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Review

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Quantum Field Theory in the Weyl–Wigner Representation

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Abstract

The Wigner representation for quantum mechanics of particles is generalized to Bose fields. The standard Hilbert space quantization becomes, via the Weyl transform, a quantization method that consists of adding a Gaussian zeropoint field distribution to the vacuum. I comment on the possible advantages of the method in order to study quantum fields in curved spaces. I study a unified formulation of non-relativistic quantum electrodynamics in the Weyl–Wigner formalism, in terms of (classical-like) c-numbers.

Keywords: Wigner representation; Bose fields; quantum vacuum fields

1. Introduction

The aim of the article is to generalize the Wigner representation, initially proposed for quantum mechanics, to Bose fields. Thus, I review and extend previous works [1,2]. The formalism provides an alternative to the canonical field quantization, in terms of operators on a Hilbert space, that might have advantages for the study of quantum fields in curved spacetimes and for the search of a satisfactory quantum gravity theory. In the last section of the article, I present a formulation of non-relativistic quantum electrodynamics involving particles and electromagnetic field in a unified treatment. In order that the reader is aware of the background, main developments, motivations, and relevance of the study, it is appropriate to put the paper in a wider context as follows:

The article is a contribution to a research program attempting to elaborate a realistic interpretation of quantum theory [3]. This fits in the aim of theoretical physics, as it was well summarized in the initial paragraph of the celebrated EPR article [4]: "Any serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts *we picture this reality to ourselves* " (my emphasis). I will call "realistic" those interpretations of a physical theory which provide a picture of reality, although the name might be disputed because "realism" is too broad a concept. The choice is justified because, in recent times, the denomination "local realism" has become popular in relation with Bell's work [5], see, e.g., [6].

The distinction between reality and theoretical concepts by EPR [4] means that different theories may exist for the description of a given domain of reality. Therefore, a specific theory may be very efficient in order to derive predictions for the results of experiments, but other theories, or different formulations of the same theory, may be more suitable to obtain a picture of reality. This fact is especially relevant for quantum theory, which has had a large variety of interpretations [7]. I believe that obtaining a picture of reality is an essential aspect of physics or, at least, an essential ingredient of the philosophical view of the natural world. In fact, a theory should deal with what nature does and not only with

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the results of human observations or measurements. Hence, a purely pragmatic approach to physics is not sufficient for our understanding of the world, say an approach which only appreciates the prediction of empirical results, as is the Copenhagen interpretation. In my view, this is not just a matter of taste because the picture of reality provided by a physical theory may be a clue for future progress.

Quantum field theory (QFT) with canonical quantization has had spectacular success in the quantitative prediction of observable properties, especially in the domain of quantum electrodynamics. However, there are difficulties in other domains like quantum gravity or curved spaces. The former success shows that canonical QFT provides an extremely efficient algorithm for the calculations; in particular, renormalization techniques allow for the removal of divergences, but the subtraction of infinities is not satisfactory from a fundamental point of view. I believe that the difficulty to extend QFT to gravity or curved spaces has to do with the fact that neither canonical quantization nor path integrals [8] provide pictures of reality. Thus, the aim of the present article is *to show* that the Wigner representation may be generalized to Bose fields and *to suggest* that further work along the same line of research might solve the present problems.

The Wigner representation is known from the early period of quantum mechanics. The problem is that it was applied to non-relativistic quantum mechanics, where it does not provide a "picture of reality", as commented at the end of the next section. Indeed, the success of the standard theory of fundamental particles shows, or strongly suggests, that the inhabitants of our world are quantum fields, while quantum particles consist of many interacting fields, which is expressed saying that physical particles are dressed by fields. The situation strongly contrasts with classical mechanics, where point particles are usually good representations of bodies. In my view, the attempt to interpret quantum theory starting with non-relativistic quantum mechanics has been a persistent error. I believe that we should start with the interpretation of fields.

In summary, I think that the problems of QFT might be solved or alleviated with a method of quantization different from the standard ones (either canonical of path integrals), and I propose the Wigner representation together with a yet unknown extension to Fermi fields appropriate for a realistic interpretation, which I hope is possible.

2. The Wigner Representation in Quantum Mechanics

In the following, I revisit the Wigner representation (or Weyl–Wigner, WW) of quantum mechanics for non-relativistic particles. It is well known [9,10], but I shall summarize the most relevant features for convenience in later sections.

2.1. Weyl Transform and Wigner Function

In 1927, Weyl proposed a quantization method for systems of particles via a transform that converts classical (c-number) coordinates and momenta into operators in a Hilbert space [11] as follows: Let us consider a system of N particles whose state is characterized by their position coordinates, x_j, and momenta, p_j, j = 1,2,...3 N. Weyl transform converts any polynomial function of$\left(\{x_j, p_j\}\right)$ into a function of the quantum operators $\{\hat{x}_j, \hat{p}_j\}$. (For the sake of clarity, I will write quantum operators with a 'hat', e.g., $\hat{x}_j, \hat{p}_j$, and numerical, c-number, quantities without 'hat', e.g., x_j, p_j). In one dimension, the Weyl transform T_W may be written as follows:

$$\begin{aligned} & T_W[f(\{x,p\})] \equiv f_W(\{x,p\}) \\ = \; & \frac{1}{4\pi^2}\int d\lambda \int d\mu \int dx \int dp \exp[i\lambda\cdot(x-\hat{x})+i\mu\cdot(p-\hat{p})]f(x,p). \end{aligned} \tag{1}$$

The generalization to 3N coordinates and momenta of N particles is straightforward.

