EXCEPTIONAL GROUPS AND ELEMENTARY PARTICLES[+*]

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ABSTRACT If the exceptional observables introduced by Jordan, von Neumann and Wigner are identified with charge states corresponding to internal degrees of freedom of elementary particles, then one is led to the classification of quarks and leptons by means of exceptional groups. It is shown that the groups of the E series are likely candidates and a gauge field theory based on E_7 is given as an example. The hierarchy of symmetry breaking is linked to the hierarchy of stability groups for geometries that are associated with the exceptional groups.

I. Introduction: the New World Picture

Thanks to far reaching recent developments in gauge field theories[1,2] (unification of weak and electromagnetic interactions, renormalizability of spontaneously broken gauge field theories, asymptotic freedom in non-abelian gauge theories) and momentous experimental discoveries (scaling[3] in lepton-hadron, hadron-hadron scattering and e^+e^- annihilation into hadrons; existence of weak interactions mediated by neutral currents[3]; new hadron families associated with a new quantum number[4,5]) we have now the possibility of describing electromagnetic, weak and strong interactions in a unified way: local renormalizable field theory is applicable to the whole world of elementary particles. The fields that occur in the local Lagrangian are leptons, quarks (which come in three colors and at least four flavors), gauge vector bosons which mediate interactions among these fundamental fermions and scalar mesons (Higgs fields) with self interactions and interactions with the fermions and vector bosons.

The Lagrangian of the theory is determined by Poincaré invariance, the principle of local (gauge) invariance with respect to a compact internal symmetry group G, and finally the principle of renormalizability (smooth high energy behavior). For a truly unified theory G is a simple

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or semi-simple Lie group. It must have $SU(2)xU(1)xSU^c(3)$ as a subgroup. Here $SU^c(3)$ is the exact group of strong interactions (the color group) leading to asymptotic freedom at high energies and hence to scaling (up to logarithmic terms) in scattering involving hadrons. The $SU(2)xU(1)$ subgroup is associated with the electromagnetic and weak parts of the interaction (including neutral currents).

The vector bosons belong to the adjoint representation of G. The fermions and the Higgs scalars may belong to other irreducible representations. Some of the Higgs fields develop non zero vacuum expectation values through the minimization of the Higgs potential which describes their self interactions. Then the vacuum is no longer symmetrical with respect to a subgroup of G and we have the phenomenon of spontaneous symmetry breaking which plays a fundamental role in all the major fields of physics as beautifully explained by Professor Michel in his contribution to this conference. The vacuum expectation values of the Higgs fields give masses to the fermions and to the vector bosons. Those vector bosons associated with the unbroken subgroup of G (color gluons and the photon) remain massless.

Three of the quarks are carriers of isotopic spin and hypercharge. A fourth one carries the new hadronic quantum number called charm. There may be other quarks associated with other flavors and new leptons corresponding to new degrees of freedom that characterize the group G. Thus the fermion multiplet must include at least 4 colored quarks (12 charge states) and 4 leptons (e, μ, ν^e and ν^μ). There are strong indications that there exist another heavy charged lepton[6] E^- and its neutral companion N^E. Then, in a unified theory the basic fermions will belong to an irreducible representation of G which has dimension $d>16$ or $d>18$. The dimension of the adjoint representation (number of parameters of G) must be at least 12 (8 for color, 4 for the weak and electromagnetic group).

For a physical interpretation we require another principle: color confinement. According to this principle the only observable scattering states are those that are invariant under the exact color group $SU^c(3)$. This includes leptons, weak bosons and the photon associated with $SU(2)xU(1)$ and those bound states of quarks that are color singlets. These will include mesons ($q\bar{q}$), baryons (qqq), antibaryons ($\bar{q}\bar{q}\bar{q}$) and resonances or bound states of mesons and baryons. Hadrons being bound states will be extended objects held together by gluons. They may have string or bag structure. Field theory is directly applicable not to these hadrons but to their point like constituents (quarks), to leptons and gluons.

The picture of the world we have outlined has already received impressive experimental support. The evidence for the new approach has been eloquently spelled out at this conference by Professor Iliopoulos. There are some serious problems: namely, the difficulty of incorporating gravitation in our world picture (provisionally excluded by the principle of renormalizability), the absence of a rigorous mathematical justification for color confinement, the necessity for the introduction of very high mass bosons (of the order of the Planck mass) for a truly unified theory based on a simple group G, the stability of the proton, the need for too many Higgs scalars to explain the hierarchy of masses and coupling constants, the difficulty of introducing the Higgs fields as bound states (dynamical symmetry breaking) and the absence of a principle for selecting the gauge group G.

