Hydrodynamics: from effective field theory to holography

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Abstract

Hydrodynamics is an effective theory that is extremely successful in describing a wide range of physical phenomena in liquids, gases and plasmas. However, our understanding of the structure of the theory, its microscopic origins and its behaviour at strong coupling is far from complete.

To understand how an effective theory of dissipative hydrodynamics could emerge from a closed microscopic system, we analyse the structure of effective Schwinger-Keldysh Closed-Time-Path theories. We use this structure and the action principle for open systems to derive the energy-momentum balance equation for a dissipative fluid from an effective CTP Goldstone action. Near hydrodynamical equilibrium, we construct the first-order dissipative stress-energy tensor and derive the Navier-Stokes equations. Shear viscosity is shown to vanish, while bulk viscosity and thermodynamical quantities are determined by the form of the effective action.

The exploration of strongly interacting states of matter, particularly in the hydrodynamic regime, has been a major recent application of gauge/string duality. The strongly coupled theories involved are typically deformations of large-N SUSY gauge theories with exotic matter that are unusual from a low-energy point of view. In order to better interpret holographic results, an understanding of the weak-coupling behaviour of such gauge theories is essential. We study the exact and SUSY-broken $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-QED with finite densities of electron number and R-charge, respectively. Despite the fact that fermionic fields couple to the chemical potentials, the strength of scalar-fermion interactions, fixed by SUSY, prevents a Fermi surface from forming. This is important for hydrodynamical excitations such as zero sound. Intriguingly, in the absence of a Fermi surface, the total charge need not be stored in the scalar condensates alone and fermions may contribute.

Gauss-Bonnet gravity is a useful laboratory for non-perturbative studies of the higher derivative curvature effects on transport coefficients of conformal fluids with holographic duals. It was previously known that shear viscosity can be tuned to zero by adjusting the Gauss-Bonnet coupling, λ_{GB}, to its maximal critical value. To understand the behaviour of the fluid in this limit, we compute the second-order transport coefficients non-perturbatively in λ_{GB} and show that the fluid still produces entropy, while diffusion and sound attenuation are suppressed at all order in the hydrodynamic expansion. We also show that the theory violates a previously proposed universal relation between three of the second order transport coefficients. We further compute the only second-order coefficient thus far unknown, λ_2, in the $\mathcal{N} = 4$ super Yang-Mills theory with the leading-order 't Hooft coupling correction. Intriguingly, the universal relation is not violated by these leading-order perturbative corrections. Finally, by adding higher-derivative photon field terms to the action, we study charge diffusion and non-perturbative parameter regimes in which the charge diffusion constant vanishes.

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Statement of Originality

This thesis is based on research and contains no material that has already been accepted, or is concurrently being submitted, for any degree or diploma or certificate or other qualification in this university or elsewhere. To the best of my knowledge and belief this thesis contains no material previously published or written by another person, except where due reference is made in the text.

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Sašo Grozdanov　　　　August, 2014

To my parents.

Contents

Chapter 1

Introduction

Hydrodynamics is an enormously successful theory that describes the collective dynamics of gases, liquids and plasmas. Its applicability to physical systems extends through vast ranges of energy scales, from low-energy dynamics of fluids seen in nature every day, to a successful description of quark-gluon plasmas in the early universe and heavy-ion collision experiments at RHIC and the LHC.

From a microscopic quantum field theory (QFT) point of view, hydrodynamics should be understood as an effective theory of low-energy degrees of freedom, valid up to some energy scale Λ_{hydro}.[1] However, the relevant degrees of freedom that such an effective theory must describe can often be complicated bound states of various interacting nucleons and electrons; think of, for example, water molecules made of two hydrogen atoms covalently bonded to an oxygen atom. Given that the standard model of particle physics, phrased in terms of a local microscopic QFT, is in excellent agreement with almost all experiments done on Earth, one should, in principle, be able to derive and classify different types of hydrodynamics from *first principles*, i.e. from QFT.

This *thesis* is aimed at presenting a few small steps in that direction. It will present an effective field theory approach to dissipative hydrodynamics, as well as a stringy, supersymmetric point of view on hydrodynamics. We will further outline and study some issues, which generically arise in attempts to draw analogies between supersymmetric and non-supersymmetric low-energy systems.

Given enormous difficulties in treating realistic non-perturbative objects and strong coupling in QFT, it is not surprising that a systematic derivation of hydrodynamics from QFT is destined to be extremely complicated. Rather than deriving an effective field the-

[1] The exact scale of applicability of hydrodynamics, the hydrodynamisation scale, is not known and is a question of active research. It is believed to be somewhat higher, but comparable to the scale of thermalisation and often significantly higher than the scale one would naïvely expect. Models of the dynamics of quark-gluon plasma at RHIC and the LHC show its thermalisation length scale to be on the order of one Fermi, i.e. 10^{-15} m [1, 2]. Hence, Λ_{hydro} could in some cases be comparable to the inverse proton size.

ory, standard approaches to hydrodynamics combine phenomenological and microscopic inputs. One uses the notion of local thermodynamic equilibrium and the existence of conserved quantities to write down a gradient expanded stress-energy tensor and other relevant currents in terms of near-equilibrium fields, i.e. the generalisations of velocity, $u^\mu(x)$, temperature, $T(x)$, and chemical potential, $\mu(x)$. The procedure leaves hydrodynamic coefficients, e.g. shear and bulk viscosity, of various tensor structures undetermined. They must be computed using microscopic techniques, such as kinetic theory or lattice computations.

Beyond microscopic derivation, a problem concerning phenomenological hydrodynamics is that the most general gradient expansion is not known beyond second order in derivatives of hydrodynamics variables. Hence, the convergence properties of the hydrodynamic expansion, as well as higher-order dispersion relations are not known. In order to study these problems and to have a systematic view of different hydrodynamic models, it would be extremely beneficial to have a Lagrangian approach to hydrodynamics, where well-understood techniques of effective field theory could be employed.

Recently, work has been done showing that long-range Goldstone modes can be successfully used to encode hydrodynamic excitations in an effective theory [3–5], analogously with the ideas behind the extremely successful phenomenological chiral pion Lagrangian [6, 7]. The benefit of this approach is clear; if one could write down a hydrodynamic Lagrangian, then all conserved tensors and equations of motion would follow from variational principle, including all hydrodynamic coefficients. However, problems arise when one tries to include dissipative terms, which could encode viscosity. Standard variational methods are unable to describe dissipation in the absence of an explicit environment. For example, a harmonic oscillator must be explicitly coupled to an external system, which can serve as a forcing term and cause damping. This appears to be incompatible with the philosophy of effective field theory, where one would desire a theory of *only* the relevant, macroscopic degrees of freedom in the system.

If one adopts the view that dissipation is the energy loss of macroscopic hydrodynamic degrees of freedom to the integrated-out, microscopic degrees of freedom, then an effective field theory of only long-range Goldstone modes can be constructed using the Schwinger-Keldysh Closed-Time-Path (CTP) formalism [8, 9]. This approach to hydrodynamics will be discussed in Chapter 3. The CTP formalism was initially designed for the computation of expectation values in out-of-equilibrium QFT involving mixed states, but it can also be used to formulate an effective field theory of an *open system*. By analysing the possible structures of effective CTP Lagrangians, we will construct a classical effective theory with dissipation and non-zero bulk viscosity. We will also study its thermodynamic properties.

The second approach to hydrodynamics to be addressed in this thesis is through the gauge/string duality, also known as the AdS/CFT correspondence or simply, holography [10–12]. By duality we mean that all information about the physics in one theory is encoded in the dual theory, and vice versa. AdS/CFT is a holographic duality between gauge field theories and higher-dimensional string theory that reduces to an effective theory of supergravity in the low-energy limit. Computational control over the gravity side of the duality demands that we suppress stringy and quantum gravity corrections by taking the two limits, $N \to \infty$, as well as the 't Hooft coupling $\lambda = g_{YM}^2 N \to \infty$.[2] Gravity calculations can then provide a window into the low-energy, hydrodynamical limit of certain strongly coupled field theories.

The problem is that so far AdS/CFT has only given us computational access to strongly coupled theories, which are not directly observable in nature. They are usually conformal supersymmetric (SUSY) theories with a large number of colours. It is therefore important to understand similarities and differences between field theories with known gravity duals and reality. Many AdS/CFT calculations, for example, give thermodynamical scalings and transport properties that have never been observed. One would therefore like to understand whether those predictions are a result of strong coupling, unusual matter content or other reasons.

Deriving from this motivation, Chapter 4 will study perturbative low-energy behaviour of supersymmetric QED at finite density. This theory, which is a SUSY extension of quantum electrodynamics, is the minimal SUSY theory containing many of the features of more complicated theories with known duals. It is therefore a physically rich and natural starting point for investigations of thermal and hydrodynamic properties in SUSY, previously rarely investigated, yet essential for the understanding of AdS/CFT.

A ubiquitous feature of condensed matter systems at finite density are Fermi surfaces. For example, Landau's Fermi liquid theory is derived by considering quasiparticle excitations around a Fermi surface, giving rise to hydrodynamic transport properties of the system, such as zero sound. However, in a SUSY counterpart of such a theory, there are always extra Majorana fermions and, most importantly, scalar fields with a non-trivial moduli space of flat directions in the field space, which minimise the potential. In the high-density regime with $T/\mu \ll 1$, this may easily lead to instabilities.[3] In Chapter 4, we will discuss the moduli space stabilisation in super QED. This will lead to non-trivial scalar vevs breaking the symmetries and a complicated fermion mass matrix due to Yukawa

[2] The 't Hooft coupling λ, rather than just the Yang-Mills coupling g_{YM}, turns out to be an important expansion parameter in field theories with a large number of colours N. More details on Yang-Mills theories and the AdS/CFT correspondence will be presented in Chapter 2.

[3] T is the temperature and μ the chemical potential of the system.

interactions with a schematic form of $\langle\phi\rangle \times \text{fermion}^2$. We will see that scalar-fermion interactions cause the fermions to *not* form a Fermi surface while scalars undergo condensation. This is thus in sharp contrast with the behaviour of a realistic Fermi gas in the absence of fundamental scalars. To approach reality, we will tune the SUSY Yukawa interaction and show that at weaker coupling, compared to the scalar self-interaction, a Fermi surface begins to form. We will also analyse fermionic contributions to the total charge density, finding that fermionic modes can contribute even in the absence of a Fermi surface, which differs from the usual systems protected by Luttinger's theorem.

In Chapter 5, we will turn our attention to a holographic, gravitational analysis of hydrodynamic properties of a field theory dual to the Einstein-Gauss-Bonnet gravity in five space-time dimensions. Even though a string theoretic construction of this duality is not known, and hence we do not know the details of the CFT, this theory serves as a great laboratory for explorations of higher curvature effects on dual hydrodynamics.[4] The reason for this investigation is that holography can provide us with information about different classes of fluids based on the behaviour of their transport coefficients. The power of holography is precisely in its ability to determine them microscopically at all orders in the hydrodynamic derivative expansion.

We will be particularly interested in *non-dissipative* fluids with *non-trivial* second-order hydrodynamic transport properties.[5] To study such fluids, we will make use of large, non-perturbative corrections of the Gauss-Bonnet coupling to results that follow from pure Einstein theory.[6] They will allow us to analyse the conformal theory near a point where shear viscosity η vanishes, i.e. as $\lambda_{GB} \to 1/4$. Analytical and numerical studies of hydrodynamic dispersion relations will point towards the suppression of dissipation at all hydrodynamic orders. In spite of that, we will analytically compute second-order conformal hydrodynamic coefficients, which will remain non-trivial and produce entropy. The fluid will thus still be dissipative. Using the values of the second-order transport coefficients, we will show that a previously proposed universal holographic relation between three of the coefficients is violated. Furthermore, we will compute the leading 't Hooft correction to the last unknown coefficients in the $\mathcal{N} = 4$ theory to show another example

[4] While the Einstein-Hilbert action includes two derivatives of the metric tensor, the Gauss-Bonnet term includes four. However, its structure is such that it only gives non-trivial contributions with *at most* two derivatives of the metric to the gravitational equations of motion. The absence of higher derivatives in the equations of motion allows for convenient calculations with a non-perturbative value of the Gauss-Bonnet coupling λ_{GB}.

[5] Note that viscosity is a first-order transport coefficient as it multiplies single derivatives of hydrodynamic fields, u^μ, T and μ, in the stress-energy tensor as well as the Navier-Stokes equations. Similarly, second-order coefficients accompany terms with two derivatives.

[6] Pure Einstein theory on $AdS_5 \times S^5$ is itself a limit of Type IIB supergravity and is dual to the stress-energy tensor sector of the $\mathcal{N} = 4$ superconformal Yang-Mills theory. The CFT dual of the Einstein-Gauss-Bonnet theory is presently unknown.

of a holographic fluid, which violates the proposed relation. Finally, we will analyse charge diffusion properties by adding the most general four-derivative photon and photon-graviton terms to the Lagrangian that again only produce second-order differential equations.

The thesis is divided into four chapters. In Chapter 2, background material will be presented, which is required to make the ideas in this work self-contained. We will begin by discussing the concept of renormalisation group and effective field theory. This will be followed by the construction of phenomenological hydrodynamics and the motivation for the necessity of the Schwinger-Keldysh CTP formalism in out-of-equilibrium quantum field theory. We will then move on to the discussion of supersymmetry and its power in enabling strong/weak dualities in four-dimensional theories. A presentation of string theory and gauge/string duality will follow. In particular, we will focus on the usefulness of the AdS/CFT correspondence in computing transport properties of strongly coupled hydrodynamical and condensed matter systems.

In Chapter 3, we will discuss in detail the CTP formalism and its applications to an effective theory of hydrodynamics. In Chapter 4, we will analyse $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric QED at finite density, focusing on the issues of stability of the moduli space and the existence, or rather, the lack of Fermi surfaces. Chapter 5 will be devoted to a holographic analysis of second-order hydrodynamics in a field theory dual to the Einstein-Gauss-Bonnet gravity. Although each chapter will contain a discussion of results and an outline of interesting future research directions, we will use Chapter 6 to present a summary of the thesis's main contributions to the field of study.[7]

[7] The thesis's main body chapters will be using different sign conventions for the metric tensor in order to remain consistent with the majority of the modern literature related to the subjects of study. Thus, QFT calculations in Chapter 4 and the first part of Chapter 3, where we present the CTP formalism in QFT, will use the signature $\eta_{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1)$. Topics related to general relativity and string theory in Chapter 2 as well as the entire Chapter 5 will use $\eta_{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, +1)$. Same metric conventions will be used in classical theory of hydrodynamics in Chapter 3.

Chapter 2

Effective field theory and the gauge/string duality

2.1 Effective field theory

2.1.1 Quantum field theory and renormalisation group

Field theory is a language used in physics to describe a variety of subjects ranging across all energy scales: from hydrodynamics and condensed matter, to cosmology, particle physics and string field theory. Quantum field theory (QFT) [13–15] is the relativistic, second-quantised theory, which can describe processes that occur in particle physics. It provides a consistent, predictive quantum description of all observed forces, with the exception of gravity. Some of its fundamental concepts that will be used throughout this work are introduced in this chapter.

The main ingredients for specifying a QFT are its particle content, i.e. *fields*, and *symmetries*, which leave the theory invariant. A theory can then be used to compute *correlation functions* from a generating functional $Z[J]$, which is a path integral over an action defined on a $d+1$-dimensional space-time manifold $\mathcal{M}$ as

$$Z\left[J_n\right] = \int \mathcal{D}\Phi \exp\left\{ iS[\Phi] + i \sum_n \int_{\mathcal{M}} J_n \Phi_n \right\}. \tag{2.1}$$

The symbol Φ denotes an arbitrary set of quantum fields $\{\Phi_1, \ldots, \Phi_n\}$ and J_n are sources with respect to which functional derivatives are taken.

Bare correlation functions, which naïvely follow from (2.1) are normally divergent and thus un-physical. In renormalisable theories, e.g. QED and QCD, these divergences can be successfully removed at an expense of introducing a new scale μ into the problem. This is done by regulating the bare quantities through an introduction of a cut-off Λ, or by using some other regulator such as dimensional regularisation or the Pauli-Villars procedure. *Renormalisation conditions* fixing values of certain vertices, mass poles and propagator residues are then specified an at arbitrary scale μ. The momentum scale $p^2 = \mu^2$ can be either time-like or space-like, noting that in the time-like cases, additional singularities may appear. Divergences can be absorbed into *counter-terms*, which yields finite *renormalised* correlation functions that depend on μ and obey the RG conditions.

Coupling constants in a renormalised theory, for example the Yang-Mills coupling, become scale-dependent, $g(\mu)$, and their running is described by a *beta function* $\beta_g \equiv dg/d\log\mu$. By introducing the *wave-function renormalisation* Z, the bare and renormalised correlation functions become related by

$$\langle \Omega\,|\Phi_1(x_1)\ldots\Phi_n(x_n)|\,\Omega\rangle_{\text{bare}} = (Z_1\ldots Z_n)^{1/2}\,\langle \Omega\,|\Phi_1(x_1)\ldots\Phi_n(x_n)|\,\Omega\rangle_{\text{ren}}\,. \tag{2.2}$$

Great physical insight can be gained by noting that bare correlators that depend on Λ cannot depend on μ, hence

$$\frac{d}{d\mu}\,\langle \Omega\,|\Phi_1(x_1)\ldots\Phi_n(x_n)|\,\Omega\rangle_{\text{bare}} = 0. \tag{2.3}$$

The right-hand side of Eq. (2.2) then leads to the Callan-Symanzik equation [16–18],

$$\left[\frac{\partial}{\partial\mu} + \sum_m \beta_{g_m}\frac{\partial}{\partial g_m} + \sum_n \gamma_n\right]\langle \Omega\,|\Phi_1(x_1)\ldots\Phi_n(x_n)|\,\Omega\rangle_{\text{ren}} = 0, \tag{2.4}$$

where g_m are the couplings in the theory and γ_n are the anomalous dimensions $\gamma_n \equiv (1/2)(d\log Z_n/d\log\mu)$. The same considerations apply to correlators of composite operators, $\mathcal{O}_k$, made of the fundamental fields Φ. Each one of them acquires a γ_n-independent anomalous dimension $\gamma_k^{(\mathcal{O})}$. Furthermore, the inclusion of composite operators requires us to introduce further counter-terms into the theory.

Let us for simplicity focus on a theory with a single coupling g, e.g. the Yang-Mills theory. The Callan-Symanzik equation can most easily be solved at a *fixed point*, which is a point in the space of couplings where the beta function vanishes, $\beta(g^*) = 0$. Eq. (2.3) implies that a two-point correlator of an operator $\mathcal{O}$ scales as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \propto \frac{1}{|x_1 - x_2|^{2\Delta}}, \tag{2.5}$$

at the scale-invariant fixed point, where Δ is the dimension of $\mathcal{O}$. The dimension Δ is the sum of the operator's "engineering" dimension and the anomalous dimension $\gamma^{\mathcal{O}}$.

As an example, a two-loop beta function in a Yang-Mills theory with N_c colours and N_f massless flavours is given by $\beta(g) = -b_1 g^3 + b_2 g^5$, where

$$b_1 = \frac{1}{(4\pi)^2}\left[\frac{11}{3}N_c - \frac{2}{3}N_f\right], \qquad b_2 = -\frac{1}{(4\pi)^4}\left[\frac{34}{3}N_c^2 - N_f\left(\frac{N_c^2-1}{N_c} + \frac{10}{3}N_c\right)\right]. \tag{2.6}$$

For a well-defined perturbation expansion, the two-loop contribution must be sub-leading, i.e. $|b_2 g^2| \ll |b_1|$. The theory has $\beta(g) < 0$ and is said to be *asymptotically free* if $b_1 > 0$, which enforces the condition $N_f < 11N_c/2$. In such theories, the coupling $g(p)$ runs to zero in the UV, i.e. at large momenta p. If $b_2 > 0$ is also satisfied, then there exists an IR fixed point $g_*^2 = b_1/b_2 \ll 1$, known as the Caswell-Banks-Zaks fixed point for asymptotically free theories in the *conformal window*,

$$\frac{68N_c^2}{16+20N_c} < N_f < \frac{11}{2}N_c. \tag{2.7}$$

In theories with a positive beta function, e.g. QED, ϕ^4 theory, the coupling diverges at a *Landau pole* in the UV [19]. Such theories are called *trivial* because it is formally impossible to remove the UV cut-off without tuning the coupling to zero. This occurs in the standard model within the Higgs and the $U(1)$ hypercharge sectors. Although the Landau pole could signal the breakdown of perturbative expansion, it is believed that QED and ϕ^4 theory are also trivial non-perturbatively.

Another important scenario that will be of relevance in later chapters, are theories with strongly coupled UV fixed points. For example, this may occur in theories, which run into the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in the UV. This super-conformal theory is known to have a vanishing beta function at all energy scales and will be widely discussed in the context of the AdS/CFT correspondence.

The trace of the stress-energy tensor $T^{\mu\nu}$ vanishes in a scale-invariant theory, such as in massless *QED*. However, a theory may possess a *trace anomaly*, which makes $T^{\mu}_{\ \mu} \neq 0$. Generically, the trace becomes proportional to the beta functions of the couplings. Although the proofs of the following fact have not been fully accepted, it is believed that a scale-invariant, unitary theory is also conformally invariant [20–22]. The conformal group in four space-time dimensions has an $SO(4,2)$ algebra and is generated by *dilatations* D, which correspond to scaling transformations, the *special conformal transformations* K_μ, as well as the Poincaré group generators $M_{\mu\nu}$ and P_μ, which correspond to Lorentz transformations and translations, respectively. The full algebra is

$$[D, K_\mu] = -iK_\mu, \qquad [D, P_\mu] = iP_\mu, \tag{2.8}$$

$$[K_\mu, P_\nu] = 2i\eta_{\mu\nu}D - 2iM_{\mu\nu}, \qquad [K_\mu, M_{\nu\rho}] = i\left[\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu\right], \tag{2.9}$$

$$[P_\rho, M_{\mu\nu}] = i\left[\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right], \qquad [M_{\mu\nu}, M_{\rho\sigma}] = i\left[\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - (\mu \leftrightarrow \nu)\right]. \tag{2.10}$$

The algebra of conformal field theories will play an important role in the identification of the duality between conformal gauge theories and theories of gravity.

2.1.2 Effective field theory

In the 1970s, Wilson developed a new view of quantum field theory, by which theories without a UV completion should necessarily be viewed as *effective theories* with a finite range of scales over which they are applicable [23,24]. The idea can be seen by considering the path integral (2.1), setting $J_n = 0$ and splitting the fields into low- and high-momentum parts, $\Phi(k) = \Phi_<(k < \Lambda) + \Phi_>(k > \Lambda)$, with respect to some scale Λ. The measure factorises, allowing us to define the Wilsonian *effective action* S_{eff}. It is the action that only depends on the low-energy fields $\Phi_<$, after all high-energy modes $\Phi_>$ have been integrated out,

$$Z\,[J=0] = \int \mathcal{D}\Phi_< \exp\left\{iS_{\text{eff}}[\Phi_<]\right\} \equiv \int \mathcal{D}\Phi_< \left[\int \mathcal{D}\Phi_> \exp\left\{iS[\Phi_<, \Phi_>]\right\}\right]. \tag{2.11}$$

Since this procedure keeps the partition function $Z_0 \equiv Z\,[J=0]$ invariant, S_{eff} fully describes all physics below the scale Λ, which is the new UV cut-off of the theory. It should be noted that in gauge theories, a hard cut-off Λ breaks gauge invariance and one usually uses a gauge-invariant smooth cut-off to avoid inconsistencies.

The effective action includes an infinite series of all possible operators $\mathcal{O}_m$, built out of the original fields Φ_n. All $\mathcal{O}_m$ must be consistent with the symmetries of the original theory. If we further integrate out fields between Λ and $\Lambda' = \Lambda - \delta\Lambda$, defining $\chi \equiv \Lambda'/\Lambda$, the operators $\mathcal{O}_m$ evolve under the RG transformations. Consider an operator $\mathcal{O}_\Delta$ with dimension Δ and a coupling λ in $S_{\text{eff}}[\Lambda]$, made of a scalar Φ with a kinetic term $(\partial_\mu\Phi)^2$. The effective action $S_{\text{eff}}[\Lambda']$ with a cut-off Λ' then becomes

$$S_{\text{eff}}\,[\Lambda'] = \int d^dx \left\{\frac{1}{2}\left(1+\Delta Z\right)\partial_\mu\phi\partial^\mu\phi + \ldots + \left(\lambda + \Delta\lambda\right)\mathcal{O}_\Delta + \ldots\right\}. \tag{2.12}$$

We can best compare $S_{\text{eff}}[\Lambda']$ with $S_{\text{eff}}[\Lambda]$ by rescaling $x' = x\chi$ and $k' = k/\chi$, so that $k' < \Lambda$. Using $d^dx = \chi^{-d}d^dx'$, $\phi' = \phi\sqrt{\chi^{2-d}\left(1+\Delta Z\right)}$ and the operator scaling $\mathcal{O}_\Delta(x) = \chi^\Delta \mathcal{O}'_\Delta(x')$, the effective action becomes

$$S_{\text{eff}}\,[\Lambda] = \int d^dx' \left\{\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' + \ldots + \chi^{-d+\Delta}\lambda'\mathcal{O}'_\Delta + \ldots\right\}. \tag{2.13}$$

This analysis implies that only a finite subset of all operators remains important in the deep IR, where $\chi \to 0$. The operators with dimension $\Delta < d$ are thus known as (IR) *relevant*, and those with $\Delta > d$ as *irrelevant*. Operators with $\Delta = d$ are called *marginal*, and a quantum analysis is required to determine whether they are *exactly marginal*, i.e. do not scale, or whether they are *marginally relevant* or *marginally irrelevant*. Another striking result is the equivalence between irrelevant and non-renormalisable operators. This view further allows us to understand various QFTs, potentially with a Landau pole, as theories with a finite range of applicability and some *unknown* UV completion.

The effective reduction in the number of relevant operators in the IR implies *universality* of low-energy phenomena that are insensitive to the details of short-distance UV physics. As long as the right degrees of freedom are identified, the limited choice of relevant operators should give the correct long-distance physics. Indeed, scalar ϕ^4 theory can be used to predict *critical exponents* in a wide variety of physical systems at their IR critical points.

The Wilsonian approach generates a quantum theory for energies with $k \leq \Lambda$. However, there is another type of effective theory, the *1PI effective action*, which gives a *classical* action with included quantum corrections. Consider a single scalar field theory

$$Z[J] = \int \mathcal{D}\phi \, \exp\left\{ iS[\phi] + i \int d^d x J \phi \right\} \equiv \exp\left\{ -iW[J] \right\}, \tag{2.14}$$

where we have used the expression to define $W[J]$. We further define a *classical field* ,

$$\phi_{cl}(x) \equiv \langle \Omega | \phi(x) | \Omega \rangle = -\frac{\delta}{\delta J(x)} W[J]. \tag{2.15}$$

The effective action for the classical field is then the *Legendre transform*,

$$\Gamma[\phi_{cl}] = -W[J] - \int d^d x' J(x') \phi_{cl}(x'). \tag{2.16}$$

The variation $\delta\Gamma/\delta\phi_{cl} = 0$ generates the quantum-corrected equation of motion, after the source J is set to zero. It turns out that as $Z[J]$ generates correlation functions, $W[J]$ is the *generator functional of connected correlation functions.* Furthermore, $\Gamma[\phi_{cl}]$ is the *generator functional of one-particle-irreducible correlation functions.*

2.1.3 Hydrodynamics

One of the core questions this thesis addresses is how hydrodynamics can arise from UV physics in a systematic language of effective field theory. We know that gases, liquids, plasmas, etc. exhibit collective behaviour that universally follows a hydrodynamical description. Hydrodynamics with its dissipative IR effects should thus arise naturally from most QFTs. The theory of hydrodynamics is usually phrased as a phenomenological theory using *gradient expansion* in the relevant variables [25, 26]. It is certainly one of the most widely used examples of an effective theory. Its phenomenological construction is presented in this section.

Phenomenological hydrodynamics with a single conserved charge can be constructed as the gradient expansion of conserved operators, i.e. the stress-energy tensor and the conserved current, in terms of the metric tensor, velocity, temperature and chemical potential fields.[1] The three fields, $u^\mu(x)$, $T(x)$ and $\mu(x)$, should be seen as near-equilibrium generalisations of their equilibrium thermodynamical counterparts. The assumption of local equilibrium is necessary because the conservation laws in d dimensions, $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu J^\mu = 0$, provide $d+1$ equations. However, a symmetric stress-energy tensor $T^{\mu\nu}$ and a

[1] For theories with more conserved charges, one can follow exactly the same line of reasoning as the one presented in the section.

vector Noether current J^μ have $d(d+1)/2$ and d independent components, respectively, giving more unknowns than variables. The assumption of local equilibrium thus restricts the problem to solvable $d+1$ variables.

The gradient expansion in a curved space with a metric $g_{\mu\nu}$ takes the form

$$T^{\mu\nu} = T^{\mu\nu}_{(0)}\left(u, T, \mu, g\right) + T^{\mu\nu}_{(1)}\left(\partial u, \partial T, \partial \mu, \partial g\right) + \ldots + T^{\mu\nu}_{(n)}\left(\partial^n u, \ldots\right), \tag{2.17}$$

$$J^{\mu} = J^{\mu}_{(0)}\left(u, T, \mu, g\right) + J^{\mu}_{(1)}\left(\partial u, \partial T, \partial \mu, \partial g\right) + \ldots + J^{\mu}_{(n)}\left(\partial^n u, \ldots\right). \tag{2.18}$$

It is important to note that because the hydrodynamic fields, u^μ, T and μ, have no microscopic definition, they can be re-defined by a choice of *frame*,

$$(u^\mu, T, \mu) \to \left(\tilde{u}^\mu = u^\mu + \sum_n \delta^n u^\mu,\ \tilde{T} = T + \sum_n \delta^n T,\ \tilde{\mu} = \mu + \sum_n \delta^n \mu\right), \tag{2.19}$$

where $\delta^n u^\mu$, $\delta^n T$ and $\delta^n \mu$ can be arbitrary functions of n-th order derivatives of the three fields. The metric tensor cannot be used in this sense, as metric variations, i.e. coordinate changes, leave tensorial equations invariant. The simplest tensorial quantity, the Riemann tensor, $R_{\mu\nu\rho\sigma}$, will enter into the equations of second- and higher-order hydrodynamics.

$T^{\mu\nu}$ and J^μ can be decomposed in terms of different tensor structures as

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P} \Delta^{\mu\nu} + \left(q^\mu u^\nu + u^\mu q^\nu\right) + t^{\mu\nu}, \tag{2.20}$$

$$J^\mu = \mathcal{N} u^\mu + j^\mu, \tag{2.21}$$

where $\mathcal{E}$, $\mathcal{P}$ and $\mathcal{N}$ are scalars, q^μ and j^μ transverse vector and $t^{\mu\nu}$ a transverse, traceless and symmetric tensor. Each one of these is then gradient expanded as in (2.17) and (2.18). Note that we define the projector $\Delta^{\mu\nu}$ as $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$. These properties allow us to find the *constitutive relations*,

$$\mathcal{E} = u_\mu u_\nu T^{\mu\nu}, \qquad \mathcal{P} = \frac{1}{d} \Delta_{\mu\nu} T^{\mu\nu}, \qquad \mathcal{N} = -u_\mu J^\mu, \qquad q_\mu = -\Delta_{\mu\rho} u_\sigma T^{\rho\sigma}, \tag{2.22}$$

$$j_\mu = \Delta_{\mu\nu} J^\nu, \qquad t_{\mu\nu} = \frac{1}{2}\left(\Delta_{\mu\rho}\Delta_{\nu\sigma} + \Delta_{\nu\rho}\Delta_{\mu\sigma} - \frac{2}{d-1}\Delta_{\mu\nu}\right) T^{\rho\sigma}. \tag{2.23}$$

At zeroth order in the gradient expansion we find that

$$T^{\mu\nu}_{(0)} = \epsilon u^\mu u^\nu + P \Delta^{\mu\nu}, \qquad\qquad J^\mu_{(0)} = n u^\mu, \tag{2.24}$$

where $\mathcal{E}_{(0)} = \epsilon$, $\mathcal{P}_{(0)} = P$ and $\mathcal{N}_{(0)} = n$ are the energy density, pressure and the charge density, respectively. The two vectors and the tensor, $q^\mu_{(0)} = j^\mu_{(0)} = t^{\mu\nu}_{(0)} = 0$, at zeroth order.

At higher orders, frame dependence can be used to eliminate some terms from the expansion. Imposing that under the frame re-definitions (2.19), $\delta T^{\mu\nu} = 0$ and $\delta J^\mu = 0$, using the fact that $u_\mu \delta u^\mu = 0$, as well as Eqs. (2.20), (2.21), (2.22) and (2.23), we see that

$\delta\mathcal{E} = \delta\mathcal{P} = \delta\mathcal{N} = \delta t^{\mu\nu} = 0$. However, $\delta q_\mu = -\left(\mathcal{E} + \mathcal{P}\right)\delta u_\mu$ and $\delta j_\mu = -\mathcal{N}\delta u_\mu$ up to higher orders. A choice of δu_μ to any order that sets $q_\mu = 0$ is known as the *Landau frame* and $j_\mu = 0$ as the *Eckart frame*. Furthermore, it is conventional to use $\delta\mathcal{E} = 0$, which enables us to write

$$\mathcal{E}_0\left(T,\mu\right) + \ldots + \mathcal{E}_n\left(\partial^n u, \ldots\right) = \mathcal{E}_0\left(\tilde{T},\tilde{\mu}\right) + \ldots + \tilde{\mathcal{E}}_n\left(\partial^n \tilde{u}, \ldots\right). \tag{2.25}$$

Expanding the re-defined fields, $\tilde{u}^\mu$, $\tilde{T}$ and $\tilde{\mu}$, allows us to order-by-order adjust the frame choices so that $\mathcal{E} = \mathcal{E}_0 = \epsilon$. Having already used $\delta^n u^\mu$, we have two remaining freedoms, which allows us to also set $\mathcal{N} = n$.

Working in the Landau frame, we are left with a scalar $\mathcal{P}$, a vector j^μ and a tensor $t^{\mu\nu}$, which need to be gradient expanded in full generality. The only remaining source of the reduction of terms are conservation equations at lower orders. In particular, we can use the scalars, $u_\mu \partial_\nu T^{\mu\nu}_{(0)} = 0$ and $\partial_\mu J^\mu_{(0)} = 0$, to eliminate two terms from first-order hydrodynamics. At other orders in derivative expansion, we can similarly form higher-derivative scalars. Finally, each independent tensor structure is given an undetermined *transport coefficient*, which can only be computed microscopically.

With these steps in mind, the first-order hydrodynamic terms are

$$T^{\mu\nu}_{(1)} = -\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\nabla_\lambda u^\lambda, \qquad J^\mu_{(1)} = -\sigma T \Delta^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) + \chi_T \Delta^{\mu\nu}\partial_\nu T, \tag{2.26}$$

with four transport coefficients, η, ζ, σ and χ_T. Coefficient η is the shear viscosity and ζ the bulk viscosity. The tensor

$$\sigma^{\mu\nu} \equiv \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d-1} g_{\alpha\beta}\nabla_\lambda u^\lambda\right), \tag{2.27}$$

is the only one-derivative transverse, traceless and symmetric tensor.

This classification can be continued at higher orders in derivative expansion, exactly following the procedure above. We will discuss second-order hydrodynamics in Chapter 5, where we will compute second-order transport coefficients in a type of a conformal fluid, using string theory techniques. Furthermore, it should be pointed out that as of third order, the full classification of hydrodynamic coefficients is presently not known.

The above description of hydrodynamics can also be extended to include parity-violating and anomalous effects [27, 28], which will not be considered in detail here.

Beyond its conserved tensor structure, phenomenological hydrodynamics has an associated entropy current, S^μ [29], which is conserved for an ideal fluid and must satisfy the positive entropy production condition, $\nabla_\mu S^\mu \geq 0$, in the presence of dissipation. A covariant expression for S^μ, which is sufficient in first-order hydrodynamics is given by

$$TS^\mu = Pu^\mu - T^{\mu\nu}u_\nu - \mu J^\mu. \tag{2.28}$$

An important class of fluids, relevant for AdS/CFT calculations, are conformal fluids in which the trace of the stress-energy tensor vanishes, $T^{\mu}_{\ \mu} = 0$. This implies the relation $\epsilon = (d-1)P$ at zeroth order. Furthermore, conformality implies that the scalar $\mathcal{P}$ has to vanish beyond zeroth order, therefore bulk viscosity ζ must also vanish in any conformal fluid.

We conclude this section by noting that the conservation laws, $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu J^\mu = 0$, provide differential equations which govern the dynamics of fluids. For example, if we consider an uncharged fist-order fluid, then the non-relativistic Navier-Stokes equations follow from equation $\partial_\mu T^{\mu\nu} = 0$, followed by a non-relativistic scaling limit. We will further analyse these issues in Chapter 3.

2.1.4 Out-of-equilibrium and thermal field theory

In many-body physics of collective phenomena, QFT built for the analysis of transition amplitudes between pure states is insufficient for the computation of expectation values of quantum operators [8]. Furthermore, quantum processes are not the only types of fluctuations that play an important role. In fact, to make any contact between quantum field theory and hydrodynamics, it is essential to introduce the concept of temperature and density into field theory. To bridge the gap between standard QFT and non-equibrium physics, we will introduce the QFT techniques, which can successfully describe the evolution of mixed states [8, 9, 30–33].

Consider an initial state density matrix $\rho(t_i)$ in the Schrödinger picture, specified at time t_i. The initial $\rho(t_i)$ can be evolved to $\rho(t) = U_{t,t_i}\, \rho(t_i)\, U_{t_i,t}$ by a unitary evolution operator,

$$U_{t,t'} = \mathcal{T} \exp\left\{ -i \int_{t'}^{t} H(t) dt \right\}, \tag{2.29}$$

where $H(t)$ is the time-dependent Hamiltonian of the system and $\mathcal{T}$ denotes time-ordering. An expectation value of a quantum operator $\mathcal{O}(t)$ at time t, is given by

$$\langle \mathcal{O}(t) \rangle \equiv \frac{\mathrm{Tr}\left[\mathcal{O}\rho(t)\right]}{\mathrm{Tr}\left[\rho(t)\right]} = \frac{\mathrm{Tr}\left[U_{t_i,t}\, \mathcal{O}\, U_{t,t_i}\, \rho(t_i)\right]}{\mathrm{Tr}\left[\rho(t)\right]}, \tag{2.30}$$

where the cyclic property of Tr was used. Eq. (2.30) can be interpreted as time evolution from t_i to t, where the expectation value is calculated, followed by a backwards time evolution from t to t_i. This doubling of time axes and associated doubling of degrees of freedom is the central idea behind the Schwinger-Keldysh Closed-Time-Path (CTP) formalism [8, 9], which allows for out-of-equilibrium QFT computations. More details on this formalism will be presented in Chapter 3, where CTP will be used to study IR effective theories with hydrodynamic properties.

To connect these ideas with the usual single time axis QFT, let us consider an equilibrium QFT at zero temperature. In that case, one is usually interested in computing $\langle\Omega|\mathcal{O}|\Omega\rangle$, where $|\Omega\rangle$ is the *interacting* ground state, which follows by time-evolution from a *non-interacting* vacuum state at asymptotic infinity, $|\Omega\rangle = U_{t,-\infty}|0\rangle$, with $\langle 0|0\rangle = 1$. The assumption at work is that interactions only turn on *adiabatically* when the evolution reaches the state $|\Omega\rangle$ at t. After that point, the interaction is again switched off adiabatically, giving us only a phase shift factor,

$$U_{+\infty,-\infty}|0\rangle = e^{i\alpha}|0\rangle, \qquad\qquad \langle 0|U_{+\infty,-\infty} = \langle 0|e^{i\alpha}. \tag{2.31}$$

Throughout this procedure, it is necessary to assume that interacting adiabatic time evolution keeps the system in its ground state. The CTP expression for an expectation value $\langle\Omega|\mathcal{O}|\Omega\rangle = \langle 0|U_{-\infty,t}\mathcal{O}U_{t,-\infty}|0\rangle$, having used two time axes, can now be written as

$$\langle\Omega|\mathcal{O}|\Omega\rangle = e^{-i\alpha}\,\langle 0|U_{+\infty,-\infty}\,U_{-\infty,t}\,\mathcal{O}\,U_{t,-\infty}|0\rangle = \frac{\langle 0|U_{+\infty,t}\,\mathcal{O}\,U_{t,-\infty}|0\rangle}{\langle 0|U_{+\infty,-\infty}|0\rangle}, \tag{2.32}$$

which implies that only forward time evolution is required for such computations.

At finite temperature and in equilibrium, the same reasoning implies that adiabatic interactions only change the ground state up to a phase. Temperature is then encoded into the length $0 \le \tau < \beta = 1/T$ of the Euclidean time interval of a Wick-rotated theory on a compactified τ-circle. This comes at the expense of eliminating time from the theory, which is anyhow irrelevant in equilibrium. As in Eq. (2.32), only forward τ evolution is required.

The equilibrium partition function, $Z = e^{-\beta H}$, in a canonical ensemble, can be promoted to a grand-canonical, or a generalised Gibbs ensemble. This is done by identifying all mutually commuting conserved charges in the theory, Q_i, and adding them to the partition function,

$$Z = e^{-\beta H + \sum_i \mu_i Q_i}, \tag{2.33}$$

where μ_i are the chemical potentials associated with conserved quantities. In a perturbative expansion, Z can be computed by summing all vacuum bubble diagrams without external legs. A theory with a chemical potential, and thus a finite density of a charge Q, can lead to a Bose-Einstein condensate in a system of bosons and a Fermi surface in a system of fermions.

The concepts of temperature and density play a very important role in the theory of hydrodynamics, as seen in Section 2.1.3, where $T(x)$ and $\mu(x)$ were treated as near-equilibrium functions, i.e. generalisations of the equilibrium T and μ considered in this section. We will thus be forced to use the concepts presented in this section when discussing a QFT approach to hydrodynamics in Chapter 3. Furthermore, we will devote Chapter 4 to the analysis of Fermi surfaces in supersymmetric field theories.

2.1.5 Supersymmetry

Symmetries play an integral role in quantum field theory. It is therefore natural to ask what are all possible symmetries a quantum field theory can possess. In 1967 Coleman and Mandula [34] proved a powerful theorem of fundamental importance to QFT, stating that the only possible *Lie algebras* of symmetries are those of the Poincaré group generators, P_μ and $M_{\mu\nu}$, along with *internal* Hermitian symmetry generators, which must *commute* with the Poincaré generators. The Poincaré algebra may be enlarged to the conformal algebra of Eqs. (2.8) - (2.10) when theories contain only massless particles.

A way to avoid this theorem is to generalise Lie algebras to *graded Lie algebras*,

$$[t_a, t_b\} \equiv t_a t_b - (-1)^{\eta_a \eta_b} t_b t_a = i \sum_c C^c_{ab} t_c, \tag{2.34}$$

where η's equal either 1 or 0. The generators now obey the super-Jacobi identity,

$$(-1)^{\eta_a \eta_c} \left[[t_a, t_b\}, t_c\right\} + (-1)^{\eta_a \eta_b} \left[[t_b, t_c\}, t_a\right\} + (-1)^{\eta_b \eta_c} \left[[t_c, t_a\}, t_b\right\} = 0. \tag{2.35}$$

This algebra is used in introducing a new type of symmetry into quantum field theory, which transforms bosonic states into fermionic states and vice versa, i.e. *supersymmetry* [35–37]. The supersymmetry transformations are generated by complex anti-commuting spinors, which obey the algebra

$$\{Q_\alpha, Q_\beta\} = \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} = 0, \qquad \{Q_\alpha, Q^\dagger_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu, \tag{2.36}$$

where $\sigma^\mu_{\alpha\dot\alpha} = (1, \sigma^i)$ and σ^i are the Pauli matrices. SUSY generators commute with translations, P_μ. An important property is that Q_a annihilates the vacuum. Furthermore, in SUSY theories the energy of the ground state vanishes, $\langle 0|H|0\rangle = 0$.

Superspace, $y^\mu = x^\mu - i\theta\sigma^\mu\bar\theta$, is a generalisation of coordinate space, x^μ, which includes non-commuting Grassmannian coordinates, θ. It is convenient to assemble SUSY fields into various *superfield multiplets*. The chiral multiplet with a complex scalar, ϕ, and a Weyl fermion, θ, is given by $\Phi(y) \equiv \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2\mathcal{F}(y)$. A gauge theory further requires a vector multiplet, V^a, with a vector field A^a_μ transforming under a representation t^a, a gaugino, λ^a, and a field D. $\mathcal{F}$ and D are convenient *auxiliary* fields and D ensures the off-shell SUSY within the vector multiplet. A SUSY Lagrangian can in general be written as a sum of two terms,

$$S_{\text{SUSY}} = \int d^4x\, d^2\bar\theta\, d^2\theta\, K\left(\Phi^\dagger, e^{g t^a V^a}\Phi\right) + \int d^4x\, d^2\theta\, W\left(\Phi\right), \tag{2.37}$$

where K is the *real Kähler potential* that encodes kinetic terms and non-renormalisable interactions. The *holomorphic* function W is the *superpotential*, which encodes the standard interactions. Its holomorphic property is the reason for the *non-renormalisation* theorems, which simplify SUSY theories and constrain their quantum fluctuations.

Another important and generic property of SUSY theories is the classical *moduli space* manifold of flat directions along the scalar (squark) potential. This can be most easily seen from a typical D-term potential given by

$$V = \frac{1}{2}\sum_a D^a D^a = \frac{1}{2g^2}\mathrm{Tr}[\phi, \phi^\dagger]^2, \tag{2.38}$$

which has $V = 0$ for any $\phi = \phi^a t^a_C$, where t^a_C are the commuting Cartan sub-group generators of the full Lie gauge group. Hence, the scalar vev and the vacuum state are *not* fixed by the classical potential and there exist flat directions in the field space. SUSY-breaking and thermal corrections can easily stabilise or de-stabilise the theory.

In supersymmetric theories, there exists a new type of charge, the *R-charge*, with a generator R, which obeys

$$[Q_a, R] = Q_a, \qquad\qquad [Q^\dagger_{\dot\alpha}, R] = -Q^\dagger_{\dot\alpha}. \tag{2.39}$$

In theories with only one SUSY generator, the *R-symmetry* group is $U(1)_R$. We can also consider theories with *extended SUSY*, which have $\mathcal{N}$ independent supercharges. Their algebra generalises to $\{Q^a_\alpha, Q^\dagger_{\dot\alpha b}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu\delta^a_b$, and the R-symmetry group is enlarged to $U(\mathcal{N})_R$. Particle states with different spins form representations of the SUSY algebra and the largest number of generators in a four-dimensional theory with particles of at most *spin-one* is $\mathcal{N} = 4$. The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory will receive much attention in the following sections as it is the gauge theory side of the best understood AdS/CFT example.

For higher spins, theories with *local* supersymmetry can be constructed, i.e. *supergravity* [37, 38]. In four dimensions, the highest number of supercharges is $\mathcal{N} = 8$, which produces multiplets with spins of $s \leq 2$. This bound is set by the fact that no consistent theories with higher spins are known in Minkowski space. $\mathcal{N} = 8$ independent Q^a_α's in four dimensions give in total 32 real supersymmetries. This is considered to be the highest number of SUSYs of any higher dimensional theory, since a theory with more SUSY would produce spins with $s > 2$ in $\mathbb{R}^{3,1}$, after a compactification to $\mathbb{R}^{3,1} \times \mathcal{M}_{int}$.

By counting supersymmetries, an $\mathcal{N} = 1$ supergravity can be formulated in at most $D = 11$ dimensions. Remarkably, this theory is the low-energy effective action of the $D = 11$ *M-theory* with solutions describing M2 and M5 branes. In supergravity, all multiplets contain a graviton and the number $\mathcal{N}$ also equals to the number of *gravitinos* with $s = 3/2$

in the supergravity multiplet. Supergravities in $D = 10$ with either $\mathcal{N} = 1$ or $\mathcal{N} = 2$ can describe low-energy limits of *closed* string theories and will be discussed in Chapter 2.2.1.

2.1.6 Duality

The concept of *duality* has fundamental importance both in QFT and string theory. It enables us to understand the physics of one theory, by translating the problems to a different theory. We define a *strong* version of duality to mean the following: given two theories, $\mathbb{T}_1$ and $\mathbb{T}_2$, with generator functionals Z_1 and Z_2, there exists a transformation, which maps the degrees of freedom from $\mathbb{T}_1$ to $\mathbb{T}_2$, and vice-versa, so that $Z_1 = Z_2$. We define a *weak* form of duality to mean that there exists a sector of each theory, $\mathbb{T}_1$ and $\mathbb{T}_2$, in which the degrees of freedom and observables are dual to each other, but $Z_1 \neq Z_2$.

A simple example of a strong duality is the equivalence between the *Thirring model* of massive fermions and a *sine-Gordon theory* of bosons, both in $1+1$ dimensions [39–42]. The generator functional Z_T of the Thirring model in Euclidean space is

$$Z_T = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{-\int d^2x \left[-i\bar{\psi}\partial\!\!\!/\psi - \frac{g^2}{2}\left(\bar{\psi}\gamma_\mu\psi\right)^2 + izm\bar{\psi}\psi\right]\right\}, \tag{2.40}$$

where z is a cut-off dependent constant, γ_μ are the Dirac matrices in two dimensions and $\gamma_5 = i\gamma_0\gamma_1$. By using the *bosonisation* relations, along with field re-scalings,

$$-i\bar{\psi}\partial\!\!\!/\psi = \frac{1}{2}\left(\partial_\mu\varphi\right)^2, \qquad \bar{\psi}\gamma_\mu\psi = i\frac{\beta}{2\pi}\epsilon_{\mu\nu}\partial_\nu\varphi, \qquad imz\bar{\psi}\psi = -\frac{\alpha_0}{\beta^2}\cos\beta\varphi, \tag{2.41}$$

the Thirring Lagrangian $\mathcal{L}_T$ can be transformed into the sine-Gordon Lagrangian,

$$\mathcal{L}_{SG} = \frac{1}{2}\left(\partial_\mu\varphi\right)^2 - \frac{\alpha_0}{\beta^2}\left(\cos\left(\beta\varphi\right) - 1\right). \tag{2.42}$$

The two coupling constants g and β are related by the expression

$$\frac{4\pi}{\beta^2} = 1 + \frac{g^2}{\pi}. \tag{2.43}$$

However, the identification of the degrees of freedom in Eq. (2.41) is not sufficient to establish the duality on a quantum level. It is necessary to show that $Z_T = Z_{SG} \equiv \int \mathcal{D}\varphi \exp\{-\int d^2x\, \mathcal{L}_{SG}\}$, a result, which was proven in [39–41]. The relation (2.43) implies that this duality is a *weak-strong (coupling) duality*. To understand the power of such dualities, suppose we only had perturbative control over Z_T and Z_{SG}, i.e. when $g, \beta \ll 1$, but were interested in a strongly-coupled phenomenon at $g \gg 1$. We could then use the duality transformation (2.41) with (2.43), perform the calculation at $\beta \ll 1$ in Z_{SG}, and translate the result back to a prediction in Z_T at strong coupling.

The holomorphic structure of the SUSY superpotential and the enlarged SUSY algebra heavily constrain quantum corrections and gives rise to dualities in higher-dimensional theories that are absent in "realistic" non-supersymmetric theories, like QCD. A beautiful example of this is the *Seiberg-Witten* theory, which provides a solution to the quantum moduli space of the $\mathcal{N} = 2$ super Yang-Mills theory with the $SU(2)$ gauge group [43]. Analogous approaches were later used to solve theories with flavour [44], with different gauge groups and with various other extensions, see e.g. [45–47].

The field content of the original $SU(2)$ example is an $\mathcal{N} = 2$ vector supermultiplet, which contains $\mathcal{N} = 1$ chiral and $\mathcal{N} = 1$ vector multiplets in the adjoint representation. The theory is asymptotically free and has an $SU(2)_R \times U(1)_{\mathcal{R}}$ R-symmetry, where the $U(1)_{\mathcal{R}}$ is anomalous and broken by instanton effects. In a theory with N_c colours and N_f flavours, the resulting R-symmetry group is $(SU(2)_R \times \mathbb{Z}_{4N_c-2N_f})/\mathbb{Z}_2$, where the division by $\mathbb{Z}_2$ arises because the centre of $SU(2)_R$ is contained in $\mathbb{Z}_{4N_c-2N_f}$. The classical moduli space of (2.1.7) with an $SU(2)$ gauge symmetry can be parametrised by $u = \frac{1}{2}a^2$, where $\phi = \frac{1}{2}a\sigma^3$. It possesses a $\mathbb{Z}_2$ symmetry, $u \to -u$. The interactions are controlled by the holomorphic coupling,

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{2.44}$$

Because of the extended $\mathcal{N} = 2$ SUSY, the entire low-energy effective action, up to two derivatives and four fermions, can be described by a single holomorphic function, i.e. the *prepotential* $P(A)$. A is used to denote a chiral multiplet with a scalar field a. Seiberg and Witten were able to identify two non-perturbative singularities on the moduli space (due to $\mathbb{Z}_2$), the BPS monopole (or dyon) states with the mass, $M^2 = 2|Z|^2$. In this equation, Z is the *central charge* of the SUSY algebra, given by $Z = an_e + a_D n_m$, where n_e and n_m are the electric and magnetic charges of the relevant non-perturbative state. Again, a is related to the scalar vev at the dyon point on the moduli space and a_D is its *dual value*. They are related to the holomorphic coupling by $\tau = \partial a_D/\partial a$.