An important property of Weyl transform is that the function f_W consists of operators in symmetrical order, meaning a sum of terms with one possible ordering each, divided by the number of terms. For instance

$$T_W\left(x^2 p\right) = \left(\hat{x}^2 \hat{p}\right)_{sym} \equiv \frac{1}{3}\left(\hat{x}^2 \hat{p} + \hat{x}\hat{p}\hat{x} + \hat{p}\hat{x}^2\right). \tag{2}$$

Weyl transform is reversible (with caution, see below), and we are mainly interested in the inverse of Weyl transform in this article. It leads from operators, representing either states or observables, to functions in the phase space of the particles, that is, functions of their positions and momenta. In one dimension, the inverse Weyl transform is as follows:

$$\begin{aligned} W_{\hat{M}}(x,p) &= T_W^{rev}\left[\hat{M}\right] \equiv \frac{1}{4\pi^2}\int d\lambda \int d\mu \exp[-i\lambda \cdot x - i\mu \cdot p] \\ &\quad \times Tr\left\{\hat{M}\exp[i\lambda \cdot (x-\hat{x}) + i\mu \cdot (p-\hat{p})]\right\}, \end{aligned} \tag{3}$$

where $\hat{M}$ is an operator for a quantum particle, and $Tr\{\}$ means the trace operation. It is possible to show that the application of the Weyl transform followed by its inverse, or vice versa, reproduces the initial expression, with the caution that follows.

Actually, the definition Equation (3) is ambiguous and may give rise to contradictions, as shown with the following example. Let us consider the inverse Weyl transform Equation (3) of the operator products $\hat{x}\hat{p}$ and $\hat{p}\hat{x}$. In both cases, the result is $xp = px$. Then, the application of that transform to both sides of the fundamental quantum commutation rule

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar, \tag{4}$$

would lead to the absurd result

$$xp - px = 0 = i\hbar.$$

The difficulty may be solved taking into account that the Weyl transform leads always to symmetrized expressions. Therefore, we must specify that the inverse Weyl transform Equation (3) has a sense only if applied to products of operators in symmetrical order, but this constraint is not too restrictive. In fact, any operator which may be written as a product of the fundamental operators $\left\{\hat{x}_j, \hat{p}_j\right\}$ may be rewritten as a sum of products in symmetrical order by repeated application of the canonical commutation rule, Equation (4).

In 1932, Wigner [12] proposed a representation of non-relativistic quantum mechanics in terms of a formalism with classical flavor. He defined a function $W_\psi\left(\left\{x_j, p_j\right\},t\right)$ in the phase space of a set of particles from the quantum wavefunction $\psi\left(\left\{x_j\right\},t\right)$ of the state as follows (in one dimension)

$$W_\psi(x,p) = \int du\psi(x-u)\psi^*(x+u)\exp(2iup), \tag{5}$$

where $W_\psi(x,p)$ is called the Wigner function of the state ψ. The Wigner Equation (5), defined in the Schrödinger representation (that is, involving wave functions), is equivalent to the inverse Weyl transform, Equation (3), formulated in the abstract Hilbert space formalism. Weyl´s is more general because it applies to any (trace-class) operator, for instance, observables, in addition to states. Of course, the Wigner Equation (5) may, and it has, be generalized to operators representing observables.

2.2. Equivalence with Quantum Mechanics

Several properties of the Wigner representation are reported in the following. Most relevant is the fact that the expectation value of the observable $\hat{M}$ in the state ψ becomes the integral

$$\langle\psi|\hat{M}|\psi\rangle = \int W_M(x,p)W_\psi(x,p,t)dxdp. \tag{6}$$

Hence, the expectation values obtained via the Wigner function agree with those obtained from the canonical, Hilbert-space formalism of non-relativistic quantum mechanics. This fact guarantees that the predictions of the results of experiments should agree in both canonical and Wigner representations, provided that the laws of evolution are appropriated.

The evolution of the Wigner function of a state may be obtained from the time derivative of $W_\psi(r,p,t)$ Equation (5), taking the Schrödinger equation into account in order to express $\partial\psi(r-u,t)/\partial t$ and $\partial\psi^*(r+u,t)/\partial t$ as functions of $r-u$ and $r+u$, respectively. This should be followed by an integral with respect to u, but the integration requires an explicit knowledge of the dependence of $\psi(x,t)$ on x. For the particular case of a single particle moving in a potential $V(\mathbf{r})$, the evolution equation is as follows (in three dimensions):

$$\begin{aligned} \frac{\partial W(\mathbf{r},\mathbf{p})}{\partial t} &= -\frac{1}{m}\mathbf{p}\cdot\nabla W - \frac{1}{\hbar}\int\frac{d\mathbf{p}'}{(2\pi)^3}\widetilde{V}(\mathbf{r},\mathbf{p}')W(\mathbf{r},\mathbf{p}+\mathbf{p}',t), \\ \widetilde{V}(\mathbf{r},\mathbf{p}') &\equiv \int d\mathbf{u}\sin(\mathbf{p}'\cdot\mathbf{u})[V(\mathbf{r}+\hbar\mathbf{u}/2) - V(\mathbf{r}-\hbar\mathbf{u}/2)], \end{aligned} \tag{7}$$

which is clearly more involved than Schrödinger equation. A general evolution equation of the Wigner function may be written in terms of the coordinates $\{x_j\}$ and momenta $\{p_j\}$ of several particles as follows:

$$\begin{aligned} \frac{\partial W}{\partial t} &= \frac{2}{\hbar}\sin\left[\frac{\hbar}{2}\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial p_j'} - \frac{\partial}{\partial p_j}\frac{\partial}{\partial x_j'}\right)\right]\left[W(\{x_j,p_j\})H_{part}\left(\left\{x_j',p_j'\right\}\right)\right] \\ &\equiv \frac{2}{\hbar}\sum_{n=0}^{3N}\frac{(-1)^n}{(2n+1)!}\left[\frac{\hbar}{2}\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial p_j'} - \frac{\partial}{\partial p_j}\frac{\partial}{\partial x_j'}\right)\right]^{2n+1} \\ &\quad\times\left[W(\{x_j,p_j\})H_{part}\left(\left\{x_j',p_j'\right\}\right)\right] \equiv \{W,H_{part}\}_M, \end{aligned} \tag{8}$$

where we should identify $\left\{x_j',p_j'\right\} = \{x_j,p_j\}$ after performing the derivatives. $\{W,H_{part}\}_M$ is a simplified notation to be used in the following, the subindex M standing for *Moyal bracket*. In Equation (8), $\sin(x)$ stands for its expansion in powers of x.

A remarkable property of Equation (8) is that when either the Hamiltonian or the Wigner function or both are at most quadratic in the coordinates and momenta, only first and second derivatives may appear in Equation (8), whence the Moyal bracket becomes the Poisson bracket of classical dynamics. The evolutions of coordinates and momenta are governed by classical laws in the Wigner representation in that case. A typical example is a set of linearly coupled harmonic oscillators, for instance the motion of ions in a lattice crystal using a standard approximation.

The mentioned properties give rise the Wigner (or Weyl–Wigner, WW) representation of non-relativistic quantum mechanics, which is equivalent to the approach in terms of either operators or wavefunctions for the states. In fact, Equation (6) and the derivation of Equations (7) and (8) from the Schrödinger equation guarantee that the expectation values obtained via the Wigner representation agree with those got from the canonical formalism of quantum mechanics, thus proving the equivalence of the formulations.

The Wigner representation is helpful for some purposes, but the calculations are usually more involved than the standard ones, e.g., those using the Schrödinger equation. Also, it does not provide an intuitive ("realistic") interpretation of quantum mechanics because the Wigner functions of states are positively definite but for a slight fraction of quantum states, that is, when the wavefunction is Gaussian [13]. Therefore, the states cannot be interpreted as probability distributions in phase space. In sharp contrast an intuitive interpretation of the Weyl–Wigner representation for Bose fields is possible, as commented in the next section. The reason for the difference between elementary quantum mechanics and quantized electromagnetic field has been discussed elsewhere [1,2].

3. Bose Fields in the Weyl–Wigner Representation

3.1. The Massive Neutral Spin Zero Field

3.1.1. Classical Treatment

Let us start with a field fulfilling the Klein–Gordon equation, that is

$$\left(-\frac{\partial^2}{\partial t^2}+\triangle\right)\psi(\mathbf{r},t)=m^2\psi(\mathbf{r},t), \tag{9}$$

where $\triangle$ is the Laplacean operator (in units $c=\hbar=1$, but Planck constant $\hbar$ will be restored for clarity in some cases). Equation (9) appeared in the context of quantum mechanics, but it may be treated with the methods of classical field theory. It was studied for the first time during the pioneer work of Schrödinger in 1925-26. Thus, it may be appropriately named "relativistic Schrödinger equation". As is well known, Equation (9) does not predict correctly the fine details of atomic spectra, whence Schrödinger himself restricted attention to the non-relativistic approximation, which is an appropriate evolution equation for non-relativistic quantum mechanics.

The classical treatment of Equation (9) may start with the Lagrangean density

$$L=\partial_\mu\psi^*\cdot\partial^\mu\psi-m^2\psi^*\psi, \tag{10}$$

where ∂_μ are the derivatives with respect to the space-time coordinates. Indeed, Lagrange equations lead from Equation (10) to Equation (9). From Equation (10), it is also possible to obtain the energy–momentum tensor, that is

$$T_\mu^\nu=\sum\partial_\mu q\frac{\partial L}{\partial(\partial_\mu q)}-L\delta_\mu^\nu.$$

In particular, the Hamiltonian density may be written

$$T_{00}=\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t}+\nabla\psi^*\nabla\psi+m^2\psi^*\psi. \tag{11}$$

In order to obtain the canonical ("second") quantization of the field Equation (9), we shall start expanding the field in plane waves. Here, I assume that the field ψ is real, which gives

$$\psi=\sum_j\frac{1}{\sqrt{2\varepsilon_j}}\left[a_j\exp\left(i\mathbf{k}_j.\mathbf{r}-i\varepsilon_j t\right)+a_j^*\exp\left(-i\mathbf{k}_j.\mathbf{r}+i\varepsilon_j t\right)\right], \tag{12}$$

a_j^* being the complex conjugate of a_j, where the single-mode energy is

$$\varepsilon_j=\sqrt{m^2+\mathbf{p}_j^2},\mathbf{p}_j=\hbar\mathbf{k}_j, \tag{13}$$

and $\mathbf{p}_j$ is the linear momentum. In terms of the amplitudes $\left\{a_j, a_j^*\right\}$, the free-field Hamiltonian may be got from Equation (11), leading to

$$H = \int d^3x T_{00} = \sum_j \varepsilon_j a_j a_j^* = \sum_j \varepsilon_j |a_j|^2. \tag{14}$$

It is interesting to define the current density j^μ

$$j^\mu = i(\psi^* \partial_\mu \psi - (\partial_\mu \psi^*)\psi) \Rightarrow \partial_\mu j^\mu = 0, \tag{15}$$

whence we may obtain a conserved quantity Q, that is

$$Q = \int j_0 d^3x \,,\, j_0 = j^0 = i\left(\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t}\psi\right). \tag{16}$$

In the case of charged fields, Q may be interpreted as the electric charge.

