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A sedenion algebraic representation of three colored fermion generations

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Abstract. Three generations of fermions with $SU(3)_C$ symmetry are represented algebraically in terms of the algebra of sedenions, $\mathbb{S}$, generated from the octonions, $\mathbb{O}$, via the Cayley-Dickson process. Despite significant recent progress in generating the Standard Model gauge groups and particle multiplets from the four normed division algebras, an algebraic motivation for the existence of exactly three generations has been difficult to substantiate. In the sedenion model, one generation of leptons and quarks with $SU(3)_C$ symmetry is represented in terms of two minimal left ideals of $\mathbb{C}\ell(6)$, generated from a subset of all left actions of the complex sedenions on themselves. Subsequently, the finite group S_3, which are automorphisms of $\mathbb{S}$ but not of $\mathbb{O}$, is used to generate two additional generations. The present paper highlight the key aspects and ideas underlying this construction.

1. Introduction

In an attempt to establish the geometric and algebraic roots of the Standard Model (SM), several proposals have been put forth over the years which take as its essential mathematical ingredients (tensor products of) the only four normed division algebras over the reals: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. These proposals seek to unify the gauge groups together with the leptons and quarks that they act on into a single unified algebraic framework, in terms of an algebra acting on itself.

The octonions $\mathbb{O}$, the largest of the division algebras, were first considered in the 70s for their intriguing efficacy in describing quark color symmetry [1]. More recent constructions have sought to extend these earlier results by describing additional SM structure in terms of tensor products of division algebras [2, 3, 4, 5, 6, 7]. Many others have contributed to these, and related, algebraic approaches including those based on topology [8, 9, 10, 11, 12], exceptional Lie groups [13, 14, 15, 16, 17, 18], Clifford algebras [19, 20, 21, 22, 23, 24, 25, 26, 27], and Jordan algebras [28, 29, 30, 31, 32, 33].

Existing division algebraic models offer an elegant algebraic construction of the internal space of a single generation of leptons and quarks. Despite several attempts however [4, 34, 35], a clear algebraic origin for the existence of three generation is yet to be found. In [34] three generations of color states are identified directly from the algebra $\mathbb{C}\ell(6)$ generated from the left adjoint actions of $\mathbb{C}\otimes\mathbb{O}$ onto itself. In [3], the algebra $\mathbb{T}^6=\mathbb{C}\otimes\mathbb{H}^2\otimes\mathbb{O}^3$, is used to represent three generations instead. In [36], dimensional reduction is used to isolate three $\mathbb{H}$ subalgebras within $\mathbb{O}$ in order to describe three generations of leptons.

Starting with $\mathbb{R}$, each of the remaining three division algebras can be generated via the Cayley-Dickson (CD) process. This process continues beyond $\mathbb{O}$ to produce a series of 2^n-dimensional algebras. One might therefore ask if the next algebra in the series, the sedenions $\mathbb{S}$, allows for an elegant algebraic description of three generations of leptons and quarks.

In a recent paper [37] (which builds on earlier work [35, 38]), we argued that $\mathbb{S}$ exhibits the algebraic structure necessary to describe (at least part of) the internal space of three generations. A single generations of leptons and quarks is represented in terms of two minimal left ideals of the $\mathbb{C}\ell(6)$, generated from the left adjoint actions of three intersecting $\mathbb{C}\otimes\mathbb{O}$ subalgebras of $\mathbb{C}\otimes\mathbb{S}$ onto $\mathbb{C}\otimes\mathbb{S}$. Subsequently, the S_3 automorphisms of $\mathbb{S}$ (which are not automorphisms of $\mathbb{O}$) are used to generate two additional generations. The three generations transform under a single copy of the gauge group $SU(3)_C$, thereby avoiding introducing three generations of gluons.

2. Normed division algebras

A well-known result by Hurwitz [39] states that there exist only four normed division algebras (over the reals): $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, of dimensions one, two, four and eight respectively.

