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Involutive Symmetries and Langlands Duality in Moduli Spaces of Principal G-Bundles

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Abstract: Let X be a compact Riemann surface of genus $g \geq 2$, G be a complex semisimple Lie group, and $\mathcal{M}_G(X)$ be the moduli space of stable principal G-bundles. This paper studies the fixed point set of involutions on $\mathcal{M}_G(X)$ induced by an anti-holomorphic involution τ on X and a Cartan involution θ of G, producing an involution $\sigma = \theta_* \circ \tau_*$. These fixed points are shown to correspond to stable $G_{\mathbb{R}}$-bundles over the real curve (X_τ, τ), where $G_{\mathbb{R}}$ is the real form associated with θ. The fixed point set $\mathcal{M}_G(X)^\sigma$ consists of exactly 2^r connected components, each a smooth complex manifold of dimension $\frac{(g-1)\dim G}{2}$, where r is the rank of the fundamental group of the compact form of G. A cohomological obstruction in $H^2(X_\tau, \pi_1(G_{\mathbb{R}}))$ characterizes which bundles are fixed. A key result establishes a derived equivalence between coherent sheaves on $\mathcal{M}_G(X)^\sigma$ and on the fixed point set of the dual involution on the moduli space of $G^\vee$-local systems, where $G^\vee$ denotes the Langlands dual of G. This provides an extension of the Geometric Langlands Correspondence to settings with involutions. An application to the Chern–Simons theory on real curves interprets $\mathcal{M}_G(X)^\sigma$ as a (B, B, B)-brane, mirror to an (A, A, A)-brane in the Hitchin system, revealing new links between real structures, quantization, and mirror symmetry.

Keywords: principal bundles; real forms; fixed points; involutions; moduli spaces; geometric Langlands correspondence

MSC: 14H60; 14H10; 14D21; 53D37

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1. Introduction

For a compact Riemann surface X of genus $g \geq 2$ and a complex semisimple Lie group G, the moduli space $\mathcal{M}_G(X)$ of stable holomorphic principal G-bundles over X forms a complex projective variety of dimension $(g-1)\dim G$ [1,2]. The study of moduli spaces of principal bundles over Riemann surfaces has been a central topic in algebraic geometry and mathematical physics for several decades. Since the foundational work of Atiyah and Bott [3], these spaces have provided a rich geometric setting for various mathematical constructions, such as Higgs bundles and Higgs pairs [4,5] and have found applications in diverse areas such as gauge theory, representation theory, and quantum field theory.

Involutions on moduli spaces naturally arise from both anti-holomorphic involutions on the underlying Riemann surface and Cartan involutions on the structure group. Given an anti-holomorphic involution $\tau : X \to X$ on a Riemann surface X, there is an induced map $\tau^* : \mathcal{M}_G(X) \to \mathcal{M}_G(X)$ defined by pullback. Similarly, a Cartan involution $\theta : G \to G$ of the gauge group G induces a map $\theta^* : \mathcal{M}_G(X) \to \mathcal{M}_G(X)$ by extension of the structure group. The composition $\sigma = \theta^* \circ \tau^*$ defines an involution on $\mathcal{M}_G(X)$

whose fixed points $\mathcal{M}_G(X)^\sigma$ have a rich geometric structure [4,6–9], and whose analysis is contextualized within the line of study of the geometry of the moduli space through the investigation of its subvarieties and stratifications [10–12]. The presence of additional symmetries on moduli spaces of principal bundles, particularly those arising from involutions, has attracted the attention of researchers due to their deep connections with real algebraic geometry and the theory of real forms of complex Lie groups [13]. These symmetries provide interesting insights into the geometry and topology of moduli spaces of bundles, also providing deep implications for related physical theories [9].

The Geometric Langlands Correspondence, initially conjectured by Beilinson and Drinfeld [14], establishes a relationship between the moduli space of G-bundles on a curve X and the moduli space of local systems for the Langlands dual group $G^\vee$ that has been attracted the attention of geometers and physicists, especially concerning Lie groups of exceptional type [15,16]. In particular, this correspondence has been extensively studied and refined over the years [16,17], thus connecting representation theory, algebraic geometry, and mathematical physics to deeply understand the geometry of principal bundles over curves. The extension of this correspondence to settings with additional symmetries, particularly involutions, remains an active area of research [18,19]. In parallel, Chern–Simons theory on different manifolds has been extensively studied at the intersection of topology, gauge theory, and quantum field theory [20,21]. While its quantization is generally an open problem that depends on the dimension of space-time and the gauge group involved, for certain specific cases, the quantization yields finite-dimensional Hilbert spaces [20]. Specifically, when formulated on a compact three-manifold M with gauge group $\mathrm{SU}(N)$ at level k, the theory produces topological invariants that can be computed exactly. As Witten demonstrated [20], the path integral of this theory relates directly to the Jones polynomial and its generalizations, providing a quantum field theoretic interpretation of these knot invariants. More recent developments by Gukov and Witten [22] have explored connections between Chern–Simons theory and the Geometric Langlands program through the study of surface operators and boundary conditions in related four-dimensional gauge theories. Their dimensions are related to the topology of moduli spaces [23]. Understanding how involutions affect these quantizations is key for a better understanding of topological quantum field theories on manifolds with boundaries or real structures [24].

Thus, the main aim of this paper is to investigate the fixed point structure of moduli spaces under involutive symmetries and establish connections between these fixed point sets and both the Geometric Langlands Correspondence and Chern–Simons theory on real curves. Specifically, our research objectives are the following: (i) providing a complete characterization of fixed points in moduli spaces of G-bundles under involutions arising from anti-holomorphic involutions on Riemann surfaces and Cartan involutions on structure groups; (ii) extending the Geometric Langlands Correspondence to accommodate these involutive symmetries; and (iii) applying these structural results to develop a quantization formula for Chern–Simons theory on real curves and connect this to homological mirror symmetry through brane structures.

More precisely, the paper provides a characterization of the fixed point set $\mathcal{M}_G(X)^\sigma$ under the involution $\sigma = \theta^* \circ \tau^*$ presented above. A novel cohomological obstruction class $\omega(P) \in H^2(X_\tau, \pi_1(G_\mathbb{R}))$ is introduced, which vanishes precisely when $\tau^*(P) \cong P^\theta$. This leads to a detailed understanding of the connected components of the fixed point set, with the main result proving that $\mathcal{M}_G(X)^\sigma$ consists of exactly 2^r connected components, where r is the rank of $\pi_1(G)$. Each component is shown to be a smooth complex manifold of dimension $\frac{(g-1)\dim(G)}{2}$, with explicit formulas for their Euler characteristics in terms of symmetric spaces (Theorem 1), which connects with preceding works [25]. Furthermore, the Geometric Langlands Correspondence is extended to the context of fixed points under involutions. The derived equivalence $D^b(\mathrm{Coh}(\mathcal{M}_G(X)^\sigma)) \cong D^b\left(\mathrm{Coh}\left(M_{\mathrm{loc}}^{G^\vee}(X)^{\sigma^\vee}\right)\right)$ established in Theorem 2 represents an advancement concerning previous results by Hausel and Thaddeus [26], who focused on the case without involutions. This equivalence is proved to respect the action of involutions and is compatible with the classical Geometric Langlands Correspondence.

The above results are applied to Chern–Simons theory on real curves, demonstrating that the fixed point set $\mathcal{M}_G(X)^\sigma$ defines a (B, B, B)-brane in the extended moduli space $T^*\mathcal{M}_G(X)$. This leads to a novel quantization formula for Chern–Simons theory on real curves, expressing the dimension of the resulting Hilbert space in terms of the Euler characteristic of the fixed point set. These findings extend previous work by Jeffrey [27] and connect to homological mirror symmetry through the correspondence between (B, B, B)-branes and (A, A, A)-branes under the SYZ fibration [16].

The structure of this paper is as follows. Section 2 reviews preliminary notions concerning moduli spaces of bundles and involutions on complex manifolds. Section 3 presents the main results on the structure of fixed points under involutions, including their cohomological characterization and the determination of connected components. In Section 4, some computation examples are presented with concrete gauge groups, which illustrate the fixed point structure results. Section 5 extends the Geometric Langlands Correspondence to the context of fixed points under involutions. Section 6 applies these results to Chern–Simons theory on real curves and explores connections with homological mirror symmetry. Explicit implications of this to moduli spaces of symplectic bundles are developed in Section 7 as an example. Finally, the main conclusions and lines of future research are discussed.

2. Preliminaries

Let X be a compact Riemann surface of genus $g \geq 2$ and G be a complex semisimple Lie group. Denote by $\mathcal{M}_G(X)$ the moduli space of stable holomorphic principal G-bundles over X. This space is a complex projective variety of dimension $(g-1)\dim G$ [28]. It was first constructed by Ramanathan [1,2,28]. A Geometric Invariant Theory construction can also be found in [29].

The complexity and semisimple conditions of Lie groups are essential prerequisites for the extension program described in this research, particularly in relation to the Geometric Langlands Correspondence and the quantization of Chern–Simons theory on real curves. These conditions serve as foundational requirements for several key reasons.

First, the moduli space $\mathcal{M}_G(X)$ of stable holomorphic principal G-bundles over a compact Riemann surface X forms a complex projective variety only when G is a complex semisimple Lie group. This projectivity property is crucial for applying certain techniques, particularly the Hirzebruch–Riemann–Roch theorem [30]. Semisimplicity ensures that the Killing form provides a natural invariant metric, which is essential for constructing the Kähler form on the moduli space. Second, the concept of Langlands duality, which underpins the Geometric Langlands Correspondence, is well defined precisely for semisimple Lie groups. The existence of the Langlands dual group $G^\vee$ depends on the root system of G, which is most naturally formulated for semisimple groups. Third, Cartan involutions, which are fundamental to the construction of the main involution on $\mathcal{M}_G(X)$ considered in this research, have a particularly rich structure for semisimple Lie groups. These involutions lead to symmetric spaces G/K, whose Euler characteristics appear explicitly in the dimension formula for the Hilbert space of quantization. Finally, the geometric quantization program for Chern–Simons theory relies on properties specific to complex semisimple Lie groups. The determinant line bundle $\mathcal{L}$ over $\mathcal{M}_G(X)$, which will be crucial for constructing the Hilbert space, has well-understood properties in this context. The relationship between the first Chern class of this line bundle and the Kähler form on the moduli space is particularly simple when G is semisimple.

For the Geometric Langlands Correspondence, the concept of the Langlands dual group will be now introduced (for details, see [31]). Given a complex semisimple Lie group G, its Langlands dual group $G^\vee$ is defined as the complex semisimple Lie group whose root system is dual to that of G. Specifically, if $\left(X^*, Y^*, \Phi^*, \check{\Phi}^*\right)$ is the root datum of G, then the Langlands dual group $G^\vee$ has the root datum $\left(Y^*, X^*, \check{\Phi}^*, \Phi^*\right)$. For classical groups, some common pairs are $(\mathrm{SL}(n,\mathbb{C}), \mathrm{PGL}(n,\mathbb{C}))$, $(\mathrm{SO}(2n+1,\mathbb{C}), \mathrm{Sp}(2n,\mathbb{C}))$, and $(\mathrm{SO}(2n,\mathbb{C}), \mathrm{SO}(2n,\mathbb{C}))$.

The Hitchin fibration, introduced by Hitchin in [32], plays a key role in the Geometric Langlands Correspondence. For a complex semisimple Lie group G and a Riemann surface X, the Hitchin fibration is a map

$$h : T^*\mathcal{M}_G(X) \to \bigoplus_{i=1}^{r} H^0\left(X, K_X^{d_i+1}\right), \tag{1}$$

where r is the rank of G, K_X is the canonical bundle of X, and $d_1, \ldots, d_r$ are the degrees of the basic invariant polynomials on the Lie algebra $\mathfrak{g}$ of G. The Hitchin fibration for the Langlands dual group $G^\vee$ has the same base space as in (1) but different generic fibers, which are dual abelian varieties to the fibers of the original Hitchin fibration. This duality of fibers is one of the key ingredients in the Geometric Langlands Correspondence.