3.1.2. Canonical (Hilbert Space) Quantization

The canonical method to quantize a field is to promote the quantities $\left\{a_j, a_j^*\right\}$ to be operators on a Hilbert space (HS), fulfilling the commutation rules

$$\left[\hat{a}_j, \hat{a}_k\right] = \left[\hat{a}_j^\dagger, \hat{a}_k^\dagger\right] = 0, \left[\hat{a}_j, \hat{a}_k^\dagger\right] = \delta_{jk}, \tag{17}$$

δ_{jk} being Kronecker delta. The operators $\hat{a}_j$ $\left(\hat{a}_j^\dagger\right)$ are usually named annihilation (creation) operators of particles.

The commutation properties of the operators Equations (17) may be related to the standard commutation rules of (non-relativistic) quantum mechanics at equal times, that is

$$\left[\hat{x}_j(t), \hat{p}_l(t)\right] = i\hbar\delta_{jl}, \tag{18}$$

where i is the imaginary unit and δ_{jl} is Kronecker delta. In fact, introducing the following change of variables for the field amplitude a_j

$$x_j(t) \equiv \frac{c}{\sqrt{2}\omega_j}\left(a_j(t) + a_j^*(t)\right), p_j(t) \equiv \frac{i\hbar\omega_j}{\sqrt{2}c}\left(a_j(t) - a_j^*(t)\right), \tag{19}$$

it is possible to show that the free evolution of $a_j(t)$ is given by the evolution of $\left[x_j(t), p_j(t)\right]$, as if they were the coordinate and momentum of a classical harmonic oscillator. Therefore, the classical field may be treated formally as a collection of oscillators. Consequently, we may write the quantum counterpart of Equation (19) as follows:

$$\hat{x}_j(t) \equiv \frac{c}{\sqrt{2}\omega_j}\left(\hat{a}_j(t) + \hat{a}_j^\dagger(t)\right), \hat{p}_j(t) \equiv \frac{i\hbar\omega_j}{\sqrt{2}c}\left(\hat{a}_j(t) - \hat{a}_j^\dagger(t)\right). \tag{20}$$

where, taking Equation (18) into account, we obtain the commutation rules for the field Equation (17).

The Hamiltonian for the free evolution in terms of the operators $\left\{\hat{a}_j, \hat{a}_j^\dagger\right\}$ is as follows:

$$\hat{H} = \frac{1}{2}\sum_j \varepsilon_j\left(\hat{a}_j\hat{a}_j^\dagger + \hat{a}_j^\dagger\hat{a}_j\right), \tag{21}$$

and the evolution of the free quantum field from an inital state, represented by the density operator $\hat{\rho}$, is given in the HS formalism by the Heisenberg equation, that is

$$\frac{d}{dt}\hat{\rho} = \frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right]. \tag{22}$$

For details, see any book on quantum field theory. I use the notation of Berestetskii et al. [14].

3.1.3. Weyl–Wigner Quantization

Up to here, the standard (Hilbert space) quantum theory of the said Bose field has been used. The Weyl–Wigner (WW) representation of *the same quantum field* is achieved via the (inverse) Weyl transform, T_W, which becomes, for a quantum field operator $\hat{M}$,

$$\begin{aligned} T_W\left[\hat{M}\right] &= \frac{1}{(2\pi^2)^n}\prod_{j=1}^{n}\int_{-\infty}^{\infty} d\lambda_j \int_{-\infty}^{\infty} d\mu_j \exp\left[-2i\lambda_j \mathrm{Re} a_j - 2i\mu_j \mathrm{Im} a_j\right] \\ &\quad \times Tr\left\{\hat{M}\exp\left[i\lambda_j\left(\hat{a}_j + \hat{a}_j^\dagger\right) + \mu_j\left(\hat{a}_j - \hat{a}_j^\dagger\right)\right]\right\} \equiv W_{\hat{M}}. \end{aligned} \tag{23}$$

where $T_W\left[\hat{M}\right]$ stands for (inverse) Weyl transform of the operator $\hat{M}$, which may be either an observable or the density operator of a state. This generalizes the Weyl transform Equation (3) from quantum mechanics of particles to quantum fields, taking into account the fact that the real and imaginary parts of the field amplitude evolve like the position and momentum of a harmonic oscillator [1,2].

The result $W_{\hat{M}}\left(\left\{a_j, a_j^*\right\}\right)$, obtained from Equation (23), is a function of (c-number) field amplitudes. Thus, we might question whether the effect of the inverse Weyl transform is not just reversing the HS quantization of the field (involved operators) to become again a classical field (involving classical-like amplitudes). The answer to the question is negative because the WW formalism disagrees from the classical one in a fundamental aspect, that is, the definition of the ground state. In fact, the ground state of a classical field is the (empty) vacuum where all amplitudes are nil, that is, $a_j = a_j^* = 0$ for every j. In contrast, the ground state of the Hilbert-space quantized field is represented by the operator

$$\hat{\rho} = \mid 0\rangle\langle 0 \mid, \tag{24}$$

fulfilling the following equalities

$$\hat{a}_j \mid 0\rangle = 0 \Rightarrow \langle 0 \mid \hat{a}_j^\dagger = 0, \text{ for all radiation modes,} \tag{25}$$