It is to this last question that I would like to address myself in this talk. I shall summarize some results of work[7] done in collaboration with Dr. M. Günaydin,[8] Dr. P. Sikivie and Dr. P. Ramond.[9]

Some examples of universal gauge field theories have been developed and studied by Georgi and Glashow[10] (SU(5)), Pati and Salam[11] (SU(4)xSU(4)), Fritzsch and Minkowski[12] (SO(10), SO(14), SU(16)xSU(16), etc.) and others. The most economical and elegant of these schemes is the SO(10) model. They all suffer however, either from proton instability or chiral anomalies in renormalization.

II. Are charges the JNW exceptional observables?

In searching for a principle that leads to the compact group G, we are trying to single out a finite Hilbert space on which the generators of G act. Are there such finite spaces that arise naturally in the classification of Hilbert spaces? An affirmative answer was given by Jordan, von Neumann and Wigner more than four decades ago.[13] Confronted by the new puzzling phenomena of the nuclear world following the discovery of the neutron and the study of β-decay they looked for all possible extensions of Quantum Mechanics which had been so successful in understanding atomic physics. They showed that the usual formulation of Quantum Mechanics in a vector space (the kets and the bras of the dual space) with observables represented by hermitian operators which act on states represented by kets is equivalent to an algebraic formulation in which both observables and states are represented by hermitian matrices, the states being now associated with projection operators

$$P_\alpha = |\alpha><\alpha| \qquad (P_\alpha^2 = P_\alpha) \tag{2.1}$$

that are in one-to-one correspondence with kets, $|\alpha>$. A state which is not a pure state is also represented by a hermitian operator, namely a density matrix

$$D = \sum_i p_i^2 |\alpha_i><\alpha_i| \ , \quad \mathrm{Tr}\, D = \sum p_i^2 = 1 \ . \tag{2.2}$$

Instead of the expectation value $<\alpha|\Omega|\alpha>$ of a hermitian operator (observable) Ω, and its matrix elements $<\alpha|\Omega|\beta>$ we consider respectively the positive numbers

$$\mathrm{Tr}\ P_\alpha\Omega = <\alpha|\Omega|\alpha> \text{ and } \mathrm{Tr}(P_\alpha\Omega P_\beta\Omega) = |<\alpha|\Omega|\beta>|^2. \tag{2.3}$$

For states that are not pure we substitute density matrices D for the projection operators P_α. Note that in all these physically meaningful quantities the operation on hermitian matrices that is relevant is their symmetrized product since contact with experiment is made after taking the trace of matrix products. Thus, the only product which occurs in this algebraic formulation of quantum mechanics is the symmetrized (or Jordan) product of hermitian matrices which we denote by

$$A.B = \frac{1}{2}(AB + BA), \ (A = A^\dagger;\ B = B^\dagger) \ . \tag{2.4}$$

This product which defines the algebra of observables is commutative but non associative, the associator being

$$(A,X,B) = -(B,X,A) = (A.X).B - A.(X.B) \ . \tag{2.5}$$

In fact, if A, X, B are matrices over complex numbers, we have

$$(A,X,B) = \frac{1}{4}[X,[A,B]], \tag{2.6}$$

so that A and B are compatible (simultaneously measurable) observables if $(A,X,B) = 0$ for all X.

The automorphism groups of the Jordan algebra of nxn hermitian matrices are the orthogonal groups SO(n) for real matrices, the unitary groups SU(n) for complex matrices and the symplectic groups Sp(n,q) for nxn quaternionic matrices (hermitian with respect to quaternion conjugation). Thus we recover the classical groups and their associated Hilbert spaces.

An infinitesimal unitary transformation on X reads

$$X' = X + i[X,C] \ . \tag{2.7}$$

If we express the antihermitian matrix in terms of two hermitian matrices A and B by $\frac{1}{4}[A,B] = iC$, then Eq. (2.7), takes the form

$$X' = X + (A,X,B) \tag{2.8}$$

expressing the infinitesimal action of the automorphism group purely in terms of the Jordan algebra. Then A and B are also observables associated with the transformation of the observable X. The finite unitary transformation on X can be written in two ways

$$X' = e^{-iC}Xe^{iC} = 1 + \frac{i}{1!}[X,C] + \frac{i}{2!}[[X,C],C] + \ldots \quad (2.9a)$$

in terms of the Lie algebra of commutators or, alternatively

$$X' = E_{A,B}X = X + \frac{1}{1!}(A,X,B) + \frac{1}{2!}(A,(A,X,B),B) + \ldots \quad (2.9b)$$

in terms of associators that only involve the Jordan algebra.