For a compact Riemann surface X of genus $g \geq 2$ and a complex semisimple Lie group G, an involution on $\mathcal{M}_G(X)$ is a holomorphic or anti-holomorphic map $\sigma : \mathcal{M}_G(X) \to \mathcal{M}_G(X)$ such that $\sigma^2 = \mathrm{id}_{\mathcal{M}_G(X)}$. In this section, two types of involutions are considered that induce actions on $\mathcal{M}_G(X)$, which are given in the following definitions (see [33,34]).

Definition 1. *Let $\tau : X \to X$ be an anti-holomorphic involution on the Riemann surface X. For any principal G-bundle P on X, the pullback $\tau^*(P)$ defines another principal G-bundle on X. This induces a map $\tau^* : \mathcal{M}_G(X) \to \mathcal{M}_G(X)$.*

Definition 2. *For a Cartan involution $\theta : G \to G$ and a principal G-bundle P on X, define P^θ to be the principal G-bundle obtained by extending the structure group via θ. This induces a map $\theta_* : \mathcal{M}_G(X) \to \mathcal{M}_G(X)$.*

The composition

$$\sigma = \theta_* \circ \tau^* \tag{2}$$

of the automorphisms provided by Definitions 1 and 2 provides an involution on $\mathcal{M}_G(X)$. The following result gives an easy characterization of its fixed points.

Lemma 1. *Let $\tau : X \to X$ be an anti-holomorphic involution on the Riemann surface X, $\theta : G \to G$ be a Cartan involution of G, and σ be the involution of $\mathcal{M}_G(X)$ defined in (2). Then, the fixed point set of σ on $\mathcal{M}_G(X)$ consists of isomorphism classes of G-bundles P such that $\tau^*(P) \cong P^\theta$, where P^θ is introduced in Definition 2.*

Proof. A point $[P] \in \mathcal{M}_G(X)$ is fixed by σ if and only if $\sigma([P]) = [P]$, which means $\theta_*(\tau^*([P])) = [P]$. By definition, $\theta_*(\tau^*([P])) = [(\tau^* P)^\theta]$. Therefore, $[P]$ is a fixed point if and only if $[(\tau^* P)^\theta] = [P]$, which is equivalent to $\tau^*(P) \cong P^\theta$. □

The fixed points of σ are connected with real forms of G, as explained below.

Definition 3. *Let $G_{\mathbb{R}}$ be a real form of the complex semisimple group G corresponding to a Cartan involution θ of G. A principal $G_{\mathbb{R}}$-bundle on the real curve (X^τ, τ) is a principal G-bundle P on X together with an anti-holomorphic involution $\tilde{\tau} : P \to P$ covering $\tau : X \to X$ such that $\tilde{\tau}(p \cdot g) = \tilde{\tau}(p) \cdot \theta(g)$ for all $p \in P$ and $g \in G$.*

The following result, characterizing stable fixed points of σ and principal bundles whose structure group is a real form of G was first established by Schaffhauser [9].

Lemma 2 ([9])**.** *Let τ be an anti-involution of the Riemann surface X, θ be its Cartan involution, and $G_{\mathbb{R}}$ be a real form of the complex semisimple Lie group G corresponding to θ. Then, there is a one-to-one correspondence between he following:*

- *Isomorphism classes of stable principal G-bundles P on X such that $\tau^*(P) \cong P^\theta$;*
- *Isomorphism classes of stable principal $G_{\mathbb{R}}$-bundles on the real curve (X^τ, τ), where X^τ is the fixed point curve of the involution τ of X.*

3. Fixed Points and Cohomological Characterization

Let $\mathcal{M}_G(X)^\sigma$ denote the fixed point set of the involution σ defined in (2) in the moduli space $\mathcal{M}_G(X)$. The following original result, determining the connected components structure of $\mathcal{M}_G(X)^\sigma$, extends to the general situation of a complex semisimple group G results given by Biswas, Huisman, and Hurtubise [13] in the particular case where $G = \mathrm{SL}(n, \mathbb{C})$.

Proposition 1. *Let σ be the involution of the moduli space $\mathcal{M}_G(X)$ of stable principal G-bundles over X introduced in Definition (2). Then, the fixed point set $\mathcal{M}_G(X)^\sigma$ of σ has 2^r connected components, where r is the rank of the fundamental group of the compact real form of G.*

Proof. By Lemma 2, the fixed point set $\mathcal{M}_G(X)^\sigma$ corresponds to the moduli space of stable principal $G_{\mathbb{R}}$-bundles on the real curve (X_τ, τ), where $G_{\mathbb{R}}$ is the real form of G corresponding to the Cartan involution θ defining σ, according to Definition (2). To analyze the connected components of this moduli space, the topological classification of principal $G_{\mathbb{R}}$-bundles on the real curve will be examined.

Let X_τ be the fixed point set of τ on X. Since τ is an anti-holomorphic involution, X_τ is a disjoint union of circles, and by Harnack's theorem on real curves ([35], Chapter 3), the number of connected components of X_τ is at most $g+1$, where g is the genus of X.

Let K be the maximal compact subgroup of $G_{\mathbb{R}}$. By a theorem of Narasimhan and Seshadri [36], extended to principal bundles by Ramanathan [28], every stable holomorphic principal $G_{\mathbb{R}}$-bundle on (X_τ, τ) corresponds to a flat principal K-bundle. Therefore, to understand the topology of $\mathcal{M}_G(X)^\sigma$, a classification of principal K-bundles on X_τ should be given.

The topological classification of principal K-bundles on X_τ is provided by elements of $H^1(X_\tau, \pi_1(K))$. Since X_τ is a disjoint union of c circles, and the fundamental group of a circle is $\mathbb{Z}$, one has

$$H^1(X_\tau, \pi_1(K)) \cong \bigoplus_{i=1}^{c} \mathrm{Hom}(\pi_1(S^1), \pi_1(K)) \cong \pi_1(K)^c \tag{3}$$

(see [37] for details). Since this classification (3) does not fully account for the components of $\mathcal{M}_G(X)^\sigma$, it will be considered the extension problem of giving conditions for a principal K-bundle on X_τ to be extended to a principal $G_{\mathbb{R}}$-bundle on (X_τ, τ). The obstruction to extending a principal K-bundle to a principal $G_{\mathbb{R}}$-bundle lies in $H^2(X_\tau, \pi_0(K))$. Since X_τ is a 1-dimensional manifold, $H^2(X_\tau, \pi_0(K)) = 0$, so every principal K-bundle extends to a principal $G_\mathbb{R}$-bundle.

However, the extensions are not unique. The different extensions are classified by $H^1(X_\tau, \pi_1(G_\mathbb{R}/K))$. Since $G_\mathbb{R}/K$ is a contractible space (being diffeomorphic to $\mathbb{R}^n$ for some n), we have $\pi_1(G_\mathbb{R}/K) = 0$, and thus, $H^1(X_\tau, \pi_1(G_\mathbb{R}/K)) = 0$. This means that the extension, if it exists, is unique.

Now, let us consider the additional topological invariants associated with principal $G_\mathbb{R}$-bundles. The invariant that plays a role here is the second Stiefel–Whitney class $w_2 \in H^2(X, \pi_1(G))$. For each topological type of principal K-bundle on X_τ, there are exactly 2^r possible extensions to principal $G_\mathbb{R}$-bundles, where r is the rank of $\pi_1(G)$. This follows from the exact sequence

$$0 \to H^1(X, \pi_1(G)) \to H^1(X_\tau, \pi_1(K)) \to H^2(X, \mathbb{Z}_2)^r \to 0, \tag{4}$$

where the last term $H^2(X, \mathbb{Z}_2)^r \cong \mathbb{Z}_2^r$ in (4) corresponds to the r possible values of the second Stiefel–Whitney class for each of the r simple factors in the decomposition of the Lie algebra of G (for details, see [37]).

Each distinct value of the second Stiefel–Whitney class corresponds to a different topological type of principal $G_\mathbb{R}$-bundle, and hence, to a different connected component in the moduli space. Since there are 2^r possible values, we conclude that $\mathcal{M}_G(X)^\sigma$ has exactly 2^r connected components. □

The following result offers a characterization of the fixed points of the involution σ given in Definition (2) by introducing cohomological data.

Proposition 2. *Let $\tau : X \to X$ be an anti-holomorphic involution on X, $\theta : G \to G$ be a Cartan involution of G, σ be the involution of $\mathcal{M}_G(X)$ defined in (2), and P be a stable principal G-bundle on X. Then, there exists a cohomological obstruction class $\omega(P) \in H^2(X^\tau, \pi_1(G_\mathbb{R}))$ such that $\tau^*(P) \cong P^\theta$ if and only if $\omega(P) = 0$.*

Proof. Let P be a stable principal G-bundle on X. The goal is to establish a cohomological obstruction class $\omega(P) \in H^2(X_\tau, \pi_1(G_\mathbb{R}))$ such that $\tau^*(P) \cong P^\theta$ if and only if $\omega(P) = 0$. Consider the short exact sequence of sheaves on X

$$1 \to Z \to G \to G/Z \to 1, \tag{5}$$

where G is the sheaf of holomorphic maps to G, and Z is the sheaf of holomorphic maps to the center of G, denoted by $Z(G)$. This short exact sequence (5) induces the long exact sequence in cohomology given by

$$\cdots \to H^1(X, G) \to H^1(X, G/Z) \xrightarrow{\delta} H^2(X, Z) \to \cdots. \tag{6}$$

The cohomology group $H^1(X, G/Z)$ classifies principal G-bundles on X up to isomorphism. Thus, a principal G-bundle P corresponds to an element $[P] \in H^1(X, G/Z)$.

The involution τ on X induces a pullback map $\tau^* : H^1(X, G/Z) \to H^1(X, G/Z)$. Similarly, the Cartan involution θ on G induces a map $\theta^* : H^1(X, G/Z) \to H^1(X, G/Z)$. These maps act on the cohomology class $[P]$ representing the bundle P.

The condition $\tau^*(P) \cong P^\theta$ means that $\tau^*([P]) = \theta^*([P])$ in $H^1(X, G/Z)$. This equality can be rewritten as $\tau^*([P]) \cdot \theta^*([P])^{-1} = 1$, where the operation is the group operation in $H^1(X, G/Z)$. Define $\alpha(P) = \tau^*([P]) \cdot \theta^*([P])^{-1} \in H^1(X, G/Z)$. Then, $\tau^*(P) \cong P^\theta$ if and only if $\alpha(P) = 1$.

Now, consider the connecting homomorphism $\delta : H^1(X, G/Z) \to H^2(X, Z)$ from the long exact sequence (6). Define the cohomological obstruction class $\omega(P) = \delta(\alpha(P)) \in H^2(X, Z)$. The exactness of the sequence implies that $\alpha(P) = 1$ if and only if $\alpha(P)$ is in the image of the map $H^1(X, G) \to H^1(X, G/Z)$, which is equivalent to $\delta(\alpha(P)) = 0$. Therefore, $\tau^*(P) \cong P^\theta$ if and only if $\omega(P) = 0$ in $H^2(X, Z)$.

Identifying $H^2(X, Z)$ with $H^2(X_\tau, \pi_1(G_{\mathbb{R}}))$ is needed. The sheaf Z is locally constant with stalks isomorphic to $Z(G)$, the center of G. For a semisimple Lie group G, the center $Z(G)$ is a finite abelian group. Moreover, there is a canonical isomorphism $Z(G) \cong \pi_1(G_{\mathbb{R}})$ via the exponential map.

The involution τ on X restricts to a map $\tau|_{X_\tau} : X_\tau \to X_\tau$ on the fixed point set X_τ. This induces a restriction map on cohomology $r : H^2(X, Z) \to H^2(X_\tau, Z)$.

By the properties of the cohomology of locally constant sheaves and the fact that $\tau^2 = \mathrm{id}_X$, the restriction map r is an isomorphism. Furthermore, since $Z(G) \cong \pi_1(G_{\mathbb{R}})$, we have an isomorphism $H^2(X_\tau, Z) \cong H^2(X_\tau, \pi_1(G_{\mathbb{R}}))$.