0 being here the null vector in the Hilbert space. Usually, this state is called "vacuum" state, but it is not empty in the WW formalism. In fact, if the operator Equation (24) is inserted in Equation (23) in place of $\hat{M}$, after some algebra, we obtain the Wigner function of that "vacuum" state, that is

$$W_0(\{a_j\}) = \prod_j \frac{2}{\pi}\exp\left(-2\left|a_j\right|^2\right), \tag{26}$$

where we have normalized it for the integration with respect to $\prod_j dRea_j dIma_j$. Hence, the mean square value of each amplitude in the vacuum is as follows:

$$\left\langle \left|a_j\right|^2 \right\rangle = \frac{1}{2}. \tag{27}$$

Equation (26) strongly suggests an interpretation of the vacuum Wigner function as a probability distribution, while Equation (27) would be the variance of $\left|a_j\right|$. The field represented by Equation (26) is labeled the "zeropoint field" (ZPF) and corresponds to the "quantum vacuum fluctuations" of the canonical (Hilbert space) formalism for quantum fields.

There are also differences between HS and WW for other states. In the HS formalism, all pure excited states of the field may be obtained by repeated application of the creation operators to the vacuum state, and additional pure states may be achieved, which are linear combinations of the former. The Wigner functions of these states may be obtained in the WW formalism using the Weyl transform Equation (23). However, those Wigner functions cannot be interpreted as probability distributions because they are not positive in general. This fact might lead some authors to think either that the WW representation for fields does not admit a realistic interpretation, or it is not equivalent to the standard HS one.

In my opinion, the dilemma derives from the flawed assumption that all state-vectors' declared states in the HS representation are physical. For instance, a single-particle state in the form of a plane wave extended over an arbitrary large normalization volume cannot represent the *physical* state of a (localized) particle, I believe. Indeed, in the WW representation, particles (bosons) do not appear explicitly. We see just continuous fields, while the particles of the HS representation may be just useful mathematical elements that appear in intermediate stages of the calculations. The question has been discussed elsewhere [1,2], in relation to the electromagnetic field, and similar arguments apply for other Bose fields. Further comments on the question are made in subsection 5.2 below.

The evolution of the field amplitudes is given by the WW counterpart of the Heinsenberg Equation (22) in the HS formalism, that is, Moyal Equation (8). However, this should be written in terms of the field amplitudes, rather than coordinates and momenta, see Equation (19). That is

$$\begin{aligned}\frac{\partial W_{\hat{M}}}{\partial t} &= 2\{\sin\left[\frac{1}{4}\left(\frac{\partial}{\partial \mathrm{Re}a'_j}\frac{\partial}{\partial \mathrm{Im}a''_j} - \frac{\partial}{\partial \mathrm{Im}a'_j}\frac{\partial}{\partial \mathrm{Re}a''_j}\right)\right] \\ &\quad \times W_{\hat{M}}\left\{a'_j, a^{*\prime}_j, t\right\} H\left(a''_j, a^{*\prime\prime}_j\right)\}_{a_j}, \end{aligned} \tag{28}$$

where $\{\}_{a_j}$ means making $a'_j = a''_j = a_j$ and $a^{*\prime}_j = a^{*\prime\prime}_j = a^*_j$ after performing the derivatives. A remarkable result is that the free field Hamiltonian $\left\{a_j, a^*_j\right\}$ is a quadratic function of the amplitudes, where only terms with first and second derivatives remain in Equation (28). The consequence is that the Moyal bracket becomes the Poisson bracket of classical dynamics, and the WW quantized free EM field evolution is governed by classical theory (for a similar feature in the quantum electromagnetic field, see [1,2]).

In summary, we started with the classical field Equations (9) and (10). After that, we quantized the field via the canonical Hilbert space method, promoting amplitudes to be creation and annihilation operators of particles. Then, we performed an inverse Weyl transform and obtained a theory with classical flavor. Indeed, it contains (c-number) amplitudes and classical evolution, which might appear as if we returned to the classical theory Equations (9) and (10). However, there is a relevant variation. At a difference with a classical field, in the WW quantized field, we assume the existence of a non-empty vacuum filled with a random radiation Equation (26). In conclusion, I have achieved the goal of obtaining a formalism for the quantized bose field that leads to a clear picture of reality: it is just the classical field with the additional assumption that the ground state (the "vacuum") consists of a random background radiation. Particles (bosons) are mathematical elements, useful for computations, but without physical reality.

3.2. The Charged Spin-Zero Massive Field

The assumption that the field function ψ in Equation (9) is real (in the mathematical sense of being defined in the real numbers) leads to the quantum theory of *neutral* spin zero massive particles. Now, we shall study the case when ψ is complex. The expansion in plane waves is similar to Equation (12) except that now we shall introduce two different amplitudes, namely a_j and b_j, obtaining the following:

$$\psi(\mathbf{r},t) = \sum_j \frac{1}{\sqrt{2\varepsilon_j}} \left[a_j \exp\left(i\mathbf{k}_j.\mathbf{r} - i\varepsilon_j t\right) + b_j^* \exp\left(-i\mathbf{k}_j.\mathbf{r} + i\varepsilon_j t\right) \right], \tag{29}$$

where b_j^* *is not* the complex conjugate of a_j. The main difference with the canonical quantization of the neutral field is that now we must introduce two kinds of field operators, namely $\left\{\hat{a}_j, \hat{a}_j^\dagger\right\}$ and $\left\{\hat{b}_j, \hat{b}_j^\dagger\right\}$, which correspond to different kinds of particles. These operators fulfill the following commutation rules:

$$\left[\hat{a}_j, \hat{a}_k^\dagger\right] = \delta_{jk}, \left[\hat{b}_j, \hat{b}_k^\dagger\right] = \delta_{jk}, \tag{30}$$

all other commutators being nil. The operators $\hat{a}_j^\dagger$ and $\hat{b}_j^\dagger$ are interpreted as creating particles and antiparticles, respectively, while $\hat{a}_j$ and $\hat{b}_j$ annihilate them. Particles and antiparticles have the same mass but, if they are charged, opposite electric charges, see any book on quantum field theory [14].

