial X;\mu_{\partial X})$ adjoints of C, G. Define the problem

$$\begin{cases} P^* v = f & \text{on } X, \\ J\underline{\gamma} v + G^* \underline{v} = \underline{g} & \text{on } \partial X, \\ C^* \underline{v} = \underline{h} & \text{on } \partial X, \end{cases} \tag{3.38}$$

where $\underline{v} = (v_1, \ldots, v_{J+1})$, $(\underline{g}, \underline{h}) = (g_1, g_2, h_1, \ldots h_J)$ are functions on ∂X.

Although Green's formula (3.28) was previously only established for the formal adjoint of a Bessel operator on $\mathbb{T}^n_+$, it is clear that (3.28) also holds here when the appropriate μ and $\mu_{\partial X}$ inner products are substituted on X and ∂X:

$$\langle Pu, v\rangle_X + \langle Tu + C\underline{u}, \underline{v}\rangle_{(\partial X)^{J+1}} = \langle u, P^* v\rangle_X + \left\langle \underline{\gamma} u, G^*\underline{v} + J\underline{\gamma} v\right\rangle_{(\partial X)^2} + \langle \underline{u}, C^*\underline{v}\rangle_{(\partial X)^J}.$$

In light of this, the problem (3.38) is said to be the formal adjoint of (3.37). Also notice that (3.38) has the same form as (3.37). The corresponding operator is denoted by $\mathscr{P}^*$.

The Lopatinskiĭ condition

The standard Lopatinskiĭ condition for smooth elliptic boundary value problems (see [75, 84]) has a natural generalization to the situation here. Begin by choosing $c_{k,j} \in \mathbb{Z}$ (not necessarily nonnegative) such that

$$\operatorname{ord}(C_{k,j}) \le c_{k,j} \le \tau_j + \mu_k,$$

and then define the matrix $\widehat{C}_{(y,\eta)}$ with entries

$$(\widehat{C}_{(y,\eta)})_{k,j} = \sigma_{c_{k,j}}(C_{k,j})(y,\eta).$$

Thus $(y,\eta) \mapsto \widehat{C}_{(y,\eta)}$ is a function on $T^*\partial X$ with values in matrices over $\mathbb{C}$. Furthermore, define $\widehat{G}_{(y,\eta)}$ by the equality

$$\widehat{G}_{(y,\eta)}\underline{\gamma} u = \widehat{T}_{(y,\eta)} u.$$

Remark 9. *The matrix $\widehat{C}_{(y,\eta)}$ depends strongly on the choice of $c_{k,j}$, and is not necessarily obtained by calculating the principal symbol of $C_{k,j}$ entry-wise with respect to the order of $C_{k,j}$. The numbers $c_{k,j}$ in general will depend on the choice of τ_j, μ_k as well. Similarly, $\widehat{G}_{(y,\eta)}$ is in general different from the principal symbol of G calculated entry-wise with respect to the order of each entry.*

Definition 3.4.2. *Suppose P is elliptic on ∂X. The boundary operators (T, C) are said to satisfy the Lopatinskiĭ condition with respect to P if for each fixed $p \in \partial M$ and $\eta \in T^*_p\partial X \setminus 0$, the only element $(u, \underline{u}) \in \mathcal{M}_+(p, \eta) \times \mathbb{C}^J$ satisfying*

$$\widehat{T}_{(p,\eta)}u + \widehat{C}_{(p,\eta)}\underline{u} = 0$$

is the trivial solution $(u, \underline{u}) = 0$. The boundary value problem (3.37), or equivalently the operator $\mathscr{P} = \{P, T, C\}$, is said to be elliptic on ∂X if P is elliptic on ∂X in the sense of Definition 3.1.5 and (T, C) satisfy the Lopatinskiĭ condition on ∂X with respect to P.

As in Section 3.1, the one-dimensional space $\mathcal{M}_+(y, \eta)$ is spanned by the function

$$u(y, \eta; x) = \frac{2^{1-\nu}\Gamma(1-\nu)\sin(\pi\nu)}{\pi}(i\xi(y, \eta))^\nu \sqrt{x} K_\nu(i\xi(y, \eta)x). \tag{3.39}$$

The argument is chosen so that all of the quantities are positive when ξ lies on the negative imaginary axis. This choice of normalization for u is motivated by the following:

Lemma 3.4.3. *Let $0 < \nu < 1$. Then*

$$\gamma_-(u(y, \eta)) = 1, \quad \gamma_+(u(y, \eta)) = -2\nu\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{i\xi(y, \eta)}{2}\right)^{2\nu}.$$

Proof. This follows from the asymptotic behavior

$$K_\nu(i\xi x) \sim \frac{\pi}{2\sin(\pi\nu)}\left(\frac{1}{\Gamma(1-\nu)}\left(\frac{i\xi x}{2}\right)^{-\nu} - \frac{1}{\Gamma(1+\nu)}\left(\frac{i\xi x}{2}\right)^{\nu}\right)$$

as $x \to 0^+$. □

Example. It is clear from Lemma 3.4.3 that the Dirichlet condition $T = \gamma_-$ and Neumann condition $T = \gamma_+$ satisfy the Lopatinskiĭ condition with respect to any elliptic Bessel operator. The same is therefore true for the Robin condition $T = \gamma_+ + T^-\gamma_-$, where $T^- \in C^\infty(\partial X)$.

Example. Consider a boundary condition $T = \gamma_+ + T^-\gamma_-$, where T^- is a nonzero vector field on ∂X.

1. If $1/2 < \nu < 1$, then $\widehat{T}_{(y,\eta)} = \gamma_+$ for arbitrary T^-. Thus T satisfies the Lopatinskiĭ conditions with respect to any elliptic Bessel operator.

2. If $\nu = 1/2$, then T is a classical oblique boundary condition. The Lopatinskiĭ condition is satisfied if T^- is a real vector field for example, but can otherwise fail.

3. If $0 < \nu < 1/2$, then
$$\widehat{T}_{(y,\eta)} = \sigma_1(T^-)(y, \eta)\gamma_-.$$
Since $\sigma_1(T^-)(y, \eta)$ is linear in η, it must have a nontrivial zero at each $y \in \partial X$ provided the dimension of the underlying manifold X is at least four (or three if T^- is real). In that case the Lopatinskiĭ condition necessarily fails at every point on the boundary.

Example. Consider the operator $\Delta_\nu = |D_\nu|^2 + D_y^2 + D_z^2$ acting on $(0,1)\times\mathbb{T}^2$, where (y,z) are standard coordinates on $\mathbb{T}^2 = (R/2\pi\mathbb{Z})^2$. Clearly Δ_ν is an elliptic Bessel operator. Consider the boundary value problem

$$\begin{cases} \Delta_\nu u = f, \\ T_0 u = g, \quad T_1 u = 0, \end{cases}$$

where $T_1 u = u|_{x=1}$, and $T_0 = (\partial_y - \partial_z)\gamma_-$. This is not a Fredholm problem, since there is an infinite dimensional kernel: for each $n \geq 0$, consider the function

$$u_n(x,y,z) = \left(\sqrt{x} K_\nu(nx) - \frac{K_\nu(n)}{K_\nu(-n)} \sqrt{x} K_\nu(-nx) \right) e^{in(y+z)}.$$

The family of u_n is linearly independent and each u_n solves the boundary value problem. If $0 < \nu < 1/2$, then $T_0' = \gamma_+ + T_0$ is a compact perturbation of the original problem; thus the problem with T_0' replacing T_0 is not Fredholm either. If $1/2 \leq \nu < 1$ then the problem with T_0' satisfies the Lopatinskiĭ condition, so is indeed Fredholm.

Before proceeding with the next lemma, suppose that $P \in \mathrm{Bess}_\nu(X)$ and μ is a density of product type near ∂X. If P is elliptic at $p \in \partial X$, then so is P^*, since the function (3.7) is simply replaced by its complex conjugate.

Lemma 3.4.4. *Suppose that $\mathscr{P} = \{P, T, C\}$ is elliptic. If μ is a density of product type near ∂X and $\mathscr{P}^*$ is the corresponding adjoint boundary value problem, then $\mathscr{P}^*$ is also elliptic.*

Proof. Since ellipticity only depends on various "principal symbols", it is easy to see that

$$\widehat{P^*}_{(y,\eta)} = \widehat{P}^*_{(y,\eta)},$$

where the latter adjoint is calculated with respect to the standard $L^2(\mathbb{R}_+)$ inner product. Similarly

$$\widehat{T^*}_{(y,\eta)} = \widehat{T}^*_{(y,\eta)}, \quad \widehat{C^*}_{(y,\eta)} = \widehat{C}^*_{(y,\eta)},$$

where the latter adjoints are taken in the sense of matrices over $\mathbb{C}$.

Suppressing the dependence on (y,η), Green's formula (3.28) implies that

$$\langle \widehat{P}u, v\rangle_{\mathbb{R}_+} + \langle \widehat{T}u + \widehat{C}\underline{u}, \underline{v}\rangle_{\mathbb{C}^{J+1}} = \langle u, \widehat{P}^* v\rangle_X + \langle \underline{\gamma} u, \widehat{G}^* \underline{v} + J\underline{\gamma} v\rangle_{\mathbb{C}^2} + \langle \underline{u}, \widehat{C}^* \underline{v}\rangle_{\mathbb{C}^J}.$$

The goal is to prove that if the right hand side vanishes, then $(v, \underline{v}) = 0$. The proof relies on Lemma 3.4.7 below (whose proof is of course independent of the present lemma). As in the proof of Lemma 3.4.7, the Lopatinskiĭ condition implies that

$$(u, \underline{u}) \mapsto \widehat{T}u + \widehat{C}\underline{u}$$

is an isomorphism between the spaces $\mathcal{M}_+ \times \mathbb{C}^J \to \mathbb{C}^{J+1}$. So choose $(u, \underline{u}) \in \mathcal{M}_+ \times \mathbb{C}^J$ such that

$$\widehat{T}u + \widehat{C}\underline{u} = \underline{v}.$$

Since $\widehat{A}u = 0$, it follows from Green's formula that $\underline{v} = 0$. On the other hand, from Lemma 3.4.7 it is always possible to solve the inhomogeneous equation

$$\begin{cases} \widehat{A}u = v, \\ \widehat{T}u + \widehat{C}\underline{u} = 0, \end{cases}$$

whence Green's formula implies that $v = 0$ as well. □

The Dirichlet Laplacian

The results of this section are applied in one dimension to Section 3.4, and in higher dimensions to Appendix 3.7.5. Define the Bessel operator $\Delta_\nu \in \mathrm{Bess}_\nu(\mathbb{T}^n_+)$ by

$$\Delta_\nu = |D_\nu|^2 + \Delta_{\mathbb{T}^{n-1}},$$

where $\Delta_{\mathbb{T}^{n-1}} = \sum_{i=1}^{n-1} D^2_{y^i}$ is the positive Laplacian on $\mathbb{T}^{n-1}$. Consider the continuous, non-negative Hermitian form

$$\ell(u,v) := \langle D_\nu u, D_\nu v\rangle_{\mathbb{T}^n_+} + \sum_{i=1}^{n-1} \left\langle D_{y^i} u, D_{y^i} u\right\rangle_{\mathbb{T}^n_+} \tag{3.40}$$

on $\mathring{\mathcal{H}}^1(\mathbb{T}^n_+)$. Associated to this form is the unbounded self-adjoint operator L on $L^2(\mathbb{T}^n_+)$ with domain

$$D(L) = \{u \in \mathring{\mathcal{H}}^1(\mathbb{T}^n_+) : v \mapsto \ell(u,v) \text{ is continuous on } L^2(\mathbb{T}^n_+)\}.$$

Standard manipulations show that

$$D(L) = \mathring{\mathcal{H}}^1(\mathbb{T}^n_+) \cap \{u \in L^2(\mathbb{T}^n_+) : \Delta_\nu u \in L^2(\mathbb{T}^n_+)\}, \tag{3.41}$$

and $Lu = \Delta_\nu u$ in the sense of distributions for each $u \in D(L)$. The domain $D(L)$ is equipped with the graph norm.

Remark 10. *In one dimension it is obvious tht $D(L) = \mathcal{H}^2(\mathbb{R}_+) \cap \mathring{\mathcal{H}}^1(\mathbb{R}_+)$, with an equivalence of norms via the open mapping theorem. This is also true in higher dimensions, but is not immediate from the definition*

The next lemma follows from the Lax–Milgram theorem.

Lemma 3.4.5. *Let $\nu > 0$. For each $a \in \mathbb{C} \setminus (-\infty, 0]$ the inverse $(L+a)^{-1}$ exists, and maps*

$$(L+a)^{-1} : \begin{cases} \mathring{\mathcal{H}}^1(\mathbb{T}^n_+)' \to \mathring{\mathcal{H}}^1(\mathbb{T}^n_+), \\ L^2(\mathbb{T}^n_+) \to D(L). \end{cases}$$

Proof. Since $a \notin (-\infty, 0]$ the form $\ell_a(u,v) = \ell(u,v) + a\,\langle u, v\rangle_{\mathbb{T}^n_+}$ is coercive on $\mathring{\mathcal{H}}^1(\mathbb{T}^n_+)$, so $\ell_a(u,v)$ defines an inner product on $\mathring{\mathcal{H}}^1$ equivalent to the usual one. The Lax–Milgram theorem guarantees that for each $f \in \mathring{\mathcal{H}}^1(\mathbb{T}^n_+)'$ there exists a unique $u \in \mathring{\mathcal{H}}^1(\mathbb{T}^n_+)$ such that $\ell_a(u,v) = \langle f, v\rangle$, and the mapping $u \mapsto f$ is continuous $\mathring{\mathcal{H}}^1(\mathbb{T}^n_+)' \to \mathring{\mathcal{H}}^1(\mathbb{T}^n_+)$.

Furthermore, the unbounded operator associated to ℓ_a is clearly $L + a$ (acting in the distributional sense) so $L + a : D(L) \to L^2(\mathbb{T}^n_+)$ is bijective. Since this map is continuous when $D(L)$ is equipped with the graph norm, it is an isomorphism by the open mapping theorem. □

Elliptic Bessel operators on $\mathbb{R}_+$

In this section, fix an operator P on $\mathbb{R}_+$ of the form

$$P = |D_\nu|^2 + a, \quad a \in \mathbb{C}. \tag{3.42}$$

Thus $\xi \mapsto \xi^2 + a$ has no real roots precisely when $a \notin (-\infty, 0]$. In that case, P is said to be *regular*. This is distinguished from ellipticity of P since the principal symbol of multiplication by a as a second order operator is zero (in other words, the boundary symbol operator is $|D_\nu|^2$ and not $|D_\nu|^2 + a$). Furthermore, if $0 < \nu < 1$, fix boundary conditions (T, C). Thus T is just a column vector of J boundary operators $T_k = T_k^\pm \gamma_\pm$ with $T_k^\pm \in \mathbb{C}$, and C is a $(J+1) \times J$ matrix with $\mathbb{C}$-valued entries.

Regularity of the operator $\mathscr{P} = \{P, T, C\}$ is defined as just the Lopatinskiĭ condition: let $\mathcal{M}_+$ denote the space of bounded solutions to the equation $Pu = 0$. Then $\mathscr{P}$ is regular if the only element $(u, \underline{u}) \in \mathcal{M}_+ \times \mathbb{C}^J$ satisfying $Tu + C\underline{u} = 0$ is the trivial solution.

Proposition 3.4.6. *Suppose that P given by (3.42) is regular, and that $\mathscr{P} = \{P, T, C\}$ is regular if $0 < \nu < 1$.*

1. *If $0 < \nu < 1$, then $\mathscr{P}$ is an isomorphism*

$$\widetilde{\mathcal{H}}^s(\mathbb{R}_+) \times \mathbb{C}^J \to \mathcal{H}^{s-2}(\mathbb{R}_+) \times \mathbb{C}^{1+J}$$

   for each $s = 0, 1, 2$. The operator norm of $\mathscr{P}^{-1}$ depends continuously on a and the coefficients of G and C

2. *If $\nu \geq 1$, then P is an isomorphism*

$$\mathcal{H}^s(\mathbb{R}_+) \to \mathcal{H}^{s-2}(\mathbb{R}_+)$$

   for each $s = 0, 1, 2$. The operator norm of P^{-1} depends continuously on a.

The proof of this proposition is split up across several Lemmas.

Lemma 3.4.7. *Proposition 3.4.6 holds when $0 < \nu < 1$ and $s = 2$.*

Proof. Since $\widetilde{\mathcal{H}}^2(\mathbb{R}_+)$ is isomorphic to $\mathcal{H}^2(\mathbb{R}_+)$ via the map $v \mapsto (v, \underline{\gamma} v)$, it is sufficient to prove the lemma with $\mathcal{H}^2(\mathbb{R}_+)$ replacing $\widetilde{\mathcal{H}}^2(\mathbb{R}_+)$. By the regularity condition, P is injective. Indeed any solution in $\mathcal{H}^2(\mathbb{R}_+)$ to the equation $Pu = 0$ must lie in $\mathcal{M}_+$, and the Lopatinskiĭ condition implies that such a solution is unique. It remains to show surjectivity.

Fix $(f, g) \in \mathcal{H}^0(\mathbb{R}_+) \times \mathbb{C}^{J+1}$. From Lemma 3.4.5, it follows that that the equation

$$Pu = f$$

has a solution $u_1 \in \mathcal{H}^2(\mathbb{R}_+) \cap \mathring{\mathcal{H}}^1(\mathbb{R}_+)$. It then suffices to let $(u_2, \underline{u}) \in \mathcal{M}_+ \times \mathbb{C}^J$ solve

$$\begin{cases} Pu_2 = 0, \\ Tu_2 + C\underline{u} = g - Tu_1, \end{cases}$$

This is possible since

$$(u, \underline{u}) \mapsto Tu + C\underline{u}$$

as a map between the finite dimensional vector spaces $\mathcal{M}_+ \times \mathbb{C}^J \to \mathbb{C}^{J+1}$ is injective, hence an isomorphism. Setting $u = u_1 + u_2$ shows that $\mathscr{P}(u, \underline{u}) = (f, g)$. It is also easy to see that the operator norm of $\mathscr{P}$ depends continuously on a and the coefficients of G and C, which implies the same for the operator norm of $\mathscr{P}^{-1}$ via the resolvent identity. □

Lemma 3.4.8. *Proposition 3.4.6 holds when $0 < \nu < 1$ and $s = 0$.*

Proof. Since the formal adjoint operator $\mathscr{P}^*$ is also regular according to Lemma 3.4.4, the map

$$\mathcal{H}^2(\mathbb{R}_+) \times \mathbb{C}^{1+J} \to \mathcal{H}^0(\mathbb{R}_+) \times \mathbb{C}^2 \times \mathbb{C}^J$$

given by

$$(v, \underline{v}) \mapsto (P^* v,\ J\underline{\gamma} v + G^* \underline{v},\ C^* \underline{v})$$

is an isomorphism according to Lemma 3.4.7. But in that case, a direct calculation shows that $\mathscr{P}^*$ agrees with the Hilbert space adjoint $\mathscr{P}'$ of

$$\mathscr{P} : \widetilde{\mathcal{H}}^0(\mathbb{R}_+) \times \mathbb{C}^J \to \mathcal{H}^{-2}(\mathbb{R}_+) \times \mathbb{C}^{1+J}.$$

Since $\mathscr{P}'$ is an isomorphism, $\mathscr{P}$ is an isomorphism on the stated spaces as well. □

To prove Proposition 3.4.6 for $s = 1$, the following regularity result is needed.

Lemma 3.4.9. *Let $0 < \nu < 1$. Suppose that $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^0(\mathbb{R}_+)$ satisfies $P(u, \underline{\phi}) \in \mathcal{H}^{-1}(\mathbb{R}_+)$. Then $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^1(\mathbb{R}_+)$.*

Proof. Let $f \in \mathring{\mathcal{H}}^1(\mathbb{R}_+)'$ denote the restriction of the functional $P(u, \underline{\phi})$ to $\mathring{\mathcal{H}}^1(\mathbb{R}_+)$. This implies that $f = Pu$ in the sense of distributions. By Lemma 3.4.5, there exists a unique

$\tilde{u} \in \mathring{\mathcal{H}}^1(\mathbb{R}_+)$ such that $P\tilde{u} = f$ in the distributional sense. Thus in sense of distributions on $\mathbb{R}_+$,

$$P(u - \tilde{u}) = 0.$$

Since u and $\tilde{u}$ are square integrable, it follows that $u - \tilde{u} \in \mathcal{M}_+$. Thus it is certainly true that

$$u = (u - \tilde{u}) + \tilde{u} \in \mathcal{H}^1(\mathbb{R}_+).$$

It remains remains to prove that $\phi_- = \gamma_- u$. A priori $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^0(\mathbb{R}_+)$, so for each $v \in \mathcal{H}^2(\mathbb{R}_+)$,

$$\langle f, v \rangle_{\mathbb{R}_+} = \langle u, P^* v \rangle_{\mathbb{R}_+} - \phi_+(\gamma_- v) + \phi_-(\gamma_+ v).$$

Using that $u \in \mathcal{H}^1(\mathbb{R}_+)$, this may be rewritten as

$$\langle f, v \rangle_{\mathbb{R}_+} - \langle D_\nu u, D_\nu v \rangle_{\mathbb{R}_+} - a\langle u, v \rangle_{\mathbb{R}_+} + \phi_+(\gamma_- v) = (\phi_- - \gamma_- u)\gamma_+ v$$

for each $v \in \mathcal{H}^2(\mathbb{R}_+)$. But the left hand side extends to a continuous functional on $\mathcal{H}^1(\mathbb{R}_+)$, which is not true of the right hand side unless $\phi_- = \gamma_- u$, thus completing the proof. □

Lemma 3.4.10. *Proposition 3.4.6 holds when* $0 < \nu < 1$ *and* $s = 1$*.*

Proof. The regularity result of Lemma 3.4.9 combined with Lemma 3.4.8 shows that $\mathscr{P}$ defines a continuous bijection, hence an isomorphism

$$\widetilde{\mathcal{H}}^1(\mathbb{R}_+) \times \mathbb{C}^J \to \mathcal{H}^{-1}(\mathbb{R}_+) \times \mathbb{C}^{J+1}$$

as stated. □

Lemma 3.4.11. *Proposition 3.4.6 holds when* $\nu \geq 1$*.*

Proof. If $\nu \geq 1$, then $\mathcal{H}^s(\mathbb{R}_+) = \mathring{\mathcal{H}}^s(\mathbb{R}_+)$ for $s \geq 0$. Thus it suffices to apply Lemma 3.4.5 directly when $s = 1, 2$. The case $s = 0$ is handled by duality, similar to 3.4.8. □

Proof of Proposition 3.4.6. The combination of Lemmas 3.4.7, 3.4.10, 3.4.8, 3.4.11 establishes Proposition 3.4.6 □

Elliptic Bessel operators on $\mathbb{T}^n_+$ with constant coefficients

Throughout this section, P denotes a constant coefficient Bessel operator on $\mathbb{T}^n_+$,

$$P(D_\nu, D_y) = |D_\nu|^2 + A(D_y) \tag{3.43}$$

If $0 < \nu < 1$, then P is also augmented by boundary conditions (T, C) with constant coefficients: thus each boundary operator is of the form $T_k(D) = T_k^{\pm}(D_y)\gamma_\pm$, and each entry of $C(D_y)$ has constant coefficients.

The principal part of P is the operator

$$P^\circ(D_\nu, D_y) = |D_\nu|^2 + A^\circ(D_y),$$

where $A^\circ(D_y)$ is the standard principal part of A. The principal parts of $(T(D_y), C(D_y))$ are defined to be the unique boundary operators $(T^\circ(D_y), C^\circ(D_y))$ satisfying

$$T^\circ(\eta) = \widehat{T}_\eta, \quad C^\circ(\eta) = \widehat{C}_\eta$$

for each $\eta \in \mathbb{R}^{n-1}$. Finally, define $\mathscr{P}^\circ(D_\nu, D_y) = \{P^\circ(D_\nu, D_y), T^\circ(D_y), C^\circ(D_y)\}$. Ellipticity of either P or $\mathscr{P}$ depends only on these principal parts.

Lemma 3.4.12. *Assume that P and $\mathscr{P}$ are elliptic. Furthermore, assume that the one dimensional operators $P(D_\nu, q)$ (if $\nu \geq 1$) and $\mathscr{P}(D_\nu, q)$ (if $0 < \nu < 1$) are regular for each $q \in \mathbb{Z}^{n-1}$.*

1. *If $0 < \nu < 1$, then*

$$\mathscr{P} : \mathcal{H}^2(\mathbb{T}^n_+) \times H^{2+\underline{\tau}}(\mathbb{T}^{n-1}) \to \mathcal{H}^0(\mathbb{T}^n_+) \times \mathcal{H}^{2-\underline{\mu}}(\mathbb{T}^{n-1})$$

is an isomorphism.

2. *If $\nu \geq 1$, then*

$$P : \mathcal{H}^2(\mathbb{T}^n_+) \to \mathcal{H}^0(\mathbb{T}^n_+)$$

is an isomorphism.

Proof. (1) Let $0 < \nu < 1$. By ellipticity,

$$\mathscr{P}^\circ(D_\nu, \langle q\rangle^{-1} q) : \mathcal{H}^2(\mathbb{R}_+) \times \mathbb{C}^J \to \mathbb{C}^{1+J}$$

is an isomorphism for each $q \in \mathbb{Z}^{n-1}$, according to Proposition 3.4.6. Since $\langle q\rangle^{-1} q$ ranges over a compact subset of $\mathbb{R}^{n-1}$, the operator norm of $\mathscr{P}^\circ(D_\nu, \langle q\rangle^{-1} q)^{-1}$ is bounded uniformly with respect to $q \in \mathbb{Z}^{n-1} \setminus 0$. On the other hand, the homogeneity of P° implies

$$\tau^{-2} S_{-\tau} P^\circ(D_\nu, D_y) S_\tau = P^\circ(D_\nu, \tau^{-1} D_y), \quad \tau^{-\underline{\mu}+1/2} T^\circ(D_y) S_\tau = T^\circ(\tau^{-1} D_y).$$

Using $\tau = \langle q\rangle$, this implies that the operator norm corresponding to the problem

$$\begin{cases} \langle q\rangle^{-2} S_{\langle q\rangle^{-1}} P(D_\nu, q) S_{\langle q\rangle} v = \phi, \\ \langle q\rangle^{-\mu_k+1/2} T(q) S_{\langle q\rangle} v + \sum_{i=1}^J \langle q\rangle^{-\tau_j-\mu_k} C_{k,j}(q)\underline{v} = \underline{\psi} \end{cases}$$

tends to that of $\mathscr{P}^\circ(D_\nu, \langle q\rangle^{-1} D_y)$ as $|q| \to \infty$. Thus the former problem is invertible for $q \in \mathbb{Z}^{n-1}$ with operator norm uniformly bounded in q. Apply this invertibility result to the functions

$$v = S_{\langle q\rangle^{-1}} \hat{u}(q), \quad \underline{v} = (\langle q\rangle^{\tau_1+1/2} \underline{\hat{u}}_1(q), \ldots, \langle q\rangle^{\tau_J+1/2} \underline{\hat{u}}_J(q)).$$

This implies that

$$\begin{aligned} &\|S_{\langle q\rangle^{-1}} \hat{u}(q)\|^2_{\mathcal{H}^2(\mathbb{T}^n_+)} + \langle q\rangle^{1+2\underline{\tau}} \|\underline{\hat{u}}(q)\|^2_{\mathbb{C}^J} \\ &\qquad \leq C\left(\langle q\rangle^{-4} \|S_{\langle q\rangle^{-1}} P(D_\nu, q)\hat{u}(q)\|^2 + \langle q\rangle^{1-2\underline{\mu}} \|T(q)\hat{u}(q) + C_{k,j}(q)\hat{u}(q)\|^2_{\mathbb{C}^{J+1}}\right). \quad (3.44) \end{aligned}$$

From (3.44) it follows that $\mathscr{P}$ is injective. Now multiply this equation by $\langle q\rangle^{2s-1} = \langle q\rangle^3$ and sum over $q \in \mathbb{Z}^{n-1}$. Then Lemma 3.3.8 shows that the Fourier series for $(u, \underline{u})$ converges in $\mathcal{H}^2(\mathbb{T}^n_+) \times H^{2+\underline{\tau}}(\mathbb{T}^{n-1})$. Combined with the fact that $\mathscr{P}(D_\nu, q)$ is invertible for each $q \in \mathbb{Z}^{n-1}$, this shows that $\mathscr{P}$ is surjective.

(2) The proof when $\nu \geq 1$ follows as above, disregarding the boundary operators. □

Corollary 3.4.13. *Assume that P and $\mathscr{P}$ are elliptic. Furthermore, assume that the one dimensional operators $P(D_\nu, q)$ (if $\nu \geq 1$) and $\mathscr{P}(D_\nu, q)$ (if $0 < \nu < 1$) are regular for each $q \in \mathbb{Z}^{n-1}$.*

1. *If $0 < \nu < 1$, then*
$$\mathscr{P} : \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+) \times H^{s+\underline{\tau}}(\mathbb{T}^{n-1}) \to \mathcal{H}^{s-2}(\mathbb{T}^n_+) \times \mathcal{H}^{s-\underline{\mu}}(\mathbb{T}^{n-1})$$
is an isomorphism for $s = 0, 1, 2$.

2. *If $\nu \geq 1$, then*
$$P : \mathcal{H}^s(\mathbb{T}^n_+) \to \mathcal{H}^{s-2}(\mathbb{T}^n_+)$$
is an isomorphism for $s = 0, 1, 2$.

Proof. (1) It remains to handle the cases $s = 0, 1$. First consider $s = 0$. As in the proof of Lemma 3.4.8, the formal adjoint

$$\mathscr{P}^* : \mathcal{H}^2(\mathbb{T}^n_+) \times H^{\underline{\mu}}(\mathbb{T}^{n-1}) \to \mathcal{H}^0(\mathbb{T}^n_+) \times H^{\underline{\nu}-2}(\mathbb{T}^{n-1}) \times H^{-\underline{\tau}}(\mathbb{T}^{n-1})$$

agrees with the adjoint of

$$\mathscr{P} : \widetilde{\mathcal{H}}^0(\mathbb{T}^n_+) \times H^{\underline{\tau}}(\mathbb{T}^{n-1}) \to \mathcal{H}^{-2}(\mathbb{T}^n_+) \times H^{-\underline{\mu}}(\mathbb{T}^{n-1}).$$

Now $\mathscr{P}^*$ satisfies the same hypotheses as $\mathscr{P}$ in regards to the application of Lemma 3.4.12, so is an isomorphism. This implies that $\mathscr{P}'$ is an isomorphism, hence so is $\mathscr{P}$ on the stated spaces.

The case $s = 1$ follows from (3.44) combined with Lemma 3.4.10: indeed, multiplying the analogue of (3.44) by $\langle q\rangle^{2s-1} = \langle q\rangle$ and using the invertibility result from Lemma 3.4.10 shows that $\mathscr{P}$ is surjective on $\widetilde{\mathcal{H}}^1(\mathbb{T}^n_+) \times H^{1+\underline{\tau}}(\mathbb{T}^n_+)$ (as well as injective by the $s = 0$ case).

(2) As usual, when $\nu \geq 1$ the proof follows by dropping the boundary terms. □

Remark 11. *If $P(D_\nu, D_y)$ is elliptic, then $P^\circ(D_\nu, D_y + \frac{1}{2})$ satisfies the hypotheses of Lemma 3.4.12. Similarly, if $0 < \nu < 1$ and $\mathscr{P}(D_\nu, D_y)$ is elliptic, then $\mathscr{P}^\circ(D_\nu, D_y + \frac{1}{2})$ also satisfies the hypotheses of Lemma 3.4.12.*

Elliptic Bessel operators on $\mathbb{T}^n_+$ with variable coefficients

In this section, let P be a Bessel operator on $\mathbb{T}^n_+$ of the form

$$P(x, y, D_\nu, D_y) = |D_\nu|^2 + B(x, y, D_y)D_\nu + A(x, y, D_y),$$

where the coefficients of A, B are constant outside a compact subset of $\overline{\mathbb{T}^n_+}$. If $0 < \nu < 1$, then P is also augmented by boundary conditions $(T(y, D_y), C(y, D_y))$. Introduce the notation

$$P^{(0)}(D_\nu, D_y) := P^\circ(0, 0, D_\nu, D_y + \tfrac{1}{2}),$$
$$T^{(0)}(D_y) = T^\circ(0, D_y + \tfrac{1}{2}), \quad C^{(0)}(D_y) = C^\circ(0, D_y + \tfrac{1}{2}).$$

According to Lemma 3.4.12, if P and $\mathscr{P}$ are elliptic, then $P^{(0)}$ (if $\nu \geq 1$) and $\mathscr{P}^{(0)}$ (if $0 < \nu < 1$) are isomorphisms on the appropriate spaces.

Given $\rho > 0$, define the Fourier multiplier $K_\rho = 1_{|x|\geq\rho}(D_y)$. This operator acts both on Sobolev spaces $H^m(\mathbb{T}^{n-1})$, as well as on $\mathcal{H}^s(\mathbb{T}^n_+)$ (or $\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$) via the results of Section 3.3. If $m > m'$ then clearly

$$\|K_\rho \phi\|_{H^{m'}(\mathbb{T}^{n-1})} \leq \langle\rho\rangle^{m-m'} \|\phi\|_{H^m(\mathbb{T}^{n-1})} \tag{3.45}$$

for $\phi \in H^m(\mathbb{T}^{n-1})$. Similarly,

$$\|K_\rho u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} \leq \langle\rho\rangle^{-|\alpha|} \|D_y^\alpha u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} \leq \langle\rho\rangle^{-|\alpha|} \|u\|_{\mathcal{H}^{s+|\alpha|}(\mathbb{T}^n_+)} \tag{3.46}$$

for $u \in \mathcal{H}^s(\mathbb{T}^n_+)$, provided $s + |\alpha| \leq 2$. A similar statement holds for $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)$.

Lemma 3.4.14. *Assume that P and $\mathscr{P}$ are elliptic. Then there exists $\delta > 0$ such that the following hold.*

1. *Let $0 < \nu < 1$ and $s = 0, 1, 2$. Suppose that $(u, \underline{\phi}, \underline{u}) \in \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+) \times H^{s+\underline{\tau}}(\mathbb{T}^{n-1})$ satisfies*

$$\operatorname{supp} u \subseteq \{(x, y) \in \overline{\mathbb{T}^n_+} : |x| + |y| < \varepsilon\},$$
$$\operatorname{supp} \underline{\phi} \subseteq \{y \in \mathbb{T}^{n-1} : |y| < \delta\}, \quad \operatorname{supp} \underline{u} \subseteq \{y \in \mathbb{T}^{n-1} : |y| < \delta\}.$$

Then

$$\begin{aligned}\|(u, \underline{\phi}, \underline{u})\|_{\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)\times H^{s+\underline{\tau}}(\mathbb{T}^{n-1})} \leq C(&\|\mathscr{P}(u, \underline{\phi}, \underline{u})\|_{\mathcal{H}^{s-2}(\mathbb{T}^n_+)\times H^{s-\underline{\mu}}(\mathbb{T}^{n-1})} \\ &+ \|(u, \underline{\phi}, \underline{u})\|_{\widetilde{\mathcal{H}}^{s-1}(X)\times H^{s-1+\underline{\tau}}(\mathbb{T}^{n-1})}),\end{aligned} \tag{3.47}$$

where $C > 0$ does not depend on $(u, \underline{\phi}, \underline{u})$. In addition, if $s = 0, 1$ and

$$\mathscr{P}(u, \underline{\phi}, \underline{u}) \in \mathcal{H}^{s-1}(\mathbb{T}^n_+) \times \mathcal{H}^{s-\underline{\mu}+1}(\mathbb{T}^{n-1}),$$

then $(u, \underline{\phi}, \underline{u}) \in \widetilde{\mathcal{H}}^{s+1}(\mathbb{T}^n_+) \times H^{s+\underline{\tau}+1}(\mathbb{T}^n_+)$.

2. *Let* $\nu \geq 1$ *and* $s = 0, 1, 2$. *Suppose that* $u \in \mathcal{H}^2(\mathbb{T}^n_+)$ *satisfies*

$$\operatorname{supp} u \subseteq \{(x,y) \in \overline{\mathbb{T}^n_+} : |x| + |y| < \delta\}$$

Then

$$\|u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} \leq C(\|Pu\|_{\mathcal{H}^{s-2}(\mathbb{T}^n_+)} + \|u\|_{\mathcal{H}^{s-1}(\mathbb{T}^n_+)}),$$

where $C > 0$ *does not depend on* u. *In addition, if* $s = 0, 1$ *and* $Pu \in \mathcal{H}^{s-1}(\mathbb{T}^n_+)$, *then* $u \in \mathcal{H}^{s+1}(\mathbb{T}^n_+)$.

Proof. (1) For concreteness, assume that $s = 1$ and $\mathscr{P}(u, \underline{\phi}, \underline{u}) \in \mathcal{H}^0(\mathbb{T}^n_+) \times H^{2-\underline{\mu}}(\mathbb{T}^{n-1})$. If $(f, \underline{g}) = \mathscr{P}(u, \underline{\phi}, \underline{u})$, consider the identity

$$\begin{aligned}\mathscr{P}^{(0)}(u, \underline{\phi}, \underline{u}) + (\mathscr{P} - \mathscr{P}^{(0)})(K_\rho(u, \underline{\phi}), K_\rho \underline{u})& \\ = (f, \underline{g}) - (\mathscr{P} - \mathscr{P}^{(0)})((1-K_\rho)(u, \underline{\phi}), (1-K_\rho)\underline{u}).& \quad (3.48)\end{aligned}$$

Noting that the term $(P - P^{(0)})(u, \underline{\phi})$ depends only on u (and not on $\underline{\phi}$), it follows from Lemmas 3.3.13, 3.3.14 and (3.46) that

$$\begin{aligned}\|(P - P^{(0)})K_\rho u\|_{\mathcal{H}^0(\mathbb{T}^n_+)} &\leq C_1 \delta \|u\|_{\mathcal{H}^2(\mathbb{T}^n_+)} + C_2 \|K_\rho u\|_{\mathcal{H}^1(\mathbb{T}^n_+)} \\ &\leq (C_1 \delta + C_2 \langle \rho \rangle^{-1}) \|(u, \underline{\phi})\|_{\widetilde{\mathcal{H}}^2(\mathbb{T}^n_+)},\end{aligned}$$

for positive constants C_1, C_2. By standard interpolation inequalities on $H^m(\mathbb{T}^{n-1})$,

$$\begin{aligned}\|(T_k - T^{(0)})_k K_\rho \underline{\phi}\| &\leq C_3 \delta \|\underline{\phi}\|_{H^{2-\underline{\nu}}(\mathbb{T}^{n-1})} + C_4 \|K_\rho \underline{\phi}\|_{H^{1-\underline{\nu}}(\mathbb{T}^{n-1})} \\ &\leq (C_3 \delta + C_4 \langle \rho \rangle^{-1}) C_5 \|(u, \underline{\phi})\|_{\widetilde{\mathcal{H}}^2(\mathbb{T}^n_+)}.\end{aligned}$$

For this, one should consider the cases $0 < \nu < 1/2$, $\nu = 1/2$, and $1/2 < \nu < 1$ separately, but they all yield the same type of the estimate. Similarly,

$$\|(C - C^{(0)})K_\rho \underline{u}\|_{H^{2-\underline{\mu}}(\mathbb{T}^{n-1})} \leq (C_6 \delta + C_7 \langle \rho \rangle^{-1}) \|\underline{u}\|_{H^{s+\underline{\tau}}(\mathbb{T}^{n-1})}.$$

These inequalities imply that the operator norm of

$$(u, \underline{\phi}, \underline{u}) \mapsto (\mathscr{P} - \mathscr{P}^{(0)})(K_\rho u, K_\rho \underline{u})$$

can be made arbitrarily small by choosing $\delta > 0$ small and $\rho > 0$ large. Since $\mathscr{P}^{(0)}$ is invertible with domain $\widetilde{\mathcal{H}}^2(\mathbb{T}^n_+) \times H^{2+\underline{\tau}}(\mathbb{T}^{n-1})$, it follows that the operator on the left hand side of (3.48) is invertible for δ small and ρ large.

On the other hand, the map

$$(u, \underline{\phi}, \underline{u}) \mapsto (\mathscr{P} - \mathscr{P}^{(0)})((1-K_\rho)(u, \underline{\phi}), (1-K_\rho)\underline{u})$$

is bounded $\widetilde{\mathcal{H}}^1(\mathbb{T}^n_+) \times H^{1+\underline{\tau}}(\mathbb{T}^{n-1}) \to \mathcal{H}^0(\mathbb{T}^n_+) \times H^{2-\underline{\mu}}(\mathbb{T}^{n-1})$. In particular, $(u,\underline{\phi},\underline{u}) \in \widetilde{\mathcal{H}}^2(\mathbb{T}^n_+) \times H^{2+\underline{\tau}(\mathbb{T}^{n-1})}$, and the estimate (3.47) holds. Of course this also implies that (3.47) holds for arbitrary $(u,\underline{\phi},\underline{u}) \in \widetilde{\mathcal{H}}^2(\mathbb{T}^n_+) \times H^{2+\tau}(\mathbb{T}^{n-1})$ as well. The exact same argument establishes the regularity result for $s = 0$, as well (3.47) for $s = 0, 1$.

(2) As usual, the case $\nu \geq 1$ can be handled by a simpler argument not involving the boundary operators.

□

Lemma 3.4.14 can be semi-globalized via a partition of unity argument.

Corollary 3.4.15. *Assume that P and $\mathscr{P}$ are elliptic at ∂X. There exists $\delta > 0$ such that if $\varphi, \chi \in C_c^\infty([0,\delta))$ satisfy $\varphi = 1$ near $x = 0$ and $\chi = 1$ near $\operatorname{supp}\varphi$, then the following hold.*

1. *Let $0 < \nu < 1$ and $s = 0, 1, 2$. Then*
$$\|\varphi(u,\underline{\phi},\underline{u})\|_{\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)\times H^{s+\underline{\tau}}(\mathbb{T}^{n-1})} \leq C(\|\varphi\mathscr{P}(u,\underline{\phi},\underline{u})\|_{\mathcal{H}^{s-2}(\mathbb{T}^n_+)\times H^{s-\underline{\mu}}(\mathbb{T}^{n-1})} + \|\chi(u,\underline{\phi},\underline{u})\|_{\widetilde{\mathcal{H}}^{s-1}(\mathbb{T}^n_+)\times H^{s-1+\underline{\tau}}(\mathbb{T}^{n-1})}), \tag{3.49}$$
for each $(u,\underline{\phi},\underline{u}) \in \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+) \times \mathcal{H}^{s+\underline{\tau}}(\mathbb{T}^{n-1})$. In addition, if $s = 0, 1$ and
$$\varphi\mathscr{P}(u,\underline{\phi},\underline{u}) \in \mathcal{H}^{s-1}(\mathbb{T}^n_+) \times \mathcal{H}^{s-\underline{\mu}+1}(\mathbb{T}^{n-1}),$$
then $\varphi(u,\underline{\phi},\underline{u}) \in \widetilde{\mathcal{H}}^{s+1}(\mathbb{T}^n_+) \times H^{s+\underline{\tau}+1}(\mathbb{T}^{n-1})$.

2. *Let $\nu \geq 1$ and $s = 0, 1, 2$. Then*
$$\|\varphi u\|_{\mathcal{H}^s(\mathbb{T}^n_+)} \leq C(\|\varphi P u\|_{\mathcal{H}^{s-2}(\mathbb{T}^n_+)} + \|\chi u\|_{\mathcal{H}^{s-1}(\mathbb{T}^n_+)}), \tag{3.50}$$
for each $u \in \mathcal{H}^s(\mathbb{T}^n_+)$. In addition, if $s = 0, 1$ and $\varphi P u \in \mathcal{H}^{s-1}(\mathbb{T}^n_+)$, then $\varphi u \in \mathcal{H}^{s+1}(\mathbb{T}^n_+)$.

Sketch of proof for $0 < \nu < 1$. By compactness of $\mathbb{T}^{n-1}$ it is possible to choose δ and a finite cover $\mathbb{T}^{n-1} = \bigcup_i U_i$ such that Lemma 3.4.14 is valid for $(u,\underline{\phi},\underline{u})$ supported in $[0,\delta) \times U_i$. Fix a partition of unity β_i subordinate to U_i, and choose γ_i supported in U_i that $\gamma_i = 1$ on $\operatorname{supp}\beta_i$. For φ, χ as in the statement of the corollary,
$$\begin{aligned}\|\varphi(u,\underline{\phi},\underline{u})\|_{\widetilde{\mathcal{H}}^s(\mathbb{T}^n_+)\times H^{s+\underline{\tau}}(\mathbb{T}^{n-1})} &\leq C_1\|\varphi\mathscr{P}(u,\underline{\phi},\underline{u})\|_{\mathcal{H}^s(\mathbb{T}^n_+)\times H^{s-\underline{\mu}}(\mathbb{T}^{n-1})} \\ &+ \sum_i \|[\mathscr{P},\beta_i\varphi]\gamma_i\chi(u,\underline{\phi},\underline{\mu})\|_{\mathcal{H}^s(\mathbb{T}^n_+)\times H^{s-\underline{\mu}}(\mathbb{T}^{n-1})} \\ &+ C_2\|\varphi(u,\underline{\phi},\underline{u})\|_{\mathcal{H}^{s-1}(\mathbb{T}^n_+)\times H^{s-1+\underline{\tau}}(\mathbb{T}^{n-1})}.\end{aligned}$$
Writing $\varphi_i\beta_i\varphi$, the commutator $[\mathscr{P},\beta_i\varphi]$ is given by
$$(u,\underline{\phi},\underline{u}) \mapsto (P(\varphi_i u, \beta_i\underline{\phi}) - \varphi_i P(u,\underline{\phi}), [G,\beta_i]\underline{\phi} + [C,\beta_i]\underline{u}).$$
It is then straightforward to check that this operator has the requisite mapping properties. The regularity statement is established in the same way. □

Remark 12. *As usual, the norms of the lower order terms on the right hand sides of (3.53), (3.54) can be taken in less regular Sobolev spaces by iterating Corollary 3.4.15. Similarly, the regularity result can also be iterated.*

Elliptic Bessel operators on a compact manifold with boundary

The main theorem in this section establishes elliptic estimates and elliptic regularity for elliptic Bessel operators on a compact manifold with boundary $\overline{X}$.