The octonions $\mathbb{O}$ are the largest division algebra. An orthonormal basis is comprised of seven imaginary units: $i_1, ...i_7$, along with the unit $1 = i_0$. A general octonion x may then be written as $x = x_0 i_0 + x_1 i_i + ... + x_7 i_7, \quad x_0, ..., x_7 \in \mathbb{R}$, with the octonion conjugate $\overline{x}$ defined as $\overline{x} = x_0 i_0 - x_1 i_1 - ... - x_7 i_7$. The norm of an octonion $|x|$ is subsequently defined by $|x|^2 = x\overline{x} = \overline{x}x$, and the inverse $x^{-1} = \overline{x}/|x|^2$.

The multiplication of octonions[1] is captured in terms of the Fano plane, see Fig. 1. Each projective line in the Fano plane corresponds (together with the identity i_0) to a $\mathbb{H}$ subalgebra; there are seven such subalgebras.

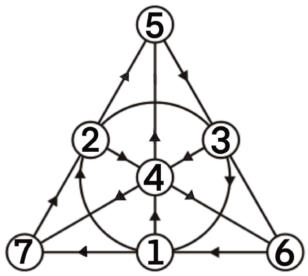

Figure 1. The Fano plane, encoding the multiplicative structure of our octonions, where $a \equiv i_a,\ a = 1, ..., 7$. Note that each line is cyclic, representing a quaternionic triple.

The automorphism group of $\mathbb{O}$ is G_2, which contains $SU(3)$ as one of its maximal subgroups, corresponding to the stabilizer subgroup of one of the octonion imaginary units, or equivalently, the subgroup of $Aut(\mathbb{O})$ that preserves the representation of $\mathbb{O}$ as the complex space $\mathbb{C}\oplus\mathbb{C}^3$. This splitting is associated with the quark-lepton symmetry [36].

3. The Cayley-Dickson algebra of sedenions

The CD process is an iterative construction that generates at each stage an algebra (with involution) of dimension twice that of the previous. A generic element of the CD algebra $\mathbb{A}_n$, can be written as $a + bu$, where $a, b \in \mathbb{A}_{n-1}$, and u is the new imaginary unit introduced by the CD process applied to $\mathbb{A}_{n-1}$. The fifth CD algebra $\mathbb{A}_4$ ($\mathbb{A}_0 = \mathbb{R}$), generated from $\mathbb{O}$, is the 16-dimensional algebra of sedenions $\mathbb{S}$. This algebra is non-commutative, non-associative, and not even alternative ($x(xy) \neq (xx)y$ and $y(xx) \neq (yx)x$ in general).

An orthonormal basis for $\mathbb{S}$ comprises 15 mutually anticommuting imaginary units $e_1, ..., e_{15}$ together with the unit $1 = e_0$. The imaginary units $e_1, .., e_7$ correspond to the original octonion units $i_1, ..., i_7$. A general sedenion may then be written as

$$w = w_0 e_0 + w_1 e_1 + ... + w_{15} e_{15}, \quad w_0, ..., w_{15} \in \mathbb{R}.$$

[1] There are different multiplication rules for $\mathbb{O}$ used by different authors in the literature. Here we follow the multiplication table used in [40]

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with the sedenion conjugate $\overline{w}$ defined as $\overline{w} = w_0 e_0 - w_1 e_1 - ... - w_{15} e_{15}$, and the sedenion norm $|w|$ defined by $|w|^2 = w\overline{w} = \overline{w}w$. Whenever $|w|^2 \neq 0$, the inverse of w is given by $w^{-1} = \overline{w}/|w|^2$. The product of two sedenions w, v can be determined using the multiplication table of the orthonormal basis units given in [40, 41], together with linearity.

3.1. *Octonion subalgebras inside the sedenions*

$\mathbb{S}$ contains seven $\mathbb{O}$ subalgebras in addition to the original $\mathbb{O}$ algebra from which $\mathbb{S}$ was generated [41]. In particular, three $\mathbb{O}$ subalgebras can be isolated by applying the CD process to the three $\mathbb{H}$ subalgebras of $\mathbb{O}$ that contain i_4

$$\mathbb{O}_i \quad : \quad \{e_0, e_i, e_4, e_{i+4}, e_8, e_{i+8}, e_{12}, e_{i+12}\}, \qquad i = 1, 2, 3. \tag{1}$$

The intersection $\mathbb{O}_1 \cap \mathbb{O}_2 \cap \mathbb{O}_3 = \mathbb{H}$ is generated by $\{e_0, e_4, e_8, e_{12}\}$.