The duality at work here is the *electric-magnetic* duality, a version of the *S-duality*. S-duality is fundamental in string theory and is a strong-weak duality generated by S and T transformations, together forming the $SL(2, \mathbb{Z})$ symmetry group,

$$\mathrm{S}: \ \tau \to -\frac{1}{\tau}, \quad \mathrm{T}: \ \tau \to \tau + 1 \quad \Longrightarrow \quad SL(2,\mathbb{Z}): \ \tau \to \frac{a\tau + b}{c\tau + d} \wedge ad - bc = 1. \tag{2.45}$$

The effective theory near the non-perturbative points on the moduli space that behave similarly to electrons is supersymmetric QED. The electric-magnetic duality ($E \to B$ and $B \to -E$) then provides a dual, perturbative theory ($\tau_D = -1/\tau$) with mixed electric and magnetic degrees of freedom. Because of this duality, one is able to find a perturbative description everywhere on the moduli space. The prepotential, and thus the full non-perturbative quantum low-energy effective theory can then be calculated. This is done using *monodromy* techniques, by a contour encircling the three singular points on the moduli space, i.e. $\mathcal{M}_{\pm u}$ at the non-perturbative BPS states and $\mathcal{M}_\infty$ at the asymptotically-free infinity, with $\mathcal{M}$ built out of the S and T transformations. Furthermore, we can find the full running coupling by using $\tau = \partial^2 P/\partial A^2$. The quantum moduli space is identified with an elliptic curve and is thus a Riemann surface.

The same theory was analysed in [48] at finite temperature. It was found that the strongly coupled monopole and dyon points on the moduli space minimise the free energy. The moduli space lifts at large ϕ, i.e. in the perturbative asymptotically-free regime.

To conclude this section, we present the *Seiberg duality* in $\mathcal{N} = 1$ theories [49, 50]. This is another example of an electric-magnetic S-duality in a weak sense, as it only relates the IR fixed points of two different theories, $\mathbb{T}_1$ and $\mathbb{T}_2$. The duality addresses the chiral $\mathcal{N} = 1$ super QCD with N_c and N_f flavours, which we call $\mathbb{T}_1$. The gauge group is $SU(N_c)$ and the internal global symmetries are $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$, where $U(1)_B$ is the baryon number and $U(1)_R$ the R-charge. The matter content is, beyond a vector multiplet, described by two chiral multiplets Q and Q^c. The phases of this theory can be classified by the ranges of numbers N_c and N_f. For $N_f < N_c$, the theory has no vacuum. For $N_f = N_c$, the vacuum degeneracy is lifted by quantum corrections and the theory has confinement and chiral symmetry breaking. Similarly, for $N_f = N_c + 1$, there is confinement but no chiral symmetry breaking.

From here on, we will focus on the regime of $N_f > N_c + 1$, which is relevant for the duality. The theory $\mathbb{T}_1$ is asymptotically free when $N_f < 3N_c$. Thus, when $N_f \geq 3N_c$, the spectrum of quarks and gluons can be understood from a weakly coupled Lagrangian. In the regime of $\frac{3}{2}N_c < N_f < 3N_c$, the theory flows to a non-trivial, strongly coupled superconformal IR fixed point in a non-Abelian Coulomb phase. For $N_f \leq \frac{3}{2}N_c$, the IR fixed point is trivial. Note that the identification of fixed points is exact because of the holomorphicity of the superpotential. The analytic structure makes all two-loop and higher-loop contributions to the beta function vanish, up to non-perturbative instanton effects.

The dual theory, $\mathbb{T}_2$, is an $\mathcal{N} = 1$ theory with the $SU(N_f - N_c)$ gauge group and the same internal global symmetries as $\mathbb{T}_1$. However, its matter content includes *three*, instead of two, chiral multiplets, $\tilde{Q}$, $\tilde{Q}^c$ and M, where M is a neutral *meson* superfield.

The conjecture, for which much evidence has been gathered, states that the strongly coupled IR fixed point of $\mathbb{T}_1$, in the regime of $\frac{3}{2}N_c < N_f < 3N_c$, is related by electric-magnetic duality to the weakly coupled IR fixed point of $\mathbb{T}_2$. Furthermore, the RG flows into the two IR fixed points, including all deformations of $\mathbb{T}_1$ and $\mathbb{T}_2$, are also believed to be dual to each other.

2.1.7 $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and S-duality

As an extension of the discussion on dualities, and in preparation for introducing the AdS/CFT duality, we devote this section to the properties of the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory. The easiest way to obtain the Lagrangian is to dimensionally reduce a $D = 10$, $\mathcal{N} = 1$ Yang-Mills theory to four dimensions. The $D = 10$ vector field, from the vector multiplet, becomes a $D = 4$ vector field and the remaining six components become *six real scalars*, ϕ^I. The fermionic content, which accounts for the eight real on-shell degrees of freedom, are *four Weyl gauginos*, λ^i. All of the fields transform under the adjoint representation of the same Lie Group G. The non-anomalous global R-symmetry is $SU(4)_R \simeq SO(6)_R$, under which the fermions transform as a **4**, and scalars as a **6**. The latter is a clear manifestation of rotations in the six internal dimensions in $D = 10$, transverse to the remaining four.

The Lagrangian has standard kinetic terms, scalar potential and Yukawa couplings with the form $g_{YM}\lambda[\phi, \lambda]$. Furthermore, we can again introduce the topological $F_{\mu\nu}\tilde{F}^{\mu\nu}$ term and define the holomorphic coupling τ, as in (2.44). The *full* $\mathcal{N} = 4$ theory is conjectured to possess the S-duality (2.45), known historically as the *Montonen-Olive duality* [51]. Beyond the strong/weak coupling transformation, the gauge group of the dual is the Langlands' dual group ${}^LG$, for example

$$ {}^LU(N) = U(N), \qquad {}^LSU(N) = SU(N)/\mathbb{Z}_N, \qquad {}^LSO(2N) = Sp(N). \tag{2.46}$$

The theory has an *exactly vanishing* beta function $\beta(g_{YM}) = 0$ to all order in perturbation theory. Furthermore, it is believed that the theory is superconformal non-perturbatively [52].

2.2 Gauge/string duality

2.2.1 Strings, branes and effective actions

Perturbative string theory involves two types of objects: *open* and *closed* strings [53–57]. Both types are described by a two-dimensional world-sheet action with different boundary conditions. The only fundamental parameter is their tension, $T = 1/(2\pi\alpha')$, where $\alpha' = \ell_s^2$ and ℓ_s is the length scale of the fundamental strings. Strings with only bosonic excitations, $X^\mu(\tau, \sigma)$, must be embedded into $D = 26$ *critical* dimensions in order to avoid the Weyl anomaly. However, their spectrum contains a tachyon, which is believed to signal an instability.

Supersymmetric strings with both bosonic, X^μ, and fermionic, ψ^μ and $\tilde{\psi}^\mu$, excitations, can have the tachyon eliminated from the spectrum by the space-time supersymmetry-preserving GSO projection. Their critical dimension is $D = 10$. There are five self-consistent types of superstring theory: *Type I* (open strings), *Type IIA* and *Type IIB* (closed strings), and two *heterotic* hybrids of bosonic and supersymmetric theories with gauge groups $SO(32)$ and $E_8 \times E_8$. Through the use of non-perturbative string dualities, all five types can be obtained as limits of *M-theory* in $D = 11$. We will mostly focus on Type IIB theory, which originally gave rise to the *gauge/string duality*.

2.2.1.1 Closed strings and p-branes

The world-sheet fermions on closed strings can either have *Ramond* (R) boundary condition or satisfy the *Neveu-Schwarz* (NS) condition:

$$\text{(R)}: \qquad \psi^\mu(0,\tau) = \tilde{\psi}^\mu(0,\tau), \qquad \psi^\mu(\pi,\tau) = \tilde{\psi}^\mu(\pi,\tau), \tag{2.47}$$

$$\text{(NS)}: \qquad \psi^\mu(0,\tau) = -\tilde{\psi}^\mu(0,\tau), \qquad \psi^\mu(\pi,\tau) = \tilde{\psi}^\mu(\pi,\tau). \tag{2.48}$$

Massless bosonic excitations of all closed superstrings have the same NS sector with the graviton $g_{\mu\nu}$, the anti-symmetric $B_{\mu\nu}$ field and the dilaton ϕ. The vev of the dilaton introduces a coupling parameter, $g_s = e^\phi$, for the perturbative world-sheet *genus* expansion. We also define $H = dB$. In Type II theories, the massless *Ramond-Ramond* fields are form fields, C, with an associated field strength, $F = dC$. In Type IIA, we have the one- and the three-form, C_1 and C_3. In Type IIB, the spectrum includes the scalar axion, C_0, and form fields, C_2 and C_4. Massless fermions in Type IIB are two Majorana-Weyl gravitini, $\psi_{\mu,\alpha}$ and two Weyl dilatini, λ_α. The five-form, $F_5 = dC_4$ is Hodge self-dual, i.e. $F_5 = \star F_5$. It is important to note that the field content of Type I strings is an $\mathcal{N} = 2$, $D = 10$ supergravity. The spectrum of Type IIB theory is *chiral*, whereas the spectrum of type IIA is *not* chiral.

To understand how supergravity arises from the closed Type II string, its low-energy effective action must be constructed. This is done by including the massless fields as generalised couplings into the Polyakov action, which takes the form,

$$S_{\text{Poly}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}\left\{G_{\mu\nu}(X)\partial_a X^\mu \partial^a X^\nu + iB_{\mu\nu}(X)\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu + \ldots\right\}, \tag{2.49}$$

where g_{ab} is the $D = 2$ world-sheet metric and μ label bosonic fields, X^μ, for which the criticality of the string demands that $\mu = 0, 1, ..., 9$. Beta functions of the new "couplings", $\beta(G)$, $\beta(B)$, $\beta(\phi)$, $\beta(C_n)$, etc., can then be computed on the world-sheet in a perturbative α'-expansion, for which the Weyl invariance of the path integrals demands that $\beta(G) = \beta(B) = \ldots = 0$. However, these equations, coming from the consistency-condition of the world-sheet theory, also have an alternative interpretation. They can be derived as the Euler-Lagrange equations of motion from an $\mathcal{N} = 2$, $D = 10$ space-time *effective* supergravity action in the string frame,

$$S_{\rm IIB} = \frac{1}{(2\pi)^7\alpha'^4}\int d^{10}x\sqrt{-G}\left[e^{-2\phi}\left(R + 4\left(\partial_\mu\phi\right)^2\right) - \frac{2}{5!}F_5^2\right] + S_{\rm B} + S_{\rm F}, \tag{2.50}$$

where $S_{\rm B}$ and $S_{\rm F}$ stand for other bosonic, including Chern-Simons, and fermionic terms in the effective supergravity action. Newton's constant G_N in ten dimensions can be found by transforming (2.50) into the Einstein frame, giving $16\pi G_N = (2\pi)^7\alpha'^4 g_s^2$. Furthermore, note that the self-dual nature of F_5 cannot be incorporated into the action, but has to be added as a condition at the level of equations of motion.

We can consistently set all fields but G, ϕ and C_4 to zero in the equations of motion at the lowest order in α'. Thus, it is sufficient to analyse the action (2.50) with $S_{\rm B} = S_{\rm F} = 0$. An important family of gravitational objects solves the equations arising form (2.50), also with a *p-form* term, $\frac{2}{(8-p)!}F_{p+2}^2$, instead of F_5. These extended objects are known as *p-branes* because they possess translationally-invariant horizons. For our purposes, it will suffice to consider branes with $p = 3$. The solution of an *extremal* 3-brane is

$$\begin{aligned} &ds^2 = H_3^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H_3^{1/2}dx^m dx^m, && \mu,\,\nu\in\{0,\dots,3\},\quad m\in\{4,\dots,9\},\\ &H_3 = 1 + \frac{L^4}{r^4},\qquad L^4 = 4\pi g_s N\alpha'^2, && r^2 = x^m x^m, \end{aligned} \tag{2.51}$$

with the dilaton and the R-R field given by

$$e^{2\phi} = g_s^2, \qquad\qquad C_4 = \left(H_3^{-1} - 1\right)g_s^{-1}dx^0\wedge dx^1\wedge dx^2\wedge dx^3. \tag{2.52}$$

This is a *conformal* brane as the value of the dilaton is constant throughout the $D = 10$ space-time. The space with coordinates $\{x^5,\, x^6, \dots,\, x^9\}$ is a 5-sphere, S^5, with the line element $d\Omega_5$. The integer N arises from the Dirac quantisation of the R-R five-form flux through the 5-sphere,

$$\int_{S^5}\star F_5 = N. \tag{2.53}$$

The brane is thus magnetically charged under the R-R field.

Branes with $p+1$ flat space-time dimensions, instead of four, can easily be generated in Type IIA and Type IIB supergravity by using the F_{p+2} R-R flux mentioned above. In Type IIA, these are F_2 and F_4, while Type IIB supports F_1, F_3 and F_5. Furthermore, branes can be electrically charged under the Hodge-dual $dA_{7-p} = \star dA_{p+1}$. To summarise, we see that Type IIA (IIB) theory supports p-branes with p even (odd). The dilaton is r-dependent for all non-conformal branes, i.e. for all $p \neq 3$.

2.2.1.2 Open strings and D-branes

As supergravity is the low-energy effective theory of closed strings, open strings give rise to (non-Abelian) gauge theories on hypersurfaces known as D-branes [58–60]. A variation of the world-sheet string action for an open string demands that the boundary condition, $\partial_\sigma X^\mu \delta X_\mu = 0$, be satisfied at the two ends, $\sigma = 0$ and $\sigma = \pi$. We can thus select the *von Neumann boundary conditions*, $\partial_\sigma X^\mu = 0$, for $\mu = \{0, \ldots, p\}$, and the *Dirichlet boundary conditions*, $X^\mu = C^\mu$, for $\mu = \{p+1, \ldots, D-1\}$. C^μ is a constant vector. Open strings can thus be thought of as attached to a $p+1$-dimensional hypersurface, the *Dirichlet (D)-brane*, on which they can move freely. The two types of boundary conditions are interchanged by the *T-duality*, which is a duality between string spectra when a compactified dimension with radius R is exchanged for radius $1/R$. Many properties of D-branes follow from imposing this duality, which is furthermore a duality within the M-theory relating Type IIA and IIB theories, as well as the two heterotic theories.

The massless excitation of the open bosonic string is the vector field, A_μ, and supersymmetric open strings include additional fermionic partners. The boundary conditions break half of the 32 supersymmetries of the Type II theory, hence D-branes are BPS states in the non-perturbative theory with 16 supersymmetries. The effective action of D-branes can be derived in a similar manner as the supergravity action in Section 2.2.1.1: by coupling the string to massless background fields and computing the world-sheet beta functions [61]. It is important to note that D-branes interact with closed strings in the bulk. The effective action is then a sum of the Dirac-Born-Infeld, Wess-Zumino and fermionic contributions, i.e. $S_{\text{D-brane}} = S_{\text{DBI}} + S_{\text{WZ}} + S_{\text{F}} +$ anomalous curvature, where

$$S_{\text{DBI}} = -\tau_p \int_{\mathcal{M}_{p+1}} d^{p+1}\xi \, e^{-\phi} \sqrt{-\det\left[G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right]}, \tag{2.54}$$

$$S_{\text{WZ}} = \mu_p \int_{\mathcal{M}_{p+1}} C_{p+1}. \tag{2.55}$$

Tensor $G_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}$ is the metric pull-back onto the hypersurface. Similarly, B_{ab} is the pull-back of the NS closed string spectrum 2-form field $B_{\mu\nu}$. The action has reparametrisation invariance of its world-volume coordinates ξ^a. It is convenient to choose the *static gauge*, $X^\mu = \xi^\mu$, for $\mu = \{0, \ldots, p\}$, which removes longitudinal fluctuations of the brane from G_{ab}.

The ends of open strings can further be equipped with *Chan-Patton factors*, so that $F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$ becomes the field strength of the non-Abelian vector field transforming under the $U(N)$ group. From the string theory point of view, this means that we are describing the world-volume theory of N coincident D-branes. D-branes can be further interpreted as sources of N units of the R-R charge flux in Type II theory [62], which establishes the connection between p-branes and D-branes. A p-brane should be thought of as a classical supergravity description of the gravitational field sourced by a heavy non-perturbative D-brane with $p+1$ extended dimensions. Similarly as with p-branes, Dp

branes with even (odd) p exist in Type IIA (B) theory.

Let us now focus on the D3 brane. According to the above analysis, the world-volume effective theory has 16 supersymmetries in four dimensions, which implies that the low-energy limit of $S_{\rm D3}$ should be the maximally supersymmetric $\mathcal{N}=4$ Yang-Mills theory with the gauge group $U(N)$, presented in 2.1.7. Indeed, this can be seen by expanding $S_{\rm DBI}$, as well as the supersymmetric fermion contributions, $S_{\rm F}$ in powers of the massless world-volume fields. The terms with real scalars ϕ^I, with $I \in \{1,\dots,6\}$, follow from the expansion of the pull-back metric G_{ab} around a flat metric η_{ab} in the static gauge,

$$G_{ab} = \eta_{ab} + \frac{\partial X^I}{\partial \xi^a}\frac{\partial X^J}{\partial \xi^b}\delta_{ab}, \tag{2.56}$$

where $\phi^I \equiv X^i/(2\pi\alpha')$. Each of the six scalars describes a transverse fluctuation of the brane. The Yang-Mills coupling g_{YM} and the 't Hooft coupling λ for the D3 brane are given in terms of the string parameters

$$g_{YM} = \sqrt{4\pi g_s}, \qquad\qquad \lambda = g_{YM}^2 N = 4\pi g_s N. \tag{2.57}$$

2.2.2 AdS/CFT correspondence

The AdS/CFT correspondence is a *holographic gauge/string duality* formulated by Maldacena in [10], where he considered the 3-brane solution (2.51) in the limit of $\alpha' \to 0$, while holding the quantity $u = r/\alpha'$ fixed.[2] This is the *near-horizon* ($r \to 0$) limit of an asymptotically-flat brane background in which the metric becomes that of $AdS_5 \times S^5$, where AdS_5 stands for the anti-de Sitter space, given in Poincaré coordinates by

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + L^2\frac{dr^2}{r^2} + L^2 d\Omega_5^2. \tag{2.58}$$

The scale $L = (4\pi g_s N \alpha'^2)^{1/4}$, known as the *AdS radius*, characterises the scale of curvature of the gravitational p-brane solution.

In Sections 2.2.1.1 and 2.2.1.2, we established two different descriptions of the same extended object, i.e. the $D=10$ supergravity description and the $D=4$, D-brane effective world-volume theory. In order for supergravity to be a good effective description of the underlying string dynamics, $L \gg \ell_s$ must be true. This implies that $L^4/\alpha'^2 = 4\pi g_s N = \lambda \gg 1$. Hence, the 't Hooft coupling, as defined in (2.57), must be *large* for the supergravity limit of closed string theory to be a suitable description of N D3-branes.

Alternatively, when N D-branes coincide, the relevant parameter in the open string perturbative expansion is $g_s N$. The D-brane effective world-volume description, i.e. $\mathcal{N}=4$ SYM gauge theory, is thus a good description of the string spectrum when $g_s N \ll 1$. This

[2]See references [52, 63–70] for various summaries and lectures on AdS/CFT.

implies that the 't Hooft coupling must be *small*, $\lambda \ll 1$. The two different descriptions are thus applicable in exactly the opposite limits of the 't Hooft coupling.

Before stating the duality, let us consider more carefully the low-energy Wilsonian effective action, $S_{\text{IIB-IR}} = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{int}}$, of massless excitation of the full Type IIB theory in the "open string picture" ($g_s N \ll 1$). The key observation is that in the low energy limit, the interaction term $S_{\text{int}} \sim \mathcal{O}(\omega^8 G_N) \to 0$, hence the bulk (closed string) and brane (open string) excitations decouple. Furthermore, since gravitational interactions are irrelevant, the open string gauge theory with a marginal coupling in $D = 4$ dominates the low-energy spectrum. S_{bulk} is simply classical gravity in $D = 10$ Minkowski space.

Similarly, in the "closed string picture" ($g_s N \gg 1$), the near-horizon and far from horizon gravitational theories decouple. The region far away is again $D = 10$ Minkowski space gravity, while the near-horizon spectrum includes higher energy closed-string excitations in $AdS_5 \times S^5$, because all energies are red-shifted by the warp factor $g_{00} \to 0$. See figure 2.1 for a graphical representation of the geometry of the bulk.

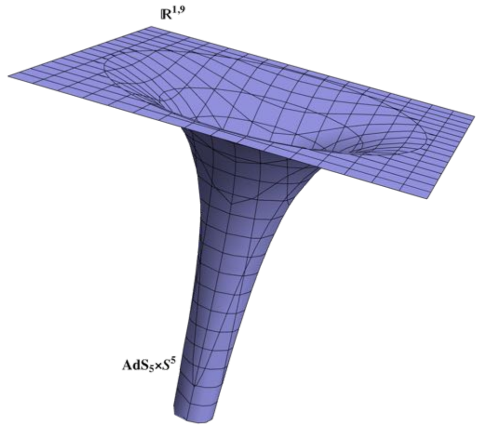

Figure 2.1: A representation of the bulk geometry in the closed string picture. The near-horizon anti de-Sitter throat extends towards the asymptotically flat *far* region.

The two near-horizon descriptions are then identified via the AdS/CFT duality conjecture, stating that the $\mathcal{N} = 4$, $U(N)$ superconformal Yang-Mills theory in $D = 4$ is dual to the Type IIB string theory on $AdS_5 \times S^5$ [10]. Although this statement has not been proven or explicitly constructed, there is much evidence supporting it. The simplest check is the comparison of symmetries, namely the isometry group of $AdS_5 \times S^5$, which is $SO(4, 2) \times SO(6)$, matches the conformal and R-symmetry groups in the $\mathcal{N} = 4$ theory. Among other evidence, it has been shown that the spectra of supersymmetric states, as well as many scattering amplitudes, match.

The strong/weak duality is *holographic* in the sense that all information about a theory with gravity is encoded in a lower-dimensional field theory. Usually, one only considers the five-dimensional, asymptotically-AdS part of the bulk, which is a dimensionally reduced solution of Einstein's equations with a negative cosmological constant. In this sense, AdS/CFT is an explicit example of the holographic principle in gravity [71,72], which stems from Bekenstein's observation that the entropy of a black hole that scales as the *area* of its event horizon, should be the maximal entropy in a *volume* of space-time [73].

AdS/CFT has been extended to numerous other examples of string theory and M-theory brane constructions, including probe branes [74]. The brane-bulk decoupling can also be achieved by a large density of smeared brane defects, as opposed to a large number of colour branes [75]. The duality can be extended to bulks, which include a gravitational object with an event horizon, such as a black hole or a black non-extremal brane. The dual theory then has non-zero temperature. Its temperature is identified with the Hawking temperature T_H of the black hole by resolving the conical space-time singularity at the horizon. Furthermore, we can include finite density with a chemical potential through an introduction of charge into the bulk [52, 63–70].

In Euclidean space, the duality can be made precise by the GKPW formula [11, 12],

$$Z_{\text{string}}\left[\Phi(x,r)\big|_{r\to\infty} = \phi_0(x)\right] = \left\langle \exp\left\{\int d^4x\,\phi_0(x)\mathcal{O}(x)\right\}\right\rangle_{\text{CFT}}. \tag{2.59}$$

The formula states that a generic supergravity field Φ (dilaton, graviton, etc.), which propagates in AdS_5, sources a dual operator on the CFT side. By matching quantum numbers, a scalar is dual to a scalar operator $\mathcal{O}$, a graviton to the conserved stress-energy tensor $T_{\mu\nu}$, a vector field to a conserved current J_μ, etc.

In order to make calculations possible, a further limit needs to be taken. This is the limit of classical gravity in which all graviton loops are suppressed, $L \gg \ell_P$, where ℓ_P is the Planck length in $D = 10$, i.e. $\ell_P \sim G_N^{1/8}$. Hence, $L^4/\ell_P^4 \sim (g_s\alpha'^2 N)/(g_s\alpha'^2) = N \gg 1$. Eq. (2.59) then implies that classical gravity in AdS space gives us access to the strongly coupled QFT with a large number of colours, $N_c \gg 1$,

$$\lim_{\lambda,N\to\infty} Z_{\text{string}}\left[\phi(x,r)|_{r\to\infty} = \phi_0(x)\right] = \exp\left\{-S_{\text{grav}}[\phi_0]\right\}. \tag{2.60}$$

We conclude this section by noting that there exists a direct connection between effective field theory, as interpreted in the Wilsonian renormalisation group picture, and AdS/CFT correspondence. A fundamental feature of AdS/CFT is the IR/UV duality [76, 77]. More precisely, the extra radial dimension r is related to the energy scale of the field theory; the near-boundary and the deep bulk regions correspond to the UV and IR regimes of the dual field theory, respectively. This statement can be motivated from various points of view. The divergence of the metric tensor near AdS infinity corresponds to the UV divergence of the field theory, whereas the IR is controlled by the black hole thermodynamics. Furthermore, the longer the distance between two boundary points, the deeper the geodesic between them extends into the bulk. We also know that, for example, the radial dependence of the dilaton $\phi(r)$, in non-conformal scenarios, can be interpreted as the beta function of the gauge theory coupling [63]. Lastly, if we slice the bulk along the radial direction and integrate the slices out by starting from the boundary, we can show that this corresponds to the Wilsonian integration of high-momentum modes in the boundary field theory.[3]

2.2.3 Hydrodynamics from AdS/CFT

The GKPW prescription (2.59) enables us to compute Euclidean correlation functions by taking functional derivatives of the supergravity action with respect to the boundary ($r \to \infty$) values of the bulk fields, $\delta/\delta\phi_0$. It is easiest to compute a connected two-point function, $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$, by using the generator functional of connected correlation functions, $W[\phi_0]$, where $Z = \exp\{-W\}$, as discussed in Section 2.1.2. Using Eq. (2.60), we see that $W = S_{\text{grav}}$. Hence, the *on-shell* classical gravity action evaluated at the boundary of the asymptotically *AdS* space gives the holographic connected two-point function.

Consider, for example, a probe scalar field ϕ in a $d+1$ dimensional Euclideanised *AdS* background with a Poincaré-patch metric and the *AdS*-radius set to $L = 1$,

$$ds^2 = r^2 \left(dt^2 + d\boldsymbol{x}_{d-1}^2\right) + \frac{dr^2}{r^2}. \tag{2.61}$$

The action can be written purely in terms of space-time boundary contributions,

$$S_{\text{grav}} = \frac{1}{\kappa^2}\int d^{d+1}x\sqrt{g}\left[\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \dots\right] = \int \frac{d^dk d^dk'}{(2\pi)^{2d}}\phi_0(k)\phi_0(k')\mathcal{F}(r,k,k')\Big|_0^{1/\epsilon}, \tag{2.62}$$

where $\phi(r,x) = \int \frac{d^dk}{(2\pi)^d} e^{ikx} f_k(r)\phi_0(k)$ and we are using the Dirichlet boundary conditions. All other bulk terms vanish when S_{grav} is evaluated on ϕ, which satisfies the equation of motion. We have cut off the bulk near the *AdS* infinity by introducing a cut-off surface at $r = 1/\epsilon$, for $\epsilon \ll 1$. This regularisation scheme is required because the boundary action diverges as $r \to \infty$. A procedure of holographic renormalisation [81–83] can then be employed, which precisely cancels off divergent terms in (2.62). The scheme it employs is the minimal subtraction scheme, which only subtracts the purely divergent terms. All holographic counter-terms can be written in a covariant form, which manifestly preserves the bulk diffeomorphisms.

Beyond imposing the Dirichlet boundary condition at $r = 1/\epsilon$, we demand that ϕ vanishes deep in the bulk, which makes $\mathcal{F}(0,k,k') = 0$. The Euclidean two-point function

[3] For details on the correspondence between Wilsonian RG and holography, the discussion of the IR/UV correspondence and the issues related to the identification of the momentum cut-off $\Lambda(r)$ in terms of the radial coordinate, see [78–80] and references therein.

is then given by expression

$$\langle \mathcal{O}(k)\mathcal{O}(k')\rangle = \lim_{\epsilon\to 0}\left[2\mathcal{F}\left(1/\epsilon, k, k'\right) + \text{counter-terms}\right]. \tag{2.63}$$

It can be shown that $\langle \mathcal{O}(k)\mathcal{O}(k')\rangle$ has exactly the right position-space scaling of a CFT, as in the fixed-point Eq. (2.5), i.e. $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim |x-y|^{-2\Delta_{\mathcal{O}}}$. The operator dimension $\Delta_{\mathcal{O}}$ is given in terms of the scalar field's mass, $\Delta_{\mathcal{O}} = d/2 + \sqrt{(d/2)^2 + m^2}$. Furthermore, expression (2.63) may include contact terms that can also be eliminated.[4]

If we consider a general asymptotically AdS background, it is very instructive to study the solution of the massive scalar bulk equation of motion in a near-boundary expansion. Using the Fourier decomposition of $\phi(x,r)$, as before, and introducing a new radial variable $z = 1/r$, we find that $f_k(z)$ scales as $f_k(z) = z^{\Delta}$, as $z \to 0$. The two solutions of Δ are

$$\Delta_{\pm} = d/2 \pm \sqrt{(d/2)^2 + m^2}. \tag{2.64}$$

The full Frobenius series solution of the second-order can then be written as

$$\phi(x,z) = z^{\Delta_-}\phi_0(x)\left(1 + \mathcal{O}(z^2)\right) + z^{\Delta_+}\phi_1(x)\left(1 + \mathcal{O}(z^2)\right). \tag{2.65}$$

In the *standard* quantisation with the Dirichlet boundary conditions, ϕ_0 is the source of the dual operator $\mathcal{O}$, and ϕ_1 is proportional to the vev, $\langle \mathcal{O}\rangle$. Notice that Δ_+ is precisely the dimension of $\mathcal{O}$, i.e. $\Delta_{\mathcal{O}}$. In the *alternative* quantisation, von Neumann boundary conditions are used and ϕ_0 and ϕ_1 reverse their roles in relation to the source and the vev of $\mathcal{O}$. It is clear from the form of Eq. (2.64) that $m^2 \geq -(d/2)^2$, which is known as the "Breitenlohner-Freedman" bound on the allowed range of tachyon masses in AdS [84, 85]. There exists a further, lower *unitarity bound* on the operator dimension, $\Delta_{\mathcal{O}} \geq (d-2)/2$, which is relevant for the alternative quantisation where $\Delta_{\mathcal{O}} = \Delta_-$.

Similarly to the above procedure, we can find the current J^μ correlators by considering vector fields A_μ in the bulk, and the stress-energy tensor $T^{\mu\nu}$ correlators by perturbing the background metric with a spin-two $h_{\mu\nu}$. The dimensions of the dual J^μ and $T^{\mu\nu}$ are $\Delta_J = d-1$ and $\Delta_T = d$, respectively. Furthermore, fermionic boundary operators are sourced by bulk fermions, ψ. In gravity, the $z \ll 1$ expansion of the metric is known as the Fefferman-Graham expansion [86]. For each of the bulk fields, there is an associated holographic renormalisation procedure, which renders n-point functions finite. It is important to note that for dynamical graviton fields, one must add the Gibbons-Hawking counter-term, which renders the variational principle well-defined and allows for the Dirichlet boundary conditions. The term is proportional to the trace of the extrinsic curvature K of the boundary hyper-surface with the induced metric γ, $S_{\text{GH}} = -2\int d^dx\sqrt{\gamma}K$.

[4] Contact terms are terms analytic in momentum. Therefore, contact terms can only give contributions proportional to derivatives of the Dirac delta function, after $\langle \mathcal{O}\mathcal{O}\rangle$ is Fourier transformed to position space.

In order to facilitate the computation of higher connected n-point correlation functions, we must add interaction terms into the S_{grav} action, e.g. $\int d^{d+1}x\sqrt{g}\lambda\phi^3$. The correlation functions can then be computed using standard diagrammatic techniques with external legs fixed to the boundary of the bulk space-time. Such diagrams are often referred to as the *Witten diagrams* [11].

In order to be able to use AdS/CFT for computations in the hydrodynamic limit of strongly-coupled field theories with holographic duals, one must first understand how correlation functions with Lorentzian signature can be recovered from gravity. A prescription for the calculation of *retarded* and *advanced* two-point Green's function was given in [87] and will be summarised here. Consider for example a five dimensional asymptotically AdS black brane metric with a horizon at $z = z_h$,

$$ds^2 = \frac{L^2}{z^2}\left(\left(1 - z^4/z_h^4\right)dt^2 + d\boldsymbol{x}^2 + \frac{dz^2}{\left(1 - z^4/z_h^4\right)}\right). \tag{2.66}$$

The dual field theory now has finite temperature, $T = \pi/z_h$.

If we consider a scalar momentum space mode, $\phi_k(z)$, propagating in Lorentzian background (2.66), we can no longer demand for ϕ_k to vanish in the interior of the geometry, as we did in the Euclidean case. We find that the two solutions at the horizon correspond to *in-going* and *out-going* modes, $\phi_k(z) = (1 - z/z_h)^{\pm i\mathfrak{w}/2} F_k(z)$, where we have used dimensionless frequency and momentum, $\mathfrak{w} \equiv \omega/(2\pi T)$ and $\mathfrak{q} \equiv |\boldsymbol{k}|/(2\pi T)$.

Since a black hole should absorb all information, the authors of [87] proposed the prescription whereby a holographic *retarded* two-point Green's function can be computed by imposing the in-going boundary condition on $\phi_k(z)$ and using the expression $G_R(k, k') = \lim_{\epsilon\to 0}\left[2\mathcal{F}(\epsilon, k, k') + \text{c.t.}\right]$, in analogy with (2.63). The non-vanishing $\mathcal{F}(z_h, k, k')$ does not enter into the two-point function, hence this prescription cannot be derived from the Lorentzian version of the GKPW formula (2.59). Similarly, the *advanced* Green's function G_A requires us to impose the out-going boundary condition on $\phi_k(z)$, or a mode of any other spin.

The prescription for the calculation of higher Lorentzian n-point functions was established in [88], where the authors promoted the holographic bulk calculation to the Schwinger-Keldysh formalism[5], which allows one to have control over all real-time correlation functions. The doubling of the time axes was shown to correspond to the maximal extension of the black brane's (hole's) Penrose diagram in Kruskal-Szekeres coordinates. This naturally mixes black and white hole regions, giving access to mixed retarded and advanced correlation functions.

[5]See Section 2.1.4 and Chapter 3 for a detailed discussion of the Schwinger-Keldysh Closed-Time-Path formalism in QFT.

We are now ready to study the $\mathfrak{w} \to 0$ and $\mathfrak{q} \to 0$ limit of strongly coupled thermal and dense systems, i.e. hydrodynamics of holographic gases, fluids and plasmas [89–91]. As discussed in Section 2.1.3, hydrodynamic properties of a system can be understood from the gradient expansion ($\mathfrak{w}, \mathfrak{q} \ll 1$) of the stress-energy tensor $T^{\mu\nu}$ and the conserved current J^{μ}. To understand the hydrodynamic behaviour of the $\mathcal{N} = 4$ theory, or other theories with holographic duals, we must perturb the background, $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, as well as the vector field A_{μ}, when the fluid is charged [90, 92]. Here, we will only consider the simplest case of an uncharged $\mathcal{N} = 4$ fluid at finite temperature, and hence only perturb the metric $g_{\mu\nu}$, which was stated in Eq. (2.66).

Following [90, 91, 93], it is convenient to pick momentum to flow in the z-direction and write the rotationally invariant metric perturbations as $h_{\mu\nu}(r)e^{i\omega t - ikz}$. Note that r is the radial coordinate (as in (2.61), where $r = 1/z$ of (2.66)) and that z is one of the three flat spatial boundary coordinates, $\boldsymbol{x} = (x, y, z)$. The metric perturbations decompose into three independent sectors according to the remaining $SO(2)$ symmetry,

$$\text{Spin 0 (sound channel)}: \qquad h_{tt},\, h_{tz},\, h_{zz},\, h,\, h_{rr},\, h_{tr},\, h_{zr} \tag{2.67}$$

$$\text{Spin 1 (shear channel)}: \qquad h_{tx},\, h_{ty},\, h_{zx},\, h_{zy},\, h_{rx},\, h_{ry} \tag{2.68}$$

$$\text{Spin 2 (scalar channel)}: \qquad h_{xy} \tag{2.69}$$

The scalar channel transforms as a rank 2 tensor and we have defined $h \equiv h_{xx} + h_{yy}$. Beyond the fact that the equations of motion in each of the sectors are decoupled from the remaining two sectors, the fields can be assembled into *three* gauge invariant variables, $Z_{1,2,3}$. We thus end up with the total of three independent scalar second order differential equations. Furthermore, the on-shell action S_{grav} can be written solely in terms of Z_i. As a result of this decomposition, there are three independent two-point Green's functions of $T^{\mu\nu}$, of which the poles give the gradient expanded hydrodynamic dispersion relations for the shear and the sound modes. They have the form

$$\text{Shear:} \qquad \omega = -i\,\frac{\eta}{\varepsilon + T}k^2 - \mathcal{O}(k^4), \tag{2.70}$$

$$\text{Sound:} \qquad \omega = \pm c_s k - \frac{2}{3}\frac{\eta}{\varepsilon + P}k^2 \pm \mathcal{O}(k^3), \tag{2.71}$$

where c_s is the speed of sound fixed by conformal invariance to $c_s = 1/\sqrt{3}$. The retarded Green's function in the scalar sector is

$$\langle T^{xy}(-\omega, k) T^{xy}(\omega, k)\rangle_R = P - i\eta\omega + \mathcal{O}(\omega^2) + \mathcal{O}(k^2), \tag{2.72}$$

and has no hydrodynamic poles. It is easy to see from this expression that shear viscosity can be computed using the Kubo formula[6],

$$\eta = \lim_{\omega \to 0} \left[\frac{i}{\omega} \lim_{k \to 0} \langle T^{xy}(-\omega, k) T^{xy}(\omega, k) \rangle_R \right]. \tag{2.73}$$

Holography thus enables us to determine the transport coefficients η, and others, as a function of the microscopic parameters of a strongly coupled field theory.

These and similar gravitational methods were first used to show the universality of $\eta/s = \hbar/(4\pi k_B)$ in strongly coupled holographic theories with two-derivative bulk actions [94]. This value gives the correct order-of-magnitude prediction for η/s in strongly coupled theories, consistent with various experimental measurements [95]. The authors of [94] further conjectured the $\eta/s \geq 1/(4\pi)$ inequality for realistic fluids. Although no experimental fluid has been found that would violate this bound, there exist phenomenological holographic, as well as top-down string theoretic constructions that do violate the inequality [96, 97]. These issues will be discussed in detail in Chapter 5.

As a final comment on AdS/CFT in this chapter, we would like to stress that holography is deeply connected with the details of black hole physics. Thermodynamic properties of the bulk black hole are those of the dual field theory. Beyond thermodynamics, hydrodynamic transport can be understood in terms of the black hole's quasi-normal modes [93]. To better understand this fact, let us write a gauge-invariant $Z(r)$ as in Eq. (2.65), i.e. $Z(r) = \mathcal{A} r^{-\Delta_-} + \mathcal{B} r^{-\Delta_+} + \ldots$. Imposing the Dirichlet boundary conditions, whereby $Z = 1$ at the boundary, the retarded two-point function becomes

$$\langle \mathcal{O}\mathcal{O} \rangle_R \sim \frac{\mathcal{B}}{\mathcal{A}} + \text{contact terms}. \tag{2.74}$$

The poles of the Green's function correspond to zeros of $\mathcal{A}$, subject to in-going boundary conditions at the horizon. Setting $\mathcal{A} = 0$ (ϕ_0 in our previous notation) as the second boundary condition therefore precisely corresponds to the way the quasi-normal modes are computed in a black hole background. The quasi-normal modes are seen to equal dispersion relations, $\omega(k)$, which solve the $\lim_{r\to\infty} Z(r) = 0$ equation for in-going $Z(r)$ solutions of the background fluctuation equations. Taylor expanded lowest quasi-normal modes in terms of k precisely reproduce the forms of the hydrodynamic dispersion relations (2.70) and (2.71). Furthermore, the explicit coefficients of k determine the transport properties of the strongly coupled field theory, dual to the gravitational setup.

All of these methods will be used in Chapter 5, where we will analyse transport in theories with higher derivative gravity. An alternative approach to holographic hydrodynamics, i.e. the fluid/gravity correspondence [98, 99], which relies on a direct computation of the holographic stress-energy tensor [100] will also be presented and employed in Chapter 5.

[6] See the end of Section 3.2 of Chapter 3 for a discussion on how Kubo formulas can be derived from the CTP formalism of QFT.

Chapter 3

Hydrodynamics from quantum field theory

3.1 Motivation

Effective field theories combine a set of tools, which are extremely useful in describing physical systems of which the full microscopic details are either too complicated, or simply irrelevant for questions under consideration. Since this thesis is motivated by a desire to pursue a systematic understanding of hydrodynamics, we will devote this chapter to the use of powerful effective field theory techniques to discuss hydrodynamics. Our main focus will be the inclusion of dissipation into classical effective field theory of hydrodynamics, which is a longstanding and difficult problem.

Effective theories of Goldstone modes have recently been shown to be the appropriate framework to systematically derive hydrodynamics [3–5]. The equations of non-dissipative hydrodynamics have previously been generated using this description at the zeroth order in the gradient expansion for relativistic fluids that are insensitive to static, non-compressional deformations [3, 4] and at second order by [101]. This was achieved by constructing a gradient-expanded action describing the long-range scalar modes that correspond to spatial excitations around the equilibrium state of a fluid. The form of the action was restricted by the identification of appropriate symmetries, with the volume-preserving diffeomorphisms playing the central role in the reduction of potential Lagrangian terms.

A serious limitation of this scheme is that dissipative forces cannot be derived from the variational principle. Our goal is, however, to develop a systematic scheme for the construction of hydrodynamics at all orders - including dissipation. One approach to this problem is to rely on linear response theory [102]. A different approach aimed at computing hydrodynamic correlation functions from an effective action was recently proposed in [103]. In this chapter, we will present another method, which will enable us to describe dissipative fluids using the variational principle. This will be done by considering a classical effective action with the characteristics of open-system effective field theories, which emerge in the Schwinger-Keldysh Closed-Time-Path (CTP) formalism [8,9], first introduced by Schwinger [8]. The formalism was invented to describe retarded time-evolution of operator expectation values acting on mixed states, which are specified by density matrices.

We will begin this chapter by presenting the details of the CTP formalism as an extension of the usual quantum field theory used to compute scattering amplitudes between asymptotic pure states. We will discuss CTP in quantum field theory as well as its importance to low-energy classical field theory, which will be of direct relevance to hydrodynamics. We will focus on the matrix structure of the CTP propagators, which arises from the doubling of the degrees of freedom and the introduction of *two time axes*; one evolving from past to future and the other evolving backwards in time. Effective theories emerge when the unobserved degrees of freedom, called the *environment*, are eliminated. The remaining degrees of freedom, called the *system*, follow more involved effective dynamics than in a theory of pure states. This requires the use of the CTP formalism, which is able to incorporate the entanglement between the system and the environment. We will argue that, generically, effective field theories can include couplings between the two time axes, expressed within the *influence functional* considered first by Feynman and Vernon in [104], which includes all effective interactions. The coupling of the two time axes corresponds to system-environment interactions that make the state of the system mixed. This in turn leads to a theory with excited environment states at asymptotically long time. If the spectrum of these states has no gap, then the system experiences dissipative dynamics.

The double axes structure of effective CTP theory descends into a classical low-energy theory, which we will use to derive dissipative hydrodynamical equations of motion from the variational principle. This will be done at a phenomenological level, directly in terms of an effective classical CTP field theory without a microscopic derivation, in accordance with the logic used in [3, 4]. By varying the fields on only one of the two CTP time axes, we will obtain the energy-momentum balance equation containing a two-tensor that will not be conserved because of interactions between the fluid and the environment. Near hydrodynamical equilibrium, however, we will show that this tensor becomes approximately conserved. We will, therefore, identify it as the fluid's stress-energy tensor. Using the energy-momentum balance equation, we will also derive the Navier-Stokes equations. Shear viscosity will be shown to vanish and a possible cause of this restriction will be discussed, i.e. the theory's invariance under volume-preserving diffeomorphisms. Thermodynamical quantities and bulk viscosity will be identified in terms coefficient functions of the effective Lagrangian. Finally, we will discuss entropy production and conclude this chapter by summarising our results.

3.2 Schwinger-Keldysh CTP formalism

In Section 2.1.4, we motivated the necessity for the Schwinger-Keldysh Closed-Time-Path (CTP) formalism for understanding a variety of different physical systems, which evolve from either a pure or a mixed state to a non-vacuum state at some later time. In fact, the CTP formalism is generically required for all types of QFT computations with the exception of transition amplitudes between two asymptotic vacua. We will devote this section to introducing the formalism in more detail, of which the first part will follow the presentation of reference [30].

To simplify the discussion, let us consider a simple many-body system of bosons described by the free Hamiltonian,

$$H(a^\dagger, a) = \omega\, a^\dagger a, \tag{3.1}$$

where $a^\dagger$ and a are the creation and annihilation operators satisfying $[a, a^\dagger] = 1$. We can further introduce the concept of *coherent states* $|\phi\rangle$, which are defined by

$$a|\phi\rangle = \phi|\phi\rangle, \qquad\qquad \langle\phi|a^\dagger = \phi^*\langle\phi|. \tag{3.2}$$

This over-complete set of states, with $\langle\phi|\phi'\rangle = \exp\{\phi^*\phi'\}$, provides a convenient resolution of the unity operator,

$$1 = \int \mathcal{D}\phi^*\mathcal{D}\phi\, e^{-|\phi|^2}|\phi\rangle\langle\phi|, \tag{3.3}$$

similarly to the usual $1 = \sum_{n=0}^{\infty}|n\rangle\langle n|$. We can use Eq. (3.3) to write

$$\mathrm{Tr}\,[\mathcal{O}] \equiv \sum_{n=0}^{\infty}\langle n|\mathcal{O}|n\rangle = \int \mathcal{D}\phi^*\mathcal{D}\phi\, e^{-|\phi|^2}\langle\phi|\mathcal{O}|\phi\rangle. \tag{3.4}$$

Let us now consider evaluating the partition function,

$$Z = \frac{\mathrm{Tr}\,[U_{\mathcal{C}}\rho]}{\mathrm{Tr}[\rho]}, \tag{3.5}$$

where $U_{\mathcal{C}}$ is the operator, which takes the state around the entire *discretised* time contour, introduced in 2.1.4. We will evolve the state $|\phi_1\rangle$ from $t_1 = -\infty$ to $|\phi_N\rangle$ at $t_N = +\infty$, where the state $|\phi_N\rangle$ will be identified with $|\phi_{N+1}\rangle = |\phi_N\rangle$ at $t_{N+1} = t_N$ and taken backwards in time to $|\phi_{2N}\rangle$ at $t_{2N} = t_1 = -\infty$. We will assume the physics to be the same on both time axes, which trivially implies $Z = 1$, and evolve the equilibrium bosonic density matrix, $\rho_0 = \left[1 - e^{-\beta(\omega-\mu)}\right]^{-1}$. The expectation value $\langle\phi_{2N}|\rho_0|\phi_{2N}\rangle$ takes the form

$$\langle\phi_{2N}|U_{-\delta t}|\phi_{2N-1}\rangle \ldots \langle\phi_{N+1}|1|\phi_N\rangle\langle\phi_N|U_{+\delta t}|\phi_{N-1}\rangle \ldots \langle\phi_1|\rho_0|\phi_{2N}\rangle, \tag{3.6}$$

which has to be integrated over each discrete time site with the weight $\exp\{-|\phi_i|^2\}$. For infinitesimal time steps $\pm\delta t$, we can show that $U_{\pm\delta t} = \exp\{\mp i H(a^\dagger, a)\delta t\}$ gives

$$\langle \phi_i | U_{\pm\delta t} | \phi_{j-1} \rangle \approx e^{\phi_i^* \phi_{i-1}} e^{\mp i\omega \phi_i^* \phi_{i-1} \delta t}. \tag{3.7}$$

Furthermore, $\langle \phi_1 | \rho_0 | \phi_{2N} \rangle = \exp\left\{\phi_1^* \phi_{2N} \cdot \exp\{-\beta(\omega-\mu)a^\dagger a\}\right\}$, which allows us to write

$$Z = \frac{1}{\mathrm{Tr}[\rho_0]} \prod_{k=1}^{2N} \int \mathcal{D}\phi_k^* \mathcal{D}\phi_k \, \exp\left\{ i \sum_{i,j}^{2N} \phi_i^* G_{ij}^{-1} \phi_j \right\}, \tag{3.8}$$

where iG_{ij}^{-1} is a $2N\times 2N$ CTP matrix. Its non-zero components are $iG_{ii}^{-1} = -1$, $iG_{1,2N}^{-1} = \rho_0$, $iG_{21}^{-1} = iG_{32}^{-1} = \ldots = iG_{N,N-1}^{-1} = h_-$, $iG_{N+1,N}^{-1} = 1$ and $iG_{N+2,N+1}^{-1} = iG_{N+3,N+2}^{-1} = \ldots = iG_{2N,2N-1}^{-1} = h_+$, where $h_\pm = 1 \pm i\omega\delta t$. Because the inverse propagator G^{-1} includes off-diagonal entries, the propagator $\langle \phi_i \phi_i^* \rangle$ has a matrix CTP structure as well.

We are interested in the continuum limit of the CTP partition function (3.8) with an arbitrary initial density matrix, $\rho_i(t_i)$, in four space-time dimensions. The integrand inside the path integral for Z always takes the form of an exponentiated action, $\exp\{iS\}$, as in Eq. (3.8). The main feature of this expression are the doubled *microscopic* physical degrees of freedom, $\varphi(t, \boldsymbol{x})$. It is therefore convenient to introduce the doubling notation,

$$\varphi \to \hat{\varphi} = (\varphi^+, \varphi^-), \tag{3.9}$$

where φ^+ is thought of as propagating on the positive time axis, from initial time t_i to the final t_f, and φ^- propagating on the backwards time axis. Note that the length of the axes may be finite depending on the details of the problem. The matching of the two axes demands that we set $\varphi^+(t_f) = \varphi^-(t_f)$. The CTP action can now be written in terms of a single time integral,

$$S_{CTP}[\hat{\phi}] = S[\varphi^+] - S^*[\varphi^-] = \int_{t_i}^{t_f} dt \int d^3x \left[\mathcal{L}(\varphi^+) - \mathcal{L}^*(\varphi^-)\right], \tag{3.10}$$

where the Lagrangian could be complex. There always exists the *CTP symmetry*,

$$S_{CTP}[\varphi^+, \varphi^-] = -S_{CTP}^*[\varphi^-, \varphi^+], \tag{3.11}$$

which plays an important role in restricting the structure of the Green's functions and effective actions.

By introducing two sources, $j^\pm(x)$, we can generate n-point correlation functions and facilitate the perturbation expansion. To develop the necessary tools for the evaluation of expectation values at the final time t_f, given some initial ρ_i, it is useful to write the generator functional in the Heisenberg representation,

$$e^{iW[j^+,j^-]} = \mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int dx[H(x)-j^+(x)\varphi(x)]}\right\}\rho_i\,\mathcal{T}^*\left\{e^{i\int dx[H(x)+j^-(x)\varphi(x)]}\right\}\right], \tag{3.12}$$

where $\mathcal{T}$ and $\mathcal{T}^*$ denotes the time ordering and anti-time ordering, respectively. The sources $j^\pm(x)$ then generate observables through functional differentiation $\delta/\delta j^\pm(x)$. At the end of the calculation, both sources are set to the same physical value, $j^+(x) = -j^-(x) = j(x)$. As in standard QFT, however, it is most convenient to write down the path integral representation of Eq. (3.12), which gives the generator functional

$$e^{iW[j^+,j^-]} = \int\limits_{\varphi^+(t_f,\boldsymbol{x})=\varphi^-(t_f,\boldsymbol{x})} \mathcal{D}\varphi^+\mathcal{D}\varphi^-\rho_i\left[\varphi^+(t_i,\boldsymbol{x}),\varphi^-(t_i,\boldsymbol{x})\right] e^{i\left\{S[\varphi^+]+\int j^+\varphi^+-S^*[\varphi^-]+\int j^-\varphi^-\right\}}. \tag{3.13}$$

On the level of the whole system, unitarity of the time evolution is expressed by the preservation of total probability. The trace of the density matrix, (3.12), calculated for a physical source, $j^+ = -j^- = j$, gives $\exp\{iW\} = 1$, as $W[j,-j] = 0$. This is completely equivalent to finding that $Z = 1$ in the case of free bosons, which we discussed above.

We should note that the continuum notation of Eq. (3.10) is misleading as it would seem to imply that φ^+ and φ^- are uncorrelated. This is not true, as we saw in the discrete CTP analysis above. From a continuum point of view, there exists a zero mode that is sensitive to the boundaries of the two time axes.