Theorem 1. *Let $\overline{X}$ be a compact manifold with boundary as in Section 3.1. Assume that $P \in \mathrm{Bess}_\nu(X)$ is elliptic at ∂X in the sense of Section 3.1. If $0 < \nu < 1$, then assume P is augmented by boundary conditions (T, C) such that $\mathscr{P} = \{P, T, C\}$ is elliptic at ∂X. There exists $0 < \delta < \varepsilon$ such that if $\varphi, \chi \in C_c^\infty(\{0 \le x < \delta\})$ satisfy $\varphi = 1$ near ∂X and $\chi = 1$ near $\operatorname{supp}\varphi$, then the following hold.*

1. *Let $0 < \nu < 1$ and $s = 0, 1, 2$. Then*

$$\begin{aligned} \|\varphi(u, \underline{\phi}, \underline{u})\|_{\widetilde{\mathcal{H}}^s(X) \times H^{s+\underline{\tau}}(\partial X)} \le C(&\|\varphi \mathscr{P}(u, \underline{\phi}, \underline{u})\|_{\mathcal{H}^{s-2}(X) \times H^{s-\underline{\mu}}(\partial X)} \\ &+ \|\chi(u, \underline{\phi}, \underline{u})\|_{\widetilde{\mathcal{H}}^{s-1}(X) \times H^{s-1+\underline{\tau}}(\partial X)}), \end{aligned} \tag{3.51}$$

   for each $(u, \underline{\phi}, \underline{u}) \in \widetilde{\mathcal{H}}^s(X) \times \mathcal{H}^{s+\underline{\tau}}(\partial X)$. In addition, if $s = 0, 1$ and

$$\varphi \mathscr{P}(u, \underline{\phi}, \underline{u}) \in \mathcal{H}^{s-1}(X) \times \mathcal{H}^{s-\underline{\mu}+1}(\partial X),$$

   then $\varphi(u, \underline{\phi}, \underline{u}) \in \widetilde{\mathcal{H}}^{s+1}(X) \times H^{s+\underline{\tau}+1}(\partial X)$.

2. *Let $\nu \ge 1$ and $s = 0, 1, 2$. Then*

$$\|\varphi u\|_{\mathcal{H}^s(X)} \le C(\|\varphi P u\|_{\mathcal{H}^{s-2}(X)} + \|\chi u\|_{\mathcal{H}^{s-1}(X)}), \tag{3.52}$$

   for each $u \in \mathcal{H}^s(X)$. In addition, if $s = 0, 1$ and $\varphi P u \in \mathcal{H}^{s-1}(X)$, then $\varphi u \in \mathcal{H}^{s+1}(X)$.

Proof. The global problem may be reduced to a local problem on $\mathbb{T}^n_+$ via coordinate charts and a partition of unity. Indeed, cover the collar neighborhood $\overline{W}$ by finitely many coordinate charts (U_i, ψ_i) of the form

$$U_i = \psi([0, \varepsilon) \times Y_i), \quad \psi_i = (1 \times \theta_i) \circ \phi^{-1},$$

where (Y_i, θ_i) is a coordinate chart on ∂X. Choose a partition of unity β_i subordinate to Y_i and a function $\alpha \in C_c^\infty(0 \le x < \varepsilon)$ such that $\alpha = 1$ near $\{0 \le x < \delta\}$. Then set $\chi_i = \alpha\beta_i$ and apply the results of Corollary 3.4.15 to each element

$$((\chi_i u) \circ \psi_i^{-1}, (\beta_i \underline{\phi}) \circ \theta_i^{-1}, (\beta_i \underline{u}) \circ \theta_i^{-1}) \in \widetilde{\mathcal{H}}^s(\mathbb{T}^n_+) \times H^{s+\underline{\tau}}(\mathbb{T}^{n-1}).$$

Again there will be various commutator terms which give only lower order contributions, as in Corollary 3.4.15. ◻

As in the remark following Corollary 3.4.15, the error terms in Theorem 1 can taken in weaker Sobolev spaces by iteration.

Recall the definition of $\mathcal{H}^s_P(X)$ in Section 3.3. Theorem 1 can be used to show that $\mathcal{H}^s_P(X)$ (or equivalently $\widetilde{\mathcal{H}}^s(X)$) may be identified with the space of all pairs $(u,f) \in \mathcal{H}^s(X) \times \mathcal{H}^{s-2}(X)$ such that $Pu = f$ in the weak sense, see also [84, Chapter 6.1].

Lemma 3.4.16. *Let $0 < \nu < 1$, and suppose that P is elliptic at ∂X. Then for $s = 0, 1, 2$,*

$$\mathcal{H}^s_P(X) = \{(u,f) \in \mathcal{H}^s(X) \times \mathcal{H}^{s-2}(X) : Pu = f \text{ weakly}\},$$

where the space on the right hand side is equipped with the $\mathcal{H}^s_P(X)$ norm.

Proof. As in the remark following Lemma 3.3.26, $\mathcal{H}^s_P(X)$ is contained in the space on the right hand side. For the converse, suppose that $u \in \mathcal{H}^s(X)$ and $f = Pu \in \mathcal{H}^{s-2}(X)$ weakly. Similar to the proof of Lemma 3.3.26, consider the functional

$$\ell(\underline{\psi}) = \langle u, P^* v\rangle_X - \langle f, v\rangle_X, \quad \underline{\psi} \in C^\infty(\partial X) \times C^\infty(\partial X)$$

where v is any member of $\mathcal{F}_\nu(X)$ such that $\underline{\gamma} v = \underline{\psi}$. Since $Pu = f$ weakly, ℓ is well defined (namely it does not depend on the choice of v). In particular, one may take $v = \mathcal{K}(\underline{\psi})$, where $\mathcal{K}$ is a bounded right inverse mapping $C^\infty(\partial X) \times C^\infty(\partial X) \to \mathcal{F}_\nu(X)$. Thus

$$\ell(\underline{\psi}) \leq C_1 \|u\|_{\mathcal{H}^s_P(X)} (\|\mathcal{K}(\underline{\psi})\|_{\mathcal{H}^{2-s}(X)} + \|\underline{\psi}\|_{H^{2-s-\underline{\nu}}(\partial X)}) \leq C_2 \|u\|_{\mathcal{H}^s_P(X)} \|\underline{\psi}\|_{H^{2-s-\underline{\nu}}(\partial X)}$$

By Hahn–Banach and the Riesz theorem, there exists a unique $\underline{\phi} \in H^{s-\underline{\nu}}(\partial X)$ such that

$$\langle u, P^* v\rangle_X - \langle f, v\rangle_X = \left\langle J\underline{\phi}, \underline{\gamma} v\right\rangle_{\partial X}.$$

for each $v \in \mathcal{F}_\nu(X)$. Consider the pair $(u, \underline{\phi})$; a priori this is an element of $\widetilde{\mathcal{H}}^0(X)$. On the other hand, for each $v \in \mathcal{F}_\nu(X)$,

$$\begin{aligned} \left\langle P(u,\underline{\phi}), v\right\rangle_X &= \langle u, P^* v\rangle_X + \left\langle \underline{\phi}, J\underline{\gamma} v\right\rangle_{\partial X} \\ &= \langle f, v\rangle_X, \end{aligned}$$

so $P(u,\underline{\phi}) = f$. Since $f \in \mathcal{H}^{s-2}(X)$ and $\phi_- \in H^{s-1+\nu}(\partial X)$, Theorem 1 implies that $(u,\underline{\psi}) \in \widetilde{\mathcal{H}}^s(X)$ since the the boundary value problem $\{P, \gamma_-\}$ is elliptic at ∂X. According to Lemma 3.3.26, this means that the pair (u,f) can be identified with an element of $\mathcal{H}^s_P(X)$. ☐

Suppose that P is elliptic at ∂X and let $s = 0, 1$. If $u \in \mathcal{H}^s(X)$ and $Pu \in \mathcal{H}^0(X)$ in distributions, then there is a canonical $f \in \mathcal{H}^{s-2}(X)$ such that $Pu = f$ weakly, namely the element $Pu \in \mathcal{H}^0(X) \hookrightarrow \mathcal{H}^{s-2}(X)$ itself. According to Lemma 3.4.16, to this choice of f there is a uniquely associated $\phi \in \mathcal{H}^{s-\underline{\nu}}(\partial X)$ such that $P(u,\underline{\phi}) = Pu$ and the norm $\|(u,\underline{\phi})\|_{\widetilde{\mathcal{H}}^s(X)}$ is equivalent to $\|u\|_{\mathcal{H}^s(X)} + \|Pu\|_{\mathcal{H}^{2-s}(X)}$. Adding $\|Pu\|_{\mathcal{H}^0(X)}$ to both of these norms shows that the spaces

$$\{u \in \mathcal{H}^s(X) : Pu \in \mathcal{H}^0(X)\} \text{ and } \{(u,\underline{\phi}) \in \widetilde{\mathcal{H}}^s(X) : P(u,\underline{\phi}) \in \mathcal{H}^0(X)\}$$

coincide, with an equivalence between the natural graph norms. This will be exploited in Section 3.5.

Parameter-elliptic boundary value problems

This section concerns elliptic estimates for parameter-dependent Bessel operators. The exposition is deliberately brief, since most of the definitions and facts in this section are straightforward adaptations from the non-parameter-dependent setting. In particular the main theorem of this section, Theorem 2, is stated without proof. The interested reader is referred to [84, Chap. 9] for an indication of how the proofs should be modified in the parameter-dependent setting.

Fix a compact manifold with boundary $\overline{X}$ with the usual data of a boundary defining function and collar diffeomorphism. Let $P(\lambda) \in \mathrm{Bess}^{(\lambda)}_\nu(X)$ be a parameter-dependent Bessel operator; if $0 < \nu < 1$, then $P(\lambda)$ is augmented by boundary conditions as in Section 3.4. The boundary conditions themselves may depend on the spectral parameter λ, namely one considers $(T(\lambda), C(\lambda))$ where

$$T_k(\lambda) = T^{\pm}(\lambda)\gamma_{\pm}, \quad T_-(\lambda) \in \mathrm{Diff}^1_{(\lambda)}(\partial X),\, T^+(\lambda) \in \mathrm{Diff}^0(\partial X).$$

and $C_{k,j} \in \mathrm{Diff}^*_{(\lambda)}(\partial X)$. It is necessary to formulate a parameter-dependent Lopatinskiĭ condition for $(T(\lambda), C(\lambda))$. Suppose that $\mu \in \{1-\nu, 2-\nu, 1+\nu\}$ and $\mathrm{ord}^{(\lambda)}_\nu(T(\lambda)) \leq \mu$. Here the order of T with respect to ν is defined in the parameter-dependent sense, namely factors of λ are given the same weight as a derivative tangent to ∂X. Define the family of operators

$$\widehat{T}_{(y,\eta;\lambda)} = \sigma^{(\lambda)}_{\lceil \mu-1+\nu \rceil}(T^-(\lambda))\gamma_- + \sigma^{(\lambda)}_{\lceil \mu-1-\nu \rceil}(T^+(\lambda))\gamma_+,$$

indexed by $(y, \eta, \lambda) \in T^*\partial X \times \mathbb{C}$. Thus each $(y, \eta, \lambda) \in T^*\partial X \times \mathbb{C}$ gives rise to a one-dimensional boundary operator $\widehat{T}_{(y,\eta;\lambda)}$.

Next, choose $c_{k,j} \in \mathbb{Z}$ such that

$$\mathrm{ord}^{(\lambda)}(C_{k,j}(\lambda)) \leq c_{k,j} \leq \tau_j + \mu_k,$$

and then define the matrix $\widehat{C}_{(y,\eta)}$ with entries

$$(\widehat{C}_{(y,\eta;\lambda)})_{k,j} = \sigma^{(\lambda)}_{c_{k,j}}(C_{k,j}(\lambda))(y, \eta; \lambda).$$

Again the order of $C_{k,j}(\lambda)$ is taken in the parameter-dependent sense.

Definition 3.4.17. *Suppose that $P(\lambda)$ is parameter-elliptic on ∂X with respect to an angular sector Λ. The boundary operators $(T(\lambda), C(\lambda))$ are said to satisfy the parameter-dependent Lopatinskiĭ condition with respect to P and Λ if for each $p \in \partial X$ and $(\eta, \lambda) \in T^*_p\partial X \times \Lambda \setminus 0$, the only element $(u, \underline{u}) \in \mathcal{M}_+(p, \eta, \lambda) \times \mathbb{C}^J$ satisfying*

$$\widehat{T}_{(p,\eta,\lambda)}u + \widehat{C}_{(p,\eta,\lambda)}\underline{u} = 0$$

is the trivial solution $(u, \underline{u}) = 0$. The operator $\mathscr{P}(\lambda) = \{P(\lambda), T(\lambda), C(\lambda)\}$, is said to be parameter elliptic if $P(\lambda)$ is parameter-elliptic and $(T(\lambda), C(\lambda))$ satisfy the parameter-dependent Lopatinskiĭ condition on ∂X with respect to $P(\lambda)$ and Λ.

Example. If $P(\lambda)$ is parameter-elliptic in an angular sector Λ and T is a λ-independent boundary condition satisfying the Lopatinskiĭ condition, then $\{P(\lambda), T\}$ is parameter-elliptic with respect to Λ.

Example. If $1/2 < \nu < 1$, then any boundary condition of the form $T = \gamma_+ + T^-(\lambda)\gamma_-$, where $T^-(\lambda) \in \mathrm{Diff}^1_{(\lambda)}(\partial X)$, satisfies the parameter-dependent Lopatinskiĭ condition with respect to any angular sector.

In the notation of Theorem 1, the main theorem of this section is the following. As remarked in the introduction to this section, it is provided without proof.

Theorem 2. *Let $\overline{X}$ be a compact manifold with boundary as in Section 3.1. Assume that $P(\lambda) \in \mathrm{Bess}^{(\lambda)}_\nu(X)$ is parameter-elliptic at ∂X with respect to an angular sector Λ in the sense of Section 3.1. If $0 < \nu < 1$, then assume $P(\lambda)$ is augmented by parameter-dependent boundary conditions $(T(\lambda), C(\lambda))$ such that $\mathscr{P}(\lambda) = \{P(\lambda), T(\lambda), C(\lambda)\}$ is elliptic with respect to Λ. There exists $0 < \delta < \varepsilon$ such that if $\varphi, \chi \in C^\infty_c(\{0 \le x < \delta\})$ satisfy $\varphi = 1$ near ∂X and $\chi = 1$ near $\mathrm{supp}\,\varphi$, then the following hold.*

1. *Let $0 < \nu < 1$ and $s = 0, 1, 2$. Then*

$$\begin{aligned}\interleave \varphi(u, \underline{\phi}, \underline{u}) \interleave_{\widetilde{\mathcal{H}}^s(X) \times H^{s+\underline{\tau}}(\partial X)} \le C(&\interleave \varphi \mathscr{P}(\lambda)(u, \underline{\phi}, \underline{u}) \interleave_{\mathcal{H}^{s-2}(X) \times H^{s-\underline{\mu}}(\partial X)} \\ &+ \interleave \chi(u, \underline{\phi}, \underline{u}) \interleave_{\widetilde{\mathcal{H}}^{s-1}(X) \times H^{s-1+\underline{\tau}}(\partial X)}),\end{aligned} \tag{3.53}$$

   for each $(u, \underline{\phi}, \underline{u}) \in \widetilde{\mathcal{H}}^s(X) \times \mathcal{H}^{s+\underline{\tau}}(\partial X)$ and $\lambda \in \Lambda$.

2. *Let $\nu \ge 1$ and $s = 0, 1, 2$. Then*

$$\interleave \varphi u \interleave_{\mathcal{H}^s(X)} \le C(\interleave \varphi P(\lambda) u \interleave_{\mathcal{H}^{s-2}(X)} + \interleave \chi u \interleave_{\mathcal{H}^{s-1}(X)}), \tag{3.54}$$

   for each $u \in \mathcal{H}^s(X)$ and $\lambda \in \Lambda$.

Conormal regularity

So far only regularity at the $\mathcal{H}^2$ level has been discussed. Higher order regularity is defined in terms of a scale of conormal Sobolev spaces relative to $\mathcal{H}^s$. Let $\overline{X}$ be a compact manifold with boundary with a fixed boundary defining function x and collar neighborhood $\overline{W}$. Then let $\overline{X}_{\mathrm{even}}$ denote the manifold $\overline{X}$ equipped with a new smooth structure: on the collar $\overline{W} \simeq [0, \varepsilon)_x \times \partial X$, functions are smooth if in the normal direction they depend on x^2 (rather than just x).

Define the Lie algebra $\mathcal{V}_b(\overline{X}_{\mathrm{even}})$ of smooth vector fields on $\overline{X}_{\mathrm{even}}$ which are tangent to ∂X. In local coordinates $x, y_1, \ldots, y_{n-1}$ on the collar, elements of $\mathcal{V}_b(\overline{X}_{\mathrm{even}})$ are $C^\infty(\overline{X}_{\mathrm{even}})$ linear combinations of $x\partial_x$ and ∂_{y_i}.

Lemma 3.4.18. *If $P \in \mathrm{Bess}_\nu(X)$ and $V \in \mathcal{V}_b(\overline{X}_{\mathrm{even}})$ satisfies $Vx = x + \mathcal{O}(x^3)$, then $[P, V] \in \mathrm{Bess}_\nu(X)$.*

Proof. The hypothesis implies that in local coordinates,

$$V(x,y) = a(x^2,y)x\partial_x + b^i(x^2,y)\partial_{y_i}$$

where $a(0,y) = 1$. Note that

$$[|\partial_\nu|^2, x\partial_x] = -2|\partial_\nu|^2, \quad [\partial_\nu, x\partial_x] = -\partial_\nu.$$

Also from (3.20), if $a \in C^\infty(\overline{X}_{\mathrm{even}})$, then

$$[|\partial_\nu|^2, a] = \hat{a}x\partial_\nu + \tilde{a}$$

for $\hat{a}, \tilde{a} \in C^\infty(\overline{X}_{\mathrm{even}})$, as well as $[x\partial_\nu, a] \in C^\infty(\overline{X}_{\mathrm{even}})$. The result follows immediately from these observations. □

Given $k \in \mathbb{N}$ and $s = 0,1,2$, the space $\mathcal{H}^{s,k}(X)$ is defined as

$$\mathcal{H}^{s,k}(X) = \{u \in \mathcal{H}^s(X) : V_1 \cdots V_k u \in \mathcal{H}^s(X) \text{ for any } V_1,\ldots,V_k \in \mathcal{V}_b(\overline{X}_{\mathrm{even}})\}.$$

Fixing a finite generating set $\mathscr{V}$ for $\mathcal{V}_b(\overline{X}_{\mathrm{even}})$, this space can be given the topology of a Hilbert space by inductively defining the norms

$$\|u\|^2_{\mathcal{H}^{s,k}(X)} = \sum_{V \in \mathscr{V}} \|Vu\|^2_{\mathcal{H}^{s,k-1}(X)}.$$

A different choice of generating set yields an equivalent norm. Note that over any compact $K \subseteq X$, there is an equivalence between functions in $\mathcal{H}^{s,k}(X)$ and $H^{s+k}(X)$ which are supported on K. In addition, all of the density results which hold for $\mathcal{H}^s(X)$ also hold for $\mathcal{H}^{s,k}(X)$.

Remark 13. *If $s = 0,1$, then in fact*

$$\mathcal{H}^{s,k}(X) = \{u \in \mathcal{H}^s(X) : V_1 \cdots V_k u \in \mathcal{H}^s(X) \text{ for any } V_1,\ldots,V_k \in \mathcal{V}_b(\overline{X})\}.$$

Thus only $\mathcal{H}^{2,k}(X)$ necessitates the introduction of a new smooth structure on $\overline{X}$.

Lemma 3.4.19. *Let $P \in \mathrm{Bess}_\nu(X)$ and $k \in \mathbb{N}$.*

1. *If $\nu > 0$, then $P : \mathcal{H}^{2,k}(X) \to \mathcal{H}^{0,k}(X)$ is bounded.*

2. *If $0 < \nu < 1$ and T is a boundary operator such that $\mathrm{ord}_\nu(T) \le \mu$, then $T : \mathcal{H}^{2,k}(X) \to H^{k+2-\mu}(\partial X)$ is bounded.*

Proof. (1) Any $V \in \mathcal{V}_b(\overline{X}_{\mathrm{even}})$ can be written as $V = aV_1$, where $a \in C^\infty(\overline{X}_{\mathrm{even}})$ and $V_1 x = x + \mathcal{O}(x^3)$. The result can now be deduced from Lemma 3.4.18.

(2) Given a vector field Z on ∂X, there exists $V \in \mathcal{V}_b(\overline{X}_{\mathrm{even}})$ such that $V|_{\partial X} = Z$. Then $Z(\gamma_\pm u) = \gamma_\pm(Vu)$ for each $u \in \mathcal{H}^{2,k}(X)$; this is certainly true on $\mathcal{F}_\nu(X)$ and extends by density. □

Fix a generating set $\mathscr{V} = \{V_0, V_1, \ldots, V_N\}$ for $\mathcal{V}_b(\overline{W}_{\mathrm{even}})$ as follows: set $V_0 = x\partial_x$, and then choose a collection of vector fields $V_1, \ldots, V_N$ on ∂X which span $T\partial X$. Then $V_0, \ldots, V_N$ may be considered as vector fields on $[0,\varepsilon)_x \times \partial X$, hence on $\overline{W}$. Note that the flow of V_0 is given by $\exp(hV_0)(x,y) = (e^h x, y)$, where $(x,y) \in [0,\varepsilon)_x \times \partial X$. Given $V \in \mathscr{V}$, let

$$\varrho_V^h u = (u \circ \exp(hV) - u)/h$$

denote the associated difference quotient.

Suppose that $u \in \mathcal{H}^{2,k}(X)$ is supported in $\{0 \leq x < \delta\}$, where $0 < \delta < \varepsilon$. Observe that there exists $h_0 > 0$ depending on δ such that $\varrho_{V_0}^h u$ is well defined for $0 < h_0 < h$; the difference quotients corresponding to $V_1, \ldots, V_N$ are defined for all h. The first step is to calculate the commutator of P with $\varrho_{V_0}^h$; this is illustrated for $[|D_\nu|^2, \varrho_{V_0}^h]$. First note that

$$[\partial_x, \varrho_{V_0}^h]u = h^{-1}(e^h - 1)(\partial_x u) \circ \exp(hV_0).$$

A short calculation gives

$$[|D_\nu|^2, \varrho_{V_0}^h]u = h^{-1}(1 - e^{2h})(|D_\nu|^2 u) \circ \exp(hV_0),$$

which shows that

$$\|[|D_\nu|^2, \varrho_{V_0}^h]u\|_{\mathcal{H}^{0,k}(X)} \leq C\|u\|_{\mathcal{H}^{2,k}(X)}$$

for $0 < h_0 < h$, where $C > 0$ does not depend on u or h. Continuing this calculation shows that $\|[P, \varrho_V^h]u\|_{\mathcal{H}^{0,k}(X)} \leq C\|u\|_{\mathcal{H}^{2,k}(X)}$ for any $V \in \mathscr{V}$. As for the boundary operators, one has

$$\gamma_-(u \circ \exp(hV_0)) = \gamma_- u, \quad \gamma_+(u \circ \exp(hV_0)) = e^{(1/2+\nu)s}\gamma_+ u,$$

so $\gamma_- \circ \varrho_{V_0}^h = 0$ and $\gamma_+ \circ \varrho_{V_0}^h = (e^{(1/2+\nu)s} - 1)\gamma_+ u$. Similarly,

$$\|[T, \varrho_{V_i}^h]u\|_{H^{k+2-\mu}(\partial X)} \leq C\|u\|_{\mathcal{H}^{2,k}(X)}$$

for $i = 1, \ldots, N$, uniformly in h.

Theorem 3. *Let $\overline{X}$ be a compact manifold with boundary as in Section 3.1. Assume that $P \in \mathrm{Bess}_\nu(X)$ is elliptic at ∂X in the sense of Section 3.1. If $0 < \nu < 1$, then assume P is augmented by a boundary condition T such that $\mathscr{P} = \{P, T\}$ is elliptic at ∂X. There exists $0 < \delta < \varepsilon$ such that if $\varphi, \chi \in C_c^\infty(\{0 \leq x < \delta\})$ satisfy $\varphi = 1$ near ∂X and $\chi = 1$ near $\operatorname{supp}\varphi$, then the following hold.*

1. *Let $0 < \nu < 1$. If $\chi u \in \mathcal{H}^2(X)$ and $\chi P u \in \mathcal{H}^{0,k}(X)$, $Tu \in H^{k+2-\mu}(\partial X)$ for some $k \in \mathbb{N}$, then $\varphi u \in \mathcal{H}^{2,k}(X)$. Furthermore,*

   $$\|\varphi u\|_{\mathcal{H}^{2,k}(X)} \leq C\left(\|\chi \mathscr{P} u\|_{\mathcal{H}^{0,k}(X)\times H^{k+2-\mu}(\partial X)} + \|\chi u\|_{\mathcal{H}^0(X)}\right),$$

   where $C > 0$ does not depend on u.

2. *Let $\nu \geq 1$. If $\chi u \in \mathcal{H}^2(X)$ and $\chi P u \in \mathcal{H}^{0,k}(X)$ for some $k \in \mathbb{N}$, then $\varphi u \in \mathcal{H}^{2,k}(X)$. Furthermore,*
$$\|\varphi u\|_{\mathcal{H}^{2,k}(X)} \leq C \left(\|\chi P u\|_{\mathcal{H}^{0,k}(X)} + \|\chi u\|_{\mathcal{H}^0(X)} \right),$$
where $C > 0$ does not depend on u.

Proof. The proof is by induction; the case $k = 0$ is Theorem 1. Suppose that the result holds for $k \in \mathbb{N}$; combined with the calculations preceeding the theorem, this gives that $\varrho_V^h \varphi u \in \mathcal{H}^{2,k}(X), V \in \mathscr{V}$ is well defined and uniformly bounded for h sufficiently small. Standard functional analysis (extracting a weakly convergent subsequence, etc.) proves that $V\varphi u \in \mathcal{H}^{2,k}(X)$ for every $V \in \mathscr{V}$, with a corresponding estimate. This allows one to conclude the result for $k+1$. □

Asymptotic expansions

Using Mellin transform techniques, it is straightforward to give asymptotic expansions for solutions of certain Bessel equations. This section is a special case of far more general expansions; see [77, Section 7] for example. The approach taken here is essentially the same as [97, Lemma 4.13]. The space $\dot{C}^\infty(X)$ refers to smooth functions on X which vanish to infinite order at ∂X.

Lemma 3.4.20 ([97, Lemma 4.13]). *Suppose that $P \in \mathrm{Bess}_\nu(X)$ for $\nu > 0$, and $g_\pm \in C^\infty(\partial X)$. Then there exist $v_\pm$ such that $P(x^{1/2+\nu} v_+ + x^{1/2-\nu} v_-) \in \dot{C}^\infty(X)$ and $v_\pm|_{\partial X} = g_\pm$ with the following properties.*

1. *If $2\nu \notin \{3,5,7,\ldots\}$, then $v_\pm \in C^\infty(\overline{X})$. In addition $v_\pm - g_\pm \in x^2 C^\infty(\overline{X})$.*

2. *If $2\nu \in \{3,5,7,\ldots\}$, then $v_+ \in C^\infty(\overline{X})$ and*
$$v_- \in C^\infty(\overline{X}) + x^{2\nu}(\log x) C^\infty(\overline{X}),$$
where $a_j \in C^\infty(\partial X)$.

Suppose that $P \in \mathrm{Bess}_\nu(X)$ is elliptic at ∂X. Write P in the form
$$P = |D_\nu|^2 - E$$
near ∂X, where $E \in \mathrm{Diff}_b^2(\overline{X})$ and $\mathrm{Diff}_b^m(\overline{X})$ are the operators of order at most m generated by vector fields in $\mathcal{V}_b(\overline{X})$. If $0 < \nu < 1$, then also fix a boundary condition T such that $\{P, T\}$ is elliptic at ∂X. The equation $Pu = f$ can be expressed as
$$x^2 |D_\nu|^2 u = x^2 (Eu + f) \tag{3.55}$$
Formally, the Mellin transform of the left hand side of (3.55) is
$$(s + 1/2 - \nu)(s + 1/2 + \nu) M u(s, \cdot), \quad s \in \mathbb{C}.$$

Now suppose that $u \in \mathcal{H}^0(X)$ and $f \in \dot{C}^\infty(X)$. Also suppose that $Tu \in C^\infty(\partial X)$ when $0 < \nu < 1$. If u is supported sufficiently close to ∂X, then $u \in \mathcal{H}^{2,k}(X)$ by Theorem 3 for any $k \in \mathbb{Z}$. In that case, the left hand side of (3.55) is square integrable with respect to the measure $d(\operatorname{Im} s)$ along the line $\{\operatorname{Re} s = 1/2\}$. Furthermore, since $u \in \mathcal{H}^{2,k}(X) \subseteq \mathcal{H}^{0,k}(X)$ and $E \in \operatorname{Diff}^2_b(\overline{X})$, the right hand side of (3.55) is an element of $x^2\mathcal{H}^0(X)$. Define $\mathcal{H}^{s,\infty}(X) = \cap_{k\geq 0}\mathcal{H}^{s,k}(X)$.

Proposition 3.4.21. *Suppose that P and $\{P, T\}$ are elliptic at ∂X. If $u \in \mathcal{H}^0(X)$ and*

$$Pu \in \dot{C}^\infty(X), \quad Tu \in C^\infty(\partial X),$$

then the following hold.

1. *Let $0 < \nu < 1$. Then there exist $u_\pm \in C^\infty(\overline{X})$ such that*

$$u = x^{1/2+\nu}u_+ + x^{1/2-\nu}u_-.$$

In addition $u_\pm - g_\pm \in x^2 C^\infty(\overline{X})$, where $g_- = \gamma_- u$ and $2\nu g_+ = \gamma_+ u$.

2. *Let $\nu \geq 1$. If $2\nu \notin \{3, 5, 7, \ldots\}$, then there exists $u_\pm \in C^\infty(\overline{X})$ such that*

$$u = x^{1/2+\nu}u_+ + x^{1/2-\nu}u_-,$$

where $u_- \in x^k C^\infty(\overline{X})$ for some $k \in \mathbb{N}$ satisfying $k > \nu - 1$. If $2\nu \in \{3, 5, 7, \ldots\}$, then the same statement holds but with $u_- \in x^k C^\infty(\overline{X}) + x^{2\nu}(\log x)C^\infty(\overline{X})$.

Proof. For a more details, again see [97, Lemma 4.13]. By cutting off u (which does not affect the condition $Pu \in \dot{C}^\infty(X)$), it may be assumed that u is supported near ∂X, hence a function on $(0, \varepsilon) \times \partial X$. Let

$$h_1 = Eu + f \in \mathcal{H}^0((0, \varepsilon) \times \partial X).$$

Write $l_\nu(s) = (s + 1/2 + \nu)(s + 1/2 - \nu)$ and then take the Mellin transform to yield

$$Mu(s, \cdot) = l_\nu(s)^{-1} Mh(s + 2, \cdot).$$

First suppose that 2ν is not an integer, so the roots of $l_\nu(s)$ are simple. Since $Mh(s + 2, \cdot)$ is holomorphic for $\operatorname{Re} s > 1/2$, this provides a meromorphic extension of $Mu(s, \cdot)$ from $\{\operatorname{Re} s > 1/2\}$ to $\{\operatorname{Re} s > -3/2\}$ with simple poles at the roots of $l_\nu(s)$ in the strip $\{-3/2 < \operatorname{Re} s < 1/2\}$. Since $u \in \mathcal{H}^{0,\infty}(X)$ the residues are smooth functions on ∂X.

Now take the inverse Mellin transform by deforming the contour to any line $\{\operatorname{Re} s = -3/2 + \varepsilon\}$. Note that $Mu(s, \cdot)$ has two poles in $\{-3/2 < \operatorname{Re} s < 1/2\}$ if $0 < \nu < 1$, and no poles in this region if $\nu \geq 1$. In the former case,

$$u = x^{1/2+\nu}g_+ + x^{1/2-\nu}g_- + u_1$$

for $g_\pm \in C^\infty(\partial X)$ and $u_1 \in x^2\mathcal{H}^{0,\infty}(X)\cap\mathcal{H}^{2,\infty}(X)$, while in the latter case $u \in x^2\mathcal{H}^{0,\infty}(X)\cap\mathcal{H}^{2,\infty}(X)$. In the first case, choose $v_\pm$ as in Lemma 3.4.20, where $v_\pm = g_\pm + x^2 C^\infty(\overline{X})$. Thus

$$u - x^{1/2-\nu}v_+ - x^{1/2+\nu}v_- \in x^2\mathcal{H}^{0,\infty}(X)\cap\mathcal{H}^{2,\infty}(X)$$

and $P(u - x^{1/2+\nu}v_+ - x^{1/2-\nu}v_-) \in \dot{C}^\infty(X)$. Applying the same argument gives the next terms in the expansion, which come from the poles at $s = -5/2 \pm \nu$. This may be continued indefinitely, but note that after this second step there may appear powers of the form $x^{r+1/2\pm\nu}$ (for $r > 2$) rather than just $x^{2r+1/2\pm\nu}$ (unless additional evenness assumptions are made on E). A similar argument applies in the second case, where $Mu(s,\cdot)$ is first continued further to left until an indicial root is crossed. When 2ν is an integer, one picks up a logarithmic factor when taking the inverse Mellin transform, corresponding to a pole of multiplicity two. □

3.5 The Fredholm alternative and unique solvability

Global assumptions

Let $\overline{X}$ denote a compact manifold with boundary as in Section 3.1. Consider a pseudodifferential operator $P \in \Psi^2(X)$ of the form $P = P_1 + P_2$, where

$$P_1 \in \mathrm{Bess}_\nu(X), \quad P_2 \in \Psi^2(X) \text{ is compactly supported.}$$

Assume that P_1 is elliptic at ∂X in the sense of Section 3.1. Furthermore, if $0 < \nu < 1$, fix a scalar boundary condition T with $\mathrm{ord}_\nu(T) \le \mu$; this is just for simplicity, whereas matrix boundary conditions arise in the adjoint problem. Assume that $\mathscr{P} = \{P, T\}$ is elliptic at ∂X as well. Since P_2 has a compactly supported Schwartz kernel, this is just a statement about the operator $\{P_1, T\}$.

Without any assumptions on the behavior of P away from ∂X, there is no reason to expect that P or $\mathscr{P}$ are Fredholm. This section outlines some additional global assumptions which guarantee a Fredholm problem. The simplest of these assumptions is that P is everywhere elliptic (in the standard sense) on X, but in view of applications to general relativity, this is overly restrictive. Indeed, operators which arise in the study of quasinormal modes on black holes spacetimes have the property that their ellipticity degenerates at the event horizon. Moreover, rotating Kerr–AdS black holes contain an ergoregion, so that the corresponding operator is not everywhere elliptic even in the black hole exterior.

The global assumptions on P presented next are motivated by recent work of Vasy [95], which applies to the setting of rotating black holes. More generally, these assumptions are typical for situations where coercive estimates are proved via propagation results.

Given $\nu > 0$, define the space

$$\mathcal{Y} = \begin{cases} \{u \in \mathcal{H}^1(X) : Pu \in \mathcal{H}^0(X),\, Tu \in H^{2-\mu}(\partial X)\} & \text{if } 0 < \nu < 1, \\ \{u \in \mathcal{H}^1(X) : Pu \in \mathcal{H}^0(X)\} & \text{if } \nu \ge 1 \end{cases}$$

where Pu is taken as a distribution on X. That Tu is well defined follows from Lemma 3.4.16. Equip $\mathcal{Y}$ with the norm

$$\|u\|_{\mathcal{Y}} = \begin{cases} \|u\|_{\mathcal{H}^1(X)} + \|Pu\|_{\mathcal{H}^0(X)} + \|Tu\|_{H^{2-\mu}(\partial X)} & \text{if } 0 < \nu < 1 \\ \|u\|_{\mathcal{H}^1(X)} + \|Pu\|_{\mathcal{H}^0(X)} & \text{if } \nu \geq 1. \end{cases}$$

According to the discussion following Lemma 3.4.16, the space $\mathcal{Y}$ is equivalent to

$$\{(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^1(X) : \mathscr{P}(u, \underline{\phi}) \in \mathcal{H}^0(X) \times H^{2-\mu}(\partial X)\} \tag{3.56}$$

for $0 < \nu < 1$ when the latter space is equipped with the norm $\|(u, \underline{\phi})\|_{\widetilde{\mathcal{H}}^1(X)} + \|\mathscr{P}(u, \underline{\phi})\|_{\mathcal{H}^0(X)\times H^{2-\mu}(\partial X)}$.

Lemma 3.5.1. *The space $\mathcal{Y}$ has the following properties.*

1. *$\mathcal{Y}$ is complete.*
2. *$\mathcal{F}_\nu(X)$ is dense in $\mathcal{Y}$*
3. *If $0 < \nu < 1$, then $\mathscr{P} : \mathcal{Y} \to \mathcal{H}^0(X) \times H^{2-\mu}(\partial X)$ is bounded.*
4. *If $\nu \geq 1$, then $P : \mathcal{Y} \to \mathcal{H}^0(X)$ is bounded.*
5. *If $\zeta \in C_c^\infty(X)$ and $K = \operatorname{supp} \zeta$, then for each $m < 1$ the map $\mathcal{Y} \to H_K^m(X)$ given by $u \mapsto \zeta u$ is compact.*

Proof. (1) For $0 < \nu < 1$, use the alternative description (3.56) of $\mathcal{Y}$: suppose that $(u_n, \underline{\phi_n}) \in \mathcal{Y}$ is a Cauchy sequence. This implies that there exists

$$(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^1(X), \quad (w, \underline{w}) \in \mathcal{H}^0(X) \times H^{2-\mu}(\partial X)$$

such that

$$(u_n, \underline{\phi_n}) \to (u, \underline{\phi}) \text{ in } \widetilde{\mathcal{H}}^1(X), \quad \mathscr{P}(u_n, \underline{u_n}) \to (w, \underline{w}) \text{ in } \mathcal{H}^0(X) \times H^{2-\mu}(\partial X).$$

Certainly $(u_n, \underline{\phi_n}) \to (u, \underline{\phi})$ in $\widetilde{\mathcal{H}}^1(X)$, and then by continuity $\mathscr{P}(u_n, \underline{\phi_n}) \to \mathscr{P}(u, \underline{\phi})$ in $\mathcal{H}^{-1}(X) \times H^{1-\mu}(\partial X)$. This implies that $\mathscr{P}(u, \underline{\phi}) = (w, \underline{w})$ since the natural map

$$\mathcal{H}^0(X) \times H^{2-\mu}(\partial X) \hookrightarrow \mathcal{H}^{-1}(X) \times H^{1-\mu}(\partial X),$$

is injective. Thus $\mathcal{Y}$ is complete. A simpler proof works when $\nu \geq 1$.

(2) Again assume that $0 < \nu < 1$. Fix a cutoff χ such that $\chi = 1$ in a neighborhood of ∂X. If χ is supported sufficiently close to ∂X, then $\chi u \in \mathcal{H}^2(X)$ by Theorem 1. Thus there is certainly a sequence $u_n \in \mathcal{F}_\nu(X)$ such that $u_n \to \chi u$ in $\mathcal{H}^1(X)$ and $Pu_n \to P\chi u$ in $\mathcal{H}^0(X)$, along with $Tu_n \to Tu$. If $\varphi = 1$ near ∂X and $\chi = 1$ near $\operatorname{supp} \varphi$, then also $\varphi u_n \to \varphi u$ and $\varphi P u_n \to \varphi P u$. This also implies that $P(\varphi u_n) \to P(\varphi u)$ since

$$[P, \varphi] u_n \to [P, \varphi] u = [P, \varphi] u$$

in $\mathcal{H}^0(X)$ by continuity. On the other hand, the same reasoning above combined with the mollification argument in [95, Section 2.6] shows the existence of a sequence $v_n \in C^\infty(X)$ such that $(1-\varphi)v_n \to (1-\varphi)u$ in $\mathcal{H}^1(X)$ and $P((1-\varphi)v_n) \to P((1-\varphi)u)$ in $\mathcal{H}^0(X)$. It then suffices to take the sequence $\varphi u_n + (1-\varphi)v_n \in \mathcal{F}_\nu(X)$ which converges to u in $\mathcal{Y}$.

(3) The boundedness of $u \mapsto Pu$ as a map $\mathcal{Y} \to \mathcal{H}^0(X)$ holds by construction of $\mathcal{Y}$. As in the previous part, $\chi u \in \mathcal{H}^2(X)$ so $T : \mathcal{Y} \to H^{2-\mu}(\partial X)$ is also bounded. This establishes the boundedness of $\mathscr{P} = \{P, T\}$.

(4) See (3) above.

(5) The map $u \to \zeta u$ is bounded $\mathcal{H}^1(X) \to H^1_K(X)$, which embeds compactly in $H^m_K(X)$. □

Typically, one constructs a partition of unity of the form

$$1 = \varphi + \sum_{i=1}^N A_i + R,$$

where $A_i \in \Psi^0(X)$, $R \in \Psi^{-\infty}(X)$ are compactly supported pseudodifferential operators, and $\varphi \in C^\infty(\overline{X})$ satisfies $\varphi = 1$ near ∂X. Under various hypotheses on P (now considered as an element of $\Psi^2(X)$), it is often the case that there exists compactly supported $B_i, X_i \in \Psi^0(X)$ such that

$$\|A_i u\|_{H^1(X)} \leq C\|B_i P u\|_{H^0(X)} + \|X_i u\|_{H^m(X)}, \quad m < 1 \tag{3.57}$$

for each $u \in C^\infty(X)$. Since the operators B_i, X_i and R are compactly supported, it is possible to combine (3.57) with the results of Theorem 1 to conclude the following type of a priori estimate: if $0 < \nu < 1$, then

$$\|u\|_{\mathcal{H}^1(X)} \leq C\left(\|\mathscr{P}u\|_{\mathcal{H}^0(X)\times H^{2-\mu}(X)} + \|u\|_{\mathcal{H}^0(X)} + \|\chi u\|_{H^m(X)}\right) \tag{AP0}$$

for each $u \in \mathcal{F}_\nu(X)$, while if $\nu \geq 1$ then

$$\|u\|_{\mathcal{H}^1(X)} \leq C\left(\|Pu\|_{\mathcal{H}^0(X)} + \|u\|_{\mathcal{H}^0(X)} + \|\chi u\|_{H^m(X)}\right) \tag{AP1}$$

for each $u \in \mathcal{F}_\nu(X)$. Since $\mathcal{F}_\nu(X)$ is dense in $\mathcal{Y}$, the estimate (AP0) implies

$$\|u\|_{\mathcal{Y}} \leq C\left(\|\mathscr{P}u\|_{\mathcal{H}^0(X)\times H^{2-\mu}(X)} + \|u\|_{\mathcal{H}^0(X)} + \|\chi u\|_{H^m(X)}\right)$$

for each $u \in \mathcal{Y}$, and similarly for (AP1). It is standard that (AP0), (AP1) imply $\mathscr{P} : \mathcal{Y} \to \mathcal{H}^0(X) \times H^{2-\mu}(\partial X)$ and $P : \mathcal{Y} \to \mathcal{H}^0(X)$ have finite dimensional kernels provided $m < 1$ — see Lemma 3.5.4.

Suppose that $0 < \nu < 1$. In order to prove that $\mathscr{P}$ has finite dimensional cokernel, it is necessary to introduce spaces associated with the formal adjoint $\mathscr{P}^*$ and Hilbert space adjoint $\mathscr{P}'$. Fix a density μ on $\overline{X}$ of product type near ∂X. A priori, $\mathscr{P}^*$ is bounded

$$\widetilde{\mathcal{H}}^0(X) \times H^{\mu-2}(\partial X) \to \mathcal{H}^{-2}(X) \times H^{\underline{\nu}-2}(\partial X).$$

Recall that if $(f, \underline{g}) = \mathscr{P}^*(v, \underline{\psi}, \underline{v})$, then

$$\langle u, f \rangle_X + \langle \underline{w}, \underline{g} \rangle_{\partial X} = \langle Pu, v \rangle_X + \langle \underline{w} - \underline{\gamma} u, J\underline{\psi} \rangle_{\partial X} + \langle G\underline{w}, \underline{v} \rangle_{\partial X}, \tag{3.58}$$

where the dualities on X and ∂X are induced by μ and $\mu_{\partial X}$. Now define the space

$$\widetilde{\mathscr{X}} = \{(v, \underline{\psi}, \underline{v}) \in \widetilde{\mathcal{H}}^0(X) \times H^{\mu-2}(\partial X) : \mathscr{P}^*(v, \underline{\psi}, \underline{v}) \in \mathcal{H}^{-1}(X) \times H^{\underline{\nu}-1}(\partial X)\}.$$

The corresponding space for $\nu \geq 1$ is defined to be

$$\mathscr{Z} = \{u \in \mathcal{H}^0(X) : Pu \in \mathcal{H}^{-1}(X)\}.$$

The spaces $\widetilde{\mathscr{X}}$ and $\mathscr{Z}$ have properties similar to those in Lemma 3.5.1. In particular, the set of all $(v, \underline{\gamma} v, \underline{v})$ such that $v \in \mathcal{F}_\nu(X)$ and $\underline{v} \in C^\infty(\partial X)$ is dense in $\widetilde{\mathscr{X}}$. Similarly, $\mathcal{F}_\nu(X)$ is dense in $\mathscr{Z}$ for $\nu \geq 1$.

The analogue of (AP0), (AP1) is formulated next for the adjoint problems. First suppose that $0 < \nu < 1$. The a priori estimate is

$$\begin{aligned} \|(v, \underline{\gamma} v, \underline{v})\|_{\mathcal{H}^0(X) \times H^{\mu-2}(\partial X)} \leq C(&\|\mathscr{P}^*(v, \underline{v})\|_{\mathcal{H}^{-1}(X) \times H^{\underline{\nu}-1}(\partial X)} \\ &+ \|(v, \underline{v})\|_{\widetilde{\mathcal{H}}^{-1}(X) \times H^{\mu-3}(\partial X)} + \|\chi v\|_{H^m(X)}) \end{aligned} \tag{AP0*}$$

for each $(v, \underline{v}) \in \mathcal{F}_\nu(X) \times C^\infty(\partial X)$. By density this implies the same estimate for $(v, \underline{\psi}, \underline{v}) \in \widetilde{\mathscr{X}}$. When $\nu \geq 1$ the estimate is

$$\|v\|_{\mathcal{H}^0(X)} \leq C(\|P^* v\|_{\mathcal{H}^{-1}(X)} + \|v\|_{\mathcal{H}^{-1}(X)} + \|\chi v\|_{H^m(X)} \tag{AP1*}$$

for each $v \in \mathcal{F}_\nu(X)$. For (AP0*), (AP1*) to be useful, one should require $m < 0$.