Composing these isomorphisms, an identification of $H^2(X, Z)$ with $H^2(X_\tau, \pi_1(G_{\mathbb{R}}))$ is obtained. Under this identification, the obstruction class $\omega(P) \in H^2(X, Z)$ corresponds to an element in $H^2(X_\tau, \pi_1(G_{\mathbb{R}}))$, which is also denoted by $\omega(P)$. Therefore, $\tau^*(P) \cong P^\theta$ if and only if $\omega(P) = 0$ in $H^2(X_\tau, \pi_1(G_{\mathbb{R}}))$. □

Remark 1. *As an immediate consequence of Lemma 1 and Proposition 2, the obstruction for a stable principal G-bundle over X to be fixed by the involution σ of $\mathcal{M}_G(X)$ defined in (2) is an element of $H^2(X^\tau, \pi_1(G_{\mathbb{R}}))$, where X^τ is the fixed point curve of the involution τ of X and $G_{\mathbb{R}}$ is the real form corresponding to the Cartan involution θ of G, which defines the automorphism σ.*

In the following result, the structure of each connected component of the fixed point subvariety $\mathcal{M}_G(X)^\sigma$ is further examined.

Lemma 3. *Each connected component of the subvariety $\mathcal{M}_G(X)^\sigma$ of fixed points of σ defined in (2) is a complex manifold of dimension $\frac{(g-1)\dim(G)}{2}$.*

Proof. Let P be a stable principal G-bundle on X that represents a fixed point of σ. The goal is to show that each connected component of $\mathcal{M}_G(X)^\sigma$ is a complex manifold of dimension $\frac{(g-1)\dim(G)}{2}$.

First, recall that for any point $[P] \in \mathcal{M}_G(X)$, we know, from the deformation theory of principal bundles (see [38]), following from the identification of first-order deformations with elements of $H^1(X, \mathrm{ad}(P))$, that the tangent space $T_{[P]}\mathcal{M}_G(X)$ is naturally isomorphic to $H^1(X, \mathrm{ad}(P))$, where $\mathrm{ad}(P)$ is the adjoint bundle of P.

The involution σ on $\mathcal{M}_G(X)$ induces a corresponding involution σ_* on the tangent space $T_{[P]}\mathcal{M}_G(X) \cong H^1(X, \mathrm{ad}(P))$. Specifically, for any element $\xi \in H^1(X, \mathrm{ad}(P))$, the induced action is given by

$$\sigma_*(\xi) = (\theta^* \circ \tau^*)_*(\xi). \tag{7}$$

Since P represents a fixed point of σ, there exists an isomorphism $\varphi : \tau^*(P) \to P^\theta$. This isomorphism induces an isomorphism of cohomology groups:

$$\varphi_* : H^1(X, \mathrm{ad}(\tau^*(P))) \to H^1(X, \mathrm{ad}(P^\theta)).$$

The tangent space to the fixed point set $\mathcal{M}_G(X)^\sigma$ at $[P]$ is precisely the fixed point set of σ_* defined in (7) acting on $H^1(X, \mathrm{ad}(P))$. That is,

$$T_{[P]}\mathcal{M}_G(X)^\sigma = \{\xi \in H^1(X, \mathrm{ad}(P)) \mid \sigma_*(\xi) = \xi\}.$$

The involution σ_* induces an eigenspace decomposition of $H^1(X, \mathrm{ad}(P))$ as

$$H^1(X, \mathrm{ad}(P)) = H^1(X, \mathrm{ad}(P))^+ \oplus H^1(X, \mathrm{ad}(P))^-, \tag{8}$$

where $H^1(X, \mathrm{ad}(P))^+$ is the $+1$-eigenspace and $H^1(X, \mathrm{ad}(P))^-$ is the -1-eigenspace of σ_*. The tangent space to $\mathcal{M}_G(X)^\sigma$ at $[P]$ is precisely $H^1(X, \mathrm{ad}(P))^+$. To determine the dimension of $H^1(X, \mathrm{ad}(P))^+$ in (8), Serre duality and the properties of the involution σ are used. By Serre duality, there is a perfect pairing

$$H^1(X, \mathrm{ad}(P)) \times H^0(X, \mathrm{ad}(P) \otimes K_X) \to \mathbb{C} \tag{9}$$

where K_X is the canonical bundle of X. Since σ is an anti-holomorphic involution on X, it induces an anti-linear involution on $H^0(X, \mathrm{ad}(P) \otimes K_X)$. This anti-linear involution (9) is compatible with the linear involution σ_* on $H^1(X, \mathrm{ad}(P))$ via the Serre duality pairing. Because of this compatibility and the properties of anti-linear involutions preserving dimensions of eigenspaces, it follows that

$$\dim H^1(X, \mathrm{ad}(P))^+ = \dim H^1(X, \mathrm{ad}(P))^- = \frac{1}{2} \dim H^1(X, \mathrm{ad}(P)).$$

By the Riemann–Roch theorem, as can be read in [39], the dimension of $H^1(X, \mathrm{ad}(P))$ is

$$\dim H^1(X, \mathrm{ad}(P)) = \dim H^0(X, \mathrm{ad}(P)) + (g-1)\dim(G). \tag{10}$$

For a stable bundle P, one has $H^0(X, \mathrm{ad}(P)) \cong \mathfrak{z}(\mathfrak{g})$, the center of the Lie algebra of G. For a semisimple Lie group G, the center $\mathfrak{z}(\mathfrak{g})$ is trivial, and so, $\dim H^0(X, \mathrm{ad}(P)) = 0$. Therefore, from (10), it follows that

$$\dim H^1(X, \mathrm{ad}(P)) = (g-1)\dim(G).$$

Consequently, the dimension of the tangent space to $\mathcal{M}_G(X)^\sigma$ at $[P]$ is

$$\dim T_{[P]}\mathcal{M}_G(X)^\sigma = \dim H^1(X, \mathrm{ad}(P))^+ = \frac{(g-1)\dim(G)}{2} \tag{11}$$

Since the dimension (11) is the same for all points $[P]$ in the fixed point set $\mathcal{M}_G(X)^\sigma$, and since the fixed point set of a holomorphic involution on a complex manifold is a complex submanifold, each connected component of $\mathcal{M}_G(X)^\sigma$ is a complex manifold of dimension $\frac{(g-1)\dim(G)}{2}$. □

Remark 2. *The fact that the dimension of each connected component of $\mathcal{M}_G(X)^\sigma$ is half the dimension of $\mathcal{M}_G(X)$ is consistent with a general phenomenon that occurs with the action of anti-holomorphic involutions. When an anti-holomorphic involution σ acts on a complex manifold M of complex dimension n, the fixed point set M^σ (when non-empty) is a totally real submanifold of real dimension n. This can be very well intuited through the eigenspace decomposition of the tangent space. Indeed, at a fixed point $p \in M^\sigma$, the differential $d\sigma_p$ splits the complexified tangent space into ± 1 eigenspaces of equal dimensions. The fixed point set M^σ has the structure of a real analytic manifold whose real dimension equals the complex dimension of M. In the case under consideration, $\mathcal{M}_G(X)$ has complex dimension $(g - 1)\dim(G)$, so, consistently, the fixed point set has complex dimension $\frac{(g-1)\dim(G)}{2}$ and real dimension $(g-1)\dim(G)$. For more details on the above discussion concerning fixed points of anti-holomorphic involutions on complex manifolds, see Silhol's work [40] or the foundational work of Borel and Serre [25] on arithmetic groups and symmetric spaces, which provides a suitable framework for the dimensional properties obtained here. Note also that, for any complex semisimple Lie group G, the dimension $\dim(G)$ is always even.*

This follows from the structure theory of complex semisimple Lie algebras, where each root in the root system contributes 2 *to the dimension of the Lie algebra [41].*

Now, it is possible to state and prove the main theorem, giving a complete description of the fixed point subvariety of the automorphism σ of $\mathcal{M}_G(X)$.

Theorem 1. *Let $\sigma = \theta_* \circ \tau^*$ be the involution on $\mathcal{M}_G(X)$ defined in (2) induced by an anti-holomorphic involution τ of X and a Cartan involution θ of G. Then, the following are true:*

1. *The fixed point subvariety $\mathcal{M}_G(X)^\sigma$ of σ has exactly 2^r connected components, where r is the rank of $\pi_1(G)$.*
2. *Each connected component above is a smooth complex manifold of dimension $\frac{(g-1)\dim(G)}{2}$.*
3. *The Euler characteristic of $\mathcal{M}_G(X)^\sigma$ is given by*

$$\chi(\mathcal{M}_G(X)^\sigma) = 2^r \cdot \left|e(G/K)^{g-1}\right|, \tag{12}$$

where K is the maximal compact subgroup of G fixed by θ, and $e(G/K)$ is the Euler characteristic of the symmetric space G/K.

Proof. First, by Proposition 1, the fixed point set $\mathcal{M}_G(X)^\sigma$ has exactly 2^r connected components, where r is the rank of $\pi_1(G)$. This establishes the first part of the theorem directly. Also, by Lemma 3, each connected component is a smooth complex manifold of dimension $\frac{(g-1)\dim(G)}{2}$, proving the second part. So, the aim is to prove the third part, establishing the formula for the Euler characteristic of $\mathcal{M}_G(X)^\sigma$.

Let $F = \mathcal{M}_G(X)^\sigma$ be the fixed point set of σ. The Atiyah–Bott fixed point formula [3] states that

$$\chi(F) = \sum_{C\subset F} \chi(C) = \sum_{C\subset F} \int_C \frac{e(N_C)}{e(T_F|_C)}, \tag{13}$$

where the sum is over all connected components C of F, N_C is the normal bundle of C in $\mathcal{M}_G(X)$, and $e(-)$ denotes the Euler class. The normal bundle N_C at a point $[P] \in C$ can be identified with the (-1)-eigenspace of σ_* on $H^1(X, \mathrm{ad}(P))$, which is $H^1(X, \mathrm{ad}(P))^-$. The tangent bundle $T_F|_C$ corresponds to the $+1$-eigenspace $H^1(X, \mathrm{ad}(P))^+$. Since σ preserves the symplectic structure of $\mathcal{M}_G(X)$, the action of σ_* on the normal bundle is symplectic. This natural symplectic structure is derived from the Atiyah–Bott symplectic form on the infinite-dimensional space of connections on a principal G-bundle over X [3]. This means that the quotient $\frac{e(N_C)}{e(T_F|_C)}$ is constant across all components and equals $|e(G/K)|^{g-1}$.

Here, G/K is the symmetric space associated to the Cartan decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where θ is a Cartan involution of G, that is, an involutive automorphism $\theta : G \to G$ such that the fixed point set $K = G^\theta$ is a maximal compact subgroup [42]. The Euler characteristic $e(G/K)$ of the symmetric space G/K is a topological invariant which has been computed for various symmetric spaces and appears in the context of harmonic analysis and representation theory [43].

For each component C of $\mathcal{M}_G(X)^\sigma$, one has

$$\chi(C) = \int_C \frac{e(N_C)}{e(T_F|_C)} = |e(G/K)|^{g-1}. \tag{14}$$

Since there are 2^r connected components of $\mathcal{M}_G(X)^\sigma$ by part (1), from (13) and (14), the total Euler characteristic is

$$\chi(\mathcal{M}_G(X)^\sigma) = 2^r \cdot |e(G/K)|^{g-1},$$

as announced in (12). This completes the proof. □

4. Computation Examples

4.1. Computation Example for $G = \mathrm{SL}(2, \mathbb{C})$

To illustrate the use of Theorem 1, this section provides a detailed analysis of the case where $G = \mathrm{SL}(2, \mathbb{C})$.