One of the powers of the CTP formalism is that it allows us to set up perturbation expansion for retarded Green's functions. They are completely encoded by the full continuum CTP propagator,

$$iD^{\sigma\sigma'}(x,x') = \mathrm{Tr}\left[\bar{\mathcal{T}}\left\{\varphi^\sigma(x)\varphi^{\sigma'}(x')\right\}\rho_i\right], \tag{3.14}$$

where σ and σ' indices can be either $+$ or $-$. The generalised time ordering, $\bar{\mathcal{T}}$, corresponds to $\mathcal{T}$ on the positive time axis and $\mathcal{T}^*$ on the negative time axes. In vacuum, D^{++} is the Feynman propagator,

$$\mathrm{Tr}\left[\mathcal{T}\left\{\varphi^+(x)\varphi^+(x')\right\}|0\rangle\langle 0|\right] = \langle 0|\mathcal{T}\{\varphi(x)\varphi(x')\}|0\rangle. \tag{3.15}$$

The action of $\bar{\mathcal{T}}$ is trivial if the two operators belong to different time axes,

$$\mathrm{Tr}\left[\bar{\mathcal{T}}\left\{\varphi^-(x)\varphi^+(x')\right\}|0\rangle\langle 0|\right] \;=\; \mathrm{Tr}\left[\varphi(x)\varphi(y)|0\rangle\langle 0|\right] = \langle 0|\varphi(x)\varphi(y)|0\rangle, \tag{3.16}$$

hence the off-diagonal components of the propagator give the Wightman function without time ordering. The other components of the CTP propagator can be found by complex conjugation, leading to the block matrix form,

$$i\hat{D}(x,y) = \begin{pmatrix} \langle\mathcal{T}\left[\varphi(x)\varphi(y)\right]\rangle & \langle\varphi(y)\varphi(x)\rangle \\ \langle\varphi(x)\varphi(y)\rangle & \langle\mathcal{T}\left[\varphi(y)\varphi(x)\right]\rangle^* \end{pmatrix}. \tag{3.17}$$

The CTP propagators for free bosons in momentum space is given by

$$\begin{aligned}\hat{D}(k) = &\begin{pmatrix} \frac{1}{k^2-m^2+i\epsilon} & -2\pi i\,\delta(k^2-m^2)\Theta(-k^0) \\ -2\pi i\,\delta(k^2-m^2)\Theta(k^0) & -\frac{1}{k^2-m^2-i\epsilon}\end{pmatrix} \\ &- i2\pi\delta\left(k^2-m^2\right)n_B(k)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix},\end{aligned} \tag{3.18}$$

where n_B is the Bose-Einstein distribution function,

$$n_B(k) = \frac{\Theta(-k^0)}{e^{\beta(\epsilon_{\boldsymbol{k}}+\mu)} - 1} + \frac{\Theta(k^0)}{e^{\beta(\epsilon_{\boldsymbol{k}}-\mu)} - 1}. \tag{3.19}$$

The inversion of the propagator yields for $\hat{\Delta} = \hat{D}^{-1}$ the structure

$$\Delta^n(k) = k^2 - m^2, \quad \Delta^i(k) = \epsilon, \quad \Delta^f(k) = i\mathrm{sign}(k^0)\epsilon. \tag{3.20}$$

The free fermionic propagator, defined by the generator functional

$$e^{\frac{i}{\hbar}W[\hat{j},\hat{\bar{j}}]} = \int D[\hat{\psi}]D[\hat{\bar{\psi}}]e^{\frac{i}{\hbar}\hat{\bar{\psi}}\hat{G}^{-1}\hat{\psi}+\frac{i}{\hbar}\hat{\bar{j}}\hat{\psi}+\frac{i}{\hbar}\hat{\bar{\psi}}\hat{j}}, \tag{3.21}$$

can be written as

$$\hat{G}^{\alpha\beta}(x,y) = \begin{pmatrix} \langle 0|T[\psi^\alpha(x)\bar{\psi}^\beta(y)]|0\rangle & -\langle 0|\bar{\psi}^\beta(y)\psi^\alpha(x)|0\rangle \\ \langle 0|\psi^\alpha(x)\bar{\psi}^\beta(y)|0\rangle & \langle 0|T[(\gamma^0\psi(y))^\beta(\bar{\psi}(x)\gamma^0)^\alpha]|0\rangle^* \end{pmatrix}, \tag{3.22}$$

which gives the momentum space expression $\hat{G}(k) = (\not{k} + m)\,\hat{D}_k$, written here in terms of the scalar propagator $\hat{D}_k$. In case of finite temperature and density,

$$\hat{G}(k) = (\not{k} + m)\left[\hat{D}(k) + 2\pi i\,\delta(k^2 - m^2)n_F(k)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right]. \tag{3.23}$$

where n_F is the Fermi-Dirac distribution,

$$n_F(k) = \frac{\Theta(k^0)}{e^{\beta(\epsilon_{\boldsymbol{k}}-\mu)} + 1} + \frac{\Theta(-k^0)}{e^{\beta(\epsilon_{\boldsymbol{k}}+\mu)} + 1}. \tag{3.24}$$

The CTP identity,

$$\mathcal{T}\left\{A(t_A)B(t_B)\right\} + \mathcal{T}^*\left\{A(t_A)B(t_B)\right\} = A(t_A)B(t_B) + B(t_B)A(t_A), \tag{3.25}$$

valid for bosonic operators, restricts the propagator to the standard CTP form,

$$\hat{D} = \begin{pmatrix} D^n + iD^i & -D^f + iD^i \\ D^f + iD^i & -D^n + iD^i \end{pmatrix}, \tag{3.26}$$

where the functions D^n, D^f and D^i appearing in the matrix elements are real. The exchange symmetry $(\sigma, x) \leftrightarrow (\sigma', x')$ imposes $D^n(x,y) = D^n(y,x)$, $D^f(x,y) = -D^f(y,x)$, and $D^i(x,y) = D^i(y,x)$ in the bosonic case.

The Fourier transform of the Wightman function,

$$iD^{-+}(p) = \Theta(p^0)S(p), \tag{3.27}$$

is the spectral function of excitations, which are generated by $\varphi(p)$ in a translationaly invariant system with $S(p) \geq 0$. The relation allows us to express both D^f and D^i in terms of the spectral function, which leads to the *spectral condition*, $D^f(p) = \text{sign}(p^0) i D^i(p)$, and the CTP propagator can thus be specified by only two real functions,

$$\hat{D}(p) = \begin{pmatrix} D^n(p) + \text{sign}(p^0) D^f(p) & -2\Theta(-p^0) D^f(p) \\ 2\Theta(p^0) D^f(p) & -D^n(p) + \text{sign}(p^0) D^f(p) \end{pmatrix}, \tag{3.28}$$

where the positive definiteness of the norm imposes the bound, $i\Theta(p^0)D^f(p) > 0$.

The inverse of the propagator (3.26) is given by

$$\hat{D}^{-1} = \hat{\sigma} \begin{pmatrix} \Delta^n + i\Delta^i & -\Delta^f + i\Delta^i \\ \Delta^f + i\Delta^i & -\Delta^n + i\Delta^i \end{pmatrix} \hat{\sigma}, \tag{3.29}$$

where $\hat{\sigma}$ is a diagonal "metric tensor" of the form $\hat{\sigma} = \text{diag}\,(1, -1)$. Furthermore,

$$\Delta^{r,a} = 1/D^{r,a}, \qquad \Delta^i = -\Delta^r D^i \Delta^a, \tag{3.30}$$

where $\Delta^n(x,y) = \Delta^n(y,x)$, $\Delta^f(x,y) = -\Delta^f(y,x)$, $\Delta^i(x,y) = \Delta^i(y,x)$, $\Delta^r = \Delta^n + \Delta^f$ and $\Delta^a = \Delta^n - \Delta^f$. We also note that the spectral condition yields $\Delta^f(p) = \text{sign}(p^0) i \Delta^i(p)$.

Even though the preceding discussion applies to interacting fields, it is instructive to consider free fields in a harmonic model. The action is given by

$$S_{\text{harm}}[\hat{\phi}] = \frac{1}{2}(\varphi^+, \varphi^-) \begin{pmatrix} \Delta^n + i\Delta^i & \Delta^f - i\Delta^i \\ -\Delta^f - i\Delta^i & -\Delta^n + i\Delta^i \end{pmatrix} \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}. \tag{3.31}$$

The external source generates a non-trivial expectation value, which can be obtained from either time axes,

$$\langle \varphi(x) \rangle \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\int dy\, \hat{D}(x,y) \begin{pmatrix} j(y) \\ -j(y) \end{pmatrix}, \tag{3.32}$$

showing that $D^r = D^n + D^f$ and $D^a = D^n - D^f$ are the retarded and advanced Green's function, respectively. Since these Green's functions are real in position space and complex in momentum space, $D^n(p) = \Re D^r(p)$ and $D^f(p) = i\Im D^r(p)$.

In terms of the generator functional, the expectation value of the field φ can be expressed as

$$\langle \varphi(x) \rangle = \sum_\sigma \sigma \int dy \frac{\delta^2 W[\hat{j}]}{\delta j^\pm(x) \delta j^\sigma(y)} j(y), \tag{3.33}$$

which follows from Eq. (3.32). The quadratic approximation of the generator functional (3.12) then reproduces the linear response formalism [105], including Kubo formulae, because

$$D^r(x,y) = \sum_\sigma \sigma \frac{\delta^2 W[\hat{j}]}{\delta j^\pm(x) \delta j^\sigma(y)}, \tag{3.34}$$

is the retarded Green's function.

Hydrodynamics addresses the inverse problem. There, we are interested in the equations (of motion), satisfied by the expectation values, where the external sources appear linearly. It is easy to find the equation in question for the linear response,

$$j(x) = \int dy\, (D^r)^{-1}(x,y) \langle \varphi(y) \rangle. \tag{3.35}$$

The only subtlety is the necessity to exclude the null-space from the domain of the inverse Green's function, when necessary.

The generalisation of such an inverse linear response formula is provided by the functional Legendre transform of $W[\hat{j}]$, the effective action,

$$\Gamma[\hat{\phi}] = W[\hat{j}] - \hat{j}\hat{\varphi}, \tag{3.36}$$

where

$$\hat{\varphi} = \frac{\delta W[\hat{j}]}{\delta \hat{j}}. \tag{3.37}$$

In fact, the inverse Legendre transform is given by Eq. (3.36) and

$$\hat{j} = -\frac{\delta \Gamma[\hat{\varphi}]}{\delta \hat{\varphi}}. \tag{3.38}$$

This expression plays the role of the equation of motion and produces a non-linear extension of the hydrodynamical equations. The inverse Legendre transform generates the non-linearity necessary to close the equation without auxiliary variables, such as thermodynamical functions.

3.3 CTP Wilsonian effective action

In this section, we will analyse the characteristic behaviour of a Wilsonian effective action in the framework of the CTP formalism. Let us consider a microscopic real scalar field φ, with the single time axis action $S_s[\varphi]$. The doubling of the degrees of freedom leads to the quantum generator functional,

$$Z_{CTP} = \int \mathcal{D}\hat{\varphi} \exp\left\{ iS_s[\varphi^+] - iS_s[\varphi^-] + i\int \hat{J}\hat{\varphi} \right\}. \tag{3.39}$$

The full CTP action of $\hat{\varphi}$, $S_{CTP}[\varphi^+, \varphi^-] = S_s[\varphi^+] - S_s^*[\varphi^-]$, possesses the CTP symmetry (3.11). The generator functional with two sources, $\hat{J} = (J^+, J^-)$, leads to the free 2×2 matrix propagator $\hat{D}$, discussed in detail in Section 3.2. $\hat{D}$ contains the Feynman propagator as the diagonal block, D^{++}, and the off-diagonal Wightman function,

$$\langle \varphi(y)\varphi(x) \rangle = -2\pi i \delta(k^2 - m^2)\Theta(-k^0). \tag{3.40}$$

The off-diagonal pieces of $\hat{D}$ induce interactions between φ^+ and φ^-. Finite temperature, $T = 1/\beta$, and density with a chemical potential μ, in cases when φ is complex, further modify the free propagator to give the full expression of Eq. (3.18).

Let us now consider a scalar $\lambda\varphi^4$ theory in which we follow the Wilsonian approach to effective field theory and integrate out UV-degrees of freedom. We introduce a scheme with two cut-offs in the original bare theory, one for frequency, $|k_0| < \Lambda_0$, and one for momentum, $\sqrt{\boldsymbol{k}^2 + m^2} = \varepsilon_{\boldsymbol{k}} < \Lambda_\varepsilon$. We then split the fields, $\hat{\varphi} = \hat{\varphi}_< + \hat{\varphi}_>$, and integrate out $\hat{\varphi}_>$, with frequency and energy intervals, given by

$$\xi\Lambda_0 \leq |k_0| < \Lambda_0, \qquad\qquad \zeta\Lambda_\varepsilon \leq \varepsilon_{\boldsymbol{k}} < \Lambda_\varepsilon. \tag{3.41}$$

The UV-mode integrals must therefore run over three regions,

$$I_1: \quad \{\xi\Lambda_0 \leq k_0 < \Lambda_0,\, 0 \leq \varepsilon_{\boldsymbol{k}} < \Lambda_\varepsilon\}\,, \tag{3.42}$$

$$I_2: \quad \{-\Lambda_0 < k_0 \leq -\xi\Lambda_0,\, 0 \leq \varepsilon_{\boldsymbol{k}} < \Lambda_\varepsilon\}\,, \tag{3.43}$$

$$I_3: \quad \{-\Lambda_0 < k_0 < \Lambda_0,\, \zeta\Lambda_\varepsilon \leq \varepsilon_{\boldsymbol{k}} < \Lambda_\varepsilon\}\,. \tag{3.44}$$

In a perturbative expansion of (3.39), we find various couplings between the two axes, for example $\lambda^2 \left(\varphi^+_<\right)^2 \left(\varphi^+_>\right)^2 \left(\varphi^-_<\right)^2 \left(\varphi^-_>\right)^2$. In the process of integrating out $\varphi^+_>$ and $\varphi^-_>$, the on-shell Wightman functions can connect vertices on different time axes, and give rise to non-trivial $\varphi^+_<\varphi^-_<$ couplings in the effective theory, $S_{eff}[\hat{\phi}_<]$. We find that the effective action includes the following type of terms,

$$S_{eff}[\hat{\phi}_<] = S_{CTP}[\phi^+_<, \phi^-_<] + \int d^4x \left[\mu_1\, \varphi^{+2}_< \varphi^{-2}_< + \mu_2\, \varphi^{+3}_< \varphi^-_> - \mu_2^*\, \varphi^{-3}_< \varphi^+_>\right]. \tag{3.45}$$

Due to the CTP symmetry, μ_1 has to be purely imaginary, whereas μ_2 will be complex. The equations of motion for $\hat{\phi}$ derived from a CTP effective action will thus also in general be complex. The real part of the equations of motion, coming from $\Re S_{eff}$, has the property that $\phi^+ = \phi^-$ is the solution, which is always true in real CTP actions. The imaginary terms from $\Im S_{eff}$ will be complex conjugates of each other in the equations for ϕ^+ and ϕ^-. We should note that the same structure as in the Wilsonian effective action arises in a 1PI effective action, which we introduced in Section 2.1.2. In both effective actions the real part of the action is important for physical Hermitian expectation values, whereas the imaginary part controls decoherence.

Beyond this proof of principle, which shows that coupling between ϕ^+ and ϕ^- generically arise in effective actions, we will discus the significance of such effective coupling in the following section. Furthermore, note that this type of effective theory, which is constructed with the full CTP machinery, is able to account for the time-evolution of any pure or mixed state in a closed or open field theory system. The details of the system we are describing are determined by the degrees of freedom that were integrated out, i.e the *environment*. The remaining *reduced density matrix* of the sub-*system* encodes all of the information about the entanglement with the environment and dissipation of energy from the sub-system. The sub-system can thus either preserve or break various symmetries of the full closed system. This fact will play an important role in our construction of dissipative hydrodynamics.

3.4 CTP formalism in classical field theory

3.4.1 Closed system

To see how all of the features presented in section (3.3) apply to classical effective theory, let us consider a classical field theory for an isolated system, described by the field $\psi(x)$, which is invariant under time inversion. Instead of deriving the effective theory from microscopic dynamics, we can directly use the CTP formalism in classical physics [106, 107]. This is necessary when considering any physical problem in which we wish to specify initial conditions for the equations of motion and to have the possibility of introducing effective interactions with dissipative forces into the Lagrangian formalism. From the microscopic point of view, the theory can be understood as an effective field theory; a special case of those considered in Section 3.3, which keep the IR dynamics closed. All of the considerations below would follow directly form such a derivation.

The procedure again begins by doubling the degrees of freedom [108],

$$\psi \to \hat{\psi} = (\psi^+, \psi^-), \tag{3.46}$$

in a way that both members of the CTP doublet satisfy the same equation of motion, initial conditions and the relation $\psi^+(t_f, \boldsymbol{x}) = \psi^-(t_f, \boldsymbol{x})$ at the final time. The action describing the dynamics of $\hat{\psi}$ is defined as in Eq. (3.39),

$$S_{CTP}[\hat{\psi}] = \int_{t_i}^{t_f} d^{d+1}x \left\{ \mathcal{L}_s \left[\psi^+\right] - \mathcal{L}_s^* \left[\psi^-\right] \right\}, \tag{3.47}$$

where $\mathcal{L}_s[\psi] = \mathcal{L}[\psi, \partial\psi] + i\epsilon\psi^2$ now differs from the original Lagrangian in that it splits the degeneracy of the CTP action for $\psi^+(x) = \psi^-(x)$. The action (3.47) possesses the CTP symmetry (3.11), related to the exchange of the two time axes, $\psi^+ \leftrightarrow \psi^-$,

$$S_{CTP}[\psi^+, \psi^-] = -S^*_{CTP}[\psi^-, \psi^+], \tag{3.48}$$

which must be obeyed by any classical CTP action.

3.4.2 Open systems

In order to describe an *open system* of IR hydrodynamical degrees of freedom in the language of classical field theory, we first need to consider a question of how to construct a general classical field theory of a subset ϕ of the degrees of freedom ψ. The effective dynamics of ϕ can be obtained by eliminating the environment degrees of freedom by using their equations of motion. Similarly, from the point of view of QFT presented in 3.3, the environment could be seen as the degrees of freedom that are integrated out. This view is consistent with what dissipation in hydrodynamics really means; it is the energy loss of the fluid's IR degrees of freedom to the UV degrees of freedom of the environment. Only the total closed system, combining all degrees of freedom, conserves energy.

In classical CTP theory, as in Section 3.3, the effective action again has a more involved structure than (3.47), namely

$$S_{eff}[\hat{\phi}] = S_1[\phi^+] - S_1^*[\phi^-] + S_2[\hat{\phi}], \tag{3.49}$$

where the indices 1 and 2 reflect the number of time axes entering the term in the action. S_1 and S_2 can be uniquely distinguished by imposing

$$\frac{\delta^2 S_2}{\delta\phi^+\delta\phi^-} \neq 0. \tag{3.50}$$

Elimination of the environment generates contributions both to S_1 and S_2. We would like to point out that in the original terminology of Feynman and Vernon [104], *all* effective contributions to S_{eff} were collected into the influence functional S_i,

$$S_{eff} = S_0[\phi^+] - S_0^*[\phi^-] + S_i[\hat{\phi}]. \tag{3.51}$$

In Eq. (3.51), S_0 stands for the original single time-axis action preceding the elimination of the environment. We find it is more convenient to separate the influence functional into terms entering S_1 and S_2. In this language, S_0 will be included in S_1. This separation is useful because the terms in S_1 preserve energy and momentum, while terms in S_2 represent dissipative forces. The inclusion of S_2 into the classical action for hydrodynamics, discussed in Section 3.6, will thus be our addition to the previous works on deriving hydrodynamics from an action principle [3, 4, 101].

In the classical picture, the couplings between ϕ^+ and ϕ^- appear due to the boundary conditions for the environment coordinates at the final time. These contributions arise from asymptotic long-time excitations of the environment and are usually approximated by gradient expansion. We will assume that the imaginary part of the effective action obtained by eliminating the environment remains small, as in the case of an isolated system. It will be ignored below.

Let us assume that the gradient expansion in terms of space-time derivatives is applicable in the effective action (3.49). We impose identical initial conditions on the two time axes, $\partial_t^n \phi^+(t_i, \boldsymbol{x}) = \partial_t^n \phi^-(t_i, \boldsymbol{x})$, together with the auxiliary conditions $\partial_t^n \phi^+(t_f, \boldsymbol{x}) = \partial_t^n \phi^-(t_f, \boldsymbol{x})$, for all orders of derivatives labeled by $n \geq 0$.

Variational equations can thus still be derived in the CTP theory because the boundary contributions arising from partial integration cancel, due to the above conditions. Furthermore, the solutions of the open system's Euler-Lagrange equations of motion give

$$\phi^+(x) = \phi^-(x). \tag{3.52}$$

The classical effective action must again obey the CTP symmetry,

$$S_{eff}[\phi^+, \phi^-] = -S^*_{eff}[\phi^-, \phi^+]. \tag{3.53}$$

From the point of view of effective field theory, relation (3.53) can be seen as a constraint on the form of terms one can write down in the effective action.

As an example of this formalism, it is instructive to consider a non-relativistic one-dimensional particle whose effective theory is defined by the Lagrangians

$$\mathcal{L}_1 = \frac{1}{2}\left(m\dot{x}^2 - m\omega^2 x^2\right), \tag{3.54}$$

$$\mathcal{L}_2 = \frac{\gamma}{2}\left(x^-\dot{x}^+ - x^+\dot{x}^-\right). \tag{3.55}$$

The corresponding equations of motion describe a damped harmonic oscillator,

$$m\ddot{x}^\pm + \gamma\dot{x}^\mp + m\omega^2 x^\pm = 0 \qquad \Longrightarrow \qquad x^+ = x^-. \tag{3.56}$$

The conservation of energy is obviously violated by $\mathcal{L}_i$.

In CTP, the naïve application of the Noether theorem to the action (3.49) gives, due to the CTP symmetry, an identically vanishing stress-energy tensor for fields that satisfy the equations of motion. However, the trivial cancellation between the time axes can be avoided and the energy-momentum balance equation can be derived by varying only one of the CTP doublet fields,

$$\begin{aligned} \phi^+(x) &\to \phi^+(x + a(x)), \\ \phi^-(x) &\to \phi^-(x). \end{aligned} \tag{3.57}$$

The equation of motion for $a(x)$, *the balance equation*, can then be written in the form of a tensor divergence as

$$\partial_\mu T^{\mu\nu} = R^\nu. \tag{3.58}$$

Note that the dynamics of ϕ^+ and dynamics of the ϕ^- degrees of freedom on the two time axes are related to each other by the CTP symmetry, (3.53). Either time axis could thus have been used for the variation. In this work, we will always choose to treat the positive axis with ϕ^+ fields as the one directly relevant to physical observations.

3.5 Hydrodynamics as effective field theory

An effective field theory describing hydrodynamics has recently been developed in terms of a gradient expansion of Goldstone modes arising from the broken spatial boost invariance [3, 4]. Reference [109] used the coset construction of a space-time symmetry breaking pattern to show that three scalar modes were sufficient in parametrising the low energy effective theory. The dynamics of the scalar modes ϕ^I, with $I = \{1, 2, 3\}$, in flat $3 + 1$ dimensional space-time with the metric $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1)$[1], is chosen to display internal symmetries under rigid translations,

$$\phi^I \to \phi^I + \alpha^I, \quad \text{with } \alpha^I = const., \tag{3.59}$$

rotations,

$$\phi^I \to R^I_J \phi^J, \quad \text{with } R^I_J \in SO(3), \tag{3.60}$$

and volume-preserving diffeomorphisms (reparametrisations), abbreviated by $S\mathrm{Diff}\,(\mathbb{R}^{1,3})$,

$$\phi^I \to \xi^I(\phi), \quad \text{with } \det\left(\frac{\partial \xi^I}{\partial \phi^J}\right) = 1. \tag{3.61}$$

The SDiff symmetry, which is imposed here, deserves special attention. Arnold showed that non-dissipative ideal hydrodynamical equation on a manifold $\mathcal{M}$, i.e. the Euler equation, can be generated as the co-adjoint orbit on the Lie group manifold of $S\mathrm{Diff}(\mathcal{M})$ [110, 111]. This symmetry should broken by dissipation, but this mechanism has not been understood. We will proceed by making use of it and comment at the end on why this symmetry is most likely too restrictive to construct the full equations of viscous fluids.

Returning to the setup of [3, 4], we note that in equilibrium, the fields equal spatial coordinates, $\phi^I = const. \cdot x^I$. Furthermore, relativistic hydrodynamics also requires the Poincaré symmetry. The gradient expansion is constructed by counting the number of derivatives acting on the vector field,

$$K^\mu = \frac{1}{6}\epsilon^{\mu\alpha_1\alpha_2\alpha_3}\epsilon_{IJK}\partial_{\alpha_1}\phi^I\partial_{\alpha_2}\phi^J\partial_{\alpha_3}\phi^K \equiv P^{\mu\alpha}_K\partial_\alpha\phi^K, \tag{3.62}$$

which is a combination of gradients of the Goldstone modes allowed by the symmetries in three spatial dimensions. The vector field is conserved because of its anti-symmetric structure,

$$\partial_\mu K^\mu = 0, \tag{3.63}$$

and keeps the comoving coordinates constant along its direction, $K^\mu\partial_\mu\phi^I = 0$. We can introduce a scalar field b, such that

$$K^\mu \equiv b u^\mu. \tag{3.64}$$

The norm of the velocity vector, $u^\mu u_\mu = -1$, then implies that $b^2 = -K^\mu K_\mu$.

[1]This metric signature is normally used in string theory-motivated texts on hydrodynamics.

Two useful projector identities can be derived for $P_K^{\mu\alpha}$, as defined in (3.62), by using the properties of K^μ,

$$P_K^{\mu\nu}\partial^\lambda\phi^K = \frac{1}{3}\left(K^\mu\Delta^{\nu\lambda} - K^\nu\Delta^{\mu\lambda}\right), \tag{3.65}$$

$$P_K^{\mu\nu}\partial^\lambda\partial_\nu\phi^K = \frac{1}{3}\partial^\lambda K^\mu, \tag{3.66}$$

with $\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$. The zeroth and first-order Lagrangians for the uncharged fluid are then

$$\mathcal{L}^{(0)} + \mathcal{L}^{(1)} = F(b) + g(b)K^\mu K^\nu\partial_\mu K_\nu. \tag{3.67}$$

At zeroth order [4], the conserved stress-energy tensor of the closed system takes the form of an ideal fluid,

$$T_{(0)}^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu}, \tag{3.68}$$

where the energy density ϵ and pressure p are

$$\epsilon = -F, \qquad\qquad p = F - b\partial_b F. \tag{3.69}$$

Further thermodynamic analysis reveals that the temperature is given by

$$T = -\partial_b F. \tag{3.70}$$

Finally, vector field K^μ can be interpreted at this order as the conserved entropy current

$$K^\mu = bu^\mu \equiv S^\mu = su^\mu, \tag{3.71}$$

with

$$s = b, \tag{3.72}$$

playing the role of the entropy density. This identification was performed in [4] because K^μ is parallel to u^μ and is by construction conserved, which is consistent with the entropy conservation in an ideal fluid, i.e. in zeroth-order hydrodynamics. In reference [101], the authors considered non-dissipative second-order hydrodynamics using the same identification of the entropy current, noting that the construction should be understood as being done in the *entropy frame*, in which $S^\mu = su^\mu$ to all orders. In standard phenomenological hydrodynamics, one instead of the entropy frame usually chooses either the Landau frame or the Eckart frame [25]. As discussed in Chapter 2.1.3, the physical meaning of the Landau frame is that there is no energy flow in the local rest frame of the fluid. The Eckart frame, useful for a description of charged fluids, means that there is no charge flow in the local rest frame.

The first-order contribution to the Lagrangian (3.67) can be rewritten as a total derivative and hence does not contribute to $T^{\mu\nu}$. As a final point in this construction, note that the chemical potential is vanishing in the absence of a conserved $U(1)$ Noether current [4], which we will not consider in this work.

3.6 Hydrodynamics with dissipation

3.6.1 The setup

Variational methods in the usual effective theory formalism cannot describe dissipation. However, this limitation can be avoided by using the CTP scheme as introduced above. Firstly, the degrees of freedom are doubled, giving us six Goldstone fields $\phi^{\pm I}$. The action must be invariant under pairs of translations, rotations and volume-preserving diffeomorphisms, each acting independently on ϕ^{I+} and ϕ^{I-}. The diffeomorphisms act as

$$\phi^{\pm I} \to \xi^{\pm I}\left(\phi^{\pm}\right), \tag{3.73}$$

with conditions on the determinants

$$\det\left(\frac{\partial \xi^{+I}}{\partial \phi^{+J}}\right) = 1, \qquad \det\left(\frac{\partial \xi^{-I}}{\partial \phi^{-J}}\right) = 1. \tag{3.74}$$

The field content and symmetries allow for two independent currents $K^{i\mu}$, both with the same Lorentz structure as before, where $\{i, j, k, ...\} \in \{0, 3\}$ correspond to the number of ϕ^+ fields inside $K^{i\mu}$. We write

$$K^{i\mu} = \frac{1}{6}\epsilon^{\mu\alpha_1\alpha_2\alpha_3}\epsilon_{IJK}\partial_{\alpha_1}\phi^{\sigma_1 I}\partial_{\alpha_2}\phi^{\sigma_2 J}\partial_{\alpha_3}\phi^{\sigma_3 K}, \tag{3.75}$$

with $(\sigma_1\sigma_2\sigma_3) = \{(---), (+++)\}$ for $i = \{0, 3\}$. Both $K^{i\mu}$ are still conserved,

$$\partial_\mu K^{i\mu} = 0, \tag{3.76}$$

and both $K^{i\mu} = K^\mu$ after $\phi^{+K} = \phi^{-K}$ is imposed. It is useful to define, as in Eq. (3.62),

$$K^{3\mu} \equiv P_K^{3\mu\alpha}\partial_\alpha\phi^{+K}. \tag{3.77}$$

Furthermore, we can introduce

$$P_K^{0\mu\alpha} \equiv 0, \tag{3.78}$$

which will make it clear that the transformation δ^+ acting on $K^{0\mu}$ gives a vanishing contribution.

We can now write down the CTP action for the first two orders in the gradient expansion of $K^{i\mu}$,

$$\mathcal{L}^{(0)}_{CTP} = F(K^3_\gamma K^{3\gamma}) - F(K^0_\gamma K^{0\gamma}) + G(K^i_\gamma K^{j\gamma}), \tag{3.79}$$

$$\mathcal{L}^{(1)}_{CTP} = \sum_{i,j,k} f_{ijk}(K^l_\gamma K^{m\gamma}) K^{i\mu} K^{j\nu} \partial_\mu K^k_\nu. \tag{3.80}$$

Small latin indices are always summed over $\{0, 3\}$. The single axis contributions, i.e. S_1, to $\mathcal{L}^{(0)}$ remain the same as in (3.67) and the zeroth-order action S_2, which includes couplings between the two time axes, is parametrised by G. It mixes $K^{i\mu}$'s with different CTP indices. We include no single axis action at first order, as it would be a total derivative [4], so $\mathcal{L}^{(1)}$ is purely a part of S_2, as classified by Eq. (3.49). This means that f_{333} cannot be a function of only $K^{3\mu}$ and f_{000} not of only $K^{0\mu}$. The real coefficient functions F, G and f_{ijk} can depend on any Lorentz-contracted combination of $K^{i\mu}$, but may include no derivatives. At first order, we thus have $2^3 = 8$ coefficient functions f_{ijk}, which are reduced to 4 independent functions by the CTP symmetry (3.53).

3.6.2 Energy-momentum balance equation

The variation of the current $K^{i\mu}$ with respect to ϕ^+ results in an expression that is weighted by the number of ϕ^+ fields inside of $K^{i\mu}$,

$$\delta^+_\phi K^{i\mu} = i P^{i\mu\alpha}_K \partial_\alpha \delta\phi^{+K}. \tag{3.81}$$

The zeroth-order Euler-Lagrange equations of motion are

$$\partial_\lambda \sum_{i\leq j} \frac{\partial\,(F+G)}{\partial(K^i_\alpha K^{j\alpha})} \left(i P^{i\mu\lambda}_K K^j_\mu + j K^i_\mu P^{j\mu\lambda}_K \right) = 0. \tag{3.82}$$

To find the energy-momentum balance equation for the open system, we vary the space-time dependence of ϕ^+ by $x \to x + a(x)$. This results in $\delta^+_x \phi^{+K} = a^\mu \partial_\mu \phi^{+K}$, while leaving $\delta^+_x \phi^{-K} = 0$. By using the definitions of $K^{i\mu}$ as stated in Eqs. (3.75), (3.77) and (3.78), it follows that

$$\delta^+_x K^{i\mu} = i P^{i\mu\alpha}_K \left(\partial_\alpha a_\lambda \partial^\lambda \phi^{+K} + a_\lambda \partial^\lambda \partial_\alpha \phi^{+K} \right). \tag{3.83}$$

After we identify $\phi^{+K} = \phi^{-K}$, which is implied by the equations of motion, and use projector identities (3.65) and (3.66), the form of the left-hand-side of (3.58) remains that of $T^{\mu\nu}_{(0)}$ in (3.68). The energy density and pressure are now

$$\epsilon = -F, \tag{3.84}$$

$$p = F - b\partial_b F + \frac{b^2}{3} \sum_{i\leq j} \bar{G}'_{ij}\,(i+j)\,, \tag{3.85}$$

and the non-conserved part of the balance equation is

$$R^{\nu}_{(0)} = \sum_{i\leq j} \bar{G}'_{ij}\,(i+j)\,b\partial^{\nu}b/3. \tag{3.86}$$

Throughout this work, we define the barred functions as being evaluated on the equations of motion $\phi^{+K} = \phi^{-K}$,

$$\bar{G}'_{ij} \equiv G'_{ij}\big|_{\phi^{+K}=\phi^{-K}}. \tag{3.87}$$

Furthermore, we have defined the derivatives of G_{ij} by

$$G'_{ij} \equiv \frac{\partial G}{\partial\left(K^{l}_{\delta}K^{m\delta}\right)}. \tag{3.88}$$

The first-order equations of motion for $S^{(1)}_{CTP}$ are

$$\begin{aligned}
&\partial_{\lambda}\sum_{i,j,k}\Big\{if_{ijk}P^{i\mu\lambda}_{K}K^{j\nu}\partial_{\mu}K^{k}_{\nu} + jf_{ijk}K^{i\mu}P^{j\nu\lambda}_{K}\partial_{\mu}K^{k}_{\nu} - kf_{ijk}K^{i\mu}\partial_{\mu}K^{j}_{\nu}P^{k\nu\lambda}_{K}\\
&+\sum_{l\leq m} f'_{ijk,lm}\Big[\left(lP^{l\gamma\lambda}_{K}K^{m}_{\gamma} + mK^{l}_{\gamma}P^{m\gamma\lambda}_{K}\right)K^{i\mu}K^{j\nu}\partial_{\mu}K^{k}_{\nu}\\
&-k\partial_{\mu}\left(K^{l}_{\gamma}K^{m\gamma}\right)K^{i\mu}K^{j}_{\nu}P^{k\nu\lambda}_{K}\Big]\Big\} = 0,
\end{aligned} \tag{3.89}$$

where

$$f_{ijk,lm} \equiv \frac{\partial f_{ijk}}{\partial K^{l}_{\delta}K^{m\delta}}. \tag{3.90}$$

The calculation of $T^{\mu\nu}_{(1)}$ goes through as it did for $T^{\mu\nu}_{(0)}$, resulting in a non-symmetric tensor $T^{\mu\nu}$ on the left-hand-side of (3.58),

$$T^{\mu\nu} = \epsilon u^{\mu}u^{\nu} + p\Delta^{\mu\nu} - \eta_1 u^{\mu}u^{\lambda}\partial_{\lambda}u^{\nu} + \left(\chi_1\eta^{\mu\nu} + \chi_2 u^{\mu}u^{\nu}\right)\partial_{\lambda}u^{\lambda} + \beta u^{\mu}\partial^{\nu}b, \tag{3.91}$$

where the coefficient functions are given by

$$\eta_1 = \frac{b^3}{3}\sum_{i,j,k}(j-k)\,\bar{f}_{ijk}, \tag{3.92}$$

$$\chi_1 = \chi_2 + b\beta, \tag{3.93}$$

$$\chi_2 = \frac{b^3}{3}\sum_{i,j,k}\sum_{l\leq m}\left[(j-k)\,\bar{f}_{ijk} - C_{ijk,lm}\right], \tag{3.94}$$

$$\beta = \frac{b^2}{3}\sum_{i,j,k} i\bar{f}_{ijk}, \tag{3.95}$$

with $C_{ijk,lm} \equiv b^2\bar{f}'_{ijk,lm}\,(l+m-2k)$. The contribution to the non-conserving R^{ν} from the first-order action is

$$R^{\nu}_{(1)} = \eta_1 u^{\alpha}\partial_{\alpha}u_{\lambda}\partial^{\nu}u^{\lambda} + \frac{\chi_1}{b}\partial_{\lambda}u^{\lambda}\partial^{\nu}b - \frac{\beta}{b}\partial_{\lambda}b\partial^{\nu}u^{\lambda}. \tag{3.96}$$

3.6.3 Stress-energy tensor, the Navier-Stokes equations and bulk viscosity

The remaining question is how the energy-momentum balance equation (3.58) relates to viscous phenomenological hydrodynamics that can be obtained from the symmetric stress-energy tensor

$$T^{\mu\nu}_{ph} = T^{\mu\nu}_{(0)ph} + T^{\mu\nu}_{(1)ph}. \tag{3.97}$$

The form of $T^{\mu\nu}_{(0)ph}$ equals that of $T^{\mu\nu}_{(0)}$ in Eq. (3.68) and

$$T^{\mu\nu}_{(1)ph} = -\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\partial_\lambda u^\lambda + (q^\mu u^\nu + q^\nu u^\mu)\,. \tag{3.98}$$

The tensor $\sigma^{\mu\nu}$ is the transverse traceless symmetric tensor

$$\sigma^{\mu\nu} \equiv \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\eta_{\alpha\beta}\partial_\lambda u^\lambda\right). \tag{3.99}$$

Hydrodynamics is constructed (see e.g. [25, 112]) as a gradient expansion in temperature, chemical potential and velocity fields: $T(x)$, $\mu(x)$ and $u^\mu(x)$. In our discussion, $\mu(x) = 0$. As discussed in Chapter 2.1.3, the stress-energy tensor is then written as

$$T^{\mu\nu}_{ph} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (u^\mu q^\nu + u^\nu \tilde{q}^\mu) + t^{\mu\nu}, \tag{3.100}$$

with q^μ, $\tilde{q}^\mu$ and $t^{\mu\nu}$ all being transverse. In phenomenological hydrodynamics, the stress-energy tensor is symmetric, hence $\tilde{q}^\mu = q^\mu$.

Despite the fact that the tensor $T^{\mu\nu}$ we derived in (3.91) is not conserved, we can write it in the form of (3.100). It is important to note that $T^{\mu\nu}_{(1)}$ is *not* symmetric, thus $\tilde{q}^\mu \neq q^\mu$. At this point, the fact that $T^{\mu\nu}$ is not symmetric means that we cannot interpret it as a stress-energy tensor, in the absence of the Belinfante-Rosenfeld procedure [113, 114]. However, we will see below that within an approximate scheme, a simple symmetrisation of $T^{\mu\nu}$ can lead to a hydrodynamic stress-energy tensor, which reproduces exactly the same physical equations as the ones we have derived from the energy-momentum balance equation (3.58).

The tensor structure of (3.100) allows us to identify the coefficient functions of (3.91) as

$$\mathcal{E} = u_\mu u_\nu T^{\mu\nu} = \epsilon, \tag{3.101}$$

$$\mathcal{P} = \Delta_{\mu\nu}T^{\mu\nu}/3 = p + \chi_1\partial_\lambda u^\lambda, \tag{3.102}$$

$$q_\mu = -\Delta_{\mu\beta}u_\alpha T^{\alpha\beta} = \beta\partial_\mu b - b\beta u_\mu\partial_\lambda u^\lambda - \eta_1 u^\lambda\partial_\lambda u_\mu, \tag{3.103}$$

$$\tilde{q}_\mu = -\Delta_{\mu\alpha}u_\beta T^{\alpha\beta} = 0, \tag{3.104}$$

$$t_{\mu\nu} = \frac{1}{2}\left[\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\mu\beta}\Delta_{\nu\alpha} - \frac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right]T^{\alpha\beta} = 0. \tag{3.105}$$

Phenomenological stress-energy tensor is by construction conserved. Its conservation equations,

$$\partial_\mu T^{\mu 0}_{ph} = 0, \qquad\qquad \partial_\mu T^{\mu i}_{ph} = 0, \tag{3.106}$$

respectively give the continuity equation and the Navier-Stokes equation. They can be reduced to their standard compressible form by using the non-relativistic scaling [115]: $t \to t/\epsilon_{nr}^2$, $x \to x/\epsilon_{nr}$, $v^i \to \epsilon_{nr} v^i$ and $p \to \epsilon_{nr}^2 p$,

$$\partial_0 \rho + \partial_i \left(\rho v^i\right) = 0, \tag{3.107}$$

$$\rho \left(\partial_0 + v^j \partial_j\right) v^i = -\partial^i p + \eta \partial^2 v^i + (\zeta + \eta/3)\, \partial^i \partial_j v^j, \tag{3.108}$$

where v^i is the velocity field, $\rho = \epsilon + p$ and $\partial^2 = \partial^j \partial_j$.

To show how (3.107) and (3.108) arise in our construction, we first note that the effective Goldstone action (3.79), (3.80) for $\phi^\pm$ fields describes an *out-of-equilibrium* theory in which the gradient expansion is organised by counting derivatives of currents $K^{i\mu}$ at some IR hydrodynamic scale Λ_h. To understand the *near-equilibrium* limit, we study the energy-momentum balance equation (3.58) by introducing a near-equilibrium parameter ℓ, so that

$$\phi^I(x) = b_0^{1/3}\left(x^I + \ell \pi^I(x)\right). \tag{3.109}$$

Expanding around a constant equilibrium current

$$K_0^\mu = (b_0, 0, 0, 0)\,, \tag{3.110}$$

it follows that

$$b = b_0 + \ell \Delta b + \dots, \tag{3.111}$$

$$u^\mu = u_0^\mu + \ell v^\mu + \dots, \tag{3.112}$$

with

$$u_0^\mu = (1, 0, 0, 0)\,, \qquad\qquad v^\mu = \left(0, v^i\right). \tag{3.113}$$

In terms of the fluctuation fields π^i, we find that

$$\Delta b = b_0 \partial_i \pi^i, \tag{3.114}$$

$$v^i = -\partial_0 \pi^i. \tag{3.115}$$

Conservation equation (3.63) then implies the order-ℓ relation

$$b_0 \partial_i v^i = -\partial_0 \Delta b. \tag{3.116}$$

At the leading order in ℓ, the force of the environment acting on the fluid that is encoded in the non-conserving $R^{\nu}_{(1)}$, vanishes. The first-order $T^{\mu\nu}_{(1)}$ is thus approximately conserved near equilibrium and can be treated as the viscous contribution to the total fluid's stress-energy tensor.

Since first-order contributions are suppressed in the double expansion by ℓ as well as a derivative acting on v^i, we expand the zeroth-order energy-momentum balance equation to order ℓ^2. The contribution from $R^{\nu}_{(0)}$ remains non-vanishing, but it can be absorbed into the small $\mathcal{O}(\ell)$-suppressed shifts of the fluid's energy and pressure,

$$\epsilon \to \epsilon + \ell p_0, \qquad\qquad p \to p - \ell p_0, \tag{3.117}$$

where the un-shifted expressions are those of Eqs. (3.84) and (3.85). Furthermore, p_0 is given by the expression

$$p_0 = \frac{(i+j)}{3}\Delta b\left[b_0 \bar{G}'_{ij} + \frac{1}{2}\ell\left(\bar{G}'_{ij} + b_0\partial_b\bar{G}'_{ij}\right)\Delta b\right], \tag{3.118}$$

with G'_{ij} evaluated at $b = b_0$ and the expression summed over i and j.

With this re-definition of ϵ and p, the tensor $T^{\mu\nu}$ in (3.91) becomes approximately conserved near equilibrium and mimics the expected behaviour of a *stress-energy tensor*,

$$\partial_\mu T^{\mu\nu} \approx 0. \tag{3.119}$$

A further requirement for a genuine identification of $T^{\mu\nu}$ with the hydrodynamic stress-energy tensor of the fluid described by our CTP construction, is that $T^{\mu\nu}$ needs to be symmetric. We can show that to the order of ℓ we are working at, a symmetrised $T^{(\mu i)}$ obeys

$$\partial_\mu T^{\mu i} = \partial_\mu T^{(\mu i)} = \frac{1}{2}\partial_\mu\left(T^{\mu i} + T^{i\mu}\right) + \mathcal{O}(\ell^2) \approx 0. \tag{3.120}$$

The symmetrisation of $T^{\mu 0}$ does not work in the same way. However, in the non-relativistic limit, only zeroth-order, ideal hydrodynamic terms of the $T^{\mu 0}$ components contribute to the continuity equation (3.107). Thus, for a non-relativistic, near-equilibirum Navier-Stokes fluid, we can identify the symmetrised version of our tensor $T^{\mu\nu}$ with the phenomenological stress-energy tensor,

$$T^{(\mu\nu)} \approx T^{\mu\nu}_{ph}. \tag{3.121}$$

One should be aware that beyond the aesthetic desire to exactly match the phenomenological stress-energy tensor, what is important for the physics are dynamical equations of motion. Those follow from Eq. (3.58), which is approximately conserved and does not require $T^{\mu\nu}$ to be symmetric. Dynamical equations derived in the near-equilibrium limit of our CTP construction are equivalent to those derived from phenomenological hydrodynamics with the use of conservation laws.

The Navier-Stokes equations (3.107) and (3.108) again follow from the near-equilibrium expansion to $\mathcal{O}(\ell^2)$ at zeroth order, and $\mathcal{O}(\ell)$ at first order in gradient expansion, followed by a non-relativistic scaling limit $\epsilon_{nr} \to 0$. From this expansion, or directly from (3.105), we find that shear viscosity η vanishes while bulk viscosity is non-zero,

$$\eta = 0, \qquad\qquad \zeta = -\chi_1|_{b=b_0}. \tag{3.122}$$

Note that the vanishing of shear viscosity was most likely caused by the very large symmetry group of volume-preserving diffeomorphisms, under which our fluid is invariant. In fact, viscosity in [102] resulted from a Lagrangian term that explicitly broke this symmetry. Because of the near-equilibrium expansion, the hydrodynamic coefficient ζ becomes an equilibrium b_0-dependent constant. In terms of the four undetermined coefficient functions in Lagrangian (3.80),

$$\begin{aligned}\zeta = &- b_0^3\left(\bar{f}_{333} + \bar{f}_{300} - f_{303} + 3\bar{f}_{330}\right)|_{b=b_0} - 2b_0^5\left(\bar{f}'_{333,03} + \bar{f}'_{303,03} - \bar{f}'_{330,03} - \bar{f}'_{300,03}\right)|_{b=b_0}\\ &- 4b_0^5\left(\bar{f}'_{333,00} + \bar{f}'_{303,00}\right)|_{b=b_0} + 4b_0^5\left(\bar{f}'_{330,33} + \bar{f}'_{300,33}\right)|_{b=b_0}.\end{aligned} \tag{3.123}$$

Lastly, the entropy current S^μ, which can be associated with the system, must satisfy the covariant thermodynamic relation [25],

$$TS^\mu = pu^\mu - T^{\nu\mu}u_\nu, \tag{3.124}$$

as well as the positive entropy production condition $\partial_\mu S^\mu \geq 0$. Eq. (3.124) then implies that in our theory the first-order correction to the entropy current, as identified in Eq. (3.71), takes the frame-invariant form

$$S^\mu = \left(\frac{\epsilon + p}{T}\right)u^\mu + \frac{q^\mu}{T} = K^\mu + \frac{q^\mu}{T}. \tag{3.125}$$

Since the zeroth-order entropy current K^μ is conserved, the positivity of the divergence of (3.125) requires us to impose

$$\partial_\mu\left(\frac{q^\mu}{T}\right) \geq 0. \tag{3.126}$$

This statement is frame-dependent and applies only in our frame with $\mathcal{E} = \varepsilon$, cf. Eq. (3.101), and for a conserved zeroth-order entropy current K^μ. We can see that in our effective CTP theory, temperature T can be identified with the expression

$$T = \sum_{i\leq j}\left[-\partial_b F + \frac{b}{3}\bar{G}'_{ij}\,(i+j)\right], \tag{3.127}$$

with $i, j \in \{0, 3\}$. The entropy density relation, $s = b$, remains valid at first order. At the leading order in ℓ and in the non-relativistic limit, we find that positive entropy production condition (3.126) demands that

$$\beta(b_0)\partial^i \partial_i \Delta b \geq 0. \tag{3.128}$$

This expression is consistent with the following fact pertaining to incompressible fluids, which are characterised by the condition

$$\partial_i v^i = 0. \tag{3.129}$$

According to definitions (3.112) and (3.113), the incompressibility condition (3.129) implies the relativistic relation, $\partial_\mu u^\mu = 0$, to first order in ℓ. Conservation of K^μ, cf. Eq. (3.63), then implies that b must be a space-time independent constant. Given definition (3.111) of b to order ℓ, the fact that b must be constant means that we may absorb a constant value Δb into b_0, and set $\Delta b = 0$. Finally, Eq. (3.128) shows that incompressibility implies conservation of entropy. These findings are therefore consistent with the fact that an incompressible non-relativistic fluid with $\eta = 0$ behaves as an ideal fluid without any entropy production. In such cases, the presence of bulk viscosity ζ alone cannot influence the solutions of the Navier-Stokes equation (3.108).

3.7 Discussion

In this chapter, we showed that phenomenological relativistic hydrodynamics with dissipation can be constructed using classical CTP effective action. We were able to derive closed-form equations describing the fluid from an action principle, containing dissipative effects triggered by the presence of non-zero bulk viscosity.

Of central importance were terms collected into S_2, which coupled fields living on the two time axes and reflected quantum and classical interactions between the open (sub)-system and the integrated-out, UV degrees of freedom of the environment. Dissipation thus manifested itself in the energy loss of the low-energy degrees of freedom to the UV microscopic degrees of freedom, which was represented by the system-environment interactions. We note that this physical interpretation is in accordance with the usual phenomenological view of dissipation. However, in that approach one is able to maintain all conservation laws. The relation between the approach presented in this chapter and phenomenology should be understood in a more precise and quantitative manner.

Despite the lack of energy conservation, the stress-energy tensor was shown to be conserved in the near-equilibrium regime. This enabled us to identify bulk viscosity of the family of fluids, which could be described by the action we constructed. Shear viscosity, however, vanished in this setup, which is most likely the result of a large amount of symmetry, namely the volume preserving diffeomorphisms that were used to construct the effective action. A further study of this important problem, i.e. the identification of the correct symmetries of dissipative fluids, as well as classification of different fluids described by the presented formalism should be returned to in the future.

Chapter 4

Fermi surfaces in supersymmetric field theories

4.1 Motivation

Understanding the behaviour of quantum matter at finite temperature T and density μ is a major challenge in many areas of physics, ranging from traditional condensed matter topics to quark-gluon plasmas as explored at RHIC and the LHC, to the behaviour of super-dense QCD matter in the cores of neutron stars. Developing such an understanding is especially difficult when the systems are strongly coupled and traditional perturbative techniques are not useful. One powerful non-perturbative technique which has attracted a great deal of attention in recent years is gauge/gravity duality, introduced in Chapter 2.2.2, which maps questions about some special strongly-coupled field theories to questions about weakly-coupled theories of gravity, which are much easier to work with.[1] This has led to many interesting results for the study of finite-density quantum matter, but also a number of puzzles, such as the fate of Fermi surfaces in the strongly-coupled systems which have gravity duals.

The ability to do controlled calculations on the gravity side of the duality comes with several conditions and costs. To justify treating the gravity side of the duality classically, which is in general the only tractable limit, one needs the field theory to be (1) strongly coupled, typically in the sense of having a tunable 't Hooft coupling which is taken to be large and (2) to be in some kind of large N limit. Indeed, in all of the cases where the dual field theory Lagrangian is explicitly known, the field theory is a non-Abelian gauge theory, and the parameter N is associated with the rank of the gauge group.[2] Finally, the class of theories which have strong-coupling limits and a large N limit is clearly rather

[1]For an additional review focusing on Fermi surfaces see reference [116].

[2]Finding such a large parameter in the known phenomenologically-relevant examples is a challenge, especially in the examples from condensed matter.

special,[3] and in all of the cases where the dual field theory Lagrangians are known, they are supersymmetric gauge theories or deformations thereof, see *e.g.* [10, 74, 117–119] for some prototypical examples.

These considerations make it difficult to tell a priori which of the many interesting results gauge/gravity duality has yielded are due to strong coupling, large N, the special nature of the field content and interactions in the theories which have gravity duals, or some combination of these. In this sense, gauge/gravity duality is essentially a black box, since it is only tractable in a limit where the field theory description is fundamentally difficult to work with. Moreover, while the duality has yielded many striking results, it has also produced many mysteries, such as the fate of Fermi surfaces at strong coupling, explored in e.g. [120–138]. The 'microscopic' field content of the theories with gravity duals generally includes gauge bosons, fermions, and scalars, with the number of degrees of freedom for all of these scaling as $\mathcal{O}(N^2)$ in the 4D field theory examples. In these theories chemical potentials for conserved charges usually couple to *both* the scalars and fermions at the microscopic level. Hence if intuition derived from studies of weakly-coupled non-supersymmetric theories were to be boldly applied to the strong coupling limit of the kind of theories which have gravity duals, then one might have expected that Fermi surfaces would be ubiquitous in systems with gravity duals.