Remark 14. *As with the direct problem, it is frequently possible to combine the local estimates of Theorem 1 with interior estimates via a pseudodifferential partition of unity to show that the adjoint estimates (AP0*), (AP1*) hold.*

When $0 < \nu < 1$, the formally adjoint operator $\mathscr{P}^*$ should be compared with the Hilbert space adjoint

$$\mathscr{P}' : \mathcal{H}^0(X) \times H^{\mu-2}(\partial X) \to \widetilde{\mathcal{H}}^2(X)',$$

defined by

$$\langle (u, \underline{\phi}), \mathscr{P}'(v, \underline{v}) \rangle_X = \langle Pu, v \rangle_X + \langle Tu, \underline{v} \rangle_{\partial X}.$$

Recall that the inclusion of $\widetilde{\mathcal{H}}^2(X) \hookrightarrow \widetilde{\mathcal{H}}^1(X)$ is dense. Consequently $\widetilde{\mathcal{H}}^1(X)'$ may be identified with a dense subspace of $\widetilde{\mathcal{H}}^2(X)'$, where this identification is induced by the μ–inner product. In order to describe $\widetilde{\mathcal{H}}^1(X)'$, note that that there is an isomorphism

$$\Phi : \widetilde{\mathcal{H}}^1(X) \to \mathcal{H}^1(X) \times H^{-\nu}(\partial X)$$

given by $\Phi(u,\underline{\phi}) = (u,\phi_+)$; the inverse of Φ is $\Phi^{-1}(u,\phi_+) = (u,\gamma_- u,\phi_+)$. Thus for each $\alpha \in \widetilde{\mathcal{H}}^1(X)'$ there exist unique $f \in \mathcal{H}^{-1}(X)$, $g_+ \in H^{\nu}(\partial X)$ such that

$$\alpha(u,\underline{\phi}) = \langle f,u\rangle_X + \langle g_+,\phi_+\rangle_{\partial X}.$$

Furthermore, note that if $g_- \in H^{-\nu}(\partial X)$, then the functional given by $u \mapsto \langle g_-,\gamma_- u\rangle_{\partial X}$ is an element of $\mathcal{H}^1(X)'$. Thus it may be represented in the form $u \mapsto \langle f_-,u\rangle_X$ for a unique $f_- \in \mathcal{H}^{-1}(X)$. The next lemma summarizes this discussion.

Lemma 3.5.2. *Each $\alpha \in \widetilde{\mathcal{H}}^1(X)'$ admits a representation*

$$\alpha(u,\underline{\phi}) = \langle f,u\rangle_X + \langle \underline{g},\underline{\phi}\rangle_{\partial X}, \tag{3.59}$$

where $f \in \mathcal{H}^{-1}(X)$ and $\underline{g} \in H^{\underline{\nu}-1}(\partial X)$. Furthermore, $\|\alpha\|_{\widetilde{\mathcal{H}}^1(X)'}$ is equivalent to the norm

$$\inf\{\|f\|_{\mathcal{H}^{-1}(X)} + \|\underline{g}\|_{H^{\underline{\nu}-1}(\partial X)}\},$$

where the infimum is taken over all $f,\underline{g}$ such that (3.59) holds.

Now define $\widetilde{\mathscr{Z}} = \{(v,\underline{v}) \in \mathcal{H}^0(X)\times H^{\mu-2}(\partial X) : \mathscr{P}'(v,\underline{v}) \in \widetilde{\mathcal{H}}^1(X)'\}$.

Lemma 3.5.3. *Suppose that (AP0*) holds. Then*

$$\begin{aligned}\|(v,\underline{v})\|_{\mathcal{H}^0(X)\times H^{\mu-2}(\partial X)} \leq C(&\|\mathscr{P}'(v,\underline{v})\|_{\widetilde{\mathcal{H}}^1(X)'} \\ &+ \|(v,\underline{v})\|_{\mathcal{H}^{-1}(X)\times H^{\mu-3}(\partial X)} + \|\chi v\|_{H^m(X)})\end{aligned} \tag{AP0'}$$

for each $(v,\underline{v}) \in \widetilde{\mathscr{Z}}$.

Proof. Since $\mathscr{P}'(v,\underline{v}) \in \widetilde{\mathcal{H}}^1(X)'$, there exists $f \in \mathcal{H}^{-1}(X)$ and $\underline{g} \in H^{\underline{\nu}-1}(\partial X)$ such that the action of $\mathscr{P}'(v,\underline{v})$ on $(u,\underline{\phi}) \in \widetilde{\mathcal{H}}^1(X)$ is given by

$$(u,\underline{\phi}) \mapsto \langle f,u\rangle_X + \langle g,\underline{\phi}\rangle_{\partial X}. \tag{3.60}$$

Now let $\underline{\psi} = JG^*\underline{v} - J\underline{g}$, so that $J\underline{\psi} + G^*\underline{v} = \underline{g}$. Furthermore, note that $\underline{\psi} \in H^{-\underline{\nu}}(\partial X)$, so $(v,\underline{\psi})$ may be considered as an element of $\widetilde{\mathcal{H}}^0(X)$. Referring back to (3.58), it follows that $\mathscr{P}^*(u,\underline{\psi},\underline{v}) = (f,g)$. This shows that $(u,\underline{\psi},\underline{v}) \in \widetilde{\mathscr{X}}$, so

$$\begin{aligned}\|(v,\underline{v})\|_{\mathcal{H}^0(X)\times H^{\mu-2}(\partial X)} \leq C(&\|f\|_{\mathcal{H}^{-1}(X)} + \|g\|_{H^{\underline{\nu}-1}(\partial X)} \\ &+ \|(v,\underline{v})\|_{\mathcal{H}^{-1}(X)\times H^{\mu-3}(\partial X)} + \|\chi v\|_{H^m(X)})\end{aligned}$$

by (AP0*). In the last line, this used the fact that

$$\|\underline{\psi}\|_{H^{-1-\underline{\nu}}(\partial X)} \leq C(\|\underline{v}\|_{H^{\mu-3}} + \|g\|_{H^{\underline{\nu}-1}(\partial X)}).$$

It now suffices to take the infimum over all $f,\underline{g}$ satisfying (3.60), and then appeal to Lemma 3.5.2. ∎

The Fredholm property

In this section, the Fredholm property is established whenever (AP0), (AP1), (AP1), (AP1*) hold. A complete proof is given for the more complicated case $0 < \nu < 1$.

Lemma 3.5.4. *Let $0 < \nu < 1$.*

1. *If (AP0) holds with $m < 1$, then the operator*

$$\mathscr{P} : \mathcal{Y} \to \mathcal{H}^0(X) \times H^{2-\mu}(\partial X)$$

has a finite dimensional kernel.

2. *If (AP0*) holds with $m < 0$, then the operator*

$$\mathscr{P}' : \mathcal{H}^0(X) \times H^{\mu-2}(\partial X) \to \mathcal{H}^{-2}(X) \times H^{\underline{\nu}-2}(\partial X)$$

has a finite dimensional kernel

Proof. (1) This is immediate from the compactness of the inclusion $\mathcal{Y} \hookrightarrow \mathcal{H}^0(X)$ and the multiplication operator $\chi : \mathcal{Y} \to H^m_{\operatorname{supp}\chi}(X)$, combined with (AP0).

(2) Clearly the kernel of $\mathscr{P}'$ restricted to $\mathcal{H}^0(X) \times H^{\mu-2}(\partial X)$ is equal to the kernel of $\mathscr{P}'$ restricted to $\widetilde{\mathcal{Z}}$. The result follows from the same type of compactness considerations as in (1), using (AP0'). □

In light of Lemma 3.5.4, let $\mathcal{K}$ denote the finite dimensional kernel of $\mathscr{P}'|_{\widetilde{\mathcal{Z}}}$.

Lemma 3.5.5. *Let $0 < \nu < 1$, and assume that (AP0') holds. Suppose that*

$$(h, \underline{k}) \in \mathcal{H}^0(X) \times H^{2-\mu}(\partial X)$$

lies in the annihilator of $\mathcal{K}$ via the duality between $\mathcal{H}^0(X) \times H^{\mu-2}(\partial X)$ and $\mathcal{H}^0(X) \times H^{2-\mu}(\partial X)$. Then there exists $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^1(X)$ such that $\mathscr{P}(u, \underline{\phi}) = (h, \underline{k})$.

Proof. This fact is more or less standard, but a complete proof is included for the readers convenience.

(1) Fix a (closed) subspace V of $\mathcal{H}^0(X) \times H^{\mu-2}(\partial X)$ which is complementary to the finite-dimensional space $\mathcal{K} \subset \widetilde{\mathcal{Z}}$. Then there exists $C' > 0$ such that

$$\|(v, \underline{v})\|_{\mathcal{H}^0(X)\times H^{\mu-2}(\partial X)} \leq C' \|\mathscr{P}'(v, \underline{v})\|_{\widetilde{\mathcal{H}}^1(X)'} \tag{3.61}$$

for each $(v, \underline{v}) \in V \cap \widetilde{\mathcal{Z}}$. If this were not true, there would exist a sequence $(v_n, \underline{v_n}) \in V \cap \widetilde{\mathcal{Z}}$ such that

$$\|(v_n, \underline{v_n})\|_{\mathcal{H}^0(X)\times H^{\mu-2}(\partial X)} = 1, \quad \|\mathscr{P}'(v_n, \underline{v_n})\|_{\widetilde{\mathcal{H}}^1(X)'} \to 0.$$

By weak compactness of the $\mathcal{H}^0(X) \times H^{\mu-2}(\partial X)$-unit ball, it may be assumed that $(v_n, \underline{v_n})$ is weakly convergent. Since V is closed, it follows that $(v_n, \underline{v_n}) \to (v, \underline{v})$ weakly for some $(v, \underline{v}) \in V$. Thus

$$\mathscr{P}'(v_n, \underline{v_n}) \to \mathscr{P}'(v, \underline{v})$$

weakly in $\widetilde{\mathcal{H}}^2(X)'$. This means $\mathscr{P}'(v, \underline{v}) = 0$ since $\mathscr{P}'(v_n, \underline{v_n}) \to 0$ (in norm) in $\widetilde{\mathcal{H}}^1(X)'$. Thus $(v, \underline{v}) \in V \cap \mathcal{K}$, which implies $(v, \underline{v}) = 0$, since V complements the kernel.

Now by compactness there exists a subsequence $(v_{n_j}, \underline{v_{n_j}})$, such that χv_{n_j} is convergent in $H^m_{\operatorname{supp}\chi}(X)$ and $(v_{n_j}, \underline{v_{n_j}})$ is convergent in $\mathcal{H}^{-1}(X) \times \overline{H^{\mu-3}}(\partial X)$. Then (AP0') implies that $(v_{n_j}, \underline{v_{n_j}})$ is Cauchy, hence convergent in $\mathcal{H}^0(X) \times H^{\mu-2}(\partial X)$. This limit must be $(v, \underline{v})$, but that contradicts $(v, \underline{v}) = 0$ since $(v_n, \underline{v_n})$ has unit norm in $\mathcal{H}^0(X) \times H^{\mu-2}(\partial X)$. This completes the proof of 3.61.

(2) Now suppose that $(h, k) \in \mathcal{H}^0(X) \times H^{2-\mu}(\partial X)$ is in the annihilator of $\mathcal{K}$. Define the antilinear functional ℓ on the range of $\mathscr{P}'|_{\widetilde{\mathcal{Z}}}$ by the formula

$$\mathscr{P}'(v, \underline{v}) \mapsto \langle h, v\rangle_X + \langle k, \underline{v}\rangle_{\partial X},$$

where $(v, \underline{v}) \in \widetilde{\mathcal{Z}}$. This is well defined, since if $\mathscr{P}'(v, \underline{v}) = 0$ and $(v, \underline{v}) \in \widetilde{\mathcal{Z}}$, then $(v, \underline{v}) \in \mathcal{K}$, hence the right hand side vanishes.

For each $(v, \underline{v}) \in V \cap \widetilde{\mathcal{Z}}$, one has by (3.61)

$$\langle h, v\rangle_X + \langle k, \underline{v}\rangle_{\partial X} \leq C \left(\|(h, \underline{k})\|_{\mathcal{H}^0(X)\times H^{2-\mu}(\partial X)}\right) \left(\|\mathscr{P}^*(v, \underline{v})\|_{\widetilde{\mathcal{H}}^1(X)'}\right).$$

Since this is invariant under adding elements of $\mathcal{K}$, it is in fact true for $(v, \underline{v}) \in \widetilde{\mathcal{Z}}$. Thus ℓ is bounded on the range of $\mathscr{P}'|_{\widetilde{\mathcal{Z}}}$.

Now extend ℓ to an antilinear functional on $\widetilde{\mathcal{H}}^1(X)'$ by the Hahn–Banach theorem. Then there exists a unique $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^1(X)$ such that $\ell(\alpha) = \alpha(u, \underline{\phi})$, and furthermore

$$\ell(\mathscr{P}'(v, \underline{v})) = \langle h, v\rangle_X + \langle \underline{k}, \underline{v}\rangle_{\partial X}$$

whenever $(v, \underline{v}) \in \widetilde{\mathcal{Z}}$.

The claim is that $\mathscr{P}(u, \underline{\phi}) = (h, \underline{k})$. To see this, approximate $(u, \underline{\phi})$ in $\widetilde{\mathcal{H}}^1(X)$ by a sequence $(u_n, \underline{\phi_n}) \in \widetilde{\mathcal{H}}^2(X)$. Certainly $\mathscr{P}(u_n, \underline{\phi_n}) \to \mathscr{P}(u, \underline{\phi})$ in $\mathcal{H}^{-1}(X) \times H^{1-\mu}(\partial X)$. Furthermore, the pairing between $(u_n, \underline{\phi_n})$ and $\mathscr{P}'(v, \underline{v})$ is given by

$$\langle Pu_n, v\rangle_{\partial X} + \langle Tu_n, \underline{v}\rangle_{\partial X}.$$

for each $(v, \underline{v}) \in \mathcal{H}^1(X) \times H^{\mu-1}(\partial X)$. Thus for $(v, \underline{v}) \in \mathcal{F}_\nu \times C^\infty(\partial X) \subseteq \mathcal{H}^1(X) \times H^{\mu-1}(\partial X)$, this converges to $\langle Pu, v\rangle_X + \langle Tu, \underline{v}\rangle_{\partial X}$. But on the other hand it converges to $\langle h, v\rangle_X + \langle \underline{k}, \underline{v}\rangle_{\partial X}$ since $\mathcal{F}_\nu(X) \times C^\infty(\partial X) \subseteq \widetilde{\mathcal{Z}}$ as well. Thus $\mathscr{P}(u, \underline{\phi}) = (h, \underline{k})$, since $\mathcal{F}_\nu(X) \times C^\infty(\partial X)$ is dense in $\mathcal{H}^1(X) \times H^{\mu-1}(\partial X)$. □

Theorem 4. *Let* $\nu > 0$ *and* P *as in Section 3.5 be elliptic at* ∂X*. If* $0 < \nu < 1$*, then let* T *denote a scalar boundary operator satisfying* $\mathrm{ord}_\nu(T) \leq \mu$*, such that* $\mathscr{P} = \{P, T\}$ *is elliptic at* ∂X*.*

1. *Suppose that* $0 < \nu < 1$*. If* $\mathscr{P}$ *satisfies* (AP0) *with* $m < 1$ *and* (AP0*) *with* $m < 0$*, then*
$$\mathscr{P} : \mathcal{Y} \to \mathcal{H}^0(X) \times H^{2-\mu}(\partial X)$$
is Fredholm.

2. *Suppose that* $\nu \geq 1$*. If* P *satisfies* (AP1) *with* $m < 1$ *and* (AP1*) *with* $m < 0$*, then*
$$P : \mathcal{Y} \to \mathcal{H}^0(X)$$
is Fredholm.

Proof. (1) Lemma 3.5.4 shows the kernel is finite dimensional. On the other hand, Lemma 3.5.5 shows that the equation $\mathscr{P}(u, \underline{\phi}) = (h, \underline{k})$ has a solution $(u, \underline{\phi}) \in \widetilde{\mathcal{H}}^1(X)$ for $(h, \underline{k})$ in a space of finite codimension in $\mathcal{H}^0(X) \times H^{2-\mu}(X)$; clearly this $(u, \underline{\phi})$ can be identified with a unique element of $\mathcal{Y}$, namely u.

(2) When $\nu \geq 1$, there is a natural analogue of Lemma 3.5.5. Since the arguments are simpler when there is no boundary operator, the proofs are omitted. □

Unique solvability

In this section, again let $\overline{X}$ denote a compact manifold with boundary as in Section 3.1. This time, consider a pseudodifferential operator $P(\lambda) \in \Psi^{2,(\lambda)}(X)$ of the form $P(\lambda) = P_1(\lambda) + P_2(\lambda)$, where

$$P_1(\lambda) \in \mathrm{Bess}_\nu^{(\lambda)}(X), \quad P_2(\lambda) \in \Psi^{2,(\lambda)}(X) \text{ is compactly supported (uniformly on } \lambda).$$

Assume that $P_1(\lambda)$ is parameter-elliptic at ∂X with respect to an angular sector Λ in the sense of Section 3.1. If $0 < \nu < 1$, fix a scalar boundary condition $T(\lambda)$ with $\mathrm{ord}_\nu^{(\lambda)}(T(\lambda)) \leq \mu$, and assume that $\mathscr{P}(\lambda) = \{P(\lambda), T(\lambda)\}$ is parameter-elliptic at ∂X with respect to Λ. It is also assumed that the 'principal parts' of $P(\lambda), T(\lambda)$ do not depend on λ, so the spaces $\mathcal{Y}$ are independent of λ.

The parameter-dependent versions of (AP0), (AP1), (AP0*), (AP1*) are obtained by replacing the norms $\|\cdot\|$ with their uniform counterparts $|||\cdot|||$, and insisting that the estimates hold for all $\lambda \in \Lambda$.

Theorem 5. *Let* $\nu > 0$ *and* $P(\lambda), \mathscr{P}(\lambda), \Lambda$ *be as above. Suppose that the parameter-dependent versions of* (AP0), (AP1), (AP0*), (AP1*) *hold.*

1. *Let $0 < \nu < 1$. There exists $R > 0$ such that*

$$\mathscr{P}(\lambda) : \mathcal{Y} \to \mathcal{H}^0(X) \times H^{2-\mu}(\partial X)$$

is an isomorphism for $\lambda \in \Lambda$ satisfying $|\lambda| > R$.

2. *Let $\nu \geq 1$. Then there exists $R > 0$ such that*

$$P(\lambda) : \mathcal{Y} \to \mathcal{H}^0(X)$$

is an isomorphism for $\lambda \in \Lambda$ satisfying $|\lambda| > R$.

Proof. The parameter-dependent versions of (AP0), (AP0*) show that $\mathscr{P}(\lambda)$ and $\mathscr{P}(\lambda)'$ respectively are injective on the appropriate spaces (for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large). This implies that $\mathscr{P}(\lambda)$ is an isomorphism for $|\lambda|$ sufficiently large. Similar remarks hold for P when $\nu \geq 1$. □

3.6 Completeness of generalized eigenfunctions

In this section, sufficient conditions are given which guarantee an elliptic parameter-dependent Bessel operator has a complete set of generalized eigenvectors. Completeness of eigenvectors for non-self adjoint boundary value problems has a long history, going back to classic works of Keldysh [68], Browder [15], Schechter [86], Agmon [1], among many others. The results of this section apply to large classes Bessel operator pencils with a spectral parameter in the boundary condition, and two-fold completeness is established (which is stronger than just completeness).

One application of this section is to describe a class of boundary conditions for which linearized scalar perturbations of global anti-de Sitter space have complete sets of normal modes. Recent numerical and perturbative studies have hinted at a relationship between the linear spectra of such perturbations and possible nonlinear instability mechanisms [7, 9, 11, 10, 16, 24, 26]. These normal modes have been studied by separation of variable techniques, but there has not appeared a general criterion guaranteeing completeness of normal modes (nor even the discreteness of normal frquencies) for general, possibly time-periodic, boundary conditions. The results of this section also apply to more general stationary aAdS spacetimes with compact time slices, where ∂_t is Killing but the spacetime is not necessarily static.

Two-fold completeness

The main reference for this section is [76, Chapter II] Let $\overline{X}$ be a manifold with boundary, and let $P(\lambda) \in \mathrm{Bess}_\nu^{(\lambda)}(X)$ be a parameter-dependent Bessel operator such that $P(\lambda)$ is parameter-elliptic at ∂X in the sense of Section 3.1, and $P(\lambda)$ is parameter-elliptic on X in the usual sense. If $0 < \nu < 1$, let $T(\lambda)$ be a scalar parameter-dependent boundary operator such that $\mathscr{P}(\lambda) = \{P(\lambda), T(\lambda)\}$ is parameter-elliptic at ∂X.

Parameter-ellipticity of $P(\lambda)$ implies a decomposition

$$P(\lambda) = P_2 + \lambda P_1 + \lambda^2 P_0,$$

where $P_0 \in C^\infty(\overline{X})$ does not vanish. Dividing by P_0, it may be assumed that $P(\lambda)$ is of the form $P(\lambda) = P_2 + \lambda P_1 + \lambda^2$. The boundary operator $T(\lambda)$ is written as $T(\lambda) = T_1 + \lambda T_0$.

If $0 < \nu < 1$, a complex number $\lambda_0 \in \mathbb{C}$ is said to be an eigenvalue of $\mathscr{P}(\lambda)$ if there exists $u_0 \in \mathcal{H}^2(X)$ such $\mathscr{P}(\lambda_0)u_0 = 0$. Corresponding to an eigenvalue λ_0, a sequence $(u_0, \ldots, u_k)$ with $u_0 \neq 0$ is said to be a chain of generalized eigenvectors if

$$\begin{gathered} P(\lambda_0)u_p + \frac{1}{1!}\partial_\lambda P(\lambda_0)u_{p-1} + \frac{1}{2!}\partial_\lambda P(\lambda_0)u_{p-2} = 0, \\ T(\lambda_0)u_p + \frac{1}{1!}T(\lambda_0)u_{p-1} = 0 \end{gathered}$$

for $p = 0, \ldots, k$. Thus $(u_0, \ldots, u_k)$ is a chain of generalized eigenvectors with eigenvalue λ_0 if and only if the function

$$u(t) = e^{\lambda_0 t} \sum_{j=0}^{k} \frac{t^k}{k!} u_{k-j}$$

solves the (time-dependent) equation $\mathscr{P}(\partial_t)u(t) = 0$. Such a solution $u(t)$ is called elementary. To each elementary solution is associated the Cauchy data $(u(0), \partial_t u(0))$. The set of generalized eigenvectors (for all possible eigenvalues) is said to be two-fold complete in a Hilbert space H continuously embedded in $\mathcal{H}^0(X) \times \mathcal{H}^0(X)$ if the span of all Cauchy data $(u(0), \partial_t u(0))$ corresponding to elementary solutions (for all eigenvalues) is dense in H. The same definition holds if $\nu \geq 1$, this time replacing $\mathscr{P}(\lambda)$ with $P(\lambda)$.

A general criterion concerning two-fold completeness is given by [107, Theorem 3.4]; that theorem is a refinement of the standard reference [29, Corollary XI.9.31].

Proposition 3.6.1. *Let $P(\lambda), T(\lambda)$ be defined as above. Fix rays $\Gamma_1, \ldots, \Gamma_s$ through the origin of the complex plane such the angle between any two adjacent rays is less than or equal to π/n, where $\dim X = n$.*

1. *Let $0 < \nu < 1$. If $\mathscr{P}(\lambda)$ is elliptic with respect to $\Gamma_1, \ldots, \Gamma_s$, then the eigenvalues of $\mathscr{P}(\lambda)$ are discrete and the set of generalized eigenvectors is two-fold complete in the space $\{(v_1, v_2) \in \mathcal{H}^2(X) \times \mathcal{H}^1(X) : T_0 v_2 + T_1 v_1 = 0\}$.*

2. *Let $\nu \geq 1$. If $P(\lambda)$ is elliptic with respect to $\Gamma_1, \ldots, \Gamma_s$, then the eigenvalues of $P(\lambda)$ are discrete and the set of generalized eigenvectors is two-fold complete in the space $\mathcal{H}^2(X) \times \mathcal{H}^1(X)$.*

Proof. (1) To apply [107, Theorem 3.4], it must be verified that the singular values of the embeddings $J_k : \mathcal{H}^k(X) \hookrightarrow \mathcal{H}^{k-1}(X)$ satisfy $s_j(J_k) \leq Cj^{-1/n}$ for $k = 1, 2$, and that the space $\{(v_1, v_2) \in \mathcal{H}^2(X) \times \mathcal{H}^1(X) : T_0 v_2 + T_1 v_1 = 0\}$ is dense in $\mathcal{H}^1(X) \times \mathcal{H}^0(X)$.

The claim about the singular values follows from Lemma 3.8.4. To verify the density claim, first fix a sequence $\lambda_n \in \mathbb{C}$ such that $|\lambda_n|$ tends to infinity along one of the rays of ellipticity, say Γ_1. Given $(u_1, u_2) \in \mathcal{H}^1(X) \times \mathcal{H}^0(X)$, take a sequence $(v_1^n, v_2^n) \in \mathcal{H}^2(X) \times \mathcal{H}^1(X)$ such that $(v_1^n, v_2^n) \to (u_1, u_2)$ in $\mathcal{H}^1(X) \times \mathcal{H}^0(X)$ as $n \to \infty$. According to Theorem 5, the operator

$$\mathscr{P}(\lambda_n)^{-1} : \mathcal{H}^0(X) \times H^{2-\mu}(X) \to \mathcal{H}^2(X)$$

exists for n sufficiently large, where $\mu = \mathrm{ord}_\nu^{(\lambda)}(T(\lambda))$. Note that $T_0 v_2^n + T_1 v_1^n \in H^{2-\mu}(\partial X)$. Let

$$w_1^n = \mathscr{P}(\lambda_n)^{-1}(0, -T_0 v_2^n - T_1 v_1^n),$$

so w_1^n lies in $\mathcal{H}^2(X)$, and set $w_2^n = \lambda w_1^n$. Then

$$(v_1^n + w_1^n, v_2^n + w_2^n) \in \{(v_1, v_2) \in \mathcal{H}^2(X) \times \mathcal{H}^1(X) : T_0 v_2 + T_1 v_1 = 0\}.$$

Furthermore, according to Theorems 2, 5 the solution w_1^n satisfies

$$|\lambda|^{2-s} \|w_1^n\|_{\mathcal{H}^s(X)} \leq C \|T_0 v_2^n + T_1 v_1^n\|_{H^{2-\nu}(\partial X)}$$

for $s = 0, 1$. The right hand side is uniformly bounded in $H^{2-\mu}(X)$ as $n \to \infty$, so $(w_1^n, w_2^n) \to 0$ in $\mathcal{H}^1(X) \times \mathcal{H}^0(X)$. This shows that $(v_1^n + w_1^n, v_2^n + w_2^n) \to (u_1, u_2)$, establishing the density.

(2) For $\nu \geq 1$ the singular value estimates remain the same, and the density result is trivial. □

3.7 Density

The proof of Lemma 3.3.6 is broken up into several stages. Recall in this section that $\gamma_\pm$ are defined as in the beginning of Section 3.3 without any mention of the space $\mathcal{F}_\nu$.

Lemma 3.7.1. *Let $\nu > 0$.*

1. *If $u \in \mathcal{H}^1(\mathbb{T}^n_+)$ and $\gamma_- u = 0$, then for a.e. $y \in \mathbb{T}^{n-1}$,*

$$u(x, y) = x^{1/2-\nu} \int_0^x t^{\nu-1/2} \partial_\nu u(t, y)\, dt.$$

2. *Suppose in addition that $0 < \nu < 1$. If $u \in \mathcal{H}^2(\mathbb{T}^n_+)$, and $\underline{\gamma} u = 0$, then for a.e. $y \in \mathbb{T}^{n-1}$,*

$$u(x, y) = x^{1/2-\nu} \int_0^x t^{-2\nu+1} \int_0^t s^{1/2-\nu} \partial_\nu^* \partial_\nu u(s, y)\, ds\, dt.$$

Proof. These two facts follow from the Sobolev embedding for weighted spaces, as in Section 3.3. In the first case, for a.e. $y \in \mathbb{T}^{n-1}$ the function $x \mapsto x^{\nu-1/2} u(x, y)$ is absolutely continuous on $\overline{\mathbb{R}_+}$, and $\gamma_- u = 0$ implies that $x^{\nu-1/2} u(x, y) \to 0$ as $x \to 0$ for a.e. $y \in \mathbb{T}^{n-1}$. The the result follows from the fundamental theorem of calculus. A similar argument applies in the second case, in which the functions $x \mapsto x^{\nu-1/2}u(x,y)$, $x \mapsto x^{1/2-\nu}\partial_\nu u(x,y)$ are absolutely continuous on $\overline{\mathbb{R}_+}$ for a.e. $y \in \mathbb{T}^{n-1}$, and vanish at $x = 0$.

□

Lemma 3.7.2. *Let $0 < \nu < 1$. Then $\mathring{\mathcal{H}}^1(\mathbb{T}^n_+) = \ker \gamma_-$, and $\mathring{\mathcal{H}}^2(\mathbb{T}^n_+) = \ker \underline{\gamma}$.*

Proof. The first equality comes from [47, Proposition 1.2]. It remains to show the second equality.

(1) First show that if $u \in \mathcal{H}^2(\mathbb{T}^n_+)$ and $\underline{\gamma}u = 0$, then $u \in \mathring{\mathcal{H}}^2(\mathbb{T}^n_+)$. Begin by assuming that u has compact support in $\overline{\mathbb{T}^n_+}$; this is possible, since if $\chi \in C_c^\infty(\overline{\mathbb{R}_+})$ satisfies $\chi = 1$ near $x = 0$, then it is easy to see that u is approximated in $\mathcal{H}^2(\mathbb{T}^n_+)$ by the functions $\chi(x/n)u$ as $n \to \infty$.

Next, fix $\chi \in C_c^\infty(\mathbb{R}_+)$ satisfying (i) $0 \le \chi \le 1$, (ii) $\chi(x) = 0$ for $0 \le x \le 1$, (iii) $\chi(x) = 1$ for $x \ge 2$. Then let $\chi_n(x) = \chi(nx)$ and consider the sequence $u_n = \chi_n u$. Then u_n has compact support in $\mathbb{T}^n_+$, which implies that $u_n \in H^2(\mathbb{T}^n_+)$ since the $\mathcal{H}^2(\mathbb{T}^n_+)$ and $H^2(\mathbb{T}^n_+)$ norms are comparable on compact subsets of $\mathbb{T}^n_+$. But the compact support also implies that $u_n \in \mathring{H}^2(\mathbb{T}^n_+)$ by the well known characterization of $\mathring{H}^2(\mathbb{T}^n_+)$. This implies u_n can be approximated by compactly supported functions in the $H^2(\mathbb{T}^n_+)$ norm, all of whose supports are contained in a fixed compact subset of $\mathbb{T}^n_+$. Again by the comparability of norms, this implies $u_n \in \mathring{\mathcal{H}}^2(\mathbb{T}^n_+)$.

It now suffices to prove that $u_n \to u$ in $\mathcal{H}^2(\mathbb{T}^n_+)$, since $\mathring{\mathcal{H}}^2(\mathbb{T}^n_+)$ is closed. This is deduced from Lemma 3.7.1, imitating the proof of [35, Chapter 5.5, Theorem 2] for instance.

(2) The inclusion $\mathring{\mathcal{H}}^2(\mathbb{T}^n_+) \subseteq \ker \underline{\gamma}$ is clear, since $\underline{\gamma} = 0$ for each $u \in C_c^\infty(\mathbb{T}^n_+)$, and hence $\underline{\gamma} = 0$ for each $u \in \mathring{\mathcal{H}}^2(\mathbb{T}^n_+)$ by density and continuity. □

Lemma 3.7.3. *There exists a map*

$$\mathcal{K} : C^\infty(\mathbb{T}^{n-1}) \times C^\infty(\mathbb{T}^{n-1}) \to \mathcal{F}_\nu$$

such that $\underline{\gamma} \circ \mathcal{K} = 1$ on $C^\infty(\mathbb{T}^{n-1}) \times C^\infty(\mathbb{T}^{n-1})$ and $\mathcal{K}$ extends by continuity

$$\mathcal{K} : H^{s-\underline{\nu}}(\mathbb{T}^{n-1}) \to \mathcal{H}^s(\mathbb{T}^n_+)$$

for each $s = 0, \pm 1, \pm 2$. In particular, if $s = 2$ then $\mathcal{K}$ is a right inverse for $\underline{\gamma}$

Proof. Let $\varphi \in C_c^\infty(\overline{\mathbb{R}_+})$ be such that $\varphi = 1$ near $x = 0$, and set

$$v_-(x) = x^{1/2-\nu}\varphi(x^2), \quad v_+(x) = (2\nu)^{-1}x^{1/2+\nu}\varphi(x^2),$$

so $v_\pm \in \mathcal{F}_\nu$. Given $(f_-, f_+) \in C^\infty(\mathbb{T}^{n-1}) \times C^\infty(\mathbb{T}^{n-1})$, define $u_\pm(x,y)$ by its Fourier coefficients,

$$\hat{u}_\pm(x,q) = \langle q \rangle^{-(1/2\pm\nu)} \hat{f}_\pm(q) v_\pm(\langle q \rangle x).$$

Then $u_\pm \in \mathcal{F}_\nu$ and $\gamma_\pm(u_- + u_+) = f_\pm$ in the sense of Lemma 3.3.5. Appealing to Section 3.3 shows that the map defined by

$$\mathcal{K}(f_-, f_+) := u_- + u_+$$

extends by continuity to a map $\mathcal{K} : H^{s-\underline{\nu}}(\mathbb{T}^{n-1}) \to \mathcal{H}^s(\mathbb{T}^n_+)$. If $s = 2$, then $\underline{\gamma}$ is bounded on $\mathcal{H}^2(\mathbb{T}^n_+)$, and $\underline{\gamma} \circ \mathcal{K}$ is the identity on $H^{s-\underline{\nu}}(\mathbb{T}^{n-1})$ by Lemma 3.3.5 and continuity. □

Lemma 3.7.4. *Suppose that* $0 < \nu < 1$. *Then* $\mathcal{F}_\nu$ *is dense in* $\mathcal{H}^2(\mathbb{T}^n_+)$ *for each* $s = 0, 1, 2$.

Proof. This is clear when $s = 0$. The proof is given here in the case $s = 2$; the case $s = 1$ is simpler, and can be handled similarly. Suppose that $u \in \mathcal{H}^2(\mathbb{T}^n_+)$, and let $\tilde{u} = \mathcal{K}(\underline{\gamma} u)$. Then $\underline{\gamma}(u - \tilde{u}) = 0$, so $u - \tilde{u} \in \mathring{\mathcal{H}}^2(\mathbb{T}^n_+)$ by Lemma 3.7.2. It follows that there exists a sequence $u_j \in C^\infty_c(\mathbb{T}^n_+)$ such that $u_j \to u - \tilde{u}$ in $\mathcal{H}^2(\mathbb{T}^n_+)$.

On the other hand, approximate $\underline{\gamma} u$ by a sequence $\underline{v}_j \in C^\infty(\mathbb{T}^{n-1}) \times C^\infty(\mathbb{T}^{n-1})$, and hence $\tilde{u}_j = \mathcal{K}\underline{v}_j$ satisfy $\tilde{u}_j \in \mathcal{F}_\nu$ and $\tilde{u}_j \to \tilde{u}$ in $\mathcal{H}^2(\mathbb{T}^n_+)$. Therefore, $u_j + \tilde{u}_j \in \mathcal{F}_\nu$ and $u_j + \tilde{u}_j \to u$, which shows that $\mathcal{F}_\nu$ is dense in $\mathcal{H}^{2,k}(\mathbb{T}^n_+)$. □

Lemma 3.7.5. *Suppose that* $\nu \geq 1$. *Then* $C^\infty_c(\mathbb{T}^n_+)$ *is dense in* $\mathcal{H}^s(\mathbb{T}^n_+)$ *for each* $s = 0, 1, 2$.

Proof. This result clearly holds for $s = 0$, and by the results of Section 3.3, it also holds for $s = 1$. For $s = 2$, Lemma 3.4.5 implies that $\Delta_\nu + 1$ is an isomorphism from $D(L)$ onto $L^2(\mathbb{T}^n_+)$. The first step is write down an explicit formula for the inverse $(\Delta_\nu + 1)^{-1}$ acting on $L^2(\mathbb{T}^n_+)$. Introduce the Zemanian space $\mathcal{Z}_\nu(\mathbb{R}_+)$ [108, Chapter 5] by

$$\mathcal{Z}_\nu(\mathbb{R}_+) = \{v(x) = x^{1/2+\nu} v_+(x^2) : v_+(x) \in \mathcal{S}(\mathbb{R})\},$$

where $\mathcal{S}(\mathbb{R})$ is the space of Schwartz functions on $\mathbb{R}$. Note that $\mathcal{Z}_\nu(\mathbb{R}_+)$ is contained in $\mathcal{H}^s(\mathbb{R}_+)$ for each $s = 0, 1, 2$. Given $v \in \mathcal{Z}_\nu(\mathbb{R}_+)$, define the Hankel transform

$$(\mathscr{H}_\nu v)(\xi) = \int_{\mathbb{R}_+} (\xi x)^{1/2} J_\nu(\xi x) v(x) dx.$$

Referring to [108, Chapter 5], it is well known that (i) $\mathscr{H}_\nu$ is an automorphism of $\mathcal{Z}_\nu(\mathbb{R}_+)$, (ii) $\mathscr{H}_\nu^2 = I$, (iii) $\mathscr{H}_\nu$ is isometric with respect to the $L^2(\mathbb{R}_+)$ norm.

If $f = v \otimes w$ where $v \in \mathcal{Z}_\nu(\mathbb{R}_+)$ and $w \in C^\infty(\mathbb{T}^{n-1})$, then

$$f = (2\pi)^{-(n-1)/2} \sum_{q \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}_+} e^{i\langle q, y\rangle} (\xi x)^{1/2} J_\nu(\xi x) (\mathscr{H}_\nu v)(\xi) \hat{w}(q)\, d\xi,$$

and

$$\|f\|^2_{L^2(\mathbb{T}^n_+)} = (2\pi)^{-(n-1)/2} \sum_{q \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}_+} |(\mathscr{H}_\nu v)(\xi) \hat{w}(q)|^2 d\xi.$$

Given $f = v \otimes w$ as above, let

$$u = (2\pi)^{-(n-1)/2} \sum_{q\in\mathbb{Z}^{n-1}} \int_{\mathbb{R}_+} (1+|q|^2+|\xi|^2)^{-1} e^{i\langle q,y\rangle} (\xi x)^{1/2} J_\nu(\xi x)(\mathscr{H}_\nu v)(\xi)\hat{w}(q)\, d\xi. \tag{3.62}$$

Then u has an expansion

$$u(x,y) = x^{1/2+\nu} u_+(x^2,y), \tag{3.63}$$

where $u_+(x,y) \in \mathcal{S}(\mathbb{R}\times\mathbb{T}^{n-1})$ is rapidly decaying in the x variable — this can be shown by the same method as [108]. Denote this space by $\mathcal{Z}_\nu(\mathbb{T}^n_+)$, which is contained in $\mathcal{H}^2(\mathbb{T}^n_+)\cap\mathring{\mathcal{H}}^1_0(\mathbb{T}^n_+)$. Note that there is a continuous inclusion

$$\mathcal{H}^2(\mathbb{T}^n_+)\cap\mathring{\mathcal{H}}^1_0(\mathbb{T}^n_+) \hookrightarrow D(L).$$

Since $(\Delta_\nu+1)u = f$, it follows that u is the unique solution in $D(L)$ to the equation $(\Delta_\nu+1)u = f$. Furthermore,

$$\|u\|_{\mathcal{H}^2(\mathbb{T}^n_+)} \leq C\|f\|_{L^2(\mathbb{T}^n_+)},$$

where $C>0$ does not depend on u or f. Lemma 3.4.5 and the open mapping theorem imply that $D(L) = \mathcal{H}^2(\mathbb{T}^n_+)\cap\mathring{\mathcal{H}}^1(\mathbb{T}^n_+)$ with an equivalence of norms; since functions $v\otimes w$ as above are dense in $L^2(\mathbb{T}^n_+)$, the space $\mathcal{Z}_\nu(\mathbb{T}^n_+)$ is dense in $\mathcal{H}^2(\mathbb{T}^n_+)\cap\mathring{\mathcal{H}}^1(\mathbb{T}^n_+)$. Finally, if $\nu\geq 1$ then $\mathcal{H}^2(\mathbb{T}^n_+) = \mathcal{H}^2(\mathbb{T}^n_+)\cap\mathring{\mathcal{H}}^1(\mathbb{T}^n_+)$ by the density result for $s=1$.

Given $u\in\mathcal{Z}_\nu(\mathbb{T}^n_+)$, there exists a sequence $v_n\in C^\infty_c(\mathbb{T}^n_+)$ such that $v_n\to u$ weakly in $\mathcal{H}^2(\mathbb{T}^n_+)$. To see this, fix $\chi\in C^\infty_c(\mathbb{R}_+)$ satisfying (i) $0\leq\chi\leq 1$, (ii) $\chi(x)=0$ for $0\leq x\leq 1$, (iii) $\chi(x)=1$ for $x\geq 2$, and then let $\chi_n(x)=\chi(nx)$. The claim is that $\chi_n u\to u$ weakly in $\mathcal{H}^2(\mathbb{T}^n_+)$ after passing to a subsequence if necessary. Take for example

$$\begin{aligned}|D_\nu|^2(\chi_n(x)-1)u(x,y) &= (\chi_n(x)-1)|D_\nu|^2u(x)\\ &\quad + 2n\chi_n'(x)\left((1/2+\nu)x^{-1/2+\nu}u_+(x^2,y)+2x^{3/2+\nu}\partial_x u_+(x^2,y)\right)\\ &\quad + n^2\chi_n''(x)u(x,y).\end{aligned}$$

The first term tends to zero in $L^2(\mathbb{T}^n_+)$ norm. The $L^2(\mathbb{T}^n_+)$ norm squared of the third term is bounded by a constant times

$$n^4\int_{\mathbb{T}^{n-1}}\int_{1/n}^{2/n} x^{1+2\nu}|u_+(x^2,y)|^2\,dx\,dy \leq Cn^4\int_{1/n}^{2/n} x^{1+2\nu} = \mathcal{O}(n^{2\nu-2}).$$

So $n^2\chi_n''(x)u(x,y)$ is bounded in $L^2(\mathbb{T}^n_+)$ for $\nu\geq 1$ (and converges to zero if $\nu>1$). The second term is similarly bounded in $L^2(\mathbb{T}^n_+)$. It is also clear the second and third terms converge to zero in $\mathscr{D}'(\mathbb{R}_+)$. Extracting a weakly convergent subsequence, this implies that $|D_\nu|^2(\chi_n-1)u(x,y)$ tends to zero weakly along a subsequence. Repeating this argument for the other terms whose $L^2(\mathbb{T}^n_+)$ norms define the $\mathcal{H}^2(\mathbb{T}^n_+)$ norm (as in (3.13)), it follows that

$\chi_n u \to u$ weakly in $\mathcal{H}^2(\mathbb{T}^n_+)$. Furthermore, by truncating $\chi_n u$ at successively larger values of $x > 0$, one may find a sequence of $C_c^\infty(\mathbb{T}^n_+)$ functions v_n such that $v_n \to u$ weakly in $\mathcal{H}^2(\mathbb{T}^n_+)$.

Now suppose that $u \in \mathcal{H}^2(\mathbb{T}^n_+)$ satisfies

$$\langle u, v\rangle_{\mathcal{H}^2(\mathbb{T}^n_+)} = 0$$

for all $v \in C_c^\infty(\mathbb{T}^n_+)$. Choose a sequence $u_m \in \mathcal{Z}_\nu(\mathbb{T}^n_+)$ such that $u_m \to u$ in $\mathcal{H}^2(\mathbb{T}^n_+)$. But if $v_{m,n}$ is an associated sequence of $C_c^\infty(\mathbb{T}^n_+)$ functions converging weakly to u_m (as constructed above), then

$$\langle u, u_m\rangle_{\mathcal{H}^2(\mathbb{T}^n_+)} = \lim_{n\to\infty} \langle u, v_{m,n}\rangle_{\mathcal{H}^2(\mathbb{T}^n_+)} = 0.$$

Passing to the limit $m \to \infty$ gives $u = 0$, so $C_c^\infty(\mathbb{T}^n_+)$ is dense.

□

3.8 Compactness and embeddings of Schatten class

Let $\mathbb{T}^n_\sharp = \mathbb{T}^{n-1} \times (0,1)$. The spaces $\mathcal{H}^s(\mathbb{T}^n_\sharp)$, $\mathring{\mathcal{H}}^s(\mathbb{T}^n_\sharp)$ are defined as before. The goal of this section is to study Schatten class properties of the embeddings of these spaces into $L^2(\mathbb{T}^n_\sharp)$ for $s = 1, 2$. First, compactness properties are examined — this is done differently in [58, Section 6], but the approach taken here immediately yields the Schatten property.

The first observation is that the embedding $\mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp) \hookrightarrow L^2(\mathbb{T}^n_\sharp)$ is compact for $\nu > 0$, since $\mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp) = \mathring{H}^1(\mathbb{T}^n_\sharp)$ according to Lemma 3.3.3. To prove compactness of the embedding $\mathcal{H}^1(\mathbb{T}^n_\sharp) \hookrightarrow L^2(\mathbb{T}^n_\sharp)$ requires slightly more work.

Recall the following facts: first,

$$\sqrt{x}K_\nu(x) \sim (\pi/2)^{1/2}e^{-x}, \quad \sqrt{x}I_\nu(x) \sim (1/2\pi)^{1/2}e^{x},$$

valid for real $x \to \infty$. Furthermore,

$$|\sqrt{x}K_\nu(x)| \le C\left(\frac{1+x^\nu}{1+x^{1/2}}\right)x^{1/2-\nu}e^{-x}$$

for all $x > 0$. Combined with the equation satisfied by $\sqrt{x}K_\nu(x)$, this gives

$$\int_0^1 |\sqrt{x}K_\nu(\tau x)|^2\,dx \le C\tau^{-2}, \quad \int_0^1 |(|D_\nu|^2\sqrt{x}K_\nu(\tau x))|^2\,dx \le C\tau^2 \tag{3.64}$$

for $\tau > 0$. Combining (3.64) with Lemma 3.4.3, an integration by parts shows that

$$\int_0^1 |(\partial_\nu\sqrt{x}K_\nu(\tau x))|^2\,dx \le C. \tag{3.65}$$

By Laplace's method, integrals of $\sqrt{x}I_\nu(\tau x)$ may be evaluated as well. In particular,

$$\int_0^1 |\sqrt{x}I_\nu(\tau x)|^2\,dx \sim \frac{1}{4\pi}\tau^{-2}e^{2\tau}, \quad \int_0^1 |(\partial_\nu\sqrt{x}I_\nu(\tau x))|^2\,dx \sim \frac{1}{4\pi}e^{2\tau}, \tag{3.66}$$

$$\int_0^1 |(|D_\nu|^2\sqrt{x}I_\nu(\tau x))|^2\,dx \sim \frac{1}{4\pi}\tau^2 e^{2\tau}. \tag{3.67}$$

as $\tau \to \infty$. The following construction of a Poisson operator is a refinement of Lemma 3.7.3.