The quantization in the WW representation is quite similar to the case of neutral particles discussed above. The Weyl transform Equation (23) now consists of the product of two similar terms involving the particle´s and antiparticle´s operators, respectively. Then, the vacuum consists of two fields with opposite charge and a Wigner function similar to Equation (1) each. That is, the Wigner function of the vacuum state for the two fields should be

$$W_0(\{a_j, b_k\}) = \prod_j \frac{2}{\pi} \exp\left(-2|a_j|^2\right) \times \prod_k \frac{2}{\pi} \exp\left(-2|b_k|^2\right). \tag{31}$$

If the fields are charged, the positive and negative electric charges of the vacuum may cancel on the average, being associated to the amplitudes a_j and b_k, respectively. However, the vacuum field Equation (31) may give rise to fluctuations of both energy and charge in the vacuum. Actually, particles and antiparticles interact via the electromagnetic force, where the actual vacuum should take the electromagnetic interaction of the charges into account. Consequently, the vacuum field will be more involved than Equation (31), where the electromagnetic interactions are neglected.

3.3. Spin-One Fields: Electromagnetism

Massive spin-one fields, either neutral or charged, are studied via the Proca equation. The Weyl–Wigner quantization is similar to that for spin-zero fields and will not be studied further here [14].

The most relevant spin-one field is the quantized electromagnetism, whose Weyl–Wigner representation has been studied elsewhere [1,2], and I will briefly revisit it below. It shall be studied in the Coulomb gauge, as is appropriate for the study of the interaction of the field with particles in non-relativistic motion, which will be made in the last Section 5.

The free EM field Hamiltonians in the HS and WW formalisms are related by the inverse Weyl transform and are defined as follows [1]:

$$\hat{H}_{HS} = \hbar \sum_j \omega_j (\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2}), H_{WW} = \hbar \sum_j \omega_j |a_j|^2, \tag{32}$$

respectively.

When there are charged particles, the evolution of both the particles and the field is modified by the interactions, the interaction Hamiltonian being given in WW by

$$H_{int} = -\sum_k e_k \mathbf{p}_k \cdot \mathbf{A}\left(\mathbf{x}_k, \left\{a_j, a_j^*\right\}\right), \tag{33}$$

where e_k is the charge of the particle, $\mathbf{p}_k$ its momentum, and $\mathbf{A}$ is the potential vector of the field at the position $\mathbf{x}_k$ of the particle. The subindex k runs for all charged particles, but we might sum for all particles in Equation (33), substituting j for k and putting $e_j = 0$ for neutral particles. I note that $\mathbf{p}_k$ is a 3D vector that corresponds to three generalized one-dimensional momenta. I omit the expression for the dependence of the potential vector $\mathbf{A}$ in terms of the field amplitudes $\left\{a_j, a_j^*\right\}$, which is well known. Actually, the work in the Coulomb gauge requires taking also into account the instantaneous electrostatic interaction between charged particles, which is not included in the interaction Hamiltonian Equation (33). Here, I skip this term but it should be included in actual calculations.

In the WW formalism, the field evolution is given by the Moyal Equation (28), but it is governed by the classical Maxwell theory when the total Hamiltonian is at least most quadratic in the amplitudes, as is usually the case. This is similarly to other Bose fields studied in section 3. The most relevant difference between the canonical, HS, and the WW formalism of the field is that the WW-quantized field assumes the existence of a radiation in the vacuum with a distribution of amplitudes (Equation (26).) That is, the zeropoint field (ZPF), which corresponds to the "quantum vacuum fluctuations" of the standard (Hilbert space) formalism for the quantized electromagnetic field.

The main result is that both formalisms for the quantized EM field are equivalent, that is, canonical (HS) and WW. Both would predict the same results for experiments.The field E_{ZPF} is random (mathematically, it is a stochastic process), and its most relevant property is the energy density, $\rho(\omega)$, defined as the energy per unit time, unit volume and unit frequency interval. This may be derived from the distribution of the amplitudes of the radiation modes (Equation (26)), which have a mean energy $\frac{1}{2}\hbar\omega$ per normal mode [1]. Hence, the total energy density per unit frequency interval, $\rho(\omega)$, is the product of $\frac{1}{2}\hbar\omega$ times the number density of modes per unit frequency interval in a large normalization volume, which gives

$$\rho(\omega) = \frac{1}{2\pi^2 c^3}\hbar\omega^3. \tag{34}$$

This expression for the vacuum energy density (usually named zeropoint field, ZPF) goes back to the early period of quantum theory, when it was studied by Planck, Einstein, Nernst, and other people (see, e.g., [15,16]). Equation (34) implies that the electric and magnetic fields of the ZPF are very strong for high frequencies. However, the fields fluctuate rapidly then, and the clean effect on charges is relatively small.

4. Discussion and Applications

In the previous section, I presented a quantization procedure of Bose fields, which is physically equivalent to the canonical one. The quantization consists of adding to the classical field a Gaussian distribution of field amplitudes that correspond to the quantum vacuum fluctuations in the canonical formalism, but the evolution remains classical. In this section, I discuss the realistic interpretation, related works, and possible applications.