However, while Fermi surfaces have shown up in some examples of gauge/gravity duality, they do not seem to be at all ubiquitous. Signs of Fermi surfaces for the $\mathcal{O}(N^2)$ degrees of freedom have recently shown up in e.g. [125] in correlation functions of fermionic operators in electron star geometries [124,139], and in some top-down calculations in [137,138] for 4D theories. But in other examples Fermi surfaces appear to be absent [126, 127]. Meanwhile, Fermi surfaces have been observed in fermionic correlators of $\mathcal{O}(N^0)$ densities of probe fermions in work initiated in [120, 121]. To make the situation more complicated, naïvely — that is, based on expectations from weak-coupling studies of systems familiar from condensed matter — Fermi surfaces should have an imprint on bosonic correlation functions as well, showing up as e.g. momentum-space singularities in density-density correlation functions leading to Friedel oscillations. Indeed, in holography one only has access to gauge-invariant observables, while Fermi surfaces for the quarks in a gauge theory would not be gauge-invariant. So such Fermi surfaces might be 'hidden' [130] in the gravity duals, and hence singularities in gauge-invariant charge density correlation functions may seem to be especially promising places to look for traces of Fermi surface physics. But such density-correlator signatures of underlying Fermi surfaces have not been seen in many holographic

[3] For instance, large N QCD is not such a theory, since its 't Hooft coupling runs, and is thus not a tunable control parameter.

systems.[4]

These considerations motivate our belief that in order to better understand the results of gauge/gravity duality calculations, it would be very useful to reexamine some observables for which strong coupling results from holography are available at weak coupling using conventional field-theory techniques, where one can see all of the moving pieces. In particular, one would have direct access to any 'hidden' Fermi surfaces, since at weak coupling it makes sense to work in a gauge-fixed formalism. We will focus on $D = 3+1$ dimensional theories for simplicity, and confine our attention to the $T = 0$ limit. The metric signature convention used in this chapter will be the one used in QFT, i.e. $\eta_{\mu\nu} = \text{diag}\,(+1, -1, -1, -1)$.

An example of the kind of theory one might want to study at weak coupling is $\mathcal{N} = 4$ super-Yang-Mills theory with a chemical potential for R-charge, where the number of charged degrees of freedom scales as N^2, originally studied in [142–145]. Another example, where the number of charged degrees of freedom scales as N^1, is the $\mathcal{N} = 2$ gauge theory dual to N_f D7 branes intersecting N_c D3 branes in the 'quenched' $N_f/N_c \ll 1$ approximation [74]. The study of this latter flavoured $\mathcal{N} = 2$ system at finite quark number density was initiated in [146]. Calculations using the gravity side of the duality predict unusual thermodynamical features for this theory which are not known to arise from any weakly-coupled theory, with *e.g.* a specific heat with the temperature scaling $c_V \sim T^6$ [147], in contrast to what one might expect from a Fermi liquid where $c_V \sim T$. Moreover, [147] found a gapless quasiparticle mode in the system which was argued to be Landau's zero sound mode (see also [148–152] for some further exploration of this identification). But the c_V scaling shows that the system is clearly a non-Fermi liquid, and to the extent that the dual field theory is a gauge theory with gapless gauge interactions, a zero sound mode would be surprising, at least at weak coupling, as we discuss further in Section 4.2. What is the origin of the curious thermodynamic properties of this system and what is the true identity of the quasiparticles modes? It is possible that the puzzling thermodynamics is driven by some intrinsically strongly-coupled physics, or — as explored recently in *e.g.* [152–154] — were the calculations of [147] done in some metastable vacuum? Another possibility, which can be explored using weak-coupling techniques, is that at least some of these properties are a consequence of the unusual field content and interactions of the field theory.

However, as with the other theories with known field theory Lagrangians and gravity duals, the $\mathcal{N} = 4$ super-Yang-Mills field theory examined in [10] is quite complicated, as are its cousins discussed in the many follow-up works, and we will not address field theories

[4] In [135] it is observed that density-density correlation functions in theories with dual Lifshitz geometries [140, 141] with $z = \infty$ have momentum-space singularities which suggest the presence of a Fermi surface, but $z < \infty$ examples do not.

with gravity duals directly in this work. Instead, as a first step we will study a few simpler toy-model supersymmetric gauge theories. Specifically, we will explore the behaviour of $\mathcal{N} = 1$ super-QED (sQED) and $\mathcal{N} = 2$ sQED in the presence of chemical potentials at zero temperature. Even these simple toy models show some curious features, since from a condensed-matter point of view they have unusual field content and interactions, with the chemical potential coupling to both scalar fields and fermions, which are in turn coupled to each other by the demands of supersymmetry.

Perhaps the simplest questions one can ask about such systems concern the nature of their ground states. Do the bosons condense, and do the fermions develop a Fermi surface? It seems natural to expect weakly-coupled scalars to condense at $T = 0$ in response to a chemical potential, and we find that this is indeed what happens in our examples. One might expect Fermi surfaces to be a generic consequence of turning on chemical potentials that couple to weakly-interacting fermions based on a naïve application of the standard Landau Fermi liquid picture, and intuitions derived from thinking about non-supersymmetric electron plasmas. But we find that dense plasmas based on $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sQED fail to be Fermi liquids in a fairly dramatic way, already at weak coupling. While the chemical potential couples to the fermions in all of our examples, it does not lead to a Fermi surface in most of them. This suggests another possible reason for the mysterious cases of missing Fermi surfaces encountered in holographic studies, aside from strong coupling.

In this chapter, we will use Section 4.2 to give an overview of our toy models, explain their unusual features from a condensed-matter perspective, and discuss what one might expect for their behaviour at finite density. The findings will be summarised in Section 4.2.1. In Section 4.3, we explore $\mathcal{N} = 1$ sQED at finite electron number density. Then in Section 4.4 we discuss $\mathcal{N} = 2$ sQED with a finite electron number density, where we are forced to introduce some soft SUSY-breaking terms to stabilize the scalar sector. Next, in Section 4.5 we look at $\mathcal{N} = 2$ sQED with a finite R-charge density. Algebraically, the $\mathcal{N} = 2$ R-charged theory and its SUSY-broken cousins are our cleanest examples, and we evaluate the fermion contribution to the charge density for some examples in this class of theories. The somewhat surprising result of this investigation is described in Section 4.6. Finally, in Section 4.7 we present an extended discussion of our findings and sketch some of the many possible directions for future work.

We also make a brief comment on the scarce existing literature on SUSY gauge theories at finite density using field-theoretic techniques. The works most closely related to the approach of this chapter are [48, 155] and [129]. Ref. [155] studied $\mathcal{N} = 4$ SYM theory with R-charge chemical potentials compactified on a 3-sphere, with a focus mostly on the high-T limit, while [48] studied the finite-T properties of $\mathcal{N} = 2$ super-Yang-Mills (SYM) theory.

Ref. [130] studied physics related to Fermi surfaces in non-supersymmetric theories inspired by 4D $\mathcal{N} = 4$ SYM, among other examples, but with their choice of models they did not run into many of the issues we deal with here. We also note the important work [156] exploring the interplay between Luttinger's theorem, Fermi surfaces, and Bose-Einstein condensation in the context of cold atomic gases.

Also, the study of super-QCD at finite quark-number was initiated in [157] for $\mathcal{N} = 1$ supersymmetry and in [158] for $\mathcal{N} = 2$ supersymmetry, with an aim of understanding colour superconductivity in a supersymmetric context. However, the issue of the existence of Fermi surfaces in supersymmetric gauge theories at finite density was not examined in these papers. Finally, the interesting recent works [159, 160] constructed a supersymmetric version of 'BCS theory', without dynamical gauge fields, and engineered things such that there is no scalar condensation but there are Fermi surfaces.

4.2 What should we expect?

The standard example of a finite-density relativistic system involving fermions and gauge fields is a QED plasma, which we now briefly describe before considering supersymmetric theories. We do this because much of our intuition for what to expect for finite-density physics is based on experience with this non-supersymmetric system.

The Lagrangian describing an electron plasma is just that of QED, involving the electron field ψ and the photon gauge field A_μ, and is very simple:

$$\mathcal{L}_{\mathcal{N}=0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i \not{D} - m \right) \psi + A_\mu J^\mu, \tag{4.1}$$

where $D_\mu = \partial_\mu - i\mu\delta_{\mu 0} - igA_\mu$, g is the gauge coupling, μ is a chemical potential which couples to the charge of the electrons, and J^μ encodes the effects of other matter which provides a neutralising background, such as some ions.

The requirement of having a neutralising background is essential. While the addition of the chemical potential term is a gauge-invariant deformation of the theory, it couples to a gauged charge. If one wants a finite density of matter in the vacuum in the infinite-volume limit, with a finite free energy density, then any negative charge density carried by the electrons must be compensated by a positive charge density carried by the ions. Otherwise one would pay an infrared-divergent energy cost for having long-range electric fields. This is a textbook observation for QED plasmas [161], and is also true for non-Abelian gauge theories like QCD at high densities.[5] As is explained in *e.g.* Section 2 in [163], neutrality must be imposed even if the gauged charge is spontaneously broken, which will be relevant

[5]For some seminal papers exploring this issue *e.g.* see [162–164], for a review see [165].

for our discussion of sQED. Otherwise a finite size chunk of the degenerate matter would have electric fields outside of it which grow in strength with its size, again causing problems with the infinite-volume limit.

Before beginning a discussion of supersymmetric plasmas, and exploring to what extent they can be thought of as Fermi liquids, it is important to note that a standard dense low-temperature electron gas described by Eq. (4.1) is already *not* a Fermi liquid. The issue is the long range of the electromagnetic interactions, and the subtle nature of screening due to the degenerate electrons. While Coulomb photons pick up a screening mass in the static (zero-frequency) limit due to medium effects, the transverse ('magnetic') photons do not get a static screening mass so long as the photons do not become Higgsed. Consequently, the magnetic photons continue to mediate long-range interactions, and this drives the breakdown of Fermi liquid theory [166–169]. This leads to subtle effects such as a non-Fermi-liquid scaling of the specific heat with temperature, $c_v \sim T \ln T$, among others. At a more pedestrian level, the non-trivial momentum and energy dependence of the Coulomb screening effects in an electron plasma are such that the residual Coulomb interaction obliterates the would-be gapless Fermi zero-sound mode present in Fermi liquids, turning it into the gapped plasmon mode of the dense electron gas as explained in *e.g.* Chapter 16 of the textbook [161].

Given these results for non-supersymmetric gauge theories at finite density, we clearly cannot assume that the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sQED plasmas should be Fermi liquids. Nevertheless, while non-supersymmetric degenerate plasmas are not Fermi liquids, the fermions populating the plasma still have a Fermi surface, at least before considering the standard sort of pairing (superconducting) instabilities which can lead to its breakdown. This remains true[6] even in more exotic non-supersymmetric systems, such as degenerate quark matter, and generalisations of Eq. (4.1) to include condensed dynamical scalar fields in J^μ [170–173], or some types of Yukawa interactions [174]. As we will see, however, even the very existence of a Fermi surface cannot be taken for granted in the supersymmetric case.

For a final observation about non-supersymmetric plasmas, we note that having $g \ll 1$ is necessary but *not* sufficient for a QED plasma to be weakly coupled. The reason is that Coulomb interactions are, in a sense, strong at low energies, and tend to lead to the formation of bound states — atoms — if the interaction energy dominates over the characteristic momenta of the electrons and ions. Indeed, if we define $l \equiv [3/(4\pi n)]^{1/3}$ as

[6]Since electron and quark fields are not gauge invariant, the notion of a Fermi surface is easiest to discuss in a gauge-fixed setting, and understanding its effects in gauge-invariant language requires more work. Fortunately, at weak coupling, where our attention will be confined, the use of such gauge-fixed notions will be very useful, as it is in *e.g.* the standard discussions of gauge symmetry 'breaking' in the Standard Model's Higgs mechanism.

the inter-electron 'spacing' and denote the Bohr radius by $a_0 \equiv 1/(\alpha m)$, then it is well-known that in an electron gas the physical expansion parameter is $r_s \equiv l/a_0$, rather than $\alpha \equiv g^2/4\pi$, and one must have $r_s \ll 1$ for calculability. We expect that our results in the supersymmetric examples below will be reliable in a similar high density limit, but it will be important to verify this in future work by doing higher-order calculations. For this work, we simply assume that our number densities are large enough that we do not have to worry about the formation of supersymmetric atoms, which were studied recently in [175–177]. In the terminology often used in the AdS/CMT literature, our focus on high density fully ionised plasmas means that we work in the *'fractionalised'* regime of super QED, as opposed to the low-density atomic gas regime, which could be thought of as *'confined'*.

We now turn to a discussion of the subtleties particular to supersymmetric plasmas. To keep the discussion streamlined, we use $\mathcal{N} = 1$ sQED as our example. The action of $\mathcal{N} = 1$ sQED is significantly more complicated than that of QED. In addition to ψ and A_μ, supersymmetry requires the addition of selectron fields ϕ_+, ϕ_-, as well as the gaugino λ, along with interaction terms amongst all of these mandated by the supersymmetrisation of the gauge interaction. The resulting action is

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=1} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\bar{\lambda}i\partial\!\!\!/\lambda \\ &+ \bar{\psi}\left(iD\!\!\!\!/^{-} - m\right)\psi + \left|D_\mu^-\phi_-\right|^2 + \left|D_\mu^+\phi_+\right|^2 - |m\phi_-|^2 - |m\phi_+|^2 \\ &+ \sqrt{2}ig\left(\phi_+^\dagger\bar{\psi}P_-\lambda - \phi_-^\dagger\bar{\lambda}P_-\psi - \phi_+\bar{\lambda}P_+\psi + \phi_-\bar{\psi}P_+\lambda\right) - \frac{g^2}{2}\left(|\phi_+|^2 - |\phi_-|^2\right)^2 \\ &+ \text{Ions},\end{aligned} \tag{4.2}$$

where λ is a Majorana fermion, $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$, $\phi_\pm$ are complex scalar fields,

$$D_\mu^\pm = \partial_\mu \pm i\mu\delta_{\mu 0} \pm igA_\mu \tag{4.3}$$

and the +Ions term encodes couplings to neutralising 'ion' fields. We assume the ion sector is supersymmetric as well, and defer writing out the relevant contributions to the action for now. The physical motivation for assuming that the ion sector is supersymmetric is that the theories we are really interested in — the ones with gravity duals — usually do not include dynamical non-supersymmetric sectors. The action describing $\mathcal{N} = 2$ matter at finite density is even more complex, and we do not write it out here; the general comments about $\mathcal{N} = 1$ sQED below also apply to $\mathcal{N} = 2$ sQED.

Before launching a search for Fermi surfaces in $\mathcal{N} = 1$ sQED, and then $\mathcal{N} = 2$ sQED, we should emphasise a few features of Eq. (4.2) which make the analysis tricky. First, note that before considering the 'ions', there is only *one* continuous symmetry in Eq. (4.2), under which the fields have the transformation properties $\psi \to e^{-i\alpha}\psi$, $\phi_+ \to e^{i\alpha}\phi_+$, $\phi_- \to$

$e^{-i\alpha}\phi_-$, $A_\mu \to A_\mu$, $\lambda \to \lambda$. So there are no *separate* fermion number or scalar number symmetries, in a striking contrast to familiar non-supersymmetric theories, even ones studied in [156]. The fields are tied together by the Yukawa interactions in such a way that only a *single* $U(1)$ remains. Second, we observe that as usual, the chemical potential enters the Lagrangian as the time component of a background gauge field. So *both* the selectron and the electron fields directly experience the chemical potential. We note that in this situation, one should interpret any expectations based on Luttinger's theorem [178] or the theory of 'compressible quantum matter' [129] with care, since the assumptions underlying these frameworks do not apply in general to the systems we consider once the scalar fields condense.

The issue we explore in this work concerns the response of the selectrons and electrons to the chemical potential. Let us start by considering the behaviour of the scalar fields of $\mathcal{N} = 1$ sQED. The scalar effective potential V_{eff} is the sum of the classical potential

$$V_{\text{eff}}^{(0)} = (|m|^2 - \mu^2)(|\phi_+|^2 + |\phi_-|^2) + \frac{g^2}{2}(|\phi_+|^2 - |\phi_-|^2)^2 \tag{4.4}$$

plus quantum corrections. Interactions with the electrons and photons will contribute new terms to the bosonic effective potential starting at one loop level. But so long as the theory is weakly coupled, and the classical potential is non-vanishing, the selectron ground state should be determined by $V_{\text{eff}}^{(0)}$, since quantum corrections to $V_{\text{eff}}^{(0)}$ should be comparatively small.

From the form of $V_{\text{eff}}^{(0)}$, one might think that once $\mu > m$, the scalars should condense, breaking the $U(1)$ gauge symmetry and making the system a superconductor. Moreover, since the masses of the electrons and selectrons are fixed to be identical due to supersymmetry, the fermions should naïvely start populating a Fermi surfaces at the same time that the scalars start condensing.

But there is an immediate subtlety we must deal with: supersymmetric gauge theories typically have moduli spaces protected by supersymmetry at $\mu = 0$. In the current context, the moduli space for $m = 0$, $\mu = 0$ is isomorphic to $\mathbb{C}$, and is parametrised by the value of $\phi_+ = \phi_-$. For any set of vacuum expectation values for the selectrons satisfying $\phi_+ = \phi_-$, the potential energy vanishes. But as soon as we make $\mu > m$, $V_{\text{eff}}^{(0)}$ develops a runaway direction along $\phi_+ = \phi_-$. That is, the effective potential becomes unbounded from below, and the theory as defined in Eq. (4.2) does not make sense for $\mu > m$.[7]

This should not be especially surprising. For a system comprised of weakly-interacting

[7]If both a finite μ and finite temperature T are turned on, things may be different, since the finite temperature breaks supersymmetry, and should help lift the moduli space at $\mu = 0$. For an interesting recent exploration of finite-T physics in a supersymmetric gauge theory using field-theoretic techniques, see [48].

bosonic particles to be stable at finite chemical potential, the bosons must have sufficiently repulsive interactions. If the interactions of the bosons were attractive, then the system would be unstable against a collapse towards arbitrarily high densities, and there would not be any equilibrium finite-density ground state. This is precisely the issue that one faces in $\mathcal{N} = 1$ sQED, where supersymmetry demands the presence of an attractive interaction between the positive and negative selectrons $-\frac{g^2}{2}|\phi_+|^2|\phi_-|^2$. The arguments above imply that this issue indeed causes an instability which is unavoidable without deforming the theory in some way.

Fortunately, in $\mathcal{N} = 1$ sQED, it is possible to dodge this problem by turning on a Fayet-Iliopulous term, which does not explicitly break the supersymmetry of the action, and has the effect of modifying the potential to

$$V_{\text{eff}}^{(0)} = (|m|^2 - \mu^2)(|\phi_+|^2 + |\phi_-|^2) + \frac{g^2}{2}(|\phi_+|^2 - |\phi_-|^2 - \xi^2)^2, \tag{4.5}$$

where ξ^2 can be either positive or negative, and has mass dimension two. At $\mu = 0$, this lifts the moduli space, and indeed supersymmetry becomes spontaneously broken for $\xi > 0$ so long as $m \neq 0$. With ξ turned on, we will argue that the selectrons of the theory have a stable non-trivial ground state for μ in a certain range. Hence the naïve expectation that the $U(1)$ gauge symmetry is broken at finite density is borne out, and the system is a superconductor.

One might have hoped that so long as $g \ll 1$, and the system is weakly coupled, the response of the electrons to the chemical potential should resemble that of the free limit $g = 0$. This is true in a QED plasma. However, one should not expect it to be true in general for supersymmetric plasmas, as we now explain.

First, it is clear from the structure of the Yukawa terms in Eq. (4.2), which include terms of the form

$$g\, \phi_+^\dagger \bar{\psi} P_- \lambda, \tag{4.6}$$

that turning on scalar VEVs leads to mixing between the electron and gaugino fields, and this makes it difficult to guess what the fermionic fields will do in response to a chemical potential for the electrons just by looking at Eq. (4.2). The way to deal with this is obvious in principle, since one just has to rotate to an eigen-basis where the kinetic terms for the fermions become diagonal in the in-medium 'flavours', but in practice actually doing such a rotation can be algebraically involved. Since the coefficient of the mixing term is proportional to g, however, one might have hoped that when $g \ll 1$, the mixing would be small, and the response of the fermions to a chemical potential would be close to that of the $g = 0$ system.

To see why this expectation is overly naïve, note that the coefficient of the Yukawa terms is forced to be the gauge coupling g by supersymmetry. But the coefficient controlling the strength of the self-interaction of the selectrons in Eq. (4.5) is g^2, which is also fixed by supersymmetry. So unlike in a non-supersymmetric system, here the strengths of the Yukawa interactions and the selectron self-interactions cannot be tuned independently. In particular, given the form of the selectron potential it is obvious that a non-zero selectron VEV $\langle\phi\rangle$ must scale as

$$\langle\phi\rangle \sim \frac{1}{g}. \tag{4.7}$$

So since the size of the electron-gaugino mixing terms is controlled by $g\langle\phi\rangle$, we see that the fermion mixing will be essentially independent of g. The mixing alters the dispersion relations of the fermion fields at the quadratic level, and so we cannot assume that the response of the electrons to a chemical potential at $g = 0$, which involves the formation of a Fermi surface, will necessarily persist to *any* $g > 0$, no matter how small. This observation is generic, and applies to essentially any supersymmetric gauge theory in which one turns on a chemical potential for selectrons or squarks which can also cause selectron or squark condensation.

4.2.1 Summary of expectations

For the reasons discussed above, we expect that:

- The chemical potentials we will consider couple to both fermions and scalars, and so long as the theory is supersymmetric we expect the scalars to condense at the same time as the fermions begin to feel the chemical potential. This means the $U(1)$ gauge symmetry will be broken, and the supersymmetric plasmas will be superconductors.

- It is essential to take into account the electric neutrality constraint. In a related context, this was also emphasised in [129].

- We assume that the densities are large enough that we do not have to worry about the formation of supersymmetric atoms, so that we deal with a completely ionised plasma. This means we are focusing on the *fractionalized* regime of the plasma, as opposed to the low-density atomic gas *confined* regime.

- Achieving a stable finite-density ground state may be tricky due to possible run-away directions in the scalar potential due to supersymmetry.

- We should not expect the behaviour of the fermions to be close to that of a conventional free system once there is scalar condensation, because of the structure of the Yukawa interactions and the scalar self-interactions dictated by supersymmetry.

- If the scalars condense, the fact that the $U(1)$ electron number symmetry is shared between the scalars and the fermions means that the resulting quantum liquids will not be 'compressible quantum matter' as it is defined in [130]. Moreover, the assumptions of Luttinger's theorem [178], which ties the charge density carried by a fermionic system to the volume of the Fermi surface, will not apply to such a liquid. So we should not expect the existence of Fermi surfaces to be automatic for finite-density supersymmetric QED.

With these observations in mind, we turn to a more detailed examination of these issues in our $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sQED toy models.

4.3 $\mathcal{N} = 1$ sQED at finite electron number density

4.3.1 Scalar Ground State

We begin by writing down the complete $\mathcal{N} = 1$ action that will be considered below. We include two chiral superfields Φ_+ and Φ_- which supply the matter fields for the 'electron' sector: the electron Dirac spinor field ψ, as well the bosonic selectrons ϕ_+, ϕ_-. We also include two other chiral superfields Q_+ and Q_-, which supply the matter fields for the 'ion' sector: the ion Dirac spinor field η, as well as the bosonic sion fields q_+, q_-. We consider a superpotential of the simplest possible form

$$\mathcal{W} = m(\Phi_+\Phi_- + Q_+Q_-), \tag{4.8}$$

so that the ions and the electrons have the same mass m. The tree-level Kahler potential is

$$\mathcal{K} = \Phi_+^\dagger e^V \Phi_+ + \Phi_-^\dagger e^{-V} \Phi_- + Q_+^\dagger e^V Q_+ + Q_-^\dagger e^{-V} Q_-, \tag{4.9}$$

and V is the vector superfield, which includes the photon and photino fields A_μ, λ. We also allow a Fayet-Iliopoulos term

$$\mathcal{L}_\xi = -\xi^2 \int d^4\theta V. \tag{4.10}$$

The Lagrangian of the version of $\mathcal{N} = 1$ SQED that we will consider is thus

$$\mathcal{L}_{\mathcal{N}=1} = \left(\frac{1}{4g^2}\int d^2\theta\, W^2 + \text{h.c.}\right) + \int d^4\theta\, \mathcal{K} + \left(\int d^2\theta\, \mathcal{W} + \text{h.c.}\right) + \mathcal{L}_\xi, \tag{4.11}$$

and W_α is the photon field strength chiral super field.

Transformation Properties in $\mathcal{N} = 1$ sQED

Fields:	ψ	ϕ_+	ϕ_-	λ	η	q_+	q_-
$U(1)_e$	$e^{-i\alpha}\psi$	$e^{+i\alpha}\phi_+$	$e^{i\alpha}\phi_-$	λ	η	q_+	q_-
$U(1)_i$	ψ	ϕ_+	ϕ_-	λ	$e^{-i\alpha}\eta$	$e^{+i\alpha}q_+$	$e^{-i\alpha}q_-$

Table 4.1: Matter field transformation properties under the $U(1)_e$ and $U(1)_i$ symmetries.

The matter sector has two obvious $U(1)$ symmetries, $U(1)_e$ and $U(1)_i$, which act on the component fields as shown in Table 4.1. The diagonal $U(1)_e \times U(1)_i$ symmetry (acting as $\psi \to e^{-i\alpha}\psi, \eta \to e^{-i\alpha}\eta$, and so on) is gauged, and we will refer to the gauged charge as the 'electric' charge.

We want to have a net density of electron-sector fields — electrons, selectrons, or both — in the ground state. To do this we turn on a chemical potential μ_e for the $U(1)_e$ symmetry, which appears in the action as the time component of a background gauge field coupling only to $U(1)_e$ charge. At the same time, we wish to maintain charge neutrality. To do this, we also turn on a chemical potential μ_i for the conserved charge associated with the ion $U(1)_i$ symmetry. Then the μ_e chemical potential can be viewed as the parameter controlling the matter density of the system, while μ_i is an auxiliary parameter determined by the requirement of charge neutrality.

It turns out that setting $\mu \equiv \mu_e = -\mu_i$ will be sufficient to maintain charge neutrality. Heuristically, turning on $\mu > 0$ gives an equal energetic subsidy to the *particles* created by the field operators ψ, ϕ_-, ϕ_+ and the *antiparticles* created by η, q_-, q_+. Since these two sets of particles and antiparticles have the same masses but opposite electric charges, this will create a ground state which is electrically neutral. To see this in a more quantitative way, recall that we can read off the expression for the charge density from the part of the action which is linear in A_0:

$$gA_0 Q \in \mathcal{L}, \tag{4.12}$$

since A_0 is, by definition, the source for Q. This yields

$$\begin{aligned} Q = &-\bar{\psi}\gamma^0\psi + i[\phi_+^\dagger(\partial_0 + i\mu)\phi_+ - ((\partial_0 + i\mu)\phi_+)^\dagger\phi_+] \\ &+ i[\phi_-^\dagger(\partial_0 - i\mu)\phi_- - ((\partial_0 - i\mu)\phi_-)^\dagger\phi_-] \\ &+ \bar{\eta}\gamma^0\eta + i[q_+^\dagger(\partial_0 - i\mu)q_+ - ((\partial_0 - i\mu)q_+)^\dagger q_+] \\ &+ i[q_-^\dagger(\partial_0 + i\mu)q_- - ((\partial_0 + i\mu)q_-)^\dagger q_-]. \end{aligned} \tag{4.13}$$

If $Q \neq 0$ in the ground state, the system would not be electrically neutral. As explained above, this would not be physically sensible, since the infinite-volume limit would come with a divergent energetic cost. More formally, one can see that the situation when $\langle Q \rangle \neq 0$

would be problematic because then the action for A_μ would involve a tadpole term for A_0. Once one adjusts μ_i to set $\langle Q \rangle = 0$, so that the ground state is electrically neutral, the action for A_μ becomes quadratic.

We start by considering the scalar sector, and look for ground states in which the bosonic fields get time-independent vacuum expectation values, so that $\partial_0 \phi_\pm = \partial_0 q_\pm = 0$. We use unitary gauge in our analysis, so that if any of the scalars (which are all charged under $U(1)_Q$) condense, the gauge bosons pick up a mass via the Higgs mechanism. If two scalars condense in such a way that both $U(1)_e$ and $U(1)_i$ are broken, then one of the would-be Goldstone bosons will be eaten by the gauge field in unitary gauge, but the other will remain as a bona-fide physical gapless Goldstone mode.

If we take $\mu_e = -\mu_i \equiv \mu$, then we get the tree-level matter sector scalar potential

$$\begin{aligned} V_{\text{eff}}^{(0)} &= \left(|m|^2 - \mu^2\right)\left(|\phi_+|^2 + |\phi_-|^2 + |q_+|^2 + |q_-|^2\right) \\ &+ \frac{g^2}{2}\left(|\phi_+|^2 - |\phi_-|^2 + |q_+|^2 - |q_-|^2 - \xi^2\right)^2. \end{aligned} \tag{4.14}$$

To develop a heuristic understanding of the scalar field ground states, it is instructive to rewrite the potential as

$$\begin{aligned} V_{\text{eff}}^{(0)} &= \left(|m|^2 - \mu^2 - g^2\xi^2\right)\left(|\phi_+|^2 + |q_+|^2\right) + \left(|m|^2 - \mu^2 + g^2\xi^2\right)\left(|\phi_-|^2 + |q_-|^2\right) \\ &+ \frac{g^2}{2}\left(|\phi_+|^2 + |q_+|^2 - |\phi_-|^2 - |q_-|^2\right)^2 + \frac{g^2}{2}\xi^4. \end{aligned} \tag{4.15}$$

Now suppose that $\xi^2 > 0$, and consider $m^2_{\phi_+,q_+}$ and $m^2_{\phi_-,q_-}$ while we slowly increase μ from 0. (What would happen if $\xi^2 < 0$ can be read off from the following discussion by exchanging ϕ_+, q_+ with ϕ_-, q_-.) When $m^2 - \mu^2 > g^2\xi^2 > 0$, we have $m^2_{\phi_+,q_+} > 0$ and $m^2_{\phi_-,q_-} > 0$, so none of the scalars condense. That is, all of the scalar VEVs are zero. This regime of the theory is not interesting for our purposes, since the scalar sector does not respond non-trivially to the chemical potential. Moreover, given that in this regime $\mu^2 < |m|^2$ and there is no scalar condensation to leading order in g, the fermion sector responds to μ in the same way as a free theory would - which is to say, no spinor electrons or ions populate the vacuum either.

Next, suppose that $-g^2\xi^2 < m^2 - \mu^2 < g^2\xi^2$. Then $m^2_{\phi_+,q_+} < 0$ while $m^2_{\phi_-,q_-} > 0$, and ϕ_+, q_+ will develop non-trivial VEVs, and minimization of the scalar potential naïvely implies that they must satisfy

$$|\phi_+|^2 + |q_+|^2 = \frac{\mu^2 - m^2 + \xi^2 g^2}{g^2}, \qquad |\phi_-|^2 = |q_-|^2 = 0, \tag{4.16}$$

Plugging these VEVs back into the potential to get a feeling for what happens to ϕ_-, q_-, we find that $m^2_{\phi_-,q_-}$ vanishes due to contributions from cross-terms in the potential linking

ϕ_+, q_+ with ϕ_-, q_-. This means that one should do a more careful analysis to understand the regime in which it is consistent to assume that ϕ_+ and q_+ are condensed, but ϕ_- and q_- are not. Computing the eigenvalues λ_1, λ_2 of the Hessian matrix describing the fluctuations around the VEVs in Eq. (4.16) yields

$$\lambda_1 = m^2 - \mu^2 - g^2\xi^2 + g^2(|\phi_+|^2 + |q_+|^2), \tag{4.17}$$

$$\lambda_2 = m^2 - \mu^2 + g^2\xi^2 - g^2(|\phi_+|^2 + |q_+|^2). \tag{4.18}$$

Demanding that $\lambda_1, \lambda_2 > 0$, so that our field configuration is stable, implies that we must ensure that $m^2 > \mu^2$. Hence we learn that so long as $0 < m^2 - \mu^2 < g^2\xi^2$, ϕ_+, q_+ are condensed and must obey Eq. (4.16), but ϕ_-, q_- do not condense. In this regime we expect a non-trivial scalar ground state, and we do not have to worry about run-away directions in the potential. But once $m^2 - \mu^2 < 0$, all of the scalar fields are free to develop non-zero VEVs. Given the form of the potential, there is clearly a run-away direction in the potential along $\phi_+ = q_+ = \phi_- = q_-$, so the system has no stable ground state once $\mu^2 > m^2$.

Given the remarks above, we can simplify the discussion without loss of generality by assuming that $\xi^2 > 0$ from here onwards. We still have to take the constraint of charge neutrality into account. The scalar contribution to Q is

$$Q|_{\text{scalar}} = 2\mu_e|\phi_+|^2 - 2\mu_e|\phi_-|^2 + 2\mu_i|q_+|^2 - 2\mu_i|q_-|^2, \tag{4.19}$$

which becomes

$$Q|_{\text{scalar}} = 2\mu\left(|\phi_+|^2 - |q_+|^2\right). \tag{4.20}$$

If we now demand that $Q|_{\text{scalar}} \stackrel{!}{=} 0$, we find that

$$|\phi_+|^2 = |q_+|^2 = \frac{\mu^2 - m^2 + \xi^2 g^2}{2g^2}, \quad \phi_- = q_- = 0. \tag{4.21}$$

Although here we have focused on the selectrons and sions, it is clear that the symmetric way μ enters the action guarantees that if the fermionic electron and ion fields contribute to the charge density, they do so in such a way that the sum of their electric charges is separately zero. This is the reason that we are able to demand that the scalar contribution to the electric charge vanishes separately from the one from the fermions.

4.3.2 Search for a Fermi surface

We now examine the fermionic part of the action to see whether the fermions organize into a Fermi sphere at $\mu > 0$. Of course, in view of the discussion above, while looking for a Fermi surface, we have to always assume the condition

$$0 < m^2 - \mu^2 < g^2\xi^2. \tag{4.22}$$

In particular, we emphasise that $\mu^2 < m^2$ throughout this range. If we were to consider $\mu^2 > m^2$, the scalar sector would have no stable ground state. On the other hand, if $\mu^2 < m^2$ but μ were to go outside the bound in Eq. (4.22), the scalars would have vanishing VEVs. But then because at the same time μ would be smaller than the fermion mass, the ground state could not possibly carry any $U(1)_e$ charge. So insisting on the condition in Eq. (4.22) is essential to keep things interesting.

We recall that to see whether a system has a Fermi surface to leading order in perturbation theory, one can examine the dispersion relations for the fermions. For instance, for a free Dirac fermion with Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m + \mu\gamma^0\right)\psi = \bar{\psi}M\psi, \tag{4.23}$$

this can be done by finding the momentum-space eigenvalues $\lambda_i(p_0, p)$ of M, and then solving $\lambda_i(p) = 0$ for p_0 in terms of p. This yields the dispersion relations $p_0 = \epsilon(p)$, where ϵ is the energy density, for the fermion and anti-fermion modes determined in the free case by $(p_0 - \mu)^2 = p^2 + m^2$. A Fermi surface can be defined as the solution to $0 = p_0 = \epsilon(p)$ for some $p = p_F > 0$. For a free fermion, we obtain $p_F^2 = \mu^2 - m^2$. Our task in this section is to carry out this simple procedure for the somewhat baroque fermion sector of sQED.

In four-component spinor notation, the fermion part of the $\mathcal{N} = 1$ sQED Lagrangian is

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=1}|_{\text{fermion}} = {} & \frac{1}{2}\bar{\lambda}i\partial\!\!\!/\lambda + \bar{\psi}\left(iD\!\!\!\!/^{-} - m\right)\psi + \bar{\eta}\left(iD\!\!\!\!/^{-} - m\right)\eta \\ & + \sqrt{2}ig\left(\phi_+^\dagger\bar{\psi}P_-\lambda - \phi_-^\dagger\bar{\lambda}P_-\psi - \phi\bar{\lambda}P_+\psi + \phi_-\bar{\psi}P_+\lambda\right) \\ & + \sqrt{2}ig\left(q_+^\dagger\bar{\eta}P_-\lambda - q_-^\dagger\bar{\lambda}P_-\eta - q\bar{\lambda}P_+\eta + q\bar{\eta}P_+\lambda\right),\end{aligned} \tag{4.24}$$

where

$$D_\mu^-\psi = \partial_\mu - i\mu\delta_{\mu,0} - igA_\mu, \qquad D_\mu^-\eta = \partial_\mu + i\mu\delta_{\mu,0} - igA_\mu \tag{4.25}$$

In view of our discussion in Section 4.2 and the response of the scalar sector to the chemical potential, once the scalar fields develop non-trivial VEVs in Eq. (4.21) all of the fermionic fields mix with each other, with the mixing between electron and ion fields mediated by the photino. Moreover, if for simplicity we scale ξ as $\xi \sim 1/g$, the mixing is g-independent. It is thus difficult to understand the response of the fermions to the chemical potential through a visual examination of Eq. (4.24), in contrast to the free case in Eq. (4.23).

To look for a Fermi surface, we want to compute the dispersion relations of the fermionic eigenmodes described by Eq. (4.24). This is easier if we switch to two-component spinor notation,

$$\psi = \begin{pmatrix}\psi_{L\alpha} \\ \psi_R^{\dagger\dot{\alpha}}\end{pmatrix}, \qquad \eta = \begin{pmatrix}\eta_{L\alpha} \\ \eta_R^{\dagger\dot{\alpha}}\end{pmatrix}, \qquad \lambda = \begin{pmatrix}\lambda_\alpha \\ \lambda^{\dagger\dot{\alpha}}\end{pmatrix}, \tag{4.26}$$

where λ is the Majorana photino and we introduce the standard matrices $\sigma^{\mu}_{\alpha\dot{\alpha}} = (I_2, \boldsymbol{\sigma})$ and $\bar{\sigma}^{\mu\dot{\alpha}\alpha} = (I_2, -\boldsymbol{\sigma})$.

The fact that the VEVs are given by Eq. (4.21) means that one can write the quadratic fermion action in terms of a 5×5 matrix, without the need to introduce Nambu-Gorkov-type spinors. Defining

$$\Psi = \begin{pmatrix} \psi_{L\alpha} & \psi_R^{\dagger\dot{\alpha}} & \lambda_\alpha & \eta_{L\alpha} & \eta_R^{\dagger\dot{\alpha}} \end{pmatrix}^T, \tag{4.27}$$

we can now write

$$\mathcal{L}_{\mathcal{N}=1}|_{\text{fermion}} = \Psi^\dagger \cdot M_{\mathcal{N}=1} \cdot \Psi, \tag{4.28}$$

where

$$M_{\mathcal{N}=1} = \begin{pmatrix} i\bar{\sigma}^\mu\left(\partial_\mu - i\mu\delta_{\mu 0}\right) & -mI_2 & 0 & 0 & 0 \\ -mI_2 & i\sigma^\mu\left(\partial_\mu - i\mu\delta_{\mu 0}\right) & ig\sqrt{2}\phi_+^\dagger & 0 & 0 \\ 0 & -ig\sqrt{2}\phi_+ & i\bar{\sigma}^\mu\partial_\mu & 0 & -ig\sqrt{2}q_+ \\ 0 & 0 & 0 & i\bar{\sigma}^\mu\left(\partial_\mu + i\mu\delta_{\mu 0}\right) & -mI_2 \\ 0 & 0 & ig\sqrt{2}q_+^\dagger & -mI_2 & i\sigma^\mu\left(\partial_\mu + i\mu\delta_{\mu 0}\right) \end{pmatrix}. \tag{4.29}$$

After going to momentum space we can compute the determinant of $M_{\mathcal{N}=1}$. Lorentz invariance is broken by μ, but rotational invariance is unbroken and hence $\det(M_{\mathcal{N}=1})$ must depend on p_0 and $p = \sqrt{p_1^2 + p_2^2 + p_3^2}$. The dispersion relations may be found by solving $\det(M_{\mathcal{N}=1}) = 0$ for p_0 as a function of p, but they are complicated and their form unilluminating. Fortunately, once we set $p_0 = 0$, as is needed in the search for the Fermi surface, they simplify and give

$$\det(M_{\mathcal{N}=1})|_{p_0=0} = -p^2\left(-\mu^2 + m^2 + p^2\right)^4. \tag{4.30}$$

The contribution of the selectron and sion VEVs to $\det(M_{\mathcal{N}=1})|_{p0=0}$ cancels thanks to charge neutrality. Amusingly, what is left has the form which we would have obtained by dropping the Yukawa terms in the first place! We emphasise that this dramatic simplification happens only at $p_0 = 0$.

Looking for values of $p = p_F > 0$ which make $\det(M_{\mathcal{N}=1})|_{p_0=0}$ vanish, at first glance $p = \sqrt{\mu^2 - m^2}$ may seem work. But as we have seen, the scalar sector is under control only for $\mu < m$, and indeed we have assumed the condition in Eq. (4.22) at the start of the fermion analysis. So $p = \sqrt{\mu^2 - m^2}$ is *not* a legitimate solution of $\det(M_{\mathcal{N}=1})|_{p_0=0} = 0$. But there are no other solutions to $\det(M_{\mathcal{N}=1})|_{p_0=0} = 0$.

Thus we conclude that within the domain of validity of our analysis, there is no $p = p_F > 0$ for which $\det(M_{\mathcal{N}=1})|_{p_0=0}$ vanishes, and hence there is no Fermi surface in finite-density $\mathcal{N} = 1$ sQED at weak coupling. Note also that changing the strength of the Yukawa couplings (which would break supersymmetry) would not change this result due to the structure of the determinant above.

4.3.3 Non-supersymmetric cousin of $\mathcal{N} = 1$ sQED

Before proceeding to $\mathcal{N} = 2$ sQED, it is instructive to discuss what would have happened if we had not insisted on charge neutrality, for instance by working with only the electron-sector fields. The point of considering this example is to emphasise that $U(1)$ breaking does *not* necessarily lead to the obliteration of Fermi surfaces. One way to make this reasonable would be to modify the Lagrangian by erasing the gauge field while leaving everything else untouched. Then the Lagrangian would be

$$\begin{aligned}\mathcal{L}_{\text{no ions}} = \mathcal{L}_{\mathcal{N}=1}|_{\text{fermion}} + \left|D^-_\mu \phi_-\right|^2 + \left|D^+_\mu \phi_+\right|^2 - |m\phi_-|^2 - |m\phi_+|^2 \\ - \frac{g^2}{2}\left(|\phi_+|^2 - |\phi_-|^2 - \xi^2\right)^2,\end{aligned} \tag{4.31}$$

with the ion fields deleted from $\mathcal{L}_{\mathcal{N}=1}|_{\text{fermion}}$. Deleting the gauge fields breaks SUSY.

Going through the same analysis as above, we now obtain

$$\begin{aligned}\det(M_{\text{no ions}})_{p_0=0} = 4g^4 |\phi_+|^4 \left(\mu^2 - p^2\right) \\ - p^2 \left(-\mu^2 + m^2 + p^2\right)\left(4g^2|\phi_+|^2 - \mu^2 + m^2 + p^2\right),\end{aligned} \tag{4.32}$$

with $g^2|\phi_+|^2 = m^2 - \mu^2 + g^2\xi^2$. Solving $\det(M_{\text{no ions}})|_{p_0=0} = 0$, we obtain a solution for the Fermi momentum:

$$p_F = \frac{\left[c + 27\mu\left(g^2\xi^2 - \mu^2 + m^2\right)\right]^{2/3} - 6g^2\xi^2 + 9(\mu^2 - m^2)}{3\left[c + 27\mu\left(g^2\xi^2 - \mu^2 + m^2\right)\right]^{1/3}}, \tag{4.33}$$

where $c \equiv \sqrt{\left(6g^2\xi^2 - 9\mu^2 + 9m^2\right)^3 + 729\mu^2\left(g^2\xi^2 - \mu^2 + m^2\right)^2}$. Since these expressions are rather complicated, we plot Eq. (4.33) in Fig. 4.1. The plot shows this non-supersymmetric system *does* have a Fermi surface, in contrast to the supersymmetric system we considered above. Note that the Yukawa terms are essential to this result, since here we are still considering $\mu < m$, so that without the mixing terms the fermions would be free to leading order, and would not develop a Fermi surface until $\mu > m$.

However, as we have seen, when electric neutrality is taken into account, as it *must* be in $\mathcal{N} = 1$ sQED, the story is very different.

4.4 Softly broken $\mathcal{N} = 2$ sQED at finite electron number density

4.4.1 Scalar Ground State

We start by attempting to work with the most obvious $\mathcal{N} = 2$ generalisation of our $\mathcal{N} = 1$ toy model. As the field content of our $\mathcal{N} = 2$ sQED model, we will use essentially the

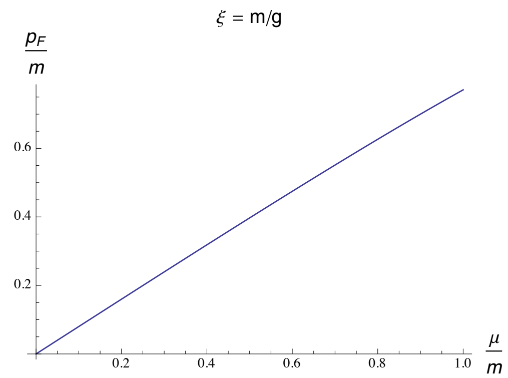

Figure 4.1: A solution for the Fermi momentum p_F which would have been obtained if we had ignored the constraint of charge neutrality, and worked with a system including electrons only. For simplicity we set $\xi = m/g$. In this case the infinite volume does not make sense, unless one modifies the theory by removing the gauge fields but keeping all else fixed.

same chiral 'electron' and 'ion' super fields as the $\mathcal{N} = 1$ model, with the following changes. First, we must add an extra 'adjoint' $\mathcal{N} = 1$ chiral multiplet Λ which contains an extra Majorana photino χ and a scalar a, which combines with the $\mathcal{N} = 1$ vector multiplet to form the $\mathcal{N} = 2$ vector hypermultiplet. Second, the scalar fields from the $\mathcal{N} = 1$ chiral multiplets, ϕ_+ and $\phi_-^\dagger$, combine to form a single $\mathcal{N} = 2$ matter hypermultiplet.[8] The same goes for the sion fields. Finally, to be consistent with $\mathcal{N} = 2$ supersymmetry, the superpotential must be modified to (in $\mathcal{N} = 1$ language)

$$\mathcal{W} = m\left(\Phi_+\Phi_- + Q_+Q_-\right) + \sqrt{2}\Lambda\left(\Phi_+\Phi_- + Q_+Q_-\right), \tag{4.34}$$

where Φ_+, Φ_- are the electron-multiplet superfields and Q_+, Q_- are the ion-sector superfields. The tree-level Kahler potential is the same as before with the obvious changes to account for the discussion above. We continue to include the FI term in the theory. This $\mathcal{N} = 2$ gauge theory has the scalar potential

$$\begin{aligned} V_{\text{eff}}^{(0)} &= \left|\sqrt{2}ga + m\right|^2 \left(|\phi_+|^2 + |\phi_-|^2 + |q_+|^2 + |q_-|^2\right) + 2g^2\left(\phi_+\phi_- + q_+q_-\right)\left(\phi_+^\dagger\phi_-^\dagger + q_+^\dagger q_-^\dagger\right) \\ &+ \frac{g^2}{2}\left(|\phi_+|^2 - |\phi_-|^2 + |q_+|^2 - |q_-|^2 - \xi^2\right)^2 - \mu^2\left(|\phi_+|^2 + |\phi_-|^2 + |q_+|^2 + |q_-|^2\right), \end{aligned} \tag{4.35}$$

and has the same $U(1)_e \times U(1)_i$ symmetry as the $\mathcal{N} = 1$ theory, but also has an $SU(2)_R$ non-anomalous R-symmetry. We explore the response of $\mathcal{N} = 2$ sQED to an R-charge chemical potential in Section 4.5, and focus on the $U(1)_e \times U(1)_i$ symmetries here. The transformation properties of the matter fields are given in Table 4.2. Recalling the comments about the way ϕ_+ and $\phi_-^\dagger$ enter the $\mathcal{N} = 2$ theory above, and noting that the fields a and ξ do not contribute to the electric charge density, we find that the gauged (electric)

[8]The hypermultiplet contains the *conjugate* of ϕ_- since the gauge generators commute with the supersymmetry generators and hence all fields within a multiplet have the same gauge charges.

Transformation Properties in $\mathcal{N} = 2$ sQED

Fields:	ψ	ϕ_+	ϕ_-	η	q_+	q_-	λ	χ	a
$U(1)_e$	$e^{-i\alpha}\psi$	$e^{+i\alpha}\phi_+$	$e^{-i\alpha}\phi_-$	η	q_+	q_-	λ	χ	a
$U(1)_i$	ψ	ϕ_+	ϕ_-	$e^{-i\alpha}\eta$	$e^{+i\alpha}q_+$	$e^{-i\alpha}q_-$	λ	χ	a

Table 4.2: Matter field transformation properties under the $U(1)_e$ and $U(1)_i$ symmetries.

charge density is unchanged from Eq. (4.13),

$$\begin{aligned} Q = & -\bar{\psi}\gamma^0\psi + i[\phi_+^\dagger(\partial_0 + i\mu)\phi_+ - ((\partial_0 + i\mu)\phi_+)^\dagger\phi_+] \\ & + i[\phi_-^\dagger(\partial_0 - i\mu)\phi_- - ((\partial_0 - i\mu)\phi_-)^\dagger\phi_-] \\ & + \bar{\eta}\gamma^0\eta + i[q_+^\dagger(\partial_0 - i\mu)q_+ - ((\partial_0 - i\mu)q_+)^\dagger q_+] \\ & + i[q_-^\dagger(\partial_0 + i\mu)q_- - ((\partial_0 + i\mu)q_-)^\dagger q_-]. \end{aligned} \tag{4.36}$$

Unfortunately, it turns out that once μ is turned on the scalars do not have a stable ground state, since there are run-away directions in the scalar potential. The quickest way to see this is to observe that minimizing V_{eff} for a implies that a picks up a VEV

$$\langle a \rangle = -\frac{m}{\sqrt{2}g}. \tag{4.37}$$

Heuristically, apart from the surviving group of terms in the first line of Eq. (4.35), the potential for $\phi_{1,2}$, $q_{1,2}$ is the same as the massless limit of the potential in the $\mathcal{N} = 1$ case, for which there would be no stable solutions once $\mu > 0$, even when a FI term is present. The new terms demanded by $\mathcal{N} = 2$ do not save the day if there is more than one flavour hypermultiplet.

We have not figured out a way to prevent the emergence of run-away directions in the scalar potential in two-flavour $\mathcal{N} = 2$ sQED, but it is possible to get some insight into what the supersymmetric interactions do to Fermi surfaces by modifying the theory above in two simple ways:

A: Work with $\mathcal{N} = 2$ sQED with only one flavour. This means giving up on electric neutrality, and requires a hard breaking of supersymmetry to be sensible in the infinite-volume limit, much as in Section 4.3.3. We defer a discussion of this case in Section 4.4.3.

B: Keep the ion fields, but add some soft SUSY-breaking terms.

Given the title of this section, we proceed with option B, and work with a theory defined by

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=2} + m_s^2\left(|\phi_+|^2 + |\phi_-|^2 + |q_+|^2 + |q_-|^2\right), \tag{4.38}$$

where $\mathcal{L}_{\mathcal{N}=2}$ is the Lagrangian of $\mathcal{N} = 2$ sQED with electron and ion superfields we presented above, and m_s is the soft SUSY-breaking mass.