Lemma 3.8.1. *Let $0 < \nu < 1$. There exists a map $\mathcal{K}_0 : C^\infty(\mathbb{T}^{n-1}) \to \mathcal{F}_\nu(\mathbb{T}^n_\sharp)$ such that*

$$(\gamma_- \circ \mathcal{K}_0)\phi = \phi, \quad (\mathcal{K}_0\phi)(1,\cdot) = 0, \quad (\Delta_\nu + 1)\mathcal{K}_0\phi = 0$$

for each $\phi \in C^\infty(\mathbb{T}^{n-1})$. The map $\mathcal{K}_0$ extends by continuity

$$\mathcal{K}_0 : H^{s-1+\nu}(\mathbb{T}^{n-1}) \to \mathcal{H}^s(\mathbb{T}^n_\sharp)$$

for each $s = 0, \pm 1, \pm 2$. Similarly, for each $\nu > 0$ there exists a map $\mathcal{K}_1 : C^\infty(\mathbb{T}^{n-1}) \to \mathcal{F}_\nu(\mathbb{T}^n_\sharp)$ such that

$$(\gamma_- \circ \mathcal{K}_1)\phi = 0, \quad (\mathcal{K}_1\phi)(1,\cdot) = \phi, \quad (\Delta_\nu + 1)\mathcal{K}_1\phi = 0,$$

and $\mathcal{K}_1$ extends by continuity

$$\mathcal{K}_1 : H^{s-1/2}(\mathbb{T}^{n-1}) \to \mathcal{H}^s(\mathbb{T}^n_\sharp)$$

for $s = 0, \pm 1, \pm 2$.

Proof. Only $\mathcal{K}_1$ is constructed in detail; the construction of $\mathcal{K}_0$ is simpler. For each $q \in \mathbb{Z}^{n-1}$, consider the function

$$v(x,q) = \Gamma(1-\nu)\left(\frac{\langle q\rangle}{2}\right)^\nu \frac{\sqrt{x}K_\nu(\langle q\rangle\, x)I_\nu(\langle q\rangle) - \sqrt{x}I_\nu(\langle q\rangle\, x)K_\nu(\langle q\rangle)}{\frac{\pi}{2\sin\pi\nu}I_\nu(\langle q\rangle) - K_\nu(\langle q\rangle)}.$$

Note that $v(q) \in \mathcal{F}_\nu(\mathbb{T}^n_\sharp)$ by asymptotics of Bessel functions. Furthermore,

$$(|D_\nu|^2 + \langle q\rangle^2)v(x,q) = 0,$$

and $v(1,q) = 0, \gamma_- v(q) = 1$. Given $f \in C^\infty(\mathbb{T}^{n-1})$, let

$$\hat{u}(x,q) = \hat{f}(q)v(x,q).$$

According to (3.64), (3.65), (3.66), (3.67), the map sending f to the function u with Fourier coefficients $\hat{u}(q)$ is bounded $H^{s-1+\nu}(\mathbb{T}^{n-1}) \to \mathcal{H}^s(\mathbb{T}^n_+)$ for $s = 0, \pm 1, \pm 2$. □

Recall that $\mathcal{H}^1(\mathbb{T}^n_\sharp)$ is a Hilbert space, equipped with the scalar product

$$\langle u, v\rangle_{\mathcal{H}^1(\mathbb{T}^n_\sharp)} = \langle D_\nu u, D_\nu v\rangle_{\mathbb{T}^n_\sharp} + \sum_{i=1}^{n-1} \langle D_{y_i} u, D_{y_i} v\rangle_{\mathbb{T}^n_\sharp} + \langle u, v\rangle_{\mathbb{T}^n_\sharp}.$$

Suppose that $0 < \nu < 1$. Lemma 3.8.1 implies that $\langle \mathcal{K}_0\phi_0 + \mathcal{K}_1\phi_1, v\rangle_{\mathcal{H}^1(\mathbb{T}^n_\sharp)} = 0$ for all $\phi_0, \phi_1 \in C^\infty(\mathbb{T}^{n-1})$ and $v \in C^\infty_c(\mathbb{T}^n_\sharp)$. By continuity this holds true for $\phi_0 \in H^\nu(\mathbb{T}^{n-1})$, $\phi_1 \in H^{1/2}(\mathbb{T}^{n-1})$ and $v \in \mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp)$. Consider the orthogonal decomposition

$$\mathcal{H}^1(\mathbb{T}^n_\sharp) = \mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp) \oplus \mathcal{X},$$

where $\mathcal{X}$ is the orthogonal complement of $\mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp)$ in $\mathcal{H}^1(\mathbb{T}^n_\sharp)$. Let γ_1 denote the restriction to $\{x = 1\}$. Lemma 3.8.1 shows that the map

$$\Gamma : u \mapsto (\gamma_- u, \gamma_1 u)$$

is an isomorphism from $\mathcal{X}$ onto $H^\nu(\mathbb{T}^{n-1}) \times H^{1/2}(\mathbb{T}^{n-1})$. The inverse of Γ is given by $\mathcal{K}_0 + \mathcal{K}_1 : (\phi_0, \phi_1) \mapsto \mathcal{K}_0\phi_0 + \mathcal{K}_1\phi_1$.

A similar discussion applies if $\nu \geq 1$. In that case, $\gamma_1 : \mathcal{X} \to H^{1/2}(\mathbb{T}^{n-1})$ is an isomorphism, with inverse $\mathcal{K}_1$.

Lemma 3.8.2. *If $\nu > 0$, then the embeddings*

$$\mathcal{H}^2(\mathbb{T}^n_\sharp) \hookrightarrow \mathcal{H}^1(\mathbb{T}^n_\sharp), \quad \mathcal{H}^1(\mathbb{T}^n_\sharp) \hookrightarrow L^2(\mathbb{T}^n_\sharp)$$

are compact.

Proof. (1) First suppose that $0 < \nu < 1$. Write the orthogonal decomposition $\mathcal{H}^1(\mathbb{T}^n_\sharp) = \mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp) \oplus \mathcal{X}$. The inclusion of $\mathcal{X}$ into $L^2(\mathbb{T}^n_\sharp)$ can be factored as

$$\mathcal{X} \xrightarrow{\Gamma} H^\nu(\mathbb{T}^{n-1}) \times H^{1/2}(\mathbb{T}^{n-1}) \hookrightarrow H^{-1+\nu}(\mathbb{T}^{n-1}) \times H^{-1/2}(\mathbb{T}^{n-1}) \xrightarrow{\mathcal{K}_0+\mathcal{K}_1} L^2(\mathbb{T}^n_\sharp), \tag{3.68}$$

noting that $\mathcal{K}_0 + \mathcal{K}_1 : H^{-1+\nu}(\mathbb{T}^{n-1}) \times H^{-1/2}(\mathbb{T}^{n-1}) \to L^2(\mathbb{T}^n_\sharp)$ is an extension by continuity of the same map acting $H^\nu(\mathbb{T}^{n-1}) \times H^{1/2}(\mathbb{T}^{n-1}) \to \mathcal{X}$. This is compact since the inclusion of $H^\nu(\mathbb{T}^{n-1}) \times H^{1/2}(\mathbb{T}^{n-1})$ into $H^{-1+\nu}(\mathbb{T}^{n-1}) \times H^{-1/2}(\mathbb{T}^{n-1})$ is compact.

The space $\mathcal{H}^2(\mathbb{T}^n_\sharp)$ may be identified with a closed subspace H of $\mathcal{H}^1(\mathbb{T}^n_\sharp)^{n+1} \times \mathcal{H}^1_*(\mathbb{T}^n_\sharp)$ via the mapping

$$u \mapsto (u, \partial_{y_1} u, \ldots, \partial_{y_{n-1}} u, \partial_\nu u).$$

With this in mind, the embedding $\mathcal{H}^2(\mathbb{T}^n_\sharp) \hookrightarrow \mathcal{H}^1(\mathbb{T}^n_\sharp)$ is identified with the embedding

$$\mathcal{H}^1(\mathbb{T}^n_\sharp)^{n+1} \times \mathcal{H}^1_*(\mathbb{T}^n_\sharp) \hookrightarrow L^2(\mathbb{T}^n_\sharp)^{n+2}, \tag{3.69}$$

restricted to H. But the inclusion $\mathcal{H}^1_*(\mathbb{T}^n_\sharp) \hookrightarrow L^2(\mathbb{T}^n_\sharp)$ is compact by the first part as well (since $0 < 1 - \nu < 1$), so (3.69) is compact.

(2) Now suppose that $\nu \geq 1$. The same argument as in the first part shows that $\mathcal{H}^1(\mathbb{T}^n_\sharp) \hookrightarrow L^2(\mathbb{T}^n_\sharp)$ is compact. Next, by the same reductions as above, it suffices to consider the inclusion $\mathcal{H}^2(\mathbb{T}^n_\sharp) \cap \mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp) \hookrightarrow \mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp)$. As in Section 3.4, let L denote the self-adjoint operator with distributional action Δ_ν and form domain $\mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp)$. Now the embedding of $D((L+1)^{1/2}) = \mathring{\mathcal{H}}^1(\mathbb{T}^{n-1})$ into $L^2(\mathbb{T}^n_\sharp)$ is compact, hence so is each embedding $D((L+1)^N) \hookrightarrow D((L+1)^n)$ for $N > n$. But as in Lemma 3.7.5, the domain $D(L) = D(L+1)$ (with the graph norm) is equivalent to $\mathcal{H}^2(\mathbb{T}^n_\sharp) \cap \mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp)$ (with its usual norm). □

Let L denote the self-adjoint operator with distributional action given by Δ_ν and form domain $\mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp)$ (see Section 3.4). Lemma 3.8.2 and Lax–Milgram guarantee that this operator has discrete spectrum. The eigenvalues and eigenvectors are well known. The eigenvalues are $|q|^2 + j^2_{\nu,n} + 1$, where $q \in \mathbb{Z}^{n-1}$ and $j_{\nu,n}$ is the n'th positive root of the Bessel function J_ν. The corresponding eigenfunction is

$$\sqrt{x} J_\nu(j_{\nu,n} x) \otimes e^{i\langle q,y\rangle}.$$

The zeros $j_{\nu,n}$ satisfy the asymptotic formula

$$j_{\nu,n} = \left(n + \tfrac{1}{2}\nu - \tfrac{1}{4}\right)\pi + \mathcal{O}(n^{-1})$$

as $n \to \infty$. The eigenvalues of the compact operator $(L+1)^{-1/2}$ are therefore $(1 + |q|^2 + j^2_{\nu,n})^{-1/2}$, and if they are listed in descending order $\lambda_1 > \ldots > \lambda_j > \ldots > 0$ (with multiplicity) then

$$\lambda_j \leq C j^{-1/n}.$$

for some $C > 0$.

Lemma 3.8.3. *If $\nu > 0$, then the singular values of the embedding*

$$J_1 : \mathcal{H}^1(\mathbb{T}^n_\sharp) \hookrightarrow L^2(\mathbb{T}^n_\sharp), \quad J_2 : \mathcal{H}^2(\mathbb{T}^n_\sharp) \hookrightarrow \mathcal{H}^1(\mathbb{T}^n_\sharp)$$

satisfy $s_j(J_i) < C j^{-1/n}$.

Proof. (1) First suppose that $0 < \nu < 1$. Let Π denote the orthogonal projection onto $\mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp)$, so $J_1 = J_1\Pi + J_1(1-\Pi)$. Since $\mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp)$ is the form domain of L, the operator $(L+1)^{-1/2}$ is an isomorphism acting $L^2(\mathbb{T}^n_\sharp) \to \mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp)$. Write

$$J_1\Pi = J_1(L+1)^{-1/2}(L+1)^{1/2}\Pi.$$

The composition $J_1(L+1)^{-1/2}$ is self-adjoint and positive definite on $L^2(\mathbb{T}^n_\sharp)$, so its singular values are the λ_j which satisy $\lambda_j \leq C j^{-1/n}$. Furthermore, $(L+1)^{1/2}\Pi$ is bounded $\mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp) \to L^2(\mathbb{T}^n_\sharp)$, so the inequality $s_j(AB) \leq s_j(A)\|B\|$ shows that $s_j(J_1\Pi) \leq C j^{-1/n}$. On the other hand, the term $J_1(1-\Pi)$ factors through the map in (3.68). It is well known that the inclusion

$$H^\nu(\mathbb{T}^{n-1}) \times H^{1/2}(\mathbb{T}^{n-1}) \hookrightarrow H^{\nu-1}(\mathbb{T}^{n-1}) \times H^{-1/2}(\mathbb{T}^{n-1})$$

has singular values bounded by $Cj^{-1/(n-1)}$. The inequality $s_{i+j-1}(A+B) \leq s_i(A) + s_j(B)$ applied to the sum $J_1 = J_1\Pi + J_1(1-\Pi)$ shows that $s_j(J_1) \leq Cj^{-1/n}$. The same method of proof applies when $\nu \geq 1$

(2) Now consider J_2. In the case $0 < \nu < 1$, the space $\mathcal{H}^2(\mathbb{T}^n_\sharp)$ is identified with the closed subspace $H \subseteq \mathcal{H}^1(\mathbb{T}^n_\sharp)^{n+1} \times \mathcal{H}^1_*(\mathbb{T}^n_\sharp)$ as in the proof of Lemma 3.8.2. Since the singular values of the embedding $\mathcal{H}^1(\mathbb{T}^n_\sharp)^{n+1} \times \mathcal{H}^1_*(\mathbb{T}^n_\sharp) \hookrightarrow L^2(\mathbb{T}^n_\sharp)^{n+2}$ are bounded by $Cj^{-1/n}$, the same is true of the embedding $\mathcal{H}^2(\mathbb{T}^n_\sharp) \hookrightarrow \mathcal{H}^1(\mathbb{T}^n_\sharp)$. For $\nu \geq 1$ it would also suffice to bound the singular values of $J_2' : \mathcal{H}^2(\mathbb{T}^n_\sharp) \cap \mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp) \hookrightarrow \mathring{\mathcal{H}}^1(\mathbb{T}^n_\sharp)$. But $J_2' = J_2'(L+1)^{-1/2}(L+1)^{1/2}$, and the singular values of $J_2'(L+1)^{-1/2}$ are again the $\lambda_j \leq Cj^{-1/n}$. □

Lemma 3.8.3 easily extends to the case of a compact manifold with boundary.

Lemma 3.8.4. *Let $\overline{X}$ be a compact manifold with boundary. If $\nu > 0$, then the embeddings*

$$J_1 : \mathcal{H}^1(X) \hookrightarrow \mathcal{H}^0(X), \quad J_2 : \mathcal{H}^2(X) \hookrightarrow \mathcal{H}^1(X)$$

are compact, and the singular values of J_i satisfy $s_j(J_i) \leq Cj^{-1/n}$.

Chapter 4

A global definition of quasinormal modes

4.1 Introduction

Let $\mathcal{M}_0$ denote the exterior of a Kerr–AdS spacetime with metric g, determined by parameters (Λ, a, M). It is convenient to use units in which $|\Lambda| = 3$. After modifying the time slicing (originally defined via Boyer–Lindquist coordinates), there always exists an extension of g across the event horizon to a larger spacetime $\mathcal{M}_\delta$, such that the time slice $X_\delta = \{t^\star = 0\}$ is spacelike. For more on this, refer back to the discussion in Section 2.5.

The stationary Klein–Gordon operator $P(\lambda)$ is defined on X_δ by replacing T with a spectral parameter $-i\lambda \in \mathbb{C}$ in the operator $\varrho^2(\Box_g + \nu^2 - 9/4)$.

Remark 15. *It is convenient to multiply the Klein–Gordon equation by a positive prefactor $\sim r^2$, as we can then prove estimates for $P(\lambda)$ acting between L^2 based spaces with the same r-weight. We already did this in Chapter 3 (in particular, see 3.2.1), but the ultimate motivation comes from when we study $P(\lambda)$ via energy estimates in Section 6.3. The choice $\varrho^2 \sim r^2$ simplies many formulae.*

The effective mass is required to satisfy the Breintenlohner–Freedman bound $\nu > 0$. This restriction has a variety of consequences for the study of massive waves on asymptotically AdS spaces; here, the bound must be satisfied in order to apply the results of Chapter 3 on certain singular elliptic boundary value problems.

The purpose of this chapter is to prove that $P(\lambda)^{-1}$ (augmented by boundary conditions when $0 < \nu < 1$) forms a meromorphic family of operators on appropriate function spaces. Recall from Section 2.5 that the surface gravity of the Killing horizon $\mathcal{H}^+ = \{r = r_+\}$ is given by

$$\kappa = \frac{\partial_r \Delta_r(r_+)}{2(1-\alpha)(r_+^2 + a^2)},$$

and that we require $\kappa > 0$. The first result, valid for $\nu \geq 1$, is the following.

Theorem 6. *If $\nu \geq 1$ and $k \in \mathbb{N}$, then*

$$P(\lambda) : \mathcal{X}^k \to \mathcal{H}^{0,k}(X_\delta)$$

is Fredholm for λ in the half-plane $\{\operatorname{Im}\lambda > -\kappa(1/2+k)\}$. Furthermore, given any angular sector $\Lambda \subseteq \mathbb{C}$ in the upper half-plane, there exists $R > 0$ such that $P(\lambda)$ is invertible for $\lambda \in \Lambda$ and $|\lambda| > R$.

In this theorem, $\mathcal{H}^{s,k}(X_\delta)$ and $\mathcal{X}^k \subseteq \mathcal{H}^{1,k}(X_\delta)$ are certain Hilbert spaces. This is a slight abuse of notation if compared to Chapter 3, since we slightly redefine $\mathcal{H}^{s,k}(X_\delta)$ to account for the different normalization of $P(\lambda)$ (we did not conjugate by r^{-1}) — see Section 4.2 for the definitions. Also observe that X_δ has a non-compact end at $r = r_+ - \delta$.

By analytic Fredholm theory, the family $\lambda \mapsto P(\lambda)^{-1}$ is meromorphic. QNFs in the half-plane $\{\operatorname{Im}\lambda > -\kappa(1/2+k)\}$ are defined as poles of $P(\lambda)^{-1} : \mathcal{H}^{0,k}(X) \to \mathcal{X}^k$. The poles are discrete and the corresponding residues are finite rank operators. Furthermore, any element u in the kernel of $P(\lambda) : \mathcal{X}^k \to \mathcal{H}^{0,k}(X)$ is smooth up to the artificial boundary $H = \{r = r_+ - \delta\}$ provided the threshhold condition $\operatorname{Im}\lambda > -\kappa(1/2+k)$ is satisfied, and u has a conormal asymptotic expansion at $Y = X_\delta \cap \mathcal{I}$.

The analogous statement when $0 < \nu < 1$ is more involved since boundary conditions (in the sense Chapter 3) must be imposed at the conformal boundary Y to obtain a Fredholm problem. Fix a weighted trace $T(\lambda)$ whose 'principal part' is independent of λ and let

$$\mathscr{P}(\lambda) = \begin{pmatrix} P(\lambda) \\ T(\lambda) \end{pmatrix}.$$

The operator $\mathscr{P}(\lambda)$ is required to satisfy the parameter-dependent Lopatinskiĭ condition (in the sense of Section 3.4) with respect to an angular sector $\Lambda \subseteq \mathbb{C}$ in the upper half-plane.

Theorem 7. *If $0 < \nu < 1, k \in \mathbb{N}$, and μ is the order of $T(\lambda)$ with respect to ν, then*

$$\mathscr{P}(\lambda) : \{u \in \mathcal{X}^k : T(0)u \in H^{k+2-\mu}(Y)\} \to \mathcal{H}^{0,k}(X) \times H^{k+2-\mu}(Y)$$

is Fredholm for λ in the half-plane $\{\operatorname{Im}\lambda > -\kappa(1/2+k)\}$. Furthermore, there exists $R > 0$ such that $\mathscr{P}(\lambda)$ is invertible for $\lambda \in \Lambda$ and $|\lambda| > R$.

QNFs in the half-plane $\{\operatorname{Im}\lambda > -\kappa(1/2+k)\}$ are again defined as poles of the meromorphic family $\lambda \mapsto \mathscr{P}(\lambda)^{-1}$. The observations following Theorem 6 are also applicable.

A natural question is to what extent QNFs depend on how the original metric is extended across the event horizon. The following answer was suggested to the author by Peter Hintz; unlike the other results in this chapter, it strongly uses axisymmetry of the exact Kerr–AdS metric. Given $m \in \mathbb{Z}$, let

$$\mathcal{D}'_m = \{u \in \mathcal{D}' : (D_\phi - m)u = 0\}.$$

The axisymmetry of g implies that $\mathcal{D}'_m(X)$ is invariant under $P(\lambda)$ for each λ. When $0 < \nu < 1$, the trace $T(\lambda)$ is said to be axisymmetric if $T(\lambda)D_\phi u = D_\phi T(\lambda)u$ for each λ, which implies the mapping property

$$T(\lambda) : \mathcal{X}^k \cap \mathcal{D}'_m(X) \to H^{k+1-\mu}(Y) \cap \mathcal{D}'_m(Y).$$

Let $P_0(\lambda)$ denote the restriction of $P(\lambda)$ to X_0.

Theorem 8. *Let $\nu > 0$ and suppose that $T(\lambda)$ is axisymmetric. If $\nu \geq 1$, then $\lambda_0 \in \mathbb{C}$ is a QNF if and only if there exists $m \in \mathbb{Z}$ and a nonzero function*

$$u \in C^\infty(X_0) \cap \mathcal{H}^0(X_0) \cap \mathcal{D}'_m(X_0),$$

smooth up to $r = r_+$, such that $P_0(\lambda_0)u = 0$. The same is true for $0 < \nu < 1$ under the additional condition $T(\lambda_0)u = 0$.

Of course it is possible that a different method of proof could establish Theorem 8 without making use of any additional symmetries.

Relation to previous works

The mathematical study of QNMs for AdS black holes began slightly later than their nonnegative cosmological constant counterparts. QNMs of Schwarzschild black holes were rigorously studied by Bachelot [4] and Bachelot–Motet-Bachelot [6]. Meromorphy of the scattering resolvent for Schwarzschild–de Sitter black holes was established by Sá Barreto–Zworski [85], who also described the lattice structure of QNFs. Expansions of scattered waves in terms of QNMs was established for Schwarzschild–de Sitter space by Bony–Häfner [12]. Later, Dyatlov constructed a meromorphic continuation of the scattering resolvent for Kerr–de Sitter metrics and analysed the distribution of QNFs [31, 30].

All of the aforementioned works used delicate separation of variables techniques to study QNMs, hence are not stable under perturbations. In a landmark paper [95], Vasy proved meromorphy of a family of operators whose poles define QNFs of Kerr–de Sitter metrics. This method depends only on certain microlocal properties of the geodesic flow, which are stable under perturbations. Additionally, resolvent estimates, expansions in terms of QNMs, and wavefront set properties of the resolvent were also established (not to mention other applications, for instance to asymptotically hyperbolic spaces). For non-rotating Schwarzschild–AdS black holes, QNMs were treated mathematically by the author in [42] using the Regge-Wheeler formalism [43] (separation of variables). The Regge–Wheeler equations at a fixed angular momentum ℓ in the nonrotating case fit into the framework of classical one-dimensional scattering theory. Using a "black-box" approach, it was shown that the scattering resolvent exists and its restriction to a fixed space of spherical harmonics forms a meromorphic family of operators [42, Section 4]. Therefore discreteness of QNFs for ℓ fixed is solved by identifying them as poles of this resolvent. Furthermore, there exist sequences of QNFs converging exponentially to the real axis, with a precise description of their real parts. In [42], only Dirichlet boundary conditions were considered at the conformal boundary.

For general black hole backgrounds with asymptotically AdS ends, a global definition and discreteness of QNFs was first studied by Warnick [99]. There, QNFs are defined as eigenvalues of an infinitesimal generator whose associated semigroup solves a mixed initial boundary value problem for the linear wave equation. When applied to the special class of Kerr–AdS metrics, there are two main results:

1. QNFs at a *fixed* axial mode m are discrete. This holds for all rotation speeds satisfying the regularity condition $|a| < 1$. More generally, it holds for a more general class of "locally stationary" asymptotically AdS black holes, once the notion of a Fourier mode is appropriately generalized — these spacetimes have some additional symmetries.

2. The set of all QNFs is discrete provided the rotation speed satisfies the Hawking–Reall bound $|a| < \min\{1, r_+^2\}$. These Kerr–AdS metrics admit a globally causal Killing field; this remarkable property is not shared by either the Kerr or Kerr-de Sitter family of metrics as soon as $a \neq 0$.

Furthermore, self-adjoint boundary conditions of Dirichlet or Robin type may be imposed at the conformal boundary. As mentioned above, this paper generalizes [99] in two ways: the QNF spectrum is shown to be discrete for rotation speeds satisfying $|a| < 1$, and when $0 < \nu < 1$ this discreteness holds for a broader class of boundary conditions than considered in [99].

4.2 Kerr–AdS spacetime

Recall the definition of the Kerr–AdS definition from Section 2.5. We work in units where $\Lambda = -3$, so the metric is uniquely determined by a rotation speed $|a| < 1$ and mass $M > 0$. As usual, we make the non-degeneracy assumption $\kappa > 0$ for the surface gravity, which is equivalent to $\Delta_r'(r_+) > 0$.

By choosing an appropriate function f_+ in (2.20), we consider the extended space $\mathcal{M}_\delta$ foliated by spacelike surfaces of constant $t^\star$. Each of these surfaces is diffeomorphic to

$$X_\delta = (r_+ - \delta, \infty) \times \mathbb{S}^2.$$

Furthermore, we assume that each surface meets the conformal boundary $\mathcal{I}$ orthogonally.

The manifold with boundary

It is natural to view X_δ as the interior of a compact manifold with two boundary components. Let

$$\overline{X}_\delta = X_\delta \cup Y \cup H,$$

where $H = \{r = r_+ - \delta\}$ and $Y = \{s = 0\}$, recalling that $s = r^{-1}$ is a boundary defining function for $\mathcal{I}$. The crucial observation is that dr is timelike in the region $\{r_+ - \delta \leq r < r_+\}$.

Klein–Gordon equation

The main object of study is the Klein–Gordon equation

$$\left(\Box_g + \nu^2 - 9/4\right)\phi = 0 \text{ on } \mathcal{M}_\delta. \tag{4.1}$$

Consider the Fourier transformed operator

$$P(\lambda) = e^{i\lambda t^\star}\varrho^2(\Box_g + \nu^2 - 9/4)e^{-i\lambda t^\star}.$$

on X_δ. Up to a positive bounded multiple and a conjugation by a non-vanishing weight, this operator is the same as (2.14). If dS_t is the volume density induced on X_δ by the metric, then we define the density

$$\mu = \varrho^{-2}A\cdot dS_t, \quad A = g^{-1}(dt^\star, dt^\star)^{-1/2}.$$

If we define $\mathcal{L}^2(X_\delta)$ with respect to μ, then the formal adjoint of $P(\lambda)$ satisfies

$$P(\lambda)^* = P(\bar{\lambda}).$$

This follows from the relationship $\det g = A^2 \det h$, where h is the induced metric on X_δ, and the self-adjointness of $\Box_g$ with respect to the volume density on $\mathcal{M}_\delta$.

Function spaces

Observe that $\varrho^{-2}A \sim r^{-1}$ as $r \to \infty$, so $\mathcal{L}^2(X_\delta)$ is the space of distributions $u \in \mathcal{D}'(X_\delta)$ for which

$$\|u\|^2_{\mathcal{L}^2(X_\delta)} = \int_{X_\delta} |u|^2\, r^{-1}\, dS_t < \infty.$$

This is different than the L^2 spaces employed throughout Chapter 3, since we did not conjugate $\Box_g$ by $r^{(1-d)/2} = r^{-1}$. As usual, we will sometimes write $\mathcal{H}^0(X_\delta) = \mathcal{L}^2(X_\delta)$. This also means that we must modify the Sobolev spaces introduced in Chapter 3. To avoid adding additional notation, we redefine $\mathcal{H}^1(X_\delta)$ as the space of distributions $u \in \mathcal{D}'(X_\delta)$ for which

$$\|u\|^2_{\mathcal{H}^1(X_\delta)} = \int_{X_\delta} \left(\, |u|^2 + r^{2\nu-1}|d(r^{3/2-\nu}u)|^2_g \right)\, r^{-1}dS_t < \infty.$$

Similarly, we could redefine $\mathcal{H}^2(X_\delta)$ to account for the different measure. We also define $\mathcal{F}_\nu(X_\delta)$ as follows. If $0 < \nu < 1$, then $\mathcal{F}_\nu(X_\delta)$ is the space of all $u \in C^\infty(X_\delta)$ such that

1. u is smooth up to $H = \{r = r_+ - \delta\}$,

2. u has the form
$$u = r^{\nu-3/2}u_-(r^{-2}, y) + r^{-\nu-3/2}u_+(r^{-2}, y)$$
near $Y = X_\delta \cap \mathcal{I}$, where $u_\pm(s, y) \in C^\infty([0, \varepsilon)_s \times Y)$.

If $\nu \geq 1$, then we let $\mathcal{F}_\nu(X_\delta) = C_c^\infty(\overline{X}_\delta \setminus Y)$. In other words, these are functions smooth up to H and vanishing in a neighborhood of Y. Then $\mathcal{F}_\nu(X_\delta)$ is dense in $\mathcal{H}^s(X_\delta)$ for each $\nu > 0$ and $s = 0, 1, 2$.

When needed to avoid confusion, we also write $\overline{\mathcal{H}}^s(X_\delta) = \mathcal{H}^s(X_\delta)$ for $s = 0, 1, 2$. This notation emphasizes that elements of $\overline{\mathcal{H}}^s(X_\delta)$ are extendible as distributions across H, see [60, Appendix B.2]. This is in contrast to the space $\dot{\mathcal{H}}^s(X_\delta)$ for $s = 0, 1, 2$, which is the closure in $\overline{\mathcal{H}}^s(X_\delta)$ of those functions $u \in \mathcal{F}_\nu(X)$ which vanish to infinite order at H; the latter subspace will be denoted $\dot{\mathcal{F}}_\nu(X_\delta)$.

For duality purposes, we use the notation

$$\dot{\mathcal{H}}^{-s}(X_\delta) := \left[\overline{\mathcal{H}}^s(X_\delta)\right]', \quad \overline{\mathcal{H}}^{-s}(X_\delta) := \left[\dot{\mathcal{H}}^s(X_\delta)\right]'$$

for $s = 0, 1, 2$.

Suppose that X is a manifold without boundary containing $X_\delta \cup H$ as an open subset, such that $X \setminus X_\delta$ is compact. In other words X is a "compact" extension of X_δ across H, and thus $\overline{X} = X \cup Y$ is the type of manifold studied in Chapter 3. We can then concretely characterize $\overline{\mathcal{H}}^{-s}(X_\delta)$ for $s = 0, 1, 2$ as restrictions to $\dot{\mathcal{H}}^s(X_\delta)$ of elements in $\mathcal{H}^{-s}(X)$, viewing $\dot{\mathcal{H}}^s(X_\delta)$ as a closed subspace of $\mathcal{H}^s(X)$. On the other hand, any $f \in \dot{\mathcal{H}}^{-s}(X_\delta)$ is naturally an element of $\mathcal{H}^{-s}(X)$, since f can be paired with $u \in \mathcal{H}^s(X)$ by

$$\langle f, u \rangle_X := \langle f, u|_{X_\delta} \rangle_{X_\delta}.$$

In this sense f is supported on $X_\delta \cup H$, and the inclusion into $\mathcal{H}^{-s}(X)$ is an isometry.

We also have for $s = 0, 1, 2$ and $k \in \mathbb{N}$ the spaces $\mathcal{H}^{s,k}(X_\delta)$. Here, elements of $u \in \mathcal{H}^{s,k}(X_\delta)$ are stable under applications of any k many vector fields tangent to Y. Finally, introduce the space

$$\mathcal{X}^k = \{u \in \overline{\mathcal{H}}^{1,k}(X) : P(0)u \in \overline{\mathcal{H}}^{0,k}(X)\},$$

equipped with the norm $\|u\|_{\overline{\mathcal{H}}^{1,k}(X_\delta)} + \|P(0)u\|_{\overline{\mathcal{H}}^{0,k}(X_\delta)}$. This space is complete, and in fact $\overline{\mathcal{F}}_\nu(X_\delta)$ is dense in $\mathcal{X}$ according to Lemma 3.4.16.

For each $m \in \mathbb{Z}$ let $P(\lambda, m)$ denote the operator obtained from $P(\lambda)$ by replacing D_ϕ with m. Since D_ϕ is also Killing, $P(\lambda, m)$ preserves the space of distributions

$$\mathcal{D}'_m(X_\delta) = \{u \in \mathcal{D}'(X_\delta) : (\Phi - im)u = 0\}.$$

We therefore have $P(\lambda, m) : \mathcal{X}^k \cap \mathcal{D}'_m(X_\delta) \to \mathcal{H}^{0,k}(X_\delta) \cap \mathcal{D}'_m(X_\delta)$.

4.3 Microlocal study of $P(\lambda)$

The purpose of this section is to understand the microlocal structure of $P(\lambda)$. In the notation of Section 2.2, the homogeneous principal symbol of $P(\lambda)$ is

$$p_0(x, \xi) = -\varrho^2 G(x, \xi), \quad (x, \xi) \in T^*X.$$

Explicitly, we have

$$p_0 = \Delta_r \xi_r^2 + 2a(1-\alpha)\xi_r \xi_{\phi^\star} + \Delta_\theta {\xi_\theta}^2 + \frac{(1-\alpha)^2}{\Delta_\theta \sin^2\theta}\xi_{\phi^\star}^2, \tag{4.2}$$

where $\xi = \xi_r\, dr + \xi_\theta\, d\theta + \xi_{\phi^\star}\, d\phi^\star$.

Characteristic set

Since p_0 is homogeneous, its characteristic set $\{p_0 = 0\}$ is conic. We may therefore view it as a subset $\widehat{\Sigma}$ of the cosphere bundle $S^*X_\delta = (T^*X_\delta \setminus 0)/\mathbb{R}_+$, where $\mathbb{R}_+$ acts by positive dilations on the fibers. In the notation of Section 2.2, we have that $\widehat{\Sigma}$ is the image of

$$\Sigma \cap T^*X_\delta$$

under the projection $\kappa : T^*X_\delta \setminus 0 \to S^*X_\delta$. In addition, $\widehat{\Sigma}$ is divided into two components $\widehat{\Sigma}_\pm$, where

$$\widehat{\Sigma}_\pm = \kappa(\{\mp g^{-1}(\xi, dt) > 0\}).$$

According to Lemma 2.2.1, the projection of $\widehat{\Sigma}$ to the base X_δ does not intersect the region where T is timelike. For $\Delta_r > 0$, it is easily checked using Boyer–Lindquist coordinates that T is timelike provided

$$\Delta_r > a^2 \Delta_\theta \sin^2\theta,$$

at least away from the poles of $\mathbb{S}^2$. Changing to Cartesian coordinates, it is also easy to see that T is always timelike at the poles. In particular, $\widehat{\Sigma} \subseteq \{\Delta_r \le a^2\}$.

Null-geodesic flow

The analysis in this section closely follows [95, Section 6.3]. Let $\Lambda = N^*(\{r = r_+\}) \setminus 0 \subseteq T^*X_\delta \setminus 0$ denote the conormal bundle to $\{r = r_+\} \subseteq X_\delta$, less the zero section. Since $\xi_r \neq$ on Λ, we have the splitting

$$\Lambda_\pm = \{\Delta_r = 0;\ \xi_\theta = \xi_{\phi^\star} = 0;\ \pm\xi_r > 0\} \subset T^*X_\delta \setminus 0,$$

Let $L_\pm \subseteq S^*X$ denote the image of $\Lambda_\pm$ under the projection $\kappa : T^*X_\delta \setminus 0 \to S^*X_\delta$. From (4.2),

$$L_\pm \subseteq \widehat{\Sigma}_\pm,$$

If $\xi_r = 0$, then p_0 cannot vanish on $T^*X_\delta \setminus 0$, since then

$$p_1 = \Delta_\theta {\xi_\theta}^2 + \frac{(1-\alpha)^2}{\Delta_\theta \sin^2\theta}\xi_{\phi^\star}^2,$$

would vanish, implying that we are at the zero section. With this in mind, let us define

$$\widetilde{\Sigma}_\pm = \kappa(\{\xi_r > 0\}) \cap \widehat{\Sigma}.$$

It is clear that $\widehat{\Sigma}_\pm \cap L_\pm = \widetilde{\Sigma}_\pm \cap L_\pm$, but their relationship away from $L_\pm$ is not yet obvious.

Since p_0 is homogeneous of degree two, the rescaled Hamilton vector field $|\xi|^{-1}H_{p_0}$ is homogeneous of degree zero; here $|\cdot|$ is any norm on the fibers. Therefore $|\xi|^{-1}H_{p_0}$ is well defined on S^*X_δ, and its integral curves on S^*X_δ are reparametrizations of those of H_{p_0} on $T^*X_\delta \setminus 0$ projected onto S^*X_δ. Near $\widehat{\Sigma}$, we can also replace $|\xi|^{-1}H_{p_0}$ with $|\xi_r|^{-1}H_{p_0}$, and this again just reparametrizes the integral curves; furthermore, these two vector fields agree at $L_\pm$. The two sets $\widehat{\Sigma}_\pm$ are then invariant under the $|\xi|^{-1}H_{p_0}$ flow.

We now work near $\widehat{\Sigma}$. If $\rho = |\xi_r|^{-1}$ (which is homogeneous of degree -1), then $H_{p_0}\rho$ is homogeneous of degree zero, hence a function on S^*X_δ. We have

$$(H_{p_0}\rho)\,|_{\widetilde{\Sigma}_\pm} = \pm\Delta_r'(r).$$

We also have that $H_{p_0}p_1 = 0$ — indeed, p_1 is the well known Carter constant [18] (with the momentum dual to $t^\star$, also conserved under the flow, set to zero). Therefore

$$\left(|\xi_r|^{-1}H_{p_0}(\rho^2 p_1)\right)|_{\widetilde{\Sigma}_\pm} = \pm 2\Delta_r'(r)\rho^2 p_1. \tag{4.3}$$

Finally, observe that the vanishing of $\rho_1 := \rho^2 p_1$ within $\widetilde{\Sigma}_\pm$ defines $L_\pm$.

Lemma 4.3.1. *There exists a neighborhood $U_\pm$ of $L_\pm$ in $\widetilde{\Sigma}$ such that for each $(x,\xi) \in U_\pm$,*

$$\exp(\mp t|\xi|^{-1}H_{p_0})(x,\xi) \to L_\pm$$

as $t \to \infty$.

Proof. As noted above, the restriction of ρ_1 to a sufficently small neighborhood of $L_\pm$ within $\widetilde{\Sigma}_\pm$ vanishes precisely on $L_\pm$. It follows from (4.3) that flow lines of $|\xi|^{-1}H_{p_0}$ in a small neighborhood of $L_\pm$ within $\widetilde{\Sigma}_\pm$ converge to $L_\pm$ as $\mp t \to \infty$; this is because $\Delta_r'(r) > 0$ near $r = r_+$. □

For Lemma 4.3.1 to be useful, one needs a (mild) global nontrapping condition implying that all bicharacteristics starting at $\widetilde{\Sigma}$ either tend $L_\pm$ or otherwise reach $\{r = r_+ - \delta\}$ in appropriate time directions.

Lemma 4.3.2. *If $\gamma(t)$ is an integral curve of $|\xi|^{-1}H_{p_0}$ on S^*X, then the following hold.*

1. *If $\gamma(0) \in \widetilde{\Sigma}_\pm$, then $\gamma(\mp t) \to L_\pm$ as $t \to \infty$.*

2. *If $\gamma(0) \in \widehat{\Sigma} \cap \kappa(\{\pm\xi_r > 0\}) \setminus L_\pm$, then there exists $T > 0$ such that $\gamma(\pm T) \in \{r \le r - \delta\}$.*

Proof. (1) This statement is already implied by (4.3), since actually $\Delta'(r) > 0$ is bounded away from zero uniformly for $r \ge r_+ - \delta$.

(2) This follows from the same argument as in [95, Section 6.3]: recall that $\widehat{\Sigma}$ is contained in $\{r : -\delta < \Delta_r < (1+\varepsilon)a^2\}$, and so

$$((1+\varepsilon)a^2 - \Delta_r) \ge \frac{\varepsilon}{1+\varepsilon}\rho_1.$$

Combined with the first part, this shows that eventually $r \le r_+ - \delta$ along the flow. □

As a corollary, we can now see that $\widetilde{\Sigma}_\pm \subseteq \widehat{\Sigma}_\pm$. First, they are both invariant under the flow since $\widehat{\Sigma}$ is invariant. Now any flow line in $\widetilde{\Sigma}_\pm$ eventually enters an arbitrarily small neighborhood of $L_\pm$, where $\widehat{\Sigma}_\pm$ and $\widetilde{\Sigma}_\pm$ coincide. Therefore the flow line must be entirely contained in $\widehat{\Sigma}_\pm$.

For the density μ defined in Section 4.2, one has $P(\lambda)^* = P(\bar{\lambda})$, and

$$|\xi_r|^{-1}\sigma_1(\operatorname{Im} P(\lambda))|_{L_\pm} = \mp(1-\alpha)(r_+^2 + a^2)\operatorname{Im}\lambda = \kappa^{-1}(\operatorname{Im}\lambda)(H_{p_0}\rho)|_{L_\pm},$$

where $\kappa > 0$ is the surface gravity. This factorization of the subprincipal symbol at $L_\pm$ gives a threshold value for $\operatorname{Im}\lambda$ in the radial point estimates of Melrose [81], and adapted to this setting by Vasy [95]. The following microlocal result says we can propagate regularity away from $L_\pm$ along null-bicharacteristics provided one works with high regularity Sobolev spaces. See [95, Section 2.1] as well as [32, Appendix E] for a discussion of the microlocal notions used below.

Proposition 4.3.3 ([95, Proposition 2.3])**.** *Given a compactly supported $G \in \Psi^0(X_\delta)$ such that $L_\pm \subseteq \operatorname{ell}(G)$, there exists a compactly supported $A \in \Psi^0(X_\delta)$ such that $L_\pm \subseteq \operatorname{ell}(A)$ with the following properties.*

Suppose $u \in \mathcal{D}'(X_\delta)$ and $GP(\lambda)u \in H^{s-1}(X_\delta)$ for $s \geq m$, where $m > 1/2 - \kappa^{-1}\operatorname{Im}\lambda$. If there exists $A_1 \in \Psi^0(X)$ with $L_\pm \subseteq \operatorname{ell}(A_1)$ such that if $A_1 u \in H^m(X_\delta)$, then $Au \in H^s(X_\delta)$. Moreover, there exists $\chi \in C_c^\infty(X_\delta)$ such that

$$\|Au\|_{H^s(X_\delta)} \leq C\left(\|GP(\lambda)u\|_{H^{s-1}(X_\delta)} + \|\chi u\|_{H^{-N}(X_\delta)}\right)$$

for each N.

Similarly, there is a propagation result towards $L_\pm$ provided one works with sufficiently low regularity Sobolev norms.

Proposition 4.3.4 ([95, Proposition 2.4])**.** *Given a compactly supported $G \in \Psi^0(X_\delta)$ with $L_\pm \subseteq \operatorname{ell}(G)$, there exist compactly supported $A, B \in \Psi^0(X_\delta)$ such that $L_\pm \subseteq \operatorname{ell}(A)$ and $\operatorname{WF}(B) \subseteq \operatorname{ell}(G) \setminus L_\pm$, with the following properties.*

Suppose $u \in \mathcal{D}'(X_\delta)$ and $GP(\lambda)u \in H^{s-1}(X_\delta)$, $Bu \in H^s(X_\delta)$ for $s < 1/2 + \kappa^{-1}\operatorname{Im}\lambda$. Then $Au \in H^s(X_\delta)$, and moreover there exists $\chi \in C_c^\infty(X_\delta)$ such that

$$\|Au\|_{H^s(X_\delta)} \leq C\left(\|GP(\lambda)u\|_{H^{s-1}(X_\delta)} + \|Bu\|_{H^s(X_\delta)} + \|\chi u\|_{H^{-N}(X_\delta)}\right)$$

for each N.

Remark 16. *In the Kerr–de Sitter case, an additional restriction must be placed on a to ensure that the appropriate Δ_r in that case has derivative which is bounded away from zero in the region $\{\Delta_r \leq a^2\}$, see [95, Eq. 6.13]. This is needed to show the above nontrapping condition, which in turn is crucial to showing discreteness of QNFs. This does not present a problem here since $\partial_r \Delta_r$ is always strictly positive for $r \geq r_+ - \delta$.*

Energy estimates I

The next step is to estimate u near the artificial boundary H in terms of $P(\lambda)u$. This may be done by observing that $P(\lambda)$ is strictly hyperbolic with respect to the hypersurfaces $\{r = \text{constant}\}$ for $r \in (r_+ - 2\delta, r_+)$ for δ sufficiently small. Given $R_1 < R_2$, let $X_{[R_1,R_2]} = \{R_1 \le r \le R_2\}$

Lemma 4.3.5. *Fix $R_1 < R_2$ such that $r_+ - \delta < R_i < r_+$, and let $R_0 = r_+ - \delta$. Then*

$$\|u\|_{H^{k+1}(X_{[R_0,R_1]})} \le C\left(\|P(\lambda)u\|_{H^k(X_{[R_0,R_2]})} + \|u\|_{H^{k+1}(X_{[R_1,R_2]})}\right)$$

for each $u \in C^\infty(X_{[R_0,R_2]})$ and $k \in \mathbb{N}$.

Since $X_{[R_0,R_1]}$ in our application is compact, the H^s norms are well defined — note that these are norms on spaces of extendible H^s distributions [60, Appendix B.2]. Lemma 4.3.5 follows for example from the results of [60, Chapter 23]; a different proof can be found in [95, Proposition 3.8].