4.1. The Realistic Interpretation

In my view, the most interesting feature of the WW formalism for Bose fields is that it allows for *realistic interpretations* in the sense of Section 1. Actually, after one century of quantum mechanics, there is no consensus about the interpretation [7]. Furthermore, there is a widespread opinion that a realistic interpretation is not possible, but I do not agree [3,6]. The realistic interpretation of Bose fields quantized via the WW representation is as follows: Quantization means assuming that the classical laws are valid, but also that the vacuum is not empty but is filled with *real fields*, having a Gaussian probability distribution (Equation (26)). That field corresponds to the quantum vacuum fluctuations of the canonical formulation.

As commented on Section 3.1, in WW-quantized Bose fields, the bosons are not physical (localized) particles, just mathematical elements useful for calculations. This is, in particular, the case for photons. In fact, many quantum phenomena taken as proofs of the existence of photons (more properly a discrete character of the electromagnetic field) may be understood in terms of a continuous field, at least qualitatively. This is the case in particular for the photoelectric or the Compton effects [17].

4.2. Stochastic Electrodynamics

The essential ingredient of WW quantization is the existence of a vacuum field (the zero-point field, ZPF) with a Gaussian distribution of field amplitudes (see Equation (26)). Historically, the idea of ZPF appeared in the second Planck radiation theory of 1912. As said above, Einstein and other people were interested in it. Later on, Walter Nernst suggested that the ZPF might be relevant in order to explain the stability of atoms and molecules. For a good account of these works, see the book by Milonni [15]. The ZPF line of research was abandoned due to the appearance of Bohr atomic model in 1913, which led the community to a different research program named "old quantum theory" that culminated in 1925 with the standard form of quantum mechanics. However, around 1960, the early idea of Nernst reappeared as a theory named "stochastic (or random) electrodynamics" (SED) [18]. Many articles have been published on the subject from that time. For a review of the work made before 1996, see [19]; more recent reviews are [16] and Chapter 5 of the book [3] reproduced in [20].

Initially, the purpose of SED was to explain some quantum non-relativistic phenomena as being due to the ZPF. Indeed, there is an analogy between SED and non-relativistic QED [21]. Later on, the aim was extended to the relativistic domain, defining SED more generally as "classical electron theory with classical electromagnetic zero-point radiation" [22], a definition that might be applied to the WW-quantized field.

4.3. The Lack of a WW Representation for Fermi Fields

The quantization formalism studied in this paper has the shortcoming that it cannot be easily extended to Fermi fields. If the generalization was achieved it would provide a new and interesting formulation of the whole quantum field theory. However in the restricted non-relativistic domain Fermi fields may be treated as sets of particles, specially those with spin-1/2 like electrons. In addition in that domain electromagnetism is the unique quantum field of interest. Hence it is possible to formulate the whole non-relativistic quantum electrodynamics in the Weyl–Wigner representation because both the EM field and the particles may be WW quantized. The resulting theory, i.e., non-relativistic QED will be sketched in the next Section 5.

4.4. The Divergence Problem

A possible difficulty of the approach studied in this paper is the divergence of the vacuum fields, a problem that actually appears also in the canonical Hilbert space formulation of quantum fields, although less explicitly. As is well known the difficulty has been treated in practice with renormalization techniques that have achieved a spectacular success. However from a fundamental point of view the problem remains. A practical solution is to state that the present treatment of fields cannot be correct beyond Planck´s density, but this gives rise to a difficulty known as the "cosmological constant problem" [23]. I may especulate about the complete solution of the divergence problem, rather than just a practical one, suggesting two possibilities. The first one is that the vacuum ZPF of Fermi fields might provide a negative energy that could balance the positive divergent energy of Bose fields. Another possibility is that the set of all fields and interactions provide energy and pressure in free space with the "equation of state" of the vacuum, that is a negative pressure p and positive energy ε such that $p = -\varepsilon$, which might be counterbalanced by a cosmological constant in Einstein equation of general relativity. Anyway the cancelation might not be exact, the remaining energy and pressure giving rise to the cosmological dark energy [24].

4.5. Fields in Curved Spacetime

The study of quantum fields in curved spacetime presents a difficulty, namely that the commutation relations at neighbour points are well defined in flat (Minkowski) space but not so clearly in curved space. The standard method to avoid the problem is to define normal modes of the quantum fields in a curved spacetime [25,26], then studying field excitations via the creation of virtual particles. However this gives rise to the non-uniqueness of canonical field quantization [27]. I believe that the WW quantization might avoid, or diminish, these difficulties of the canonical quantization.

Several predictions have been achieved with the study of quantum fileds in curved space, the most popular being the Hawking radiation by black holes. Here I will comment on another one, the Unruh effect. It is currently interpreted stating that photon detectors accelerating through the quantum vacuum behave as if they were located in an inertial frame in a thermal bath with temperature

$$T = \frac{\hbar a}{2\pi c k_B}, \tag{35}$$

a being the acceleration and k_B Boltzmann constant [28–30]. On the other side Timothy Boyer has derived Equation (35) as due to the modification of the ZPF spectrum of the vacuum, Equation (34), when it is seen in an accelerating frame, see [22] and references therein. On the other hand the "detection of individual photons" could be explained as the action of a continuous field on a photocounter, say via a photoelectric effect [2,17]. Boyer attributes the agreement to the fact that "classical electron theory with classical electromagnetic ZPF" is close to the quantum theory of the EM field. In this article I have shown that it is not just close, but identical. Indeed the sentence within inverted commas is actually an appropriate definition for *WW quantized electromagnetic field*.