Now, Jordan, von Neumann and Wigner found one case in which the algebra of observables is obeyed by operators that are not matrices over an associative division algebra. They are 3x3 octonionic matrices hermitian with respect to octonionic conjugation. Segal and Sherman[14] later showed that idempotent matrices of this kind can be made to correspond to pure states and a general matrix of the form

$$X = \begin{pmatrix} \alpha & c & \bar{b} \\ \bar{c} & \beta & a \\ b & \bar{a} & \gamma \end{pmatrix}, \quad (2.10)$$

with Tr X = 1 can represent a quantum mechanical density matrix. Here α, β, γ are real and a,b,c are octonions with the bar denoting octonionic conjugation. A determinant form can be uniquely associated with X by the formula

$$I \text{ Det } X = X^{\cdot 3} - X^2 \text{ Tr } X + \tfrac{1}{2} X \{(\text{Tr } X)^2 - \text{Tr } X^2\}, \quad (2.11)$$

where I is the 3x3 unit matrix and $X^{\cdot 3} = X.X^2\ldots$ Chevalley and Schafer[15] showed that these observables can be transformed by the generalization of the unitary transformations in the form (2.9b) with A and B matrices elements of the set (2.10) with zero trace. Thus the group has 2x26=52 parameters and is the exceptional group F_4 acting on the exceptional observable X. The generators of F_4 correspond to observable charges. The automorphism group of the octonions was already known to be the exceptional group G_2. Its infinitesimal action on the octonion x is given by

$$x' = x - \tfrac{1}{4}[[a,b],x] + \tfrac{3}{4}[a,b,x] \quad (2.11)$$

where [a,b,x] is the completely antisymmetrical associator with respect to the octonion product which is non commutative and non associative. Here a and b are such that their scalar parts $\frac{1}{2}(a+\bar{a})$ and $\frac{1}{2}(b+\bar{b})$ vanish so that G_2 has 2x7=14 parameters. It has a maximal subgroup $SU^c(3)$ which leaves one of the 7 octonionic imaginary units invariant.

Another subgroup of F_4 that acts on X and commutes with $SU^c(3)$ is an SU(3) associated with the 3x3 matrix structure of X in which the octonion unit left invariant by $SU^c(3)$ plays the role of the imaginary unit of complex numbers. Thus, F_4 admits $SU(3)xSU^c(3)$ as a maximal subgroup. We also note that the group F_4 leaves invariant Tr X, Tr X^2 and Det X.

We have arrived at the remarkable conclusion that there exists a finite Hilbert space associated with an exceptional realization of the algebra of observables with a transformation law that involves two different SU(3) groups, one associated with octonions which has the same structure as the color group $SU^c(3)$ and the other with the SU(3) of unitary symmetry. Furthermore these observables arise only in the JNW formulation,as the ket associated with an idempotent X does not transform linearly under F_4.

Since the discovery of these exceptional observables there has been a case of the missing observables in physics as this exceptional Hilbert space could not be related to any known states in Physics. Now with the new discoveries of a rich charge structure for elementary particles, the above considerations strongly suggest the identification of the charge space with the space of exceptional observables.

The 26 and 52 dimensional representations of F_4 have the following decomposition in terms of $SU(3)xSU^c(3)$:

$$26: (8,1) + (3,3^c) + (\bar{3},\bar{3}^c) \qquad (2.12)$$

$$52: (8,1) + (1,8^c) + (6,\bar{3}^c) + (\bar{6},3^c) \ . \qquad (2.13)$$

Hence if we take G = F_4 the (26) representation unifies 8 Majorana (4 Dirac) leptons with integer charges together with 3 fractionally charged quarks and 3 antiquarks. The adjoint representation contains color singlet vector bosons together with octet colored gluons and twelve lepto quark bosons.

The other exceptional groups E_6, E_7 and E_8 act also on finite Hilbert spaces with elements that are partly real or complex numbers and partly exceptional observables. Hence if we take as our guiding principle for finding G the identification of internal degrees of freedom with exceptional observables we are uniquely led to exceptional groups. The need for charm singles out the E series as natural candidates for the universal gauge group G.