We do not make any claims about uniformity in λ (although this can also be arranged, see [95, Proposition 3.8]). Lemma 4.3.5 also holds for $P(\lambda)^* = P(\bar{\lambda})$, and we are also free to propagate in the opposite direction. In particular, suppose that $u \in C^\infty(X_{[R_0,R_1]})$ vanishes to infinite order at $r = R_0$. Then we may consider u as an element of $C^\infty(X_{[R_0-\delta,R_1]})$ with $\operatorname{supp} u \subseteq \{r \ge R_0\}$, and hence

$$\|u\|_{H^{k+1}(X_{[R_0,R_1]})} \le C\|P(\lambda)^*u\|_{H^k(X_{[R_0,R_1]})}$$

for each $k \in \mathbb{N}$.

Energy estimates II

We also use energy estimates to prove that $P(\lambda)$ is invertible in the upper half-plane. This is based on the divergence theorem

$$\partial_{t^\star} \int_{X_\delta} g(V, N_t)\, dS_t + \int_H g(V, N_r) A\, dS_H = \int_{X_\delta} (\operatorname{div}_g V)\, A\, dS_t, \tag{4.4}$$

provided that V is a sufficiently smooth vector field which has compact support in $\overline{X}_\delta \setminus Y$. Here dS_H is the induced measure on H, and N_r is outward pointing unit normal to H (which observe is timelike). As usual, $A = g^{-1}(dt^\star, dt^\star)^{-1/2}$. Note that both N_t and N_r are timelike, and they lie in the same lightcone on their common domain of definition.

The covariant stress-energy tensor $\mathbb{T} = \mathbb{T}[v]$ associated to the wave equation is

$$\mathbb{T}(Y, Z) = \operatorname{Re}(Yv \cdot Z\overline{v}) - \tfrac{1}{2} g(Y, Z) g^{-1}(dv, d\overline{v})$$

Here v is a sufficiently smooth function on $\mathcal{M}_\delta$ and Y, Z are real vector fields on $\mathcal{M}_\delta$. It is well known that $\mathbb{T}$ is positive definite in dv provided Y, Z are both timelike in the same lightcone [60, Lemma 24.1.2].

Given a real vector field Y, let $\mathbb{J}^Y = \mathbb{J}^Y[v]$ be the unique vector field such that $g(\mathbb{J}^Y, Z) = \mathbb{T}(Y, Z)$. Then

$$\operatorname{div}_g \mathbb{J}^Y = \operatorname{Re}\left((\Box_g v \cdot Y\overline{v}) + \mathbb{K}^Y(dv, \overline{dv}\right) \tag{4.5}$$

for some $(0,2)$ tensor $\mathbb{K}^Y$. In particular, if we let $F = (\Box_g + \nu^2 - 9/4)v$, then

$$\operatorname{div}_g \mathbb{J}^Y = \operatorname{Re}\left(F \cdot Y\overline{v}\right) + R,$$

where R is a quadratic form in (v, dv). Apply (4.4) to the vector field $\mathbb{J}^Y$, where we assume that v vanishes for r sufficiently large. This yields the identity

$$\partial_{t^\star} \int_{X_\delta} \mathbb{T}(Y, N_t)\, dS_t + \int_H \mathbb{T}(Y, N_r) A\, dS_H = \int_{X_\delta} \left(\operatorname{Re}\left(F \cdot Y\overline{v}\right) + R\right) A\, dS_t. \tag{4.6}$$

Now suppose that Y, Z are stationary in the sense that $[Y, T] = [Z, T] = 0$. Let d_λ denote the covector

$$d_\lambda u := \lambda u\, dt^\star + du,$$

where du is the differential of u on X_δ, and define $Y(\lambda)u := e^{i\lambda t} Y e^{-i\lambda t} u$. Then the stress-energy tensor associated to $e^{-i\lambda t^\star} u$ satisfies

$$e^{-2\operatorname{Im}\lambda t^\star} \mathbb{T}[e^{-i\lambda t^\star} u](Y, Z) = \operatorname{Re}\left(Y(\lambda)u \cdot \overline{Z(\lambda)u}\right) - \tfrac{1}{2} g(Y, Z) g^{-1}(d_\lambda u, \overline{d_\lambda u}\,)$$

and the right hand side is positive definite in $d_\lambda u$ if Y, Z are timelike in the same light cone.

On the other hand, if $v = e^{-i\lambda t^\star} u$ then the integrand on the right hand side of (4.6) can be written as

$$e^{2(\operatorname{Im}\lambda)t^\star} \left(\operatorname{Re}(\varrho^{-2} P(\lambda)u \cdot \overline{Y(\lambda)u}) + \widetilde{R}^Y\right),$$

where now $\widetilde{R}$ is a quadratic form in $(u, d_\lambda u)$.

Lemma 4.3.6. *Let $u \in C_c^2(X_\delta \cup H)$. There exists $C_0, C_1 > 0$ and $C > 0$ depending only on the support of u such that*

$$|\lambda| \|u\|_{L^2(X_\delta)} + \|du\|_{L^2(X_\delta)} \leq \frac{C}{\operatorname{Im}\lambda} \|P(\lambda)u\|_{L^2(X_\delta)}$$

for each $\lambda \in (C_0, \infty) + i(C_1, \infty)$.

Proof. Apply (4.6) with the multipler $Y = rN_t$ and $v = e^{-i\lambda t^\star} u$. The resulting identity is independent of $t^\star$ after multiplying by $e^{-2(\operatorname{Im}\lambda)t^\star}$. First, observe that the integral over H is nonnegative, since N_r and N_t are both timelike in the same lightcone. With $f = P(\lambda)u$ we have

$$2\operatorname{Im}\lambda \left(|\lambda|^2 \|u\|^2_{L^2(X_\delta)} + \|du\|^2_{L^2(X_\delta)}\right) \leq \int_{X_\delta} \left(\operatorname{Re}(f \cdot \varrho^{-2} r \cdot \overline{N_t(\lambda)u}) + \widetilde{R}\right) A\, dS_t.$$

Since u has compact support and therefore the asymptotics at infinity don't matter, we are writing $L^2(X_\delta)$ for its norm. Note that A is bounded by a constant depending on the size of $\operatorname{supp} u$, and hence the quadratic form $\widetilde{R}$ in $(u, d_\lambda u)$ can be absorbed into the left hand side for $\operatorname{Im}\lambda > 0$ and $|\lambda|$ both sufficiently large. On the right hand side, Cauchy–Schwarz gives

$$r|f \cdot \overline{N_t(\lambda)u}| < \frac{|f|^2}{2\varepsilon \operatorname{Im}\lambda} + \frac{\varepsilon \operatorname{Im}\lambda\, |rN_t(\lambda)u|^2}{2}$$

for $\operatorname{Im}\lambda > 0$, and the second term can be absorbed into the left hand side for $\varepsilon > 0$ sufficiently small. This finally gives

$$2\operatorname{Im}\lambda \left(|\lambda|^2 \|u\|^2_{L^2(X_\delta)} + \|du\|^2_{L^2(X_\delta)} \right) \leq \frac{C}{\operatorname{Im}\lambda} \|f\|^2_{L^2(X_\delta)}$$

as desired.

□

In Chapter 6 we will consider functions not vanishing for r large, hence we will need to be much more about the different r weights and measures. In the same way we obtain the following:

Lemma 4.3.7. *Let $u \in C_c^2(\overline{X}_\delta \setminus Y)$ such that $u|_H = 0$. There exists $C_0, C_1 > 0$ and $C > 0$ depending only on the support of u such that*

$$|\lambda| \|u\|_{L^2(X_\delta)} + \|du\|_{L^2(X_\delta)} \leq \frac{C}{\operatorname{Im}\lambda} \|P(\lambda)^* u\|_{L^2(X_\delta)}$$

for each $\lambda \in (C_0, \infty) + i(C_1, \infty)$.

Proof. Since $P(\lambda)^* = P(\bar{\lambda})$, we now apply (4.6) to $v = e^{-i\bar{\lambda}t^\star} u$ with the multiplier $Y = rN_t$. Since $\operatorname{Im}\bar{\lambda} = -\operatorname{Im}\lambda$, the two integrals on the left hand side of (4.6) have opposite signs for $\operatorname{Im}\lambda > 0$. However, if u vanishes at H, then the same argument as in Lemma 4.3.6 applies. □

4.4 The anti-de Sitter end

Near the conformal boundary we use the elliptic theory developed in Chapter 3. By a slight abuse of notation we refer to $P(\lambda)$ as Bessel operator even though we did not conjugate $P(\lambda)$ to the precise form in Section 3.1.

Lemma 4.4.1. *$P(\lambda)$ is a parameter-elliptic Bessel operator with respect to any angular sector $\Lambda \subseteq \mathbb{C}$ disjoint from $\mathbb{R} \setminus 0$.*

Proof. This follows from the timelike nature of T and $dt^\star$ at Y, as discussed in Section 3.2. □

When $0 < \nu < 1$, the operator $P(\lambda)$ must be augmented by elliptic boundary conditions. Thus assume that $T(\lambda)$ is a parameter-dependent boundary operator of the form

$$T(\lambda) = (T_1^- + \lambda T_0^-)\gamma_- + T_1^+\gamma_+,$$

where the weighted restriction $\gamma_\pm$ are given by

$$\gamma_- u = \lim_{r\to\infty} r^{3/2-\nu} u, \quad \gamma_+ u = \lim_{r\to\infty} r^{2\nu+1}\partial_r(r^{3/2-\nu}u). \tag{4.7}$$

Here $\gamma_\pm$ are redefined from Chapter 3 since we did not conjugate our operator by r^{-1}. It is assumed that the "principal part" of $T(\lambda)$ (in the sense of Section 3.4) is independent of λ. We furthermore assume that

$$\mathscr{P}(\lambda) = \begin{pmatrix} P(\lambda) \\ T(\lambda) \end{pmatrix}$$

is parameter-elliptic with respect to an angular sector Λ contained in $\mathbb{C}_+$.

4.5 Fredholm property and meromorphy

In this section, the Fredholm property for $P(\lambda)$ and meromorphy of $P(\lambda)^{-1}$ is derived from estimates on $P(\lambda)$, combined with some standard arguments from functional analysis. Of course $P(\lambda)$ should be replaced by $\mathscr{P}(\lambda)$ when $0 < \nu < 1$. For a brief review of the microlocal notions used in this section, see [95, Section 2] and also [32, Appendix E] for the semiclassical perspective. More thorough expositions can be found in [19, 60, 87].

The case $\nu \geq 1$

The simpler case $\nu \geq 1$ is considered first. Our first task is to prove that

$$P(\lambda) : \mathcal{X}^k \to \mathcal{H}^{0,k}(X_\delta)$$

has closed range and finite dimensional kernel for each $k \in \mathbb{N}$, provided $\operatorname{Im}\lambda$ lies in an appropriate half-plane.

Proposition 4.5.1. *If $C_0 < \kappa(k + 1/2)$, then there exists a constant $C = C(\lambda) > 0$, a compactly supported function $\chi \in C_c^\infty(X_\delta)$, and $\varphi \in C^\infty(\overline{X}_\delta)$ supported arbitrarily close Y such that*

$$\|u\|_{\overline{\mathcal{H}}^{1,k}(X)} \leq C\left(\|P(\lambda)u\|_{\overline{\mathcal{H}}^{0,k}(X)} + \|\chi u\|_{H^{-N}(X)} + \|\varphi u\|_{\overline{\mathcal{H}}^{0,k}(X)}\right) \tag{4.8}$$

for any N and $u \in \mathcal{F}_\nu(X_\delta)$, provided $\operatorname{Im}\lambda > -C_0$.

Proof. Let $u \in \mathcal{F}_\nu(X)$ and $f := P(\lambda)u$. Begin by choosing two functions $\zeta, \psi \in C^\infty(\overline{X}; [0,1])$ subject to the following conditions:

1. $\operatorname{supp}\psi \subseteq \{0 \le s < \delta'\}$ and $\psi = 1$ near $\{0 \le s < \delta\}$, where $0 < \delta' < \delta$ and δ is sufficiently small (recall that $s = r^{-1}$)
2. $\operatorname{supp}\zeta \subseteq \{r_+ - \delta \le r < r_+ - 2\delta/3\}$ and $\zeta = 1$ near $\{r_+ - \delta \le r < r_+ - 3\delta/4\}$.

It is possible to find a microlocal partition of unity

$$1 = \zeta + \psi + \sum_{j=1}^{J} A_j + R,$$

where the operators $A_j \in \Psi^0(X_\delta),\ R \in \Psi^\infty(X_\delta)$ are compactly supported, and each $A \in \{A_1, \ldots, A_J\}$ has one of the following properties:

1. $\mathrm{WF}(A) \subseteq \mathrm{ell}(P(\lambda))$. By microlocal elliptic regularity,

   $$\|Au\|_{H^{s+1}(X_\delta)} \le C\|Gf\|_{H^{s-1}(X_\delta)} + \|\chi u\|_{H^{-N}(X_\delta)}$$

   for G microlocalized near $\mathrm{WF}(A)$ and some $\chi \in C_c^\infty(X_\delta)$.

2. $\mathrm{WF}(A)$ is contained in a small neighborhood of $L_\pm$. In order to apply Proposition 4.3.3, the imaginary part of λ must satisfy $\operatorname{Im}\lambda > \kappa(\frac{1}{2} - s)$. In that case,

   $$\|Au\|_{H^s(X)} \le C\left(\|Gf\|_{H^{s-1}(X)} + \|\chi u\|_{H^{-N}(X)}\right)$$

   for some G microlocalized near $\mathrm{WF}(A)$ and some $\chi \in C_c^\infty(X_\delta)$.

3. $\mathrm{WF}(A)$ is contained a neighborhood of a point $(x_0, \xi_0) \in \widehat{\Sigma}_+ \setminus L_+$. By shrinking $\mathrm{WF}(A)$ if necessary, for any neighborhood $U_+ \supseteq L_+$ there exists $T > 0$ such that

   $$\exp(-T|\xi|^{-1}H_{p_0})(\mathrm{WF}(A)) \subseteq \mathrm{ell}(B),$$

   for any $B \in \Psi_0(X)$ such that $\mathrm{WF}(B) \subseteq U_+$ — this is Lemma 4.3.2. It is now possible to combine propagation of singularities ([95, Section 2.3]) with the previous item (2). For some G_1 microlocalized near the set

   $$\bigcup_{t\in[0,T]} \exp(t|\xi|^{-1}H_{p_0})(\mathrm{WF}(A))$$

   and G as in (2),

   $$\|Au\|_{H^s(X_\delta)} \le C\left(\|Gf\|_{H^{s-1}(X_\delta)} + \|G_1 f\|_{H^{s-1}(X_\delta)} + \|\chi u\|_{H^{-N}(X_\delta)}\right)$$

   for some $\chi \in C_c^\infty(X_\delta)$. The same argument applies if $(x_0, \xi_0) \in \widehat{\Sigma}_- \setminus L_-$, making sure to reverse the direction of the flow (by considering $-P(\lambda)$) and noting that the sign of the subprincipal is also reversed.

The estimates on Au are applied with Sobolev index $s = 1+k$ where $k \in \mathbb{N}$. Thus $C_0 > 0$ is subject to the condition $C_0 < \kappa(k+1/2)$. The term ψu is then estimated in $\overline{\mathcal{H}}^{1,k}(X)$ using Theorem 1, provided $\delta > 0$ is sufficiently small. In the region where $r < r_+$ we apply Lemma 4.3.5. We can estimate

$$\|\zeta u\|_{\overline{H}^{k+1}(X_\delta)} \leq C \left(\|P(\lambda)u\|_{\overline{\mathcal{H}}^{0,k}(X_\delta)} + \|\zeta' u\|_{H^{k+1}(X_\delta)} \right),$$

where ζ' has compact support in $\{r_+ - \delta/2 < r < 0\}$. In particular,

$$\zeta'\zeta = \zeta\psi = 0,$$

and hence $A_1 + \cdots + A_J = 1$ on $\operatorname{supp} \zeta'$ modulo the compactly supported smoothing operator R. Therefore $\zeta' u$ is controlled by the $A_j u$ terms handled above. □

Proposition 4.5.1 implies that $P(\lambda) : \mathcal{X}^k \to \mathcal{H}^{0,k}(X_\delta)$ has closed range and finite-dimensional kernel for $\operatorname{Im} \lambda > -C_0 > -\kappa(1/2+k)$. This can be seen from the same argument as in Lemma 3.5.4, using the density of $\overline{\mathcal{F}}_\nu(X_\delta)$ in $\mathcal{H}^{0,k}(X_\delta)$ — see Lemma 3.5.1

Since $\lambda \mapsto P(\lambda)$ is norm continuous, it now suffices to prove that $P(\lambda_0)$ is actually invertible on $\mathcal{X}^k$ for some $\lambda_0 \in \mathbb{C}$. This will show that $P(\lambda)$ is Fredholm of index zero in the half-plane $\operatorname{Im} \lambda > \kappa(1/2 + k)$.

Lemma 4.5.2. *There exists $\lambda_0 \in \mathbb{C}$ such that $P(\lambda_0) : \mathcal{X}^0 \to \mathcal{L}^2(X_\delta)$ is invertible.*

Proof. (1) If $\operatorname{Im} \lambda > -\kappa/2$, then $u \in \mathcal{X}^0$ and $P(\lambda)u = 0$ implies that $u \in C^\infty(\overline{X}_\delta \setminus Y)$; this follows from the analysis at $L_\pm$ and propagation of singularities. Fix $\varphi \in C_c^\infty(\overline{X}_\delta \setminus Y)$ such that $\varphi = 1$ near $\{\Delta_r < a^2\}$. Then there exists $C_0, C_1 > 0$ and $C > 0$ such that whenever $|\operatorname{Re} \lambda| > C_0$, $\operatorname{Im} \lambda > C_1$ and $P(\lambda)u = 0$, then

$$|\lambda| \|\varphi u\|_{L^2(X_\delta)} + \|d(\varphi u)\|_{L^2(X_\delta)} \leq \frac{C}{\operatorname{Im} \lambda} \|[P(\lambda), \varphi]u\|_{L^2(X_\delta)}.$$

This follows from Lemma 4.3.6, since $P(\lambda)\varphi u = [P(\lambda), \varphi]u$.

Next, observe that the commutator is supported in the region where T is timelike. Therefore $P(\lambda)$ is parameter-elliptic with respect to any angular sector disjoint from $\mathbb{R}\setminus 0$ according to Lemma 2.2.2. Either from the results of [87, Section 9.3] or the closely related semiclassical formulation [110, Theorem E.32], we obtain

$$\|[P(\lambda), \varphi]u\|_{L^2(X_\delta)} \leq C_2 \|\psi u\|_{L^2(X_\delta)}$$

uniformly in some sector intersecting $|\operatorname{Re} \lambda| > C_0, \operatorname{Im} \lambda > C_1$, where $\psi = 1$ near $\operatorname{supp} d\varphi$. On the other hand, if $\operatorname{supp} \varphi$ is sufficiently large, then since $P(\lambda)$ is a parameter-elliptic Bessel operator at Y,

$$|\lambda| \|(1-\varphi)u\|_{\mathcal{L}^2(X_\delta)} + \|(1-\varphi)u\|_{\mathcal{H}^1(X_\delta)} \leq C_3 \|\psi' u\|_{\mathcal{L}^2(X_\delta)}$$

in the same sector, where ψ' is supported near Y. Combining these two estimates shows that $P(\lambda)$ is injective on $\mathcal{X}^0$ for $|\operatorname{Re}\lambda|$ and $\operatorname{Im}\lambda$ sufficiently positive.

(2) The same type of argument also applies to the adjoint

$$P(\lambda)^* : \mathcal{L}^2(X_\delta) \to \dot{\mathcal{H}}^{-2}(X_\delta).$$

Suppose that $P(\lambda)^* u = 0$. Let $X = (r_+ - 2\delta) \times \mathbb{S}^2$, and extend u by zero to $\tilde{u} \in \mathcal{L}^2(X_\delta)$. Now $P(\lambda)^*$ is still defined on X, and $P(\lambda)^*\tilde{u} = 0$ in distributions on X. Since $\tilde{u}$ vanishes for $r < r_+ - \delta$, propagation of singularities and the analysis at $L_\pm$ allows us to conclude that $\tilde{u} \in H^s_{\mathrm{loc}}(X)$ for any fixed s provided $\operatorname{Im}\lambda > 0$ is sufficiently large. Since $\operatorname{supp}\tilde{u} \subseteq \{r \geq r_+ - \delta\}$, we can apply Lemma 4.3.7 and Theorem 1 as in the first part of the proof to conclude that $u = 0$.

□

We are now ready to prove Theorem 6

Proof of Theorem 6. Write $P^{(k)}(\lambda)$ for the operator

$$P(\lambda) : \mathcal{X}^k \to \overline{\mathcal{H}}^{0,k}(X).$$

Proposition 4.5.1 shows that $P^{(k)}(\lambda)$ has closed range and finite dimensional kernel in the half-plane $\operatorname{Im}\lambda > \kappa(k + 1/2)$ for any $k \in \mathbb{N}$. According to Lemma 4.5.2, we can choose λ_0 with sufficiently large imaginary part so that $P^{(0)}(\lambda)$ is invertible. Clearly injectivity of $P^{(0)}(\lambda_0)$ implies injectivity of $P^{(k)}(\lambda_0)$. Furthermore, suppose that $f \in \overline{\mathcal{H}}^{0,k}(X) \subseteq \overline{\mathcal{H}}^{0}(X)$. Let $u \in \mathcal{X}^0$ denote the unique solution to

$$P(\lambda_0)u = f.$$

The claim is that actually $u \in \mathcal{X}^k$. This is proved in a local fashion similar to Proposition 4.5.1. Near Y, the elliptic regularity in [40, Theorem 3] implies u is locally in $\mathcal{H}^{1,k}(X)$. At elliptic points in the interior X, it suffices to apply standard elliptic regularity. Next, since $u \in H^1$ microlocally near $L_\pm$ and $\operatorname{Im}\lambda_0 > 0$, the threshhold condition in Proposition 4.3.3 is satisfied; thus u is in H^{1+k} microlocally near $L_\pm$. This regularity is then propagated along null bicharacteristics using the nontrapping condition 4.3.2.

This shows that $P^{(k)}(\lambda)$ is invertible at $\lambda = \lambda_0$, hence of index 0. On the other hand the index of left semi-Fredholm operators (namely those with closed range and finite dimensional kernel) is constant on connected components, noting that the index may take the value $-\infty$. This implies that $P^{(k)}(\lambda)$ is Fredholm of index zero provided $\operatorname{Im}\lambda > -\kappa(k + 1/2)$, and is invertible sufficiently far up in the upper half-plane.

□

The case $0 < \nu < 1$

Fix a boundary operator $T(\lambda)$ as in Section 4.4 such that

$$\mathscr{P}(\lambda) = \begin{pmatrix} P(\lambda) \\ B(\lambda) \end{pmatrix}$$

is elliptic with respect to an angular sector $\Lambda \subseteq \mathbb{C}$ disjoint from $\mathbb{R} \setminus 0$. Assume that the principal part of $T(\lambda)$ is independent of λ.

Proof of Theorem 7. Proposition 4.5.1 has a natural analogues in this setting: the microlocal estimates on X_δ and hyperbolic estimate near H are unchanged. Near Y we apply Theorem 2 for the case $0 < \nu < 1$. Invertibility of $\mathscr{P}(\lambda)$ for $k = 0$ follows as in Lemma 4.5.2, and the same argument as in the proof of Theorem 6 handles larger values of k. □

4.6 Proof of Theorem 8

Theorem 8 is a corollary of the following proposition.

Proposition 4.6.1. *Let $m \in \mathbb{Z}$. Given $f \in C^\infty(X_\delta) \cap \mathcal{D}'_m(X_\delta)$ such that* $\operatorname{supp} f \subseteq X_\delta \setminus X_0$*, there exists a unique solution to the problem*

$$P(\lambda)u = f, \quad \operatorname{supp} u \subseteq X_\delta \setminus X_0,$$

such that $u \in C^\infty(X_\delta) \cap \mathcal{D}'_m(X_\delta)$.

Delaying the proof of Proposition 4.6.1 for a moment, Theorem 8 is now established by precisely the same argument as [53, Lemma 2.1]:

Proof of Theorem 8. See also [53, Lemma 2.1]. First suppose that $\nu \geq 1$. If λ_0 is a pole of $P(\lambda)^{-1}$, then there exists $m \in \mathbb{Z}$ and a nonzero $v \in C^\infty(X_\delta) \cap \mathcal{H}^0(X_\delta) \cap \mathcal{D}'_m(X_\delta)$ such that $P(\lambda_0)v = 0$. The restriction of v to X_0 is nonzero, since otherwise v would be supported in $X_\delta \setminus X_0$ which implies $v = 0$ according to Proposition 4.6.1. Thus $u := v|_{X_0}$ is nonzero and $P_0(\lambda_0)u = 0$.

Conversely, assume that λ_0 is not a pole of $P(\lambda)$. Suppose that there exists nonzero $u \in C^\infty(\overline{X}_0) \cap \mathcal{H}^0(X_0) \cap \mathcal{D}'_m(X_0)$ such that $P_0(\lambda_0)u = 0$. Extend u arbitrarily to X as an element $\widetilde{u} \in C^\infty(X_\delta) \cap \mathcal{H}^0(X_\delta) \cap \mathcal{D}'_m(X_\delta)$; according to Proposition 4.6.1, the equation $P(\lambda_0)v = P(\lambda_0)\widetilde{u}$ has a solution $v \in C^\infty(X_\delta) \cap \mathcal{D}'_m(X_\delta)$ such that $\operatorname{supp} v \subseteq X_\delta \setminus X_0$. Then $\widetilde{u} - v$ is nonzero and $P(\lambda)(\widetilde{u} - v) = 0$, which is contradiction.

The same argument applies when $0 < \nu < 1$ since $T(\lambda)$ is axisymmetric, replacing $P(\lambda)$ with $\mathscr{P}(\lambda)$. □

Although Proposition 4.6.1 is closely related to the results of [97] on asymptotically de-Sitter spacetimes, a direct proof is outlined here — see also [110, Lemma 1] for the same type of result (at least for the uniqueness part).

Define the Riemannian metric

$$h = \frac{1}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{(1-\alpha)^2} d\phi^2,$$

which extends smoothly across the poles to $\mathbb{S}^2$. Let d_y denote the differential on $\mathbb{S}^2$ and $|d_y u|_h$ the magnitude of d_y with respect to h. The idea is to apply an energy identity in the region where $\Delta_r < 0$. First define $\rho = r_+ - r$, which is positive in that region. Then for any $N \in \mathbb{R}$ and $u \in C^\infty(X)$

$$\begin{aligned}\partial_\rho\left(\rho^N\left(-\Delta_r|\partial_\rho u|^2 + |d_y u|_h^2\right)\right) &= 2\rho^N \operatorname{Re}\left(\partial_\rho \bar{u}\left(-\Delta_r \partial_\rho^2 u\right) + h^{-1}(d_y\partial_\rho u, d_y \bar{u})\right) \\ &\quad + N\rho^{N-1}\left(-\Delta_r|\partial_\rho u|^2 + |d_y u|_h^2\right) \\ &\quad + \rho^N R(u, du),\end{aligned}$$

where $R(u, du)$ is a quadratic form in (u, du) which is independent of N (at this stage $R(u, du)$ is just $-(\partial_r\Delta_r)|\partial_\rho u|^2$). Given $0 < \varepsilon < \delta$, integrate over the region $[\varepsilon, \delta]_\rho \times \mathbb{S}^2$ and apply Green's theorem to obtain

$$\begin{aligned}\rho^N E(\delta) - \rho^N E(\varepsilon) = 2\int_{[\varepsilon,\delta]_\rho\times\mathbb{S}^2} &\rho^N \operatorname{Re}\left(\partial_\rho\bar{u}\left(-\Delta_r\partial_\rho^2 u + \Delta_h u\right)\right) d\rho\, dh \\ &+ N\int_\varepsilon^\delta \rho^{N-1}E(\rho)\, d\rho + \int_{[\varepsilon,\delta]_\rho\times\mathbb{S}^2} \rho^N R(u, du)\, d\rho\, dh,\end{aligned}$$

where

$$E(\rho) = \int_{\mathbb{S}^2}\left(-\Delta_r|\partial_\rho u|^2 + |d_y u|_h^2\right)\, dh.$$

In general, $-\Delta_r\partial_\rho^2 + \Delta_h$ differs from $-P(\lambda)$ by a second order operator. On the other hand, after restricting to $\mathcal{D}'_m(X)$ this difference is of first order and can be absorbed in $R(u, du)$. Thus

$$\begin{aligned}\rho^N E(\delta) - \rho^N E(\varepsilon) = -2\int_{[\varepsilon,\delta]_\rho\times\mathbb{S}^2} &\rho^N \operatorname{Re}\left(\partial_\rho\bar{u}\, P(\lambda)u\right) d\rho\, dh \\ &+ N\int_\varepsilon^\delta \rho^{N-1}E(\rho)\, d\rho + \int_{[\varepsilon,\delta]_\rho\times\mathbb{S}^2} \rho^N R(u, du)\, d\rho\, dh \quad (4.9)\end{aligned}$$

for each $u \in C^\infty(X_\delta) \cap \mathcal{D}'_m(X_\delta)$, where now $R(u, du)$ is a real quadratic form in (u, du) which depends on λ and m.

Proof of Proposition 4.6.1. To prove the uniqueness statement, suppose $u \in C^\infty(X_\delta) \cap \mathcal{D}'_m(X_\delta)$ satisfies $P(\lambda)u = 0$ and $\operatorname{supp} u \subseteq X_\delta \setminus X_0$. Apply (4.9) with N large and negative. Since Δ_r vanishes to first order at $\mathcal{H}^+$, the sum of the last two integrals is nonpositive for N sufficiently negative. Furthermore, $\rho^N E(\varepsilon)$ tends to zero as $\varepsilon \to 0$ for any N since u

vanishes to infinite order at $\mathcal{H}^+$. Again by the nonnegativity of $E(\delta)$ this gives $E(\delta) = 0$ for each $\delta > 0$. Since $E(\rho)$ controls $\|u(\rho,\cdot)\|_{H^1(\mathbb{S}^2)}$, this implies $u = 0$ by Poincaré inequality.

For the existence part of the proof, note that the adjoint of $P(\lambda)$ with respect to $d\rho\, dh$ is $P(\bar{\lambda})$, so (4.9) also applies to $P(\lambda)^*$. The error (the quadratic form R) can still be dominated by $N\rho^{N-1}E(\rho)$, but now the sum of the last two terms in (4.9) is nonnegative. In particular, assume that $v \in C^\infty(X_\delta) \cap \mathcal{D}'_m(X_\delta)$ satisfies $\operatorname{supp} v \subseteq \{\rho < \delta/2\}$ for some $\delta > 0$ fixed. Then $E(\delta) = 0$ in (4.9) while $\rho^N E(\varepsilon) \to 0$ as $\varepsilon \to 0$ in light of the ρ^N factor. Combined with Cauchy–Schwarz and Poincaré inequality,this implies

$$N\int_0^\delta \rho^{N-1}\|v(\rho,\cdot)\|^2_{H^1(\mathbb{S}^2)}\, d\rho \leq C\int_0^\delta \rho^N\|P(\lambda)^* v(\rho,\cdot)\|^2_{H^0(\mathbb{S}^2)}\, d\rho$$

for $N > 0$ sufficiently large. Furthermore, by commuting with an elliptic pseudodifferential operator on $\mathbb{S}^2$ of negative order and absorbing the commutator into the left hand side by possibly increasing N,

$$N\int_0^\delta \rho^{N-1}\|v(\rho,\cdot)\|^2_{H^{-s+1}(\mathbb{S}^2)}\, d\rho \leq C\int_0^\delta \rho^N\|P(\lambda)^* v(\rho,\cdot)\|^2_{H^{-s}(\mathbb{S}^2)}\, d\rho. \tag{4.10}$$

Thus N depends on λ, m, and s.

Now suppose that $f \in C^\infty(X_\delta \setminus X_0) \cap \mathcal{D}'_m(X_\delta \setminus X_0)$ vanishes to infinite order at $r = r_+$, so in particular

$$f \in \rho^{(K-1)/2}L^2((0,\delta); H^s(\mathbb{S}^2)) \cap \mathcal{D}'_m(X_\delta \setminus \overline{X}_0)$$

for each $K > 0$ and $s \in \mathbb{R}$. Define the form ℓ mapping

$$\ell : P(\lambda)^* v \mapsto \langle f, v\rangle_{L^2((0,\delta)\times\mathbb{S}^2)}$$

where $v \in C^\infty(X_\delta \setminus X_0) \cap \mathcal{D}'_m(X_\delta \setminus \overline{X}_0)$ and $\rho < \delta/2$ on the support of v. The estimate (4.10) shows that ℓ is bounded on the set of all such $P(\lambda)^* v$. Then Hahn-Banach and the Riesz representation imply the existence $u \in \rho^{K/2}L^2((0,\delta); H^s(\mathbb{S}^2)) \cap \mathcal{D}'_m(X \setminus \overline{X}_+)$ such that

$$\langle f, v\rangle_{L^2(0,\delta)\times\mathbb{S}^2} = \langle u, P(\lambda)^* v\rangle$$

where the pairing on the right is duality between

$$\rho^{K/2}L^2((0,\delta); H^s(\mathbb{S}^2)) \cap \mathcal{D}'_m(X_\delta \setminus \overline{X}_0) \Longleftrightarrow \rho^{-K/2}L^2((0,\delta); H^{-s}(\mathbb{S}^2)) \cap \mathcal{D}'_m(X_\delta \setminus \overline{X}_0).$$

and v is as above. In particular $P(\lambda)u = f$ in $\mathcal{D}'_m(X \setminus \overline{X}_+)$.

Since K and s are arbitrary, this implies the existence of a solution u such that

$$u \in \rho^N L^2((0,\delta); C^\infty(\mathbb{S}^2)) \cap \mathcal{D}'_m(X_\delta \setminus \overline{X}_0)$$

for each N. The Sobolev regularity of u in the ρ variable now follows from the usual "partial hypoellipticity at the boundary" argument (using the high order of vanishing of u and f to account for the derivatives in the ρ variable which degenerate at the boundary), see [60, Theorem B.2.9]. Once a sufficiently regular solutions exists with N larger than some threshhold value, we conclude it is unique by the energy estimates for $P(\lambda)$; therefore u is in fact smooth and vanishes to infinite order at $r = r_+$ □

Chapter 5

Quasimodes

In this chapter we prove the existence of quasimodes for the Klein–Gordon equation on a Schwarzschild–AdS spacetime (with Dirichlet boundary conditions at $\mathcal{I}$). Roughly speaking, these quasimodes consist of a sequence of real frequencies $\lambda_\ell \in \mathbb{R}$ (tending to infinity) functions

$$u_\ell \in \mathcal{H}^{2,\infty}(X_\delta), \quad \gamma_- u_\ell = 0$$

such that if

$$f_\ell := P(\lambda_\ell) u_\ell$$

then some norms of f_ℓ decay at a rate as $\ell \to \infty$. In the Schwarzschild–AdS setting, we can take $\|f_\ell\|_{\mathcal{L}^2(X_\delta)} = \mathcal{O}(e^{-\ell/C})$. The functions u_ℓ are also localized in space, namely there exists $R > r_+$ such that $\operatorname{supp} u_\ell \subseteq \{r > R\}$. For a more precise statement, see Theorem 9 of this chapter.

A classic argument due to Ralston [83] shows that in the presence of quasimodes satisfying

$$\operatorname{supp} u_\ell \subseteq \{r > R\}, \quad \|u_\ell\|_{\mathcal{L}^2(X_\delta)} = 1, \quad \|f_\ell\|_{\mathcal{L}^2(X_\delta)} \to 0 \text{ as } \ell \to \infty$$

there is no uniform local energy decay. This means that for each function $p(t) \to 0$ as $t \to \infty$, there is no $t_0 \geq 0$ such that

$$\begin{aligned} \|\phi(t^\star, x)\|_{\mathcal{H}^1(\{r>R\})} + \|\partial_{t^\star}\phi(t^\star, x)\|_{\mathcal{L}^2(\{r>R\})} \\ \leq p(t)\left(\|\phi(0,x)\|_{\mathcal{H}^1(X_\delta)} + \|\partial_{t^\star}\phi(0,x)\|_{\mathcal{L}^2(X_\delta)}\right) \end{aligned} \tag{5.1}$$

for all $t^\star \geq t_0$ and all solutions ϕ to the Klein–Gordon equation (whith Dirichlet boundary conditions at $\mathcal{I}$). Now suppose that f_ℓ decays exponentially. A more refined result shows that even by losing k additional derivatives on the right hand side of (5.1), no uniform decay rate of the form $p(t)\log(t^\star)^{-k}$ is possible. For a proof of this in the Kerr–AdS setting, see [56]. In that paper, an independent construction of exponentially quasimodes is given for Kerr–AdS black holes (rather than just the non-rotating case studied here). The basic idea behind the proof is that by Duhamel's formula, quasimodes approximate actual solutions to the Klein–Gordon equation up to times $t^\star \sim e^\ell$. We should also remark that a logarithmic decay rate (with loss of derivatives) has indeed been established for slowly rotating Kerr–AdS spacetimes [55].

The construction of quasimodes is motivated by the existence of a potential well near $\mathcal{I}$ separated from the black hole horizon by a barrier. We consider a related problem supporting bound states by imposing an additional Dirichlet boundary condition in the barrier. By systematically employing the exponential decay of these states in the barrier, we construct quasimodes for the original problem close to the bound states.

We prove the existence of bounds states near the bottom of the well using the harmonic approximation (see Section 5.2). This consists of identifying the Schrödinger operator as a harmonic oscillator plus a perturbation. Although the perturbation is not globally small, we can make use of the semiclassical concentration of these eigenvalues to the bottom of the well. We also give a full asymptotic expansion in powers of $\ell^{-1/2}$ for these eigenvalues; the coefficients in the expansion are ordinary Rayleigh–Schrödinger coefficients. Finally, we address a conjecture of Dias et al. [26] on the vanishing of certain coefficients in this expansion.

5.1 Harmonic oscillator

In this section we collect some useful facts about the operator

$$H = |D_\nu|^2 + x^2 \tag{5.2}$$

on $\mathbb{R}_+$. In analogy with the smooth setting, we will refer to H as the harmonic oscillator.

Maximal domain

To discuss its spectral properties, we need to fix a self-adjoint realization of H. We begin by characterizing its maximal realization $H_{\max}$, which acts on

$$D(H_{\max}) = \{u \in L^2(\mathbb{R}_+) : Hu \in L^2(\mathbb{R}_+)\}$$

in the sense of distributions on $\mathbb{R}_+$. To decouple the behavior near infinity and the origin, we can also consider the standard harmonic oscillator on $\mathbb{R}$,

$$\mathfrak{H} = D_x^2 + x^2.$$

It is a basic fact that $\mathfrak{H}$ is self-adjoint on the domain

$$D(\mathfrak{H}) = \{u \in L^2(\mathbb{R}) : \mathfrak{H}u \in L^2(\mathbb{R})\},$$

and $(\mathfrak{H}, D(\mathfrak{H}))$ satsifies the following properties.

Lemma 5.1.1 ([52]). *There is equality*

$$D(\mathfrak{H}) = \langle x\rangle^{-2} L^2(\mathbb{R}) \cap H^2(\mathbb{R}).$$

Furthermore, if $f \in \mathcal{S}(\mathbb{R})$ and $u \in D(\mathfrak{H})$ solves $(\mathfrak{H} - E)u = f$, then $u \in \mathcal{S}$.

Now suppose that $u \in D(H_{\max})$. Observe that by elliptic regularity, $u \in \mathcal{H}^2(\Omega)$ for any bounded interval $\Omega = (0, a)$. Choose a smooth cutoff φ on $\mathbb{R}_+$ such that $\varphi(x) = 1$ for x large and $\varphi(x) = 0$ in a neighborhood of $x = 0$. Then

$$H(\phi u) = \phi H u + [D_x^2, \phi]u \in L^2(\mathbb{R}_+)$$

has compact support. Also $H(\phi u) = \mathfrak{H}(\phi u)$ in the sense of distributions, so by the previous lemma, $\phi u \in \langle x \rangle^{-2} L^2(\mathbb{R}) \cap H^2(\mathbb{R})$. This shows that

$$D(H_{\max}) = \langle x \rangle^{-2} L^2(\mathbb{R}_+) \cap \mathcal{H}^2(\mathbb{R}_+).$$

In the same way, local elliptic regularity for H combined with the global properties of $\mathfrak{H}$ shows that if $f \in \langle x \rangle^{-\infty} \mathcal{H}^{s,\infty}(\mathbb{R}_+)$ and $u \in D(H_{\max})$ solves $Hu = f$, then $u \in \langle x \rangle^{-\infty} \mathcal{H}^{s,\infty}(\mathbb{R}_+)$.

Lemma 5.1.2. *There is equality*

$$D(H_{\max}) = \langle x \rangle^{-2} L^2(\mathbb{R}_+) \cap \mathcal{H}^2(\mathbb{R}_+).$$

Furthermore, if $f \in \langle x \rangle^{-\infty} \mathcal{H}^{s,\infty}(\mathbb{R}_+)$ and $u \in D(H_{\max})$ solve $Hu = f$, then $u \in \langle x \rangle^{-\infty} \mathcal{H}^{s,\infty}(\mathbb{R}_+)$. This holds for $s = 0, 1, 2$.

Dirichlet realization

Restricting $H_{\max}$ to the space $\{u \in D(H_{\max}) : \gamma_- u = 0\}$ is the Dirichlet realization of H. Since this is the only realization considered here, we will just write it as H with the domain

$$D(H) = \{u \in \langle x \rangle^{-2} L^2(\mathbb{R}_+) \cap \mathcal{H}^2(\mathbb{R}_+) : \gamma_- u = 0\}.$$

By Green's formulas (3.29), (3.31), for each $\nu > 1$ the operator H is seen to be self-adjoint with this domain. There is also a nice characterization of H as the unique self-adjoint operator associated to the sesquilinear form

$$B_H(u, v) = \langle D_\nu u, D_\nu v \rangle_{\mathbb{R}_+} + \langle xu, xv \rangle_{\mathbb{R}_+}, \quad u, v \in Q(B_H)$$

where $Q(B_H) = \mathcal{H}_0^1(\mathbb{R}_+) \cap \langle x \rangle^{-1} L^2(\mathbb{R}_+)$. Note that $Q(B_H)$ is a Hilbert space for the norm

$$\|u\|_{\mathcal{H}^1(\mathbb{R}_+)} + \|xu\|_{L^2(\mathbb{R}_+)}.$$

The form B_H is positive definite, since an integration by parts shows that

$$B_H(u, v) = \langle (D_\nu u + xu), (D_\nu v + xv) \rangle_{\mathbb{R}_+} + 2 \langle u, v \rangle_{\mathbb{R}_+}. \tag{5.3}$$

Standard manipulations shows that the operator H' associated to $(B, Q(B_H))$ has domain

$$D(H') = Q(B_H) \cap D(H_{\max}) = D(H),$$

and acts by $H'u = H_{\max} u$ for $u \in D(H')$. Therefore $H = H'$.

Lemma 5.1.3. *The inclusion $Q(B_H) \hookrightarrow L^2(\mathbb{R}_+)$ is compact.*

Proof. It is well known that the inclusion $H_0^1(\mathbb{R}_+) \cap \langle x \rangle^{-2} L^2(\mathbb{R}_+) \hookrightarrow L^2(\mathbb{R}_+)$ is compact. Since $\mathcal{H}_0^1(\mathbb{R}_+) = H_0^1(\mathbb{R}_+)$ by Lemma 3.3.3, the result follows. □

Therefore the spectrum of H is purely discrete and bounded from below. In fact, (5.3) shows that $H \geq 2$ in the sense of forms, so $\operatorname{spec}(H) \subseteq [2, \infty)$. Furthermore, each eigenfunction of H lies in $\langle x \rangle^{-\infty} \mathcal{H}^{2,\infty}(\mathbb{R}_+)$.

Spectrum

Up to a scalar multiple there is a unique solution to the equation

$$(H - E)u = 0$$

satisfying the boundary condition $\gamma_- u = 0$. In fact $(H - E)u = 0$ can be recast as a hypergeometric equation, and the recessive solution satisfying $\gamma_- u = 0$ is

$$u(x) = x^{-1/2} M_{\kappa,\mu}(x^2), \quad \kappa = E/4, \quad \mu = \nu/2,$$

where $M_{k,\mu}(\cdot)$ is the first Whittaker function [102, Chapter XVI]. In general this function grows exponentially as $x \to \infty$. However, when $\frac{\nu+1}{2} - \frac{E}{4} = -n$ for an integer $n \in \mathbb{N}$, the hypergeometric series defining the Whittaker function truncates. For each $n \in \mathbb{N}$, the resulting function is of the form

$$u_n(x) = x^{\nu+1/2} e^{-x^2/2} L_n^{(\nu)}(x^2)$$

where $L_n^{(\nu)}(\cdot)$ is a Laguerre polynomial [49]. Clearly $u_n \in \langle x \rangle^{-\infty} \mathcal{H}^{2,\infty}(\mathbb{R}_+)$, and in fact the decay is of Gaussian type. Therefore the spectrum of H is given by the sequence of eigenvalues

$$E_n = 2(2n + 1 + \nu), \quad n \in \mathbb{N}.$$

If $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is the h-dependent unitary dilation

$$(Uu)(x) = h^{1/4} u(h^{1/2} x),$$

and $H(h) = |hD_\nu|^2 + x^2$, then

$$hH = UH(h)U^{-1}.$$

Therefore $H(h)$ is selfadjoint with domain $D(H(h)) = D(H)$, and $\operatorname{spec}(H(h)) = \operatorname{spec}(hH)$.

Remark 17. *The familiar isotropic harmonic oscillator is the Hamiltonian*

$$Q = -\Delta + |\mathbf{x}|^2, \quad \mathbf{x} \in \mathbb{R}^d.$$

Specializing to dimension $d = 2$ and introducing polar coordinates $\mathbf{x} = r \cdot \theta$,

$$Q = (D_r)^2 - ir^{-1} D_r + r^2 D_\theta^2 + r^2.$$

Spherical harmonics on $\mathbb{S}^1$ are the functions $\theta \mapsto \exp(i\nu\theta)$, where $\nu \in \mathbb{Z}$. Therefore Q can be decomposed into a family of ordinary differential operators

$$Q_\nu = D_r^2 - ir^{-1}D_r + r^2\nu^2 + r^2$$

on $\mathbb{R}_+$, indexed by $\nu \in \mathbb{Z}$. Conjugating Q_ν by $r^{-1/2}$ yields exactly the Hamiltonian (5.2).

5.2 Harmonic approximation

In this section we discuss the harmonic approximation for operators of the form

$$Q(h) = |hD_\nu|^2 + V$$

where V has a nondegenerate minimum at $x = 0$. The situation on $\mathbb{R}^d$ was studied originally in [20, 51, 88] and Simon [88].