3.2. *Automorphisms of sedenions*

The automorphism group of $\mathbb{S}$ is known to be $Aut(\mathbb{S}) = Aut(\mathbb{O}) \times S_3 = G_2 \times S_3$ [42, 43]. Explicitly, the automorphisms of $\mathbb{S}$ are given by

$$\theta' : a + be_8 \quad \rightarrow \quad a\theta + (b\theta)e_8, \tag{2}$$

$$\epsilon : a + be_8 \quad \rightarrow \quad a - be_8, \tag{3}$$

$$\psi : a + be_8 \quad \rightarrow \quad \frac{1}{4}[a + 3\overline{a} + \sqrt{3}(b - \overline{b})] + \frac{1}{4}[b + 3\overline{b} - \sqrt{3}(a - \overline{a})]e_8 \tag{4}$$

where $a, b \in \mathbb{O}$, $\theta \in Aut(\mathbb{O})$, and ϵ and ψ satisfy $\epsilon^2 = \psi^3 = 1,\ \epsilon\psi = \psi^2\epsilon$, generating S_3.

The explicit action of ψ on the sedenion basis elements can be written as:

$$\psi(e_i) = -\frac{1}{2}e_i - \frac{\sqrt{3}}{2}e_{i+8}, \;\; \psi(e_{i+8}) = -\frac{1}{2}e_{i+8} + \frac{\sqrt{3}}{2}e_i, \;\; \psi(e_8) = e_8, \tag{5}$$

where $i = 1, ..., 7$. The automorphism ψ corresponds to a simultaneous rotations in the seven $e_i - e_{i+8}$ planes by $2\pi/3$, and therefore does not correspond to an automorphism of $\mathbb{C} \otimes \mathbb{O}$.

From the action of θ' on the sedenion units above it is immediately clear that e_8 is stabilized by the G_2 automorphisms. Furthermore, these G_2 automorphisms map $e_i, i < 8$ to $e_j, j < 8$, and e_{i+8} to e_{j+8}. They therefore do not mix the new sedenion elements e_{i+8} with the original octonion elements e_i, $i = 1, ..., 7$. Only the S_3 automorphism ψ of order three mixes the original octonion units with the new sedenion units.

Given that the stabilizer of e_4 in G_2 is $SU(3)$, and e_8 is likewise an $SU(3)$ singlet (as it is fixed by G_2), it follows that the $SU(3)$ subgroup of G_2 fixes the entire quaternion generated by $\{e_0, e_4, e_8, e_4 e_8 = e_{12}\}$, corresponding to $\mathbb{O}_1 \cap \mathbb{O}_2 \cap \mathbb{O}_3$. The S_3 automorphisms on the other hand do not fix this quaternion, although they do stabilize it.

3.3. *The left multiplication algebra of* $\mathbb{C} \otimes \mathbb{S}$

Despite the algebras $\mathbb{O}$ and $\mathbb{S}$ being non-associative, one can consider the left adjoint actions of these algebras on themselves as linear operators, thereby generating an associative algebra. Due to the non-associativity of $\mathbb{O}$, this left adjoint algebra of $\mathbb{C} \otimes \mathbb{O}$ contains genuinely new maps. The 64 distinct left-acting complex-linear maps provide a faithful representation of $\mathbb{C}\ell(6)$.

The generalization to $\mathbb{C} \otimes \mathbb{S}$ is not immediately obvious since compositions of sedenion left actions, unlike compositions of octonions left actions, do not in general anti-commute. Closer inspection reveals that all the left adjoint actions of the original octonion elements $e_0 = i_0, ..., e_7 = i_7$ together with e_8 onto a general element of $\mathbb{S}$ do anticommute, and generate $\mathbb{C}\ell(8)$ (as do the composed left actions of $\{e_8, ..., e_{15}\}$). A generating basis for $\mathbb{C}\ell(8)$ is therefore given by the left adjoint actions of $\{e_1, ..., e_8\}$. Another possible generating basis is $\{e_8, ..., e_{15}\}$.