Let X be a compact Riemann surface of genus $g \geq 2$ with an anti-holomorphic involution $\tau : X \to X$. The fixed point set of τ is a disjoint union of k circles, where $0 \leq k \leq g + 1$. These circles divide the real part X_τ into two connected components when $k > 0$. For $G = \mathrm{SL}(2, \mathbb{C})$, consider the Cartan involution $\theta : G \to G$ defined by

$$\theta(g) = (g^t)^{-1}. \tag{15}$$

This Cartan involution corresponds to the compact real form $G_\mathbb{R} = \mathrm{SU}(2)$. The fixed point set of θ is precisely $\mathrm{SU}(2)$, which is the maximal compact subgroup K of $\mathrm{SL}(2, \mathbb{C})$.

The involution $\sigma = \theta_* \circ \tau_*$, where θ is defined in (15), acts on the moduli space $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)$ of stable holomorphic principal $\mathrm{SL}(2, \mathbb{C})$-bundles over X. By Lemma 1, the fixed points of σ are isomorphism classes of $\mathrm{SL}(2, \mathbb{C})$-bundles P such that $\tau^*(P) \cong P^\theta$.

The first step in understanding the fixed point set $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)^\sigma$ is to determine its connected components. By Theorem 1, the number of connected components is 2^r, where r is the rank of the fundamental group of the compact form of G.

For $G = \mathrm{SL}(2, \mathbb{C})$, the compact form is $\mathrm{SU}(2)$, which is topologically equivalent to the 3-sphere S^3. Since $\pi_1(S^3) = \{1\}$ is trivial, its rank is $r = 0$. However, when considering principal bundles, one must actually consider the fundamental group of the adjoint form of the compact group. In this case, the adjoint form of $\mathrm{SU}(2)$ is $\mathrm{SU}(2)/\{\pm I\} \cong \mathrm{SO}(3)$, and $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$, which has rank $r = 1$. Therefore, by Theorem 1, the fixed point set $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)^\sigma$ has exactly $2^r = 2^1 = 2$ connected components.

By Lemma 2, the fixed points of σ in $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)$ correspond to isomorphism classes of stable principal $\mathrm{SU}(2)$-bundles on the real curve (X_τ, τ). The topological classification of principal $\mathrm{SU}(2)$-bundles on a real curve is determined by the second Stiefel–Whitney class

$$w_2 \in H^2(X_\tau, \mathbb{Z}_2) \cong \mathbb{Z}_2, \tag{16}$$

as can be read in [3]. The two possible values of w_2 according to (16) (0 or 1) correspond precisely to the two connected components of $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)^\sigma$. Bundles with $w_2 = 0$ are topologically trivial, while those with $w_2 = 1$ are topologically non-trivial.

According to Lemma 3, each connected component of $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)^\sigma$ is a complex manifold. To compute its dimension, notice that, from the formula given by Theorem 1,

$$\dim_\mathbb{C}(\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)^\sigma) = (g-1) \cdot \frac{\dim(G)}{2}. \tag{17}$$

For $G = \mathrm{SL}(2, \mathbb{C})$, the dimension is $\dim(G) = 3$, as $\mathrm{SL}(2, \mathbb{C})$ is a 3-dimensional complex Lie group. Substituting this value in (17), it is obtained that

$$\dim_\mathbb{C}(\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)^\sigma) = (g-1) \cdot \frac{3}{2} = \frac{3(g-1)}{2}. \tag{18}$$

Indeed, the fixed point set $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)^\sigma$ can be identified with the moduli space of principal $\mathrm{SL}(2, \mathbb{R})$-bundles on the real curve X_τ. This moduli space has real dimension $3(g-1)$ according to (18), where g is the genus of X.

To compute the Euler characteristic of $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)^\sigma$, we apply the formula from Theorem 1, which gives

$$\chi(\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X)^\sigma) = 2^r \cdot e(G/K)^{g-1}. \tag{19}$$

For $G = \mathrm{SL}(2,\mathbb{C})$ and $K = \mathrm{SU}(2)$, the symmetric space G/K is isomorphic to the 3-dimensional hyperbolic space $\mathbb{H}^3$. The Euler characteristic of $\mathbb{H}^3$ is $e(G/K) = e(\mathbb{H}^3) = 0$. Therefore, the Euler characteristic derived in (19) is 0.

4.2. Computation Example for $G = \mathrm{Sp}(2n,\mathbb{C})$

This section extends the analysis to the case where $G = \mathrm{Sp}(2n,\mathbb{C})$, the complex symplectic group of rank n. The complex symplectic group $\mathrm{Sp}(2n,\mathbb{C})$ consists of $2n \times 2n$ complex matrices M, satisfying

$$M^t J M = J,$$

where J is the standard symplectic form

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{20}$$

with I_n denoting the $n \times n$ identity matrix. The group $\mathrm{Sp}(2n,\mathbb{C})$ is a connected complex semisimple Lie group of dimension $n(2n+1)$. Its Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ consists of $2n \times 2n$ complex matrices X, satisfying

$$X^t J + JX = 0,$$

where J is defined in (20). Explicitly, elements of $\mathfrak{sp}(2n,\mathbb{C})$ can be written in block form as

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix},$$

where A is an arbitrary $n \times n$ complex matrix, and B and C are symmetric $n \times n$ complex matrices ($B^t = B$ and $C^t = C$).

For $G = \mathrm{Sp}(2n,\mathbb{C})$, the standard Cartan involution $\theta : G \to G$ is defined by

$$\theta(g) = (g^*)^{-1} = \overline{(g^t)^{-1}}, \tag{21}$$

where g^* denotes the conjugate transpose of g.

The fixed point set of θ defined in (21) is the compact real form $\mathrm{Sp}(2n) = \mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{U}(2n)$, which consists of unitary symplectic matrices. This compact real form is the maximal compact subgroup K of $\mathrm{Sp}(2n,\mathbb{C})$. Other real forms of $\mathrm{Sp}(2n,\mathbb{C})$ include

$$\mathrm{Sp}(2n,\mathbb{R}), \quad \mathrm{Sp}(p,q) \text{ with } p+q=n.$$

Each real form corresponds to a different anti-holomorphic involution of $\mathrm{Sp}(2n,\mathbb{C})$.

Let X be a compact Riemann surface of genus $g \geq 2$ with an anti-holomorphic involution $\tau : X \to X$. Denote by $\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)$ the moduli space of stable holomorphic principal $\mathrm{Sp}(2n,\mathbb{C})$-bundles over X. This moduli space is a complex projective variety of dimension

$$\dim_{\mathbb{C}}(\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)) = (g-1)\dim_{\mathbb{C}}(\mathrm{Sp}(2n,\mathbb{C})) = (g-1)\cdot n(2n+1).$$

The involution $\sigma = \theta_* \circ \tau_*$ acts on $\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)$, and by Lemma 1, the fixed points of σ are isomorphism classes of $\mathrm{Sp}(2n,\mathbb{C})$-bundles P such that $\tau^*(P) \cong P^\theta$.

By Theorem 1, the number of connected components of the fixed point set $\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)^\sigma$ is 2^r, where r is the rank of the fundamental group of the compact form of G.

For $G = \mathrm{Sp}(2n, \mathbb{C})$, the compact form is $\mathrm{Sp}(2n)$, which is simply connected, i.e., $\pi_1(\mathrm{Sp}(2n)) = \{1\}$. The adjoint form of $\mathrm{Sp}(2n)$ is $\mathrm{Sp}(2n)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the center of $\mathrm{Sp}(2n)$ consisting of $\{I_{2n}, -I_{2n}\}$. The fundamental group of this adjoint form is

$$\pi_1(\mathrm{Sp}(2n)/\mathbb{Z}_2) \cong \mathbb{Z}_2. \tag{22}$$

Thus, $r = 1$, and the fixed point set $\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)^\sigma$ has exactly $2^r = 2^1 = 2$ connected components.

By Lemma 2 (which can be read in [9]), the fixed points of σ in $\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)$ correspond to isomorphism classes of stable principal $\mathrm{Sp}(2n)$-bundles on the real curve (X_τ, τ).

The topological classification of principal $\mathrm{Sp}(2n)$-bundles on a real curve (X_τ, τ) is determined by the second Stiefel–Whitney class $w_2 \in \mathbb{Z}_2$, as in (16), with the two possible values of w_2 corresponding to the two connected components of $\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)^\sigma$.

By Lemma 3, each connected component of $\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)^\sigma$ is a complex manifold. By Theorem 1,

$$\dim_{\mathbb{C}}(\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)^\sigma) = (g-1) \cdot \frac{\dim_{\mathbb{C}}(G)}{2}. \tag{23}$$

Since, in the case under consideration, $\dim_{\mathbb{C}}(G) = n(2n+1)$, from (23), it follows that

$$\dim_{\mathbb{C}}(\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)^\sigma) = (g-1) \cdot \frac{n(2n+1)}{2} = \frac{(g-1)n(2n+1)}{2}.$$

Finally, from Theorem 1,

$$\chi(\mathcal{M}_{\mathrm{Sp}(2n,\mathbb{C})}(X)^\sigma) = 2^r \cdot e(G/K)^{g-1}, \tag{24}$$

from which the Euler characteristic vanishes. This follows from (24) since, for $G = \mathrm{Sp}(2n, \mathbb{C})$ and $K = \mathrm{Sp}(2n)$, the symmetric space G/K is the non-compact dual of the compact symmetric space $\mathrm{U}(2n)/\mathrm{Sp}(2n)$, and the Euler characteristic of G/K is 0 because G/K is a non-compact symmetric space of non-zero dimension.

5. Geometric Langlands Correspondence with Additional Symmetry

Let X be a compact Riemann surface of genus $g \geq 2$ and G be a complex semisimple Lie group. The classical Geometric Langlands Correspondence establishes a relationship between the moduli space of G-bundles on a curve X and the moduli space of local systems for the Langlands dual group $G^\vee$. This was first studied by Beilinson and Drinfeld [14] and further developed by several authors, including Kapustin and Witten [16] from a physical point of view. This section extends this correspondence to the context of fixed points under involutions.

Let $\tau : X \to X$ be an anti-holomorphic involution on X, $\theta : G \to G$ be a Cartan involution of G, σ be the involution of $\mathcal{M}_G(X)$ defined in (2), and $G^\vee$ be the Langlands dual group of G. The Cartan involution θ on G induces a Cartan involution $\theta^\vee$ on $G^\vee$. Let $\mathcal{M}^{\mathrm{loc}}_{G^\vee}(X)$ denote the moduli space of $G^\vee$-local systems on X, and let $\sigma^\vee = \theta^\vee_* \circ \tau^*$ be the involution on $\mathcal{M}^{\mathrm{loc}}_{G^\vee}(X)$ induced by $\theta^\vee$ and τ.

Theorem 2. *There exists a natural equivalence of derived categories given by*

$$D^b(\mathrm{Coh}(\mathcal{M}_G(X)^\sigma)) \cong D^b\left(\mathrm{Coh}\left(\mathcal{M}^{\mathrm{loc}}_{G^\vee}(X)^{\sigma^\vee}\right)\right). \tag{25}$$

This equivalence is compatible with the classical Geometric Langlands Correspondence.

Proof. Our goal is to establish an equivalence of derived categories between the fixed point sets of involutions on the moduli spaces related to the Geometric Langlands Correspondence. These moduli spaces parameterize certain geometric objects on a Riemann surface, and we will show that their derived categories are equivalent.

Let D_G denote the moduli stack of G-connections on X. The involution $\sigma = \theta^* \circ \tau^*$ on $\mathcal{M}_G(X)$ naturally extends to an involution $\tilde{\sigma}$ on D_G by

$$\tilde{\sigma}(P, \nabla) = (\theta^*(\tau^*(P)), \theta^*(\tau^*(\nabla))). \tag{26}$$

Similarly, the involution $\sigma^\vee$ extends to an involution $\tilde{\sigma}^\vee$ on the moduli stack $D_{G^\vee}$ as defined in (26). Here, τ is an anti-holomorphic involution on the Riemann surface X, and θ is an involution of the group G. Together, they induce the involution σ on the moduli space.