The idea is to approximate the spectrum of $Q(h)$ in an h-dependent neighborhood of the minimum by that of the harmonic oscillator $H(h)$ discussed in the previous section. More precisely, let $\Omega = (0, a)$ for some $a \in (0, \infty]$, and assume that $V \in C^\infty(\overline{\Omega})$ satisfies

$$V(0) = 0, \quad V'(0), \quad V''(0) > 0. \tag{5.4}$$

By rescaling, we may always assume that $V''(0) = 2$. The potential does not need to be independent of h; in fact, the weakest assumption we can make is that V has an asymptotic expansion on Ω in powers of $h^{1/2}$,

$$V(x, h) \sim \sum_{k=0}^{\infty} h^{k/2} V_k(x), \tag{5.5}$$

where $V_k \in C^\infty(\overline{\Omega})$ is independent of h. In that case we require (5.4) to hold for V_0. The expansion is required to be uniform on compact subsets of $\overline{\Omega}$ in the sense that for each $K \subseteq \overline{\Omega}$ compact,

$$\left| V(x, h) - \sum_{k=0}^{N} h^{k/2} V_k(x) \right| \leq C_{K,N} h^{(N+1)/2}, \quad x \in K.$$

To discuss the spectrum of $Q(h)$ some global assumptions are needed, but at first only properties of V near $x = 0$ are used.

Rescaling

Given a ring R, write $R[x]$, $R[[x]]$ for the spaces of polynomials and formal powers series in the variable x, respectively. We let

$$\tau : C^\infty(\overline{\Omega}) \to \mathbb{C}[[x]]$$

denote the map which associates to a function $f \in C^\infty(\overline{\Omega})$ its formal Taylor series

$$\tau(f) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot x^k \in \mathbb{C}[[x]].$$

By the Borel lemma τ is surjective, and τ is injective modulo the ideal of smooth functions vanishing to infinite order at $x = 0$. The action of τ also extends

$$C^\infty(\overline{\Omega})[[h^{1/2}]] \to \mathbb{C}[[x]][[h^{1/2}]],$$

In particular, we can treat the asymptotic expansion (5.5) as an element of $C^\infty(\overline{\Omega})((h^{1/2}))$. The key observation is that after the rescaling $x = h^{1/2}y$, an element of $\mathbb{C}[[x]][[h^{1/2}]]$ becomes a formal power series in $h^{1/2}$ with coefficients which are polynomials in y. More precisely, define a map $\mathbb{C}[[x]][[h^{1/2}]] \to \mathbb{C}[y][[h^{1/2}]]$ by

$$\sum_{k\geq 0} h^{k/2} \sum_{i=0}^{\infty} a_i x^i \mapsto \sum_{k\geq 0} \sum_{i=0}^{\infty} h^{(k+i)/2}\, a_i y^i. \tag{5.6}$$

Since there are only finitely many $k, i \in \mathbb{N}$ such that $k + i = j$ for a fixed $j \in \mathbb{N}$, we can write the right hand side of (5.6) as

$$\sum_{k\geq 0} \sum_{i=0}^{\infty} h^{(k+i)/2}\, a_i y^i = \sum_{j=0}^{\infty} h^{j/2} P_j(y),$$

where $P_j \in \mathbb{C}[y]$ has degree at most j. Under the change of variables $x = h^{1/2}y$, we also have

$$(hD_x)^2 + h^2(\nu^2 - 1/4)x^{-2} + x^2 = h\left(D_y^2 + (\nu^2 - 1/4)y^{-2} + y^2\right).$$

Formally then, under the rescaling $x = h^{1/2}y$ we have

$$h^{-1}\left(|D_\nu|^2 + \tau(V)\right) \mapsto H + \sum_{j=1}^{\infty} h^{j/2} P_j(y), \tag{5.7}$$

where P_j is a polynomial of degree at most $2j + 2$ (as we have factored out two powers of $h^{1/2}$). The right hand side is treated as an operator in the purely formal sense.

Rayleigh–Schrödinger coefficients

We now formally look for an eigenfunction v with eigenvalue E of (5.7) in the form

$$v \sim \sum_{k=0}^{\infty} h^{k/2} v_j, \quad E \sim \sum_{k=0}^{\infty} h^{k/2} E_k.$$

Collecting powers of $h^{1/2}$, we obtain the sequence of equations

$$
\begin{aligned}
&(H - E_0)v_0 = 0, \\
&(H - E_0)v_k = -\sum_{r=0}^{k-1}(P_{k-r} - E_{k-r})v_r.
\end{aligned} \tag{5.8}
$$

The first equation is solved in $D(H)$ by fixing $n \in \mathbb{N}$ and letting $v_{n,0}$ be an eigenfunction of H with eigenvalue $E_{n,0} = 2(2n + 1 + \nu)$. Since H is self-adjoint, the subsequent equations can be solved if the right hand side is orthogonal to the kernel of $H - E_{n,0}$. This kernel is one-dimensional, spanned by $v_{n,0}$. Furthermore, since H is Fredholm of index zero, the solution to any such equation is unique up to a multiply of $v_{n,0}$ (since the cokernel is then also one-dimensional). The orthogonality condition is imposed by arranging

$$
E_{n,k} = \sum_{r=1}^{k-1} \langle (P_{k-r} - E_{n,k-r})v_{n,r}, v_{n,0}\rangle_{\mathbb{R}_+} + \langle P_k v_{n,0}, v_{n,0}\rangle_{\mathbb{R}_+}.
$$

In other words, $E_{n,k}$ are just standard Rayleigh–Schrödinger coefficients. Since $v_{n,0}(y)$ is an eigenfunction of H, it is in $\mathcal{H}^{2,\infty}(\mathbb{R}_+)$ — in fact, it decays exponentially along with all of its derivatives as $y \to \infty$, and is smooth in y^2. By lemma 5.1.2 each $v_{n,k} \in \mathcal{H}^{2,\infty}(\mathbb{R}_+)$. This is true since each P_{k-r} is a polynomial, hence can be treated as part of the inhomogeneity when multiplied by $v_{n,r}$.

Lemma 5.2.1. *The numbers $E_{n,k}$ with $k \geq 1$ depend only on $E_{n,0}$ and P_j for $j \leq k$.*

Proof. This is obvious from the recurrence equations 5.8. □

In particular, suppose that $V = V_0 + h^2 V_1$. Let

$$
\tau(V_0) = \sum_{j=0}^{\infty} a_j x^j, \quad \tau(V_1) = \sum_{j=0}^{\infty} b_j x^j
$$

with $a_1 = 0$ and $a_2 = 1$. Then $E_{n,k}$ for $k \geq 1$ depends on $a_0, \dots, a_{k+2}$ and $b_0, \dots, b_{k-2}$ (note that $E_{n,1}$ depends only on a_3).

Quasimodes

Having constructed sequences $\{E_{n,k}\}$ and $\{v_{n,k}\}$ for each fixed $n \in \mathbb{N}$ in the previous section, we can now produce quasimodes of arbitrary polynomial order for $Q(h)$. For any finite index J, let

$$
v_n^J(y) = v_{n,0}(y) + \cdots + h^{J/2} v_{n,J}(y),
$$

and then set

$$
u_{n,J}(x) = h^{-1/4}\, v_n^J(h^{-1/2}x)
$$

We may also assume that $\|v_{n,0}\|_{L^2(\mathbb{R}_+)} = 1$. By Borel summation, choose $E_n \in \mathbb{R}$ such that

$$E_n \sim h \sum_{j=0}^{\infty} h^{j/2} E_{n,j}.$$

The operator $Q(h) = |D_\nu|^2 + V$ is a priori only defined on Ω. Fortunately, the rapid decay of u_n^J allows us to localize to arbitrarily small (but fixed with respect to h) neighborhoods of $x = 0$. If U has compact closure in $(0, \infty)$, then by Lemma 5.1.2,

$$\|u_n^J\|_{H_h^s(U)} = \mathcal{O}(h^\infty)$$

for any s. Furthermore, for each $N \geq 0$,

$$\int_{\mathbb{R}_+} |x^N u_n^J(x)|^2 \, dx = \mathcal{O}(h^N)$$

after making the change of variable $y = h^{1/2}x$. We now go back to the original asymptotic expansion (5.5) for V, and let

$$\widetilde{V}^J = \sum_{k=0}^{J} h^{k/2} V_k.$$

We may also write

$$V_k = \widetilde{V}_k^J + \widetilde{R}_k^J,$$

where $\widetilde{R}_k^J = \mathcal{O}(x^{j+3-k})$. If $V^J = \widetilde{V}_0^J + \cdots + \widetilde{V}_J^J$, then

$$V = V^J + R^J,$$

where $V^J \in \mathbb{C}[y, h^{1/2}]$ and R^J is a finite sum of terms of order $\mathcal{O}(h^k x^{J+3-k})$ for $k = 0, \ldots, J$. Furthermore,

$$(|hD_\nu|^2 + V^J - E_n)u_n^J = \mathcal{O}(h^{(J+3)/2})u_n^J,$$

where the error is due to

$$E_n = hE_{n,0} + \cdots + h^{(J+2)/2}E_{n,J} + \mathcal{O}(h^{(J+3)/2}).$$

If χ is a cutoff to any neighborhood of $x = 0$, we have

$$(Q(h) - E_n)(\chi u_n^J) = [Q(h), \chi]u_n^J + (R^J + \mathcal{O}(h^{(J+3)/2}))\chi u_n^J,$$

Therefore

$$\|(Q(h) - E_n)(\chi u_n^J)\|_{L^2(\mathbb{R}_+)} = \mathcal{O}(h^{(J+3)/2}).$$

Since $\|\chi u_n^J\|_{L^2(\mathbb{R}_+)} = 1 + \mathcal{O}(h^{1/2})$, we can normalize χu_n^J without affecting the error. To summarize, we have established the following.

Lemma 5.2.2. *For each $n \in \mathbb{N}\, J \in \mathbb{N}$ there exists E_n and $u_n^J \in D(Q(h))$ such that $\|u_n^J\|_{L^2(\Omega)} = 1$ and*

$$\|(Q(h) - E_n)u_n^J\|_{L^2(\Omega)} = \mathcal{O}(h^{(J+3)/2}).$$

Furthermore, if $x_0 > 0$ is fixed, then u_n^J can be chosen such that $\operatorname{supp} u_n^J \subseteq [0, x_0)$.

Spectrum

If $\Omega = (0, a)$ is a bounded interval, we assume that $x = 0$ is a global minimum for V on $\overline{\Omega}$, and we impose an additional Dirichlet boundary condition at $x = a$. The second condition guarantees $Q(h)$ is a self-adjoint operator with discrete spectrum. If $\Omega = \mathbb{R}$, we make the assumption that

$$V_{\text{inf}} = \liminf_{x \to \infty} V(x) > 0,$$

in which case $Q(h)$ has discrete spectrum below $E = V_{\text{inf}}$.

By the spectral theorem, Lemma 5.1.2 implies that for each $n, J \in \mathbb{N}$ we have

$$\operatorname{dist}(\operatorname{spec} Q(h), E_n) \leq C_{n,J} h^{(J+3)/2},$$

where E_n is constructed in Section 5.2. In other words for each n there exists an eigenvalue e_n of $Q(h)$ such that $e_n = E_n + \mathcal{O}(h^\infty)$. The constant $C_{n,J}$ is not uniform in n, so this is only a uniform statement when considering finitely many n.

Fix $C_0 > 0$ such that $C_0 \notin 2(2\mathbb{N} + 1 + \nu)$ and let N be such that

$$2(2(N-1) + 1 + \nu) < C_0 < 2(2N + 1 + \nu).$$

By the above, we know there exist at least N eigenvalues in the interval $(-\infty, C_0 h)$, counting multiplicity.

Lemma 5.2.3. *Given $N > 0$, there exists $h_0 > 0$ such that if $h \in (0, h_0)$, then $Q(h)$ has exactly N eigenvalues $e_0, \ldots, e_{N-1}$ in the interval $(-\infty, C_0 h)$, counting multiplicity. Furthermore, $e_n = 2(2n + 1 + \nu) + \mathcal{O}(h^{3/2})$.*

Proof. The proof is essentially the same as [27, Theorem 4.23]. Fix a quadratic partition of unity $\psi_0^2 + \psi_1^2 = 1$ on $\mathbb{R}_+$ such that $\psi_0 = 1$ near $[0, 1)$ and $\operatorname{supp} \psi_0 \subseteq [0, 2)$ Then set $\chi_i(x) = \psi_i(R^{-1} h^{-1/2} x)$ for some $R > 0$ to be determined. Now for $u \in \mathcal{H}^2(\Omega)$,

$$\begin{aligned} |D_\nu|^2 u = (\chi_0^2 + \chi_1^2)|D_\nu|^2 u = \chi_0 |D_\nu|^2 (\chi_0 u) + \chi_1 |D_\nu|^2 (\chi_1 u) \\ - \chi_0 [D_x^2, \chi_0] u - \chi_1 [D_x^2, \chi_1] u \end{aligned}$$

Now

$$-\chi_0 [D_x^2, \chi_0] u - \chi_1 [D_x^2, \chi_1] u = 2\chi_0 \, \chi_0' \, \partial_x + 2\chi_1 \, \chi_1' \, \partial_x + \chi_0 \chi_0'' + \chi_1 \chi_1''.$$

The first two terms on the right hand side are $\partial_x(\chi_0^2 + \chi_1^2) = 0$. Therefore we have

$$\begin{aligned} \left\langle |D_\nu|^2 u, u \right\rangle_\Omega = \left\langle |D_\nu|^2 (\chi_0 u), \chi_0 u \right\rangle_\Omega + \left\langle |D_\nu|^2 (\chi_1 u), \chi_1 u \right\rangle_\Omega \\ + \langle \chi_0 \, \chi_0'' \, u, u \rangle_\Omega + \langle \chi_1 \, \chi_1'' \, u, u \rangle_\Omega \end{aligned}$$

Now $\chi_i'' = \mathcal{O}(R^{-2} h^{-1})$. Multiplying through by h^2 and adding $\langle V u, u \rangle_\Omega$ yields

$$\langle Q(h) u, u \rangle_\Omega = \langle Q(h)(\chi_0 u), \chi_0 u \rangle_\Omega + \langle Q(h)(\chi_1 u), \chi_1 \rangle_\Omega + \mathcal{O}(h R^{-2}) \|u\|_{L^2(\Omega)}^2.$$

Since V has a global minimum at $x = 0$ with $V''(0) = 2$, we have

$$\langle Q(h)(\chi_1 u), \chi_1 u\rangle_\Omega \geq \langle V\chi_1 u, \chi_1 u\rangle_\Omega \geq Kh\|\chi_1 u\|^2_{L^2(\Omega)}$$

for any fixed $K > 0$ by choosing R sufficiently large. In particular, we can take $K = 2(2N + 1 + \nu)$. To finish the proof, observe that

$$\langle Q(h)(\chi_0 u), \chi_0 u\rangle_\Omega = \langle H(h)(\chi_0 u), \chi_0 u\rangle_{\mathbb{R}_+} + \mathcal{O}(h^{3/2}R^3)\|\chi_0 u\|^2_{L^2(\Omega)}.$$

To apply the min-max principle, we assume that u is orthogonal to $\chi_0 w_n$ on Ω for $n = 0, \dots, N-1$, where w_n is the nth eigenvector of $H(h)$. Equivalently, χ_0 is orthogonal to w_n on $\mathbb{R}_+$, so

$$\langle H(h)(\chi_0 u), \chi_0 u\rangle_{\mathbb{R}_+} \geq 2(2N_0 + 1 + \nu)h\|\chi_0 u\|^2_{L^2(\Omega)}.$$

Since χ_i form a quadratic partition of unity,

$$\langle Q(h)u, u\rangle_\Omega \geq 2(2N + 1 + \nu)h - (R^{-2} + h^{1/2}R^3)h\|u\|^2_{L^2(\Omega)}/C$$

for some $C > 0$ independent of R, h. Taking R sufficiently large and then h small shows that

$$\langle Q(h)u, u\rangle_\Omega \geq 2\left(2N + 1 + \nu - \varepsilon\right)h\|u\|^2_{L^2(\Omega)},$$

on the orthogonal complement of an N dimensional space, therefore the $(N+1)$th eigenvalue is greater than $C_0 h$. This shows there are exactly N eigenvalues in $(-\infty, C_0 h)$. □

5.3 Agmon estimates

In this section we review the semiclassical Agmon estimates. A textbook reference for this section is [27, Chapter 6], which handles the smooth setting; replacing D_x^2 with $|D_\nu|^2$ does not have any significant effect. These estimates allow one to quantify the statement that semiclassical eigenfunctions at an energy level E decay exponentially as $h \to 0$ in regions where $V > E$. Fix a finite interval

$$\Omega = (0, a),$$

where $a > 0$. Let $V \in C^\infty(\overline{\Omega})$, and consider the operator $Q(h) = h^2|D_\nu|^2 + V$ with Dirichlet boundary conditions at $x = 0, a$.

Energy identities

First assume that $\Phi \in C^\infty(\overline{\Omega})$. Then for each $u \in D(Q(h))$ the following integration by parts is justified:

$$\mathrm{Re}\int_\Omega e^{2\Phi/h}Q(h)u \cdot \bar{u}\,dx = \int_\Omega |hD_\nu\left(e^{\Phi/h}u\right)|^2\,dx + \int_\Omega e^{2\Phi/h}\left(V - (\Phi')^2\right)|u|^2\,dx \qquad (5.9)$$

If $u \in D(Q(h))$ solves $Q(h)u = 0$, then we can control the L^2 norm of $e^{\Phi/h}u$ in the region where $V - (\Phi')^2$ is positive. To obtain an optimal result, first we need to relax the smoothness assumption on Φ. In higher dimensions it is appropriate to consider Φ Lipschitz continuous, but in one dimension we may work with Φ absolutely continuous on $\overline{\Omega}$. If Φ' is understood to exist almost everywhere, then (5.9) continues to hold for $u \in D(P)$.

Agmon distance

For each $E \in \mathbb{R}$, the Agmon metric on $\overline{\Omega}$ is given by $(V - E)_+ \, dx^2$, where $f_+ = \max(f, 0)$. Given $x, y \in \overline{\Omega}$, let $d(x, y) = d_E(x, y)$ denote the distance between x, y in this metric. Explicitly, if $x \leq y$, then

$$d(x, y) = \int_x^y (V(t) - E)_+^{1/2} \, dt,$$

This is not a distance in the usual sense, since distinct points $x \neq y$ may have $d(x, y) = 0$. On the other hand, $d(x, y)$ certainly satisfies the triangle inequality. By symmetry the triangle inequality implies.

$$|d(x, y) - d(z, y)| \leq d(x, z)$$

Since

$$d(x, z) \leq |x - z| \sup_{\overline{\Omega}} (V - E)_+^{1/2},$$

we certainly have that for each $y \in \overline{\Omega}$ the map $x \mapsto d(x, y)$ is Lipschitz on $\overline{\Omega}$, hence differentiable almost everywhere In fact, for each ε there is $\delta > 0$ such that

$$d(x, z) \leq |x - z| \left((V(x) - E)_+ + \varepsilon \right)$$

for $|x - z| < \delta$, hence

$$\partial_x d(x, y) \leq (V(x) - E)_+^{1/2}$$

whenever the left hand side exists. We can also define the distance to any subset $U \subseteq \overline{\Omega}$ by

$$d(x, U) = d_E(x, U) = \inf\{d(x, y) : y \in U\}.$$

Just as above, we have

$$|d(x, U) - d(y, U)| \leq d(x, y),$$

and so $x \mapsto d(x, U)$ is Lipschitz whose derivative satisfies

$$\partial_x d(x, U) \leq (V(x) - E)_+^{1/2} \tag{5.10}$$

almost everywhere on $\overline{\Omega}$.

Decay estimates

Suppose that $V = V_0 + h^2 V_1$, where $V_i \in C^\infty(\overline{\Omega})$. We then consider the Dirichlet realization of $Q(h) = |D_\nu|^2 + V$ on Ω. Given $E \in \mathbb{R}$, let

$$A_E = \{x \in \overline{\Omega} : V_0(x) \leq E\}, \quad F_E = \{x \in \overline{\Omega} : V_0(x) > E\}$$

denote the classically allowed and forbidden regions, respectively. For a fixed $\delta > 0$, let

$$\Phi(x) = (1-\delta) d(x, A_E)$$

From (5.10),

$$V - E - (\Phi')^2 \geq V - E - (1-\delta)^2 (V_0 - E)_+$$

almost everywhere.

Lemma 5.3.1. *Fix a compact interval $[a, b] \subseteq \mathbb{R}$, and let $d(x) = d_E(x, A_E)$. For each $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\|e^{d/h} h D_\nu u\|^2_{L^2(\Omega)} + \|e^{d/h} u\|^2_{L^2(\Omega)} \leq C_\delta \, e^{\delta/h} \|u\|_{L^2(\Omega)}$$

for each $u \in D(Q(h))$ satisfying $(Q(h) - E)u = 0$, provided $E \in [a, b]$ and $h > 0$ is sufficiently small.

Proof. Consider the set $U_\delta = \{x \in \overline{\Omega} : V(x) > E + \delta\}$. Then for almost every $x \in U_\delta$,

$$V - E - (\Phi')^2 > \delta^2$$

provided $\delta > 0$ and $h > 0$ are sufficiently small. Suppose that $(Q(h) - E)u = 0$ for $u \in D(Q(h))$ and $E \in [a, b]$. Then

$$\int_\Omega |h D_\nu \left(e^{\Phi/h} u\right)|^2 \, dx + \delta^2 \int_{\Omega_\delta} e^{2\Phi/h} |u|^2 \, dx \leq C \int_{\Omega \setminus \Omega_\delta} e^{2\Phi/h} |u|^2 \, dx,$$

where

$$C = \sup_{x \in \Omega \setminus \Omega_\delta} |(\Phi')^2 + E - V|$$

Observe that C itself is uniformly bounded by another constant uniformly for for $E \in [a, b]$ and $\delta \in (0, 1)$, $h \in (0, 1)$. Replacing C by this constant and adding δ^2 times the integral over $\Omega \setminus \Omega_\delta$ gives

$$\int_\Omega |h D_\nu \left(e^{\Phi/h} u\right)|^2 \, dx + \delta^2 \int_\Omega e^{2\Phi/h} |u|^2 \, dx \leq (C + \delta^2) \int_{\Omega \setminus \Omega_\delta} e^{2\Phi/h} |u|^2 \, dx,$$

Let

$$\alpha(E, \delta) = \sup_{x \in \Omega \setminus \Omega_\delta} 2\Phi,$$

and observe that $\alpha(E,\delta)\to 0$ as $\delta\to 0$ uniformly for $E\in[a,b]$. The derivative term can also be expanded:

$$h^2\int_\Omega e^{2\Phi/h}\left(|D_\nu u|^2+2h^{-1}\Phi'\operatorname{Re}(\partial_\nu u\cdot\bar u)+h^{-2}(\Phi')^2|u|^2\right)\,dx$$

The second term in the integrand is bounded below

$$2h^{-1}\Phi'\operatorname{Re}(\partial_\nu u\cdot\bar u)\geq -\varepsilon h^{-2}|D_\nu u|^2-\frac{1}{\varepsilon}(\Phi')^2|u|^2$$

almost everywhere. Since the integrand is multiplied by h^2 and Φ' is uniformly bounded in $\delta\in(0,1)$ and $E\in[a,b]$, these terms can be absorbed by first choosing ε and then h sufficiently small. Therefore

$$\int_\Omega e^{2\Phi/h}|hD_\nu u|^2\,dx+\int_\Omega e^{2\Phi/h}|u|^2\,dx\leq C_\delta\,e^{\alpha/h}\int_{\Omega\setminus\Omega_\delta}|u|^2\,dx.$$

We can replace Φ with $d_E(x,A_E)$ at the cost of multiplying through by $e^{2\delta K/h}$, where K is the supremum of $d(x,A_E)$ over $x\in\Omega$ and $E\in[a,b]$. Therefore there exists $C_\delta>0$ independet of $E\in[a,b]$ such that

$$\|e^{d/h}hD_\nu u\|^2_{L^2(\Omega)}+\|e^{d/h}u\|^2_{L^2(\Omega)}\leq C_\delta\,e^{\delta/h}\int_\Omega|u|^2\,dx.$$

for each $\delta>0$. □

To obtain exponential decay estimates we therefore need to look for regions where $d(x,A_E)$ is bounded from below. In our one-dimensional case, this is easy since geodesics in the Agmon distance are straight lines.

5.4 The Schwarzschild–AdS spacetime

In this section we apply the results from Sections 5.2, 5.3 to the Regge–Wheeler equations in the Schwarzschild–AdS setting. Recall from Section 2.4 that the Schwarzschild–AdS metric (with AdS radius $l=1$) in Schwarzschild coordinates is given by

$$g=f\,dt^2-f^{-1}\,dr^2-r^2d\omega^2,$$

where $f(r)=r^2+1-\mu r^{2-d}$. The exterior region refers to

$$M=\mathbb{R}_t\times(r_+,\infty)\times\mathbb{S}^{d-1}_\omega,$$

where $r_+>0$ is the unique positive root of f. In these coordinates we may fully separate the wave equation.

Regge-Wheeler coordinate

Introduce the Regge–Wheeler (or tortoise) coordinate $z : (r_+, \infty) \to (0, \infty)$ according to the equation

$$\partial_r z = -f^{-1}(r), \quad z(+\infty) = 0.$$

In other words,

$$z(r) = \int_r^\infty f(\rho)^{-1}\, d\rho. \tag{5.11}$$

We are interested in the asymptotics of z as $r \to r_+$ and $r \to \infty$. Recall in the Schwarzschild–AdS setting that the Killing horizon $\{r = r_+\}$ is always nondegenerate, with surface gravity given by $\kappa = f'(r_+)/2 > 0$ (see (2.17) and (2.23)).

Lemma 5.4.1. *There exists a real analytic function $F(w)$ in a neighborhood of $w = 0$ such that*

$$r = r_+ + F(e^{-2\kappa z})$$

and $F(0) = 0$.

Proof. Since f vanishes simply at r_+, we may write

$$f(r) = 2\kappa(r - r_+) + h(r),$$

where $h(r)$ is analytic near $r = r_+$. Upon integration,

$$-2\kappa z(r) = \log(r - r_+) + H(r),$$

where $H(r)$ is again analytic near $r = r_+$. Equivalently, $(r - r_+) \exp(H(r)) = \exp(-2\kappa z(r))$. Since the left hand side vanishes simply at $r = r_+$, we may solve for r as an analytic function of $\exp(-2\kappa z)$ near $z = \infty$ by the implicit function theorem. Thus

$$r = r_+ + F\left(e^{-2\kappa z}\right), \quad F(0) = 0. \qquad \square$$

Next, we analyze the relationship between z and r as $r \to \infty$.

Lemma 5.4.2. *There exists a real analytic function $G(s)$ in a neighborhood of $s = 0$ such that*

$$z = \operatorname{arccot}(r) + (d + 1)\mu r^{-d-1} + G(1/r)$$

and $G(s)$ vanishes to order $d + 3$ at $s = 0$.

Proof. First let

$$R = (2\mu)^{1/d},$$

so $\mu r^{-d} \in (0, 1/2)$ for $r \in (R, \infty)$. Then we may write

$$f(r)^{-1} = \frac{1}{r^2 + 1} + \mu r^{-d-2} + g(1/r),$$

where $g(s)$ is analytic near $s = 0$ and vanishes to order $d + 4$. Integration yields

$$z = \operatorname{arccot}(r) + (d+1)\mu r^{-d-1} + G(1/r)$$

with $G(s)$ vanishing to order $d + 3$ at $s = 0$. □

For future use, it is useful to express $f(r)$ and $r^{-2}f(r)$ in terms of $\csc(z)^2$ and $\sec(z)^2$. Observe that

$$\sin^2(z) \sim \frac{1}{1+r^2}\left(1 + 2(d+1)\mu r^{-d}\right).$$

Since $r = (1/z) + \mathcal{O}(1)$, we have

$$\sin^{-2}(z) \sim f(r) - (2d+1)\mu z^{d-2}. \tag{5.12}$$

Similarly,

$$\cos(z)^2 \sim \frac{1}{r^{-2}+1}(1 - 2(d+1)\mu r^{-d-2}),$$

and therefore

$$\cos^{-2}(z) \sim r^{-2}f(r) + \mu z^d. \tag{5.13}$$

These formulas will be used to compute the leading term in the harmonic approximation for the separated Regge–Wheeler equations introduced in the next section.

Separation of variables

Because of its spherical symmetry, it is possible to decouple the stationary Klein–Gordon equation into a family of ordinary differential equations in the radial variable. More precisely, we let $T = \partial_t$ and choose as our initial hypersurface $X = \{t = 0\}$. In Schwarzschild coordinates,

$$\widehat{\Box}_g(\lambda) + \nu^2 - d^2/4 = r^{1-d}D_r(r^{d-1}fD_r) - r^{-2}\Delta_\omega - \lambda^2 f^{-1} + \nu^2 - d^2/4.$$

As usual we can conjugate this operator by $r^{(1-d)/2}$ and multiply through by f. Then the stationary Klein–Gordon equation is equivalent to an eigenvalue problem

$$\left((fD_r)^2 - r^{-2}f\cdot\Delta_\omega + f\cdot V - \lambda^2\right)v = 0, \tag{5.14}$$

where the function V depends only on r:

$$V(r) = \frac{(d-1)}{2r^2}\cdot\partial_r f(r) + \frac{(d-3)(d-1)}{4r^2}\cdot f(r) + \nu^2 - 1/4.$$

Suppose that $v(r,\omega) = S_{m,\ell}(\omega)u(r)$, where $S_{m,\ell}$ is a spherical harmonic on $\mathbb{S}^{d-1}$ with eigenvalue $\ell(\ell + d - 2)$. Then (5.14) becomes

$$(P_\ell - \lambda^2)u = 0,$$

where P_ℓ is the second order differential operator on (r_+, ∞) given by

$$P_\ell = (fD_r)^2 + (\ell(\ell + d - 2)r^{-2} + V) \cdot f$$

Plugging in the expression for f, we may also write

$$P_\ell = (fD_r)^2 + (\sigma^2 - 1/4)r^{-2}f(r) + (\nu^2 - 1/4)f(r) + \frac{\mu(d-1)^2}{4r^d}f(r),$$

where we defined $\sigma = (2\ell + d - 2)/2$.

Semiclassical rescaling

We are interested in the spectral behavior of P_ℓ for large values of ℓ. For that, we consider the rescaled operator

$$Q(h) = h^2 P_\ell, \quad h = \sigma^{-1},$$

which we write in the form

$$Q(h) = (fhD_r)^2 + h^2(\nu^2 - 1/4)f + V_0 + h^2 V_1. \tag{5.15}$$

Here $V_0 = r^{-2}f$ and

$$V_1 = \mu(d-1)^2 r^{2-d}\left(\frac{V_0}{4}\right)$$

In particular there is a constant $C > 0$ such that $|V_1| \leq C|V_0|$ uniformly over (r_+, ∞).

Analysis of the potential

We now study the behavior of V_0 for $r \in (r_+, \infty)$. Observe that $r^{-2}f(r) \to 1$ as $r \to \infty$, and of course $r_+^{-2}f(r_+) = 0$.

Lemma 5.4.3. *The potential V_0 has a unique nondegerate local maximum satisfying*

$$r_{\max} = \left(\frac{\mu d}{2}\right)^{\frac{1}{d-2}}, \qquad V_0(r_{\max}) = 1 + \left(\frac{2}{\mu d}\right)^{\frac{2}{d-2}}\left(\frac{d-2}{d}\right),$$

and no other local extrema for $r \in (r_+, \infty)$.

Proof. To find the extrema of V_0, find the roots of

$$\partial_r V_0 = -\frac{2}{r^3} + \frac{\mu d}{r^{d+1}}$$

for $r \in (r_+, \infty)$. □

The existence of this local maximum is related to the trapping of null-geodesics on the background. In standard terminology $r_{\max}$ is the location of the photon sphere. We also write $V_{\max} = V_0(r_{\max})$. By the previous lemma, for any real $E \in (1, V_{\max})$ the equation $V_0 = E$ has two solutions.

Reference operator

Under the change of variables $z = z(r)$ we have $f D_r = D_z$. Furthermore, we have

$$f(r) = z^{-2} + \mathcal{O}(1)$$

uniformly on any set of the form (R, ∞) with $R > r_+$. Therefore we can write

$$Q(h) = |hD_\nu|^2 + V_0 + h^2 \widetilde{V}_1$$

with respect to the variable $z \in \mathbb{R}_+$, where now $|\widetilde{V}_1| \leq C|V_0|$ uniformly on any compact subset of $\overline{\mathbb{R}}_+$ in z.

Define a reference operator $Q^\sharp(h)$ as the restriction of $Q(h)$ to the interval $\Omega = (0, z_{\max})$, where $z_{\max} = z(r_{\max})$. We impose Dirichlet boundary conditions at $z = 0$ and $z = z_{\max}$. Then $Q\sharp(h)$ is a self-adjoint operator with discrete spectrum. We would like a detailed understanding of $\operatorname{spec}(Q^\sharp(h))$ in the spectral window $E \in (1, V_{\max})$ as $h \to 0$. Before proving the existence of spectrum in this window, we can use the Agmon estimates to prove exponential decay. Recall the definitions of the classically allowed and forbidden regions A_E, F_E from Section 5.3.

Lemma 5.4.4. *Let $E_0 \in (1, V_{\max})$. Then there exists ε and $C > 0$ depending on E_0 such*

$$d_E(F_{E_0}, A_E) \geq C$$

for each $E \in (1, E_0 - \varepsilon)$,

Proof. The result is obvious since geodesics in the Agmon distance are straight lines. □

The following is then a corollary of Lemma 5.3.1

Proposition 5.4.5. *Let $\varepsilon > 0$ and $E_0 \in (1, V_{\max})$. If $u \in D(Q^\sharp(h))$ satisfies $(Q^\sharp(h) - E)u = 0$ for $E \in (1, E_0 - \varepsilon)$ and h sufficiently small, then*

$$\int_{F_{E_0}} |hu'|^2 + |u|^2 \, dz \leq C e^{-2C/h} \int_\Omega |u|^2 \, dz,$$

where $C > 0$ is uniform for $(1, E_0 - \varepsilon)$.

Low-lying eigenvalues

Using the harmonic approximation, we can study the spectrum of $Q^\sharp(h)$ near the bottom of the potential well, now located at $E = 1$. Indeed, we have

$$V_0(0) = 1, \quad V_0'(0) = 0, \quad V_0''(0) = 2$$

where we are treating V_0 as a function of z. Let $C_0 \notin 2(2\mathbb{N}+1+\nu)$, and let $N \in \mathbb{N}$ be such that

$$2(2(N-1)+1+\nu) < C_0 < 2(2N+1+\nu).$$

According to Lemma 5.2.3, for h sufficiently small depending on N_0, there exist exactly N eigenvalues $e_0^\sharp, \ldots, e_{N-1}^\sharp$ of $Q^\sharp(h)$ in the interval $(-\infty, C_0)$ satisfying

$$e_n^\sharp = 2(2n+1+\nu)h + \mathcal{O}(h^{3/2}).$$

Moreover, each $e_n^\sharp$ has a full asymptotic expansion in powers of $h^{1/2}$. In the next section we will compute the first d terms in this expansion, where $d+1$ is the spacetime dimension; these terms do not depend on the black hole mass, but rather only on the asymptotically AdS nature of the metric.

The case of global AdS_{d+1}

When $\mu = 0$, the Schwarzschild–AdS metric reduces to that of global AdS_{d+1}; this is singular limit, since AdS_{d+1} does not have a horizon. The separation of variables applies equally well in this case, see Section 2.1. In terms of the variable $z = \operatorname{arccot}(r)$, we have an operator

$$P_\ell^{\mathrm{AdS}} = D_z^2 + (\nu^2 - 1/4)\sin^{-2}(z) + (\sigma^2 - 1/4)\cos^{-2}(z)$$

on the interval $(0, \pi/2)$ with Dirichlet boundary conditions (now the boundary conditions are twisted at $z = \pi/2$ as well). The harmonic approximation also applies to the rescaled operator $P_{\mathrm{AdS}}(h) = h^2 P_\ell^{\mathrm{AdS}}$, where $h = \sigma^{-1}$:

$$P_{\mathrm{AdS}}(h) = (hD_z)^2 + h^2(\nu^2 - 1/4)\sin^{-2}(z) + (1 - h^2/4)\cos^{-2}(z). \tag{5.16}$$

Applying Lemma 5.2.2, we therefore obtain for the nth eigenvalue of $P_{\mathrm{AdS}}(h)$

$$e_n^{\mathrm{AdS}} \sim 1 + 2(2n+1+\nu)h + \sum_{k=1}^{\infty} h^{(k+2)/2} E_{n,k}^{\mathrm{AdS}} \tag{5.17}$$

On the other hand, the eigenfrequencies of P_ℓ^{AdS} are explicitly known [65]: the equation

$$(P_\ell^{\mathrm{AdS}} - \lambda^2)u = 0$$

has solutions

$$\lambda_n^2 = (2n + \nu + \sigma + 1)^2.$$

Since $h^2\lambda_n^2 = e_n^{\mathrm{AdS}}$, we can compute the higher order Rayleigh–Schrödinger coefficients $E_{n,k}^{\mathrm{AdS}}$ explicitly: they are all zero except

$$E_{n,2}^{\mathrm{AdS}} = (1 + 2n + \nu)^2.$$

In the next section we use this information to calculate the leading behavior for the eigenvalues $e_n^\sharp$ of the reference operator.

Asymptotic expansions

Let us now return to the Schwarzschild–AdS case $\mu > 0$. Recall from (5.15) the expression

$$Q^\sharp(h) = (hD_z)^2 + h^2(\nu^2 - 1/4)f + (1 - h^2/4)r^{-2}f(r). \tag{5.18}$$

According to Lemma (5.12), (5.13), we have

$$\begin{cases} h^2(\nu^2 - 1/4)f = h^2(\nu^2 - 1/4)\sin^{-2}(z) + \mathcal{O}(h^2 z^{d-2}), \\ (1 - h^2/4)r^{-2}f(r) = (1 - h^2/4)\cos(z)^{-2} - \mu z^d + \mathcal{O}(h^2 z^d) + \mathcal{O}(z^{d+2}). \end{cases}$$

From this, we can calculate the leading behavior of $e_n^\sharp$ in dimensions $d = 3, 4$.

Lemma 5.4.6. *Let v_n denote the nth eigenvector of H, normalized so that $\|v_n\|_{L^2(\mathbb{R}_+)} = 1$. The numbers $E_{n,k}$ satisfy the following.*

1. *If $d = 3$, then*
$$E_{n,1} = -\mu \int_0^\infty y^3 \, |v_n(y)|^2 \, dy.$$

2. *If $d = 4$, then $E_{n,1} = 0$ and*
$$E_{n,2} = \frac{\nu^2 - 1}{3} + \left(\frac{2}{3} - \mu\right) \int_0^\infty y^4 \, |v_n(y)|^2 \, dy.$$

Proof. (1) If $d = 3$, then $-\mu z^3$ is the only monomial of the form $a_j z^{3-j} h^{j/2}$ with $j = 0, \ldots, 3$ in the Laurent expansion of

$$h^2(\nu^2 - 1/4)f + (1 - h^2/4)r^{-2}f - \left[h^2(\nu^2 - 1/4)z^{-2} + z^2\right] \tag{5.19}$$

Thus $-\mu z^3$ is the leading order perturbation in the harmonic approximation.

(2) If $d = 4$ then we look for monomials of the form $a_j z^{4-j} h^{j/2}$ for $j = 0, \ldots, 4$. There, we find the terms

$$\frac{\nu^2 - 1}{3} h^2 + \left(\frac{2}{3} - \mu\right) z^4.$$

□

For the analogous computations in dimensions $d \geq 5$, we use the following observation: by Lemma 5.2.1 and the discussion which followed, the numbers $E_{n,k}$ in the asymptotic expansion of

$$e_n^\sharp \sim 1 + h \sum_{j=0}^\infty h^{j/2} E_{n,j}$$

satisfy

$$E_{n,k} = E_{n,k}^{\mathrm{AdS}}, \quad k = 0, \ldots, d-3 \tag{5.20}$$

in $d+1$ spacetime dimensions. For $d = 5$, we therefore have $E_{n,1} = 0$ and $E_{n,2} = (1+2n+\nu)^2$. This can be continued for $d > 5$, but this requires actually computing the $v_{n,k}$ as well. However, if we define now

$$\lambda^\sharp_{n,\ell} = h^{-1} \left(e^\sharp_n\right)^{1/2},$$

then from (5.20) alone we see that

$$\lambda^\sharp_{n,\ell} = \sigma + 2n + \nu + 1 + \mathcal{O}(\sigma^{(2-d)/2}).$$

Writing this in terms of $\ell = \sigma + 1 - (d/2)$, we obtain the following.

Proposition 5.4.7. *In $d+1$ spacetime dimensions, the frequencies $\lambda^\sharp_{n,\ell}$ admit a full asymptotic expansion in powers $\ell^{-1/2}$, satisfying*

$$\lambda^\sharp_{n,\ell} = \ell + 2n + \nu + d/2 + \mathcal{O}(\ell^{(2-d)/2})$$

In general (unless $d = 3, 4$) we have not verified that the term $\ell^{(2-d)/2}$ has a non-vanishing coefficient in $d + 1$ dimensions. This non-vanishing was observed numerically by Dias et al. [26], who conjectured that

$$\lambda^\sharp_{n,\ell} - (\ell + 2n + \nu + d/2) = c_d \ell^{(2-d)/2} + \mathcal{O}(\ell^{(1-d)/2}), \quad c_d \neq 0.$$

Proposition 5.4.7 gives a partial answer to this conjecture in the affirmative.

5.5 Quasimodes

We are finally ready to construct exponentially accurate quasimodes for the stationary Klein–Gordon operator $P(\lambda)$ on a Schwarzschild–AdS spacetime.

Theorem 9. *For each $n \in \mathbb{N}$ there exists a sequence $\lambda^\sharp_{n,\ell} \in \mathbb{R}$ and*

$$u^\sharp_{n,\ell,m} \in \mathcal{H}^1(X_\delta) \cap C^\infty(X_\delta) \cap \mathcal{D}'_m(X_\delta),$$

for ℓ sufficiently large (depending on n) with the following properties.

1. *$\lambda^\sharp_{n,\ell}$ has a full asymptotic expansion in powers of $\ell^{-1/2}$, and moreover in $(d+1)$ dimensions,*
$$\lambda^\sharp_{n,\ell} = \ell + 2n + \nu + d/2 + \mathcal{O}(\ell^{(2-d)/2}).$$

2. *There exists $[r_1, r_2] \subseteq (r_+, \infty)$ such that $\operatorname{supp} u^\sharp_{n,\ell,m} \subseteq (r_1, \infty)$ and $\operatorname{supp} P(\lambda^\sharp_{n,\ell}) u^\sharp_{n,\ell,m} \subseteq (r_1, r_2)$.*

3. *The functions $u^\sharp_{n,\ell,m}$ are normalized exponentially accurate quasimodes in the sense that*
$$\|u^\sharp_{n,\ell,m}\|_{\mathcal{L}^2(X_\delta)} = 1, \quad \|P(\lambda^\sharp_{n,\ell}) u^\sharp_{n,\ell,m}\|_{\mathcal{L}^2(X_\delta)} \leq e^{-\ell/C}$$
for some $C > 0$. Furthermore, $\gamma_- u^\sharp_{n,\ell,m} = 0$.

4. *Each* $u^\sharp_{n,\ell,m}$ *is of the form*

$$u^\sharp_{n,\ell,m}(r,\omega) = v_{n,\ell}(r)\cdot S_{m\ell}(\omega),$$

where $S_{m\ell}$ *is a spherical harmonic and* $v_{n,\ell}$ *a function of* r *only.*

Proof. Let $\Omega = (0, z_{\max})$. Fix a cutoff χ in the Regge–Wheeler coordinate to a small neighborhood of $z = 0$ such that $\chi = 1$ near $z = 0$. If h is sufficiently small depending on n, then $V - e^\sharp_n > \varepsilon$ on the support of $d\chi$. By Lemma 5.3.1, if $v^\sharp_n$ is the $L^2(\Omega)$-normalized nth eigenfunction of $Q^\sharp(h)$, then

$$\|(Q^\sharp(h) - e^\sharp_n)(\chi v^\sharp_n)\|_{L^2(\Omega)} \le e^{-C/h}.$$

Now we must transport $\chi v^\sharp_n$ to X_δ: recall that up until now we took the Fourier transform in t rather than $t^\star$, where

$$t^\star = t + F_t(r)$$

for some F_t. Writing $h = \sigma^{-1} = (\ell - 1 + d/2)^{-1}$, we let

$$u^\sharp_{n,\ell,m}(r,\omega) = \exp(i\lambda^\sharp_{n,\ell}F_t(r))r^{(1-d)/2}\chi(r)v^\sharp_n(r;\sigma^{-1})\cdot S_{m\ell}(\omega).$$ □

A similar quasimode construction also applies in the Kerr–AdS setting, see [56]. This will be reviewed in the next chapter, where we use the quasimodes to construct sequences of QNFs converging exponentially to the real axis. The difference in the Kerr–AdS case is that there is no refined description of the frequencies, since the harmonic approximation cannot be applied directly.

Chapter 6

Existence of quasinormal modes

6.1 Introduction

In this chapter we prove the existence of QNFs converging exponentially to the real axis.

Theorem 10. *Fix a cosmological constant $\Lambda < 0$, black hole mass $M > 0$, rotation speed $a \in \mathbb{R}$ satisfying $|a|^2 < 3/|\Lambda|$, and Klein–Gordon mass $\nu > 0$. Let $X_\delta = (r_+ - \delta, \infty) \times \mathbb{S}^2$ for $\delta > 0$ sufficiently small, and $t^\star$ be the Kerr-star time coordinate. Then exists a sequence of complex numbers and smooth functions*

$$\lambda_\ell \in \mathbb{C}, \quad u_\ell \in C^\infty(X_\delta), \quad \ell \geq L$$

for some $L \geq 0$ with the following properties.