4. One generations of color states from $\mathbb{C}\otimes\mathbb{S}$

In [5] it is shown that one generation of leptons and quarks with $SU(3)_C$ symmetry can be elegantly represented in terms of two eight dimensional minimal left ideals of $\mathbb{C}\ell(6)$ generated from the left adjoint actions of $\mathbb{C}\otimes\mathbb{O}$ on itself. The construction of the minimal ideals depends on a choice of Witt basis of $\mathbb{C}\otimes\mathbb{O}$ (or equivalently $\mathbb{C}\ell(6)$). This Witt basis constitutes three pairs of fermionic ladder operators, each one defined in terms of one of the three $\mathbb{H}$ subalgebras of $\mathbb{O}$ that contain i_4. The dimensions of the minimal ideals then dictates the number of distinct physical states, whereas the gauge symmetries are those unitary symmetries that preserve the ideals (or equivalently, the Witt basis).

We generalize this construction of a single generation from $\mathbb{C}\otimes\mathbb{O}$ to $\mathbb{C}\otimes\mathbb{S}$. For each $\mathbb{C}\otimes\mathbb{O}_i\subset\mathbb{S}$, $i=1,2,3$ we construct a pair of fermionic ladder operators. The three pairs of ladder operators via their left adjoint action onto $\mathbb{C}\otimes\mathbb{S}$ then generate a $\mathbb{C}\ell(6)$ subalgebra of $\mathbb{C}\ell(8)$. Two minimal left ideals of this $\mathbb{C}\ell(6)$ can subsequently be used to represent one generation of leptons and quarks with $SU(3)_C$ symmetry, in what corresponds to a direct generalization of the construction in [5]. The novelty of this approach is that subsequently (see the next section) the order three S_3 automorphism of $\mathbb{S}$ can be used to generate exactly two additional generations.

For each of the three octonion subalgebras $\mathbb{O}_1,\mathbb{O}_2,\mathbb{O}_3$ of $\mathbb{S}$ above, define a pair of ladder operators:

$$A_j^\dagger = \tfrac{1}{2\sqrt{2}}(e_j+ie_{j+4}+e_{j+8}+ie_{j+12}),\quad A_j = \frac{1}{2\sqrt{2}}(-e_j+ie_{j+4}-e_{j+8}+ie_{j+12}), \tag{6}$$

where $j=1,2,3$. It is readily checked that $A_j(A_jw)=A_j^\dagger(A_j^\dagger w)=0$ and $A_j(A_kw)=-A_k(A_jw)$, $\forall w\in\mathbb{C}\otimes\mathbb{S}$, and therefore these ladder operators satisfy (as left actions on a general $w\in\mathbb{C}\otimes\mathbb{S}$) the usual anticommutation relations:

$$\{A_j,A_k\}w=\{A_j^\dagger,A_k^\dagger\}w=0,\qquad \{A_j,A_k^\dagger\}w=\delta_{jk}w,\quad \forall w\in\mathbb{C}\otimes\mathbb{S}. \tag{7}$$

Subsequently we can proceed to construct two minimal left ideals (see [5] for details)

$$\begin{aligned} S_1^u =&\nu_e\omega_1\omega_1^\dagger+\\ &\overline{d^r}A_1^\dagger\omega_1\omega_1^\dagger+\overline{d^g}A_2^\dagger\omega_1\omega_1^\dagger+\overline{d^b}A_3^\dagger\omega_1\omega_1^\dagger+\\ &u^rA_3^\dagger A_2^\dagger\omega_1\omega_1^\dagger+u^gA_1^\dagger A_3^\dagger\omega_1\omega_1^\dagger+u^bA_2^\dagger A_1^\dagger\omega_1\omega_1^\dagger+\\ &e^+A_3^\dagger A_2^\dagger A_1^\dagger\omega_1\omega_1^\dagger, \end{aligned}\qquad \begin{aligned} S_1^d =&\overline{\nu}_e\omega_1^\dagger\omega_1+\\ &d^rA_1\omega_1^\dagger\omega_1+d^gA_2\omega_1^\dagger\omega_1+d^bA_3\omega_1^\dagger\omega_1+\\ &\overline{u^r}A_3A_2\omega_1^\dagger\omega_1+\overline{u^g}A_1A_3\omega_1^\dagger\omega_1+\overline{u^b}A_2A_1\omega_1^\dagger\omega_1+\\ &e^-A_3A_2A_1\omega_1^\dagger\omega_1, \end{aligned}$$