III. Examples of schemes based on E_6 and E_7. Spontaneous Symmetry Breaking and Geometry.

The smallest Hilbert space E_6 acts on is provided by a pair

(X_1, X_2) of exceptional observables which can be combined in a single complex matrix J which is hermitian with respect to octonionic conjugation only. The infinitesimal E_6 transformation on this 27 dimensional representation is

$$T_{A,B,C}\, J = J + (A,J,B) + i\, C.J \quad , \qquad (3.1)$$

where A,B,C are traceless real matrices belonging to the set (2.10). Then the group has 3x26=78 parameters. J^* corresponds to an inequivalent $\overline{27}$ representation for which the transformation law is as in (3.1) with $C \to -C$.

We now introduce the symmetric Freudenthal product

$$JxK = J.K - \frac{1}{2} J\, \mathrm{Tr}\, K - \frac{1}{2} K\, \mathrm{Tr}\, J - \frac{1}{2} \mathrm{Tr}(J.K - \frac{1}{2} J\, \mathrm{Tr}\, K - \frac{1}{2} K\, \mathrm{Tr}\, J). \qquad (3.2)$$

If J_1 and J_2 transform like 27, $J_1 x J_2$ transforms as $\overline{27}$ and we have

$$I\, \mathrm{Det}\, J = (J \times J).J \quad . \qquad (3.3)$$

Under E_6 we have the invariants

$$I_1 = \frac{1}{2} \mathrm{Tr}\, JJ^*, \quad I_2 + iI_3 = \mathrm{Det}\, J, \quad I_4 = \frac{1}{2} \mathrm{Tr}\, \{(JxJ).(J^*xJ^*)\} \qquad (3.4)$$

If J represents Higgs scalar fields, then a linear combination of these invariants forms a Higgs potential the minimization of which gives a spontaneous breaking of E_6 into SO(10)xSO(2). This is just the gauge group considered by Fritzsch and Minkowski. This last group leaves invariant an idempotent J which obeys JxJ = 0, Tr J = 1. This has a geometric interpretation: SO(10)xSO(2) is the stability group of the geometrical object J which moves in a generalized projective space under the action of E_6. Thus, spontaneous symmetry breaking is intimately connected with the hierarchy of subgroups that are stability groups of various geometrical objects.[16]

E_7 acts in the 56 dim. space $(\alpha_1, J_1, J_2, \alpha_2)$ where J_1 and J_2 transform respectively like 27 and $\overline{27}$ of the E_6 subgroup and the complex numbers α_1 and α_2 are E_6 invariant. The parameters of E_7 can be grouped into A,B,C, namely the generators of E_6, a real parameter λ and a complex Jordan matrix K giving 133 real parameters. Under the maximal subgroup $SU(6) x SU^c(3)$ we have

$$56 = (20,1) + (6,3) + (\bar{6},\bar{3}) \qquad (3.4)$$

$$133 = (35,1) + (1,8) + (15,3) + (\overline{15},\bar{3}) \qquad (3.5)$$

giving 10 Dirac leptons (4 charged, 6 neutral), 6 colored quarks and six antiquarks for the 56. The adjoint representation has 35 color singlet vector bosons, 8 colored gluons and 90 leptoquark colored bosons. Under spontaneous breaking $SU(6) \to SU(4)xSU(2)$. Thus we have the $SU(4)xSU(2)xSU^c(3)$ decomposition.

$$56 = (4,1,1^c) + (\bar{4},1,1^c) + (6,2,1^c) + (1,2,3^c) + (4,1,\bar{3}^c) + (1,2,\bar{3}^c) \qquad (3.6)$$

We identify $(4,1,1^c)$ with left handed leptons, $(\bar{4},1,1^c)$ with right handed leptons, $(6,2,1)$ with heavy leptons, $(4,1,3^c)$ with the colored quarks $\mathcal{P}$, $\mathcal{N}$, λ and $\mathcal{P}'$, $(1,2,3^c)$ with two new heavy quarks $\mathcal{N}'$ and λ' and the elements involving $\bar{3}^c$ with antiquarks. The charged current is taken to be a Cabibbo rotated $j^1{}_2+j^4{}_3$. Then there are no $\Delta S = 1$ neutral currents. The ratio $R = (e^+e^- \to \text{hadrons})/*e^+e^- \to \mu^+\mu^-)$ turns out to be 4, but with one charged heavy lepton with decay products counted as hadrons becomes 5 and with the contribution of two can go up to 6. The experimental value is between 5 and 6.

The E_6 and E_7 schemes are worked out in more details in two forthcoming papers[9].

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