1. *The functions $v_\ell := e^{-i\lambda_\ell t^\star} u_\ell$ solve the Klein–Gordon equation*
$$\Box_g v_\ell + \frac{|\Lambda|}{3}(\nu^2 - 9/4)v_\ell = 0.$$
2. *The complex frequencies λ_ℓ satisfy*
$$\ell/C < \operatorname{Re}\lambda_\ell < C\ell, \quad 0 < -\operatorname{Im}\lambda_\ell < e^{-\ell/D}$$
for some $C, D > 0$.
3. *Each u_ℓ is smooth up to $\{r = r_+ - \delta\}$, and the restriction of u_ℓ to $X_0 = (r_+, \infty) \times \mathbb{S}^2$ is nonzero.*
4. *Each u_ℓ satisfies*
$$\int_{X_\delta} |u_\ell|^2 \, r^{-1} \, dS_t < \infty,$$
where dS_t is the surface measure induced on X_δ by g, and moreover
$$\lim_{r\to\infty} r^{3/2-\nu} u_\ell = 0.$$

5. *Each u_ℓ is axisymmetric in the sense that $\partial_\phi u = 0$, where ϕ is the azimuthal angle on $\mathbb{S}^2$.*

The frequencies λ_ℓ in Theorem 10 are QNFs, and u_ℓ are associated QNMs. The theorem is deduced from the existence of *real* frequencies $\lambda_\ell^\sharp \in \mathbb{R}$ and nonzero functions $u_\ell^\sharp \in C_c^\infty(X_0)$ for which (2), (4), (5) hold, such that (1) is approximately satisfied (see Theorem 13 below for a more precise statement regarding these quasimodes).

Remark 18. *The functions v_ℓ are smooth solutions to the Klein–Gordon equation in a region extending past the event horizon. This reflects the outgoing (into the horizon) nature of QNMs. Since u_ℓ does not vanish outside the event horizon one also obtains a nonzero solution to the Klein–Gordon equation by restricting to the black hole exterior.*

As will be clear from the proof, Theorem 10 is a *black box* in the sense that any sequence of quasimodes satisfying the conditions of Theorem 13 can be plugged into the machinery to obtain a corresponding sequence of QNFs. Furthermore, there is a relationship

$$|\lambda_\ell - \lambda_\ell^\sharp| \leq e^{-\ell/C} \tag{6.1}$$

for some $C > 0$. Thus any description of $\lambda_\ell^\sharp$ modulo $\mathcal{O}(\ell^{-\infty})$ gives a corresponding description for $\operatorname{Re}\lambda_\ell$ as $\ell \to \infty$. The imprecise localization of $\operatorname{Re}\lambda_\ell$ in Theorem 10 (for the rotating case) is therefore only due to the inexact nature of the quasimodes constructed in [56].

This should be compared to Theorem 9 of Chapter 5 in the simpler Schwarzschild–AdS setting; using the quasimodes constructed there, we get an improved result.

Theorem 11. *Let $a = 0$. For each $N \in \mathbb{N}$ there exists $L = L(N) \geq 0$ and sequences of QNFs*

$$\lambda_{0,\ell}, \ldots, \lambda_{N,\ell}, \quad \ell \geq L$$

converging exponentially to the real axis satisfying

$$\sqrt{\frac{3}{|\Lambda|}} \cdot \operatorname{Re}\lambda_{n,\ell} = \ell + 2n + \nu + 3/2 + \mathcal{O}(\ell^{-1/2}). \tag{6.2}$$

Furthermore, there are associated QNMs $u_{n,\ell}$ of the form

$$u_{n,\ell}(r,\omega) = w_{n,\ell}(r) \cdot S_{0\ell}(\omega),$$

where $S_{0\ell}$ is an axisymmetric spherical harmonic on $\mathbb{S}^2$.

We now make some remarks about the different hypotheses in Theorem 10.

Remark 19. *1. Exponential accuracy of the quasimodes is not necessary to deduce the existence of QNFs — see the proof of Theorem 10, as well as [92, 91, 94] for more general* *results in the Euclidean setting. Less accurate quasimodes could potentially result in slower convergence to the real axis, as well as a weaker version of* (6.1).

2. Theorem 10 still applies if the quasimodes are supported on finitely many eigenspaces of D_ϕ (uniformly in ℓ).

3. Quasimodes satisfying more general self-adjoint boundary conditions also yield a version of Theorem 10 — see the discussion in Section 6.2, as well as the statements of Propositions 6.2.1, 6.2.2, 6.2.3.

A natural question is to what extent the passage from quasimodes to QNFs depends on the exact form of the Kerr–AdS metric. Observe already that axial symmetry of the Kerr–AdS metric plays an important role in the statement of Theorem 10. This allows one to compensate for the fact that the Killing field ∂_t is not timelike near the event horizon for $a \neq 0$; see Propositions 6.2.1, 6.2.2 below. For the full range of parameters $|a|^2 < 3/|\Lambda|$, these propositions apply to stationary, *axisymmetric* perturbations of the metric (throughout, perturbations are assumed to be small).

On the other hand observe that the final ingredient in the proof of Theorem 10, Proposition 6.2.3 below, is always stable under stationary perturbations of the metric. The analysis is based on a general microlocal framework developed by Vasy [95], which is highly robust — see [95, Section 2.7] for a precise discussion.

6.2 Proof of Theorem 10

The proof of Theorem 10 relies on three key results, Propositions 6.2.1, 6.2.2, 6.2.3, stated in the next section. The proof of Proposition 6.2.3 is delayed until Section 6.3, while Proposition 6.2.1 is proved in Section 6.4 at the end of the chapter.

Quasinormal modes

We make two simplifications as compared to Chapter 4. First, since we are only concerned with QNFs near the real axis, we can work in the strip $\{\operatorname{Im} \lambda > -\kappa/2\}$. Thus we can take $k = 0$ in Theorems 6, 7, and therefore work with the simplest function spaces. This makes the application of energy estimates particularly easy. Secondly, in this chapter only self-adjoint Dirichlet or Robin boundary conditions are considered. The trace operator B is therefore

$$B = \gamma_- \text{ or } B = \gamma_+ + \beta\gamma_-,$$

where $\beta \in C^\infty(Y; \mathbb{R})$ is a real valued on the conformal boundary of X_δ. Then define the domain

$$\mathcal{X} = \begin{cases} \{u \in \mathcal{H}^1(X_\delta) : P(0)u \in \mathcal{L}^2(X_\delta)\} & \text{if } \nu \geq 1, \\ \{u \in \mathcal{H}^1(X_\delta) : P(0)u \in \mathcal{L}^2(X_\delta) \text{ and } Bu = 0\} & \text{if } \nu \in (0,1). \end{cases} \tag{6.3}$$

This is a Hilbert space for the norm $\|u\|_{\mathcal{X}} = \|u\|_{\mathcal{H}^1(X_\delta)} + \|P(0)u\|_{\mathcal{L}^2(X_\delta)}$.

QNFs are defined in the following theorem, which was proved in Chapter 4.

Theorem 12. *For each $\nu > 0$ the operator $P(\lambda) : \mathcal{X} \to \mathcal{L}^2(X_\delta)$ is Fredholm of index zero in the half-plane $\{\operatorname{Im}\lambda > -\frac{1}{2}\kappa\}$, where the surface gravity $\kappa > 0$ is given by (2.23). Furthermore,*

$$R(\lambda) := P(\lambda)^{-1} : \mathcal{L}^2(X_\delta) \to \mathcal{X}$$

is a meromorphic family of operators in $\{\operatorname{Im}\lambda > -\frac{1}{2}\kappa\}$, which is holomorphic in any angular sector of the upper half-plane provided $|\lambda|$ is sufficiently large.

QNFs are defined as poles of the meromorphic family $R(\lambda)$. More information is needed about possible QNFs in the upper half-plane — this is closely related to the boundedness of solutions to the Klein–Gordon equation. One remarkable property of rotating Kerr–AdS metrics is that for $|a| < r_+^2$ there exists a timelike Killing field K on $\mathcal{M}_0$ which is null on the horizon:

$$K = T + \Omega\Phi, \quad \Omega = \frac{a}{r_+^2 + a^2}.$$

The existence of such a vector field eliminates possible superradiant phenomena. For black holes satisfying the Hawking–Reall bound $|a| < r_+^2$, boundedness [58] (and in fact logarithmic decay [55]) is known for solutions to the Klein–Gordon under Dirichlet boundary conditions. If the condition $|a| < r_+^2$ is violated, then it is possible to construct mode solutions $e^{-i\lambda t^\star}u$ which grow exponentially in time, namely $\operatorname{Im}\lambda > 0$ [28].

Remark 20. *Interestingly, even for Neumann boundary conditions boundedness has not been established for the expected range of black hole parameters $|a| < r_+^2$, see the conjecture in [58, Section 5].*

For mode solutions $e^{-i\lambda t^\star}u$ many of the delicate issues involving lower order terms and boundary conditions are overcome in the high frequency limit. In fact by working at a fixed axial mode it is not even necessary to restrict below the Hawking–Reall bound; this is of course only possible because of the axisymmetry of the Kerr–AdS metric. In that case the Robin function β should satisfy $\Phi\beta = 0$ as well.

Note that $R(\lambda)$ decomposes as direct sum of operators

$$R(\lambda, k) : \mathcal{L}^2(X_\delta) \cap \mathcal{D}'_k(X_\delta) \to \mathcal{X} \cap \mathcal{D}'_k(X_\delta),$$

where $R(\lambda, k)$ is just the restriction of $R(\lambda)$ to $\mathcal{L}^2(X_\delta) \cap \mathcal{D}'_k(X_\delta)$. In particular, λ_0 is a QNF if and only if there exists $k_0 \in \mathbb{Z}$ such that λ_0 is a pole of $R(\lambda, k_0)$. The following crucial proposition quantifies the absence of QNFs at a fixed axial mode in the upper half-plane at high frequencies.

Proposition 6.2.1. . *Given $k \in \mathbb{Z}$ there exists $C_0 > 0$ such that if $\lambda \in (C_0, \infty) + i(0, \infty)$, then*

$$\|u\|_{\mathcal{L}^2(X_0)} \leq \frac{C}{|\lambda| \operatorname{Im}\lambda} \|P(\lambda, k)u\|_{\mathcal{L}^2(X_0)} \tag{6.4}$$

for each $u \in \mathcal{H}^1(X_0) \cap \mathcal{D}'_k(X_0)$ such that: i) $P(0)u \in \mathcal{L}^2(X_0)$, and ii) if $\nu \in (0, 1)$, then $Bu = 0$.

Proposition 6.2.1 is stated for functions on the exterior time slice X_0 rather than the extended region X_δ. On the other hand, Theorem 8 combined with Proposition 6.2.1 implies that $R(\lambda, k)$ has no poles in the region $\lambda \in (C_0, \infty) + i(0, \infty)$. Moreover, if $\mathcal{R} : \mathcal{L}^2(X_\delta) \to \mathcal{L}^2(X_0)$ is the restriction operator, then (6.4) implies

$$\|\mathcal{R}R(\lambda, k)f\|_{\mathcal{L}^2(X_0)} \leq \frac{C}{|\lambda| \operatorname{Im} \lambda} \|f\|_{\mathcal{L}^2(X_\delta)} \tag{6.5}$$

for each $f \in \mathcal{L}^2(X_\delta)$ and $\lambda \in (C_0, \infty) + i(0, \infty)$. The constant $C > 0$ in (6.5) a priori may depend on $k \in \mathbb{Z}$.

Remark 21. *Proposition 6.2.1 is also valid without restricting to a fixed axial mode provided the Hawking–Reall bound holds (as will be evident from the proof).*

It is also important to know that there are no QNFs on the real axis. The proof also exploits axisymmetry of the Kerr–AdS metric, hence the Robin function should satisfy $\Phi\beta = 0$ as well.

Proposition 6.2.2. *Let B be as in Proposition 6.2.1. Suppose that*

$$u \in \mathcal{H}^1(X_0) \cap C^\infty(X_0)$$

satisfies: i) u is smooth up to $\{r = r_+\}$, ii) $P(\lambda_0)u = 0$, iii) if $\nu \in (0, 1)$, then $Bu = 0$. Under these conditions, if $\lambda_0 \in \mathbb{R} \setminus 0$, then $u = 0$.

Proof. According to [99, Lemma A.1] the solution u vanishes in a neighborhood of $\{r = r_+\}$ within X_0. As in that lemma, one would like to apply some type of unique continuation result to conclude that once away from the horizon, u must vanish everywhere. This is known to be a difficult problem in view of possible trapping within the ergoregion where $P(\lambda)$ fails to be elliptic [64]. To work around this, define the Riemannian metric

$$h = \varrho^2 \left(\frac{1}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{(1 - \alpha)^2} d\phi^2 \right),$$

and observe that the difference between $P(\lambda)$ and the operator $\varrho^{-2}\Delta_r D_r^2 - \Delta_h$ is of first order modulo the second order term $2a\varrho^{-2}(1 - \alpha)D_r D_\phi$. Set

$$\widetilde{P}(\lambda, k) = \varrho^{-2}\Delta_r D_r^2 - \Delta_h + e^{-ik\phi^\star}(P(\lambda) - (\varrho^{-2}\Delta_r D_r^2 - \Delta_h))e^{ik\phi^\star}.$$

Thus $\widetilde{P}(\lambda, k)$ is elliptic on $\{r > r_+\}$, and furthermore

$$\widetilde{P}(\lambda, k)u = P(\lambda, k)u \tag{6.6}$$

for each $u \in \mathcal{D}'_k(X_0)$.

Now decompose u as an $\mathcal{L}^2(X_0)$ orthogonal sum $u = \sum_{k\in\mathbb{Z}} u_k$, where $u_k \in \mathcal{H}^1(X_0) \cap C^\infty(X_0) \cap \mathcal{D}'_k(X_0)$ is smooth up to $\{r = r_+\}$, satisfies $P(\lambda_0, k)u_k = 0$, and $Bu_k = 0$ if $\nu \in (0, 1)$. Unique continuation holds for $\widetilde{P}(\lambda, k)$, hence each u_k vanishes by (6.6) since it satisfies $\widetilde{P}(\lambda, k)u_k = 0$ and u_k vanishes in a neighborhood of the horizon. □

Again referring to Theorem 8, the previous proposition implies that $R(\lambda)$ does not have any poles for $\lambda \in \mathbb{R} \setminus 0$.

The crucial ingredient used to prove Theorem 10 is an exponential bound on $R(\lambda)$ in a strip away from suitable neighborhoods of the poles of $R(\lambda)$. At this stage it is convenient to introduce the semiclassical rescaling. Given a parameter $h > 0$, set $z = h\lambda$ and

$$P_h(z) = h^2 P(h^{-1}z), \quad R_h(z) = P_h(z)^{-1},$$

and similarly for $P_h(z,k)$ and $R_h(z,k)$. For the next proposition the Robin function β is not required to satisfy $\Phi\beta = 0$; in fact, no integrability properties of the metric are used at all. Fix intervals $[a,b] \subseteq (0,\infty)$ and $[C_-, C_+] \subseteq (-\kappa/2, \infty)$. Given $\varepsilon > 0$, define

$$\Omega_\varepsilon(h) = [a - \varepsilon h, b + \varepsilon h] + ih[C_- + \varepsilon, C_+ + \varepsilon]. \tag{6.7}$$

Also write $\Omega(h) := \Omega_0(h)$.

Proposition 6.2.3. *Fix $\varepsilon > 0$ and let $\{z_j\}$ denote the poles of $R_h(z)$ in $\Omega_\varepsilon(h)$. Then there exists $A > 0$ such that given any function $0 < S(h) = o(h)$,*

$$\|R_h(z)\|_{\mathcal{L}^2(X_\delta) \to \mathcal{L}^2(X_\delta)} < \exp\left(Ah^{-12}\log(1/S(h))\right), \tag{6.8}$$

for

$$z \in \Omega(h) \setminus \bigcup_j B(z_j, S(h))$$

and h sufficiently small.

Proposition 6.2.3 is proved in Section 6.3. Finally, the quasimode construction of [56] is reviewed:

Theorem 13 ([56, Theorem 1.2])**.** *There exists a sequence $\lambda_\ell^\sharp \in \mathbb{R}$ and*

$$u_\ell^\sharp \in \mathcal{H}^1(X_\delta) \cap C^\infty(X_\delta) \cap \mathcal{D}'_0(X_\delta)$$

with the following properties.

1. *There exists $C > 0$ such that $\ell/C < \lambda_\ell^\sharp < C\ell$.*

2. *There exists $[r_1, r_2] \subseteq (r_+, \infty)$ such that $\operatorname{supp} u_\ell^\sharp \subseteq (r_1, \infty)$ and $\operatorname{supp} P(\lambda_\ell^\sharp, 0)u_\ell^\sharp \subseteq (r_1, r_2)$.*

3. *The functions u_ℓ are normalized exponentially accurate quasimodes in the sense that*

$$\|u_\ell\|_{\mathcal{L}^2(X_\delta)} = 1, \quad \|P(\lambda_\ell^\sharp, 0)u_\ell\|_{\mathcal{L}^2(X_\delta)} \leq e^{-\ell/C}$$

for some $C > 0$. Furthermore, if $\nu \in (0,1)$, then $\gamma_- u_\ell = 0$.

As remarked in [56, Footnote 8], the boundary condition $B = \gamma_-$ in [56, Theorem 1.2] could also be replaced by a Robin boundary condition of the form $B = \gamma_+ + \beta\gamma_-$, where $\beta \in \mathbb{R}$ is a real constant. The only difference is that the Hardy inequality used in [56] now holds modulo a boundary term which is negligible in the semiclassical limit (after applying a trace identity [99, Lemma B.1]).

The proof of Theorem 10 can now be finished by the same arguments as in the work of Tang–Zworski [94] with refinements by Stefanov [92, 91].

Proof of Theorem 10. Define a semiclassical parameter h by $h^{-1} := \lambda_\ell^\sharp$ and set $u(h) := u_\ell$. Then, $h \to 0$ as $\ell \to \infty$.

Suppose that $R_h(z, 0)$ was holomorphic in the rectangle

$$[1 - 2w(h) - S(h), 1 + 2w(h) + S(h)] + i[-Ah^{-12}\log(1/S(h))S(h), S(h)], \tag{6.9}$$

where $A > 0$ is as in Proposition 6.2.3, and $w(h)$, $S(h)$ are to be specified. Then

$$F(z) := \mathcal{R}R_h(z, 0) : \mathcal{L}^2(X_\delta) \to \mathcal{L}^2(X_0)$$

is holomorphic in the smaller rectangle

$$\Sigma(h) = [1 - 2w(h), 1 + 2w(h)] + i[-Ah^{-12}\log(1/S(h))S(h), S(h)],$$

and satisfies the operator norm estimates

$$\|F(z)\| < \begin{cases} C/\operatorname{Im} z & \text{for } z \in \Sigma(h) \cap \{\operatorname{Im} z > 0\}, \\ e^{Ah^{-12}\log(1/S(h))} & \text{for } z \in \Sigma(h). \end{cases}$$

from Propositions 6.2.1, 6.2.3. Applying the semiclassical maximum principle [91, Lemma 1], it follows that

$$\|F(z)\| < e^3/S(h) \text{ for } z \in [1 - w(h), 1 + w(h)]$$

provided $w(h)$, $S(h)$ are chosen so that $e^{-B/h} < S(h) < 1$ for some $B > 0$ and

$$2Aph^{-12}\log(1/h)\log(1/S(h))S(h) \le w(h). \tag{6.10}$$

Define $\varepsilon(h) = \|P_h(1, 0)u(h)\|_{L^2(X_\delta)}$. Then,

$$1 = \|u(h)\|_{\mathcal{L}^2(X_\delta)} = \|F(1)P_h(1, 0)u(h)\|_{\mathcal{L}^2(X_0)} < e^3\varepsilon(h)/S(h).$$

This is a contradiction if $S(h) = 2e^3\varepsilon(h)$ for example. Thus there must exist a pole in the rectangle (6.9), which furthermore must lie in $\{\operatorname{Im} z < 0\}$ because of Propositions 6.2.1, 6.2.2. Defining $w(h)$ by the left hand side of (6.10), it follows that $R_h(z, 0)$ has a pole $z(h)$ such that

$$|z(h) - 1| < Ch^{-13}\log(1/h)\varepsilon(h), \quad 0 < -\operatorname{Im} z(h) < Ch^{-13}\varepsilon(h).$$

This is an even stronger statement than that of Theorem 10. □

The rest of the paper is dedicated to the proving Propositions 6.2.1 and 6.2.3.

6.3 Exponential bounds on the resolvent

The first goal is to prove Proposition 6.2.3. For this an approximate inverse is constructed for $P(\lambda)$ modulo an error of Schatten class. The approximate inverse is built up from local parametrices which invert $P(\lambda)$ near the event horizon and near the conformal boundary. This is similar to the black-box approach of Sjöstrand–Zworski [89].

Microlocal analysis of the stationary operator

In order to construct a local parametrix in a neighborhood of the event horizon using methods of Vasy [95], one needs to understand the Hamilton flow of the symbol of $P_h(z)$ near its characteristic set.

It is convenient to view the (rescaled) flow on a compactified phase space $\overline{T}^*X_\delta$. The fibers of $\overline{T}^*X_\delta$ are obtained by gluing a sphere at infinity to the fibers of T^*X_δ, so $\overline{T}^*X_\delta$ is a disk bundle whose interior is identified with T^*X_δ — see [95, Section 2.1], [81], as well as [32, Appendix E.1] for more details. If $|\cdot|$ is a smooth norm on the fibers of T^*X_δ, then a function on $\overline{T}^*X_\delta$ is smooth in a neighborhood of $\partial\overline{T}^*X_\delta$ if it is smooth in the polar coordinates $(x, \rho = |\xi|^{-1}, \omega = |\xi|^{-1}\xi)$, where $(x, \xi) \in T^*X_\delta$.

Throughout, $\operatorname{Im} z$ will satisfy $|\operatorname{Im} z| < Ch$ for some $C > 0$. In the semiclassical regime one may therefore assume that z is in fact real. The semiclassical principal symbol $p = \sigma_h(P_h(z))$ is given by

$$p(x, \xi; z) = -\varrho^2 g^{-1}(\xi \cdot dx - z\, dt^\star, \xi \cdot dx - z\, dt^\star),$$

where $\xi \cdot dx$ is the canonical one-form on X_δ. Explicitly, p is given by the expression

$$\begin{aligned} \varrho^{-2} p = {} & \Delta_r (\xi_r - f_+ z)^2 + \Delta_\theta \xi_\theta{}^2 + 2(1-\alpha)(\xi_r - f_+ z)\left(a\xi_\phi - (r^2 + a^2)z\right) \\ & + \frac{(1-\alpha)^2}{\Delta_\theta \sin^2\theta}\left(\xi_\phi - a\sin^2\theta\, z\right)^2, \end{aligned} \tag{6.11}$$

where $\xi_r, \xi_\theta, \xi_{\phi^\star}$ are the momenta conjugate to $r, \theta, \phi^\star$.

Note that $\langle\xi\rangle^{-2} p$ extends smoothly to a function on $\overline{T}^*X_\delta$ — here $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ with respect to the fixed norm $|\cdot|$ on T^*X_δ. Furthermore, the rescaled Hamilton vector field $\langle\xi\rangle^{-1} H_p$ extends to a smooth vector field on $\overline{T}^*X_\delta$ which is tangent to $\partial\overline{T}^*X_\delta$. The Hamilton flow of p will refer to integral curves of $\langle\xi\rangle^{-1} H_p$ in this compactified picture.

The characteristic set of p is given by $\Sigma = \{\langle\xi\rangle^{-2} p = 0\} \subseteq \overline{T}^*X_\delta$, and its complement is the elliptic set. Since $X_\delta \subseteq \mathcal{M}_\delta$ is spacelike, it is a general fact about Lorentzian metrics that for each $z \in \mathbb{R} \setminus 0$ the characteristic set is the union of two disjoint sets $\Sigma = \Sigma_\pm$, where

$$\Sigma_\pm = \Sigma \cap \{\pm \langle\xi\rangle^{-1} g(\xi \cdot dx - z\, dt^\star, dt^\star) < 0\}$$

are the backwards and forwards light cones [95, Section 3.2]. Each of these sets is invariant under the Hamilton flow. Finally, let

$$\widehat{\Sigma} = \Sigma \cap \partial\overline{T}^*X_\delta, \quad \widehat{\Sigma}_\pm = \Sigma_\pm \cap \partial\overline{T}^*X_\delta.$$

If $\partial\overline{T}^* X_\delta$ is identified with the unit sphere bundle S^*X_δ via a norm on the fibers, then $\widehat{\Sigma}$ corresponds to the characteristic set of the homogeneous principal symbol of $P(\lambda)$ (as a non-semiclassical differential operator) within S^*X. This agrees with the definition in Section 4.3.

If $x \in X_\delta$ and the vector field T is timelike at x, then p is elliptic near the fiber $\partial\overline{T}^*_x X_\delta$. Therefore the projection of $\widehat{\Sigma}$ onto the base space X_δ is contained within the ergoregion, namely the set where T is not timelike. It is easily checked in Boyer–Lindquist coordinates that the ergoregion is described by the inequality $\Delta_r \leq a^2\Delta_\theta \sin^2\theta$.

Let $\Lambda_\pm \subseteq T^*X$ denote the two components of the conormal bundle to $\{r = r_+\}$,

$$\Lambda_\pm = \{r = r_+,\, \pm\xi_r > 0,\, \xi_\theta = \xi_{\phi^\star} = 0\},$$

and let $L_\pm$ denote the images in $\partial\overline{T}^* X_+$ of $\Lambda_\pm$ under the canonical projection $T^*X_+ \setminus 0 \to \partial\overline{T}^* X$. These sets are invariant under the Hamilton flow of p, and $L_\pm \subseteq \widehat{\Sigma}_\pm$. The flow is denoted by

$$\varphi_t = \exp(t\,\langle\xi\rangle^{-1} H_p)$$

From the dynamical point of view, L_+ is a source and L_- a sink for the Hamilton flow. We analyzed this property for the flow within $\widehat{\Sigma}_\pm$ in Lemma 4.3.1. We now work in a full neighborhood of $L_\pm$ in $\overline{T}^* X_\delta$. Note that $\xi_r \neq 0$ in a neighborhood of $L_\pm$. In particular, the projective coordinates

$$\rho = |\xi_r|^{-1}, \quad \hat{\xi}_\theta = |\xi_r|^{-1}\xi_\theta, \quad \hat{\xi}_{\phi^\star} = |\xi_r|^{-1}\xi_{\phi^\star},$$

are valid near $\widehat{\Sigma}$. Then ρ is a (locally defined) boundary defining function for S^*X_δ, viewed as the boundary of the fiber-radial compactification $\overline{T}^* X_\delta$ (see [95, Section 2.1])

For now, we assume that $\lambda \in \mathbb{R}$. The vector field ρH_p extends smoothly up to S^*X_δ, and is tangent to S^*X_δ. To calculate this vector field, note that for $\pm\xi_r > 0$,

$$\partial_{\xi_r} = \mp\rho(\rho\partial_\rho + \hat{\xi}_\theta\partial_{\hat{\xi}_\theta} + \hat{\xi}_\phi\partial_{\hat{\xi}_\phi}), \quad \partial_{\xi_\theta} = \rho\partial_{\hat{\xi}_\theta}, \quad \partial_{\xi_\phi} = \rho\partial_{\hat{\xi}_\phi}.$$

The simultaneous vanishing of $\rho, \hat{\xi}_\theta, \hat{\xi}_\phi, \Delta_r$ defines $L_\pm$ in a neighborhood $L_\pm$. The rescaled Hamilton vector field for $\pm\xi_r > 0$ satisfies

$$\begin{aligned}|\xi_r|^{-1}H_p = &\pm 2(\Delta_r \pm a(1-\alpha)\hat{\xi}_\phi)\partial_r \pm (\partial_r\Delta_r)(\hat{\xi}_\theta\partial_{\hat{\xi}_\theta} + \hat{\xi}_\phi\partial_{\hat{\xi}_\phi} + \rho\partial_\rho)\\ &+ \mathcal{O}(\rho)\partial_r + \mathcal{O}(\hat{\xi}_\theta)\partial_\theta + \mathcal{O}(\hat{\xi}_\phi)\partial_\phi + \mathcal{O}(\hat{\xi}_\theta^2 + \hat{\xi}_\phi^2 + \Delta_r^2 + \rho^2).\end{aligned}$$

Define the functions

$$p_1 = \Delta_\theta{\xi_\theta}^2 + \frac{(1-\alpha)^2}{\Delta_\theta\sin^2\theta}\xi_\phi^2, \quad p_2 = (\Delta_r\xi_r + 2(1-\alpha)a\xi_\phi)^2.$$

Since p_1 is positive definite, the nonnegative quantities $\Delta_r^2 + \hat{\xi}_\theta^2 + \hat{\xi}_\phi^2$ and $\rho^2(p_1 + p_2)$ are comparable near $L_\pm$. In particular, the $\mathcal{O}(\hat{\xi}_\theta^2 + \hat{\xi}_\phi^2 + \Delta_r^2 + \rho^2)$ terms above can be replaced by $\mathcal{O}(\rho^2(1 + p_1 + p_2))$. Furthermore, if $\rho_1 = \rho^2(1 + p_2 + p_2)$, then

$$\pm\,|\xi_r|^{-1}H_p(\rho_1) \geq 2(\partial_r\Delta_r)\rho_1 + \mathcal{O}(\rho_1^{3/2}) \tag{6.12}$$

near $L_\pm$. The source/sink nature of L_+/L_- follows immediately from (6.12):

Lemma 6.3.1. *There exist neighborhoods $U_\pm$ of $L_\pm$ in $\overline{T}^*X_\delta$ such that if $(x,\xi) \in U_\pm \setminus L_\pm$, then $\varphi_t(x,\xi) \to L_\pm$ as $\mp t \to \infty$ and $\varphi_{\pm T}(x,\xi) \notin U_\pm$ for some $T > 0$.*

To analyze the flow more closely near $r = r_+$, observe that $H_p r$ evaluated at a point $(x, \xi; z)$ where $r = r_+$ is given by

$$H_p r = -g^{-1}(\xi \cdot dx - z\, dt^\star, dr).$$

If $r = r_+$, then

$$\varrho^2 H_p r = (1-\alpha)(a\xi_{\phi^\star} - (r_+^2 + a^2)z).$$

This cannot vanish for $(x,\xi) \in \Sigma \cap T^*X_\delta$ and $z \in \mathbb{R} \setminus 0$, since the condition $p = 0$ implies $\xi_{\phi^\star} = a \sin^2\theta\, z$. Hence

$$a\xi_{\phi^\star} - (r_+^2 + a^2)z = -(r_+^2 + a^2\cos^2\theta)z$$

which contradicts $H_p r = 0$. Furthermore, dr is null at $r = r_+$ and

$$g^{-1}(dr, dt^\star) = -\varrho^{-2}(1-\alpha)(r^2+a^2) < 0$$

at $r = r_+$, so dr lies in the opposite light cone as $dt^\star$. Since Σ_+ is the backwards light cone and Σ_- is the forward light cone, it follows that

$$\pm H_p r < 0 \text{ on } \Sigma_\pm \cap T^*X_\delta \cap \{r = r_+\} \tag{6.13}$$

for $z \in \mathbb{R} \setminus 0$.

Lemma 6.3.2. *There exists $\delta > 0$ such that φ_t satisfies the following conditions.*

1. *If $(x,\xi) \in \widehat{\Sigma}_\pm \setminus L_\pm$, then $\varphi_t(x,\xi) \to L_\pm$ as $\mp t \to \infty$, and $\varphi_T(x,\xi) \in \{r \leq r_+ - \delta\}$ for some $\pm T > 0$.*

2. *Suppose $z \in \mathbb{R} \setminus 0$. If $(x,\xi) \in (\Sigma_\pm \setminus L_\pm) \cap \{|r - r_+| \leq \delta\}$, then either $\varphi_t(x,\xi) \to L_\pm$ as $\mp t \to \infty$ and*

$$\varphi_T(x,\xi) \in \{|r - r_+| \geq \delta\}$$

for some $\pm T > 0$, or

$$\varphi_{\pm T_1}(x,\xi) \in \{r \leq r_+ - \delta\}, \quad \varphi_{\mp T_2}(x,\xi) \in \{r \geq r_+ + \delta\}$$

for some $T_1, T_2 > 0$.

Proof. 1. The first part is proved Lemma 4.3.2, which in turn follows from the same calculations as [95, Section 6.3].

2. The sets $U_\pm \supseteq L_\pm$ from Lemma 4.3.1 have the property that

$$K = (\Sigma \setminus (U_+ \cup U_-)) \cap \{|r - r_+| \leq \delta\}$$

is a compact subset of $\overline{T}^*X_\delta$ for $\delta > 0$ sufficiently small. If a flow line enters $U_\pm$, then it must tend to $L_\pm$ in one direction and leave $U_\pm$ in the other according to Lemma 4.3.1. As it leaves $U_\pm$, either $|r - r_+| \geq \delta$ or otherwise it enters the compact set K, at which point $H_p r$ has a definite sign according to (6.13). On the other hand, if a flow line never enters $U_\pm$, then it must escape K according to the sign of $H_p r$. □

Localizing the problem

As remarked in the introduction to this section, an approximate inverse for $P(\lambda)$ is contructed from local inverses, both near the event horizon and near infinity. Begin by decomposing

$$X_\delta = \{r_+ - \delta < r < R\} \cup \{r > R/2\},$$

where R will be fixed later. Dealing with the boundaries located at $r = R$ and $r = R/2$ is not convenient, and for this reason these cylinders are embedded in larger manifolds without these boundary components.

Let X_+ denote the larger cylinder $\{r_+ - \delta < r < 3R\}$ capped off with a 3-disk at $\{r = 3R\}$. The operator $P(\lambda)$, defined near $\{r_+ - \delta < r < R\}$, must be extended to X_+ in a suitable way. To do this, define the auxiliary manifold $\mathcal{M}_+ = \mathbb{R}_{t^\star} \times X_+$. The goal is to extend the original metric g to $\mathcal{M}_+$, and then consider the stationary Klein–Gordon operator with respect to the extended metric.

The extended metric will be chosen so that $\mathcal{M}_+$ is foliated by spacelike surfaces $\{t^\star = \text{constant}\}$. Such a metric is uniquely determined by its ADM decomposition: begin with a smooth function $A > 0$, an arbitrary vector field W, and a Riemannian metric h, all defined initially on X_+. These objects are extended to $\mathcal{M}_+$ by requiring them to be independent of $t^\star$.

The data (A, W, h) determines a stationary Lorentzian metric g_+ on $\mathcal{M}_+$ by defining

$$\begin{aligned} g_+(T,T) = A^2 - h(W,W), \quad g_+(T,V) = -h(W,V), \\ g_+(V,V) = -h(V,V), \end{aligned} \tag{6.14}$$

where V is a vector tangent to $\{t^\star = \text{constant}\}$.

In standard terminology, A is the lapse function, and W is the shift vector associated to g_+. If N_t denotes the unit normal to $\{t^\star = \text{constant}\}$, then A and W can be uniquely recovered from g_+ via the formulas

$$A = g_+(N_t, T) = g_+^{-1}(dt^\star, dt^\star)^{-1}, \quad W = \partial_{t^\star} - A\, N_t.$$

Similarly, h is recovered as the induced metric on $\{t^\star = \text{constant}\}$. The idea is to extend g to $\mathcal{M}_+$ by extending the data (A, W, h) originally defined near $\mathbb{R}_{t^\star} \times \{r_+ - \delta < r < R\}$.

Lemma 6.3.3. *Let $R > 0$ be such that T is timelike for g near $\{r > R/10\}$. There exists a stationary Lorentzian metric g_+ on $\mathcal{M}_+$ such that*

1. *$g_+ = g$ near $\mathbb{R}_{t^\star} \times \{r_+ - \delta < r < R\}$,*

2. *$dt^\star$ is everywhere timelike for g_+,*

3. *T is timelike for g_+ near $\mathbb{R}_{t^\star} \times \{r > R/10\}$.*

Proof. Let (A, W, h) denote the ADM data for g originally defined near $\mathbb{R}_{t^\star} \times \{r_+ - \delta < r < R\}$. Fix any function $\widehat{A} > 0$ and Riemannian metric $\widehat{h}$ on X_+ such that

$$\widehat{A} = A, \quad \widehat{h} = h$$

near $\{r_+ - \delta < r < 2R\}$. Choose a cutoff $\chi = \chi(r)$ with values $0 \leq \chi \leq 1$, such that $\operatorname{supp} \chi \subseteq \{r_+ - \delta \leq r < 2R\}$ and $\chi = 1$ near $\{r_+ - \delta \leq r < R\}$. Then, define $\widehat{W} = \chi W$.

The metric g_+ determined by $(\widehat{A}, \widehat{W}, \widehat{h})$ satisfies the requirements of the lemma; the only part which is not immediate is the last part. If $\chi > 0$, then $A = \widehat{A}$ and $h = \widehat{h}$, so

$$g_+(T, T) = A^2 - \chi^2 h(W, W) \geq A^2 - h(W, W) > 0.$$

On the other hand, if $\chi = 0$, then $\widehat{W} = 0$ and so $g_+(T, T) = \widehat{A}^2 > 0$. □

Given Lemma 6.3.3, define the stationary operator

$$P_+(\lambda) = e^{i\lambda t^\star} \varrho^2 \left(\Box_{g_+} + \nu^2 - 9/4\right) e^{-i\lambda t^\star}$$

on X_+, and similarly for the semiclassical version $P_{h,+}(z)$. Here we arbitrarily extend ϱ^2 as a positive function to X_+; since this function is uniformly bounded, it does not play a meaningful role.

The same argument can also be applied to the infinite part $\{r > R/2\}$: in that case X_∞ is obtained by capping $\{r \geq R/4\}$ at $r = R/4$. Then g is extended to a stationary metric g_∞ on $\mathcal{M}_\infty = \mathbb{R}_{t^\star} \times X_\infty$, and the analogue Lemma 6.3.3 holds in such a way that T is everywhere timelike for the extended metric. Thus it is possible to define an extended operator $P_\infty(\lambda)$ and $P_{h,\infty}(z)$.

An approximate inverse near the event horizon

The manifold X_+ can be bordified by adding the artificial boundary

$$H := \{r = r_+ - \delta\}$$

within X_+. The space of distributions $\overline{H}^1(X_+)$ which are extendible at H is therefore well defined. If $L^2(X_+)$ is defined with respect to any positive density μ on the compact manifold with boundary $X_+ \cup H$, then $\overline{H}^1(X_+)$ agrees with the space of distributions $u \in L^2(X_+)$ such that $du \in L^2(X_+)$ (with respect to any smooth norm on covectors). Then define

$$\mathcal{X}_+ = \{u \in \overline{H}^1(X_+) : P_+(0)u \in L^2(X_+)\},$$

equipped with the norm $\|u\|_{L^2(X_+)} + \|P_+(0)u\|_{L^2(X_+)}$.

Proposition 6.3.4. *The operator $P_+(\lambda) : \mathcal{X}_+ \to L^2(X_+)$ is Fredholm of index zero in the half-plane $\{\operatorname{Im}\lambda > -\frac{1}{2}\kappa\}$. Furthermore,*

$$R_+(\lambda) := P_+(\lambda)^{-1} : L^2(X_+) \to \mathcal{X}_+$$

is a meromorphic family of operators in $\{\operatorname{Im}\lambda > -\frac{1}{2}\kappa\}$ which is holomorphic in any angular sector of the upper half-plane, provided $|\lambda|$ is sufficiently large.

Proof. Let p_+ denote the semiclassical principal symbol of $P_{h,+}(z)$. Note that $p = p_+$ near the characteristic set of p_+ at fiber infinity (whose projection to X_+ is contained in a neighborhood of $\{r < R/10\}$). Therefore the hypotheses of [95, Section 2.6] at fiber infinity of also hold for p_+, since they were verified for p in Section 4.3. That suffices to show that $P_+(0)$ is Fredholm on $\mathcal{X}_+$.

The invertibility statement follows from the ellipticity of $P_{h,+}(z)$ on compact subsets of phase space for $\operatorname{Im} z > 0$, which in turn is a corollary of the fact that $dt^\star$ is timelike for g_+ — see [95, Sections 3.2, 7]. □

A more refined invertibility statement for $R_+(\lambda)$ will be proved at the end of the section. Before doing this, one must also consider a nontrapping model where $P_{h,+}(z)$ is modified by a complex absorbing operator $-iQ$. This will be a compactly supported semiclassical pseudodifferential operator which is compactly microlocalized in the sense that $\mathrm{WF}_h(Q)$ is a compact subset of $T^*X_+ \subset \overline{T}^*X_+$ — see [32, Appendix E.2] for more about microlocalization. We denote the space of such operators on a manifold X by $\Psi_h^{\mathrm{comp}}(X)$.

In the next lemma, $\Sigma_\pm$ will denote the two components of the characteristic set Σ of p_+ as in Section 6.3 (replacing p with p_+). This separation is possible since $dt^\star$ is timelike for g_+. Furthermore φ_t will denote the Hamilton flow of p_+.

Lemma 6.3.5. *Fix $[a,b] \subseteq (0,\infty)$. Then there exists compactly supported $Q_\pm \in \Psi_h^{\mathrm{comp}}(X_+)$ such that if $q_\pm = \sigma_h(Q_\pm)$, then for each $z \in [a,b]$,*

1. *$\mathrm{WF}_h(Q_\pm) \subseteq \Sigma_\pm \cap T^*X_+$ and $\pm q_\pm \leq 0$,*
2. *If $(x,\xi) \in \Sigma_\pm \setminus L_\pm$, then either $\varphi_t \to L_\pm$ as $\mp t \to \infty$, or $\varphi_T(x,\xi) \in \{q_\pm \neq 0\}$ for some $\mp T > 0$.*

Proof. According to Lemma 6.3.2, there exist compact sets

$$K_\pm \subseteq \Sigma_\pm \cap \{r \geq r_+ + \delta\} \cap T^*X_+$$

such that if $(x,\xi) \in \Sigma_\pm \setminus K_\pm$ then either $\varphi_t(x,\xi) \to L_\pm$ as $\mp t \to \infty$, or $\varphi_T(x,\xi) \in K_\pm$ for some $\mp T > 0$. It then suffices to quantize functions $\pm q_\pm \leq 0$ which satisfy $\pm q_\pm < 0$ near $K_\pm$ and have compact support within $\Sigma_\pm \cap T^*X_+$. This may be done uniformly for z in any compact subset of $\mathbb{R} \setminus 0$. □

Fix an interval $[a,b] \subseteq (0,\infty)$ and define $Q = Q_+ + Q_-$, where $Q_\pm$ are given by Lemma 6.3.5. Then the modified operator $P_{h,+}(z) - iQ$ satisfies the nontrapping condition of [95, Definition 2.12], and hence [95, Theorem 2.14] is valid:

Proposition 6.3.6. *Fix $[a,b] \subseteq (0,\infty)$ and $[C_-, C_+] \subseteq (-\kappa/2, \infty)$. Then there exists $C > 0$ and $h_0 > 0$ such that*

$$\mathcal{R}_{h,+}(z) := (P_{h,+}(z) - iQ)^{-1} : L^2(X_+) \to \mathcal{X}_+$$

exists and satisfies the non-trapping bounds

$$\|\mathcal{R}_{h,+}(z)u\|_{L^2(X_+)\to \overline{H}^1_h(X_+)} \leq Ch^{-1}$$

for $z \in [a,b] + ih[C_-, C_+]$ and $h \in (0, h_0)$.

Remark 22. *The semiclassical Sobolev space $\overline{H}^1_h(X_+)$ appearing in Proposition 6.3.6 is $\overline{H}^1(X_+)$ as a set, but equipped with the h-dependent norm*

$$\|u\|_{\overline{H}^1_h(X_+)} = \|u\|_{L^2(X_+)} + h\|du\|_{L^2(X_+)}.$$

The next task is to prove that $R_{h,+}(z)$ exists for $z \in [a,b] + i[Mh, \infty)$ and $M > 0$ sufficiently large, with a corresponding estimate. This does not follow from Proposition 6.3.4 since there one must take $\operatorname{Im} z > 0$ independent of h. The simplest way to prove this result is by energy estimates.

If h is the induced metric on X_+, then $|\det g_+| = A^2 \det h$, where $A = g_+^{-1}(dt^\star, dt^\star)^{-1/2}$ is the lapse function associated with g_+. Then differentiating the usual divergence theorem with respect to $t^\star$ yields the following: let N_t be the unit normal to X_+ and N_r be the unit normal to $\mathbb{R}_{t^\star} \times H$; these are oriented so that N_t points in the direction of increasing $t^\star$, while N_r points in the direction of decreasing r. Then for any vector field V on $\mathcal{M}_+$,

$$\partial_{t^\star} \int_{X_+} g_+(V, N_t)\, dS_t + \int_H g_+(V, N_r) A\, dS_H = \int_{X_+} \left(\operatorname{div}_{g_+} V\right) A\, dS_t. \tag{6.15}$$

Here dS_t is the induced measure on X_+, and dS_H is the induced measure on H. Note that both N_t and N_r are timelike, and they lie in the same lightcone on their common domain of definition.

The covariant stress-energy tensor $\mathbb{T} = \mathbb{T}[v]$ associated to the Klein–Gordon equation is

$$\mathbb{T}(Y,Z) = \left[\operatorname{Re}(Yv \cdot Z\overline{v}) - \tfrac{1}{2} g_+(Y,Z) g_+^{-1}(dv, d\overline{v})\right] + \tfrac{1}{2} g_+(Y,Z)(\nu^2 - 9/4)|v|^2.$$

Here v is a sufficiently smooth function on $\mathcal{M}_+$ and Y, Z are real vector fields on $\mathcal{M}_+$. It is well known that the sum in square brackets is positive definite in dv provided Y, Z are both timelike in the same lightcone [60, Lemma 24.1.2].

Given a real vector field Y, let $\mathbb{J}^Y = \mathbb{J}^Y[v]$ be the unique vector field such that $g_+(\mathbb{J}^Y, Z) = \mathbb{T}(Y, Z)$. Then

$$\operatorname{div}_{g_+} \mathbb{J}^Y = \operatorname{Re}\left((\Box_g + \nu^2 - 9/4)v \cdot Y\overline{v}\right) + \mathbb{K}^Y(dv, d\overline{v}), \tag{6.16}$$

for a certain tensor $\mathbb{K}^Y$ whose precise properties are not used in this paper. Apply (6.15) to the vector field $\mathbb{J}^Y$. This yields the identity

$$\partial_{t^\star} \int_{X_+} \mathbb{T}(Y, N_t)\, dS_t + \int_H \mathbb{T}(Y, N_r) A\, dS_H = \int_{X_+} \left(\operatorname{Re}\left((\Box_{g_+} + \nu^2 - 9/4)v \cdot Y\overline{v}\right) + \mathbb{K}^Y(dv, d\overline{v})\right) A\, dS_t. \tag{6.17}$$

Now suppose that Y, Z are stationary in the sense that $[Y, T] = [Z, T] = 0$. Let d_λ denote the covector

$$d_\lambda u := \lambda u\, dt^\star + du,$$

where du is the differential of u on X_+, and define $Y(\lambda)u := e^{i\lambda t} Y e^{-i\lambda t} u$. Then the stress-energy tensor associated to $e^{-i\lambda t^\star} u$ satisfies

$$e^{-2\operatorname{Im}\lambda t^\star}\mathbb{T}[e^{-i\lambda t^\star}u](Y, Z) = \left[\operatorname{Re}\left(Y(\lambda)u \cdot \overline{Z(\lambda)u}\right) - \tfrac{1}{2}g_+(Y, Z)g_+^{-1}(d_\lambda u, \overline{d_\lambda u})\right] + \tfrac{1}{2}(\nu^2 - 9/4)g_+(Y, Z)|u|^2,$$

and the term inside the square brackets is positive definite in $d_\lambda u$ if Y, Z are timelike in the same light cone.