The two mentioned interpretations of the Unruh effect illustrate the difference beween the canonical and the WW quantization. In the former approach the current interpretation is that in curved spacetime the vacuum is excited via the creation of particles (photons) with a thermal distribution. In the latter the spectrum of the ZPF differs from Equation (34) in an accelerated frame, that is the distribution of energy amongst the frequencies of the radiation modes.

4.6. Quantum Gravity

As is well known the gravitational field (general relativity) cannot be quantized via the standard method of expanding the field in plane waves because the fundamental (Einstein) equation is not linear. The nonlinear character does not fit in the quantum commutation rules. The Unruh effect has been sometimes taken as a "quantum gravity" effect. Indeed it combines a quantum element (the electromagnetic ZPF) with gravity (or acceleration).

However quantum gravity is mainly devoted to allow calculations in cases when both quantum theory and gravity are relevant (typically involving mass densities of order Planck´s). A large amount of work has been made on the problem [31]. Both canonical and path integrals quantization have been used without achieving a completely satisfactory quantum gravity theory till now.

The gravitational field is bosonic, and therefore, I believe that it could be quantized via the addition of a vacuum field to the classical theory. This would amount to assuming the existence of Gaussian spacetime fluctuations at short distances. I do not know how the WW quantization might help with the formulation of a quantum gravity theory. In any case, a problem would arise due to the small value of the Newton constant, which leads to a huge scale difference between gravitational and typical quantum field theory vacuum energy.

5. Non-Relativistic Quantum Electrodynamics

Relativistic quantum electrodynamics (QED) cannot be formulated within WW because for Fermi fields, a transform playing the role Weyl´s for Bose fields is not available. However, it is possible to formulate WW-quantized, non-relativistic QED, treating Fermi particles in the non-relativistic approximation where the Wigner representation is well known [9,10], see the Introduction section. Furthermore, a unified treatment of both the particles and the electromagnetic (EM) field is possible, in terms of generalized coordinates x_j and momenta p_j, as shown below. That theory has purely academic interest because the problem has been studied using most appropriate relativistic quantum electrodynamics (QED). The interest of the non-relativistic theory might be to illustrate the calculational methods of the Weyl–Wigner (WW) representation in comparison with either canonical (HS) or path integral treatments.

5.1. Weyl–Wigner Unified Treatment of Quantum Particles and EM Field

Quantum mechanics of particles in the Weyl–Wigner (WW) representation was skecthed in Section 2, and the WW-*quantized* EM field was studied in Section 3.3 and elsewhere [1,2]. Now, let us consider a system of N quantum particles, some of them charged and therefore interacting with the EM field.

The typical problem of non-relativistic QED in the Weyl–Wigner formalism is the joint evolution of the state of the particles represented by the Wigner function, in terms of the coordinates and momenta $\{x_j, p_j\}$, and the fields represented also by the Wigner function in terms of the amplitudes $\{a_l, a_l^*\}$. We need the total Hamiltonian, that is

$$H_{tot} = H_{part}(\{x_j, p_j\}) + H_{field}(\{a_l, a_l^*\}) + H_{int}(\{x_j, p_j, a_l, a_l^*\}) \tag{36}$$

The evolution of the system would follow from the application of Moyal equations, that is, Equation (8) for the particles and Equation (28) for the field, but the combination of the particles coordinates and momenta with the field amplitudes is inconvenient in practice.

A more refined method would be to use for the field (pseudo) coordinates x_j and momenta p_j of the plane waves expansion, using the change of variables Equation (19). Then, we might apply a unique Moyal Equation (8) involving both particles and field. For instance, we might label the coordinates and momenta of the N particles with subindices $j \leq 3N$ and the variables $\{x_j, p_j\}$ associated to the (classical) field amplitudes $\{a_j\}$ via Equation (19) for $j > 3N$, which would allow for using Equation (8) for both particles and field. In this case, the Hamiltonian Equation (36) could be written as follows:

$$H_{tot} = H(\{x_k, p_k\}),$$

where the generalized coordinates and momenta may now correspond to either particles or to radiation modes of the field. The Hamiltonian $H(\{x_k, p_k\})$ might be obtained from Equation (36) via a change in variables.

This WW formalism might allow for calculating non-relativistic radiative corrections, like the non-relativistic parts of the Lamb shift or the anomalous magnetic moment of the electron which have been calculated also using the canonical quantization (see, e.g., the book by Milonni [15]) .

5.2. Discussion

Here, I shall comment on three aspects of the proposed treatment of non-relativistic quantum electrodynamics within the WW formalism. That is mathematical, conceptual, and practical.

From the *mathematical* point of view, the unified treatment of particles and fields looks nice. The joint states and observables of both particles and fields are represented together by a set of generalized coordinates and momenta, $\{x_k, p_k\}$. However, the handling is not completely symmetrical because, for the field, the ground state includes the vacuum ZPF (Equation (34)), which is not the case for particles.

From the *conceptual* point of view, I mean whether the formalism provides an intuitive description of reality, that is, whether the formalism allows for a realistic interpretation, it is the case that the WW field formalism does admit a realistic interpretation, but the WW treatment of particles does not, as was commented in Section 2.2. Therefore, the non-relativistic QED in WW does not allow for a full realistic interpretation, and its interest is scarce from the conceptual point of view.

From the *practical* point of view, that is the simplicity of the calculations, the WW representation of the non-relativistic QED does not offer an advantage over the canonical HS formalism, except maybe when the particle Hamiltonian is at most quadratic in the coordinates and momenta. In that case, the WW treatment becomes actually classical, that is, a combination of Newtonian dynamics with Maxwell–Lorentz electrodynamics. In practice, this would correspond just to a harmonic oscillator or a set of coupled harmonic oscillators.

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