Minimizing the softly-broken scalar potential with respect to a, we again get $\langle a \rangle = -\frac{m}{\sqrt{2}g}$. The condition for the remaining scalars to have a stable condensate is

$$m_s^2 - g^2\xi^2 < \mu^2 < m_s^2, \tag{4.39}$$

where m_s is the soft mass we introduced above. If the lower bound is violated none of the scalars condense, while if the upper bound is violated there is a runaway direction. If Eq. (4.39) is satisfied, the scalar VEVs must obey the relations

$$|\phi_+|^2 + |q_+|^2 = \frac{\mu^2 - (m_s^2 - g^2\xi^2)}{g^2}, \qquad |\phi_-| = |q_-| = 0. \tag{4.40}$$

Taking into account the electric neutrality constraint means that the scalar VEVs become

$$|\phi_+|^2 = |q_+|^2 = \frac{\mu^2 - (m_s^2 - g^2\xi^2)}{2g^2}, \qquad |\phi_-| = |q_-| = 0. \tag{4.41}$$

4.4.2 Search for a Fermi surface

The fermionic terms in the Lagrangian are the same as in Eq. (4.24) together with the additional terms

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=2}|_{\text{fermions}} = \mathcal{L}_{\mathcal{N}=1}|_{\text{fermions}} &+ \frac{1}{2}\bar{\chi} i\partial\!\!\!/ \chi - \sqrt{2}g\left(a\bar{\psi}P_-\psi + a^\dagger\bar{\psi}P_+\psi + a\bar{\eta}P_-\eta + a^\dagger\bar{\eta}P_+\eta\right) \\ &- \sqrt{2}g\left(\phi_-\bar{\psi}P_-\chi + \phi_+\bar{\chi}P_-\psi + \phi_-^\dagger\bar{\chi}P_+\psi + \phi_+^\dagger\bar{\psi}P_+\chi\right) \\ &- \sqrt{2}g\left(q_-\bar{\eta}P_-\chi + q_+\bar{\chi}P_-\eta + q_-^\dagger\bar{\chi}P_+\eta + q_+^\dagger\bar{\eta}P_+\chi\right).\end{aligned} \tag{4.42}$$

Again, once the scalars pick up VEVs, all of the fermionic fields mix with each other, and seeing the effect of the chemical potential requires diagonalising the kinetic operator. To look for a Fermi surface, paralleling the approach of Section 4.3.2, we introduce a single column vector collecting all of our two-component spinors

$$\Psi = \left(\psi_{L\alpha} \;\; \psi_R^{\dagger\dot{\alpha}} \;\; \lambda_\alpha \;\; \chi^{\dagger\dot{\alpha}} \;\; \eta_{L\alpha} \;\; \eta_R^{\dagger\dot{\alpha}}\right)^T. \tag{4.43}$$

This allows us to rewrite Eq. (4.42) as

$$\mathcal{L}_{\mathcal{N}=2}|_{\text{fermion}} = \Psi^\dagger \cdot M_{\mathcal{N}=2} \cdot \Psi, \tag{4.44}$$

where

$$M_{\mathcal{N}=2} = \begin{pmatrix} i\bar{\sigma}^\mu(\partial_\mu - i\mu\delta_{\mu 0}) & 0 & 0 & -g\sqrt{2}\phi_+^\dagger & 0 & 0 \\ 0 & i\sigma^\mu(\partial_\mu - i\mu\delta_{\mu 0}) & ig\sqrt{2}\phi_+^\dagger & 0 & 0 & 0 \\ 0 & -ig\sqrt{2}\phi_+ & i\bar{\sigma}^\mu\partial_\mu & 0 & 0 & -ig\sqrt{2}q_+ \\ -g\sqrt{2}\phi_+ & 0 & 0 & i\sigma^\mu\partial_\mu & -g\sqrt{2}q_+ & 0 \\ 0 & 0 & 0 & -g\sqrt{2}q_+^\dagger & i\bar{\sigma}^\mu(\partial_\mu + i\mu\delta_{\mu 0}) & 0 \\ 0 & 0 & ig\sqrt{2}q_+^\dagger & 0 & 0 & i\sigma^\mu(\partial_\mu + i\mu\delta_{\mu 0}) \end{pmatrix}. \tag{4.45}$$

Going to momentum space, calculating $\det(M_{\mathcal{N}=2})$ and setting $p_0 = 0$, we obtain

$$\begin{aligned}\det(M_{\mathcal{N}=2})|_{p_0=0} = {}& p^2 \left(4g^2|\phi_+|^2 - \mu^2 + p^2\right)^2 \\ & \times \left(2g^2|\phi_+|^2(\mu+p) + (p-\mu)\left(2g^2|q_+|^2 + p(\mu+p)\right)\right) \\ & \times \left((p-\mu)\left(2g^2|\phi_+|^2 + p(\mu+p)\right) + 2g^2|q_+|^2(\mu+p)\right), \end{aligned} \tag{4.46}$$

where we have used the charge neutrality relation between the scalar VEVs. If the scalars condense, we can plug in Eq. (4.41) to get

$$\det(M_{\mathcal{N}=2})|_{p_0=0} = p^4 \left(\mu^2 - 2m_s^2 + 2g^2\xi^2 + p^2\right)^4 . \tag{4.47}$$

Looking for a value of $p \neq 0$ which would make this vanish, we find that p_F would have to satisfy the relation

$$p_f^2 = 2m_s^2 - \mu^2 - 2g^2\xi^2 \overset{!}{>} 0. \tag{4.48}$$

This relation will be satisfied if

$$\mu^2 < 2(m_s^2 - g^2\xi^2). \tag{4.49}$$

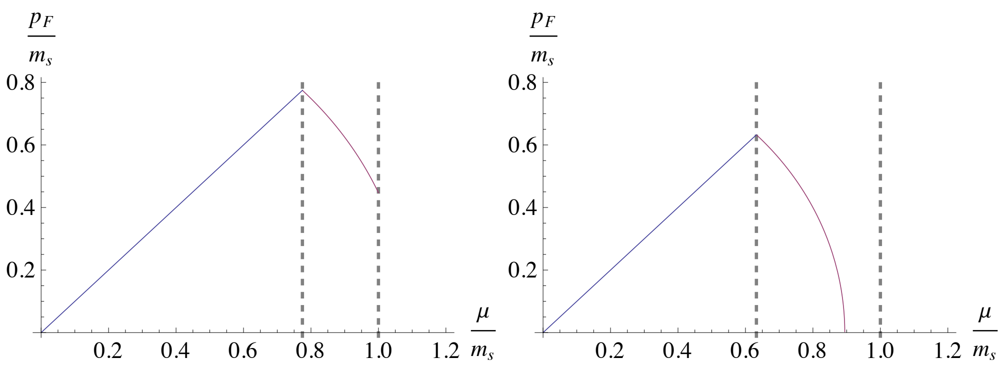

Figure 4.2: **Left:** Fermi momenta as a function of μ with $\frac{g^2\xi^2}{m_s^2} = 0.4$. **Right:** Fermi momenta as a function of μ with $\frac{g^2\xi^2}{m_s^2} = 0.6$. The area between the dashed lines is the region where the scalars are condensed and stable. Values of μ to the right of this region make the scalars unstable, while to the left, the scalars are not condensed. Note that past $g^2\xi^2 > m_s^2$, where the scalars are always condensed, there is no Fermi surface.

We are now in a position to classify all the things that can happen to the fermions in this theory. To begin with, if $\mu^2 < m_s^2 - g^2\xi^2$, then the charged scalars do not condense. The fermion sector consists of massless gauginos and massless matter fermions, which to leading order are free. Since the matter fermions feel the chemical potential, there is a fermi surface at $p_F = \mu$. Since the charged scalars are not condensed, the system is not a superconductor (before considering fermion pairing effects), and it is natural to speculate that the physics in this regime resembles that of conventional QED plasmas.

Next, if $m_s^2 - g^2\xi^2 < \mu^2 < 2m_s^2 - 2g^2\xi^2$ and $\mu^2 < m_s^2$, the theory is stable, the charged scalars are condensed, so that the quantum liquid is a superconductor, and there is a Fermi surface.

If $2m_s^2 - 2g^2\xi^2 < \mu^2 < m_s^2$, the scalar sector is stable, with the charged scalars condensed and hence a broken $U(1)_Q$, so that the system is a superconductor. But now there is no Fermi surface.

Finally, if $m_s^2 < \mu^2$, the scalar sector becomes unstable, and there does not appear to be a sensible finite-density ground state.

To help visualise the behaviour of the Fermi surfaces in this theory as a function the parameters, see Fig 4.2. As seen in the plots, as the scalar condensates get larger, the Fermi momentum decreases. naïvely, one could interpret Fig. 4.2 as implying that more and more of the charge in the system leaks from the fermions into the scalars as μ is increased enough to make the scalar condensate start growing. But see Section 4.6 for a result which suggests that this is not necessarily the case.

4.4.3 Non-supersymmetric cousin of $\mathcal{N} = 2$ sQED

We now briefly return to Option A from Section 4.4.1, where we start with $\mathcal{N} = 2$ sQED with one matter hypermultiplet, and delete the gauge fields just as in Section 4.3.3 to avoid problems with electric neutrality. This is a hard breaking of supersymmetry.

The scalar potential is now

$$V_{\text{eff}}^{(0)} = \left|\sqrt{2}ga + m\right|^2 \left(|\phi_+|^2 + |\phi_-|^2\right) + 2g^2|\phi_+|^2|\phi_-|^2 + \frac{g^2}{2}\left(|\phi_+|^2 - |\phi_-|^2 - \xi\right)^2 - \mu^2\left(|\phi_+|^2 + |\phi_-|^2\right). \tag{4.50}$$

The VEV of a is still given by Eq. (4.37), but now there is a stable minimum for the other scalar fields as well, as can be seen by rewriting the potential in the manner of Eq. (4.15). If $\xi^2 > 0$, minimizing $V_{\text{eff}}^{(0)}$ leads to

$$|\phi_+|^2 = \frac{\mu^2 + g^2\xi^2}{g^2}, \qquad |\phi_-|^2 = 0, \tag{4.51}$$

while if $\xi^2 < 0$, we get

$$|\phi_-|^2 = \frac{\mu^2 - g^2\xi^2}{g^2}, \qquad |\phi_+|^2 = 0. \tag{4.52}$$

(At $\xi = 0$, both scalar fields can condense, but for simplicity we do not consider this case further.) As we have been saying, in this case there is no way to solve the charge neutrality constraint within the scalar sector. If it were possible to adjust the chemical potential which couples to the electrons *independently* from the one which couples to the selectrons, one could imagine that this electron chemical potential could be dialed in such a way that the electrons would carry a charge density which precisely compensates that of the scalars. But the structure of our supersymmetric theory does not allow us to introduce such an independent chemical potential for the electrons, because the Yukawa interactions do not respect the $U(1)$ electron-number symmetry of the free action.

Hence, the solutions obtained in this section cannot yield an electrically-neutral background. Of course, since we have deleted the gauge fields from the theory with malice aforethought, this is not a problem.

We now start the search for a Fermi surface for this non-supersymmetric theory. Again, the diagonalisation of the fermion sector after scalar condensation is much easier if we switch to two-component notation. So long as $\xi^2 \neq 0$, $\mathcal{L}_{\mathcal{N}=2}|_{\text{fermions}}$ can be written in a matrix notation without introducing Nambu-Gorkov spinors, but at $\xi = 0$ we expect all of the scalar fields to develop non-trivial VEVs, making the analysis more involved. To keep things as simple as possible, we only discuss the $\xi^2 \neq 0$ case in this chapter. Moreover, as our previous discussion makes clear, to understand what happens for $\xi^2 \neq 0$ we can focus on $\xi^2 > 0$.

Paralleling the approach of Section 4.3.2, we introduce a single column vector collecting all of the two-component spinors

$$\Psi^{(1)} = \left(\psi_{L\alpha} \;\; \psi_R^{\dagger\dot{\alpha}} \;\; \lambda_\alpha \;\; \chi^{\dagger\dot{\alpha}}\right)^T . \tag{4.53}$$

We rewrite Eq. (4.42) as

$$\mathcal{L}_{\mathcal{N}=2}|_{\text{fermion}} = [\Psi^{(1)}]^\dagger \cdot M^{(1)}_{\mathcal{N}=2} \cdot \Psi^{(1)}, \tag{4.54}$$

with

$$M^{(1)}_{\mathcal{N}=2} = \begin{pmatrix} i\bar{\sigma}^\mu \left(\partial_\mu - i\mu\delta_{\mu 0}\right) & 0 & 0 & -g\sqrt{2}\phi_+^\dagger \\ 0 & i\sigma^\mu \left(\partial_\mu - i\mu\delta_{\mu 0}\right) & ig\sqrt{2}\phi_+^\dagger & 0 \\ 0 & -ig\sqrt{2}\phi_+ & i\bar{\sigma}^\mu\partial_\mu & 0 \\ -g\sqrt{2}\phi_+ & 0 & 0 & i\sigma^\mu\partial_\mu \end{pmatrix} . \tag{4.55}$$

Computing the determinant of $M^{(1)}_{\mathcal{N}=2}$ in frequency-momentum space, we find that the dispersion relations for the fermions are

$$p_0 = \frac{1}{2}\left(-\mu \pm \sqrt{8g^2\phi_+^2 + 4p^2 \pm 4p\mu + \mu^2}\right) . \tag{4.56}$$

But one can now check that there is no value of $p^2 = p_F^2 > 0$ such that there is a solution to the equation above for $p_0 = 0$. Thus there is no Fermi surface if we work with the non-electrically-neutral state in the $\mathcal{N} = 2$ theory with only one flavour hypermultiplet, or in the healthy but non-supersymmetric theory with the gauge fields removed. Note the contrast of this result with what we saw in Section 4.3.3, where the analogous theory had a Fermi surface.

4.5 $\mathcal{N} = 2$ sQED with a finite R-charge density

In this section, we will consider $\mathcal{N} = 2$ sQED with *one* matter hypermultiplet. As we mentioned in the previous section, $\mathcal{N} = 1$ sQED has a $U(2) = U(1)_{\mathcal{R}} \times SU(2)_R$ R-symmetry group. The $U(1)_{\mathcal{R}}$ subgroup is anomalous, whereas the $SU(2)_R$ remains anomaly free. We focus on the anomaly-free symmetry. The $SU(2)_R$ symmetry acts by matrix multiplication on the Weyl doublet $(\lambda_\alpha, \chi_\alpha)$ from the vector hypermultiplet and the charged scalars $(\phi_+, \phi_-^\dagger)$ from the matter hypermultiplet. The remaining fields in the theory are $SU(2)_R$ singlets.

We can describe a system with a net R-charge by introducing a set of chemical potentials μ_n for the R-symmetry charges. Any conserved charges Q_n that one wishes to introduce into the grand canonical partition function change the Hamiltonian by a shift, $\mathcal{H} \to \mathcal{H} - \sum_n \mu_n Q_n$. However, the Q_n charges must commute with each other in order to be simultaneously observable, as discussed in Section 2.1.4. This means that Q_n can only belong to the maximally commuting (Cartan) sub-algebra of the non-Abelian algebra of the charge operators. In our case this must pick a single $U(1)_R \subset SU(2)_R$ to which we associate the chemical potential μ_R. Furthermore, since this is a global un-gauged symmetry we do not have to worry about making the system neutral with respect to $U(1)_R$. Of course, we still have to make sure we maintain electric neutrality!

Define the $SU(2)_R$ doublet fields

$$\Phi \equiv \begin{pmatrix} \phi_+ \\ \phi_-^\dagger \end{pmatrix}, \qquad \Psi_\alpha \equiv \begin{pmatrix} \lambda_\alpha \\ \chi_\alpha \end{pmatrix}. \tag{4.57}$$

Our anomaly-free $U(1)_R$ subgroup acts on these fields as

$$\Phi \to e^{i\alpha\tau_3}\Phi, \qquad \Psi_\alpha \to e^{i\alpha\tau_3}\Psi_\alpha, \tag{4.58}$$

where $\tau_3 = \sigma_3$, the diagonal Pauli matrix.

Hence, the μ_R chemical potential enters the Lagrangian in the following way

$$\mathcal{L} = (\Psi_\alpha)^\dagger \sigma^\mu D_\mu \Psi_\alpha + |D_\mu \Phi|^2 + \ldots, \tag{4.59}$$

where we define[9]

$$D_\mu \Phi = \partial_\mu - i\mu_R \tau_3 \delta_{\mu 0} + igA_\mu, \qquad D_\mu \Psi = \partial_\mu - i\mu_R \tau_3 \delta_{\mu 0}. \tag{4.60}$$

The R-charge density is

$$Q_R = \Psi^\dagger \sigma^0 \tau_3 \Psi + i\left[\Phi^\dagger \left(\partial_0 - i\mu_R \tau_3\right) \tau_3 \Phi - \left[\left(\partial_0 - i\mu_R \tau_3\right) \tau_3 \Phi\right]^\dagger \Phi\right], \tag{4.61}$$

[9]Recall that the fields in the vector hypermultiplet transform in the adjoint representation of the gauge group, and hence are neutral under the Abelian $U(1)$ gauge symmetry, while $\phi_+, \phi_-^\dagger$ inside Φ have the same non-zero electric charge.

while the electric charge density is

$$Q_{EM} = i \left[\Phi^\dagger \left(\partial_0 - i\mu_R \tau_3 \right) \Phi - \left[\left(\partial_0 - i\mu_R \tau_3 \right) \Phi \right]^\dagger \Phi \right]. \tag{4.62}$$

For future reference, note that if ϕ_+, ϕ_- acquire identical time-independent VEVs, then $Q_R \neq 0$, while $Q_{EM} = 0$. This is the key to ensuring that a finite R-charge density does not violate the electric neutrality condition.

4.5.1 Scalar ground state

We look for time-independent scalar ground states, and work in unitary gauge, as we have done throughout the chapter. The bosonic potential with the μ_R contributions included is

$$\begin{aligned} V_{\text{eff}}^{(0)} = & \left| \sqrt{2} g a + m \right|^2 \left(|\phi_+|^2 + |\phi_-|^2 \right) + 2g^2 |\phi_+|^2 |\phi_-|^2 \\ & + \frac{g^2}{2} \left(|\phi_+|^2 - |\phi_-|^2 - \xi^2 \right)^2 - \mu_R^2 \left(|\phi_+|^2 + |\phi_-|^2 \right), \end{aligned} \tag{4.63}$$

where a is the scalar from the vector hypermultiplet. This theory always has a stable non-trivial ground state when $\mu_R \neq 0$, which can be seen from the fact that there is no attractive $|\phi_+|^2 |\phi_-|^2$ term in the potential. Just as before, a picks up the VEV

$$\langle a \rangle = -\frac{m}{\sqrt{2} g}, \tag{4.64}$$

which is independent of ξ. We will see below that charge neutrality requires that we set $\xi^2 = 0$, so we drop ξ from here onwards. Minimizing the scalar potential for the remaining fields we find the condition

$$|\phi_+|^2 + |\phi_-|^2 = \frac{\mu_R^2}{g^2}. \tag{4.65}$$

To see the consequences of electric neutrality, recall that $\phi_-^\dagger$ feels a chemical potential $-\mu_R$ compared to the field ϕ_+ which feels a chemical potential μ_R. Recalling the expression for the electric charge density, it is clear that electric neutrality in the scalar sector will be ensured if they have the same VEVs,[10] leading to

$$|\phi_+|^2 = |\phi_-|^2 = \frac{\mu_R^2}{2g^2}. \tag{4.66}$$

Since these VEVs are non-zero for $\mu_R \neq 0$, and the scalars are charged, the $U(1)$ electromagnetic symmetry is broken, and the system is a superconductor. Of course, the charged scalars also transform non-trivially under $U(1)_R$, so the R symmetry is also spontaneously broken once they develop VEVs. Indeed, since both scalars develop VEVs, the R symmetry is completely broken.

[10] If we had allowed $\xi \neq 0$, then the masses would of ϕ_- and ϕ_+ would be split, and this argument would not work.

4.5.2 Search for a Fermi Surface

Paralleling the approach of the preceding sections, we again introduce a single column vector collecting all of the two-component spinors

$$\Psi^R = \left(\psi_{L\alpha} \;\; \psi_{R\alpha} \;\; \lambda^{\dagger\dot{\alpha}} \;\; \chi^{\dagger\dot{\alpha}}\right)^T , \tag{4.67}$$

and rewriting Eq. (4.42) as

$$\mathcal{L}_{\mathcal{N}=2}|_{\text{fermion}} = [\Psi^R]^\dagger \cdot M^R_{\mathcal{N}=2} \cdot \Psi^R. \tag{4.68}$$

Now, of course, the structure of $M^R_{\mathcal{N}=2}$ is different, since the gauginos feel the R-charge chemical potential, and the matter fermions are rendered effectively massless through the VEV of a, so that

$$M^R_{\mathcal{N}=2} = \begin{pmatrix} i\bar{\sigma}^\mu\partial_\mu & 0 & ig\sqrt{2}\phi_- & -g\sqrt{2}\phi_+^\dagger \\ 0 & i\bar{\sigma}^\mu\partial_\mu & -ig\sqrt{2}\phi_+ & -g\sqrt{2}\phi_-^\dagger \\ -ig\sqrt{2}\phi_-^\dagger & ig\sqrt{2}\phi_+^\dagger & \sigma^\mu\left(i\partial_\mu - \mu_R\delta_{\mu0}\right) & 0 \\ -g\sqrt{2}\phi_+ & -g\sqrt{2}\phi_- & 0 & \sigma^\mu\left(i\partial_\mu + \mu_R\delta_{\mu0}\right) \end{pmatrix}. \tag{4.69}$$

Once we set $\phi_+ = \phi_- = \phi$ in view of Eq. (4.66), the determinant of $M^R_{\mathcal{N}=2}$ takes a relatively simple form. In fact, we find it instructive to write in two different ways. One way to write it is

$$\begin{aligned} \det M^R_{\mathcal{N}=2} &= \left(\left[p_0^2 - p^2\right]\left[(p_0+\mu_R)^2 - p^2\right] + 8g^2|\phi|^2\left[p^2 - p_0(p_0+\mu_R)\right] + 16g^4|\phi|^4\right) \\ &\times \left(\left[p_0^2 - p^2\right]\left[(p_0-\mu_R)^2 - p^2\right] + 8g^2|\phi|^2\left[p^2 - p_0(p_0-\mu_R)\right] + 16g^4|\phi|^4\right). \end{aligned} \tag{4.70}$$

This form makes it easy to see that the $g^2|\phi|^2 = 0$ consistency check is satisfied, where the determinant must reduce to one expected for four massless Weyl fermions, two without chemical potentials, and two with opposite-sign chemical potentials. But the dispersion relations for $g^2|\phi|^2 \neq 0$ are hard to see in this form.

The other way to write $\det M^R_{\mathcal{N}=2}$ is

$$\det M^R_{\mathcal{N}=2} = \prod_{i=1}^{4}\left[(p_0 - \tilde{\mu}_i)^2 - (|\boldsymbol{p}| + \kappa_i)^2 + 4g^2|\phi|^2\right], \tag{4.71}$$

where

$$\tilde{\mu}_{1,2} = \mu_R/2,\; \tilde{\mu}_{3,4} = -\mu_R/2 \qquad \text{and} \qquad \kappa_{1,3} = \mu_R/2,\; \kappa_{2,4} = -\mu_R/2. \tag{4.72}$$

This makes the form of the $g^2|\phi|^2 \neq 0$ dispersion relations for the eigenmodes manifest. These dispersion relations are simple but quite unusual.

Setting $p_0 = 0$ to look for a Fermi surface, we find

$$\det M^R_{\mathcal{N}=2}|_{p_0=0} = \left(p^4 - p^2\left(\mu_R^2 - 8g^2|\phi|^2\right) + 16g^4|\phi|^4\right)^2. \tag{4.73}$$

If $g^2|\phi|^2$ were zero, then there would be a Fermi surface at $p_F^2 = \mu^2$. For general $g^2|\phi|^2$, the Fermi momentum would have to satisfy the relation

$$p_F^2 = \frac{1}{4}\left(\mu_R \pm \sqrt{\mu_R^2 - 16g^2|\phi|^2}\right)^2 > 0. \tag{4.74}$$

In $\mathcal{N} = 2$ sQED, minimizing the scalar potential leads to a VEV $|\phi|^2 = \mu_R^2/(2g^2)$. As a result

$$\det M^R_{\mathcal{N}=2}|_{p_0=0} = \left(4\mu_R^4 + p^4 + 3\mu_R^2 p^2\right)^2, \tag{4.75}$$

which has no real zeros. Hence the fermions in $\mathcal{N} = 2$ sQED with a chemical potential for R-charge do not have a Fermi surface.

It is important to realise that the general structure of the fermion interaction terms in this theory is, in and of itself, compatible with the existence of Fermi surfaces, even after $U(1)$ breaking. What prevents a Fermi surface for the fermions from appearing is the precise relationship between the normalisation of the Yukawa terms and the scalar self-interaction terms, which is dictated by supersymmetry. To see this, consider modifying the Yukawa couplings by changing $g \to g\epsilon$ and leaving everything else, including the scalar sector, unchanged. When $\epsilon = 1$, the theory is supersymmetric, but not otherwise. The potential Fermi momenta are then modified to

$$p_F^2 = \frac{\mu_R^2}{4}\left(1 \pm \sqrt{1 - 8\epsilon^2}\right)^2 > 0. \tag{4.76}$$

Tuning $\epsilon \leq 1/(2\sqrt{2}) < 1$, a Fermi surface appears. Of course, in $\mathcal{N} = 2$ sQED, we are not allowed to vary the Yukawa couplings independently of the scalar potential, and we are stuck with $\epsilon = 1$, where there is no Fermi surface.

4.6 Fermion charge density without a Fermi surface

In the preceding sections we have seen that supersymmetric gauge theories and their cousins often do not have Fermi surfaces, despite the fact that the chemical potential couples to the fermions. How should this result be interpreted? Perhaps the simplest interpretation is that in the Fermi-surface-less examples all of the charge which would normally be stored by the fermions 'leaks out' into the scalars through the Yukawa couplings. In this scenario, when the fermions have no Fermi surface, the charge density would *only* receive contributions from the scalar fields.

In this section we show that this interpretation cannot be correct in general by explicitly computing the charge density Q in a theory with fermions and scalars where no Fermi surface develops at finite μ. The theory we consider in this section is chosen so as to simplify the calculation of the fermionic contribution to Q. We will see that this contribution is non-vanishing.

The general idea of the calculation is to evaluate the $T \to 0$ limit of the fermion contribution to the 'grand potential' $\Omega = u - Ts - \mu Q$, where u is the internal energy density, s is the entropy density, and Q is the particle number density. Ω also obeys

$$\Omega = -\frac{T}{V}\log Z, \tag{4.77}$$

where Z is the grand canonical partition function, T is the temperature, and V is the volume of the system. The contributions to Ω can generically be split into a contribution from fermionic plus a contribution from bosonic energy eigenmodes, so that

$$\Omega = -\Omega_{\text{fermions}} + \Omega_{\text{bosons}}, \tag{4.78}$$

where the minus sign accounts for fermionic statistics when evaluating the fermion determinant in Z. We write Ω_{fermions} and Ω_{bosons} as

$$\begin{aligned}\Omega_{\text{fermions}} = &\sum_{i\,\in\,\text{particles, antiparticles}} \int \frac{d^3p}{(2\pi)^3}\frac{E_{p,i}}{2} + \sum_{i\,\in\,\text{particles}} T\int \frac{d^3p}{(2\pi)^3}\log\left[1+e^{-(E_{p,i}-\mu)/T}\right] \\ &+ \sum_{i\,\in\,\text{antiparticles}} T\int \frac{d^3p}{(2\pi)^3}\log\left[1+e^{-(E_{p,i}+\mu)/T}\right], \end{aligned}\tag{4.79}$$

$$\begin{aligned}\Omega_{\text{bosons}} = &\sum_{i\,\in\,\text{particles, antiparticles}} \int \frac{d^3p}{(2\pi)^3}\frac{E_{p,i}}{2} + \sum_{i\,\in\,\text{particles}} T\int \frac{d^3p}{(2\pi)^3}\log\left[1-e^{-(E_{p,i}-\mu)/T}\right] \\ &+ \sum_{i\,\in\,\text{antiparticles}} T\int \frac{d^3p}{(2\pi)^3}\log\left[1-e^{-(E_{p,i}+\mu)/T}\right]. \end{aligned}\tag{4.80}$$

The dispersion relations $E_{p,i}$ one should use above are the ones appropriate to the interacting theory. The forms above follow from a number of formalisms, with standard statistical mechanics arguments being perhaps the most physically transparent.[11] The charge density can now be defined as

$$Q = -\frac{\partial\Omega}{\partial\mu}. \tag{4.81}$$

Note that the quantity Q defined in this way makes sense even when symmetry associated to μ is spontaneously broken, as in the case of interest below. (Essentially, in the condensed case, QV is the charge carried by a macroscopic condensate with volume V.)

[11] Another way to obtain Eq. (4.80) is to observe that e.g. $\Omega|_{\text{fermion}} = -T\log Z|_{\text{fermion}} = -\text{tr}\log M_D$, where M_D is the appropriate Dirac operator taking into account interaction corrections to the fermion propagators, compute the trace log using one's choice of finite-T formalisms, Matsubara or Schwinger-Keldysh, discussed in Chapter 3, and then take the $T = 0$ limit. Or one may use a $T = 0$ pole prescription (which is derived from the results of the finite-T approach) to evaluate the trace log directly at $T = 0$. No matter the formalism, the result is of course the same.

We define the fermionic contribution to Q as

$$Q_{\text{fermions}} = -\frac{\partial \Omega_{\text{fermions}}}{\partial \mu}. \tag{4.82}$$

So to compute Q_{fermions}, were must therefore first evaluate Ω_{fermions}.

The theory we will focus on has two Majorana fermions λ, χ, one Dirac fermion ψ, and one complex scalar ϕ, with interactions defined by the Lagrangian

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\bar{\lambda}\left(i\partial\!\!\!/ + \mu\gamma_0\right)\lambda + \frac{1}{2}\bar{\chi}\left(i\partial\!\!\!/ - \mu\gamma^0\right)\chi + \bar{\psi}i\partial\!\!\!/\psi + |\left(\partial_\mu + i\mu\delta_{\mu,0}\right)\phi|^2 \\ &+ ig\epsilon\left(\phi^\dagger\bar{\psi}P_-\lambda - \phi^\dagger\bar{\lambda}P_-\psi - \phi\bar{\lambda}P_+\psi + \phi\bar{\psi}P_+\lambda\right) \\ &- g\epsilon\left(\phi\bar{\psi}P_-\chi + \phi\bar{\chi}P_-\psi + \phi^\dagger\bar{\chi}P_+\psi + \phi^\dagger\bar{\psi}P_+\chi\right) - \frac{g^2}{2}|\phi|^4 + \mathcal{L}_{\text{CT}},\end{aligned} \tag{4.83}$$

where g and ϵ are dimensionless parameters characterizing the relative strengths of the scalar self-interactions versus the Yukawa interactions, while μ is a chemical potential for a $U(1)$ symmetry acting as $\phi \to e^{+i\alpha}\phi, \lambda \to e^{-i\alpha}\lambda, \chi \to e^{+i\alpha}\chi$. Finally, $\mathcal{L}_{\text{CT}}$ collects the counter-terms necessary to renormalise the theory

$$\mathcal{L}_{\text{CT}} = (\delta\Lambda_{cc})^4 + (\delta m)^2|\phi|^2 + \dots, \tag{4.84}$$

and we have written only the vacuum energy $(\delta\Lambda_{cc})$ and scalar mass $(\delta m)^2$ counter-terms explicitly since it turns out that they are the only ones we will need to compute Q_{fermions} to the order to which we work.

Our choice of the theory described by Eq. (4.83) is inspired by $\mathcal{N} = 2$ super-QED with a single matter hypermultiplet with mass m and a $U(1)_R$ chemical potential μ_R. Specifically, the version of Eq. (4.83) with $\epsilon = 1$ can be obtained from the $\mathcal{N} = 2$ theory by the relations $A_\mu = 0$, $\phi_+ = \phi_- = \phi/\sqrt{2}$, $a = -m/(g\sqrt{2})$, and $\mu_R = \mu$. For our purposes in this section, the case $\epsilon = 1/\sqrt{2}$ will turn out to be the easiest to analyze. From the discussion at the end of Section 4.5.2, it follows that the fermions in the theory we consider in this section have no Fermi surface so long as $\epsilon > 1/(2\sqrt{2})$, and this is the regime we focus on in this section.

Before looking at the interesting examples of what happens when $\epsilon > 1/(2\sqrt{2})$, we quickly review the textbook calculation of the charge density Q carried by a non-interacting Dirac fermion with a chemical potential μ, which help us stay oriented during calculations in the interacting theory, which work out in an unusual way. Following the discussion above, we write

$$\begin{aligned}-\Omega(T,\mu)_{\text{Dirac}} = 4\int \frac{d^3p}{(2\pi)^3}\frac{E_p}{2} + 2T\int \frac{d^3p}{(2\pi)^3}\log\left[1 + e^{-(E_p-\mu)/T}\right] \\ + 2T\int \frac{d^3p}{(2\pi)^3}\log\left[1 + e^{-(E_p+\mu)/T}\right],\end{aligned} \tag{4.85}$$

where $E_p = \sqrt{p^2 + m^2}$ is the free-fermion dispersion relation. The first term is known as the 'vacuum' contribution, while the second two terms are the 'matter' and 'anti-matter' contributions respectively. The factor of 4 on the vacuum term counts the total number of degrees of freedom (spin up and spin down particle and anti-particle modes), and the factors of 2 on the matter terms have the same origin, accounting for the spin up and down contributions. In the zero-temperature limit, and with $\mu > 0$, this reduces to

$$-\Omega(\mu)_{\text{Dirac}} = 4 \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2} + 2 \int \frac{d^3p}{(2\pi)^3} (\mu - E_p)\theta(\mu - E_p), \tag{4.86}$$

where θ is the Heaviside step function, and $\theta(\mu - E_p) = \theta(p_F)$. Of course, the anti-fermion contribution has dropped out at $T = 0$.

For the free Dirac fermion, the 'vacuum' term is obviously independent of μ, and is irrelevant for the charge density. Setting $m = 0$ for simplicity and evaluating the remaining 'matter' term we obtain

$$-\Omega(\mu)_{\text{Dirac}} = \frac{\mu^4}{12\pi^2} \implies Q_{\text{Dirac}} = \frac{\mu^3}{3\pi^2}, \tag{4.87}$$

which is the standard result [179].

We now turn to the calculation of the fermion contribution to Q in the toy theory described by Eq. (4.83). From Eq. (4.71) and Eq. (4.72), we see that we have four eigenmodes contributing to Ω, with

$$E_{p,i}^2 \equiv (|\boldsymbol{p}| + \kappa_i)^2 + 2\epsilon^2 g^2 |\phi|^2, \quad i = 1, 2, 3, 4, \tag{4.88}$$

with the i-th mode having the chemical potential $\tilde{\mu}_i$, but now we have $|\langle\phi\rangle|^2 = \mu^2/g^2$.[12] Note that $\tilde{\mu}_i$ with $i = 1, 2$ are positive, while $\tilde{\mu}_i$ with $i = 3, 4$ are negative for $\mu > 0$. Also, we observe that Eq. (4.88) describes eight fermionic degrees of freedom, since we have four Weyl fermions coupled to each other when $\epsilon \neq 0$.

These dispersion relations are highly unusual, and are a consequence of the spontaneous $U(1)$ breaking driven by scalar condensation communicated to the fermions through the Yukawa couplings with strength set by ϵ.[13] Hence in addition to exploring the behaviour of the $\epsilon = 1/\sqrt{2}$ theory, we also verify that the $\epsilon \to 0$ limit yields the expected free-fermion results.

We now write down the fermionic contribution to Ω, working with general ϵ for the moment. Note that in view of the signs on the $\tilde{\mu}_i$'s, when writing down the matter contributions to Ω at $T = 0$ we must take into account the particle contributions for the first

[12] The normalisation of ϕ used in this section differs from the one used in Section 4.5, with $\phi_{\text{here}} = \phi_{\text{there}}/2$, so that the kinetic term of ϕ_{here} in Eq. (4.83) is canonically-normalised.

[13] It would be interesting to explore what happens if the $U(1)$ symmetry is broken both spontaneously and explicitly, by $U(1)$-violating mass terms. However, the dispersion relations become very complicated in this case, and the integrals determining the fermion contribution to grand potential Ω appear to become analytically intractable.

two modes, while for the second two modes we have to take into account the antiparticle contributions. Adding up the contributions, we get

$$
\begin{aligned}
-\Omega(\mu)|_{\text{fermion}} = & \sum_{i=1}^{4} 2 \times \int \frac{d^3p}{(2\pi)^3} \frac{E_{p,i}}{2} \\
& + \sum_{i=1}^{2} \int \frac{d^3p}{(2\pi)^3} \left(\tilde{\mu}_i - E_{p,i}\right) \theta(\tilde{\mu}_i - E_{p,i}) \\
& + \sum_{i=3}^{4} \int \frac{d^3p}{(2\pi)^3} \left(-\tilde{\mu}_i - E_{p,i}\right) \theta(-\tilde{\mu}_i - E_{p,i}). \qquad (4.89)
\end{aligned}
$$

We begin by making sure that the $\epsilon \to 0$ limit of $-\Omega|_{\text{fermions}}$ behaves as expected in view of the fact that at $\epsilon = 0$ no spontaneous $U(1)$ breaking is communicated to the fermions. In the $\epsilon \to 0$, we know that the fermionic part of the theory described by Eq. (4.83) becomes a theory of a single free massless Dirac fermion that feels a chemical potential μ, and two free Weyl fermions which do not feel the chemical potential. So as $\epsilon \to 0$, we must recover get Eq. (4.87). As already noted in Section 4.5.2, the dispersion relations in Eq. (4.88) behave in a very peculiar way in this limit, so the way the consistency check is satisfied is surprisingly subtle. Evaluating Eq. (4.89) and taking the $\epsilon \to 0$ limit, and canceling the standard UV-divergent vacuum energy contribution by adjusting the $\delta\Lambda_{cc}$ counter-term, we find that

$$
-\Omega(\mu)_{\text{fermion}} = \left(\frac{\mu^4}{96\pi^2} + 2 \times \frac{7\mu^4}{192\pi^2}\right) = \frac{\mu^4}{12\pi^2}, \qquad (4.90)
$$

which matches Eq. (4.87). It is unusual that the first piece above comes from the *vacuum* term, while the second comes from the matter and anti-matter terms. The fact that the vacuum term makes a μ-dependent contribution to Ω is a consequence of the peculiar way we must write the dispersion relations at $\epsilon = 0$ to keep them diagonal when $\epsilon > 0$.

Now consider the same calculation when $\epsilon > 1/(2\sqrt{2})$. The 'matter' terms in Eq. (4.89) vanish, which is the expected signature of the lack of a Fermi surface. The 'vacuum' contributions have UV divergences, as is usually the case, and must be regulated and renormalised. For our purposes here, a simple momentum cut-off regulator Λ is sufficient, since we are considering a Yukawa theory, see Eq. (4.83), which is a classic case where cut-off regularisation is particularly efficient.[14] We obtain

[14]Dimensional regularisation (DR) is also often an efficient regulator. However, the highly unusual Lorentz-breaking dispersion relations that result after symmetry breaking make the standard DR formulas inapplicable. Rather than common Gamma functions the analytically-continued integrals have to be written in terms of Appell functions (hypergeometric functions in two variables) in DR. However, the necessary asymptotic expansions of these functions are rather complicated [180], hence, DR will not be used in this calculation. In any case, it is a standard principle of quantum field theory that if one obtains a finite and cut-off independent expression for an observable, using a systematic regularisation and renormalisation procedure, any other regulator would give the same final expression.

$$-\Omega_{\text{fermion}} = \frac{\Lambda^4}{2\pi^2} + \frac{\epsilon^2|\phi|^2\Lambda^2}{\pi^2} + \left[\frac{\mu^4}{96\pi^2} + \frac{1}{2\pi^2}\epsilon^2|\phi|^2\left(\frac{1}{2}\epsilon^2|\phi|^2 - \mu^2\right)\right.$$
$$\left. -\frac{1}{2\pi^2}\epsilon^2|\phi|^2\log(2)\left(2\epsilon^2|\phi|^2 - \mu_R^2\right) + \frac{1}{4\pi^2}\epsilon^2|\phi|^2\left(2\epsilon^2|\phi|^2 - \mu_R^2\right)\log\left(\frac{8\epsilon^2|\phi|^2}{\Lambda^2}\right)\right]. \qquad (4.91)$$

The power-law divergences above (together with any other ones coming from the non-fermion parts of Ω) are trivially cancelled off by appropriate cosmological constant and scalar mass counter-terms from Eq. (4.84). For generic ϵ, one also has to turn on $|\phi|^4$ counter-terms at this order, and this would lead to the need to renormalise g to compute Q_{fermions}.

However, if we consider a theory with $\epsilon = 1/\sqrt{2}$, then on the one hand there is still no Fermi surface since $1/\sqrt{2} > 1/(2\sqrt{2})$. On the other hand, at the order to which we work above there are no logarithmic divergences proportional to $|\phi|^2$ or $|\phi|^4$. Hence in the theory with $\epsilon = 1/\sqrt{2}$ we do not need to introduce a $|\phi|^4$ counter-term and renormalise g to compute Q_{fermions} to leading order. Since consideration of the theory described by Eq. (4.83) with $\epsilon = 1/\sqrt{2}$ is sufficient to make our point, we set $\epsilon = 1/\sqrt{2}$ from here onwards.

We are now in a position to write down the renormalised expression for Ω_{fermion}:

$$-\Omega_{\text{fermion}}\big|^{\epsilon=1/\sqrt{2}} = -\frac{17\mu^4}{96\pi^2} \qquad \Longrightarrow \qquad Q_{\text{fermion}}\big|^{\epsilon=1/\sqrt{2}} = -\frac{17\mu^3}{24\pi^2} \neq 0. \qquad (4.92)$$

Note that this has the same parametric dependence on μ as Eq. (4.87), but a different numerical coefficient. Looking back at Eq. (4.61) for the total $U(1)_R$ charge, we see that in the $\epsilon = 1/\sqrt{2}$ theory it is

$$Q\big|^{\epsilon=1/\sqrt{2}} = \frac{2\mu^3}{g^2} - \frac{17\mu^3}{24\pi^2} + \dots, \qquad (4.93)$$

where the first term is the tree-level scalar contribution from the scalars, the second is the leading fermion contribution,[15] and the ellipsis denotes the one-loop scalar contribution and higher order terms. This example shows that fermions can contribute to a charge density Q, as defined by Eq. (4.82), even when there is no Fermi surface. We emphasise that this unusual result is obtained in the unusual situation where the $U(1)$ symmetry associated to Q is spontaneously broken due to scalar condensation. For this reason, there is no conflict with Luttinger's theorem, which relates Q_{fermions} to the volume of the Fermi surface, because Luttinger's theorem assumes that the $U(1)$ symmetry is *not* spontaneously broken.

[15] As usual, the fermion contribution comes from a one-loop calculation, just as in the free case: fermions are intrinsically quantum objects.

Before closing this section, we find it illuminating to discuss how our results would be modified in a theory with a more complicated mass matrix. In any free theory with fermion-number symmetry preserving Dirac masses, the mass matrix can always be diagonalised by a linear transformation of fields with the same charge under the symmetries of the theory. After this procedure, the system is equivalent to one with free massive Dirac fermions that feel a chemical potential. The dispersion relation for the mode i, with mass $m_{D,i}$ that experiences a chemical potential μ_i is then given by

$$(E_i - \mu_i)^2 = |\boldsymbol{p}|^2 + m_{D,i}^2. \tag{4.94}$$

Consequently, the charge of the system is necessarily stored in Fermi surfaces, which would appear if there are modes with $\mu_i > m_{D,i}$. The same statement would apply in any weakly-interacting system in which the interactions do not produce effective mass terms which break the fermion number symmetries. Such systems satisfy the assumptions that go into Luttinger's theorem, and their behaviour will necessarily follow its predictions. Symmetry preserving masses can never lead to dispersion relations of the form of Eq. (4.88), as they will never produce shifts of $|\boldsymbol{p}|$, i.e. $|\boldsymbol{p}| + \kappa_i$, which lead to linear terms in $|\boldsymbol{p}|$. Hence, the $g^2\phi^2$ term in Eq. (4.88) cannot be thought of as m_D^2, where m_D is a Dirac mass.

The 'mass terms' that arise as a result of a scalar VEV in the Lagrangian (4.83) spontaneously break the U(1) R-symmetry. For example, the term $ig\epsilon \left\langle \phi^\dagger \right\rangle \bar{\psi} P_- \lambda$ couples (a component of) the state λ, which is charged under the symmetry, to ψ, which is uncharged. The only way to write down a mass term which appears in the non-standard dispersion relations in the same way as $g^2|\phi|^2$ does, without spontaneous symmetry breaking, is through *explicit* symmetry breaking. Such a mass means that the mass matrix cannot be diagonalised by a rotation of fields with the same charge, as opposed to in theories containing only Dirac masses. This is not surprising, in light of the fact that such terms in the dispersion relations break the assumptions going into Luttinger's theorem. Such mass terms may arise from symmetry-breaking Majorana mass terms, which would be an explicit rather than spontaneous breaking of the symmetry.

A potentially interesting calculation would be to find the charge stored in a system, qualitatively different from $\mathcal{N} = 2$ theories, which contain *both* symmetry-preserving Dirac and symmetry-violating masses from spontaneous symmetry breaking by a scalar (or alternatively, symmetry-breaking Majorana masses). However, the dispersion relations in such systems are extremely complicated, and even in cases where closed forms for these can be obtained, the integrals to evaluate the grand potential become very cumbersome. It would be interesting to return to this problem in future, particularly in simple non-supersymmetric theories where the dispersion relations may be tractable.

4.7 Discussion

The most familiar finite density low-temperature systems that involve chemical potentials coupling to fermions are Fermi liquids. The applicability of Landau's Fermi liquid theory requires two basic features:

1. A Fermi surface, showing up as *e.g.* the locus in spatial momentum space where the inverse fermion propagator vanishes when $p_0 = 0$.

2. Having short-range interactions amongst its degrees of freedom.

These two properties lead to the existence of well-defined quasiparticles and all of the familiar Fermi liquid phenomenology like Landau's zero sound, a specific heat linear in temperature, etc.. Examples of theories which do not fit into this paradigm are intrinsically interesting, and come about when one or both of these properties fail to hold.

Obviously, free systems satisfy both assumptions. Perhaps the simplest non-trivial example of a non-Fermi liquid, which also happens to be relevant to this chapter, is the non-supersymmetric electron plasma described by QED, which satisfies (1), but does not satisfy (2), as reviewed in Sec. 4.2. When there are strong attractive interactions among the fermions, one can also easily imagine (1) failing due to the formation of bosonic bound states. The bosonic states obviously do not have a Fermi surface, and at low temperature would typically tend to Bose condense instead. If there are only parametrically weak attractive interactions between the fermions, then while the fermionic Green's function will have a sharp Fermi surface singularity at any finite order in perturbation theory, the BCS mechanism generally leads to the formation of Cooper pairs and a non-perturbative BCS gap, $\Delta \sim \mu e^{-1/g} \ll \mu \sim p_F$. The Fermi surface then gets smeared out by a non-perturbatively small amount $\Delta/p_F \ll 1$. Systems showing both sorts of behaviour are well known, and have been explored in *e.g.* the context of the so-called BCS-BEC crossover in cold atomic gases [156, 181]. Note that in both of these examples the $U(1)$ particle number symmetry of the fermions becomes broken by *composite* scalar condensation. Luttinger's theorem does not apply once this happens.

It is much less obvious to see how a Fermi surface could disappear in perturbation theory, in the limit of arbitrarily weak interactions, where one does not expect the fermions to be able to form bosonic bound states. Indeed, so long as Luttinger's theorem is applicable, such a thing should not happen. But 4D supersymmetric theories always have elementary scalar fields, which couple to fermions, and these could condense even at arbitrarily weak coupling. So for weakly coupled supersymmetric theories, the existence of Fermi surfaces is indeed questionable. Our results indicate that at least some theories with interactions of the types found in supersymmetric gauge theories fail to satisfy (1) due to scalar condensation driving quadratic mixing between Dirac fermions, which directly feel the chemical potential μ, and Majorana fermions, which do not. Luttinger's theorem does not apply because of scalar condensation, which breaks the relevant $U(1)$ symmetry. There does not appear to be any modified Luttinger relation of sort explored in [156] that one could define in supersymmetric QED, because of the lack of separate fermionic and bosonic number symmetries.

Furthermore, as explained in Section 4.2, in supersymmetric QED, this mixing is order one, even when the gauge coupling is arbitrarily small. In a sharp contrast with the other examples in which Fermi surfaces are endangered by interactions, in supersymmetric QED there is *no* parameter which we could tune smoothly to interpolate between a regime where there is a perturbative Fermi surface to one where there is not. The physics at any $g > 0$ is sharply different from the physics at $g = 0$. After the diagonalisation which takes into account the scalar-condensate-induced mixing, the fermionic eigenmodes have highly peculiar dispersion relations with a complex dependence on μ, and when the smoke clears we do not see a Fermi surface in any of our supersymmetric examples. In our non-supersymmetric examples, with hard and soft breaking of SUSY, where Luttinger's theorem also does not apply, whether a Fermi surface appears depends on the values of the parameters. Perhaps this should not come as a surprise: just because Luttinger's theorem is not available to shield the Fermi surface from danger, this does not imply that interactions must destroy the Fermi surface. This is illustrated by our non-supersymmetric examples in Section 4.3.3 and part of Section 4.5.2, where the relevant $U(1)$ is broken, but there is nevertheless a Fermi surface. But in our supersymmetric examples, it *d*oes turn out to be the case that turning on any non-zero interaction, which results in the $U(1)$ breaking, obliterates the Fermi surface. Finally, we again emphasise that our supersymmetric examples all led to superconducting ground states, with the $U(1)$ breaking driven by charged *elementary* scalar condensation, as opposed to any sort of BCS-like fermion pairing mechanism.

Clearly, the work presented in this chapter is only the starting point of many potentially interesting research directions. With regard to super-QED, or the sort of non-supersymmetric theories we considered in this chapter, one can ask for example, what is the quasiparticle spectrum of such theories? What are their thermodynamic properties? Perhaps the most conceptually interesting question is whether the fermions manage to store any of the charge density, despite not having a Fermi surface. Relatedly, can one develop a useful heuristic understanding of the reason for the disappearance of the Fermi surface? naïvely, it may have seemed that the most natural possibility is that when there is no Fermi surface, all of the charge 'leaks out' of the fermion sector through the Yukawa terms, and gets stored by the scalars. However, in Section 4.6, we explicitly calculated the fermion contribution to the charge density in an example where there is no Fermi surface, and showed that the fermion contribution to the charge density is non-vanishing. We do not yet know a heuristic physical interpretation for this result, which seems to go against the conventional wisdom about how fermions behave at finite density. Of course, this conventional wisdom is based on Luttinger-theorem-inspired pictures, and as we have emphasised Luttinger's theorem does not apply to our condensed-scalar examples.

If one hopes to try to make direct contact with condensed matter physics, it may perhaps be of interest to start by analysing the questions we raised above in Abelian gauge theories, since examples of dynamical Abelian gauge fields coupled to fundamental and emergent matter of various statistics are ubiquitous in condensed matter. Perhaps there are condensed matter systems for which theoretical models involving Yukawa interactions of the sort seen in SUSY gauge theories may be useful.

To make contact with the results of gauge/gravity duality, it is important to generalise our analysis to include non-Abelian gauge fields, and to begin working with theories that actually have gravity duals at strong coupling. The details of the scalar stabilisation mechanisms may well be different, and presumably do not involve turning on FI terms (but see [182]), as we had to do here in a number of examples. An interesting issue is that from the weak-coupling side, it seems likely that finite density would drive squark condensation, but this would lead to gauge symmetry breaking, which has not been seen in most systems at strong coupling. (Of course, signs of breaking of *global* symmetries are ubiquitous in gauge/gravity duality.) Two other options for the stabilisation of scalars are to either turn on finite temperature or put the theory on a curved manifold, which could produce effective masses via the matter field-curvature couplings. Also, instead of electrical neutrality, colour neutrality would play a central role in the analysis of non-Abelian theories, as has been the case in studies of high density QCD. Once the generalisation to non-Abelian theories is performed, one would have the opportunity to investigate many interesting phenomenological and conceptual questions. Is the charge typically stored in fermions, or in the scalars? The possibility that in some cases it may be stored in scalar condensates has been noted in the AdS/CFT context in e.g. [150,183]. Are there actually Fermi surfaces at weak coupling in theories that do not seem to have one holographically? Are there examples of theories with the opposite behaviour — Fermi-surface like singularities at strong coupling, but no Fermi surfaces at weak coupling?

Chapter 5

Second-order hydrodynamics and dissipationless limit in the holographic Gauss-Bonnet liquid

5.1 Motivation and summary

Gauge/string duality has been used successfully to explore qualitative, quantitative and conceptual issues in fluid dynamics [184–186]. Although the number of quantum field theory models with known dual string or gravity description is limited, their transport and spectral function properties at strong coupling can be fully determined, thus giving a valuable insight into behaviour of generic strongly interacting quantum many-body systems. Moreover, relating strongly coupled fluids to gravity clarified the understanding of fluid dynamics as an effective field theory and determined the number of independent transport coefficients at first and second order in the hydrodynamic derivative expansion. For generic neutral fluids, there are two independent first-order transport coefficients (shear viscosity η and bulk viscosity ζ), and fifteen second-order coefficients[1] (see e.g. [189]). For conformal fluids, additional constraints reduce the number of transport coefficients to one at first order (shear viscosity η) and five at second order[2] (usually denoted as τ_Π, κ, λ_1, λ_2, λ_3). In the parameter regime where the dual supergravity description of conformal fluids is applicable, the six transport coefficients (in d space-time dimensions, $d > 2$) are given by [190]

$$\eta = s/4\pi\,, \tag{5.1}$$

$$\tau_\Pi = \frac{d}{4\pi T}\left(1 + \frac{1}{d}\left[\gamma_E + \psi\left(\frac{2}{d}\right)\right]\right), \qquad \kappa = \frac{d}{d-2}\,\frac{\eta}{2\pi T}\,, \tag{5.2}$$

[1]The existence of a local entropy current with non-negative divergence implies $\eta \geq 0$, $\zeta \geq 0$ [187] and constrains the number of independent coefficients at second order to ten [188].