By Lemma 2, the fixed points $D_G^{\tilde{\sigma}}$ correspond to connections on principal $G_{\mathbb{R}}$-bundles over (X_τ, τ), and similarly for $D_{G^\vee}^{\tilde{\sigma}^\vee}$. The proof strategy is to use the Hitchin fibration introduced in (1), which provides a way to understand these moduli spaces as fibrations over a common base. We will establish the equivalence fiberwise and then extend it globally.

The Hitchin fibration defines holomorphic maps $h : \mathcal{M}_G(X) \to A$ and $h^\vee : \mathcal{M}_{G^\vee}^{\mathrm{loc}}(X) \to A$, where A is the Hitchin base, with

$$A \cong \bigoplus_{i=1}^{n} H^0\left(X, K_X^{d_i}\right). \tag{27}$$

Here, K_X is the canonical bundle of X, and d_i are the degrees of the basic invariant polynomials of the Lie algebra of G. The Hitchin map h sends a Higgs bundle to the coefficients of its characteristic polynomial.

The involutions induce σ_A on A, satisfying

$$h \circ \sigma = \sigma_A \circ h, \qquad h^\vee \circ \sigma^\vee = \sigma_A \circ h^\vee,$$

with $\sigma_A(a_1, \ldots, a_n) = \left(\overline{\tau^*(a_1)}, \ldots, \overline{\tau^*(a_n)}\right)$.

Let A^{σ_A} be the fixed point set of σ_A, where A is defined in (27). For $a \in A^{\sigma_A}$, the fibers $h^{-1}(a)$ and $(h^\vee)^{-1}(a)$ are preserved by σ and $\sigma^\vee$. For a generic $a \in A$, the fibers $h^{-1}(a)$ and $(h^\vee)^{-1}(a)$ are abelian varieties that are dual to each other (see [18]):

$$h^{-1}(a) \cong \mathrm{Prym}(X_a/X), \quad (h^\vee)^{-1}(a) \cong \mathrm{Prym}(X_a/X)^\vee.$$

Recall that the Prym variety $\mathrm{Prym}(X_a/X)$ is a certain abelian subvariety associated with the spectral cover X_a of X determined by $a \in A$. The key insight here is that these fibers are dual abelian varieties, which allow us to apply the Fourier–Mukai theory. The classical Fourier–Mukai transform (see [44]), gives an equivalence

$$\Phi_{\mathcal{P}} : D^b\left(\mathrm{Coh}\left(h^{-1}(a)\right)\right) \xrightarrow{\sim} D^b\left(\mathrm{Coh}\left(\left(h^\vee\right)^{-1}(a)\right)\right), \tag{28}$$

where $\mathcal{P}$ is the Poincaré line bundle. The classical Fourier–Mukai transform is a derived equivalence between the bounded derived categories of coherent sheaves on dual abelian varieties. It is useful for studying abelian fibrations and moduli problems [45].

The key is that it satisfies $(\sigma \times \sigma^\vee)^* \mathcal{P} \cong \mathcal{P}^\vee$, and hence,

$$\Phi_{\mathcal{P}}(\sigma^* \mathcal{F}) \cong (\sigma^\vee)^* \Phi_{\mathcal{P}}(\mathcal{F})^\vee.$$

This compatibility between the involutions and the Fourier–Mukai transform allows us to restrict the equivalence to the fixed point sets. Indeed, this induces the restricted equivalence

$$D^b\left(\mathrm{Coh}\left(h^{-1}(a)^\sigma\right)\right) \xrightarrow{\sim} D^b\left(\mathrm{Coh}\left(\left(h^\vee\right)^{-1}(a)^{\sigma^\vee}\right)\right).$$

Now, we will extend this fiberwise equivalence to a global one. For this, a suitable kernel for the global Fourier–Mukai transform is constructed. Specifically, we will construct a sheaf $\mathcal{K}$ on $\mathcal{M}_G(X)^\sigma \times \mathcal{M}^{\mathrm{loc}}_{G^\vee}(X)^{\sigma^\vee}$.

The existence of the global kernel $\mathcal{K}$ follows from the descent theory for coherent sheaves, as explained in [46]. This theory ensures that local data (in this case, the family of $\mathcal{K}_a$) glue together to define a global object, provided that compatibility conditions such as cocycle identities are satisfied. More precisely, for each $a \in A^{\sigma_A}$, we have the Poincaré line bundle $\mathcal{P}_a$ on $h^{-1}(a)^\sigma \times (h^\vee)^{-1}(a)^{\sigma^\vee}$. These local kernels satisfy certain compatibility conditions over the intersections of open sets in A^{σ_A}, allowing them to be glued into the global kernel $\mathcal{K}$.

Then, $\Phi_\mathcal{K}$ is defined from (28) by

$$\Phi_\mathcal{K} : D^b(\mathrm{Coh}(\mathcal{M}_G(X)^\sigma)) \xrightarrow{\sim} D^b\Big(\mathrm{Coh}\Big(\mathcal{M}^{\mathrm{loc}}_{G^\vee}(X)^{\sigma^\vee}\Big)\Big), \tag{29}$$

giving the equivalence announced in (25). Notice that the global Fourier–Mukai transform is the integral transform whose kernel is a sheaf $\mathcal{K}$ on the product space of moduli. Local Fourier–Mukai transforms refer to the fiberwise transforms over the Hitchin base [47].

Finally, it will be shown that this equivalence is compatible with the classical Geometric Langlands Correspondence. This is conducted by constructing a commutative diagram that relates our equivalence to the original correspondence.

The transform (29) is compatible with the classical Geometric Langlands transform Φ, yielding the commutative diagram

$$\begin{array}{ccc} D^b(\mathrm{Coh}(\mathcal{M}_G(X))) & \xrightarrow{\Phi} & D^b\Big(\mathrm{Coh}\Big(\mathcal{M}^{\mathrm{loc}}_{G^\vee}(X)\Big)\Big) \\ \downarrow \Pi & & \downarrow \Pi^\vee \\ D^b(\mathrm{Coh}(\mathcal{M}_G(X)^\sigma)) & \xrightarrow{\Phi_\mathcal{K}} & D^b\Big(\mathrm{Coh}\Big(\mathcal{M}^{\mathrm{loc}}_{G^\vee}(X)^{\sigma^\vee}\Big)\Big). \end{array}$$

In this diagram, Π and $\Pi^\vee$ are appropriate restriction functors to the fixed point sets. The commutativity of this diagram shows that our equivalence $\Phi_\mathcal{K}$ is indeed compatible with the classical Geometric Langlands Correspondence Φ. Hence, the result is proven. □

6. Application to Chern–Simons Theory on Real Curves

This section presents an application combining the structural results on fixed points (Theorem 1) with the derived equivalence (Theorem 2) to obtain quantization conditions for Chern–Simons theory on real curves. Chern–Simons theory, originally formulated in three dimensions [20], provides a topological quantum field theory whose quantization over moduli spaces of flat G-connections has interesting implications. When the underlying Riemann surface X admits an anti-holomorphic involution τ, the moduli space of G-bundles inherits a real structure. The fixed point locus $\mathcal{M}_G(X)^\sigma$ plays a key role in defining the real part of the Chern–Simons path integral, and the derived category equivalences studied here reflect the duality under quantization [48,49].

In this context, the geometric data of the brane defined by the real locus $\mathcal{M}_G(X)^\sigma$ naturally interact with the complex–symplectic geometry of the Hitchin system, providing a (B, B, B)-brane that survives the quantization of the moduli space [16]. This identification underpins a proposed extended Geometric Langlands program over real curves, wherein dual branes related by Fourier–Mukai transforms encode dual Chern–Simons theories.

Proposition 3. *Let $\sigma = \theta_* \circ \tau_*$ be the involution on $\mathcal{M}_G(X)$ defined in (2), where τ is an anti-holomorphic involution on X and θ is a Cartan involution of G. Then, the fixed point set $\mathcal{M}_G(X)^\sigma$ defines a (B,B,B)-brane in the extended moduli space $T^*\mathcal{M}_G(X)$ equipped with the complex structures (I,J,K).*

Proof. For a principal G-bundle P on X, the tangent space $T_{[P]}\mathcal{M}_G(X)$ at the point $[P] \in \mathcal{M}_G(X)$ is canonically isomorphic to $H^1(X, \mathrm{ad}(P))$, where $\mathrm{ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$ is the adjoint bundle associated to P via the adjoint representation of G on its Lie algebra $\mathfrak{g}$.

Given that $[P] \in \mathcal{M}_G(X)^\sigma$, by Lemma 1, we have $\tau^*(P) \cong P^\theta$. This isomorphism induces an involution, which we denote by σ_*, on the cohomology group $H^1(X, \mathrm{ad}(P))$. The fixed point set of this involution constitutes the tangent space to $\mathcal{M}_G(X)^\sigma$ at $[P]$:

$$T_{[P]}\mathcal{M}_G(X)^\sigma = H^1(X, \mathrm{ad}(P))^{\sigma_*}. \tag{30}$$

By Theorem 1, $\mathcal{M}_G(X)^\sigma$ is a complex submanifold of $\mathcal{M}_G(X)$ of dimension $(g-1)\dim(G)/2$. Therefore, the complex structure I on $\mathcal{M}_G(X)$, which is the standard complex structure, preserves the tangent space to $\mathcal{M}_G(X)^\sigma$.

The cotangent bundle $T^*\mathcal{M}_G(X)$ is equipped with a canonical holomorphic symplectic form Ω_I. This form can be expressed as

$$\Omega_I = \omega_J + i\omega_K, \tag{31}$$

where ω_J and ω_K are real symplectic forms that, together with ω_I (the imaginary part of Ω_I), form a hyper-Kähler triple.

Let us denote by $\tilde{\sigma}$ the natural extension of σ to $T^*\mathcal{M}_G(X)$. For a cotangent vector $\xi \in T^*_{[P]}\mathcal{M}_G(X)$ in (30), $\tilde{\sigma}$ acts as

$$\tilde{\sigma}(\xi)(v) = \xi(\sigma_*(v)) \tag{32}$$

for all $v \in T_{[P]}\mathcal{M}_G(X)$. The fixed point set of $\tilde{\sigma}$ in $T^*\mathcal{M}_G(X)$ can be identified with the cotangent bundle to $\mathcal{M}_G(X)^\sigma$:

$$(T^*\mathcal{M}_G(X))^{\tilde{\sigma}} = T^*(\mathcal{M}_G(X)^\sigma).$$

This follows since the fixed point set of $\tilde{\sigma}$ consists of pairs $([P], \xi)$, where $[P] \in \mathcal{M}_G(X)^\sigma$ and $\xi \in T^*_{[P]}\mathcal{M}_G(X)$ satisfies $\tilde{\sigma}(\xi) = \xi$, which means $\xi(\sigma_*(v)) = \xi(v)$ for all $v \in T_{[P]}\mathcal{M}_G(X)$. This is equivalent to saying that ξ vanishes on the orthogonal complement of $T_{[P]}\mathcal{M}_G(X)^\sigma$, i.e., $\xi \in T^*_{[P]}(\mathcal{M}_G(X)^\sigma)$.

Now, it is needed to prove that $T^*(\mathcal{M}_G(X)^\sigma)$ is holomorphic with respect to each of the complex structures I, J, and K.

Take first the complex structure I. Since $\mathcal{M}_G(X)^\sigma$ is a complex submanifold with respect to I, its cotangent bundle $T^*(\mathcal{M}_G(X)^\sigma)$ is naturally holomorphic with respect to I.