On the other hand, if $v = e^{-i\lambda t^\star} u$ then the integrand on the right hand side of (6.17) can be written as

$$e^{2(\operatorname{Im}\lambda)t^\star}\left(\varrho^{-2}\operatorname{Re}(P_+(\lambda)u \cdot \overline{Y(\lambda)u}) + \mathbb{K}^Y(d_\lambda u, \overline{d_\lambda u})\right).$$

Lemma 6.3.7. *There exists $C_0, C_1 > 0$ and $C > 0$ such that*

$$|\lambda|\|u\|_{L^2(X_+)} + \|du\|_{L^2(X_+)} \leq \frac{C}{\operatorname{Im}\lambda}\|P_+(\lambda)u\|_{L^2(X_+)}$$

for each $\lambda \in (C_0, \infty) + i(C_1, \infty)$ and $u \in \mathcal{X}_+$.

Proof. Apply (6.17) with the multipler $Y = N_t$ and $v = e^{-i\lambda t^\star}u$. The resulting identity is independent of $t^\star$ after multiplying by $e^{-2(\operatorname{Im}\lambda)t^\star}$.

Write $f = P_+(\lambda)u$, and consider each term separately. Since N_t is timelike, the zeroth order term in the first integrand on the left can be absorbed by the part which is positive definite in $d_\lambda u$ by taking $|\lambda|$ sufficiently large. The same observation holds for the integral over H, which is positive for $|\lambda|$ sufficiently large. Therefore,

$$2\operatorname{Im}\lambda\left(|\lambda|^2\|u\|^2_{L^2(X_+)} + \|du\|^2_{L^2(X_+)}\right) \leq \int_{X_+} \varrho^{-2}\left(\operatorname{Re}(f \cdot \overline{N_t(\lambda)u}) + \mathbb{K}^Y(d_\lambda u, \overline{d_\lambda u})\right) A\, dS_t.$$

Note that $\varrho^{-2}A$ is bounded by compactness of $X_+ \cup H$, and hence the quadratic form in $d_\lambda u$ coming from the deformation tensor can be absorbed into the left hand side for $\operatorname{Im}\lambda > 0$ sufficiently large. Similarly, Cauchy–Schwarz gives

$$|f \cdot \overline{N_t(\lambda)u}| < \frac{|f|^2}{2\varepsilon \operatorname{Im}\lambda} + \frac{\varepsilon \operatorname{Im}\lambda\, |N_t(\lambda)u|^2}{2} \tag{6.18}$$

for $\operatorname{Im}\lambda > 0$, and the second term can be absorbed into the left hand side for $\varepsilon > 0$ sufficiently small (after being multiplied by $\varrho^{-2}A$). This finally gives

$$2\operatorname{Im}\lambda \left(|\lambda|^2 \|u\|^2_{L^2(X_+)} + \|du\|^2_{L^2(X_+)} \right) \leq \frac{C}{\operatorname{Im}\lambda} \|f\|^2_{L^2(X_+)}$$

as desired. The previous calculations can certainly be justified for $u \in C^\infty(X_+ \cup H)$, which is enough since $C^\infty(X_+ \cup H)$ is dense in $\mathcal{X}_+$ — see [32, Lemma E.4.2] or [95, Section 2.6] for example. □

Lemma 6.3.7 implies that $P_+(\lambda)$ is injective for $\lambda \in (C_0, \infty) + i(C_1, \infty)$. Thus $P_+(\lambda)$ is invertible for such λ since it is of index zero according Proposition 6.3.4. Furthermore, after applying the semiclassical rescaling, there exists $C > 0$ and $M > 0$ such that for any $[a, b] \subseteq (0, \infty)$,

$$\|R_{h,+}(z)\|_{L^2(X_+) \to \overline{H}^1_h(X_+)} \leq \frac{C}{Mh} \tag{6.19}$$

for $z \in [a, b] + i[Mh, \infty)$ and h sufficiently small.

An approximate inverse near infinity

The next step is to prove an analogue of Lemma 6.3.7 for the operator $P_\infty(\lambda)$ defined in Section 6.3. This will now involve boundary contributions from the conformally timelike boundary $\mathscr{I} = \{r^{-1} = 0\}$, here viewed as a subset of $\mathcal{M}_\infty$. In terms of the boundary defining function $s = r^{-1}$, the metric g_∞ on $\mathcal{M}_\infty$ has an expansion near $\mathscr{I}$ of the form

$$g_\infty = \frac{-ds^2 + \gamma + \mathcal{O}(s^2)}{s^2},$$

where γ is a Lorentzian metric on $\mathscr{I}$. In particular, $\bar{g}_\infty := s^2 g_\infty$ is smooth up to $\mathscr{I}$ and the metric induced on the boundary by $\bar{g}_\infty$ is γ. The function s is extended as a positive function to all of $\mathcal{M}_\infty$ such that $Ts = 0$, and r is extended as well via the formula $r = s^{-1}$.

If ε is sufficiently small, then $\{s < \varepsilon\}$ defines a neighborhood of $\mathscr{I}$ in $\mathcal{M}_\infty$. Let N_t and N_s be unit normals to X_∞ and $X_\infty \cap \{s = \varepsilon\}$ with respect to g_∞ such that N_t points in the direction of increasing $t^\star$, and N_s points outwards. The analogue of (6.15) becomes

$$\partial_{t^\star} \int_{X_\infty \cap \{s \geq \varepsilon\}} g_\infty(V, N_t)\, dS_t - \int_{X_\infty \cap \{s = \varepsilon\}} g_\infty(V, N_s) A\, d\mathcal{K} = \int_{X_\infty \cap \{s \geq \varepsilon\}} (\operatorname{div}_{g_\infty} V)\, A\, dS_t \tag{6.20}$$

Here dS_t and $d\mathcal{K}$ are the measures induced on X_∞ and $X_\infty \cap \{s = \varepsilon\}$ by g_∞, and A is the lapse function for g_∞.

The data associated with the foliation by surfaces of constant $t^\star$ have conformal analogues: if h is the induced metric on X_+, then

$$A = s^{-1}\bar{A}, \quad h = s^{-2}\,\bar{h},$$

where $\bar{A}, \bar{h}$ are smooth up to $\mathscr{I}$. These induce conformally related measures on X_∞ and $X_\infty \cap \{s = \varepsilon\}$ by

$$dS_t = s^{-3}d\bar{S}_t, \quad d\mathcal{K} = s^{-2}d\bar{\mathcal{K}}.$$

In addition $N_t = s\bar{N}_t$ and $N_s = s\bar{N}_s$ where $\bar{N}_t, \bar{N}_s$ are the corresponding unit normals for $\bar{g}_\infty$, smooth up to $\mathscr{I}$. The goal is to eventually let $\varepsilon \to 0$ in (6.20).

Remark 23. *In the previous paragraph, bars over certain quantities do not indicate complex conjugates.*

Written in terms of s, the traces $\gamma_\pm$ have the form

$$\gamma_- v = s^{\nu-3/2}v|_{\mathscr{I}}, \quad \gamma_+ v = -s^{1-2\nu}\partial_s(s^{\nu-3/2}v)|_{\mathscr{I}}.$$

Although $\gamma_\pm$ can be given weak formulations, for the energy estimates it is more useful to work with a space of smooth functions on which $\gamma_\pm$ are well defined. Given $\nu \in (0,1)$, let $\mathcal{F}_\nu(\mathcal{M}_\infty)$ denote the space of all $v \in C^\infty(\mathcal{M}_\infty)$ admitting a conormal expansion

$$v(s,y) = s^{3/2+\nu}v_+(s^2,y) + s^{3/2-\nu}v_-(s^2,y) \tag{6.21}$$

near $\mathscr{I}$, where $(s,y) \in [0,\varepsilon) \times \mathscr{I}$, and $v_\pm$ are smooth up to $\mathscr{I}$. If $v \in \mathcal{F}_\nu(\mathcal{M}_\infty)$ is given by (6.21), then

$$\gamma_- v(\cdot) = v_-(0,\cdot), \quad \gamma_+ v(\cdot) = (-2\nu)v_+(0,\cdot).$$

If $\nu \geq 1$, then one defines $\mathcal{F}_\nu(\mathcal{M}_\infty)$ as all smooth functions on $\mathcal{M}_\infty$ vanishing in a neighborhood of $\mathscr{I}$. The space $\mathcal{F}_\nu(X_\infty)$ is defined in the same way, simply replacing $\mathcal{M}_\infty$ with X_∞.

For general boundary conditions (of the type considered in Section 6.2), the boundary contribution on $X_\infty \cap \{s = \varepsilon\}$ arising from the usual stress-energy tensor will diverge as $\varepsilon \to 0$. This can be remedied by introducing a "twisted" stress-energy tensor as in [58, 100, 99] — the reader is referred to these works for a more complete point of view.

Fix a real, non-vanishing function f. Given a vector field Y on $\mathcal{M}_\infty$, define the operator $\widetilde{Y}$ by

$$\widetilde{Y}v = f\,Y(f^{-1}v),$$

as well as the covector $\tilde{d}v = f\,d(f^{-1}v)$. The twisted stress-energy tensor $\widetilde{\mathbb{T}} = \widetilde{\mathbb{T}}[v]$ is defined by

$$\begin{aligned}\widetilde{\mathbb{T}}(Y,Z) = &\left[\mathrm{Re}\left(\widetilde{Y}v \cdot \widetilde{Z}\bar{v}\right) - \tfrac{1}{2}g_\infty(Y,Z)g_\infty^{-1}(\tilde{d}v,\tilde{d}\bar{v})\right] \\ &+ \tfrac{1}{2}g_\infty(Y,Z)(F+\nu^2-9/4)|v|^2,\end{aligned} \tag{6.22}$$

where $F = f^{-1}\Box_{g_\infty}(f)$ is a scalar potential term. The term in square brackets is positive definite in $\tilde{d}v$ provided Y, Z are timelike in the same lightcone. The twisting function f is chosen so that

$$F + \nu^2 - 9/4 = \mathcal{O}(s^2).$$

If Y, Z are smooth up to $\mathscr{I}$, this guarantees that $g_\infty(Y,Z)(F + \nu^2 - 9/4)$ is also smooth up to $\mathscr{I}$. The simplest choice of f with this property is $f = s^{3/2-\nu}$.

Remark 24. *Since $(F + \nu^2 - 9/4)|v|^2$ is of zeroth order in v, the precise sign properties of F will not be important in the high frequency regime (just as in Lemma 6.3.7). More refined choices of f leading to positive F are discussed at length in [58].*

Next, let $\widetilde{\mathbb{J}}^Y = \widetilde{\mathbb{J}}^Y[v]$ denote the associated energy current, namely $g_\infty(\widetilde{\mathbb{J}}^Y, Z) = \widetilde{\mathbb{T}}(Y,Z)$.

Lemma 6.3.8. *Suppose that Y Killing for g_∞ and $Yf = 0$. Then*

$$\operatorname{div}_{g_\infty} \widetilde{\mathbb{J}}^Y = \operatorname{Re}\left((\Box_{g_\infty} + \nu^2 - 9/4)v \cdot Y\overline{v}\right)$$

Proof. Since Y is Killing, the condition $Yf = 0$ also implies $YF = 0$, and then the result follows from a direct calculation [99, Lemma 2.5]. □

If $f = s^{3/2-\nu}$, then $Tf = 0$ and hence Lemma 6.3.8 is valid. Now apply (6.20) to the vector field $\widetilde{\mathbb{J}}^T$, where $v \in \mathcal{F}_\nu(\mathcal{M}_\infty)$. Consider the integral over $X_\infty \cap \{s \geq \varepsilon\}$, which can be written as

$$\int_{X_\infty \cap \{s \geq \varepsilon\}} \widetilde{\mathbb{T}}(T, \bar{N}_t)\, s^{-2}\, d\bar{S}_t.$$

Checking the various powers of s, this integral has a limit as $\varepsilon \to 0$ for $v \in \mathcal{F}_\nu(\mathcal{M}_\infty)$. This also motivates the following spaces: let $\mathcal{L}^2(X_\infty)$ denote the space of distributions for which

$$\|u\|_{\mathcal{L}^2(X_\infty)} = \int_{X_\infty} |u|^2\, s\, dS_t < \infty,$$

and let $\mathcal{H}^1(X_\infty)$ denote the space of distributions for which

$$\|u\|_{\mathcal{H}^1(X_\infty)} = \int_{X_\infty} \left(|u|^2 + s^{-2} g_\infty^{-1}\left(\tilde{d}u, \tilde{d}\bar{u}\right)\right) s\, dS_t < \infty.$$

Compare these spaces with those defined in Section 4.2.

Next, consider the integral in (6.20) over $X_\infty \cap \{s = \varepsilon\}$. This is only relevant in the case $\nu \in (0,1)$ since if $\nu \geq 1$ then the integral automatically vanishes for $v \in \mathcal{F}_\nu(\mathcal{M}_\infty)$ and $\varepsilon > 0$ sufficiently small. Since T and N_s are orthogonal and $Tf = 0$, this reduces to

$$\int_{X_\infty \cap \{s = \varepsilon\}} \operatorname{Re}\left(Tv \cdot \widetilde{N}_s \bar{v}\right) s^{-3} \bar{A}\, d\bar{\mathcal{K}}. \tag{6.23}$$

Furthermore, $N_s = -s\partial_s + \mathcal{O}(s^3)$. Write

$$s^{-3}\,Tv \cdot \widetilde{N}_s\bar{v} = \left(s^{\nu-3/2}Tv\right)\left(s^{-3/2-\nu}\widetilde{N}_s\bar{v}\right),$$

and notice that this tends to $\gamma_-(Tv)\cdot\gamma_+\bar{v}$ as $\varepsilon \to 0$ for $v \in \mathcal{F}_\nu(X_\infty)$.

Taking $\varepsilon \to 0$ in (6.20), one therefore has the identity

$$\partial_{t^\star}\int_{X_\infty}\widetilde{\mathbb{T}}(T,N_t)\,dS_t - \int_{X_\infty\cap\mathscr{I}}\operatorname{Re}\left(\gamma_-(Tv)\cdot\gamma_+\bar{v}\right)\bar{A}\,d\bar{\mathcal{K}} = \int_{X_\infty}\operatorname{Re}\left((\Box_{g_\infty}+\nu^2-9/4)v\cdot T\bar{v}\right)A\,dS_t. \quad (6.24)$$

Using (6.24), it is now straightforward to prove the analogue of Lemma 6.3.7. In the following, either $B=\gamma_-$ or $B = \gamma_+ + \beta\gamma_-$, where $\beta \in C^\infty(\mathscr{I};\mathbb{R})$ satisfies $T\beta = 0$; therefore β may considered as a function on $X_\infty \cap \mathscr{I}$.

Lemma 6.3.9. *There exists $C_0 > 0$ and $C > 0$ such that*

$$|\lambda|\|u\|_{\mathcal{L}^2(X_\infty)} + \|u\|_{\mathcal{H}^1(X_\infty)} \le \frac{C}{\operatorname{Im}\lambda}\|P_\infty(\lambda)u\|_{\mathcal{L}^2(X_\infty)}$$

for each $\lambda \in (C_0,\infty)+i(0,\infty)$ and $u \in \mathcal{F}_\nu(X_\infty)$ such that $Bu = 0$ if $\nu \in (0,1)$.

Proof. The proof is close to that of Lemma 6.3.7. Apply (6.24) to a function $v = e^{-i\lambda t^\star}u$, where $u \in \mathcal{F}_\nu(X_\infty)$, and multiply the resulting identity by $e^{-2\operatorname{Im}\lambda\, t^\star}$. Since T and $\bar{N}_t$ are timelike in the same lightcone, the first integral in (6.24) controls

$$2\operatorname{Im}\lambda\left(|\lambda|^2\|u\|^2_{\mathcal{L}^2(X_\infty)} + \|u\|^2_{\mathcal{H}^1(X_\infty)}\right)$$

for $\operatorname{Im}\lambda > 0$ and $|\lambda|$ sufficiently large. Write $f = P_\infty(\lambda)u$, and write the integral on the right hand side of (6.24) in terms of the conformal measure. Recall that $\varrho^2 \sim s^{-2}$, so up to a uniformly bounded factor the integrand involving f is

$$s^{-2}\bar{A}\operatorname{Re}\left(i\overline{\lambda u}\cdot f\right)$$

Then the integrand is bounded by

$$s^{-2}\bar{A}\operatorname{Re}\left(i\overline{\lambda u}\cdot f\right) \le \frac{s^{-2}}{2\varepsilon\operatorname{Im}\lambda}\cdot|f|^2 + \frac{\varepsilon s^{-2}\operatorname{Im}\lambda\,|\lambda|^2}{2}\bar{A}^2\cdot|u|^2. \quad (6.25)$$

Now integrate (6.25) over X_∞ with respect to $d\bar{S}_t$. The integral of the first term on the right hand side is bounded by $C(\operatorname{Im}\lambda)^{-1}\|f\|^2_{\mathcal{L}^2(X_\infty)}$, while the integral of the second term can be absorbed into the left hand side for ε sufficiently small.

It remains to handle the integral over $X_\infty \cap \mathscr{I}$. If u satisfies Dirichlet boundary condition (which recall is automatic for $\nu \geq 1$), then this term vanishes. Otherwise, if $\nu \in (0,1)$ and $B = \gamma_+ + \beta\gamma_-$, then the integrand becomes (after accounting for the minus sign in (6.24))

$$2\,\mathrm{Im}\,\lambda \int_{X_\infty \cap \mathscr{I}} \beta \cdot |\gamma_- u|^2 \, \bar{A}\, d\bar{\mathcal{K}}.$$

If β is nonnegative this term can be dropped. Otherwise, note that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\int_{X_\infty \cap \mathscr{I}} |\gamma_- u|^2 \, d\bar{\mathcal{K}} \leq \varepsilon \|u\|^2_{\mathcal{H}^1(X_\infty)} + C_\varepsilon \|u\|^2_{\mathcal{L}^2(X_\infty)}.$$

This is proved directly in [99, Lemma B.1]; alternatively, it also follows from the boundedness of the trace $\gamma_- : \mathcal{H}^1(X_\infty) \to H^\nu(X_\infty \cap \mathscr{I})$ and the compactness of the inclusion $H^\nu(X_\infty \cap \mathscr{I}) \hookrightarrow L^2(X_\infty \cap \mathscr{I})$. Hence the boundary term can always be absorbed by the left hand side for any $\mathrm{Im}\,\lambda > 0$ and $|\lambda|$ sufficiently large. □

Now define

$$\mathcal{X}_\infty = \begin{cases} \{u \in \mathcal{H}^1(X_\infty) : P_\infty(0)u \in \mathcal{L}^2(X_\infty)\} & \text{if } \nu \geq 1, \\ \{u \in \mathcal{H}^1(X_\infty) : P_\infty(0)u \in \mathcal{L}^2(X_\infty) \text{ and } Bu = 0\} & \text{if } \nu \in (0,1), \end{cases} \tag{6.26}$$

equipped with the graph norm — compare to Section 6.2. Note that $P_\infty(\lambda)$ is elliptic at fiber-infinity $\partial\overline{T}^* X_\infty$, where X_∞ viewed as a noncompact manifold without boundary. Furthermore, $P_\infty(\lambda)$ is elliptic at $X_\infty \cap \mathscr{I}$ as a Bessel operator with respect to the boundary defining function $s = r^{-1}$. By elliptic regularity [40, Theorem 1 and Lemma 4.13] it is therefore possible to show that all computations in Lemma 6.3.9 for $u \in \mathcal{F}_\nu(X_\infty)$ are also valid for $u \in \mathcal{X}_\infty$. In particular,

$$|\lambda| \|u\|_{\mathcal{L}^2(X_\infty)} + \|u\|_{\mathcal{H}^1(X_\infty)} \leq \frac{C}{\mathrm{Im}\,\lambda} \|P_\infty(\lambda)u\|_{\mathcal{L}^2(X_\infty)} \tag{6.27}$$

for each $u \in \mathcal{X}_\infty$ and $\lambda \in (C_0, \infty) + i(0,\infty)$.

In addition, since B is an elliptic boundary condition in the sense of Bessel operators [40, Section 4.4], one has the following:

Lemma 6.3.10. *The operator $P_\infty(\lambda) : \mathcal{X}_\infty \to \mathcal{L}^2(X_\infty)$ is Fredholm of index zero, and is invertible outside arbitrarily small angles about the real axis for $|\lambda|$ sufficiently large.*

As a corollary of (6.27), the operator $R_\infty(\lambda) := P_\infty(\lambda)^{-1}$ exists for $\lambda \in (C_0, \infty) + i(0,\infty)$. In terms of the semiclassical rescaling, there exists $C > 0$ such that for each $[a,b] \subseteq (0,\infty)$,

$$\|R_{h,\infty}(z)\|_{\mathcal{L}^2(X_\infty) \to \mathcal{H}^1_h(X_\infty)} \leq \frac{C}{\mathrm{Im}\, z} \tag{6.28}$$

for $z \in [a,b] + i(0,\infty)$ and h sufficiently small.

Construction of a global approximate inverse

Combining the results of Sections 6.3, 6.3, it is now possible to construct a global approximate inverse for $P(\lambda)$ on X_δ.

Fix a smooth partition of unity $\chi_1 + \chi_2 = 1$ and functions ψ_1, ψ_2 on X_δ such that

- $\operatorname{supp}\chi_1 \cup \operatorname{supp}\psi_1 \subseteq \{r > R/2\}$ and $\operatorname{supp}\chi_2 \cup \operatorname{supp}\psi_2 \subseteq \{r < R\}$,
- $\psi_1 = 1$ near $\operatorname{supp}\chi_1$ and $\psi_2 = 1$ near $\operatorname{supp}\chi_2$.

Begin by fixing $M > 0$ such that (6.19) holds. Then, choose $[C_-, C_+] \subseteq (-\kappa/2, \infty)$ such that $M \in (C_-, C_+)$ and $|C_+ - M| \geq |C_- - M|$, increasing C_+ if necessary.

Given an interval $[a, b] \subseteq (0, \infty)$, the operators $\mathcal{R}_{h,+}(z), R_{h,+}(z_0), R_{h,\infty}(z_0)$ exist for $z \in [a, b] + ih[C_-, C_+]$ and $z_0 \in [a, b] + ih[M, C_+]$. Define

$$E(z, z_0) = \psi_1 R_{h,\infty}(z_0)\chi_1 \\ + \psi_2\left(\mathcal{R}_{h,+}(z) - iR_{h,+}(z_0)Q\mathcal{R}_{h,+}(z)\right)\chi_2,$$

where $z \in [a, b] + ih[C_-, C_+]$ and $z_0 \in [a, b] + ih[M, C_+]$. This is a well defined operator $\mathcal{L}^2(X_\delta) \to \mathcal{X}$ in view of the cutoffs, and it is holomorphic in z for each z_0.

Apply $P_h(z)$ to $E(z, z_0)$ on the left: the first term yields

$$\chi_1 + \psi_1\left(P_{h,\infty}(z) - P_{h,\infty}(z_0)\right)R_{h,\infty}(z_0)\chi_1 + [P_h(z), \psi_1]R_{h,\infty}(z_0)\chi_1, \tag{6.29}$$

while the second term yields

$$\chi_2 - i\psi_2\left(P_{h,+}(z) - P_{h,+}(z_0)\right)R_{h,+}(z_0)Q\mathcal{R}_{h,+}(z)\chi_2 \\ + [P_h(z), \psi_2]\left(\mathcal{R}_{h,+}(z) - iR_{h,+}(z_0)Q\mathcal{R}_{h,+}(z)\right)\chi_2. \tag{6.30}$$

Adding (6.29), (6.30), one has

$$P_h(z)E(z, z_0) = I + K_1(z, z_0) + K_2(z, z_0) + K_3(z, z_0) + K_4(z, z_0),$$

where K_1, K_2 are the second and third terms in (6.29), and K_3, K_4 are the second and third terms in (6.30). Also let $K = K_1 + K_2 + K_3 + K_4$.

Lemma 6.3.11. *There exist compactly supported pseudodifferential operators*

$$A(z) \in h\Psi_h^{-1}(X_+), \quad B(z) \in h\Psi_h^{-\infty}(X_+)$$

depending smoothly on z such that

$$[P_h(z), \psi_2]\mathcal{R}_{h,+}(z) = A(z) + B(z)\mathcal{R}_{h,+}(z)$$

for $z \in [a, b] + ih[C_-, C_+]$.

Proof. The commutator $[P_{h,+}(z), \psi_2]$ has coefficients supported near $\operatorname{supp} d\psi_2$ where it may be assumed that $P_{h,+}(z)$ is elliptic at fiber infinity $\partial \overline{T}^* X_+$. Choose $\varphi, \varphi' \in C_c^\infty(X_+)$ satisfying $\varphi = 1$ near $\operatorname{supp} d\phi_2$ and $\varphi' = 1$ near $\operatorname{supp} \varphi$. By choosing φ' with sufficiently small support it may be assumed that $P_{h,+}(z)$ is ellipic near $\pi^{-1}(\operatorname{supp} \varphi)$, where $\pi : \partial \overline{T}^* X_+ \to X_+$ is the canonical projection. By ellipticity, there exist properly supported operators

$$F(z) \in \Psi_h^{-2}(X_+), \quad Y(z) \in h^\infty \Psi_h^{-\infty}(X_+)$$

and compactly supported $C \in \Psi_h^{\mathrm{comp}}(X_+)$ such that

$$\varphi = F(z)\varphi' P_{h,+}(z) + Y(z) + C.$$

The operators $F(z)$, $Y(z)$ may be chosen to depend smoothly on z, and C can be chosen uniformly for z in a compact set. Then,

$$\begin{aligned}[P_h(z), \psi_2]\mathcal{R}_{h,+}(z) &= [P_h(z), \psi_2]\varphi \mathcal{R}_{h,+}(z) \\ &= [P_h(z), \psi_2]\left(F(z)\varphi' + (Y(z) + C)\mathcal{R}_{h,+}(z)\right).\end{aligned}$$

Since the commutator lies in $h\Psi_h^1(X_+)$, it suffices to define $A(z) = [P_h(z), \psi_2]F(z)\varphi'$ and $B(z) = [P_h(z), \psi_2](Y(z) + C)$. □

Lemma 6.3.12. *If $M > 0$ is sufficiently large, then there exists h_0 depending on M such that the following hold for $z \in [a, b] + ih[C_-, C_+]$ and $z \in [a, b] + ih[M, C_+]$.*

1. *$K(z, z_0) : \mathcal{L}^2(X_\delta) \to \mathcal{L}^2(X_\delta)$ is compact.*

2. *$I + K(z_0, z_0)$ is invertible for $h \in (0, h_0)$.*

Proof. To prove compactness, consider each term in $K(z, z_0)$ separately. For K_1, K_2, the operator $P_{h,\infty}(z)$ is an elliptic Bessel operator. Furthermore, if $\nu \in (0, 1)$, then the boundary operator B is elliptic in the sense of Section 3.4. By elliptic regularity (Theorem 1),

$$\left(P_{h,+}(z) - P_{h,+}(z_0)\right) R_{h,+}(z_0) : \mathcal{L}^2(X_+) \to \mathcal{H}^1(X_+)$$

is bounded, noting that $P_{h,+}(z) - P_{h,+}(z_0)$ is of first order. The inclusion $\mathcal{H}^1(X_+) \hookrightarrow \mathcal{L}^2(X_+)$ is compact, which shows that

$$K_1(z, z_0) : \mathcal{L}^2(X_\delta) \to \mathcal{L}^2(X_\delta)$$

is compact. A similar argument also shows that $K_2(z, z_0)$ is compact.

For $K_3(z, z_0), K_4(z, z_0)$, each of the terms containing Q are compact since any compactly supported operator in $\Psi_h^{\mathrm{comp}}(X_+)$ is smoothing. It remains to consider the commutator term $[P_h(z), \psi_2]\mathcal{R}_{h,+}(z)$. For that, compactness follows from Lemma 6.3.11.

To prove the invertibility statement, notice that for $z = z_0$,

$$K(z_0, z_0) = [P_h(z_0), \psi_1]R_{h,\infty}(z_0)\chi_1 + [P_h(z_0), \psi_2]R_{h,+}(z_0)\chi_2. \tag{6.31}$$

As first order operators the commutators are of order $\mathcal{O}(h)$, where the implicit constants are independent of any $M > 0$ for h sufficiently small depending on M. By choosing $M > 0$ sufficiently large and applying (6.19), (6.28), the operator norm of $K(z_0, z_0)$ is of order $\mathcal{O}(M^{-1})$, hence $I + K(z_0, z_0)$ is invertible by Neumann series for h sufficiently small. □

From now on it will be assumed that $M > 0$ is chosen sufficiently large so that Lemma 6.3.12 holds. This can always be achieved *before* selecting $C_\pm$ since (6.31) does not involve the operator $\mathcal{R}_{h,+}(z)$.

Since $K(z, z_0)$ is compact and $I + K(z_0, z_0)$ is invertible for an appropriate choice of z_0 with h small, it follows that $(I + K(z, z_0))^{-1} : \mathcal{L}^2(X_\delta) \to \mathcal{L}^2(X_\delta)$ is a meromorphic family of operators. If $(I + K(z, z_0))^{-1}$ exists, then $P_h(z) : \mathcal{X} \to \mathcal{L}^2(X_\delta)$ has a right inverse given by $E(z, z_0)(I + K(z, z_0))^{-1}$. In that case $P_h(z)$ is invertible, since it is of index zero by Theorem 12. Analytic continuation then shows that

$$R_h(z) = E(z, z_0)(I + K(z, z_0))^{-1}.$$

Furthermore, any pole of $R_h(z)$ is also a pole of $(I + K(z, z_0))^{-1}$.

Singular values

In order to prove (6.2.3) of Proposition 6.2.3, one must bound

$$\|R_h(z)\|_{\mathcal{L}^2(X_\delta)\to\mathcal{L}^2(X_\delta)} \leq \|E(z, z_0)\|_{\mathcal{L}^2(X_\delta)\to\mathcal{L}^2(X_\delta)}\|(I + K(z, z_0))^{-1}\|_{\mathcal{L}^2(X_\delta)\to\mathcal{L}^2(X_\delta)}.$$

Using (6.19), (6.28) and Lemma 6.3.2, the operator norm of $E(z, z_0)$ is of order $\mathcal{O}(h^{-2})$, which will be harmless compared to the exponentially growing bound on the norm of $(1 + K(z, z_0))^{-1}$.

Lemma 6.3.13 ([45, Theorem V.5.1])**.** *Let $\mathscr{H}$ be a Hilbert space. Then*

$$\|(I + A)^{-1}\|_{\mathscr{H}\to\mathscr{H}} \leq \frac{\det(I + |A|)}{|\det(I + A)|}.$$

for any operator $A : \mathscr{H} \to \mathscr{H}$ of trace class.

Lemma 6.3.13 cannot be applied directly to $I + K(z, z_0)$ since $K(z, z_0)$ is not of trace class. Instead, $K(z, z_0)$ lies in a Schatten p-class for some $p > 0$. For a compact operator $A : \mathscr{H}_1 \to \mathscr{H}_2$ between Hilbert spaces, let $s_j(A) = s_j(A; \mathscr{H}_1, \mathscr{H}_2)$, $j \in \mathbb{N}_{\geq 1}$ denote its singular values, listed in decreasing order with multiplicity.

Lemma 6.3.14. *There exists $C > 0$ such that the singular values of $K(z, z_0)$ satisfy*

$$s_j(K(z, z_0)) \leq Ch^{-3} j^{-1/3}$$

uniformly for $z \in [a, b] + ih[C_-, C_+]$ and $z_0 \in [a, b] + ih[M, C_+]$.

Proof. By Fan's inequality $s_{i+j-1}(A+B) \leq s_i(A) + s_j(B)$ applied repeatedly,

$$s_j(K(z,z_0)) \leq \sum_{i=1}^{4} s_{4j-3}(K_i(z,z_0)).$$

The operator $K_1(z,z_0) : \mathcal{L}^2(X_\delta) \to \mathcal{H}_h^1(X_\infty)$ is of order $\mathcal{O}(h^{-1})$, whereas the inclusion $\mathcal{H}_h^1(X_\infty) \hookrightarrow \mathcal{L}^2(X_\infty)$ has singular values bounded by $Ch^{-1}j^{-1/3}$ (see Section 3.8). By standard properties of singular values,

$$s_j(K_1(z,z_0)) \leq Ch^{-2}j^{-1/3}.$$

The same argument applies to $K_2(z,z_0)$, so accounting for the extra power of h coming from the commutator,

$$s_j(K_1(z,z_0)) \leq Ch^{-1}j^{-1/3}.$$

The terms K_2, K_3 can be handled similarly, using that the inclusion $\overline{H}_h^1(X_+) \hookrightarrow L^2(X_+)$ has singular values bounded by $Ch^{-1}j^{-1/3}$. The only term that requires extra care is $[P_h(z), \psi_2]\mathcal{R}_{h,+}(z)$ in K_4, but by using Lemma 6.3.11 this operator is seen to map $L^2(X_+) \to \overline{H}_h^1(X_+)$ with operator norm of order $\mathcal{O}(h)$. □

It follows from Lemma 6.3.14 that $K(z,z_0)^4$ is of trace class, and using Fan's inequality $s_{i+j-1}(AB) \leq s_i(A)s_j(B)$, one has the estimate

$$s_j(K(z,z_0)^4) \leq s_{4j-3}(K(z,z_0))^4 \leq Ch^{-12}\, j^{-4/3}. \tag{6.32}$$

This is uniform for $z \in [a,b] + ih[C_-, C_+]$ and $z_0 \in [a,b] + ih[M, C_+]$. To apply Lemma 6.3.13, write

$$(I + K(z,z_0))^{-1} = \left(\sum_{j=0}^{3} (-1)^j K(z,z_0)^j \right) (I - K(z,z_0)^4)^{-1} \tag{6.33}$$

The norm of $K(z,z_0) : \mathcal{L}^2(X_\delta) \to \mathcal{L}^2(X_\delta)$ is of order $\mathcal{O}(h^{-2})$, so the norm of the sum on the right hand side of (6.33) is polynomially bounded in h.

Note from (6.33) that any pole of $(I + K(z,z_0))^{-1}$ is a pole of $(I - K(z,z_0)^4)^{-1}$, hence the poles of $R_h(z)$ are among those of $(I - K(z,z_0)^4)^{-1}$. Now Lemma 6.3.13 is applied to $K(z,z_0)^4$.

Lemma 6.3.15. *Let $F(h)$ denote the supremum of* $\det(I + |K(z,z_0)^4|)$ *for* $z \in [a,b] + ih[C_-, C_+]$ *and* $z_0 \in [a,b] + ih[M, C_+]$. *Then*

$$F(h) \leq e^{Ch^{-12}}$$

for some $C > 0$.

Proof. The determinant is bounded by

$$\det(I + |K(z,z_0)^4|) = \prod_{j\geq 1}(1 + s_j(K(z,z_0)^4)) \leq \exp\left(Ch^{-12}\sum_{j\geq 1} j^{-4/3}\right)$$

according to (6.32). □

The next step is to bound $|f(z,z_0)|$ from below, where $f(z,z_0) = \det(I - K(z,z_0)^4)$.

Lemma 6.3.16. *The function $f(z,z_0)$ has the following properties.*

1. $|f(z,z_0)| \leq F(h)$.
2. $f(z_0,z_0) \neq 0$, *and moreover* $|f(z_0,z_0)| \geq e^{-Ch^{-12}}$ *for some* $C > 0$.

Proof. 1) The estimate $|f(z,z_0)| \leq F(h)$ follows from Weyl convexity inequalities [32, Proposition B.2.4].

2) As in the proof of Lemma 6.3.12, the norm of $K(z_0,z_0)^4$ is of order $\mathcal{O}(M^{-4})$ for h sufficiently small. By increasing $M > 0$ if necessary, the operator $I + K(z_0,z_0)^4$ is invertible, and

$$(I - K(z_0,z_0)^4)^{-1} = I + K(z_0,z_0)^4(I - K(z_0,z_0)^4)^{-1}. \tag{6.34}$$

Arguing as in Lemma 6.3.15,

$$\det(I + |K(z_0,z_0)^4(I + K(z_0,z_0)^4)^{-1}|) \leq e^{Ch^{-12}},$$

which gives $|f(z_0,z_0)| \geq e^{-Ch^{-12}}$. □

The proof of Proposition 6.2.3 can now be finished using the following lemma of Cartan [73, Theorem 11]:

Lemma 6.3.17. *Suppose that $g(z)$ is holomorphic in a neighborhood of a disk $B(z_0,R)$, such that $g(z_0) \neq 0$. Fix $r \in (0,R)$, and let $\{w_j\}$ denote the zeros of $g(z)$ in $B(z_0,R)$ for $j = 1,\ldots,n(z_0,R)$. Given any $\rho > 0$,*

$$\log|g(z)| - \log|g(z_0)| \geq -\frac{2r}{R-r}\log\left(\sup_{|z-z_0|<R}|g(z)|\right) - n(z_0,R)\log\left(\frac{(R+r)}{\rho}\right)$$

for $z \in B(z_0,r)\setminus\bigcup_j B(w_j,\rho)$.

Lemma 6.3.17 will be applied to the function $z \mapsto f(z,z_0)$ and disks of radius proportional to h. This requires a bound on the number of zeros of $f(z,z_0)$ in disks of the form $B(z_0,Rh)$.

As noted at the beginning of Section 6.3, it may always be assumed that $|C_+ - M| \geq |C_- - M|$. Recall the definitions of $\Omega_\varepsilon(h)$ and $\Omega(h)$ from (6.7). Then given $\varepsilon > 0$ there exists

$M' \geq M$ and $R > 0$ such that the union of all disks $B(w, Rh)$ with $w \in [a, b] + ih[M, M']$ covers $\Omega(h)$ and is contained in $\Omega_\varepsilon(h)$. If $\varepsilon > 0$ is sufficiently small, then

$$\Omega_{3\varepsilon}(h) \subseteq [a', b'] + ih[C'_-, C'_+],$$

where $[a', b'] \subseteq (0, \infty)$ and $[C'_-, C'_+] \subseteq (-\kappa/2, \infty)$. Applying Lemmas 6.3.15, 6.3.16 to this larger rectangle shows that certainly $|f(z, z_0)| \leq e^{Ch^{-12}}$ for $z_0 \in [a, b] + i[M, M']$ and $z \in B(z_0, (R + 2\varepsilon)h)$.

Lemma 6.3.18. *Let $z_0 \in [a, b] + ih[M, M']$. Then there exists $C > 0$ such*

$$n(z_0, (R + \varepsilon)h) \leq Ch^{-12},$$

uniformly in z_0, where $n(z_0, (R + \varepsilon)h)$ is the number of zeros of $f(z, z_0)$ in $B(z_0, (R + \varepsilon)h)$.

Proof. By Jensen's formula, the number of zeros $n(z_0, \rho)$ of $f(z, z_0)$ within $B(z_0, \rho)$ satisfies

$$\begin{aligned}\int_0^{h(R+2\varepsilon)} \frac{n(z_0, \rho)}{\rho} d\rho &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(z_0 + h(R + 2\varepsilon)e^{i\theta}, z_0)| \, d\theta - \log |f(z_0, z_0)| \\ &\leq Ch^{-12}.\end{aligned}$$

Therefore the number of zeros in a disk $n(z_0, (R + \varepsilon)h)$ is estimated by

$$\frac{\varepsilon}{R + 2\varepsilon} n(z_0, (R + \varepsilon)h) \leq \int_{h(R+\varepsilon)}^{h(R+2\varepsilon)} \frac{n(z_0, \rho)}{\rho} \, d\rho \leq Ch^{-12}.$$

□

Remark 25. *For each fixed z_0, the poles of $R_h(z)$ are among the zeros of $f(z, z_0)$. The region $\Omega(h)$ can be covered by at most $\mathcal{O}(h^{-1})$ disks of radius Rh with centers in $[a, b] + ih[M, M']$. According to Lemma 6.3.18, $R_h(z)$ has at most $\mathcal{O}(h^{-12})$ poles in each of these disks, so altogether $R_h(z)$ has at most $\mathcal{O}(h^{-13})$ poles in $\Omega(h)$.*

Combining Lemma 6.3.17 with Lemma 6.3.18 shows that for any function $0 < S(h) = o(h)$ and $z_0 \in [a, b] + ih[M, M']$,

$$|f(z, z_0)| \geq \exp(-Ch^{-12} \log(1/S(h))), \quad z \in B(z_0, Rh) \setminus \bigcup_j B(w_j, S(h)), \tag{6.35}$$

where $\{w_j\}$ are the zeros of $f(z, z_0)$ in $B(z_0, (R + \varepsilon)h)$.

Proof of Proposition 6.2.3. Combine Lemma 6.3.15 with (6.35). This shows that for any $z_0 \in [a, b] + ih[M, M']$,

$$\|R_h(z)\|_{\mathcal{L}^2(X_\delta) \to \mathcal{L}^2(X_\delta)} \leq \exp(Ch^{-12} \log(1/S(h))) \tag{6.36}$$

for

$$z \in B(z_0, Rh) \setminus \bigcup_j B(w_j, S(h)),$$

where $\{w_j\}$ are the zeros of $f(z, z_0)$ in $B(z_0, (R+\varepsilon)h)$. If w_j is not a pole of $R_h(z)$, then apply the maximum principle to the holomorphic operator-valued function $R_h(z)$ on $B(w_j, S(h))$ to see that (6.36) is valid on $B(w_j, S(h))$ as well. Thus (6.36) holds for $z \in B(z_0, Rh) \setminus \bigcup_j B(z_j, S(h))$, where now $\{z_j\}$ denote the poles of $R_h(z)$ in $B(z_0, (R+\varepsilon)h)$.

The disks $B(z_0, Rh), B(z_0, (R+\varepsilon)h)$ for $z_0 \in [a, b] + ih[M, M']$ cover $\Omega(h)$, $\Omega_\varepsilon(h)$ respectively, whence the result follows.

□

6.4 Proof of Proposition 6.2.1

To complete the proof of Theorem 10 it remains to prove Proposition 6.2.1. The proof is very similar to that of Lemmas 6.3.7 and 6.3.9. The twisted stress-energy tensor $\widetilde{\mathbb{T}} = \widetilde{\mathbb{T}}[v]$ is defined as in (6.24) using the metric g; the twisting function is again $f = r^{\nu-3/2}$.

The future pointing Killing generator K of the null surface $\{r = r_+\}$ can be normalized by requiring $Kt^\star = 1$, in which case

$$K = T + \frac{a}{r_+^2 + a^2}\Phi.$$

Let $d\sigma$ denote the measure induced on $X_0 \cap \{r = r_+\}$. With the above normalization, the analogue of (6.24) on X_0 has the form

$$\begin{aligned}\partial_{t^\star} \int_{X_0} \widetilde{\mathbb{T}}(Y, N_t)\, dS_t - \int_{X_0 \cap \mathscr{I}} \operatorname{Re}\left(\gamma_-\left(Yv\right) \cdot \gamma_+ \overline{v}\right) \bar{A}\, d\bar{\mathcal{K}} \\ = -\int_{X_0 \cap \{r=r_+\}} \widetilde{\mathbb{T}}(Y, K)\sqrt{A}\, d\sigma + \int_{X_0} \operatorname{Re}\left((\Box_g + \nu^2 - 9/4)v \cdot Y\overline{v}\right) A\, dS_t,\end{aligned}$$

where Y is a Killing field such that $Yr = 0$ and $A = g^{-1}(dt^\star, dt^\star)^{-1/2}$ is the lapse function.

This identity is applied with the vector field $Y = K$. The contribution from the horizon is the integral of

$$\widetilde{\mathbb{T}}(K, K) = |Kv|^2 \geq 0,$$

which may be dropped in view of its nonnegativity.

If $|a| < r_+^2$, then K is everywhere timelike on $\mathcal{M}_0$. In that case Proposition 6.2.1 would follow from the same proof as in Lemma 6.3.9; the only difference is that coercivity of the derivative transverse to the horizon degenerates at the horizon. This does not affect the final result since Proposition 6.2.1 only involves an L^2 bound.

Without the timelike assumption, a direct calculation in terms of the metric coefficients gives

$$\frac{2}{\sqrt{A}}\widetilde{\mathbb{T}}(\partial_{t^\star}, N_t) = g^{t^\star t^\star}|Tv|^2 - g^{rr}|\tilde{\partial}_r v|^2 - g^{r\phi^\star}\operatorname{Re}(\tilde{\partial}_r v \cdot \Phi\overline{v}) \\ - g^{\phi^\star\phi^\star}|\Phi v|^2 - g^{\theta\theta}|\partial_\theta v|^2 + (\nu^2 - n^2/4 + F)|v|^2,$$

as well as

$$\frac{1}{A}\widetilde{\mathbb{T}}(\partial_{\phi^\star}, N_t) = g^{t^\star t^\star}\operatorname{Re}(\Phi v \cdot T\overline{v}) + g^{t^\star r}\operatorname{Re}(\Phi v \cdot \tilde{\partial}_r \overline{u}) + g^{t^\star\phi^\star}|\Phi v|^2.$$

If $v \in \mathcal{D}'_0(X_0)$, then $\Phi v = 0$ and hence

$$\widetilde{\mathbb{T}}(K, N_t) = \frac{A}{2}\left(g^{t^\star t^\star}|Tv|^2 - g^{rr}|\tilde{\partial}_r v|^2 - g^{\theta\theta}|\partial_\theta v|^2 + (\nu^2 - 9/4 + F)|v|^2\right). \tag{6.37}$$

Although g^{rr} vanishes at $r = r_+$, the coefficients of $|\tilde{\partial}_r v|^2$ and $|\partial_\theta v|^2$ are certainly nonnegative on X_0, while the coefficient of $|Tv|^2$ is strictly positive.

Proof of Proposition 6.2.1. If $v = e^{-i\lambda t^\star}u$ for an axisymmetric function on u, then the contribution from $\widetilde{\mathbb{T}}(K, N_t)$ will control $|\lambda|^2\|u\|^2_{\mathcal{L}^2(X_0)}$ for $|\lambda|$ large since the coefficient of $|Tv|^2$ is strictly positive. The proof can now be finished as in Lemma 6.3.9. □

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