where $\omega_1:=A_1A_2A_3$ is nilpotent, but $\omega_1\omega_1^\dagger$ and $\omega_1^\dagger\omega_1$ are primitive idempotents.

These ideals are stable under the action of an $SU(3)_C$ symmetry, with the basis states of the ideals transforming as a single generation of leptons and quarks, thereby justifying the labeling of coefficients in the ideals. Explicitly, the $SU(3)_C$ generators are given by

$$\begin{aligned} \Lambda_1 &= -A_2^\dagger A_1 - A_1^\dagger A_2 & \Lambda_2 &= iA_2^\dagger A_1 - iA_1^\dagger A_2\\ \Lambda_3 &= A_2^\dagger A_2 - A_1^\dagger A & \Lambda_4 &= -A_1^\dagger A_3 - A_3^\dagger A_1\\ \Lambda_5 &= -iA_1^\dagger A_3 + iA_3^\dagger A_1 & \Lambda_6 &= -A_3^\dagger A_2 - A_2^\dagger A_3\\ \Lambda_7 &= iA_3^\dagger A_2 - iA_2^\dagger A_3 & \Lambda_8 &= -\frac{1}{\sqrt{3}}(A_1^\dagger A_1 + A_2^\dagger A_2 - 2A_3^\dagger A_3). \end{aligned}$$

5. Two additional generations from an S_3 automorphism

We now proceed to apply ψ to the ladder operators eqn. (6), and subsequently the basis states of the minimal left ideals, in order to generate two additional generations. It is readily checked that the S_3 automorphisms of $\mathbb{S}$ carry over the automorphisms of the left multiplication algebra $\mathbb{C}\ell(8)$, and so since the action of ψ on $e_1,\ldots,e_{15}$ as elements of $\mathbb{C}\ell(8)$ is known, we can via linearity establish the action of ψ on the ladder operators. This gives two additional sets of ladder operators:

$$B_j^\dagger = \psi(A_j^\dagger) = \frac{1}{2\sqrt{2}}(ae_j + iae_{j+4} + be_{j+8} + ibe_{j+12}) \tag{8}$$

$$B_j = \psi(A_j) = \frac{1}{2\sqrt{2}}(-ae_j + iae_{j+4} - be_{j+8} + ibe_{j+12}) \tag{9}$$

$$C_j^\dagger = \psi^2(A_j^\dagger) = \frac{1}{2\sqrt{2}}(be_j + ibe_{j+4} + ae_{j+8} + iae_{j+12}) \tag{10}$$

$$C_j = \psi^2(A_j) = \frac{1}{2\sqrt{2}}(-be_j + ibe_{j+4} - ae_{j+8} + iae_{j+12}) \tag{11}$$

where $i = 1, 2, 3$, and $a = (\sqrt{3} - 1)/2,\ b = (-\sqrt{3} - 1)/2$.

The two additional sets of ladder operators generated in this manner likewise constitute Witt bases for $\mathbb{C}\ell(6)$, satisfying the same anticommutation relations as $\{A_i^\dagger, A_i\}$, and we proceed to construct an additional pair of minimal left ideals of $\mathbb{C}\ell(6)$ for each of the two additional sets of ladder operators $\{B_i B_i^\dagger\}$ and $\{C_i, C_i^\dagger\}$. We interpret these additional ideals as representing the second and third generation of color states:

$$\begin{aligned} S_2^u =& \nu_\mu \omega_2 \omega_2^\dagger + \\ &\overline{c^r} B_1^\dagger \omega_2 \omega_2^\dagger + \overline{c^g} B_2^\dagger \omega_2 \omega_2^\dagger + \overline{c^b} B_3^\dagger \omega_2 \omega_2^\dagger + \\ &s^r B_3^\dagger B_2^\dagger \omega_2 \omega_2^\dagger + s^g B_1^\dagger B_3^\dagger \omega_2 \omega_2^\dagger + s^b B_2^\dagger B_1^\dagger \omega_2 \omega_2^\dagger + \\ &\mu^+ B_3^\dagger B_2^\dagger B_1^\dagger \omega_2 \omega_2^\dagger, \end{aligned}$$

$$\begin{aligned} S_2^d =& \overline{\nu}_\mu \omega_2^\dagger \omega_2 + \\ &c^r B_1 \omega_2^\dagger \omega_2 + c^g B_2 \omega_2^\dagger \omega_2 + c^b B_3 \omega_2^\dagger \omega_2 + \\ &\overline{s^r} B_3 B_2 \omega_2^\dagger \omega_2 + \overline{s^g} B_1 B_3 \omega_2^\dagger \omega_2 + \overline{s^b} B_2 B_1 \omega_2^\dagger \omega_2 + \\ &\mu^- B_3 B_2 B_1 \omega_2^\dagger \omega_2, \end{aligned}$$

$$\begin{aligned} S_3^u =& \nu_\tau \omega_3 \omega_3^\dagger + \\ &\overline{b^r} C_1^\dagger \omega_3 \omega_3^\dagger + \overline{b^g} C_2^\dagger \omega_3 \omega_3^\dagger + \overline{b^b} C_3^\dagger \omega_3 \omega_3^\dagger + \\ &t^r C_3^\dagger C_2^\dagger \omega_3 \omega_3^\dagger + t^g C_1^\dagger C_3^\dagger \omega_3 \omega_3^\dagger + t^b C_2^\dagger C_1^\dagger \omega_3 \omega_3^\dagger + \\ &\tau^+ C_3^\dagger C_2^\dagger C_1^\dagger \omega_3 \omega_3^\dagger, \end{aligned}$$

$$\begin{aligned} S_3^d =& \overline{\nu}_\tau \omega_3^\dagger \omega_3 + \\ &b^r C_1 \omega_3^\dagger \omega_3 + b^g C_2 \omega_3^\dagger \omega_3 + b^b C_3 \omega_3^\dagger \omega_3 + \\ &\overline{t^r} C_3 C_2 \omega_3^\dagger \omega_3 + \overline{t^g} C_1 C_3 \omega_3^\dagger \omega_3 + \overline{t^b} C_2 C_1 \omega_3^\dagger \omega_3 + \\ &\tau^- C_3 C_2 C_1 \omega_3^\dagger \omega_3, \end{aligned}$$

where $\omega_2 := B_1 B_2 B_3$ and $\omega_3 := C_1 C_2 C_3$. The automorphism ψ permutes between these three generations.

5.1. $SU(3)_C$ color symmetry of three generations

The ladder operators belonging to different generations do not satisfy the standard fermionic anticommutation relations. For example:

$$\{A_i, B_j\} w \neq 0, \quad \{B_i, C_j\} w \neq 0, \quad \{C_i, A_j\} w \neq 0, \quad \forall w \in \mathbb{C} \otimes \mathbb{S} \tag{12}$$

whenever $i \neq j$, and likewise for the remaining anticommutation relations. This is a result of the fact that the three sets of ladder operators are not linearly independent since

$$\psi^2(A_i^\dagger) + \psi(A_i^\dagger) + A_i^\dagger = 0. \tag{13}$$

It is nonetheless readily checked that

$$\{A_i, B_j + C_j\} w = 0, \quad \{A_i^\dagger, (B_j^\dagger + C_j^\dagger)\} w = 0, \quad \{A_i, (B_j^\dagger + C_j^\dagger)\} w = -\delta_{ij}, \quad \forall w \in \mathbb{C} \otimes \mathbb{S}. \tag{14}$$