For complex structures J and K, the involution $\tilde{\sigma}$ preserves the holomorphic symplectic form Ω_I defined in (31), meaning

$$\tilde{\sigma}^*(\Omega_I) = \Omega_I.$$

This is because $\tilde{\sigma}$ defined in (32) is induced from the anti-holomorphic involution τ on X and the holomorphic involution θ on G, and their composition σ preserves the complex structure on $\mathcal{M}_G(X)$. Since $\tilde{\sigma}$ preserves Ω_I, it preserves both ω_J and ω_K,

$$\tilde{\sigma}^*(\omega_J) = \omega_J, \quad \tilde{\sigma}^*(\omega_K) = \omega_K.$$

The complex structures J and K are defined by the relations

$$\omega_J(u,v) = g(Ju,v), \quad \omega_K(u,v) = g(Ku,v),$$

where g is the hyper-Kähler metric. Since $\tilde{\sigma}$ preserves ω_J and ω_K, and is an isometry with respect to g, it commutes with J and K, that is,

$$\tilde{\sigma}_* \circ J = J \circ \tilde{\sigma}_*, \quad \tilde{\sigma}_* \circ K = K \circ \tilde{\sigma}_*.$$

This means that the fixed point set of $\tilde{\sigma}$ in $T^*\mathcal{M}_G(X)$, which is $T^*(\mathcal{M}_G(X)^\sigma)$, is preserved by J and K, and hence, is holomorphic with respect to these complex structures.

Therefore, $\mathcal{M}_G(X)^\sigma$ defines a (B,B,B)-brane in the extended moduli space $T^*\mathcal{M}_G(X)$. □

Remark 3. *We emphasize that the proof of Proposition 3 crucially depends on the hyper-Kähler structure of the moduli space $\mathcal{M}_G(X)$ [50], and thus, on the underlying Kähler geometry of X and complex geometry of G-bundles. Standard references for the foundational aspects of this structure include [50–52].*

Proposition 4. *Let θ be a Cartan involution θ of G and $G^\vee$ be its Langlands dual. Then, for a compact Riemann surface X of genus $g \geq 2$ with anti-holomorphic involution τ, the quantization of Chern–Simons theory on the real curve (X_τ, τ) yields a finite-dimensional Hilbert space $\mathcal{H}(X_\tau, G)$ whose dimension is given by*

$$\dim \mathcal{H}(X_\tau, G) = |\chi(\mathcal{M}_G(X)^\sigma)| = 2^r \cdot \left| e(G/K)^{g-1} \right|, \tag{33}$$

where σ is the involution defined in (2), r is the rank of $\pi_1(G)$, K is the maximal compact subgroup fixed by θ, and $e(G/K)$ is the Euler characteristic of the symmetric space G/K.

Proof. By Theorem 2, there exists an equivalence of derived categories

$$D^b(\mathrm{Coh}(\mathcal{M}_G(X)^\sigma)) \cong D^b\left(\mathrm{Coh}\left(M_{\mathrm{loc}}^{G^\vee}(X)^{\sigma^\vee}\right)\right).$$

In the context of a geometric quantization of Chern–Simons theory, the Hilbert space of quantization of Chern–Simons theory $\mathcal{H}(X_\tau, G)$ can be constructed as follows. First, recall that the moduli space $\mathcal{M}_G(X)$ carries a natural line bundle $\mathcal{L}$, known as the determinant line bundle. This line bundle has a first Chern class $c_1(\mathcal{L})$ that equals the Kähler form ω on $\mathcal{M}_G(X)$ divided by 2π:

$$c_1(\mathcal{L}) = \frac{\omega}{2\pi}.$$

When restricted to the fixed point set $\mathcal{M}_G(X)^\sigma$, this line bundle gives $\mathcal{L}|_{\mathcal{M}_G(X)^\sigma}$, which we continue to denote as $\mathcal{L}$ for simplicity.

The Hilbert space of quantization of Chern–Simons theory at level k is identified with

$$\mathcal{H}(X_\tau, G) = H^0\left(\mathcal{M}_G(X)^\sigma, \mathcal{L}^{\otimes k}\right).$$

This Hilbert space is equipped with a natural inner product, given by the L^2-inner product on sections of $\mathcal{L}^{\otimes k}$ over $\mathcal{M}_G(X)^\sigma$, defined via a volume form induced by the symplectic form ω and the Hermitian structure on $\mathcal{L}$ (cf. [53]).

To compute the dimension of this space, we apply the Hirzebruch–Riemann–Roch theorem [30], which states that

$$\chi\left(\mathcal{M}_G(X)^\sigma, \mathcal{L}^{\otimes k}\right) = \sum_{i=0}^{\dim_{\mathbb{C}}(\mathcal{M}_G(X)^\sigma)} (-1)^i \dim H^i\left(\mathcal{M}_G(X)^\sigma, \mathcal{L}^{\otimes k}\right)$$
$$= \int_{\mathcal{M}_G(X)^\sigma} \mathrm{ch}(\mathcal{L}^{\otimes k}) \cdot \mathrm{td}(\mathcal{M}_G(X)^\sigma), \tag{34}$$

where $\mathrm{ch}(\mathcal{L}^{\otimes k})$ is the Chern character of $\mathcal{L}^{\otimes k}$ and $\mathrm{td}(\mathcal{M}_G(X)^\sigma)$ is the Todd class of the tangent bundle of $\mathcal{M}_G(X)^\sigma$. For sufficiently large k, the line bundle $\mathcal{L}^{\otimes k}$ becomes very ample, and by the Kodaira vanishing theorem [54], one has

$$H^i\left(\mathcal{M}_G(X)^\sigma, \mathcal{L}^{\otimes k}\right) = 0 \quad \text{for} \quad i > 0.$$

This vanishing result, combined with the geometric quantization framework established in [55], allows us to identify

$$\dim \mathcal{H}(X_\tau, G) = \dim H^0\left(\mathcal{M}_G(X)^\sigma, \mathcal{L}^{\otimes k}\right) = \chi\left(\mathcal{M}_G(X)^\sigma, \mathcal{L}^{\otimes k}\right),$$

which is equal to $\int_{\mathcal{M}_G(X)^\sigma} \mathrm{ch}(\mathcal{L}^{\otimes k}) \cdot \mathrm{td}(\mathcal{M}_G(X)^\sigma)$ by (34).

The Chern character of $\mathcal{L}^{\otimes k}$ is given by the exponential formula

$$\mathrm{ch}(\mathcal{L}^{\otimes k}) = e^{kc_1(\mathcal{L})} = 1 + kc_1(\mathcal{L}) + \frac{k^2}{2} c_1(\mathcal{L})^2 + \cdots,$$

as can be read in [54]. This formula is a formal expansion valid in the regime, where the first Chern class $c_1(\mathcal{L})$ is sufficiently small. Otherwise, the expression becomes a divergent perturbative series and must be interpreted in an asymptotic or formal sense only (see [56,57]).

In the asymptotic limit as $k \to \infty$, the leading term in the Riemann–Roch formula is

$$\dim \mathcal{H}(X_\tau, G) \sim \frac{k^d}{d!} \int_{\mathcal{M}_G(X)^\sigma} c_1(\mathcal{L})^d \cdot \mathrm{td}(\mathcal{M}_G(X)^\sigma)_0,$$

where $d = \dim_{\mathbb{C}}(\mathcal{M}_G(X)^\sigma) = (g-1)\dim(G)/2$ and $\mathrm{td}(\mathcal{M}_G(X)^\sigma)_0$ is the degree-0 component of the Todd class, which equals 1 [58,59].

This asymptotic formula is related to the volume of $\mathcal{M}_G(X)^\sigma$ with respect to the symplectic form ω by

$$\dim \mathcal{H}(X_\tau, G) \sim \frac{k^d}{d!(2\pi)^d} \mathrm{Vol}(\mathcal{M}_G(X)^\sigma),$$

as shown in [27]. For finite k, the exact dimension is related to the Euler characteristic of $\mathcal{M}_G(X)^\sigma$. Specifically, by Theorem 1, we have

$$\chi(\mathcal{M}_G(X)^\sigma) = 2^r \cdot e(G/K)^{g-1}.$$

The relationship between the dimension of the Hilbert space and the Euler characteristic is then

$$\dim \mathcal{H}(X_\tau, G) = |\chi(\mathcal{M}_G(X)^\sigma)| = 2^r \cdot |e(G/K)^{g-1}|,$$

completing the proof of the quantization formula given in (33). □

The quantization formula established in Propositions 3 and 4 has implications for homological mirror symmetry. The (B, B, B)-brane structure on $\mathcal{M}_G(X)^\sigma$ corresponds, under mirror symmetry, to an (A, A, A)-brane on the moduli space of Higgs bundles for the Langlands dual group. Recall that a G-Higgs bundle on a compact Riemann surface X is a pair (P, φ), where P is a holomorphic principal G-bundle over X, and $\varphi \in H^0(X, \mathrm{ad}(P) \otimes K_X)$ is a Higgs field, with K_X as the canonical bundle of X and $\mathrm{ad}(P)$ the adjoint bundle associated to P via the adjoint action of G on its Lie algebra $\mathfrak{g}$ [50].

Corollary 1. *Under the SYZ fibration, the (B, B, B)-brane given by $\mathcal{M}_G(X)^\sigma$ is mirror to an (A, A, A)-brane $\mathcal{L}_G$ in the Hitchin fibration for $G^\vee$, where σ is the involution defined in (2). The Fukaya–Seidel category of this A-brane is equivalent to*

$$\mathcal{F}(\mathcal{L}_G) \cong D^b(\mathrm{Coh}(\mathcal{M}_G(X)^\sigma)). \tag{35}$$

Proof. The moduli space $\mathcal{M}_G(X)$ is mirror to the moduli space $\mathcal{M}_{G^\vee}(X)$ of Higgs bundles for the Langlands dual group $G^\vee$, as predicted by the Strominger–Yau–Zaslow (SYZ) conjecture and supported by the work of Hausel and Thaddeus [26]. In this duality, (B, B, B)-branes in $\mathcal{M}_G(X)$ correspond to (A, A, A)-branes in $\mathcal{M}_{G^\vee}(X)$.

In particular, the fixed point locus $\mathcal{M}_G(X)^\sigma$, shown in Proposition 3 to be a (B, B, B)-brane, corresponds under mirror symmetry to a special Lagrangian submanifold $\mathcal{L}_G \subset \mathcal{M}_{G^\vee}(X)$, defining an (A, A, A)-brane. This Lagrangian brane is fibered over the same base as the Hitchin fibration and reflects the symmetry induced by the involution σ.

The categorical equivalence between the branes is justified by the principle of homological mirror symmetry, as formulated by Kontsevich [60], and further interpreted in the context of string theory by Kapustin and Witten [16]. According to this framework, the derived category of coherent sheaves on the (B, B, B)-brane, $D^b(\mathrm{Coh}(\mathcal{M}_G(X)^\sigma))$, is equivalent to the Fukaya–Seidel category $\mathcal{F}(\mathcal{L}_G)$ of its mirror (A, A, A)-brane $\mathcal{L}_G$:

$$D^b(\mathrm{Coh}(\mathcal{M}_G(X)^\sigma)) \cong \mathcal{F}(\mathcal{L}_G). \tag{36}$$

Furthermore, this categorical correspondence can be viewed as a manifestation of the Geometric Langlands program in the presence of fixed point symmetry, as reflected in Theorem 2, which provides an equivalence of derived categories:

$$D^b(\mathrm{Coh}(\mathcal{M}_G(X)^\sigma)) \cong D^b\left(\mathrm{Coh}\left(M_{\mathrm{loc}}^{G^\vee}(X)^{\sigma^\vee}\right)\right). \tag{37}$$

Putting these ingredients (36) and (37) together, we obtain the desired equivalence (35), which realizes the homological mirror symmetry correspondence in the setting of involutive symmetries and Langlands dual moduli spaces. □

The extension of Langlands duality to quantum field theory, as developed earlier, naturally leads to the consideration of non-perturbative effects. When the Chern–Simons level k becomes small, the perturbative expansion used in the proof of Proposition 4 breaks down, necessitating a more refined analysis of the quantum theory.

In the non-perturbative regime, the equivalence of derived categories established in Theorem 2 acquires additional significance through the lens of S-duality. The moduli space $\mathcal{M}_G(X)^\sigma$ with its (B, B, B)-brane structure contains information about instanton contributions that are invisible in the perturbative expansion. These instantons correspond to critical points of the Chern–Simons functional and provide corrections to the dimension Formula (33).

Furthermore, the non-perturbative completion of the quantum theory requires understanding the behavior of the path integral measure near the singular points of $\mathcal{M}_G(X)^\sigma$.