[2]There are no further constraints in addition to $\eta \geq 0$ coming from the non-negativity of the divergence of the entropy current in the conformal case, so long as the term proportional to viscosity provides the dominant contribution to the entropy current [188]. This point will be discussed in detail in Section 5.6.4

$$\lambda_1 = \frac{d\eta}{8\pi T}\,, \qquad \lambda_2 = \left[\gamma_E + \psi\left(\frac{2}{d}\right)\right]\frac{\eta}{2\pi T}\,, \qquad \lambda_3 = 0\,, \tag{5.3}$$

where s is the entropy density, $\psi(z)$ is the logarithmic derivative of the gamma function, and γ_E is the Euler-Mascheroni constant. The linear combination of the transport coefficients $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2$ was found to vanish in theories dual to two-derivative gravity [191, 192] and conjectured to vanish universally when higher-derivative terms on the gravity side of the gauge/gravity duality are taken into account [192, 193]. As we will show below, this conjecture does not hold for transport coefficients derived from Gauss-Bonnet gravity, nor does it hold for $\mathcal{N} = 4$ transport coefficients with the leading-order 't Hooft correction.

For the finite-temperature $\mathcal{N} = 4$ $SU(N_c)$ supersymmetric Yang-Mills theory in $d = 3+1$ dimensions in the limit of infinite N_c and infinite 't Hooft coupling $\lambda = g^2_{YM}N_c$, first- and second-order transport coefficients were computed, correspondingly, in [89] and [98, 185], using methods of gauge/gravity and fluid/gravity dualities[3]:

$$\eta = \frac{\pi}{8}N_c^2T^3\,, \tag{5.4}$$

$$\tau_\Pi = \frac{(2-\log 2)}{2\pi T}\,, \qquad \kappa = \frac{\eta}{\pi T}\,, \qquad \lambda_1 = \frac{\eta}{2\pi T}\,, \qquad \lambda_2 = -\frac{\eta\log 2}{\pi T}\,, \qquad \lambda_3 = 0\,. \tag{5.5}$$

Coupling constant corrections to the coefficients (5.4), (5.5) can be computed using the higher-derivative terms in the low-energy effective action of type IIB string theory [194–200]. At first order in hydrodynamic expansion,

$$\eta = \frac{\pi}{8}N_c^2T^3\left(1 + \frac{135\zeta(3)}{8}\,\lambda^{-3/2} + \dots\right)\,, \tag{5.6}$$

and at second order,

$$\tau_\Pi = \frac{(2-\log 2)}{2\pi T} + \frac{375\zeta(3)}{32\pi T}\,\lambda^{-3/2} + \dots\,, \tag{5.7}$$

$$\kappa = \frac{N_c^2T^2}{8}\left(1 - \frac{5\zeta(3)}{4}\,\lambda^{-3/2} + \dots\right)\,, \tag{5.8}$$

$$\lambda_1 = \frac{N_c^2T^2}{16}\left(1 + \frac{175\zeta(3)}{4}\,\lambda^{-3/2} + \dots\right)\,, \tag{5.9}$$

$$\lambda_2 = -\frac{N_c^2T^2}{16}\left(2\log 2 + \frac{5\,(97 + 54\log 2)\,\zeta(3)}{8}\lambda^{-3/2} + \dots\right)\,, \tag{5.10}$$

$$\lambda_3 = \frac{N_c^2T^2}{16}\,25\zeta(3)\,\lambda^{-3/2} + \dots\,. \tag{5.11}$$

Temperature T can be given in terms of the infinite-'t Hooft coupling temperature T_0 as

$$T = T_0\left(1 + \frac{15\zeta(3)}{8}\lambda^{-3/2}\right)\,. \tag{5.12}$$

[3] We use notations and conventions of [185].

Second-order coefficients τ_Π, κ, λ_1 and λ_3 were known before. The result for λ_2 is new and was to our knowledge previously unknown.

The corrections in formulae (5.6) - (5.11) can be trusted so long as they remain (infinitesimally) small relative to the leading order ($\lambda \to \infty$) result, as they are obtained by treating the higher-derivative terms in the equations of motion perturbatively. To leading order in the limit $\lambda \to \infty$, the coefficients (5.6) - (5.11) are independent of the coupling, in sharp contrast with their weak coupling behavior [201]. The coefficient λ_3 vanishes at $\lambda \to \infty$, and was argued to vanish also at $\lambda \to 0$ (this appears to be a generic property of weakly coupled theories). The full coupling constant dependance of transport coefficients (even at infinite N_c) is beyond reach. Monotonicity and other properties of various combinations of transport coefficients are of interest for studies of near-equilibrium behavior at strong coupling, in particular thermalisation, and for attempts to uncover a universality similar to the one exhibited by the ratio of shear viscosity to entropy density [94], [202], [203–205]. In $\mathcal{N} = 4$ SYM at infinite N_c, the shear viscosity to entropy density ratio appears to be a monotonic function of the coupling [94], with the correction to the universal infinite coupling result being positive [194, 196],

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + 15\zeta(3)\lambda^{-3/2} + \ldots\right) . \tag{5.13}$$

Subsequent calculations revealed that the corrections coming from higher derivative terms in the action can have either sign [96, 97]. In particular, for Gauss-Bonnet gravity with the five-dimensional bulk action

$$S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R - 2\Lambda + \frac{\lambda_{GB}}{2}L^2\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right], \tag{5.14}$$

where the cosmological constant $\Lambda = -6/L^2$, the shear viscosity to entropy density ratio in a (hypothetical) dual field theory is given by [97]

$$\frac{\eta}{s} = \frac{1 - 4\lambda_{GB}}{4\pi}, \tag{5.15}$$

non-perturbatively in the Gauss-Bonnet coupling[4] λ_{GB}, and can at least formally be driven all the way down to zero in the limit of $\lambda_{GB} \to 1/4$. Taking this limit in a hypothetical dual field theory is problematic, since it leads to causality violation for sufficiently large values of λ_{GB} [207]. On the other hand, one may hope that adding other fields to the action can cure the acausal ultraviolet behavior of the dual theory without affecting its hydrodynamic (low-frequency) limit [208].

[4] An attractive feature of Gauss-Bonnet gravity and, more generally, Lovelock gravity is that the equations of motion remain second-order and thus the higher-curvature terms can be treated non-perturbatively (implications of Lovelock gravity in the context of holography iare discussed in [206]). An obvious disadvantage of considering holography with Gauss-Bonnet action is that the quantum field theory dual to it is unknown and one remains in the realm of the bottom-up approach.

In a recent paper, Bhattacharya *et al.* [101] suggested the existence of non-trivial second-order non-dissipative hydrodynamics, i.e. a theory whose fluid dynamics derivative expansion has no contribution to the entropy production while still having some of the transport coefficients non-vanishing.[5] For conformal fluids, the classification of [101] implies the existence of a four-parameter family of non-trivial non-dissipative fluids with $\eta = 0$ and non-vanishing τ_Π, κ, $\lambda_1 = \kappa/2$, λ_2 and λ_3. Given the result (5.15), the hypothetical theory dual to Gauss-Bonnet gravity in the limit of $\lambda_{GB} \to 1/4$ is a natural candidate for a dissipationless fluid. In addition to shear viscosity, only two transport coefficients (τ_Π and κ) for the dual Gauss-Bonnet background have been previously known non-perturbatively (to leading order in λ_{GB}, all coefficients were found by Shaverin and Yarom [193]). In this chapter, we compute all Gauss-Bonnet transport coefficients non-perturbatively in λ_{GB} and analyse the $\lambda_{GB} \to 1/4$ limit. The full set of transport coefficients is given by

$$\eta = s\gamma^2/4\pi\,, \tag{5.16}$$

$$\tau_\Pi = \frac{1}{2\pi T}\left(\frac{1}{4}(1+\gamma)\left(5+\gamma-\frac{2}{\gamma}\right)-\frac{1}{2}\log\left[\frac{2(1+\gamma)}{\gamma}\right]\right), \tag{5.17}$$

$$\lambda_1 = \frac{\eta}{2\pi T}\left(\frac{(1+\gamma)(3-4\gamma+2\gamma^3)}{2\gamma^2}\right), \tag{5.18}$$

$$\lambda_2 = -\frac{\eta}{\pi T}\left(-\frac{1}{4}(1+\gamma)\left(1+\gamma-\frac{2}{\gamma}\right)+\frac{1}{2}\log\left[\frac{2(1+\gamma)}{\gamma}\right]\right), \tag{5.19}$$

$$\lambda_3 = -\frac{\eta}{\pi T}\left(\frac{(1+\gamma)(3+\gamma-4\gamma^2)}{\gamma^2}\right), \tag{5.20}$$

$$\kappa = \frac{\eta}{\pi T}\left(\frac{(1+\gamma)(2\gamma^2-1)}{2\gamma^2}\right), \tag{5.21}$$

where we have defined $\gamma \equiv \sqrt{1-4\lambda_{GB}}$. In the limit of $\lambda_{GB} \to 0$ ($\gamma \to 1$), which corresponds to the pure Einstein gravity, one recovers the standard results for strongly coupled conformal fluids, (5.4) and (5.5). The Gauss-Bonnet result for η was obtained in [97] and the relaxation time τ_Π was found numerically in [210]. Coefficients τ_Π and κ were computed analytically in [211]. The formulae for λ_1, λ_2, and λ_3 are new. To linear order in λ_{GB}, the results coincide with those found in [193]. Note, however, that

$$2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = -\frac{\eta}{\pi T}\frac{(1-\gamma)(1-\gamma^2)(3+2\gamma)}{\gamma^2} = -\frac{40\lambda_{GB}^2\eta}{\pi T} + \mathcal{O}\left(\lambda_{GB}^3\right). \tag{5.22}$$

i.e. this particular linear combination of transport coefficients vanishes only to linear order in λ_{GB}, thus disproving the universality conjecture made for two-derivative gravity in [192].

[5] The authors of [101] considered an effective field theory approach [3, 4] to non-dissipative uncharged second-order hydrodynamics. The approach relies on a classical effective action and standard variational techniques to derive the stress-energy tensor, which were discussed in Chapter 3. It is thus unable to incorporate dissipation. The inclusion of dissipation into the description of hydrodynamics, using the same effective description, was analysed in [102, 209], which was also the central topic of Chapter 3.

We observe that the inequality

$$2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 \leq 0 \tag{5.23}$$

is still obeyed by the transport coefficients of the holographic Gauss-Bonnet liquid.

It is very interesting to note that by having the knowledge of all leading-order 't Hooft-corrected $\mathcal{N} = 4$ super Yang-Mills second-order transport coefficients, we can confirm that the relation $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$ still remains valid. In spite of the higher-derivative corrections, the linear relation is not violated.

In the limit of $\lambda_{GB} \to 1/4$ ($\gamma \to 0$) we find

$$\eta\tau_\Pi = 0, \qquad \lambda_1 = \frac{3\pi^2 T^2}{2\sqrt{2}\kappa_5^2}, \qquad \lambda_2 = 0, \qquad \lambda_3 = -\frac{3\sqrt{2}\pi^2 T^2}{\kappa_5^2}, \qquad \kappa = -\frac{\pi^2 T^2}{\sqrt{2}\kappa_5^2}. \tag{5.24}$$

At first glance, this result realises the dissipationless liquid scenario outlined in [101]: the shear and bulk viscosities are zero while some of the second order coefficients are not. However, the relationship $\kappa = 2\lambda_1$, which is required for ensuring zero entropy production, does not hold among the coefficients in (5.24). We therefore conclude that the holographic Gauss-Bonnet liquid does not fall into the class of non-dissipative liquids discussed in [101].

This chapter is structured in the following way. We will begin by reviewing phenomenological second-order hydrodynamics in section 5.2. We will then move on to presenting the holographic setup in 5.3, i.e. the Gauss-Bonnet theory, in which we will perform the majority of our calculations. We will present the calculations of two-point functions and the analysis of the scalar, shear and sound modes in Section 5.4. In Section 5.5, we will study the behaviour of the theory at the extreme value of the Gauss-Bonnet coupling, $\lambda_{GB} = 1/4$. This will be followed by the calculation of three-point functions in Section 5.6, where we will present the relevant Kubo formulae, outline the steps of the calculation, present the results of the non-perturbative second-order transport coefficients and discuss their implication for the Bjorken flow and the structure of the entropy current. We will then move on to considering charge diffusion in Section 5.7. Fluid/gravity correspondence will be employed in Section 5.8 to test the validity of our non-perturbative transport coefficients and the calculation of λ_2 in the $\mathcal{N} = 4$ theory will be presented in Section 5.9. We will conclude this chapter with a discussion of results and an outline of potential future research directions.

5.2 Second-order hydrodynamics

We begin by reviewing second-order conformal hydrodynamics of uncharged fluids [98,185], which is a direct extension of our discussion in Chapter 2.1.3. The main idea, which we presented, was that phenomenological hydrodynamics can be organised as a gradient expansion of conserved tensors in velocity $u^a(x)$, temperature, $T(x)$, and chemical potential, $\mu(x)$, fields. Since our primary interest in this chapter lies with uncharged fluids, which have a vanishing chemical potential, the only conserved operator relevant for the construction of the hydrodynamic expansion is the stress-energy tensor T^{ab} [25, 187].[6]

The most general tensorial structure of the stress-energy tensor is

$$T^{ab} = \mathcal{E}u^a u^b + \mathcal{P}\Delta^{ab} + q^a u^b + u^a q^b + t^{ab}, \tag{5.25}$$

where $\mathcal{E}$ and $\mathcal{P}$ are scalars, q^a is transverse, t^{ab} transverse, symmetric and traceless, and $\Delta^{ab} \equiv g^{ab} + u^a u^b$. All $\mathcal{E}$, $\mathcal{P}$, q^a and t^{ab} are expanded in derivatives of the fields $u^a(x)$ and $T(x)$. As discussed in Chapter 2.1.3, the lack of microscopic definition of the variables results in an ambiguity (a choice of "frame"), whereby we can re-define $u^a(x)$ and $T(x)$ by any function of their derivatives. We will choose to work in the Landau frame and to set $\mathcal{E} = \varepsilon$, where ε is the energy density of the fluid. This further implies that $q^a = 0$. Since we are interested in fluids on curved manifolds, we will also include derivatives of the metric tensor into the gradient expansion.

The stress-energy tensor can then be written as

$$T^{ab} = \varepsilon u^a u^b + P\Delta^{ab} + \Pi^{ab}, \tag{5.26}$$

with the second-order conformal fluid in four dimension described by pressure $P = \varepsilon/3$, which receives no higher-order corrections. The spin-2 structure is given by

$$\begin{aligned}\Pi^{ab} = & -\eta\sigma^{ab} + \eta\tau_\Pi\left[{}^\langle D\sigma^{ab\rangle} + \frac{1}{d-1}\sigma^{ab}\left(\nabla\cdot u\right)\right] + \kappa\left[R^{\langle ab\rangle} - (d-2)u_c R^{c\langle ab\rangle d}u_d\right] \\ & + \lambda_1\sigma^{\langle a}{}_c\sigma^{b\rangle c} + \lambda_2\sigma^{\langle a}{}_c\Omega^{b\rangle c} + \lambda_3\Omega^{\langle a}{}_c\Omega^{b\rangle c},\end{aligned} \tag{5.27}$$

where $D \equiv u^a\nabla_a$. The first-order coefficient η is shear viscosity while the five second-order coefficients will sometimes be labeled by $\lambda_n = \{\eta\tau_\Pi, \lambda_1, \lambda_2, \lambda_3, \kappa\}$, where $n = \{0, 1, 2, 3, 4\}$. For convenience, we have defined

$$A^{\langle ab\rangle} \equiv \frac{1}{2}\Delta^{ac}\Delta^{bd}\left(A_{cd} + A_{dc}\right) - \frac{1}{d-1}\Delta^{ab}\Delta^{cd}A_{cd} \equiv {}^\langle A^{ab\rangle}, \tag{5.28}$$

which by construction forms tensors that are transverse, $u_a A^{\langle ab\rangle} = 0$, traceless, $g_{ab}A^{\langle ab\rangle} = 0$, and symmetric. In our case, $d = 4$. The tensor σ^{ab} is the one-derivative symmetric, transverse and traceless tensor

$$\sigma^{ab} = 2^\langle\nabla^a u^{b\rangle}. \tag{5.29}$$

[6] In this chapter, we will be using Greek letters (μ, ν, etc.) to denote five-dimensional bulk indices, Latin letter from the beginning of the alphabet (a, b, etc.) to denote four-dimensional field theory indices and Latin indices from the middle of the alphabet (i, j, etc.) to denote spatial three-dimensional field theory indices.

The vorticity $\Omega^{\mu\nu}$ is defined by the anti-symmetric, transverse and traceless one-derivative tensor

$$\Omega^{ab} = \frac{1}{2}\Delta^{ac}\Delta^{bd}\left(\nabla_c u_d - \nabla_d u_c\right). \tag{5.30}$$

Finally, R_{cabd} and R_{ab} are the Riemann and Ricci tensors with terms which include two derivatives of the metric.

5.3 Einstein-Gauss-Bonnet gravity

In this section, we begin our holographic analysis of the Einstein-Gauss-Bonnet theory, which is governed by the gravitational action (5.14) in five bulk dimensions. The coefficients of the four-derivative terms ensure that the equations of motion, which follow from the action (5.14), only contain second derivatives of the metric. They are given by

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{\lambda_{GB}L^2}{4}g_{\mu\nu}\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) - \\ - \lambda_{GB}L^2\left(RR_{\mu\nu} - 2R_{\mu\alpha}R_\nu{}^\alpha - 2R_{\mu\alpha\nu\beta}R^{\alpha\beta} + R_{\mu\alpha\beta\gamma}R_\nu^{\alpha\beta\gamma}\right). \end{aligned} \tag{5.31}$$

The black brane solution of equation (5.31) is

$$ds^2 = -f(r)N_\#^2 dt^2 + \frac{1}{f(r)}dr^2 + \frac{r^2}{L^2}\left(dx^2 + dy^2 + dz^2\right), \tag{5.32}$$

where

$$f(r) = \frac{r^2}{L^2}\frac{1}{2\lambda_{GB}}\left[1 - \sqrt{1 - 4\lambda_{GB}\left(1 - \frac{r_+^4}{r^4}\right)}\right]. \tag{5.33}$$

We will set the arbitrary constant $N_\#$ so as to normalise the speed of light at the boundary to unity,

$$N_\#^2 = \frac{1}{2}\left(1 + \sqrt{1 - 4\lambda_{GB}}\right), \tag{5.34}$$

and henceforth use this value. For convenience, we also define a shifted Gauss-Bonnet coupling,

$$\gamma \equiv \sqrt{1 - 4\lambda_{GB}}. \tag{5.35}$$

Thermodynamic quantities associated with this background are the Hawking temperature of the black brane (5.32),

$$T = N_\#\frac{r_+}{\pi L^2} = \frac{r_+}{\sqrt{2}\pi L^2}\sqrt{1+\gamma}, \tag{5.36}$$

as well as energy density and entropy density of the dual conformal theory,

$$\varepsilon = 3P = \frac{3}{4}Ts, \tag{5.37}$$

$$s = \frac{2\pi}{\kappa_5^2}\left(\frac{r_+}{L}\right)^3 = \frac{4\sqrt{2}\pi^4 L^3}{\kappa_5^2}\frac{T^3}{(1+\gamma)^{3/2}}. \tag{5.38}$$

We will set $L = 1$ in most of the sections below.

The shear viscosity to entropy density ratio was computed in [97] and found to equal the expression given in Eq. (5.15), i.e. $\eta/s = \gamma^2/4\pi^2$, which can be tuned to zero in the limit of $\lambda_{GB} \to 1/4$.

Based on various faster-than-speed of light and causality arguments, the behaviour of gravitational perturbations was argued to be pathological for large λ_{GB}, and a bound on λ_{GB} was established [97, 207, 210, 212],

$$-\frac{7}{36} \leq \lambda_{GB} \leq \frac{9}{100}. \tag{5.39}$$

However, all those arguments rely on the ultraviolet, large momentum limit of $q \to \infty$. Since we are only interested in the hydrodynamic transport properties of the Einstein-Gauss-Bonnet dual, we can interpret the theory as an effective field theory, valid only for low frequencies and momenta. These arguments led [208] to explicitly construct a theory with a low temperature phase transition, breaking the link between the hydrodynamic IR and causality breaking UV modes. Since we wish to focus on the hydrodynamic regime in which η/s goes to zero, i.e. for λ_{GB} near 1/4, we will view the above setup as a holographic dual of an effective field theory with some unspecified fields responsible for the UV completion.

5.4 Two-point function and quasi-normal modes

In this section, we will consider holographic retarded two-point functions of the stress-energy tensor, $G_R^{\mu\nu,\rho\sigma}(p_1, p_2) \equiv \langle T^{\mu\nu}(p_1), T^{\rho\sigma}(p_2)\rangle_R$, and perform an analysis of the quasi-normal mode spectrum to recover the dispersion relations of diffusive and sound modes for non-perturbative values of λ_{GB}. We will use the usual decomposition of the metric perturbations into scalar, shear and sound modes. These modes transform as spin 2, 1 and 0 tensors, respectively, after we select a specific direction for the momentum flow and make use of the remaining spatial rotational invariance [90, 91, 93].

5.4.1 Scalar mode

Let us begin by analysing the scalar sector of the metric perturbations to the second order in the hydrodynamic expansion. To first order, this was done in [97]. Without loss of generality, we choose the metric fluctuations to have momentum in the z-direction, i.e. $h_{\mu\nu}(r)e^{-it\omega+iqz}$, which identifies the relevant scalar fluctuation to be

$$Z_1 = h^x{}_y. \tag{5.40}$$

It is convenient to raise one of the indices of h_{xy} so that the mode Z_1 behaves as a minimally coupled massless scalar in the Gauss-Bonnet background. The Einstein-Gauss-Bonnet equations (5.31) governing its dynamics can be written in a convenient form,

$$A_2 Z_1'' + A_1 Z_1' + A_0 Z_1 = 0, \tag{5.41}$$

where the coefficient functions are given by

$$A_2 = rf\left(\lambda_{GB} f' - r\right), \tag{5.42}$$

$$A_1 = rf'\left(\lambda_{GB} f' - r\right) - 3rf + \lambda_{GB} f\left(rf'' + 2f'\right), \tag{5.43}$$

$$\begin{aligned} A_0 = &\frac{2}{f\left(1+\sqrt{1-4\lambda_{GB}}\right)}\Bigg[r\omega^2\left(\lambda_{GB} f' - r\right) - \left(1+\sqrt{1-4\lambda_{GB}}\right) f^2\left(\lambda_{GB} f'' - 1\right) \\ &+ \frac{1}{2}\left(1+\sqrt{1-4\lambda_{GB}}\right) f\left(f''\left(r^2 - \lambda_{GB} q^2\right) - 2\lambda_{GB} f'^2 + 4rf' + q^2 - 12r^2\right)\Bigg]. \end{aligned} \tag{5.44}$$

The function f was defined in Eq. (5.33), r is the radial variable, while ω and q are the frequency and the momentum of the background fluctuations. To solve the differential equation (5.41) for the purposes of extracting the retarded Green's function, we must impose the in-falling boundary conditions [87,88] by writing

$$Z_1 = \tilde{f}(r)^{-i\mathfrak{w}/2}\left(1 + g(r)\right), \tag{5.45}$$

where

$$\tilde{f}(r) = \frac{1}{2\lambda_{GB}}\left[1 - \sqrt{1 - 4\lambda_{GB}\left(1 - (r_+/r)^4\right)}\right]. \tag{5.46}$$

We have introduced dimensionless frequency and momentum,

$$\mathfrak{w} = \frac{\omega}{2\pi T}, \qquad \mathfrak{q} = \frac{q}{2\pi T}. \tag{5.47}$$

It is convenient to follow the discussion of [97] and introduce a new radial coordinate,

$$v = 1 - \sqrt{1 - 4\lambda_{GB}\left(1 - (r_+/r)^4\right)}, \tag{5.48}$$

so that $Z_1(v) = \left(v/(2\lambda_{GB})\right)^{-i\mathfrak{w}/2}\left(1 + g(v)\right)$. We can now rewrite the equation (5.41) as

$$v\left(1-v\right)\partial_v^2 g(v) + \left[1 + v + i\mathfrak{w}\left(v-1\right)\right]\partial_v g(v) + \mathcal{G}(v)\left[g(v) + 1\right] = 0, \tag{5.49}$$

where $\mathcal{G}$ is a function of $\mathfrak{w}$ and $\mathfrak{q}$, with the form

$$\mathcal{G}(v) = -i\mathfrak{w} + \mathfrak{w}^2\mathcal{G}_{\mathfrak{w}}(v) + \mathfrak{q}^2\mathcal{G}_{\mathfrak{q}}(v). \tag{5.50}$$

We have further defined

$$\mathcal{G}_{\mathfrak{w}}(v) = \frac{(v-1)\left[(4\lambda_{GB}+v(v-2))^{3/2} - 8\lambda_{GB}^{3/2}(v-1)^2\right]}{4v\left(4\lambda_{GB}+v(v-2)\right)^{3/2}}, \tag{5.51}$$

$$\mathcal{G}_{\mathfrak{q}}(v) = \frac{(v-1)\sqrt{\lambda_{GB}}\left(1+\sqrt{1-4\lambda_{GB}}\right)\left(1+8\lambda_{GB}+3v(v-2)\right)}{2\left(4\lambda_{GB}+v(v-2)\right)^{3/2}}. \tag{5.52}$$

To find the retarded two-point function in the hydrodynamic, low frequency and momentum limit, we may solve for $g(v)$ perturbatively in $\mathfrak{w} = \omega/(2\pi T)$ and $\mathfrak{q} = q/(2\pi T)$, assuming that the magnitudes of both $\mathfrak{w}$ and $\mathfrak{q}$ are of the same scale μ. By writing

$$g(v) = \sum_{n=1}^{\infty} \mu^n g_n(v), \tag{5.53}$$

we find that to all orders in μ, the differential equations for g_n have the form

$$v\,(1-v)\,\partial_v^2 g_n(v) + (1+v)\,\partial_v g_n(v) + H_n(v) = 0. \tag{5.54}$$

Functions H_n can be determined recursively from $\mathcal{G}$ and g_m, with $m < n$,

$$H_n(v) = i\mathfrak{w}\partial_v\left[(1-v)\,g_{n-1}(v)\right] + \left(\mathfrak{w}^2\mathcal{G}_{\mathfrak{w}}(v) + \mathfrak{q}^2\mathcal{G}_{\mathfrak{q}}(v)\right)g_{n-2}(v), \tag{5.55}$$

where $n \geq 1$. At first order, $g_0 = 1$ and $g_{-1} = 0$, which gives $H_1 = -i\mathfrak{w}$.

All functions g_n are solved by the expression given in terms of the integrals,

$$g_n(v) = D_n + \int^v dv' \frac{(1-v')^2}{v'}\left(C_n - \int^{v'} dv'' \frac{H_n(v'')}{(1-v'')^3}\right), \tag{5.56}$$

from which we can find the first-order result,

$$g_1(v) = D_1 - \frac{1}{2}C_1\,(4-v)\,v + \left(C_1 + \frac{i\mathfrak{w}}{2}\right)\log v. \tag{5.57}$$

We require all g_n to be regular at the horizon, i.e. at $v = 0$, which can be ensured by imposing a boundary condition that cancels the logarithmic divergences, i.e. terms proportional to $\log v$. In the case of g_1, the cancellation occurs when $C_1 = -i\mathfrak{w}/2$. Furthermore, we must impose that all g_n vanish at the boundary, where $v = 1 - \sqrt{1-4\lambda_{GB}}$. At first order this amounts to setting $D_1 = -\frac{i\mathfrak{w}}{2}\left(1+2\lambda_{GB} - \sqrt{1-4\lambda_{GB}}\right)$. Hence,

$$g_1(v) = -\frac{i\mathfrak{w}}{4}\left(3 - 2\gamma - \gamma^2 - 4v + v^2\right), \tag{5.58}$$

where we have used $\gamma = \sqrt{1-4\lambda_{GB}}$, as defined in Eq. (5.35).

In order to find second-order hydrodynamic contributions to the scalar channel two-point function, we need to find the solution at one order higher, i.e. g_2. This can be done by following exactly the same procedure that gave us g_1; by using Eq. (5.55) with g_1 and g_0 we first find H_2, which can be integrated using Eq. (5.56) to find g_2. The result has the form

$$\begin{aligned} g_2(v) &= \mathfrak{w}^2 g_2^{(\mathfrak{w})}(v) + \mathfrak{q}^2 g_2^{(\mathfrak{q})}(v) \\ &\quad + \frac{\mathfrak{w}^2}{4} \int^v \frac{(1-v')^2 \log\left[\gamma^2 - 1 + v' - \sqrt{(\gamma^2-1)\left(\gamma^2 - (1-v')^2\right)}\right]}{v'} dv'. \end{aligned} \tag{5.59}$$

Functions $g_2^{(\mathfrak{w})}$ and $g_2^{(\mathfrak{q})}$ have lengthy, but closed-form expression. Even though we do not have a closed-form expression for the remaining integral in $g_2(v)$, this is irrelevant for the computation of the two-point function. The form of g_2 is sufficient for fixing both boundary conditions on g_2 and for determining the near-boundary expansion of Z_1. More precisely, the undetermined integral comes from the outer integration of (5.56). Regularity at the horizon, $v = 0$, can thus be imposed without a problem to fix C_2. What remains is an integral in (5.59), which can be integrated order-by-order in the near-boundary expansion to determine D_2. By treating it as an indefinite integral, an additional undetermined constant can simply be absorbed into D_2.

The retarded two-point function can be computed by only perturbing the Einstein-Gauss-Bonnet action (5.14) in the scalar channel, $g_{xy} \to g_{xy} + r^2 Z_1$, and evaluating the on-shell contribution of Z_1. The holographic Green's function is given in terms of the variable v by

$$G^{xy,xy}_{R,hol}(\omega, \boldsymbol{q}) = \frac{r_+^4 \gamma^2 \sqrt{2(1+\gamma)}}{\kappa_5^2} \lim_{v\to 1-\gamma} \left[\frac{v}{(1-v)^2} Z_1(v, -\mathfrak{w}, -\mathfrak{q}) \frac{\partial}{\partial v} Z_1(v, \mathfrak{w}, \mathfrak{q})\right] \tag{5.60}$$

$$= \frac{r_+^4}{\kappa_5^2}(1-\gamma)\sqrt{2(1+\gamma)}\, Z_1(1-\gamma, -\mathfrak{w}, -\mathfrak{q}) \frac{\partial}{\partial v} Z_1(v, \mathfrak{w}, \mathfrak{q})\bigg|_{v=1-\gamma}. \tag{5.61}$$

From the field theory point of view, it was shown in [185] that we can use the retarded two-point function of the stress-energy tensor, $G_R^{xy,xy}(\omega, k)$, to extract two of the second-order hydrodynamical coefficients, τ_Π and κ. Up to second order in energy and momentum, the hydrodynamic correlation function takes the gradient expanded form,

$$G^{xy,xy}_{R,hydro}(\omega, \boldsymbol{q}) = P - i\eta\omega + \eta\tau_\Pi \omega^2 - \frac{\kappa}{2}\left(\omega^2 + q^2\right). \tag{5.62}$$

From the solution for Z_1 solved to second order, i.e. (5.45) with (5.58) and (5.59), we can compute the retarded Green's function (5.60) and match it with the hydrodynamic expression (5.62). Note that since we only considered fluctuations about the background in the holographic calculation, we will only find the ω- and q-dependent terms. The background itself gives the value of P. The procedure now enables us to find the shear viscosity and its ratio over the entropy density to be

$$\eta = \frac{\sqrt{2}\pi^3 T^3}{\kappa_5^2}\frac{\gamma^2}{(1+\gamma)^{3/2}}, \qquad \frac{\eta}{s} = \frac{1}{4\pi}\gamma^2, \tag{5.63}$$

as previously computed in [97], with the Hawking temperature stated in (5.36).

The second-order coefficients τ_Π and κ can now be found by matching (5.60) with (5.62), giving us expressions which were previously computed in [213] by using a different method,

$$\tau_\Pi = \frac{1}{8\pi T\gamma}\left[(1+\gamma)(\gamma(5+\gamma)-2) + 2\gamma\log\left(\frac{(1-\gamma)\gamma}{2(1-\gamma^2)}\right)\right], \tag{5.64}$$

$$\kappa = \frac{\eta}{\pi T}\frac{(1+\gamma)(2\gamma^2-1)}{2\gamma^2}. \tag{5.65}$$

In the limit of $\lambda_{GB}\to 0$, i.e. $\gamma\to 1$, we can use the $\mathcal{N}=4$ relation, $N_c^2 = 4\pi^2/\kappa_5^2$, to check that the expressions (5.63), (5.64) and (5.65) reproduce the $\mathcal{N}=4$ super Yang-Mills results found in [185],

$$\eta = \frac{\pi}{8}N_c^2 T^3, \qquad \tau_\Pi = \frac{2-\log 2}{2\pi T}, \qquad \kappa = \frac{\eta}{\pi T}. \tag{5.66}$$

At linear order in λ_{GB}, we find

$$\eta\tau_\Pi = \frac{\pi^2 T^2}{4\kappa_5^2}\left[2 - \log 2 + \frac{1}{2}\lambda_{GB}(-21+\log 32) + \mathcal{O}\left(\lambda_{GB}^2\right)\right], \tag{5.67}$$

which is in exact agreement with our three-point function calculation in section 5.6, the fluid/gravity calculation of section 5.8 and a recent paper [193].[7]

In the limit of most interest to us in this work, namely $\lambda_{GB}\to 1/4$, i.e. $\gamma\to 0$,

$$\eta = 0, \qquad \tau_\Pi = -\frac{1}{4\pi T}\left(\frac{1}{\gamma} - \log\frac{\gamma}{2} - \frac{3}{2} + \mathcal{O}(\gamma)\right), \qquad \kappa = -\frac{\pi^2 T^2}{\sqrt{2}\kappa_5^2}. \tag{5.68}$$

The expression for τ_Π becomes negative and diverges as $-1/\sqrt{1-4\lambda_{GB}}$, while κ also becomes negative but finite. The physical hydrodynamic coefficient $\eta\tau_\Pi$, however, goes to zero at $\lambda_{GB}\to 0$. Functions $\tau_\Pi(\lambda_{GB})$ and $\kappa(\lambda_{GB})$ are presented in Figure 5.1.

Causality of the theory at second order in the hydrodynamic expansion of the sound mode was analysed in [210], where they showed that one must demand the condition $\tau_\Pi T \geq 2\eta/s$ to be obeyed. The plot of the function, which determines the causality-preserving region, is presented in Figure 5.2. It reproduces the shape of the numerically obtained plot in [210]. The constraint on the coupling can be determined numerically and matches the approximate values of $-0.711 \leq \lambda_{GB} \leq 0.113$ found in [210]. However, this

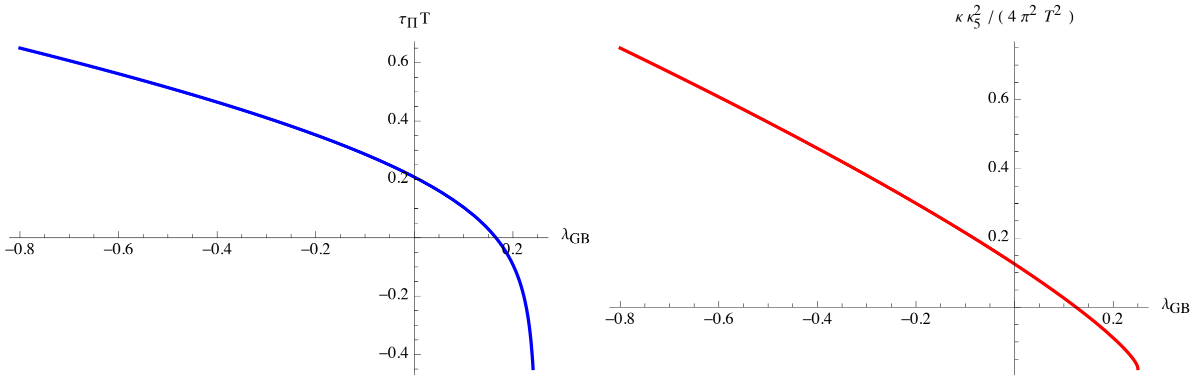

Figure 5.1: **Left:** A plot of $\tau_\Pi T$ as a function of λ_{GB}. The function diverges in the limit of $\lambda_{GB} \to 1/4$. **Right:** A plot of dimensionless $\kappa\kappa_5^2/(4\pi^2 T^2)$ as a function of λ_{GB}.

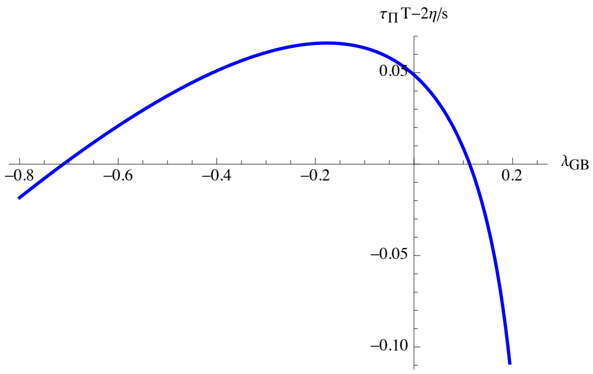

Figure 5.2: A plot of $\tau_\Pi T - 2\eta/s$ as a function of λ_{GB}.

should not be seen as a problem in our case, as we are only treating the theory as an IR effective theory.

As a final comment on the behaviour of the scalar channel, let us look at the $\gamma \to 0$ limit of the Green's function (5.60),

$$G^{xy,xy}_{R,hol}(\omega, \boldsymbol{q}) = \frac{\sqrt{2} r_+^4}{\kappa_5^2} Z_1(1, -\mathfrak{w}, -\mathfrak{q}) \frac{\partial}{\partial v} Z_1(v, \mathfrak{w}, \mathfrak{q}) \bigg|_{v=1} . \tag{5.69}$$

It was observed in [97] that (5.60) vanishes at the leading order in $\mathfrak{w}$, thus giving $\eta \to 0$. However, given that κ is non-zero in the limit, this means that the full Green's functions does not vanish in the limit of $\gamma \to 0$. In fact, this limit is very complicated as a result of an intricate interplay of zeros and infinities in various terms. As a result, the order of taking $\gamma \to 0$ does not commute with the process of computing the two-point function. This can be seen explicitly by using the exact analytic solution for Z_1 at $\gamma = 0$, which we compute in section 5.5. It is given in terms of the hypergeometric function

$$Z_1 = C_1 v^{-\frac{i\mathfrak{w}}{2}} {}_2F_1 \left[-1 - \frac{i\mathfrak{w}}{2} - \frac{\sqrt{4 - 3\mathfrak{q}^2}}{2}, -1 - \frac{i\mathfrak{w}}{2} + \frac{\sqrt{4 - 3\mathfrak{q}^2}}{2}, 1 - i\mathfrak{w}, v \right] . \tag{5.70}$$

[7] In [193], the authors used notation $\lambda_0 = \eta\tau_\Pi$ and $\lambda_{GB} = \delta/4$. By writing $\kappa_5^2 = 8\pi G_5$, our calculation reproduces their expression.

By using the expression (5.60), we now find

$$G^{xy,xy}_{R,\gamma=0}(\omega,\boldsymbol{q}) = \frac{\pi^2 T^2\left(3q^2-\omega^2\right)}{4\sqrt{2}\kappa_5^2}, \tag{5.71}$$

which would give $\kappa = -3\pi^2T^2/(2\sqrt{2}\kappa_5^2)$ and $\eta\tau_\Pi = -\pi^2T^2/(\sqrt{2}\kappa_5^2)$. This is therefore different from what we obtained by first computing $G^{xy,xy}_R$ and then taking $\gamma\to 0$.

We will defer a more detailed discussion of the Green's functions at $\lambda_{GB}=1/4$ until section 5.5. There, we will comment on a very peculiar feature of Z solutions and argue why the Green's function formula (5.69) is, in fact, most likely incorrect at $\lambda_{GB}=1/4$.

5.4.2 Shear mode

We now consider the gauge-invariant shear mode of the metric perturbation $h_{\mu\nu}(r)e^{-it\omega+iqz}$ in the radial gauge with $h_{r\mu}=0$. The relevant scalar variable is given by

$$Z_2 = \frac{q}{r^2}h_{tx} + \frac{\omega}{r^2}h_{xz}. \tag{5.72}$$

The differential equation for Z_2, written in the $u=r_+^2/r^2$ variable, is

$$Z_2'' + B_1 Z_2' + B_0 Z_2 = 0, \tag{5.73}$$

with the coefficients B_1 and B_0 given by

$$B_1(u) = -\frac{2\gamma^4(\gamma+1)\left[\frac{1}{2}\left(1-\gamma^2\right)\left(u^2-1\right)\left(U-2\right)+U-1\right]}{u(U-1)U^3\left[\gamma^2(\gamma+1)(U-1)\mathfrak{q}^2-\left(\gamma^2-1\right)U^2\mathfrak{w}^2\right]}\mathfrak{q}^2 \tag{5.74}$$

$$-\frac{\left(1-\gamma^2\right)\left(\gamma^4+\left(1-\gamma^2\right)^2u^4-2\left(1-\gamma^2\right)u^2\left(U-\gamma^2\right)-\gamma^2U\right)}{u(U-1)U\left[\gamma^2(\gamma+1)(U-1)\mathfrak{q}^2-\left(\gamma^2-1\right)U^2\mathfrak{w}^2\right]}\mathfrak{w}^2, \tag{5.75}$$

$$B_0(u) = \frac{\gamma^2(\gamma+1)(U+1)}{4u\left(u^2-1\right)U^2}\mathfrak{q}^2 + \frac{\left(U^2+2U+1\right)}{4u\left(u^2-1\right)^2}\mathfrak{w}^2. \tag{5.76}$$

We have defined $U^2\equiv u^2+\gamma^2-u^2\gamma^2$.

Quasi-normal modes are found by imposing the in-going boundary condition,

$$Z_2(u) = \left(1-u^2\right)^{-i\mathfrak{w}/2}\mathcal{Z}_2(u,\mathfrak{w},\mathfrak{q}), \tag{5.77}$$

and looking for solutions, i.e. dispersion relations, $\mathfrak{w}(\mathfrak{q})$, of the equation $\mathcal{Z}_2(u=0,\mathfrak{w},\mathfrak{q})=0$. The dispersion relation of the hydrodynamical diffusive mode that dominates the low-energy shear sector is given by

$$\mathfrak{w} = -iD\mathfrak{q}^2 + \mathcal{O}\left(\mathfrak{q}^4\right), \tag{5.78}$$

where D is the diffusion constant. The exact form of the $\mathfrak{q}^4$-order expression is unknown, as it must include contributions from third-order hydrodynamic coefficients that have not been classified. The diffusive dispersion relation is given by the hydrodynamic expression $\omega = -i\eta q^2/(\varepsilon + P)$, which translates for our conformal fluid with $\varepsilon = 3P$ into

$$D = 2\pi \frac{\eta}{s} = \frac{1}{2}\left(1 - 4\lambda_{GB}\right). \tag{5.79}$$

We are particularly interested in the behaviour of the shear mode in the limit in which λ_{GB} goes to $1/4$, i.e. the limit in which both viscosity and, hence, D vanish. In fact, we can analytically understand the effects of the hydrodynamic expansion to all orders on the diffusive dispersion relation by studying only the structure of the differential equation (5.73). We know that the shear mode's dispersion relation has the form

$$\mathfrak{w} = i \sum_{n=1}^{\infty} a_{2n}\, \mathfrak{q}^{2n}, \tag{5.80}$$

where a_{2n} are real coefficients, which depend on the hydrodynamic transport coefficients at all orders in gradient expansion. In the regime of λ_{GB} near $1/4$, i.e. $\gamma \ll 1$, by definition $\lim_{\gamma\to 0} U = u$. We can then define $\tilde{\mathfrak{q}} \equiv \gamma \mathfrak{q}$ and keep only the leading-order terms in the $\gamma \to 0$ limit. Equation (5.73) becomes

$$Z_2'' - \frac{(2-u)u\mathfrak{w}^2}{(1-u)\left(u^2\mathfrak{w}^2 - \tilde{\mathfrak{q}}^2(1-u)\right)} Z_2' + \frac{u^2\mathfrak{w}^2 - (1-u)\tilde{\mathfrak{q}}^2}{4(1-u)^2 u^3} Z_2 = 0. \tag{5.81}$$

We can immediately conclude that since the differential equation (5.81) only depends on $\tilde{\mathfrak{q}}$ and not on $\mathfrak{q}$ or γ individually, its solutions will also only depend on the product $\tilde{\mathfrak{q}} = \gamma\mathfrak{q}$. Hence, in the $\gamma \ll 1$ regime, $a_{2n} = \tilde{a}_{2n}\gamma^{2n}$, with the $\tilde{a}_{2n}$ coefficients having no dependence on γ. The shear dispersion relation thus takes the form

$$\mathfrak{w} = i \sum_{n=1}^{\infty} \tilde{a}_{2n}\, \gamma^{2n} \mathfrak{q}^{2n}, \tag{5.82}$$

and therefore hydrodynamic contributions at all orders become suppressed by powers of γ, near $\gamma \to 0$. The hydrodynamic shear mode will thus approach $\mathfrak{w} \to -i\epsilon_+$, with a real positive $\epsilon_+ \to 0$ in the limit in which viscosity vanishes. Away from this limit, $\tilde{a}_{2n}$ will of course have complicated dependences on γ.

Note that we have so far only talked about the values of λ_{GB} near $1/4$ and *not* at $\lambda_{GB} = 1/4$. We will analyse the behaviour of the quasi-normal modes at $\lambda_{GB} = 1/4$ in section 5.5. We will see that no quasi-normal mode with $\mathfrak{w} = 0$ exists at $\lambda_{GB} = 1/4$. Instead, the entire mode will vanish from the spectrum. Therefore, again, the limit of $\lambda_{GB} \to 0$ does not commute with the procedure of computing the Green's functions.

An important question about the structure of the hydrodynamic expansion can be raised at this point. Namely, whether the entire hydrodynamic tail may be controlled by shear viscosity η, such that all $a_{2n} \propto \eta$, or whether other hydrodynamic coefficients responsible for the behaviour of this dissipative mode also vanish in the limit. At present, we do not have an answer to this question.

Numerically, the quasi-normal mode spectrum is most easily found by searching for complex values of $\mathfrak{w}$ that satisfy $Z_2(u=0)=0$ at various values of $\mathfrak{q}$. We had to resort to these techniques to analyse the behaviour of the shear modes for a larger range of $\mathfrak{w}$, $\mathfrak{q}$ and for all vales of λ_{GB} between 0 and 1/4. We found that the lowest-frequency hydrodynamical mode indeed approaches $\mathfrak{w} \to 0$ in the limit of $\lambda_{GB} \to 1/4$. There exist further higher-frequency quasi-normal modes, as is usual in holography. Since their behaviour is not of direct relevance to the analysis of dissipative hydrodynamics, we will not discuss them here.

As a final comment, we note that a sequence of quasi-normal modes appears on the negative imaginary axis of the complex $\mathfrak{w}$ plane. These modes were not seen in the numerically computed spectrum of the $\mathcal{N}=4$ theory, i.e. at $\lambda_{GB}=0$, in [93]. We observe that at a fixed momentum $\mathfrak{q}$, they all travel upwards, i.e. closer to $\mathfrak{w}=0$, as λ_{GB} increases towards 1/4. Interestingly, we find that in the strict limit of $\lambda_{GB} \to 1/4$, their location precisely coincides with the quasi-normal spectrum computed from $Z_2(u)$ at $\lambda_{GB}=1/4$. The details of the calculation will be presented in 5.5. We will show that the limiting spectrum equals to the following set of quasi-normal modes,

$$\mathfrak{w} = -2i\,(1+n_1)\,, \qquad\qquad \mathfrak{w} = -2i\,(3+n_2)\,, \tag{5.83}$$

where n_1 and n_2 are non-negative integers.

5.4.3 Sound mode

We now turn our attention to the analysis of the sound channel. The relevant gauge-invariant combination of the metric perturbations for the sound mode is

$$Z_3 = \frac{2q^2}{r^2\omega^2}h_{tt} + \frac{4q}{r^2\omega}h_{tz} - \left(1 - \frac{q^2 N_\#^2\left(4r^3 - 2rf(r)\right)}{2r\omega^2\left(r^2 - 2\lambda_{GB}f(r)\right)}\right)\left(\frac{h_{xx}}{r^2} + \frac{h_{yy}}{r^2}\right) + \frac{2}{r^2}h_{zz}. \tag{5.84}$$

We have again used the $h_{r\mu}=0$ gauge. The differential equation for the sound mode, using the radial variable $u = r_+^2/r^2$, is given by

$$Z_3'' + C_1 Z_3' + C_0 Z_3 = 0, \tag{5.85}$$

where the coefficients are given by

$$\begin{aligned} C_1(u) = \frac{3}{2u} &+ \frac{3(\gamma-1)\left[\left(\gamma^2-1\right)u^2-\gamma^2\right]\left[\left(\gamma^2-1\right)u^2(5U-7)-5\gamma^2(U-1)\right]}{2u(U-1)U^2D_1}\mathfrak{w}^2 \\ &+ \frac{\left[\left(\gamma^2-1\right)^2u^4\left(-3\gamma^2+5U-7\right)+N_1\right]}{2u(U-1)U^2D_1}\mathfrak{q}^2, \end{aligned} \tag{5.86}$$

with the expressions $N_1 \equiv \gamma^2(\gamma^2-1)u^2(18\gamma^2-13U+10)-15\gamma^4(\gamma^2-2U+1)$ and $D_1 \equiv [(\gamma^2-1)u^2(3(\gamma-1)\mathfrak{w}^2+\mathfrak{q}^2)+3\gamma^2(\mathfrak{q}^2(U-1)-(\gamma-1)\mathfrak{w}^2)]$. Furthermore,

$$
\begin{aligned}
C_0(u) = \frac{(\gamma^2-1)^2}{D_0}\Big\{ & 12(\gamma-1)^2\gamma^2(\gamma+1)\mathfrak{q}^2u^5 - 4(\gamma-1)\gamma^2\mathfrak{q}^2u^3\left(3\gamma^2-7U+4\right) \\
& + \left(\gamma^2-1\right)^3\mathfrak{q}^2u^6\left(3(\gamma-1)\mathfrak{w}^2+\mathfrak{q}^2\right) \\
& - u^2\gamma^2\left(\gamma^2-1\right)\left[\mathfrak{q}^4\left(\gamma^2+2U\right)+(\gamma-1)\mathfrak{q}^2\mathfrak{w}^2\left(9\gamma^2-4U\right)-6(\gamma-1)^2U\mathfrak{w}^4\right] \\
& + \left(\gamma^2-1\right)^2u^4\left[\mathfrak{q}^4\left(3\gamma^2(U-2)+U\right)+2(\gamma-1)\mathfrak{q}^2U\mathfrak{w}^2-3(\gamma-1)^2U\mathfrak{w}^4\right] \\
& - 3\gamma^4\left[\mathfrak{q}^4\left(\gamma^2(U-2)+U\right)+2(\gamma-1)\mathfrak{q}^2\mathfrak{w}^2\left(U-\gamma^2\right)+(\gamma-1)^2U\mathfrak{w}^4\right]\Big\},
\end{aligned}
\tag{5.87}
$$

where

$$
\begin{aligned}
D_0 \equiv\; & 4(\gamma-1)u(U-1)^2U^3 \\
& \times\left[\left(\gamma^2-1\right)u^2\left(3(\gamma-1)\mathfrak{w}^2+\mathfrak{q}^2\right)+3\gamma^2\left(\mathfrak{q}^2(U-1)-(\gamma-1)\mathfrak{w}^2\right)\right].
\end{aligned}
\tag{5.88}
$$

We have again used $U^2 = u^2+\gamma^2-u^2\gamma^2$.

To find the spectrum of the sound channel, we impose the in-falling boundary condition,

$$
Z_3(u) = (1-u^2)^{-i\mathfrak{w}/2}\mathcal{Z}_3(u,\mathfrak{w},\mathfrak{q}). \tag{5.89}
$$

The lowest-frequency hydrodynamic quasi-normal mode, i.e. the sound mode, has a dispersion relation known analytically to second order in the hydrodynamic expansion [185],

$$
\mathfrak{w} = \pm\frac{\mathfrak{q}}{\sqrt{3}} - 2\pi i\Gamma T\mathfrak{q}^2 \pm 4\sqrt{3}\pi^2\Gamma T\left(\frac{1}{3}\tau_\Pi T-\frac{1}{2}\Gamma T\right)\mathfrak{q}^3+\ldots. \tag{5.90}
$$

The attenuation of the mode is controlled by

$$
\Gamma = \frac{2}{3}\frac{\eta}{sT}. \tag{5.91}
$$

Both terms proportional to q^2 and q^3 therefore vanish in the limit of $\lambda_{GB}\to 1/4$, as they are proportional to shear viscosity. In Figure 5.3, we plot the λ_{GB}-dependence of the dimensionless coefficient controlling the term proportional to q^3, $\sqrt{3}\Gamma T^2(\tau_\Pi/3-\Gamma/2)$. The question of how viscosity enters into higher-order terms again remains open in the absence of analytic understanding of higher-order hydrodynamics, as in the case of the diffusion mode.