Despite the inter-generation anticommutation relations in general being complicated expressions, it turns out that all three generations of states transform correctly under a single copy of $SU(3)$. This is because the $SU(3)_C$ generators defined in Section 4 are invariant under the action of S_3 automorphisms, as expected since $Aut(\mathbb{S}) = G_2 \times S_3$. For example:

$$\left[\Lambda_1, A_1\omega_1^\dagger\omega_1\right] = A_2\omega_1^\dagger\omega_1, \quad \left[\Lambda_1, B_1\omega_2^\dagger\omega_2\right] = B_2\omega_2^\dagger\omega_2, \quad \left[\Lambda_1, C_1\omega_3^\dagger\omega_3\right] = C_2\omega_3^\dagger\omega_3. \tag{15}$$

A single $SU(3)$ generator thus correctly transforms the equivalent states of each generation

$$\Lambda_1 : d^r \to d^g, \quad c^r \to c^g, \quad t^r \to t^g.$$

Despite having three generations of fermions, the sedenion model predicts only one generation of gauge bosons.

5.2. Linear dependence of minimal ideal states

The three pairs of minimal ideals used to represent three generations of fermions are not expected to be linearly independent, given the condition eqn. (13) above. Importantly however, the fact that only two of the three sets of ladder operators are linearly independent does not imply that only two pairs of minimal ideals are linearly independent.

We speculate that the linear dependence across the minimal ideals might provide an algebraic basis for incorporating quark mixing and/or neutrino oscillations, although a detailed investigation into the feasibility of this proposal is yet to be carried out. Nonetheless, some initial calculations (carried out using Mathematica) indicate that the (anti) down-type quarks across generations are linearly dependent. That is:

$$A_i\omega_1^\dagger\omega_1 + B_i\omega_2^\dagger\omega_2 + C_i\omega_3^\dagger\omega_3 = 0, \quad A_i^\dagger\omega_1\omega_1^\dagger + B_i^\dagger\omega_2\omega_2^\dagger + C_i^\dagger\omega_3\omega_3^\dagger = 0, \tag{16}$$

where $i = 1, 2, 3$. On the other hand, all 18 of the (anti) up-type quarks turn out to be linearly independent. This might be related to the fact that the Yukawa couplings, and subsequent mass matrix for either the up-type or down-type quarks can be taken to be diagonal, but not both. Subsequently, one can say without loss of generality that only the down-type quarks mix via the CKM matrix.

6. Discussion

We have reviewed a recent algebraic representation of three generations of leptons and quarks in terms of the Cayley-Dickson algebra of sedenions [37]. The internal symmetry of the fermions has so far been restricted to $SU(3)_C$, arising as a subgroup of the automorphism group of $\mathbb{S}$ that fixes a quaternionic structure.

This algebraic construction uses three $\mathbb{O}$ subalgebras of $\mathbb{S}$, with a common intersection isomorphic to $\mathbb{H}$, in order to construct a Witt basis for $\mathbb{C}\ell(6)$. Subsequently, two of the minimal left ideals of $\mathbb{C}\ell(6)$ are used to represent a single generation of fermions, in what is a direct generalization of the construction found in [5]. Finally, the S_3 automorphism of order three, which is an automorphism of $\mathbb{S}$ but not of $\mathbb{O}$, is then applied to the $\mathbb{C}\ell(6)$ Witt basis in order to obtain exactly two additional generations of color states.

In [4, 5], $\mathbb{C}\ell(6)$ arises as the left adjoint algebra of $\mathbb{C} \otimes \mathbb{O}$. Instead, in our approach $\mathbb{C}\ell(6)$ does not arise from the adjoint actions of a single octonion algebra, but rather from the adjoint actions of three intersecting octonion subalgebras (of $\mathbb{S}$) onto $\mathbb{S}$.

The model reviewed here is clearly far from complete as we have not considered the remaining internal symmetries nor the spacetime symmetries. Including additional internal symmetries into the sedenion model is currently work in progress. It is likewise currently being investigated if the inevitable linear dependence of some of the minimal ideal states across generations can provide an algebraic basis for the CKM or PMNS matrix.

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