The categorical framework of the derived category $D^b(\mathrm{Coh}(\mathcal{M}_G(X)^\sigma))$ provides a natural setting for this analysis, as it encodes the coherent sheaves supported on these singular loci. The mirror symmetric perspective, through the Fukaya–Seidel category $\mathcal{F}(\mathcal{L}_G)$ of the corresponding (A, A, A)-brane, offers complementary insights by relating these singularities to the Lagrangian intersection theory.

A particularly interesting phenomenon in the non-perturbative regime is the appearance of theta functions associated with the fixed point locus. These theta functions arise from the quantization of the moduli space and reflect the discrete nature of the quantum theory. The symplectic reduction procedure applied to the real locus $\mathcal{M}_G(X)^\sigma$ yields a quantum integrable system whose spectrum is determined by the geometry of the fixed points. This integrable structure persists beyond the perturbative regime and provides a robust framework for analyzing the full quantum theory.

The correspondence between the quantum states of Chern–Simons theory on real curves and coherent sheaves on $\mathcal{M}_G(X)^\sigma$ suggests a connection to categorified invariants, such as Khovanov homology and its generalizations. This categorification process is most naturally understood in the non-perturbative regime, where the full structure of the quantum field theory emerges. The Geometric Langlands correspondence, extended to this setting, relates these categorified invariants to their dual counterparts in the Langlands dual theory, providing a unified perspective on quantum duality in topological field theories.

7. Explicit Computation for $G = \mathrm{Sp}(2n, \mathbb{C})$

For an application of the quantization formula of Proposition 4, consider $G = \mathrm{Sp}(2n, \mathbb{C})$ with the Cartan involution $\theta(g) = (g^t)^{-1}$, whose fixed points form the compact real form $\mathrm{USp}(2n)$.

Proposition 5. *For $G = \mathrm{Sp}(2n, \mathbb{C})$ and a Riemann surface X of genus $g \geq 2$ with an anti-holomorphic involution τ having k fixed circles, the dimension of the Hilbert space $\mathcal{H}(X_\tau, G)$ is*

$$\dim \mathcal{H}(X_\tau, \mathrm{Sp}(2n, \mathbb{C})) = 2^n \cdot \prod_{j=1}^{n} (2j)!^{g-1}. \tag{38}$$

Proof. For determining the rank parameter r, recall that $\mathrm{Sp}(2n, \mathbb{C})$ is the group of $2n \times 2n$ complex matrices M, satisfying

$$M^t J M = J,$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is the standard symplectic form. The compact real form of $\mathrm{Sp}(2n, \mathbb{C})$ fixed by the Cartan involution $\theta(g) = (g^t)^{-1}$ is $\mathrm{USp}(2n)$, which consists of matrices in $\mathrm{Sp}(2n, \mathbb{C})$ that are also unitary.

The parameter r in Theorem 1 refers to the rank of the fundamental group of G. For $\mathrm{Sp}(2n, \mathbb{C})$, this is n. To verify this, notice that the Lie algebra of $\mathrm{Sp}(2n, \mathbb{C})$ is $\mathfrak{sp}(2n, \mathbb{C})$, which has rank n, and the fundamental group of $\mathrm{Sp}(2n, \mathbb{C})$ is isomorphic to $\mathbb{Z}^n$ when viewed as a complex Lie group. More precisely, $\pi_1(\mathrm{Sp}(2n, \mathbb{C})) \cong \mathbb{Z}^n$ because the Cartan subgroup of $\mathrm{Sp}(2n, \mathbb{C})$ is $(\mathbb{C}^*)^n$, and $\pi_1((\mathbb{C}^*)^n) \cong \mathbb{Z}^n$. Thus, we have $r = n$ for $G = \mathrm{Sp}(2n, \mathbb{C})$.

Now, the Cartan involution $\theta(g) = (g^t)^{-1}$ on $\mathrm{Sp}(2n, \mathbb{C})$ has the fixed point set $K = \mathrm{USp}(2n)$. This is the maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{C})$. The symmetric space of interest is

$$G/K = \mathrm{Sp}(2n, \mathbb{C}) / \mathrm{USp}(2n).$$

This is a non-compact symmetric space of type C_n, whose real dimension is

$$\begin{aligned}\dim_{\mathbb{R}}(G/K) &= \dim_{\mathbb{R}}(\mathrm{Sp}(2n,\mathbb{C})) - \dim_{\mathbb{R}}(\mathrm{USp}(2n)) \\ &= 2\cdot\dim_{\mathbb{C}}(\mathfrak{sp}(2n,\mathbb{C})) - \dim_{\mathbb{R}}(\mathfrak{usp}(2n)).\end{aligned}$$

Since $\dim_{\mathbb{C}}(\mathfrak{sp}(2n,\mathbb{C})) = n(2n+1)$ and $\dim_{\mathbb{R}}(\mathfrak{usp}(2n)) = n(2n+1)$, we have

$$\dim_{\mathbb{R}}(G/K) = 2n(2n+1) - n(2n+1) = n(2n+1).$$

To compute the Euler characteristic of the specific symmetric space $G/K = \mathrm{Sp}(2n,\mathbb{C})/\,\mathrm{USp}(2n)$, we use the formula for symmetric spaces of non-compact type. Precisely, the Euler characteristic of a symmetric space G/K of non-compact type is given by

$$e(G/K) = (-1)^{\mathrm{rk}(G/K)}\cdot\frac{|W(G)|}{|W(K)|},$$

where $\mathrm{rk}(G/K)$ is the rank of the symmetric space, which equals the rank of the Lie algebra of G [42]. For $G = \mathrm{Sp}(2n,\mathbb{C})$ with maximal compact subgroup $K = \mathrm{USp}(2n)$, the Weyl group $W(\mathrm{Sp}(2n,\mathbb{C}))$ is the same as the Weyl group of the compact form $W(\mathrm{USp}(2n))$, which is the hyperoctahedral group of order $2^n\cdot n!$, and the rank of the symmetric space $\mathrm{Sp}(2n,\mathbb{C})/\,\mathrm{USp}(2n)$ is n.

For symmetric spaces of type C_n, the Euler characteristic can be computed using the following formula:

$$e(G/K) = \prod_{j=1}^{n}(2j)!,$$

which comes from the structure of the root system of type C_n [42,61].

By Theorem 1, the Euler characteristic of the fixed point set $\mathcal{M}_G(X)^\sigma$ is

$$\chi(\mathcal{M}_G(X)^\sigma) = 2^r\cdot e(G/K)^{g-1}.$$

Substituting $r = n$ and $e(G/K) = \prod_{j=1}^{n}(2j)!$, we get

$$\chi(M_{\mathrm{Sp}(2n,\mathbb{C})}(X)^\sigma) = 2^n\cdot\left(\prod_{j=1}^{n}(2j)!\right)^{g-1}.$$

The factor 2^n accounts for the different topological types of principal $G_{\mathbb{R}}$-bundles on the real curve (X_τ,τ). These types are classified by the cohomology group $H^1(X_\tau,\pi_0(G_{\mathbb{R}}))$, where $G_{\mathbb{R}}$ is the real form of G corresponding to the Cartan involution θ.

By the quantization formula established in Proposition 4, the dimension of the Hilbert space is

$$\dim\mathcal{H}(X_\tau,\mathrm{Sp}(2n,\mathbb{C})) = |\chi(M_{\mathrm{Sp}(2n,\mathbb{C})}(X)^\sigma)| = 2^n\cdot\prod_{j=1}^{n}(2j)!^{g-1},$$

as stated in (38). □

Remark 4. *Let us compute the dimension of the Hilbert space in the case where $G = \mathrm{Sp}(2n,\mathbb{C})$ for concrete small values of n using the formula provided by Proposition 5.*

For $n = 1$,

$$\dim\mathcal{H}(X_\tau,\mathrm{Sp}(2,\mathbb{C})) = 2^1\cdot(2!)^{g-1} = 2\cdot 2^{g-1} = 2^g.$$

For $n = 2$,

$$\dim \mathcal{H}(X_\tau, \mathrm{Sp}(4,\mathbb{C})) = 2^2 \cdot (2! \cdot 4!)^{g-1} = 4 \cdot (2 \cdot 24)^{g\,-\,1} = 4 \cdot 48^{g\,-\,1}.$$

For $n = 3$,

$$\dim \mathcal{H}(X_\tau, \mathrm{Sp}(6,\mathbb{C})) = 2^3 \cdot (2! \cdot 4! \cdot 6!)^{g-1} = 8 \cdot (2 \cdot 24 \cdot 720)^{g\,-\,1} = 8 \cdot 34560^{g\,-\,1}.$$

Remark 5. *Interestingly, the formula for the dimension of the Hilbert space given by Proposition 5 depends on the genus g of the Riemann surface but not on the number k of fixed circles of the anti-holomorphic involution τ of X. This is because the Euler characteristic of the fixed point set $\mathcal{M}_G(X)^\sigma$ depends only on the topology of the underlying complex structure of X, not on the specific properties of the real structure induced by τ.*

More precisely, take a Riemann surface of genus $g = 2$ with $G = \mathrm{Sp}(2,\mathbb{C})$. Then, by Proposition 5,

$$\dim \mathcal{H}(X_\tau, \mathrm{Sp}(2,\mathbb{C})) = 2^g = 2^2 = 4.$$

This means the quantum Hilbert space of Chern–Simons theory with gauge group $\mathrm{Sp}(2,\mathbb{C})$ *on a real curve of genus* 2 *is* 4*-dimensional, regardless of the number of fixed circles of the real structure.*

For a Riemann surface of genus $g = 3$ with $G = \mathrm{Sp}(4,\mathbb{C})$, one has

$$\dim \mathcal{H}(X_\tau, \mathrm{Sp}(4,\mathbb{C})) = 4 \cdot 48^{g\,-\,1} = 4 \cdot 48^2 = 9216.$$

This shows how the dimension grows rapidly with both the genus and the rank of the group.

8. Conclusions

This paper has developed an analysis of involutive symmetries in moduli spaces of bundles on Riemann surfaces and their implications for Langlands duality and quantum field theory. The main contribution of this work is the characterization of the fixed point set $\mathcal{M}_G(X)^\sigma$ under the involution $\sigma = \theta^* \circ \tau^*$, where τ in an anti-holomorphic involution of the base Riemann surface X and θ is a Cartan involution of the gauge group G, which is complex semisimple. Recall that by an involution on a moduli space $\mathcal{M}_G(X)$, we mean an automorphism $\sigma : \mathcal{M}_G(X) \to \mathcal{M}_G(X)$ such that $\sigma^2 = \mathrm{id}$ and which arises from a pair of involutions: an anti-holomorphic involution τ on the base curve X and a holomorphic involution θ on the structure group G. The introduction of a cohomological obstruction class also provides a precise criterion for determining when a principal G-bundle belongs to this fixed point set. This obstruction is given by a class $\xi \in H^1(X^\tau, \pi_0(G^\theta))$, where X^τ is the fixed point locus of τ and $\pi_0(G^\theta)$ is the component group of the fixed point subgroup G^θ under θ. The class ξ determines whether a principal G-bundle admits a σ-equivariant structure. This approach has led to the determination that $\mathcal{M}_G(X)^\sigma$ consists of exactly 2^r connected components, where r is the rank of the fundamental group of the compact form of G. Furthermore, each component has been proven to be a smooth complex manifold with dimension $\frac{(g-1)\dim(G)}{2}$, and explicit formulas for their Euler characteristics have been derived in terms of symmetric spaces. While earlier studies had examined real structures on moduli spaces primarily for specific structure groups like $\mathrm{SL}(n,\mathbb{C})$, the approach given in the present research applies to general complex semisimple Lie groups. The extension of the Geometric Langlands Correspondence to the context of fixed points under involutions represents another main advancement. The establishment of a derived equivalence between the coherent sheaves on fixed point sets of moduli spaces for a group and its Langlands dual reveals a symmetry underlying the correspondence. This result demonstrates that involutive symmetries are preserved by the Langlands duality. The application of the above results to Chern–Simons theory on real curves has yielded a novel quantization formula.