To study the sound mode spectrum beyond second-order hydrodynamics, we must again resort to numerics. For better control over the numerics, we follow [214] and write

$$
Z_3(u) = \mathcal{A}\left[1+a_1u+\ldots\right]+\left(\mathcal{A}\,h\log u+\mathcal{B}\right)u^2\left[1+b_1u+\ldots\right], \tag{5.92}
$$

which is a standard Frobenius expansion result. The retarded Green's function is then proportional to $\mathcal{B}/\mathcal{A}$. Because of the logarithmic term in Z_3, it is beneficial to the precision

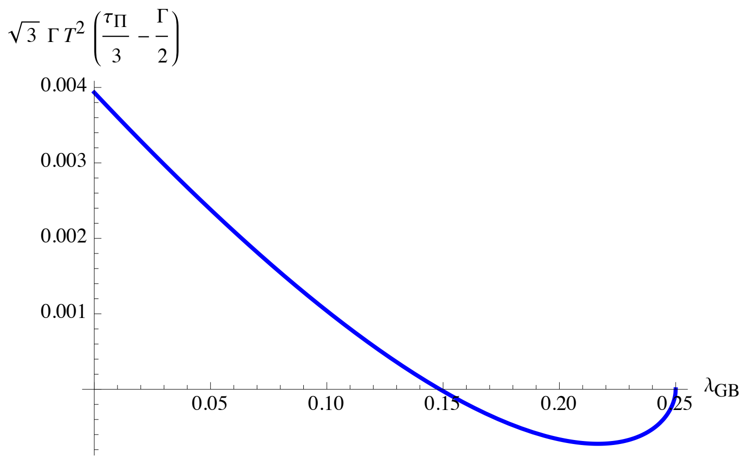

Figure 5.3: A plot of the dimensionless coefficient $\sqrt{3}\Gamma T^2\left(\tau_\Pi/3-\Gamma/2\right)$ controlling the q^3 contribution to the sound mode, as a function of λ_{GB}.

of our numerics to seek the poles of $\mathcal{B}/\mathcal{A}$ (or zeros of $\mathcal{A}/\mathcal{B}$) as opposed to the zeros of $\mathcal{A}$. Furthermore, the full Green's function includes information about the values of the residues at the poles. By writing

$$\mathcal{B}=\frac{1}{2}\lim_{u\to 0}\left(Z_3''(u)-2\mathcal{A}\,h\log u\right)-\frac{3}{2}\mathcal{A}\,h, \tag{5.93}$$

we obtain the expression, which is convenient for the computation of quasi-normal modes,

$$\frac{\mathcal{B}}{\mathcal{A}}=\lim_{u\to 0}\left[\frac{Z_3''(u)}{2Z_3(u)}-h\log u-\frac{3}{2}h\right]. \tag{5.94}$$

The coefficient h can be found analytically, $h=-8\lambda_{GB}^4\left(\mathfrak{w}^2-\mathfrak{q}^2\right)^2/\left(1-\sqrt{1-4\lambda_{GB}}\right)^4$.

Numerical results indicate that the sound pole approaches $\mathfrak{w}=\pm\mathfrak{q}/\sqrt{3}+\epsilon$, with a complex $\epsilon\to 0$, in the limit of $\lambda_{GB}\to 1/4$. We plot the real part of the dispersion relation, $\mathfrak{w}(\mathfrak{q})$, for $\lambda_{GB}=0.01$ and $\lambda_{GB}=0.2$ in Figure 5.4. Since this part of the analysis crucially depend on numerical precision, it is impossible to claim that coefficients of all terms in $\mathfrak{w}$ beyond the ideal fluid term go to zero in the limit. In fact, numerics get increasingly difficult as λ_{GB} approaches $1/4$. Similarly to the diffusion mode, the hydrodynamic sound mode also disappears from the spectrum at $\lambda_{GB}=1/4$.

The spectrum also includes the usual higher-frequency quasi-normal modes, which we will not consider here. In addition to those, there are again new poles present on the negative imaginary axis, which move towards the origin of the complex $\mathfrak{w}$ plane as λ_{GB} increases. In the limit of $\lambda_{GB}\to 1/4$, as in the shear sector, the poles coincide with the quasi-normal mode spectrum computed at $\lambda_{GB}=1/4$, which will be presented in the next section. However, there is an important difference between the behaviour of the imaginary axis sound poles and the ones found in the shear channel. Namely, the sound channel poles can cross the real axis on the complex $\mathfrak{w}$ plane, which indicates instability in the spectrum.

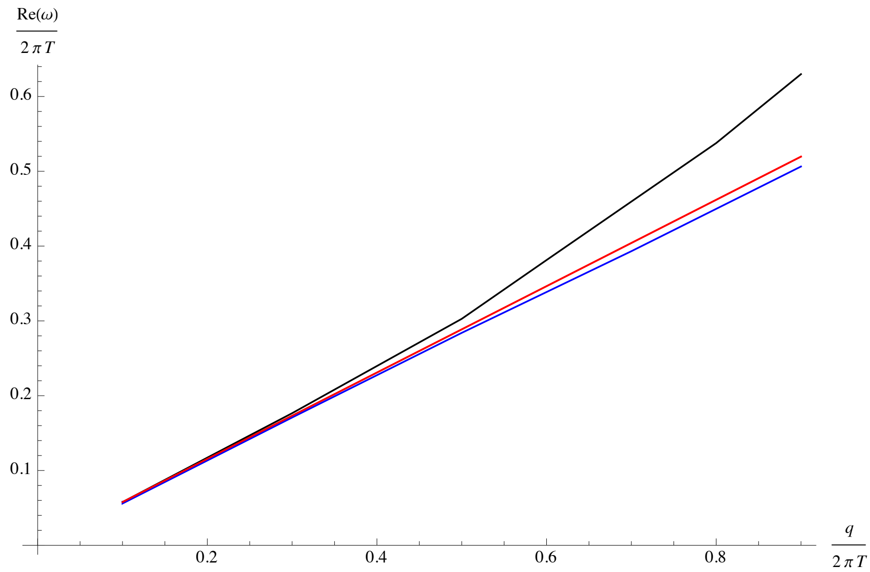

Figure 5.4: A plot of the real part of the sound mode's dispersion relation, $\Re[\mathfrak{w}(\mathfrak{q})]$, at $\lambda_{GB} = 0.01$ (black line) and $\lambda_{GB} = 0.2$ (blue line). The red line corresponds to the ideal fluid result, $\mathfrak{w} = \mathfrak{q}/\sqrt{3}$. Discrete data points are joined by lines of their respective colours. We notice that as λ_{GB} approaches 1/4, the dispersion relations become increasingly close to $\mathfrak{w} = \mathfrak{q}/\sqrt{3}$.

However, if we restrict ourselves to a finite range of momenta $\mathfrak{q}$, we can in fact avoid this instability. The value of $\mathfrak{q}$, below which the theory is stable at all λ_{GB}, will be computed analytically in section 5.5 where we will use the fact that the imaginary axis poles rise towards the origin of the complex $\mathfrak{w}$ plane (any beyond), converging towards the spectrum computed at $\lambda_{GB} = 1/4$.

5.5 Excitations at $\lambda_{GB} = 1/4$ coupling

In the previous section, we studied the behaviour of the standard hydrodynamic quasi-normal modes. However, we also observed the emergence of poles on the imaginary axis both in the shear and the sound spectrum for values of λ_{GB} approaching 1/4. In this section, we will analyse the behaviour of quasi-normal modes in the extremal $\lambda_{GB} \to 1/4$ limit, and find the spectrum analytically. Our aim is also to point out a particularly curious property of the scalar, shear and sound equations in this limit and to show that stability of the Einstein-Gauss-Bonnet theory near $\lambda_{GB} \to 1/4$ imposes a constraint on the size of allowed momenta.

Let us again perturb the metric tensor, $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, and as in section 5.4 use the momentum space metric fluctuations, $h_{\mu\nu}(r)e^{-it\omega+iqz}$, combined into gauge-invariant variables, i.e. the scalar, shear and sound modes, which we repeat for completeness.

$$\text{Scalar:} \qquad Z_1 = h^x{}_y, \tag{5.95}$$

$$\text{Shear:} \qquad Z_2 = \frac{q}{r^2} h_{tx} + \frac{\omega}{r^2} h_{xz}, \tag{5.96}$$

$$\text{Sound:} \qquad Z_3 = \frac{2q^2}{r^2\omega^2} h_{tt} + \frac{4q}{r^2\omega} h_{tz} - \left(1 - \frac{q^2 N_\#^2 \left(4r^3 - 2rf(r)\right)}{2r\omega^2 \left(r^2 - 2\lambda_{GB} f(r)\right)}\right) \left(\frac{h_{xx}}{r^2} + \frac{h_{yy}}{r^2}\right) + \frac{2}{r^2} h_{zz}. \tag{5.97}$$

We can then use the differential equations for Z_1, Z_2 and Z_3, for a general λ_{GB}, divide out potential factors of $(1-4\lambda_{GB})$, set $\lambda_{GB} = 1/4$ and study the equations for the gauge-invariant modes. The equations vastly simplify and become

$$\text{Scalar:} \qquad Z_1'' - \frac{2-u}{u(1-u)} Z_1' + \frac{\mathfrak{w}^2 - 3(1-u)\mathfrak{q}^2}{4u(1-u)^2} Z_1 = 0, \tag{5.98}$$

$$\text{Shear:} \qquad Z_2'' - \frac{2-u}{u(1-u)} Z_2' + \frac{\mathfrak{w}^2}{4u(1-u)^2} Z_2 = 0, \tag{5.99}$$

$$\text{Sound:} \qquad Z_3'' - \frac{2-u}{u(1-u)} Z_3' + \frac{\mathfrak{w}^2 + (1-u)\mathfrak{q}^2}{4u(1-u)^2} Z_3 = 0. \tag{5.100}$$

We have used the variable $u = r_+^2/r^2$, as well as the dimensionless frequency and momentum, $\mathfrak{w} = \omega/(2\pi T)$ and $\mathfrak{q} = q/(2\pi T)$.

The characteristic exponents $Z = u^\Delta$ of the leading term in the Frobenius expansion at $\lambda_{GB} = 1/4$ are $\{0,3\}$ and *not* $\{0,2\}$, which are their values for all other λ_{GB}, also arbitrarily close to $1/4$. As is usual in holography, one would expect that, since Z are graviton modes in five bulk dimensions, it is natural that they should couple to the stress-energy tensor of the four-dimensional boundary theory, thus giving $\Delta = \{0,2\}$. At $\lambda_{GB} = 1/4$, however, this is not the case. The dual theory operators scale as the spin-two stress-energy tensor in six dimensions.

All three differential equations can be solved in terms of the hypergeometric functions. Imposing the in-falling boundary conditions, we find

$$\text{Scalar:} \qquad Z_1 = (1-u)^{-\frac{i\mathfrak{w}}{2}} {}_2F_1 \left[\Omega - \frac{\sqrt{4-3\mathfrak{q}^2}}{2}, \Omega + \frac{\sqrt{4-3\mathfrak{q}^2}}{2}, 1 - i\mathfrak{w}, 1-u\right], \tag{5.101}$$

$$\text{Shear:} \qquad Z_2 = (1-u)^{-\frac{i\mathfrak{w}}{2}} {}_2F_1 \left[\Omega - 1, \Omega + 1, 1 - i\mathfrak{w}, 1-u\right], \tag{5.102}$$

$$\text{Sound:} \qquad Z_3 = (1-u)^{-\frac{i\mathfrak{w}}{2}} {}_2F_1 \left[\Omega - \frac{\sqrt{4+\mathfrak{q}^2}}{2}, \Omega + \frac{\sqrt{4+\mathfrak{q}^2}}{2}, 1 - i\mathfrak{w}, 1-u\right], \tag{5.103}$$

where $\Omega \equiv -1 - \frac{i\mathfrak{w}}{2}$. Given the three solutions, we can analytically determine the quasi-normal mode spectrum in each case to be

$$\text{Scalar:} \quad \mathfrak{w} = -i\left(4 + 2n_1 - \sqrt{4 - 3\mathfrak{q}^2}\right), \quad \mathfrak{w} = -i\left(4 + 2n_2 + \sqrt{4 - 3\mathfrak{q}^2}\right), \tag{5.104}$$

$$\text{Shear:} \quad \mathfrak{w} = -2i\,(1 + n_1)\,, \quad \mathfrak{w} = -2i\,(3 + n_2)\,, \tag{5.105}$$

$$\text{Sound:} \quad \mathfrak{w} = -i\left(4 + 2n_1 - \sqrt{4 + \mathfrak{q}^2}\right), \quad \mathfrak{w} = -i\left(4 + 2n_2 + \sqrt{4 + \mathfrak{q}^2}\right), \tag{5.106}$$

where n_1 and n_2 are non-negative integers.

As indicated by numerics in section 5.4, the imaginary part of the sound mode dispersion relation may become positive, i.e. $\text{Im}[\mathfrak{w}] > 0$, signalling an instability of the system. However, this problem can be avoided by constraining $\mathfrak{q}$ to only exist in the region of

$$\mathfrak{q} \leq 2\sqrt{3}. \tag{5.107}$$

The allowed range of q is thus set by the temperature scale, i.e. $q \leq 4\pi\sqrt{3}T$. This finding further reinforces our view that the Einstein-Gauss-Bonnet theory at large λ_{GB} may be viewed as a legitimate effective field theory, valid only at small momenta, with a well-defined hydrodynamic limit.

Another interesting feature of the behaviour of the above quasi-normal modes is that the poles of the scalar mode move off the imaginary axis for momenta in the region of $\mathfrak{q} > 2/\sqrt{3}$, i.e. their dispersion relations become complex as opposed to purely imaginary.

Finally, let us comment on the computation of retarded Green's functions at $\lambda_{GB} = 1/4$. We saw in section 5.4.1 that the two different calculations of the scalar Green's function did not coincide. We first computed $G_R^{xy,xy}$ for a general λ_{GB} and took the limit $\lambda_{GB} \to 1/4$. The second way of computing $G_R^{xy,xy}$ was to use the solution Z_1 at $\lambda_{GB} = 1/4$ from this section and plug it into the expression for the Green's function. What is unusual is not only the fact that the two limits did not commute, but that we found a transmutation from $\Delta = \{0, 2\}$ to $\Delta = \{0, 3\}$ of the characteristic exponents in all three gauge-invariant solutions. This means that the same expression for a holographic Green's function, $\mathcal{K}(u)Z(u)Z'(u)$, cannot give $\mathcal{B}/\mathcal{A}$ after the exponents change, unless $\mathcal{K}$ changes as well. It is therefore not clear what the correct prescription for the two-point function should be at $\lambda_{GB} = 1/4$. We should note that all these peculiarities arise only on the level of metric perturbations. The background metric itself has no pathological features at $\lambda_{GB} = 1/4$. It would be particularly exciting if this transmutation indicated that a five-dimensional bulk contains some knowledge about a sector of a six dimensional CFT, along the lines of [215]. However, at this point we cannot make any further claims to either substantiate or refute this speculation.

5.6 Three-point functions and second-order transport coefficients

5.6.1 Three-point functions and the Kubo formulae

To find the remaining second-order hydrodynamic transport coefficients we follow the work of [216] and perform a holographic computation of three different stress-energy tensor three-point correlation functions. We begin this section by reviewing the derivation of the necessary Kubo formulae presented in [200, 216]. Since this procedure requires a manipulation of retarded three-point functions, it is easiest to think of our microscopic CFT as defined in the Schwinger-Keldysh Closed-Time-Path formalism [9,217], which was discussed in detail in Chapter 3. However, in this case we will require the space-time to be curved and also choose a somewhat different CTP contour.

Consider a theory described by some microscopic Lagrangian $\mathcal{L}[\phi, h]$, where ϕ collectively denotes some set of matter fields and h a metric perturbation around a fixed background g. The degrees of freedom of the theory are then doubled, $\phi \to \phi^{\pm}$, $g \to g^{\pm}$, $h \to h^{\pm}$, and we use the index $\pm$ to denote whether the fields live on a "+"-time axis going from some t_0 towards the final time $t_f > t_0$, or the "−"-axis with time going from the future t_f backwards to t_0. Since our field theory is at finite temperature $T = 1/\beta$, the two separated real time contours can be joined together by a third, imaginary time axis running between t_f and $t_f - i\beta$. We use φ to denote fields living in the Euclideanised theory on the imaginary time contour. The generating functional of the stress-energy tensor correlation functions can be written as

$$W\left[h^+, h^-\right] = \log \int \mathcal{D}\phi^+ \mathcal{D}\phi^- \mathcal{D}\varphi \exp\left\{ i \int d^4x^+ \sqrt{-g^+} \mathcal{L}\left[\phi^+(x^+), h^+\right] - \int_0^\beta d^4y\, \mathcal{L}_E\left[\varphi(y)\right] - i \int d^4x^- \sqrt{-g^-} \mathcal{L}\left[\phi^-(x^-), h^-\right] \right\}. \tag{5.108}$$

It is convenient to introduce the Keldysh basis $\phi_R = \frac{1}{2}\left(\phi^+ + \phi^-\right)$ and $\phi_A = \phi^+ - \phi^-$, and similarly for the metric perturbation and the stress-energy tensor. After variation, classical expectation values always obey $\phi^+ = \phi^-$, hence all fields with an index A will vanish and we can define $T^{ab} \equiv T^{ab}_R$. Explicitly,

$$\left\langle T^{ab}_R(x) \right\rangle = - \frac{2i}{\sqrt{-g}} \frac{\partial W}{\partial h_{A\, ab}(x)} \bigg|_{h=0}. \tag{5.109}$$

The expectation value of T_R at $x = 0$ can then be expanded as

$$\left\langle T^{ab}_R(0) \right\rangle = G^{ab}_R(0) - \frac{1}{2} \int d^4x G^{ab,cd}_{RA}(0,x) h_{cd}(x) + \frac{1}{8} \int d^4x d^4y G^{ab,cd,ef}_{RAA}(0,x,y) h_{cd}(x) h_{ef}(y) + \dots, \tag{5.110}$$

where $G_{RAA\ldots}$ denote the *fully retarded* Green's functions [218], which are obtained by taking the following derivatives [216],

$$G^{ab,cd,\ldots}_{RA\ldots}(0,x,\ldots) = \frac{(-i)^{n-1}(-2i)^n \partial^n W}{\partial h_{A\,ab}(0)\partial h_{R\,cd}(x)\ldots}\bigg|_{h=0} = (-i)^{n-1}\left\langle T^{ab}_R(0)T^{cd}_A(x)\ldots\right\rangle, \tag{5.111}$$

where "..." indicate further insertions of ∂h_R in the expression with derivatives as well as the T^{ab}_A insertions into the n-point function.

All the necessary Kubo formulae for hydrodynamics up to second order are given by the following set of expressions, derived in [200, 216]. By picking momentum to flow in the z-direction, we will always perturb the scalar h_{xy} mode. On top of that, we only need to consider h_{xz} and h_{yz} perturbations to obtain

$$\eta = i \lim_{p,q\to 0}\frac{\partial}{\partial q^0}G^{xy,xz,yz}_{RAA}(p,q), \tag{5.112}$$

$$2\eta\tau_\Pi - \kappa = \lim_{p,q\to 0}\frac{\partial^2}{\partial\left(p^0\right)^2}G^{xy,xz,yz}_{RAA}(p,q), \tag{5.113}$$

$$\lambda_1 = \eta\tau_\Pi - \lim_{p,q\to 0}\frac{\partial^2}{\partial p^0\partial q^0}G^{xy,xz,yz}_{RAA}(p,q). \tag{5.114}$$

By perturbing h_{tx} and t_{ty} we find

$$\lambda_3 = 4\lim_{p,q\to 0}\frac{\partial^2}{\partial p^z\partial q^z}G^{xy,tx,ty}_{RAA}(p,q), \tag{5.115}$$

$$\kappa = \lim_{p,q\to 0}\frac{\partial^2}{\partial\left(p^z\right)^2}G^{xy,tx,ty}_{RAA}(p,q), \tag{5.116}$$

and finally, by considering only the h_{ty} and h_{xz} perturbations, we can obtain

$$\lambda_2 = 2\eta\tau_\Pi - 4\lim_{p,q\to 0}\frac{\partial^2}{\partial p^0\partial q^z}G^{xy,ty,xz}_{RAA}(p,q). \tag{5.117}$$

A consistency check on the validity of our calculations is provided by the following two Kubo formulae, which both give the expressions for pressure,

$$P = \lim_{p^0\to 0}\lim_{q^0\to 0}G^{xy,xz,yz}_{RAA}(p,q) = -\lim_{p^z\to 0}\lim_{q^z\to 0}G^{xy,tx,ty}_{RAA}(p,q). \tag{5.118}$$

Note that we have defined all hydro coefficients as in [185], which means that our λ_3 is minus the λ_3 used in [200], and our λ_2 is the negative value of the one used in [193].

The three-point functions are calculated by solving the Einstein-Gauss-Bonnet equations of motion (5.31) to second order in the relevant perturbations,

$$g_{\mu\nu} + h_{\mu\nu} = g_{\mu\nu} + \epsilon r^2 h^{(1)}_{\mu\nu} + \epsilon^2 r^2 h^{(2)}_{\mu\nu}, \tag{5.119}$$

where we impose the $h^{(2)}_{\mu\nu} = 0$ boundary condition at the AdS boundary [200]. We are using ϵ to indicate the order of perturbation. Once we find the relevant solutions, we can take three derivatives of the on-shell action with respect to the boundary value of $h_{\mu\nu}^{(b)} = h_{\mu\nu}^{(1)}(r \to \infty)$. The simplifying feature of this procedure is that since equations of motion are solved to order ϵ^2, only the boundary term contributes to the three-point function and hence no bulk-to-bulk propagators appear in the calculation.

The computation of the three different three-point functions requires us to turn on the following sets of polarisations:

$$1.) \qquad h_{xy} = h_{xy}(r) e^{-i\left(p^0+q^0\right)t}, \qquad h_{xz} = h_{xz}(r) e^{-ip^0 t}, \qquad h_{yz} = h_{yz}(r) e^{-iq^0 t}, \tag{5.120}$$

$$2.) \qquad h_{xy} = h_{xy}(r) e^{i(p^z+q^z)z}, \qquad h_{tx} = h_{tx}(r) e^{ip^z z}, \qquad h_{ty} = h_{ty}(r) e^{iq^z z}, \tag{5.121}$$

$$3.) \qquad h_{xy} = h_{xy}(r) e^{-ip^0 t + iq^z z}, \qquad h_{xz} = h_{xz}(r) e^{-ip^0 t}, \qquad h_{ty} = h_{ty}(r) e^{iq^z z}. \tag{5.122}$$

To outline the steps used in the calculation, consider calculating the $G_{RAA}^{xy,xz,yz}$ three-point function. The calculation begins by first finding solutions to $h_{xy}^{(1)}$, $h_{xz}^{(1)}$ and $h_{yz}^{(1)}$ on which we impose regular in-falling boundary conditions at the horizon and the $h_{\mu\nu}^{(1)} = h_{\mu\nu}^{(b)}$ boundary condition at the boundary. It is again most convenient, as in section 5.4.1, to perform the entire computation in the variable v (5.48).

Since we are only turning on temporal fluctuations, and turning the momentum off, it is clear that the coefficients in all three differential equation will be the same. Hence up to independent boundary values, $h_{\mu\nu}^{(b)}$ the solution of $h_{xy}^{(1)}$, $h_{xz}^{(1)}$ and $h_{yz}^{(1)}$ will have the same functional dependence on v, with the relevant frequencies, $p^0 + q^0$, p^0 and q^0, inserted, respectively. Furthermore, the solution is the same as the one obtained in section 5.4.1, with $\mathfrak{q} = 0$,

$$h_{xy}^{(1)}(v) = h_{xy}^{(b)} \left(\frac{v}{1-\gamma}\right)^{-\frac{i(p^0+q^0)}{4\pi T}} \Bigg[1 - \frac{i(p^0+q^0)}{8\pi T}\left(3 - 2\gamma - \gamma^2 - 4v + v^2\right) + \frac{(p^0+q^0)^2}{4\pi^2 T^2} g_2^{(\mathfrak{w})}(v)$$
$$+ \frac{(p^0+q^0)^2}{16\pi^2 T^2} \int^v \frac{(1-v')^2 \log\left[\gamma^2 - 1 + v' - \sqrt{(\gamma^2-1)\left(\gamma^2 - (1-v')^2\right)}\right]}{v'} dv', \tag{5.123}$$

and similarly for $h_{xz}^{(1)}$ and $h_{yz}^{(1)}$. We can deal with the remaining integral in the same way as in 5.4.1, by only integrating it order-by-order in the near-boundary expansion, $v \approx 1 - \gamma$.

Next we have to look for the second-order solution $h_{xy}^{(2)}$ with the first-order metric back-reacting on $h_{xy}^{(2)}$. The differential equation again has the form of Eq. (5.54),

$$v\,(1-v)\,\partial_v^2 h(v) + (1+v)\,\partial_v h(v) + \mathcal{H}(v) = 0, \tag{5.124}$$

with a long and complicated function $\mathcal{H}(v)$. The equation is again solved by

$$h(v) = D + \int^v dv' \frac{(1-v')^2}{v'} \left(C - \int^{v'} dv'' \frac{\mathcal{H}(v'')}{(1-v'')^3}\right). \tag{5.125}$$

In the case of $h_{xy}^{(2)}$, $\mathcal{H}$ is proportional to $p^0 q^0$. We again impose the regular in-falling boundary conditions and the $h_{xy}^{(2)} = 0$ condition at the boundary. The solution then takes the form

$$h_{xy}^{(2)} = h_{xz}^{(b)} h_{yz}^{(b)} \left(\frac{v}{1-\gamma} \right)^{-i(p^0+q^0)/(4\pi T)} \frac{p^0 q^0}{4\pi^2 T^2} h(v), \tag{5.126}$$

with a complicated and unilluminating expression for $h(v)$.

With the second-order solution in hand, we plug $g_{\mu\nu} + \epsilon r^2 h_{\mu\nu}^{(1)} + \epsilon^2 r^2 h_{\mu\nu}^{(2)}$ into the expression for the holographic stress-energy tensor [100] to compute T^{xy}. For the Einstein-Gauss-Bonnet theory, including counter-terms for a non-perturbative value of λ_{GB},

$$\langle T^{\mu\nu} \rangle = -\sqrt{-\gamma} \frac{r^2}{\kappa_5^2} \left[K^{\mu\nu} - K\gamma^{\mu\nu} + \lambda_{GB} \left(3J^{\mu\nu} - J\gamma^{\mu\nu} \right) + c_1 \gamma^{\mu\nu} + c_2 G_{(\gamma)}^{\mu\nu} \right], \tag{5.127}$$

with the counter-term coefficients,

$$c_1 = -\frac{\sqrt{2}\left(2 + \sqrt{1 - 4\lambda_{GB}}\right)}{\sqrt{1 + \sqrt{1 - 4\lambda_{GB}}}}, \qquad c_2 = \sqrt{\frac{\lambda_{GB}}{2}} \frac{\left(3 - 4\lambda_{GB} - 3\sqrt{1 - 4\lambda_{GB}}\right)}{\left(1 - \sqrt{1 - 4\lambda_{GB}}\right)^{3/2}}. \tag{5.128}$$

The induced metric on the boundary is $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, where n^μ is the vector normal to the boundary and $G_{\mu\nu}^{(\gamma)} = R_{\mu\nu}^{(\gamma)} - \frac{1}{2} R^{(\gamma)} \gamma_{\mu\nu}$ is the Einstein tensor constructed from the induced metric. The extrinsic curvature is given by

$$K_{\mu\nu} = -\frac{1}{2} \left(\nabla_\mu n_\nu + \nabla_\nu n_\mu \right), \tag{5.129}$$

K is its trace and the tensor $J_{\mu\nu}$ is defined as

$$J_{\mu\nu} = \frac{1}{3} \left(2K K_{\mu\rho} K^\rho{}_\nu + K_{\rho\sigma} K^{\rho\sigma} K_{\mu\nu} - 2K_{\mu\rho} K^{\rho\sigma} K_{\sigma\nu} - K^2 K_{\mu\nu} \right). \tag{5.130}$$

Similarly, J denotes the trace of $J_{\mu\nu}$. Note that the relative sign in front of the stress-energy tensor (5.127) as well as the signs in front of c_1 and c_2 depend on the choice of direction that n_μ is pointing in. Once we have computed the relevant component of stress-energy tensor T^{xy}, we finally need to take derivatives with respect to $h_{xz}^{(b)}$ and $h_{yz}^{(b)}$ to obtain $G_{RAA}^{xy,xz,yz}$.

The other two three-point functions are computed via exactly the same procedure, with all differential equations always taking the form of (5.124). The only difference is that we cannot impose in-falling boundary conditions on either h_{tx} or h_{ty} from (5.121), and similarly h_{ty} from (5.122), because they only fluctuate in the z-direction and not time. In fact, regularity demands that we set $h_{tx} = h_{ty} = 0$ at the horizon. Consequently, h_{xy} from (5.121) also needs to vanish at the horizon.[8]

[8]The full expressions of the three-point functions are very long and will not be presented for conciseness. For an example for a simpler calculation, going through exactly the same steps in the $\mathcal{N} = 4$ super Yang-Mills theory, see reference [200].

5.6.2 Second-order transport coefficients

Having computed all three three-point functions, we can use the Kubo formulae (5.118) to first confirm the thermodynamic result from (5.37),

$$P = \frac{\sqrt{2}\pi^4 T^4}{\kappa_5^2 (1+\gamma)^{3/2}}, \tag{5.131}$$

where we have set $L = 1$ in this calculation. The expression for the Hawking temperature T was given in (5.36). The shear viscosity is again confirmed to be

$$\eta = \frac{\sqrt{2}\pi^3 T^3}{\kappa_5^2} \frac{\gamma^2}{(1+\gamma)^{3/2}}, \tag{5.132}$$

while the second-order coefficients are given by the following functions of $\gamma = \sqrt{1-4\lambda_{GB}}$,

$$\eta\tau_\Pi = \frac{\pi^2 T^2}{4\sqrt{2}\kappa_5^2} \left(\frac{\gamma}{(1+\gamma)^{3/2}} \right) \left((1+\gamma)\left(5\gamma + \gamma^2 - 2\right) - 2\gamma \log\left[\frac{2(1+\gamma)}{\gamma}\right] \right), \tag{5.133}$$

$$\lambda_1 = \frac{\pi^2 T^2}{2\sqrt{2}\kappa_5^2} \left(\frac{3 - 4\gamma + 2\gamma^3}{\sqrt{1+\gamma}} \right), \tag{5.134}$$

$$\lambda_2 = -\frac{\pi^2 T^2}{2\sqrt{2}\kappa_5^2} \left(\frac{\gamma}{(1+\gamma)^{3/2}} \right) \left((1+\gamma)\left(2 - \gamma - \gamma^2\right) + 2\gamma \log\left[\frac{2(1+\gamma)}{\gamma}\right] \right), \tag{5.135}$$

$$\lambda_3 = -\frac{\sqrt{2}\pi^2 T^2}{\kappa_5^2} \left(\frac{3 + \gamma - 4\gamma^2}{\sqrt{1+\gamma}} \right), \tag{5.136}$$

$$\kappa = \frac{\pi^2 T^2}{\sqrt{2}\kappa_5^2} \left(\frac{2\gamma^2 - 1}{\sqrt{1+\gamma}} \right). \tag{5.137}$$

Alternatively, all the coefficients λ_n can be expressed in terms of shear viscosity η, giving us expressions (5.17) to (5.21).

In the pure Einstein theory with $\lambda_{GB} = 0$, i.e. $\gamma = 1$, all of the coefficients exactly reproduce those found in [98, 185],

$$\eta\tau_\Pi = \frac{\eta(2 - \log 2)}{2\pi T}, \qquad \lambda_1 = \frac{\eta}{2\pi T}, \qquad \lambda_2 = -\frac{\eta \log 2}{\pi T}, \qquad \lambda_3 = 0, \qquad \kappa = \frac{\eta}{\pi T}. \tag{5.138}$$

The values of the coefficients at the extreme limiting value of $\lambda_{GB} = 1/4$, i.e. $\gamma = 0$, are

$$\eta\tau_\Pi = 0, \qquad \lambda_1 = \frac{3\pi^2 T^2}{2\sqrt{2}\kappa_5^2}, \qquad \lambda_2 = 0, \qquad \lambda_3 = -\frac{3\sqrt{2}\pi^2 T^2}{\kappa_5^2}, \qquad \kappa = -\frac{\pi^2 T^2}{\sqrt{2}\kappa_5^2}. \tag{5.139}$$

All five coefficients are plotted as functions of λ_{GB} in Figure 5.5. We have represented them by dimensionless ratios, $\lambda_n \kappa_5^2/(4\pi^2 T^2)$.

While λ_1 is positive-definite for all λ_{GB} ($\lambda_1 > 0$), λ_2 and λ_3 are non-positive ($\lambda_{2,3} \leq 0$) on the interval $\lambda_{GB} \in [0, 1/4]$. Coefficients $\eta\tau_\Pi$ and κ both run from positive to negative values as λ_{GB} increases, with $\kappa = 0$ exactly half-way between the two ends of the allowed positive values of λ_{GB}, at $\lambda_{GB} = 1/8$.

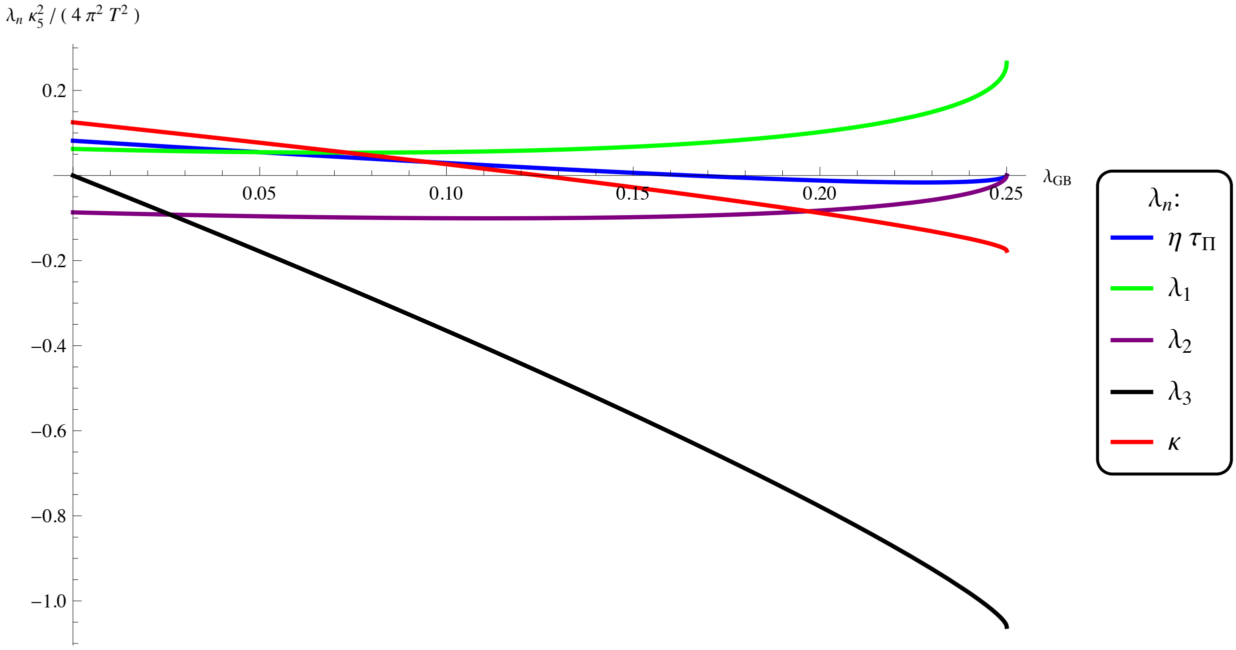

Figure 5.5: A plot of the second-order coefficients $\lambda_n = \{\eta\tau_\Pi, \lambda_1, \lambda_2, \lambda_3, \kappa\}$ in units of $4\pi^2T^2/\kappa_5^2$, as a function of $\lambda_{GB} \in [0, 1/4]$.

By further analysing the derivatives of dimensionless coefficients, $\frac{\partial}{\partial\lambda_{GB}}\left[\lambda_n\kappa_5^2/(4\pi^2T^2)\right]$, we can study the monotonicity of the hydrodynamic coefficients. The derivatives are plotted in Figure 5.6. Since all derivatives diverge at $\lambda_{GB} = 1/4$, we only plot them up to $\lambda_{GB} = 0.24$. We find that while $\eta\tau_\Pi$, λ_1 and λ_2 are not monotonic as functions of λ_{GB}, λ_3 and κ are monotonic everywhere, including for all negative λ_{GB}. This can be seen from the negative-definite expressions,

$$\frac{\kappa_5^2}{4\pi^2T^2}\frac{\partial\lambda_3}{\partial\lambda_{GB}} = -\frac{1+15\gamma+12\gamma^2}{2\sqrt{2}\gamma(1+\gamma)^{3/2}} < 0, \tag{5.140}$$

$$\frac{\kappa_5^2}{4\pi^2T^2}\frac{\partial\kappa}{\partial\lambda_{GB}} = -\frac{1+8\gamma+6\gamma^2}{4\sqrt{2}\gamma(1+\gamma)^{3/2}} < 0. \tag{5.141}$$

Both expressions (5.140) and (5.141) tend to zero in the limit of $\gamma\to\infty$, i.e. $\lambda_{GB}\to-\infty$.

It is especially interesting to note that the previously proposed and studied universality of the second-order coefficients, $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$ [191–193], is violated in the dual of the Einstein-Gauss-Bonnet theory. We find the non-perturbative result

$$2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = -\frac{\eta}{\pi T}\frac{(1-\gamma)\left(1-\gamma^2\right)(3+2\gamma)}{\gamma^2}. \tag{5.142}$$

In a perturbative λ_{GB} expansion, the right-hand-side of the expression (5.142) becomes non-zero only at the quadratic order in λ_{GB}, giving us

$$2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = -\frac{20\pi^2T^2}{\kappa_5^2}\lambda_{GB}^2 + \mathcal{O}\left(\lambda_{GB}^3\right), \tag{5.143}$$

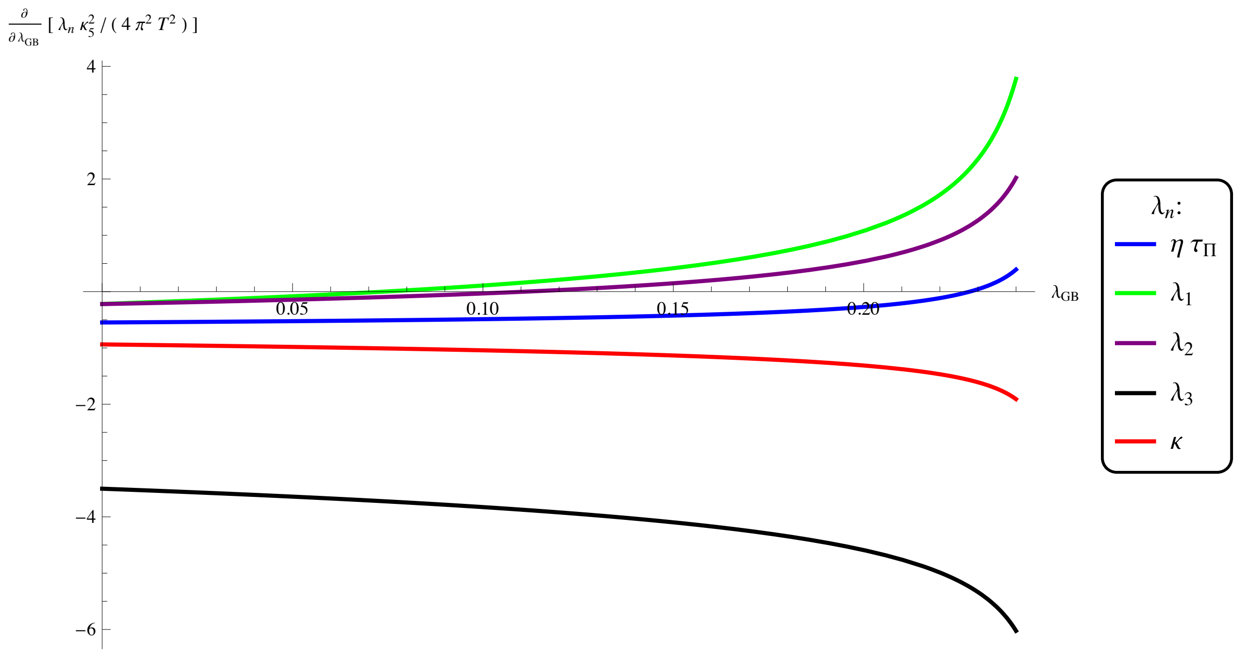

Figure 5.6: A plot of derivatives of the second-order coefficients $\lambda_n = \{\eta\tau_\Pi, \lambda_1, \lambda_2, \lambda_3, \kappa\}$ with respect to λ_{GB}, in units of $4\pi^2T^2/\kappa_5^2$, as a function of $\lambda_{GB} \in [0, 0.24]$.

which is consistent with the findings of [193]. To our knowledge, this is the first known example of the violation of the proposed universal linear combination of the three second-order coefficients.

In section 5.8, we will use the fluid/gravity correspondence to verify our expressions for the second order coefficients λ_n perturbatively in λ_{GB}. This will enable us to check the validity of the expression (5.143).

5.6.3 Boost-invariant Bjorken flow

In this section we will look at an application of the results obtained in 5.6.2 to the behaviour of boost-invariant plasmas as a function of the non-perturbative λ_{GB}. Bjorken flow is a boost-invariant solution of hydrodynamics, which is relevant to the phenomenological description of heavy-ion collisions [219]. In terms of AdS/CFT, the solution was constructed in the $\mathcal{N} = 4$ theory in [220]. The flow describes a one-dimensional motion of nuclei along a coordinate we choose as z, following the conventions of [185]. The nuclei are assumed to be infinitely large in the spatial dimensions transverse to z. In the co-moving coordinates, where the proper time is defined as $\tau = \sqrt{t^2 - z^2}$ and rapidity as $\xi = \operatorname{arctanh}(z/t)$, each fluid element is at rest, $(u^\tau, u^\xi, \boldsymbol{u}^\perp) = (1, 0, \mathbf{0})$. Furthermore, the field theory metric is given by $ds^2 = -d\tau^2 + \tau^2 d\xi^2 + d\boldsymbol{x}_\perp^2$. Because velocity u^a is constant in these coordinates, the only non-trivial derivative of u_a is $\nabla_\xi u_\xi = \tau$.

The description of a boost-invariant flow reduces to a differential equation for a single undetermined function, energy density, expressed in terms of the proper time, $\varepsilon(\tau)$,

$$u^a \nabla_a \varepsilon + (\varepsilon + P) \nabla_a u^a + \Pi^{ab} \nabla_a u_b = 0. \tag{5.144}$$

For a conformal fluid with $\varepsilon = 3P$ in $d = 4$ (boundary) dimensions, the expression simplifies to

$$\partial_\tau \varepsilon + \frac{4}{3}\frac{\varepsilon}{\tau} = -\tau \Pi^{\xi\xi}. \tag{5.145}$$

The right-hand-side of Eq. (5.145) was first written to second order in the hydrodynamic expansion in [185], and reads

$$-\tau \Pi^{\xi\xi} = 2\nu\eta\tau^{-2} + 2\nu^2 \left(\eta\tau_\Pi - \lambda_1\right) \tau^{-3} + \text{third-order hydro} + \dots, \tag{5.146}$$

where $\nu \equiv (d-2)/(d-1) = 2/3$. One can then find a solution for $\varepsilon(\tau)$ in a large-τ expansion, which is a manifestation of the gradient expansion of the hydrodynamic stress-energy tensor in derivatives of u^a. The solutions for $\varepsilon(\tau)$ takes the form [185]

$$\frac{\varepsilon}{C} = \tau^{-2+\nu} - 2\eta_0 \tau^{-2} + \left[\frac{3}{2}\eta_0^2 - \frac{2}{3}\left(\eta_0 \tau_\Pi^0 - \lambda_1^0\right)\right] \tau^{-2-\nu} + \dots, \tag{5.147}$$

where the transport coefficient functions $\eta(\tau)$, $\tau_\Pi(\tau)$ and $\lambda(\tau)$ are fixed by conformal scalings,

$$\eta = C\eta_0 \left(\frac{\varepsilon}{C}\right)^{3/4}, \qquad \tau_\Pi = \tau_\Pi^0 \left(\frac{\varepsilon}{C}\right)^{-1/4}, \qquad \lambda_1 = C\lambda_1^0 \left(\frac{\varepsilon}{C}\right)^{1/2}. \tag{5.148}$$

We would like to point out that with the knowledge of the second-order transport coefficients in the Gauss-Bonnet fluid, we know the solution of the Gauss-Bonnet Bjorken flow to the $\tau^{-2-\nu}$ order. The only relevant combination of the second-order coefficients, which enters the equations is $\eta\tau_\Pi - \lambda_1$. We plot it as a function of λ_{GB} in Figure 5.7. It is interesting to note that the combination of coefficients vanishes at two values of λ_{GB}, which can be found numerically, $\lambda_{GB} = 0.050$ and $\lambda_{GB} = -0.662$. Finally, in the limit of $\lambda_{GB} \to 1/4$, the linear combination takes a finite and negative value,

$$\eta\tau_\Pi - \lambda_1 = -\frac{3\pi^2 T^2}{2\sqrt{2}\kappa_5^2}. \tag{5.149}$$

5.6.4 Entropy current

Entropy is an important concept in the theory of relativistic hydrodynamics. As a consequence of Boltzmann's H-theorem, the divergence of the entropy current, S^a, must always be positive in order for the fluid to satisfy the positive entropy production condition,

$$\nabla_a S^a \geq 0. \tag{5.150}$$

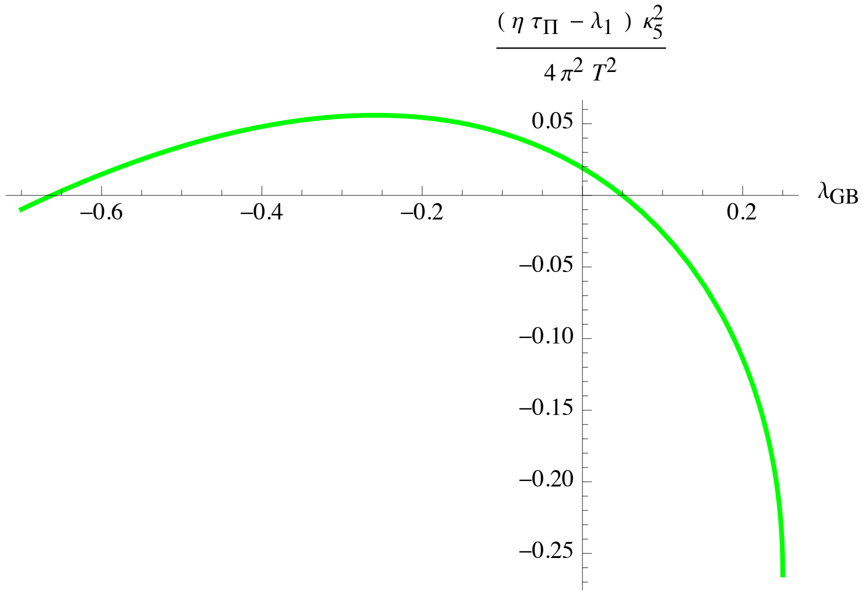

Figure 5.7: A plot of the combination of coefficients $\eta\tau_\Pi - \lambda_1$ in units of $4\pi^2T^2/\kappa_5^2$, as a function of λ_{GB}, relevant for second-order hydrodynamic contribution to the dynamics of the boost-invariant Bjorken flow.

The existence of such a current in the theory of phenomenological hydrodynamics can thus be used to constrain the structure of hydrodynamic coefficients. This avenue of research was explored in numerous papers, among others, in the recent works of [101, 188, 221, 222].

The entropy current can be constructed by using the same gradient expansion logic we used in writing down the stress-energy tensor. It can be expressed as a sum of the canonical part, S^a_{can}, and corrections,

$$S^a = S^a_{can} + S^a_{corr}, \qquad\qquad S^a_{can} = su^a - \frac{u_b \Pi^{ab}}{T}, \tag{5.151}$$

where s is the entropy density. The vectorial quantity S^a_{corr} must be written in terms of all possible tensor structures at a given order, defined by the number of derivatives acting on hydrodynamic variables. Coefficients are then introduced, multiplying each term. By imposing Eq. (5.150), the new unknown coefficients can (usually) be expressed in terms of the standard hydrodynamic transport coefficients. In second-order hydrodynamics, the entropy current was computed and analysed in [188, 221]. In the notation of [221], the divergence of the conformal entropy current is given by

$$\nabla_a S^a = \frac{\eta}{2T}\sigma_{ab}\sigma^{ab} + \frac{\kappa - 2\lambda_1}{4T}\sigma_{ab}\sigma^a{}_c\sigma^{bc} + \left(\frac{A_1}{2} + \frac{\kappa - \eta\tau_\Pi}{2T}\right)\sigma_{ab}\left[{}^{\langle}D\sigma^{ab\rangle} + \frac{1}{3}\sigma^{ab}\left(\nabla\cdot u\right)\right], \tag{5.152}$$

where A_1 is a coefficient of which the expression in terms of the second-order transport coefficients is unknown. We can see that Eq. (5.150) immediately implies that $\eta \geq 0$. Similarly, in non-conformal hydrodynamics, bulk viscosity also has to satisfy $\zeta \geq 0$.

In the usual hydrodynamics, all second-order terms are sub-leading and therefore Eq. (5.150) imposes no restrictions on the second-order hydrodynamic coefficients [188]. However, when $\eta \to 0$, as in our analysis, the signs of second-order terms matter. The difficulty in determining the constraints comes from the fact that all second-order terms include third powers of $\nabla_a u_b$, and hence the sign of tensorial quantities depends on the details of the fluid solution. Furthermore, it was pointed out in [221] that one would probably need to go to third-order hydrodynamics in order to find A_1 in terms of the second-order transport coefficients. Since A_1 is presently unknown, we cannot determine what type of fluid solutions may give positive entropy production in the Gauss-Bonnet fluid near $\lambda_{GB} \approx 1/4$, where $\eta \to 0$.

In [101], the authors studied dissipationless fluids by demanding $\nabla_a S^a = 0$. They concluded that this constraint reduced the number of independent second-order transport coefficients to *four*. Clearly, one finds that $\eta = 0$ and $\kappa = 2\lambda_1$. However, the dissipationless field theory construction of [101] left only *three* of the transport coefficients independent. The additional constraint was the proposed universal relation, $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$, of [192]. We can thus conclude that at least in the dissipationless limit, and for fluids which are invariant under volume-preserving diffeomorphisms, $S\mathrm{Diff}(\mathbb{R}^{3,1})$, $A_1 = \lambda_2/(2T)$.[9]

In the limit of $\lambda_{GB} \to 1/4$, our Gauss-Bonnet fluid does *not* behave as a dissipationless fluid. Namely, $\kappa \neq 2\lambda_1$. We plot the difference of the two coefficients, relevant for the second term in (5.150), in Figure 5.8. Furthermore, we also found that $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 \neq 0$. We can thus conclude that the Gauss-Bonnet fluid is in fact dissipative, even though its shear and sound hydrodynamic excitations behave in an approximately dissipationless manner in the limit of $\lambda_{GB} \to 1/4$. The Gauss-Bonnet fluid is thus not an example of a fluid constructed in [101]. Finally, as we do not know the expression for A_1, nor the possible solutions for the fluid's velocity profile, we cannot determine whether the fluid has positive or negative entropy production.

5.7 Gauge field action and charge diffusion

5.7.1 Four-derivative theory

In this section, we analyse the transport of charge in the theory dual to the *charge neutral* Einstein-Gauss-Bonnet theory with the black brane metric (5.32). In constructing the bottom-up vector-graviton theory, we are guided by the logic of gradient expansion, directly equivalent to the construction of the Gauss-Bonnet term. Namely, we wish to find the most

[9] The field theory construction used in [101] was discussed in detail in Chapter 3. In our analysis of dissipative first-order fluids with invariance under volume-preserving diffeomorphisms, $S\mathrm{Diff}(\mathbb{R}^{3,1})$, we similarly found a reduction in the number of independent transport coefficients, i.e. $\eta = 0$ and $\zeta \neq 0$.

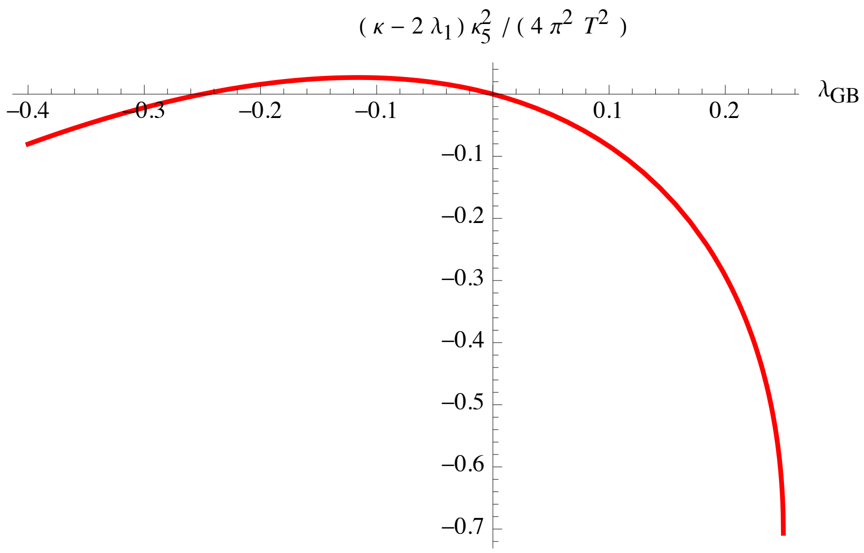

Figure 5.8: A plot of the combination of coefficients $\kappa - 2\lambda_1$ in units of $4\pi^2 T^2/\kappa_5^2$, as a function of λ_{GB}, relevant for the second-order hydrodynamic contribution to the divergence of the entropy current.

general four-derivative action of the metric $g_{\mu\nu}$ and the vector field A_μ, with the restriction that their equations of motion may involve at most second derivatives. This avoids potential problems with ghost fields and since we are interested in non-perturbative results in the new couplings, this will also avoid problems of solving higher-order differential equations. Similar theories were previously considered in [223, 224] and in the context of effective target-space heterotic string theory action in [225].