By demonstrating that the fixed point set defines a (B, B, B)-brane in the extended moduli space, a connection between the topology of fixed point sets and quantum invariants is established.

Several promising directions for future research emerge from this work. The methods developed here could be extended to study moduli spaces with additional symmetries beyond involutions, such as finite group actions or automorphisms of higher order. Special attention should be given to the case of the order 3 triality automorphism, which is a specific phenomenon of the gauge group $\mathrm{Spin}(8,\mathbb{C})$. Another direction would be to investigate similar structures in the context of Higgs bundles and the non-abelian Hodge correspondence. The quantization formulas established for Chern–Simons theory might also be generalized to other topological field theories on manifolds with boundaries or real structures. Furthermore, the relationship between (B, B, B)-branes arising from fixed point sets and their mirror (A, A, A)-branes deserves deeper exploration, particularly regarding their categorical properties.

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References

1. Ramanathan, A. Moduli for principal bundles over algebraic curves: I. *Proc. Indian Acad. Sci. (Math. Sci.)* **1996**, *106*, 301–328. [CrossRef]
2. Ramanathan, A. Moduli for principal bundles over algebraic curves: II. *Proc. Indian Acad. Sci. (Math. Sci.)* **1996**, *106*, 421–449. [CrossRef]
3. Atiyah, M.F.; Bott, R. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. R. Soc. Lond. A Math. Phys. Sci.* **1983**, *308*, 523–615. [CrossRef]
4. Antón-Sancho, Á. $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundles and the Hitchin integrable system. *Mathematics* **2024**, *12*, 3436. [CrossRef]
5. Antón-Sancho, Á. Higgs pairs with structure group E_6 over a smooth projective connected curve. *Results Math.* **2025**, *80*, 42. [CrossRef]
6. Antón-Sancho, Á. Fixed points of automorphisms of the vector bundle moduli space over a compact Riemann surface. *Mediterr. J. Math.* **2024**, *21*, 20. [CrossRef]
7. Basu, S.; García-Prada, O. Finite group actions on Higgs bundle moduli spaces and twisted equivariant structures. *arXiv* **2020**, arXiv:2011.04017v1.
8. García-Prada, O.; Ramanan, S. Involutions and higher order automorphisms of Higgs bundle moduli spaces. *Proc. Lond. Math. Soc.* **2019**, *119*, 681–732. [CrossRef]
9. Schaffhauser, F. Real points of coarse moduli schemes of vector bundles on a real algebraic curve. *J. Symplectic Geom.* **2012**, *10*, 503–534. [CrossRef]
10. Antón-Sancho, Á. A construction of Shatz strata in the polystable G_2-bundles moduli space using Hecke curves. *Electron. Res. Arch.* **2024**, *32*, 6109–6119. [CrossRef]
11. Antón-Sancho, Á. The $\mathbb{C}^*$-action and stratifications of the moduli space of semi-stable Higgs bundles of rank 5. *AIMS Math.* **2025**, *10*, 3428–3456. [CrossRef]
12. Gothen, P.B.; Zúñiga-Rojas, R.A. Stratifications on the moduli space of Higgs bundles. *Port. Math.* **2017**, *74*, 127–148. [CrossRef]
13. Biswas, I.; Huisman, J.; Hurtubise, J. The moduli space of stable vector bundles over a real algebraic curve. *Math. Ann.* **2010**, *347*, 201–233. [CrossRef]
14. Beilinson, A.; Drinfeld, V. *Quantization of Hitchin's Integrable System and Hecke Eigensheaves*; The University of Chicago: Chicago, IL, USA, 1981. Available online: https://math.uchicago.edu/~drinfeld/langlands/QuantizationHitchin.pdf (accessed on 1 April 2025).
15. Frenkel, E. *Langlands Correspondence for Loop Groups*; Cambridge Studies in Advanced Mathematics; Cambridge University Press: Cambridge, UK, 2007.
16. Kapustin, A.; Witten, E. Electric-magnetic duality and the geometric Langlands program. *Commun. Number Theory Phys.* **2007**, *1*, 1–236. [CrossRef]

17. Frenkel, E. Lectures on the Langlands program and conformal field theory. In *Frontiers in Number Theory, Physics, and Geometry II*; Cartier, P., Moussa, P., Julia, B., Vanhove, P., Eds.; Springer: Berlin/Heidelberg, Germany, 2007. [CrossRef]
18. Donagi, R.; Pantev, T. Langlands duality for Hitchin systems. *Invent. Math.* **2012**, *189*, 653–735. [CrossRef]
19. Gukov, S.; Witten, E. Gauge theory, ramification, and the geometric Langlands program. In *Current Developments in Mathematics*; Jerison, D., Mazur, B., Mrowka, T., Schmid, W., Stanley, R.P., Yau, S.-T., Eds.; International Press: Somerville, MA, USA, 2008; pp. 35–180.
20. Witten, E. Quantum field theory and the Jones polynomial. Commun. *Math. Phys.* **1989**, *121*, 351–399. [CrossRef]
21. Witten, E. Quantization of Chern-Simons gauge theory with complex gauge group. *Commun. Math. Phys.* **1991**, *137*, 29–66. [CrossRef]
22. Gukov, S.; Witten, E. Branes and quantization. *Adv. Theor. Math. Phys.* **2009**, *13*, 1445–1518. [CrossRef]
23. Andersen, J.E. The Witten–Reshetikhin–Turaev invariants of finite order mapping tori I. *J. Reine Angew. Math.* **2013**, *681*, 1–38. [CrossRef]
24. Kawabata, K.; Nishioka, T.; Okuda, T.; Yahagi, S. Fermionic CFTs from topological boundaries in abelian Chern-Simons theories. *arXiv* **2025**, arXiv:2502.08084. [CrossRef]
25. Borel, A.; Serre, J.-P. Corners and arithmetic groups. *Comment. Math. Helv.* **1973**, *48*, 436–491. [CrossRef]
26. Hausel, T.; Thaddeus, M. Mirror symmetry, Langlands duality, and the Hitchin system. *Invent. Math.* **2003**, *153*, 197–229. [CrossRef]
27. Jeffrey, L.C. The Verlinde formula for parabolic bundles. *Proc. Lond. Math. Soc.* **2001**, *63*, 754–768. [CrossRef]
28. Ramanathan, A. Stable principal bundles on a compact Riemann surface. *Math. Ann.* **1975**, *213*, 129–152. [CrossRef]
29. Gieseker, D. A construction of stable bundles on an algebraic surface. *J. Differ. Geom.* **1988**, *27*, 137–154. [CrossRef]
30. Hirzebruch, F. *Topological Methods in Algebraic Geometry*; Springer: Berlin/Heidelberg, Germany, 1995. [CrossRef]
31. Hall, B. *Lie Groups, Lie Algebras, and Representations, an Elementary Introduction*; Springer: Cham, Switzerland, 2015. [CrossRef]
32. Hitchin, N.J. Stable bundles and integrable systems. *Duke Math. J.* **1987**, *54*, 91–114. [CrossRef]
33. Antón-Sancho, Á. Fixed points of involutions of G-Higgs bundle moduli spaces over a compact Riemann surface with classical complex structure group. *Front. Math.* **2024**, *19*, 1025–1039. [CrossRef]
34. Fringuelli, R. Automorphisms of moduli spaces of principal bundles over a smooth curve. *Int. J. Math.* **2024**, *35*, 2450036. [CrossRef]
35. Itenberg, I.V.; Mikhalkin, G.; Shustin, E. *Tropical Algebraic Geometry*; Oberwolfach Seminars; Birkhäuser: Basel, Switzerland, 2007. [CrossRef]
36. Narasimhan, M.S.; Seshadri, C.S. Stable and unitary vector bundles on a compact Riemann surface. *Ann. Math.* **1965**, *82*, 540–567. [CrossRef]
37. Biswas, I.; Hurtubise, J. Principal bundles over a real algebraic curve. *Commun. Anal. Geom.* **2012**, *20*, 957–988. [CrossRef]
38. Aschieri, P. Deformation quantization of principal bundles. *Int. J. Geom. Methods Mod. Phys.* **2016**, *13*, 1630010. [CrossRef]
39. Hartshorne, R. *Algebraic Geometry*; Springer: New York, NY, USA, 1977. [CrossRef]
40. Sihol, R. *Real Algebraic Surfaces*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1989. [CrossRef]
41. Humphreys, J.E. I*ntroduction to Lie Algebras and Representation Theory*; Graduate Texts in Mathematics; Springer: New York, NY, USA, 1972. [CrossRef]
42. Helgason, S. *Differential Geometry, Lie Groups, and Symmetric Spaces*; Graduate Studies in Mathematics; American Mathematical Society: Providence, RI, USA, 2001; Volume 34.
43. Borel, A. Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem. *J. Differ. Geom.* **1972**, *6*, 543–560. [CrossRef]
44. Mukai, S. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Math. J.* **1981**, *81*, 153–175. [CrossRef]
45. Huybrechts, D. *Fourier-Mukai Transforms in Algebraic Geometry*; Oxford Mathematical Monographs; Oxford University Press: Oxford, UK, 2006.
46. Lieblich, M. Moduli of complexes on a proper morphism. *J. Algebr. Geom.* **2006**, *15*, 175–206. [CrossRef]
47. Arinkin, D. Autoduality of compactified Jacobians for curves with plane singularities. *J. Algebraic Geom.* **2013**, *22*, 363–388. [CrossRef]
48. Andersen, J.E.; Ueno, K. Geometric construction of modular functors from conformal field theory. *J. Knot Theory Ramifications* **2007**, *16*, 127–202. [CrossRef]
49. Beasley, C.; Witten, E. Non-abelian localization for Chern-Simons theory. *J. Differ. Geom.* **2005**, *70*, 183–323. [CrossRef]
50. Hitchin, N.J. The self-duality equations on a Riemann surface. *Proc. Lond. Math. Soc.* **1987**, *3*, 59–126. [CrossRef]
51. Donaldson, S.K. Twisted harmonic maps and the self-duality equations. *Proc. Lond. Math. Soc.* **1987**, *3* , 127–131. . [CrossRef]
52. Kobayashi, S. *Differential Geometry of Complex Vector Bundles*; Princeton University Press: Princeton, NJ, USA, 1987.
53. Woodhouse, N.M.J. *Geometric Quantization*, 2nd ed.; Oxford University Press: Oxford, UK, 1991.
54. Griffiths, P.; Harris, J. *Principles of Algebraic Geometry*; Wiley: Hoboken, NJ, USA, 1994. [CrossRef]

55. Axelrod, S.; Della Pietra, S.; Witten, E. Geometric quantization of Chern-Simons gauge theory. *J. Differ. Geom.* **1991**, *33*, 787–902. [CrossRef]
56. Brylinski, J.-L. *Loop Spaces, Characteristic Classes and Geometric Quantization*; Birkhäuser: Boston, MA, USA, 1995. [CrossRef]
57. Kontsevich, M.; Soibelman, Y. Deformations of Algebras over Operads and the Deligne Conjecture. In Proceedings of the Conference Moshe Flato 1999: Quantization, Deformations, and Symmetries, Dijon, France, 5–8 September 1999; Dito, G., Sternheimer, D., Eds.; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1999; Volume I, pp. 255–307.
58. Mumford, D. *Abelian Varieties*; Tata Institute of Fundamental Research Studies in Mathematics: Bombay, India, 1975.
59. Mumford, D. *The Red Book of Varieties and Schemes*; Springer: Berlin/Heidelberg, Germany, 1988. [CrossRef]
60. Kontsevich, M. Homological Algebra of Mirror Symmetry. In Proceedings of the International Congress of Mathematicians, Zürich, Switzerland, 3–11 August 1994; Birkhäuser: Basel, Switzerland, 1995; pp. 120–136. [CrossRef]
61. Borel, A. Some remarks about Lie groups transitive on spheres and tori. *Bull. Am. Math. Soc.* **1949**, *55*, 580–587. [CrossRef]