We begin by writing down the Einstein-Gauss-Bonnet theory with the most general four-derivative vector field Lagrangian,

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[R - 2\Lambda + \mathcal{L}_{GB} \right] + \int d^5x \sqrt{-g} \mathcal{L}_A, \tag{5.153}$$

where we have re-introduced the scale L. The Gauss-Bonnet Lagrangian $\mathcal{L}_{GB}$ was given in (5.14) and

$$\begin{aligned} \mathcal{L}_A = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \alpha_4 R F_{\mu\nu} F^{\mu\nu} + \alpha_5 R^{\mu\nu} F_{\mu\rho} F_\nu{}^\rho + \alpha_6 R^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \alpha_7 \left(F_{\mu\nu} F^{\mu\nu} \right)^2 \\ & + \alpha_8 \nabla_\mu F_{\rho\sigma} \nabla^\mu F^{\rho\sigma} + \alpha_9 \nabla_\mu F_{\rho\sigma} \nabla^\rho F^{\mu\sigma} + \alpha_{10} \nabla_\mu F^{\mu\nu} \nabla^\rho F_{\rho\nu} + \alpha_{11} F^{\mu\nu} F_{\nu\rho} F^{\rho\sigma} F_{\sigma\mu}. \end{aligned} \tag{5.154}$$

The modified Einstein's equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \mathcal{T}^{GB}_{\mu\nu} + 2\kappa_5^2 \mathcal{T}^A_{\mu\nu}, \tag{5.155}$$

where the gravitational stress-energy tensor term is given by

$$\begin{aligned} \mathcal{T}^{GB}_{\mu\nu} = & \frac{\lambda_{GB} L^2}{4} g_{\mu\nu} \left(R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \\ & - \lambda_{GB} L^2 \left(R R_{\mu\nu} - 2 R_{\mu\alpha} R_\nu{}^\alpha - 2 R_{\mu\alpha\nu\beta} R^{\alpha\beta} + R_{\mu\alpha\beta\gamma} R_\nu^{\alpha\beta\gamma} \right), \end{aligned} \tag{5.156}$$

and the Maxwell field contribution takes the form

$$
\begin{aligned}
\mathcal{T}^{A}_{\mu\nu} = & -\frac{1}{8}\left(g_{\mu\nu}F^{2} - 4F_{\mu\lambda}F_{\nu}{}^{\lambda}\right) \\
& + \frac{\alpha_4}{2}\left[g_{\mu\nu}RF^2 - 4RF_{\mu\alpha}F_{\nu}{}^{\alpha} - 2R_{\mu\nu}F^2 + 2\nabla_\mu\nabla_\nu F^2 - 2g_{\mu\nu}\Box F^2\right] \\
& + \frac{\alpha_5}{2}\left[g_{\mu\nu}R^{\alpha\beta}F_{\alpha\lambda}F_{\beta}{}^{\lambda} - 4R_{\mu\alpha}F_{\nu\beta}F^{\alpha\beta} - 2R_{\alpha\beta}F_{\mu}{}^{\alpha}F_{\nu}{}^{\beta} - \Box\left(F_{\mu\alpha}F_{\nu}{}^{\alpha}\right) - g_{\mu\nu}\nabla_\alpha\nabla_\beta\left(F^{\alpha}_{\ \lambda}F^{\beta\lambda}\right)\right. \\
& \qquad \left. + \nabla_\alpha\nabla_\mu\left(F_{\nu\beta}F^{\alpha\beta}\right) + \nabla_\alpha\nabla_\nu\left(F_{\mu\beta}F^{\alpha\beta}\right)\right] \\
& + \frac{\alpha_6}{2}\left[g_{\mu\nu}R^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} - 6R_{\mu\alpha\beta\gamma}F_{\nu}{}^{\alpha}F^{\beta\gamma} - 4\nabla^\beta\nabla^\alpha\left(F_{\mu\alpha}F_{\nu\beta}\right)\right] \\
& + \frac{\alpha_7}{2}\left[g_{\mu\nu}\left(F^2\right)^2 - 8F^2F_{\mu\lambda}F_{\nu}{}^{\lambda}\right] \\
& + \frac{\alpha_8}{2}\left[g_{\mu\nu}\nabla_\alpha F_{\beta\gamma}\nabla^\alpha F^{\beta\gamma} - 2\nabla_\mu F_{\alpha\beta}\nabla_\nu F^{\alpha\beta} - 4\nabla_\alpha F_{\mu\beta}\nabla^\alpha F_{\nu}{}^{\beta} + 4\nabla_\alpha\left(\nabla_\mu F^{\alpha\beta}F_{\nu\beta}\right)\right. \\
& \qquad \left. + 4\nabla_\alpha\left(\nabla^\alpha F_{\mu}{}^{\beta}F_{\nu\beta}\right) - 4\nabla_\alpha\left(\nabla_\mu F_{\nu}{}^{\beta}F^{\alpha}_{\ \beta}\right)\right] \\
& + \frac{\alpha_9}{2}\left[g_{\mu\nu}\nabla_\alpha F_{\beta\gamma}\nabla^\beta F^{\alpha\gamma} - 2\nabla_\alpha F_{\mu\beta}\nabla^\beta F_{\nu}{}^{\alpha} - 4\nabla_\mu F_{\alpha\beta}\nabla^\alpha F_{\nu}{}^{\beta} + 2\nabla_\alpha\left(\nabla^\alpha F_{\mu}{}^{\beta}F_{\nu\beta}\right)\right. \\
& \qquad \left. + 2\nabla_\alpha\left(\nabla_\mu F^{\alpha\beta}F_{\nu\beta}\right) - 2\nabla_\alpha\left(F^{\alpha}_{\ \beta}\nabla_\nu F_{\mu}{}^{\beta}\right)\right] \\
& + \frac{\alpha_{10}}{2}\left[g_{\mu\nu}\nabla_\alpha F^{\alpha\gamma}\nabla^\beta F_{\beta\gamma} - 2g_{\mu\nu}\nabla_\alpha\left(F^{\alpha\gamma}\nabla^\beta F_{\beta\gamma}\right) - 4\nabla_\mu F_{\nu\beta}\nabla_\alpha F^{\alpha\beta} - 2\nabla^\alpha F_{\mu\alpha}\nabla^\beta F_{\nu\beta}\right. \\
& \qquad \left. + 4\nabla_\mu\left(F_{\nu\beta}\nabla_\alpha F^{\alpha\beta}\right) + 4\nabla^\alpha\left(F_{\mu\alpha}\nabla^\beta F_{\nu\beta}\right)\right] \\
& + \frac{\alpha_{11}}{2}\left[g_{\mu\nu}F^{\alpha\beta}F_{\beta\gamma}F^{\gamma\delta}F_{\delta\alpha} - 8F_{\mu\alpha}F_{\nu\beta}F^{\alpha\gamma}F^{\beta}_{\ \gamma}\right]. \qquad (5.157)
\end{aligned}
$$

The modified Maxwell's equations are

$$
\begin{aligned}
\nabla_\nu F^{\mu\nu} = & \, 4\alpha_4\nabla_\nu\left(RF^{\mu\nu}\right) + 2\alpha_5\nabla_\nu\left(R^{\mu\rho}F_{\rho}{}^{\nu} - R^{\nu\rho}F_{\rho}{}^{\mu}\right) + 4\alpha_6\nabla_\nu\left(R^{\alpha\beta\mu\nu}F_{\alpha\beta}\right) \\
& + 8\alpha_7\nabla_\nu\left(F_{\alpha\beta}F^{\alpha\beta}F^{\mu\nu}\right) - 4\alpha_8\nabla_\nu\Box F^{\mu\nu} - 2\alpha_9\nabla_\nu\nabla_\rho\left(\nabla^\mu F^{\rho\nu} - \nabla^\nu F^{\rho\mu}\right) \\
& + 2\alpha_{10}\nabla_\nu\left(\nabla^\nu\nabla_\rho F^{\rho\mu} - \nabla^\mu\nabla_\rho F^{\rho\nu}\right) + 8\alpha_{11}\nabla_\nu\left(F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}\right). \qquad (5.158)
\end{aligned}
$$

To make third- and fourth-order derivatives of $g_{\mu\nu}$ and A_μ vanish in the equations of motion (5.155), we must impose the following constraints on the coefficients α_n,

$$\alpha_4 = \alpha_6, \qquad 8\alpha_4 + \alpha_5 - 4\alpha_6 = 0, \qquad (5.159)$$

$$4\alpha_4 + \alpha_5 - 2\alpha_8 - \alpha_9 = 0, \qquad 2\alpha_8 + \alpha_9 + \alpha_{10} = 0. \qquad (5.160)$$

The second constraint in (5.160) also ensures that all higher-order derivatives vanish from the Maxwell's equations. The constraints can be solved by setting

$$\alpha_6 = \alpha_4, \qquad \alpha_5 = -4\alpha_4, \qquad \alpha_9 = -2\alpha_8, \qquad \alpha_{10} = 0. \qquad (5.161)$$

Coefficients α_7 and α_{11} are left undetermined by this procedure. The reduced vector-field Lagrangian is now

$$
\begin{aligned}
\mathcal{L}_A = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \beta_1 L^2\left(RF_{\mu\nu}F^{\mu\nu} - 4R^{\mu\nu}F_{\mu\rho}F_{\nu}{}^{\rho} + R^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) \\
& + \beta_4 L^2\nabla_\mu F_{\rho\sigma}\left(\nabla^\mu F^{\rho\sigma} - 2\nabla^\rho F^{\mu\sigma}\right) + \beta_2 L^2\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \beta_3 L^2 F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}, \quad (5.162)
\end{aligned}
$$

where we have defined dimensionless couplings $\beta_1 \equiv \alpha_4/L^2$, $\beta_2 \equiv \alpha_7/L^2$, $\beta_3 \equiv \alpha_{11}/L^2$ and $\beta_4 \equiv \alpha_8/L^2$. To simplify the Lagrangian further, we notice that the term proportional to β_4 can be rewritten as

$$\nabla_\mu F_{\rho\sigma} \left(\nabla^\mu F^{\rho\sigma} - 2\nabla^\rho F^{\mu\sigma}\right) = -2\nabla^\mu \nabla^\rho A^\sigma \left(R^\lambda{}_{\mu\rho\sigma} + R^\lambda{}_{\rho\sigma\mu} + R^\lambda{}_{\sigma\mu\rho}\right) A_\lambda = 0, \tag{5.163}$$

hence the entire expression vanishes due to the cyclic property of the Riemann tensor. The vector-field theory with the desired properties is thus governed by the Lagrangian

$$\begin{aligned}\mathcal{L}_A = &-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \beta_1 L^2 \left(R F_{\mu\nu}F^{\mu\nu} - 4R^{\mu\nu}F_{\mu\rho}F_\nu{}^\rho + R^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) \\ &+ \beta_2 L^2 \left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \beta_3 L^2 F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}.\end{aligned} \tag{5.164}$$

Note that the black brane metric (5.32) is automatically a solution of this theory when $A_\mu = 0$. Although it is easy to find perturbative corrections in β_1, β_2 and β_3 to the five dimensional AdS-Reissner-Nordström metric, this is not useful for the purposes of our non-perturbative analysis. The techniques to find full non-perturbative solutions of the system under consideration are not known and hence we do not yet have the corresponding black brane metric.[10]

In the following section, we will analyse the vector field perturbation of (5.32), controlled by the action (5.164).

5.7.2 Charge diffusion

We are interested in understanding charge diffusion properties in the field theory dual of the bulk action we constructed in the previous section. To compute the charge diffusion constant, we will follow the procedure outlined in [93]. We begin by perturbing the $A_\mu = 0$ vector field as $A_\mu \to A_\mu + \epsilon a_\mu$, and writing the electromagnetic field strength corresponding to the linearised perturbation as $F = \epsilon da$.

We can immediately notice that terms proportional to α_7 and α_{11} only contribute at orders of ϵ higher than first. Hence, they will not contribute to charge diffusion. The constraint (5.160) ensures that the third- and fourth-order derivative terms in the equations of motion cancel, which enables us to treat higher-derivative contributions non-perturbatively. The vector field equations of motion simplify to

$$\nabla_\nu F^{\mu\nu} = 4\beta_1 L^2 \nabla_\nu \left(R F^{\mu\nu} + R^{\mu\nu\rho\sigma}F_{\rho\sigma} - R^{\mu\rho}F_\rho{}^\nu + R^{\nu\rho}F_\rho{}^\mu\right). \tag{5.165}$$

[10] An asymptotically AdS black hole solution to the theory considered in this section with $\beta_1 = 0$ was found in an integral form and studied in [223]. Unfortunately, the presence of a non-vanishing Lagrangian term proportional to β_1 makes the equations significantly more complicated. In particular, in the usual metric ansatz, $ds^2 = -e^{2\lambda}dt^2 + e^{2\nu}dt^2 + \ldots$, the relation $\lambda = -\nu$ is no longer true. It may nevertheless be of interest to study charge diffusion of the theory dual to the background presented in [223] in the future.

Vector field perturbations can be decomposed into transverse and longitudinal modes, with charge diffusion controlling the low-energy hydrodynamical excitations in the longitudinal sector. By selecting momentum to flow in the z-direction, the relevant gauge-invariant variable in the longitudinal sector is

$$Z_4 = \mathfrak{q} a_0 + \mathfrak{w} a_4. \tag{5.166}$$

We first define a variable $u = r_+^2/r^2$, so that the boundary is now at $u = 0$ and horizon at $u = 1$. Then we impose the in-falling boundary condition required for a calculation of retarded correlators [87, 88],

$$Z_4 = \left(1 - u^2\right)^{-i\mathfrak{w}/2} \mathcal{Z}_4(u). \tag{5.167}$$

$\mathcal{Z}_4(u)$ can be solved perturbatively in an energy-momentum expansion parameter μ, with $\mathfrak{q}$ and $\mathfrak{w}$ scaling as $\mathfrak{w} \to \mu^2\mathfrak{w}$ and $\mathfrak{q} \to \mu\mathfrak{q}$. We find it particularly useful to introduce a new variable q, so that $u = \sqrt{q^2 - \gamma^2}/\sqrt{1 - \gamma^2}$. The boundary is now at $q = \gamma$ and horizon at $q = 1$. At the order of $\mathcal{O}(\mu^0)$, the function $\mathcal{Z}_4$ can be written as $\mathcal{Z}_4 = C_1 + C_2 z(q)$, where $z(q)$ must be a solution of

$$\frac{d^2 z}{dq^2} - \frac{48\beta_1\left(q^3 - \gamma^2\right) - \gamma^2\left(1 - \gamma^2\right)}{q\left(q^2 - \gamma^2\right)\left(1 - \gamma^2 + 48\beta_1(1 - q)\right)}\frac{dz}{dq} = 0. \tag{5.168}$$

We solve for $z(q)$ and impose the $z(\gamma) = 1$ and $z(1) = 0$ boundary conditions. Constant C_2 can then be solved as a function of C_1, $\mathfrak{w}$, $\mathfrak{q}$ and other parameters of the theory by plugging $z(q)$ into the original differential equation, expanding to $\mathcal{O}(\mu^2)$ and imposing regularity at the horizon.

The diffusive quasi-normal mode can be found by solving the equation $Z_4 = 0$ at the boundary. The dispersion relation of this IR hydrodynamical mode has the form

$$\mathfrak{w} = -i\mathcal{D}\mathfrak{q}^2, \tag{5.169}$$

where $\mathcal{D}$ is the charge diffusion constant of the dual theory. We will write $\mathcal{D}$ as a function of the two Lagrangian parameters λ_{GB} and β_1 in terms of dimensionless γ and β,

$$\gamma \equiv \sqrt{1 - 4\lambda_{GB}} \qquad \text{and} \qquad \beta \equiv 1 + 48\beta_1. \tag{5.170}$$

The expression for the diffusion constant, non-perturbatively in γ and β, is then

$$\mathcal{D} = \frac{(1+\gamma)(1+2\beta)\left(\beta + \sqrt{\beta^2 - \gamma^2}\right)}{6(\beta - 1)\left[\beta\left(\beta + \sqrt{\beta^2 - \gamma^2}\right) - \gamma^2\right]} \times \left\{\sqrt{\left(1 - \gamma^2\right)\left(\beta^2 - \gamma^2\right)}\log\left[\frac{\gamma}{1 + \sqrt{1 - \gamma^2}}\right] - \left(\beta - \gamma^2\right)\log\left[\frac{\gamma}{\beta + \sqrt{\beta^2 - \gamma^2}}\right]\right\}. \tag{5.171}$$

In the pure two-derivative Maxwell limit with $\beta_1 = 0$ ($\beta = 1$), we recover the expression

$$\mathcal{D} = \frac{1}{2}\left(1 + \sqrt{1 - 4\lambda_{GB}}\right). \tag{5.172}$$

In the $\mathcal{N} = 4$ theory, where $\lambda_{GB} = 0$ and $\beta_1 = 0$, (5.172) reproduces the result of [93],

$$\mathcal{D} = 1. \tag{5.173}$$

At $\lambda_{GB} = 1/4$, Eq. (5.172) for the Einstein-Gauss-Bonnet-Maxwell theory gives

$$\mathcal{D} = 1/2. \tag{5.174}$$

In the presence of higher-derivative vector-field terms in the Lagrangian (5.164), we find the diffusion constant at the two limits of λ_{GB} to be

$$\lambda_{GB} = 0: \quad \mathcal{D} = \left(\frac{1 + 32\beta_1}{4\sqrt{6}\sqrt{\beta_1\left(1 + 24\beta_1\right)}}\right)\log\left[1 + 48\beta_1 + \sqrt{\left(1 + 48\beta_1\right)^2 - 1}\right], \tag{5.175}$$

$$\lambda_{GB} = 1/4: \quad \mathcal{D} = \left(\frac{1 + 32\beta_1}{96\beta_1}\right)\log\left(1 + 48\beta_1\right). \tag{5.176}$$

It is important to notice that $\mathcal{D}$ is not real for all values of the coefficients β_1. We will thus impose a restriction on the parameter β_1, so that $\mathcal{D} \in \mathbb{R}$. Furthermore, $\mathcal{D}$ may become negative, which would signal an instability of the diffusive mode. We will therefore also demand that $\mathcal{D} \geq 0$. From (5.171), we find that in order to have $\mathcal{D} \in \mathbb{R}$,

$$\beta_1 \geq -\frac{1}{48}\left(1 + \sqrt{1 - 4\lambda_{GB}}\right). \tag{5.177}$$

Charge diffusion of the dual theory is therefore well-defined for all values of $\lambda_{GB} \in [0, 1/4]$ if the dimensionless coupling constant β_1 accompanying the only relevant four-derivative vector field term is restricted to $-1/48 \leq \beta_1$. This parameter range also automatically ensures that $\mathcal{D} \geq 0$. Interestingly, we find that there is *no value* of β_1 that would make $\mathcal{D} = 0$. We could, however, make $\mathcal{D}$ vanish by restricting the Gauss-Bonnet coupling λ_{GB} to a smaller range. We can see from (5.171) that $\mathcal{D} = 0$ when $\beta_1 = -1/32$, which is, according to (5.177), an allowed value of the coupling so long as $\lambda_{GB} \leq 3/16$. The coupling β_1 is not bounded from above by any physical property of diffusion.

5.8 Fluid/gravity correspondence

To verify some of our above results, which all came from calculations of correlation functions, we will now turn our attention to an alternative method for extracting hydrodynamics from holography, namely the fluid/gravity correspondence [98, 186]. By using fluid/gravity correspondence in the Einstein-Gauss-Bonnet gravity, shear viscosity was calculated in [226] and second-order hydrodynamic coefficients in [193], both papers having worked perturbatively at the linear order in λ_{GB}.

Fluid/gravity uses the fact that metric perturbations $h_{\mu\nu}$ are dual, via holographic dictionary, to the stress-energy tensor of the dual boundary CFT, in the sense that $h_{\mu\nu}$ sources $T^{\mu\nu}$ in the CFT's generating functional [11, 12]. Gravitational bulk action should thus be able to capture all of the energy-momentum properties of the dual theory. The procedure for the calculation of the holographic stress-energy tensor, inspired by the prescription of Brown and York [227], was proposed by Balasubramanian and Kraus [100]. Fluid/gravity uses the fact that in appropriate variables a gradient expansion of the metric should capture the hydrodynamic gradient expansion of the CFT's stress-energy tensor. The procedure of [98] can thus be viewed as a test of the prescription established in [100].

We begin the calculation by following [98] and writing the Gauss-Bonnet black brane background solution (5.32) of the Einstein-Gauss-Bonnet equations of motion, which we repeat here for completeness,

$$\begin{aligned} E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda - \frac{\lambda_{GB} L^2}{4} g_{\mu\nu} \left(R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \\ + \lambda_{GB} L^2 \left(R R_{\mu\nu} - 2 R_{\mu\alpha} R_{\nu}^{\ \alpha} - 2 R_{\mu\alpha\nu\beta} R^{\alpha\beta} + R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} \right) = 0, \end{aligned} \tag{5.178}$$

in the Eddington-Finkelstein coordinates,

$$ds^2 = -r^2 f(br) dv^2 + 2 N_{\#} dv dr + r^2 dx^i dx^i. \tag{5.179}$$

We will again set $L = 1$ for convenience. The function $f(br)$ is still given by

$$f(br) = \frac{N_{\#}^2}{2\lambda_{GB}} \left(1 - \sqrt{1 - 4\lambda_{GB} \left(1 - \frac{1}{b^4 r^4} \right)} \right). \tag{5.180}$$

The arbitrary constant $N_{\#}^2 = \frac{1}{2}\left(1 + \sqrt{1 - 4\lambda_{GB}}\right)$ was already defined in (5.34) and set in such a way that it gives the boundary speed of light equal to unity. We also introduced b instead of $1/r_+$, to be consistent with the conventions of [98]. The stress-energy tensor is given by the expression

$$T_{\mu\nu} = \frac{r^2}{\kappa_5^2} \left[K_{\mu\nu} - K\gamma_{\mu\nu} + \lambda_{GB} \left(3J_{\mu\nu} - J\gamma_{\mu\nu} \right) + c_1 \gamma_{\mu\nu} + c_2 G_{\mu\nu}^{(\gamma)} \right], \tag{5.181}$$

where its ingredients were defined around Eq. (5.127).

The next step is to boost the brane solution (5.179) along a space-time dependent velocity four-vector u^a, where we define

$$u^a = \frac{1}{\sqrt{1 - \beta^2}} \left(1, \beta^i \right), \quad \text{with } i \in \{1, 2, 3\}. \tag{5.182}$$

Small latin indices from the beginning of the alphabet indicate four-dimensional boundary coordinates and $x^a = (v, x, y, z)$ in the Eddington-Finkelstein coordinates. The boosted black brane metric, which we denote by $g^{(0)}_{\mu\nu}$, becomes

$$ds^2_{(0)} = -2N_{\#} u_a\left(x^c\right) dx^a dr - r^2 f\left(b\left(x^c\right) r\right) u_a\left(x^c\right) u_b\left(x^c\right) dx^a dx^b + r^2 \Delta_{ab}\left(x^c\right) dx^a dx^b. \tag{5.183}$$

The metric (5.183) is no longer a solution of the Einstein-Gauss-Bonnet equations of motion and it is the essence of fluid/gravity to find corrections to the metric $g^{(0)}_{\mu\nu}$ in a gradient-expanded form so that the equations $E_{\mu\nu}$ from (5.178) are again satisfied. We will perform a gradient expansions in derivatives of $\beta^i\left(x^a\right)$ and $b\left(x^a\right)$ fields to second order, in correspondence with the boundary theory's second-order hydrodynamic gradient expansion in velocity and temperature fields described in section 5.2. The metric solution of the problem will thus take the form

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon g^{(1)}_{\mu\nu} + \epsilon^2 g^{(2)}_{\mu\nu}, \tag{5.184}$$

with $g^{(0}_{\mu\nu}$ and $g^{(1)}_{\mu\nu}$ expanded to terms with two derivatives of b and β^i. We will use powers of ϵ to denote the order of derivative expansion.

The procedure for solving (5.178), order-by-order, can be greatly simplified when one notices that it is sufficient to only solve equations of motion locally around some point $x^a = X^a$. The global metric can be obtained from this data alone [98]. The local expansions of the fields b and β^i are given by

$$b = b_{(0)}|_{X^a} + \epsilon x^a \partial_a b_{(0)}|_{X^a} + \epsilon b_{(1)}|_{X^a} + \frac{\epsilon^2}{2} x^a x^b \partial_a \partial_b b_{(0)}|_{X^a} + \epsilon^2 x^a \partial_a b_{(1)}|_{X^a}, \tag{5.185}$$

$$\beta^i = \beta^i_{(0)}|_{X^a} + \epsilon x^a \partial_a \beta^i_{(0)}|_{X^a} + \frac{\epsilon^2}{2} x^a x^b \partial_a \partial_b \beta^i_{(0)}|_{X^a}. \tag{5.186}$$

We will choose to work in a local frame at the origin, $X^a = 0$, in which

$$b_0 = 1 \qquad \text{and} \qquad \beta^i = 0. \tag{5.187}$$

Furthermore, it is consistent to choose the gauge with $\beta^i_{(1)} = 0$ at the point $x^a = X^a$.

5.8.1 First-order solution

The most general form that the first-order metric $g^{(1)}_{\mu\nu}$ can take is most conveniently written in a scalar-vector-tensor form,

$$\begin{aligned} ds^2_{(1)} = & \frac{k_1(r)}{r^2} dv^2 - 3N_{\#} h_1(r) dv dr + \frac{2}{r^2}\left(\sum_{i=1}^{3} j^i_1(r) dx^i\right) dv \\ & + r^2 h_2(r)\left(dx^2 + dy^2 + dz^2\right) + r^2 \mathcal{A}_{ab} dx^a dx^b, \end{aligned} \tag{5.188}$$

where $x^i = (x, y, z)$, k_1 and h_1 are scalars, j_1^i a three-vector and $\mathcal{A}_{ab}$ a tensor. As discussed above, we proceed by using the expanded forms of b and β^i given in (5.185) and (5.186) to write the order-ϵ metric as $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \epsilon g_{\mu\nu}^{(1)}$. We then evaluate the equations of motion (5.178), thus generating the following set of constraints and dynamical equations:

Scalar :

$$\text{Constraint 1:} \qquad r^2 f_0(r) E_{vr} + N_\# E_{vv} = 0, \tag{5.189}$$

$$\text{Constraint 2:} \qquad r^2 f_0(r) E_{rr} + N_\# E_{vr} = 0, \tag{5.190}$$

$$\text{Dynamical equation 1:} \qquad E_{rr} = 0, \tag{5.191}$$

Vector :

$$\text{Constraint 3:} \qquad r^2 f_0(r) E_{ri} + N_\# E_{vi} = 0, \tag{5.192}$$

$$\text{Dynamical equation 2:} \qquad E_{ri} = 0, \tag{5.193}$$

Tensor :

$$\text{Dynamical equation 3:} \qquad E_{ij} = 0. \tag{5.194}$$

It is easiest to first solve Dynamical equation 1 in (5.191) for $h_1(r)$. We then use Constraint 2 in (5.190), which relates $k_1'(r)$ to $h_1(r)$, to solve for $k_1(r)$. Constraints 1 and 3 in (5.189) and (5.192) give

$$\partial_v b_0 = \frac{1}{3}\partial_i \beta^i \qquad \text{and} \qquad \partial_i b_0 = \partial_v \beta^i. \tag{5.195}$$

Finally, we can solve the two remaining Dynamical equations 2 and 3 in (5.193) and (5.194) to find $j_1(r)$ and the tensor sector $\mathcal{A}_{ab}$, which contains the information about the shear viscosity.

The structure of the global first-order metric, $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \epsilon g_{\mu\nu}^{(1)}$, can be written as in [98]. It is given by the line element

$$ds^2 = \sum_{n=1}^{6} \mathcal{A}_n, \tag{5.196}$$

where we have used abbreviations $\mathcal{A}_n$ defined as

$$\mathcal{A}_1 = -2N_\# u_a dx^a dr, \qquad \mathcal{A}_2 = -r^2 f_0(br) u_a u_b dx^a dx^b, \tag{5.197}$$

$$\mathcal{A}_3 = r^2 \Delta_{ab} dx^a dx^b, \qquad \mathcal{A}_4 = 2r^2 b F_0(br) \sigma_{ab} dx^a dx^b, \tag{5.198}$$

$$\mathcal{A}_5 = \frac{2}{3} N_\# r u_a u_b \partial_c u^c dx^a dx^b, \qquad \mathcal{A}_6 = -N_\# r u^c \partial_c \left(u_a u_b\right) dx^a dx^b. \tag{5.199}$$

Our last task is to solve for $F_0(r)$, which is a part of $\mathcal{A}_4$ in (5.198). This function arises from the tensor $\mathcal{A}_{ab}$, governed by (5.194). The second-order differential equation for F_0 is

$$\frac{\partial}{\partial r}\left[\left(r^5 - \frac{r^7}{\sqrt{1-(1-r^4)\gamma^2}}\right)\frac{\partial F_0}{\partial r}\right] = \frac{(1-\gamma^2)\left(5-(5-3r^4)\gamma^2\right) r^4}{2\sqrt{2}\sqrt{1+\gamma}\left(1-(1-r^4)\gamma^2\right)^{3/2}}. \tag{5.200}$$

A pleasant feature of fluid/gravity is that, as in (5.54), the differential part of dynamical equations (the left-hand-side of (5.200)) remains the same for all unknown functions at all orders in the gradient expansion.

A solution for F_0, which is regular at the horizon and vanishes at the boundary is

$$\begin{aligned}F_0(r) = \frac{1}{8\sqrt{2}}\Bigg\{ &\frac{(1+i)\left(1-\gamma^2\right)^{1/4}\left[(1-i)\,\mathrm{arctanh}(\gamma)+\pi-(1-i)\gamma\right]}{(1-\gamma)^{1/4}(1+\gamma)^{3/4}} \\ &+ \frac{\gamma^{3/2}\Gamma\left(\frac{1}{4}\right)^2 {}_2F_1\left[\frac{1}{4},1;\frac{1}{2};\frac{1}{1-\gamma^2}\right]}{\sqrt{\pi}(1-\gamma)^{1/4}(1+\gamma)^{3/4}} + \frac{1-\gamma^2-i\pi r^4+2r^2\sqrt{1-\left(1-r^4\right)\gamma^2}}{\sqrt{\gamma}r^4} \\ &+ \frac{1}{\sqrt{1+\gamma}}\log\left[\frac{(1+r)^2\left(1+r^2\right)\left(r^2-\sqrt{1-\left(1-r^4\right)\gamma^2}\right)}{r^4\left(r^2+\sqrt{1-\left(1-r^4\right)\gamma^2}\right)}\right] - \frac{2}{\sqrt{1+\gamma}}\arctan(r) \\ &+ \frac{4r\sqrt{1-\gamma^2}}{\sqrt{1+\gamma}}F_1\left[\frac{1}{4};-\frac{1}{2},1;\frac{5}{4};-\frac{\gamma^2 r^4}{1-\gamma^2},r^4\right]\Bigg\}, \end{aligned} \tag{5.201}$$

where F_1 is the Appell hypergeometric function of two variables and ${}_2F_1$ the Gauss hypergeometric function. The power-series expansion of F_1 around r at infinity can be found from theorems presented in [180], which we can apply to find

$$\begin{aligned}&F_1\left[\frac{1}{4};-\frac{1}{2},1;\frac{5}{4};-\frac{\gamma^2 r^4}{1-\gamma^2},r^4\right] = -\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\,{}_2F_1\left[\frac{1}{4},1;\frac{1}{2};\frac{1}{1-\gamma^2}\right]}{\sqrt{\pi}}\left(\frac{\gamma^2}{1-\gamma^2}\right)^{3/4}\frac{1}{r} \\ &+\left(\frac{\gamma^2}{1-\gamma^2}\right)^{1/2}\frac{1}{r^2} + \frac{27\left(8-\gamma^2\right)\Gamma\left(-\frac{3}{4}\right)^3}{2048\sqrt{\pi}\gamma^{5/2}\left(1-\gamma^2\right)^{7/4}\Gamma\left(\frac{1}{4}\right)} \\ &\times\left\{\left(1-\gamma^2\right)\left({}_2F_1\left[-\frac{3}{4},1;\frac{1}{2};\frac{1}{1-\gamma^2}\right]+2\right)+3\gamma^2\,{}_2F_1\left[\frac{1}{4},1;\frac{1}{2};\frac{1}{1-\gamma^2}\right]\right\}\frac{1}{r^5}+\dots . \end{aligned} \tag{5.202}$$

This enables us to find the expansion of $F_0(r)$ around the boundary,

$$F_0(r) = \frac{\sqrt{1+\gamma}}{2\sqrt{2}r} - \frac{\gamma\sqrt{1+\gamma}}{8\sqrt{2}r^4} + \mathcal{O}\left(r^{-5}\right), \tag{5.203}$$

to order $\mathcal{O}(r^{-4})$, which is sufficient for the computation of the boundary stress-energy tensor. Plugging F_0 into the first-order metric $g^{\mu\nu}_{(1)}$ and computing the stress-energy tensor (5.127) with the full first-order solution, we recover the *non-perturbative* result for the shear viscosity η presented in (5.63).

5.8.2 Second-order solution

The calculation of second-order corrections to the boosted black brane metric proceeds in exactly the same way as the first-order calculation. First we perturb $g^{(0)}_{\mu\nu} + \epsilon g^{(1)}_{\mu\nu}$ to second order and then look for $g^{(2)}_{\mu\nu}$ so that the Einstein-Gauss-Bonnet equations of motion (5.178) are satisfied.

To find the second-order coefficients non-perturbatively, we would need to solve differential equations with the differential operator given in (5.200) and right-hand-sides involving integrals over the Appell function (5.202). We were not able to find useful closed-form expressions for the coefficients, but instead had to resort to a perturbative expansion in λ_{GB} to be able to at least partially verify our non-perturbative results for the second-order transport coefficients.

Since we already know from section 5.2 the tensor structure of second-order hydrodynamics, it is easiest to write down an ansatz for the line element of the second-order metric $g^{(2)}_{\mu\nu}$,

$$ds^2_{(2)} = \frac{k_2(r)}{r^2}dv^2 - 3N_\# h_2(r)dvdr + \frac{2}{r^2}\left(\sum_{i=1}^{3} j_2^i(r)dx^i\right)dv \\ + r^2 h_2(r)\left(dx^2+dy^2+dz^2\right) + r^2\sum_{n=0}^{3} P_n(r)\mathcal{B}_n, \tag{5.204}$$

where $x^i = (x,y,z)$, k_2 and h_2 are scalars, j_2^i a three-vector. We have also defined

$$\mathcal{B}_0 = \left({}_{\langle}D\sigma_{ab\rangle} + \frac{1}{3}\sigma_{ab}\left(\nabla\cdot u\right)\right)dx^a dx^b, \tag{5.205}$$

$$\mathcal{B}_1 = \sigma_{\langle a}{}^{c}\sigma_{b\rangle c}\, dx^a dx^b, \tag{5.206}$$

$$\mathcal{B}_2 = \sigma_{\langle a}{}^{c}\Omega_{b\rangle c}\, dx^a dx^b, \tag{5.207}$$

$$\mathcal{B}_3 = \Omega_{\langle a}{}^{c}\Omega_{b\rangle c}\, dx^a dx^b. \tag{5.208}$$

We may at this point focus only on the four functions P_n with $n = \{0,1,2,3\}$, which will give us the four second-order coefficients, $\lambda_0 = \eta\tau_\Pi$, λ_1, λ_2 and λ_3, respectively. The fact that the boundary theory is flat implies that this procedure does not enable us to find κ. Furthermore, we know that in the Landau frame there are no other transport coefficients coming from either the scalar or the vector sector. However, we still need to use the constraint equation $r^2 f_0(r)E_{rr} + N_\# E_{vr} = 0$ and the dynamical equation $E_{rr} = 0$ to eliminate h_2, k_2 and their derivatives from the dynamical equations for P_n.

The remaining differential equations for P_n can be solved perturbatively, to an arbitrarily high order in λ_{GB}. Here we outline the most efficient way to extract sufficient information from the functions P_n to recover the four transport coefficients. Let us expand the functions P_n in a power series near the boundary,

$$P_n(r) = \sum_{i=1}^{\infty}\frac{p_n^{(i)}}{r^i}. \tag{5.209}$$

We then plug the metric (5.204) with expanded forms of functions P_n, as in Eq. (5.209), into the full second-order metric and evaluate the stress-energy tensor (5.127). The main observation is that in the limit of $r \to \infty$, finite contributions to $T_{\mu\nu}$ only depend on coefficients of P_n proportional to r^{-4}, i.e. $T_{\mu\nu}$ depends on $p_1^{(4)}$, $p_2^{(4)}$, $p_3^{(4)}$ and $p_4^{(4)}$.

All $p_n^{(4)}$ can be found by simply plugging (5.209) into the four differential equations for P_n and expanding around $r \to \infty$. From the equation for P_0, for example, we obtain

$$\frac{p_0^{(1)}}{r^2} + \frac{2p_0^{(2)}}{r^3} + \frac{3p_0^{(3)}}{r^4} + \frac{1}{r^5}\left[4p_0^{(4)} + \left(-1 + \frac{\log(2)}{2}\right) + \left(\frac{19}{4} - \log(2)\right)\lambda_{GB} + \left(\frac{1}{8} - \log(2)\right)\lambda_{GB}^2 + \ldots\right] + \mathcal{O}\left(r^{-6}\right) = 0, \quad (5.210)$$

which enables us to find $p_0^{(4)}$. By following exactly the same procedure, we can also obtain $p_2^{(4)}$, $p_3^{(4)}$ and $p_4^{(4)}$.

With these four coefficients in hand, we can plug the metric (5.204) with (5.209) into (5.127) and take the limit $r \to \infty$. The resulting stress-energy tensor allows us to read $\eta\tau_\Pi$, λ_1, λ_2 and λ_3 from the coefficients of tensors (5.205) - (5.208). The results are in exact agreement with the λ_{GB}-expansions of the four non-perturbative second-order transport coefficients (5.17), (5.18), (5.19) and (5.20), as well as those computed in [193] to linear order. In matching these expressions, one only needs to be careful about the horizon scale r_+, which is in the fluid/gravity calculation promoted to a field $b(r)$ and fixed at $b_0 = 1$. Finally, this enables us to verify the expression (5.143), which shows the violation of the linear combination formula for the $\eta\tau_\Pi$, λ_1 and λ_2 transport coefficients.

5.9 't Hooft coupling corrections to the second-order transport in $\mathcal{N} = 4$ super Yang-Mills theory

In this section, we will analyse second-order hydrodynamics in a top-down example of a bulk theory with a higher derivative gravitational action. We will study the $\mathcal{N} = 4$ super Yang-Mills theory, which is the best understood example of the AdS/CFT correspondence [10]. In particular, we will be interested in the leading 't Hooft corrections to the hydrodynamic transport coefficients. Up to second order in gradient expansion, η, τ_Π, λ_1, λ_3 and κ have previously been computed in [194–200]. Here, we will compute the last remaining one: the λ_2 coefficient. We will again follow the procedure of [200] and use the method of thee-point functions that we already employed in Section 5.6.

It was argued in [228] that in order to compute hydrodynamic response of the original ten dimensional Type IIB supergravity with five compact dimensions, it is sufficient to consider only the reduced five dimensional action. This fact was used in [200], which we will follow in this section to compute λ_2.

The relevant five dimensional action dual to the $\mathcal{N} = 4$ theory with the leading 't Hooft correction is

$$S = \frac{N_c^2}{8\pi^2}\int d^5x\sqrt{-g}\left(R + 12 + \gamma\mathcal{W}\right), \tag{5.211}$$

where $\gamma = \alpha'^3\zeta(3)/8$, which is related to the value of the 't Hooft coupling λ. The coupling can be expressed in terms of the string scale as $\alpha'/L^2 = \lambda^{-1/2}$. The function ζ is the Riemann zeta function. $\mathcal{W}$ is given in terms of the Weyl tensor $C_{\mu\nu\rho\sigma}$ by

$$\mathcal{W} = C^{\alpha\beta\gamma\delta}C_{\mu\beta\gamma\nu}C_{\alpha}^{\ \rho\sigma\mu}C^{\nu}_{\ \rho\sigma\delta} + \frac{1}{2}C^{\alpha\delta\beta\gamma}C_{\mu\nu\beta\gamma}C_{\alpha}^{\ \rho\sigma\mu}C^{\nu}_{\ \rho\sigma\delta}. \tag{5.212}$$

The background brane solution, which we will consider in order to have a field theory dual at finite temperature, is the γ-corrected metric (5.32) (at $\lambda_{GB} = 0$), given in terms of the radial coordinate $u = r_+^2/r^2$ by

$$ds^2 = \frac{(\pi T_0)^2}{u}\left(-(1-u^2)Z_t dt^2 + dx^2 + dy^2 + dz^2\right) + Z_u\frac{du^2}{4u^2(1-u^2)}. \tag{5.213}$$

The functions Z_t and Z_u are given to order γ by

$$Z_t = 1 - 15\gamma\left(5u^2 + 5u^4 - 3u^6\right), \qquad Z_u = 1 + 15\gamma\left(5u^2 + 5u^4 - 19u^6\right). \tag{5.214}$$

The Hawking temperature is $T = T_0(1 + 15\gamma)$, where $T_0 = r_+/\pi$. Note that we have again set the AdS radius to $L = 1$.

The calculation proceeds in exactly the same way as the one in Section 5.6. The expressions involved are extremely long and will therefore not be presented here. The solutions of the relevant first-order metric fluctuations, $g_{\mu\nu} + \frac{(\pi T_0)^2}{u}\epsilon h^{(1)}_{\mu\nu}$, have the forms,

$$h^{(1)}_{xy} = e^{-iq^0t + iq^zz}\left(Z_{xy} + \gamma Z^{(\gamma)}_{xy}\right), \quad h^{(1)}_{xz} = e^{-iq^0t}\left(Z_{xz} + \gamma Z^{(\gamma)}_{xz}\right), \quad h^{(1)}_{ty} = e^{iq^zz}\left(Z_{ty} + \gamma Z^{(\gamma)}_{ty}\right), \tag{5.215}$$

where Z and $Z^{(\gamma)}$ are expanded up to second order in q^0 and q^z. All of our solutions are valid to linear order in γ. As is usual, we need to impose in-falling boundary conditions on the time-dependent fluctuations, $h^{(1)}_{xy}$ and $h^{(1)}_{xz}$. The γ-dependent exponent of $(1-u^2)^a$ is now $a = -\frac{iq^0}{4\pi T_0}(1 - 15\gamma)$. With solutions of (5.215) in hand, we can find the second-order fluctuation $h^{(2)}_{xy}$, defined by $g_{xy} + \frac{(\pi T_0)^2}{u}\epsilon h^{(1)}_{xy} + \frac{(\pi T_0)^2}{u}\epsilon^2 h^{(2)}_{xy}$.

Next we compute the holographic stress-energy tensor for the induced metric $\gamma_{\mu\nu}$,

$$T^{\mu\nu} = -\sqrt{-\gamma}\frac{N_c^2}{4\pi^2}\frac{(\pi T_0)^2}{u}\left[K^{\mu\nu} - K\gamma^{\mu\nu} + 3\left(\gamma^{\mu\nu} - \frac{1}{6}G^{\mu\nu}_{(\gamma)}\right)\right], \tag{5.216}$$

which has the same tensorial form as the one in pure Einstein theory. No higher-derivative terms contribute to its form [200]. Taking two derivatives of T^{xy} with respect to the boundary values of $h^{(1)}_{xz}$ and $h^{(1)}_{ty}$, we recover the three-point function,

$$G^{xy,ty,xz}_{RAA}(q^0, q^z) = \frac{N_c^2}{16}q^0q^zT_0^2\left(1 + 380\gamma\right). \tag{5.217}$$

Finally, by using the Kubo formula (5.117) along with the known results for η and τ_Π given in Eqs. (5.6) and (5.7), we find the expression (5.10) for λ_2. We can now show that the relation $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$ remains valid in the presence of the leading-order 't Hooft corrections, similarly to the case of the leading-order Gauss-Bonnet corrections.

5.10 Discussion

In this chapter, we discussed second-order hydrodynamics in conformal field theories with a holographic dual, particularly focusing on higher-derivative gravity corrections to the five hydrodynamic coefficients, τ_Π, λ_1, λ_2, λ_3 and κ, as well as charge diffusion. We focused on the neutral fluid dual to the Gauss-Bonnet gravity with the addition of photon fields responsible for the transport of charge. The bulk theory was constructed in such a way that the specially chosen coefficients in the four-derivative action gave equations of motion with at most two derivatives. In addition to our analysis of the Gauss-Bonnet fluid, we also completed the catalogue of 't Hooft coupling corrected second-order transport coefficients in the $\mathcal{N} = 4$ superconformal Yang-Mills theory. Namely, we found the leading-order 't Hooft coupling correction to the previously known value of λ_2 at infinite coupling.

The main motivation for this investigation was the existence of a Gauss-Bonnet limit $\lambda_{GB} \to 1/4$ in which viscosity vanishes and the possibility of finding a holographic example of a recent field theory motivated construction of fluids without dissipative viscous terms [101]. Such liquids and gases are interesting as they may posses novel types of fluid behaviour, different from those in which dissipation is controlled by viscosity. It is important to note that we have discussed a system in which no global symmetry was broken, thus the vanishing of viscosity could not be attributed to superfluidity. We showed that near $\lambda_{GB} \approx 1/4$, shear channel dissipation and sound channel attenuation were suppressed. However, the fluid still managed to produce entropy even at second order, which could be attributed to the fact that some of the second-order transport coefficients remained non-zero and that $\kappa \neq 2\lambda_1$ in the Gauss-Bonnet fluid. Furthermore, we saw that non-perturbative Gauss-Bonnet hydrodynamics violates the previously proposed universal linear combination of transport coefficients $\eta\tau_\Pi$, λ_1 and λ_2. Similarly, 't Hooft coupling corrections also broke that relation in the $\mathcal{N} = 4$ fluid.

We saw that in the limit of $\lambda_{GB} \to 1/4$, shear dissipation and sound attenuation were completely suppressed. The limit did not commute with calculations of the field theory correlation functions and spectra of excitations. More precisely, the field theory predictions calculated for a general value of λ_{GB} and analysed in the limit of $\lambda_{GB} \to 1/4$ did not agree with those obtained from setting $\lambda_{GB} = 1/4$ on the level of equations of motion and only then computing the correlation functions. In fact, the latter scenario completely eliminated hydrodynamic modes. Understanding all the intricacies of this limit remains an interesting open problem that should be addressed in the future.

Theories similar to the Einstein-Gauss-Bonnet action such as Lovelock gravity, with or without matter fields, should also be analysed in the future to see whether some of the difficulties related to causality and pathological behaviour in the disipationless limit can be avoided. It would be very interesting to analyse the *charged* fluid dual to the Gauss-Bonnet theory with the four-derivative Maxwell action discussed in Section 5.7.1. In order to pursue this goal, we would first need to find a black hole solution of the theory. And although it is easy to find perturbative corrections in β_1, β_2 and β_3 to the AdS-Reissner-Nordström black hole, such background is insufficient for studies similar to the one presented in this chapter. Unfortunately, we do not yet know of techniques able to find the solution non-perturbatively. Beyond its importance to holographic fluids in unusual and exciting regimes, the search for higher-derivative charged black holes is an important future goal in its own right.

Perhaps the most important goal for the future is to generalise second-order hydrodynamics to higher orders in derivative expansion. This could answer many open question and provide a better understanding of convergence properties of the hydrodynamics expansions. We could learn how the number of independent transport coefficients grows with the order of expansion and compute corrected dispersion relations. This could at least partially answer the question of whether vanishing η itself plays any role in suppressing higher-order contributions to diffusion and propagation of sound, or whether numerous other Gauss-Bonnet transport coefficients conspire together to suppress dissipation in the limit of extreme coupling.

Chapter 6

Conclusion

In this thesis, we approached the vast subject of hydrodynamics from three points of view, ranging from effective classical field theory to quantum field theory and string theory. We first introduced effective field theories in the language of quantum field theory and discussed the phenomenological approach to hydrodynamics, which is facilitated through the gradient expansion in derivatives of hydrodynamical variables, the velocity field and near-equilibrium generalisation of temperature and chemical potential. We then discussed why doubling the time axes and degrees of freedom within the Schwinger-Keldysh CTP formalism is necessary for computing expectation values of quantum operators acting on states that are *not* pure state at asymptotic infinity. This motivation for the fundamental importance of the CTP formalism in QFT was followed by an introduction to supersymmetry, dualities, string theory and the gauge/string duality. We also commented on the connection between holography and the Wilsonian interpretation of the renormalisation group in QFT [80]. The final part of the chapter was devoted to holographic methods for computing properties of strongly-coupled theories in the hydrodynamical limit.

Chapter 3, which was based on [209], was devoted to an important open problem of how hydrodynamics with dissipation arises as a classical effective theory, knowing that dissipation cannot be described using standard variational techniques. By adopting the view that dissipation is the energy loss of hydrodynamic macroscopic degrees of freedom to microscopic degrees of freedom, the theory of only hydrodynamic excitations should be that of an open system. To better understand such physical setups, we analysed the structure of effective actions for open systems in the CTP formalism. Establishing that such effective actions generically include terms with coupled fields from the two time axes, we used this observation to study an effective action of Goldstone modes known to describe non-dissipative hydrodynamics. The main contribution of this chapter was to show that dissipation could be incorporated into the language of CTP. We were able to recover first-order hydrodynamics with non-zero bulk viscosity. Shear viscosity vanished in this setup, which was most likely the result of the large symmetry group of volume-preserving diffeomorphism used to construct the action. The main challenge for the future is to understand how this symmetry can be relaxed to find non-zero shear viscosity, and to use the developed formalism to classify different fluids. From the point of view of QFT, the richer structure of the CTP formalism should be used in the future to rethink the structure of effective theories and to classify physical systems, which are sensitive to complicated initial states, formation of mixed states, decoherence and various other complications pertaining to open systems. To this end, the thesis began analysing the structure of the Wilsonian RG in scalar theories, something which should be built upon in future.

Gauge/string duality has provided us with invaluable insight into one of the greatest problems in theoretical physics: the analytical access to physical predictions in theories with strong coupling. However, the predictions it has thus far provided in strongly coupled hydrodynamical and condensed matter systems have been restricted to supersymmetric, large-N theories, and theories with exotic particle content. Holographic predictions for the properties of strongly interacting phases of matter are thus ofter hard to interpret and seem unusual in comparison with realistic theories. Supersymmetric low energy theories have been very rarely studied in the past. To learn about holography, as well as to uncover potential new phenomena arising from SUSY-inspired interactions and particle content, we began exploring SUSY theories in the context of low-energy condensed matter systems [229]. Chapter 4 discussed $U(1)$ super-QED theories and their deformations at finite density of electric and R-charge. We showed that, contrary to the intuition one derives from QED, scalar-fermion interactions prevent the formation of Fermi surfaces, unless SUSY is broken and the strength of interactions decreased. We also showed that, despite there being no Fermi surface, fermions were able to contribute to the total charge density. Beyond the usefulness of performing such studies to better understand the gauge/string duality, it would be particularly exciting to apply our results to potentialy realistic systems with Majorana and Dirac fermions in the presence of additional composite scalar-forming fermionic condensates. A natural hydrodynamical system, which could realise such scenario are superfluids.

In Chapter 5, we applied holographic techniques to study second-order hydrodynamics in fluids dual to theories with higher-derivative gravity [230]. Our particular goal was to analyse the Gauss-Bonnet fluid in the extreme coupling limit, $\lambda_{GB} \to 1/4$, which is dual to a fluid with (nearly)-vanishing viscosity. We also added higher-derivative vector field terms to analyse charge transport properties, finding a parameter regime of the neutral theory in which charge diffusion vanished. Because first-order hydrodynamical effects are suppressed in such a limit, we computed all five conformal second-order transport coefficients, non-perturbatively in the coupling. This computation was the chapter's main contribution to the field, along with finding the last unknown 't Hooft coupling corrected second-order coefficient, λ_2, in the $\mathcal{N} = 4$ super Yang-Mills theory. The knowledge of these coefficients enabled us to provide a counter-example to the proposed holographic universality between three of the second-order transport coefficients. The peculiar behaviour of the fluid's excitations was also analysed in this chapter. Analytically for shear, and numerically for the sound mode, we found that dissipation and attenuation were suppressed beyond second-order effects. The system's hydrodynamic excitations approached the behaviour of an ideal fluid. However, second-order effects still managed to contribute to non-trivial entropy production near $\lambda_{GB} = 1/4$, thus keeping the fluid dissipative. Precisely at $\lambda_{GB} = 1/4$, hydrodynamic modes disappeared from the spectrum and we were, fascinatingly, able to find the entire quasi-normal spectrum analytically. Several open questions regarding fluids with nearly-vanishing viscosity remain. Can one formulate a holographic dual without any pathological behaviour, which would posses dissipationless hydrodynamic modes? Can such fluids be observed in nature? What effect does third- and higher-order hydrodynamics have on the behaviour of (nearly)-dissipationless fluids? Many of these questions will be addressed in the future.

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